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Computation of the action for on-shell improved lattice gauge theories at weak coupling

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Abstract:

The coefficients in Symanzik's improved lattice action for (pure) SU(2) and SU(3) gauge theories are determined to one-loop order by requiring the absence of leading scaling violations in a set of on-shell quantities, which arise in a world where 2 dimensions are compactified in a twisted manner.

* Heisenberg foundation fellow

1. Following Symanzik's analysis of scalar lattice field theories [1] , it has been suggested [2] that the leading ultra-violet cutoff effects in lattice gauge theories can be removed by adding a few next-to-nearest neighbour terms to Wilson's action [3] (for an introduction to the improvement programme see Ref. [4]). For SU(N) lattice gauge fields $U(x,\mu)$ ($\mu = 0,1,2,3$) living on a four-dimensional hypercubic lattice with sites x and spacing "a", the improved action reads [2,5]

$$(1)\quad S[U] = \frac{2}{g_0^2} \sum_{i=0}^{3} c_i(g_0^2) \sum_{\mathcal{C} \in \mathcal{S}_i} \mathcal{L}(\mathcal{C}),$$

where g_0 denotes the bare coupling constant and the $\mathcal{S}_i$'s are sets of closed elementary loops $\mathcal{C}$ on the lattice as described by Fig. 1. Furthermore, the weight $\mathcal{L}(\mathcal{C})$ is defined by

$$(2)\quad \mathcal{L}(\mathcal{C}) = \mathrm{Re}\,\mathrm{Tr}\,[1 - U(\mathcal{C})],$$

$U(\mathcal{C})$ being the ordered product of the link variables $U(x,\mu)$ along $\mathcal{C}$. Finally, the coefficients $c_i(g_0^2)$ satisfy the normalization condition

$$(3)\quad c_0(g_0^2) + 8c_1(g_0^2) + 8c_2(g_0^2) + 16c_3(g_0^2) = 1$$

and are to be chosen such as to cancel $O(a^2)$ scaling violation terms.

In this note we describe a method to determine the coefficients $c_i(g_0^2)$ in weak coupling perturbation theory and present the results of a one-loop calculation. As explained in Ref. [5], a host of conceptual and practical problems can be avoided by requiring the removal of $O(a^2)$ scaling violations from on-shell quantities only. Furthermore, in that case one can prove that one of the terms in the improved action is redundant, i.e. one is free to choose

(4) $c_3(g_0^2) = 0,$

for example. In view of these simplifying features, the scope of improvement is restricted to on-shell quantities here too and the convention (4) is adopted.

In perturbation theory we have

(5) $c_i(g_0^2) = c_i^{(0)} + g_0^2 c_i^{(1)} + O(g_0^4),$

where $c_i^{(0)}$ can be calculated from tree amplitudes and $c_i^{(1)}$ from one-loop diagrams. At both levels, one combination of the coefficients has already been determined by evaluating the heavy quark potential at physical distances and requiring the absence of $O(a^2)$ scaling violation terms [2,6]. To obtain a second relation, one is probably forced to consider an on-shell quantity, which is proportional to the 3- or 4-point vertex function such as the (connected) 4-quark potential. Because of the many diagrams contributing at one-loop order, this particular quantity does, however, not seem to be a favourable case. Gluon scattering matrix elements are simpler in this respect, but in the absence of an infrared cutoff it is a delicate matter to define the S-matrix properly beyond the tree level. Lacking a really convincing choice for the second quantity to be calculated, we were led to consider the exotic world to which we now turn.

2. The basic idea of our approach is to look for a situation, where gluons are massive and appear as asymptotic states. Scattering amplitudes can then be defined unambiguously so that a wealth of relatively simple on-shell quantities become available for the calculation of the coefficients c_i.

By a mechanism familiar from Kaluza-Klein theories, gluons become massive when 2 of the 4 space-time dimensions are compactified in a twisted way. Thus, we assume the lattice has a finite extent L *) in the x_1- and x_2-direction with twisted periodic boundary conditions for the gauge field [7]:

(6) $U(x + L\hat{\nu}, \mu) = \Omega_\nu \, U(x,\mu) \, \Omega_\nu^{-1} \qquad (\nu = 1,2)$

($\hat{\nu}$: unit vector in the positive ν-direction). The twist matrices Ω_ν are constant, gauge field independent elements of SU(N), which satisfy the algebra

(7) $\Omega_1 \Omega_2 = e^{i\frac{2\pi}{N}} \Omega_2 \Omega_1 .$

It is well-known (e.g. [8,9]) that eq. (7) fixes the Ω_ν's up to unitary transformations. Furthermore, they are irreducible, i.e. any matrix which commutes with Ω_1 and Ω_2 is proportional to the unit matrix. Using these properties it is easy to show that the only zero action fields are pure gauge configurations:

(8a) $U(x,\mu) = \Lambda(x)\Lambda(x + a\hat{\mu})^{-1}, \qquad \Lambda(x) \in SU(N),$

(8b) $\Lambda(x + L\hat{\nu}) = \Omega_\nu \Lambda(x) \Omega_\nu^{-1} \qquad (\nu = 1,2).$

The perturbative expansion of the functional integral can thus be performed straightforwardly by substituting

(9) $U(x,\mu) = \exp g_0 a A_\mu(x),$

*) L has physical units with L/a being an integer equal to the number of lattice sites in the transverse directions.

fixing the gauge and expanding all entries in powers of g_0.

For the physical interpretation of the resulting amplitudes, it is useful to go to the momentum representation. The Fourier transformation appropriate to the boundary conditions (6) reads

$$A_\mu(x) = (L^2 N)^{-1} \sum_{k_\perp} \int_{-\pi/a}^{\pi/a} \frac{dk_0}{2\pi} \frac{dk_3}{2\pi} e^{ik\cdot x} \Gamma_k e^{\frac{i}{2} k_\mu a} \tilde{A}_\mu(k), \tag{10}$$

where the transverse momentum components k_1, k_2 run over the discrete values

$$k_\nu = \frac{2\pi}{LN} n_\nu , \qquad n_\nu \in \mathbb{Z}, \qquad -\pi/a < k_\nu \leq \pi/a . \tag{11}$$

The N x N - matrices Γ_k play the rôle of group generators and are defined by *)

$$\Gamma_k = \Omega_1^{-n_2} \Omega_2^{n_1} e^{i\frac{\pi}{N}(n_1+n_2)(n_1+n_2-1)} \tag{12}$$

The basic property of these matrices is

$$\Omega_\nu \Gamma_k \Omega_\nu^{-1} = e^{ik_\nu L} \Gamma_k , \qquad (\nu = 1,2), \tag{13}$$

which insures that the Fourier representation (10) respects the boundary conditions (6) (the amplitude $\tilde{A}_\mu(k)$ is complex valued, not a matrix).

*) The phase factor is added to make Γ_k invariant under shifts of n_ν by multiples of N.

An important observation now is that $A_\mu(x)$ is traceless and that the U(1) generator Γ_0 therefore does not occur in the Fourier decomposition of $A_\mu(x)$. In other words, we have

$$\tilde{A}_\mu(k) = 0 \text{ if } n_1 (\text{mod } N) = n_2 (\text{mod } N) = 0, \tag{14}$$

in particular, there are no modes with $k_\perp = 0$. In the complex energy plane, the physical poles of the (free) gluon propagator are given by

$$k_0 = \pm i \sqrt{k_\perp^2 + k_3^2} + O(a^2), \tag{15}$$

and since

$$k_\perp^2 = m^2(n_1^2 + n_2^2) \geq m^2 , \quad m = \frac{2\pi}{LN} , \tag{16}$$

it follows that there is a mass gap, at least to lowest order perturbation theory. Moreover, for sufficiently small L (i.e. if $m \gg \Lambda_{\overline{MS}}$) asymptotic freedom implies a small effective gauge coupling and perturbation theory is therefore expected to yield a qualitatively correct and asymptotically precise description of the model at all energies, in particular, the mass gap found at lowest order persists for a range of couplings.

If we adopt the Kaluza-Klein point of view that the theory is basically 2-dimensional with 2 extra compact dimensions, the following physical picture results. At $g_0 = 0$ there is a tower of free particles, which are labelled by the quantum numbers n_1 and n_2 and a spin quantum number $s = \pm 1$ to account for the vector degrees of freedom of $\tilde{A}_\mu(k)$. The range of the n_ν's is restricted by eqs. (11),

(14) and the masses $M(n_1,n_2,s)$ of these particles are

$$(17) \quad M(n_1,n_2,s) = m\sqrt{n_1^2 + n_2^2} + O(a^2).$$

Thus, the lowest lying particles (A mesons) have a mass equal to m, the next to lowest lying (B mesons) equal to $\sqrt{2}m$ and all other particles have masses $M \geq 2m$. When the gauge coupling is turned on, most of the latter particles must be expected to become unstable. On the other hand, the A and B mesons remain stable for small coupling and their scattering matrix can be computed without difficulty.

We finally note that although we have fixed the gauge for perturbation theory, the particle states described above can be created from the vacuum state by applying gauge invariant operators, which are well localized in x_0, x_3 and extended in the transverse directions (Wilson loops winding around the world, for example). The A and B mesons are therefore truly physical excitations whose dynamics could in principle be studied by other means such as the Monte Carlo simulation method.

3. Because there are only two independent coefficients c_i, they can be determined by requiring the absence of $O(a^2)$ scaling violation terms in two selected on-shell quantities. As the first of these, we choose the mass *) m_A of an A meson with quantum numbers $n_1 = 0$ and $n_2 = 1$. The spin s must also be specified, because the degeneracy of particles with different spin only holds if $g_0 = 0$ and $a = 0$. Fortunately, the two spin states of the A meson can be distinguished by a conserved quantum number, namely the parity under a reflection of x_1 combined with a charge conjugation (only this combined operation leaves the boundary conditions invariant).

*) i.e. the energy (= eigenvalue of the transfer matrix [11]) of a one-particle state with momentum $k_3 = 0$.

Choosing positive parity, an appropriate interpolating field is simply $\tilde{A}_1(k)$ with $k = (k_0,0,m,0)$ and the state of the A meson is thus completely characterized in a way independent of the lattice spacing.

To lowest order, m_A is easily calculated by determining the location of the pole of the propagator of $\tilde{A}_1(k)$ in the complex energy plane (see Refs. [2,10] for more details). The result is

$$(18) \quad m_A^{(o)} = m\left\{1 - a^2m^2(c_1^{(o)} - c_2^{(o)} + \frac{1}{12}) + O(a^4)\right\},$$

where eq. (3) was used to eliminate c_0 in favour of c_1 and c_2. Improvement thus requires

$$(19) \quad c_1^{(o)} - c_2^{(o)} = -\frac{1}{12},$$

a relation, which has been derived before from the heavy quark potential [2].

The second quantity which we have chosen for the calculation of the coefficients c_i derives from the scattering amplitude T for the process

$$(20) \quad A(0,1) + A(0,-1) \rightarrow A(1,0) + A(-1,0)$$

($A(n_1,n_2)$ denotes an A meson with quantum numbers n_1, n_2 and positive parity as explained above). This transmutation of A mesons can happen either via a point interaction or by exchanging a B meson. In the centre of mass system, T is a function of the momentum $\pm k_3$ of the incoming mesons. B exchange then gives rise to a

pole of T in the complex k_3 plane with a residue, which may be identified with the square of a phenomenological coupling constant λ. Being proportional to the 3-point vertex function, λ is a much simpler quantity than the full scattering amplitude and we have therefore decided to calculate λ to obtain a second relation for the coefficients c_i.

The precise definition of λ is as follows. Let $\Gamma_3(k,\mu;p,\nu;q,\omega)$ denote the full propagator amputated 3-point function of $\tilde{A}_\mu$, $Z_A(k)$ and $Z_B(k)$ the residues of the poles of the A and B meson propagators, and

$$f(k,p,q) = \frac{1}{N} \mathrm{Tr}\{\Gamma_k [\Gamma_p, \Gamma_q]\} \tag{21}$$

the U(N) structure constants. Then, λ is defined through

$$\{Z_A(k) Z_A(p) Z_B(q)\}^{1/2} \sum_{j=1}^{2} e_j \Gamma_3(k,1;p,2;q,j) = i\lambda f(k,p,q), \tag{22}$$

where $e_1 = -e_2 = 1$ is the polarization of the exchanged B meson and all momenta k,p,q are on mass shell. Explicitly,

$$k = (iE_A(ir),0,m,ir), \tag{23a}$$

$$p = (-iE_A(ir),m,0,ir), \tag{23b}$$

$$q = (0,-m,-m,-2ir), \tag{23c}$$

with $E_A(k_3)$ being the energy of an (even parity) A meson with momentum k_3 and r the solution of $E_B(-2ir) = 0$. Without proof we note that the complex point (23) is far away from the singularities of Γ_3 and λ is thus completely well-defined.

At the tree level of perturbation theory, the calculation of λ just amounts to the evaluation of the 3-point vertex functions of Ref. [6] for the momentum configuration (23) and one finds

$$\lambda = g_0 \lambda^{(0)} + g_0^3 \lambda^{(1)} + O(g_0^5), \tag{24}$$

$$\lambda^{(0)} = -8m\{1 - \tfrac{1}{2} a^2 m^2 [9(c_1^{(0)} - c_2^{(0)} + \tfrac{1}{12}) + 2c_2^{(0)}] + O(a^4)\}. \tag{25}$$

Taking into account eq. (19), improvement of λ thus implies $c_2^{(0)} = 0$ so that altogether we have

$$c_0^{(0)} = \frac{5}{3}, \quad c_1^{(0)} = -\frac{1}{12}, \quad c_2^{(0)} = 0. \tag{26}$$

While these numbers have been known for some time, our calculation is the first one to demonstrate that they follow from the requirement of on-shell improvement alone [5].

4. At one-loop order there is a total number of about 20 diagrams contributing to m_A and λ. For each diagram the $O(a^2)$ term must be isolated and the coefficients $c_i^{(1)}$ are then adjusted such as to cancel these terms in $m_A^{(1)}$ and $\lambda^{(1)}$. Except for a few momentum integrals, most steps in this calculation could be done analytically, but because of the complexity of the tree level improved propagator and vertices [6] this would be an extremely tedious way to proceed. We therefore decided to follow a different strategy, which relies more heavily on the use of a computer and which has proved to be efficient and reliable. Here, the method is only sketched in a few words, the details of the basic techniques employed being postponed to a separate publication [10].

Apart from the diagrams involving $c_i^{(1)}$ (which are tree graphs and can be done analytically), the diagrams to be calculated essentially depend on N and L/a only. Thus, the contributions to (say) $m_A^{(1)}$ can be calculated numerically on a computer for fixed N and a range of L/a, the total $O(a^2)$ piece (excluding $c_i^{(1)}$ terms) being extracted after that by comparing the result with the expected asymptotic form *)

$$(27)\quad m_A^{(1)} / m \underset{a\to 0}{\sim} a_0 + (am)^2 a_1 + (am)^4 [a_2 + b_2 \ln(am)] + \ldots$$

Experience shows that eq. (27) provides an accurate fit to the data already for small L/a, and to determine a_0 and a_1 to 4 or more significant places, it is sufficient to calculate the diagrams for $10 \leq L/a \leq 36$ (N = 2) and $6 \leq L/a \leq 30$ (N = 3).

A difficulty in our numerical approach is that straightforward programming using general purpose integration subroutines yields slow and hence expensive programs. However, using computer made vertex subprograms and an adapted exponentially convergent integration method [10], we were able to reduce the total amount of computer time needed to 53 hours on an IBM 3081K. Another difficulty is that as we are computing just a few numbers, the probability that a programming or compiler error will not be detected is sizeable. We have therefore written two independent files of programs (one per author) for all diagrams using different integration methods and subroutines. In addition, quadratic and linear divergences must cancel in the sum of diagrams and the logarithmic divergence of $\lambda^{(1)}$ must match with the Callan-Symanzik β-function, thus providing a global test of our programs. A last technical point to be mentioned is that the significance loss in the course of

*) The existence of an asymptotic expansion of this kind can be shown rigorously (cp. [1,10]).

the calculation can be substantial, especially in the last step when one extracts the $O(a^2)$ terms from the data. The use of 64 bit (and occasionally 128 bit) arithmetic is therefore indispensable, along with a careful observation of the error propagation in the fit procedure.

The outcome of the calculation is displayed in Table 1. As expected, our new value for the combination $c_1^{(1)} - c_2^{(1)}$ matches perfectly with that obtained previously from the heavy quark potential [6]. The coefficients $c_i^{(1)}$ are reasonably small so that the one-loop correction of c_1, for example, is about 20 - 30 % of the tree level value for g_0^2 around 1 (i.e. in the range where Monte Carlo simulations are usually done). However, this should not lead to the conclusion that $O(a^2)$ scaling violation terms at one-loop order are small. In eq. (27), for example, one finds for N = 2

$$(28)\quad a_0 = -0.0168, \quad a_1 = -0.0110,$$

so that for the $O(a^2)$ term to be less than 10 % of the O(1) term, L/a must be greater than 8 (or, equivalently, $am \leq 0.4$). When the same analysis is applied to the B meson mass, the bound $am \leq 0.2$ is found, thus confirming the expectation that the cutoff affects the heavier particles more strongly than the lighter ones.

5. In order to make computer simulations a precise tool for the investigation of the low energy properties of gauge theories, it is necessary to understand the structure and magnitude of scaling violations. In particular, it would be important to clarify whether indeed the $O(a^2)$ effects in ordinary lattice gauge theories are organized as suggested by perturbation theory, i.e. such that they can be simultaneously removed by improving the action. Future Monte Carlo studies of improved lattice gauge theories, using our values for the coefficients c_i as a first approximation, should help to settle this question. In addition, the reduction of scaling violations achieved by improvement may turn out to be substantial so that for a given amount of computer time more accurate results could be obtained than would be possible with Wilson's action. Investigations of $O(a^2)$ effects are certainly expensive, not so much because the improved action is complicated, but because a set of on-shell quantities must be reliably calculated for a range of lattices *). Clearly, supercomputers are needed for that as well as improved techniques to extract glueball masses and other spectral quantities from the generated ensemble of gauge field configurations.

We finally remark that a twisted world including quarks exists [12] and that our methods can therefore be applied to calculate the improved quark action [13,14].

*) We remind the reader that our improved action does not imply any improved asymptotic scaling behaviour in the sense that the lattice β-function more rapidly approaches its universal two-loop form.

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	$c_0^{(1)}$	$c_1^{(1)}$	$c_2^{(1)}$	$\Lambda_L / \Lambda_{\overline{MS}}$
SU(2)	0.1352	-0.01396	-0.00295	0.208387
SU(3)	0.2370	-0.02521	-0.00441	0.183694

Table 1

Table caption

Table 1: One-loop improvement coefficients as determined in the present work. The Λ-parameter ratio is taken from Ref. [6]. The quoted digits are significant with a tolerance of at most one unit in the last figure.

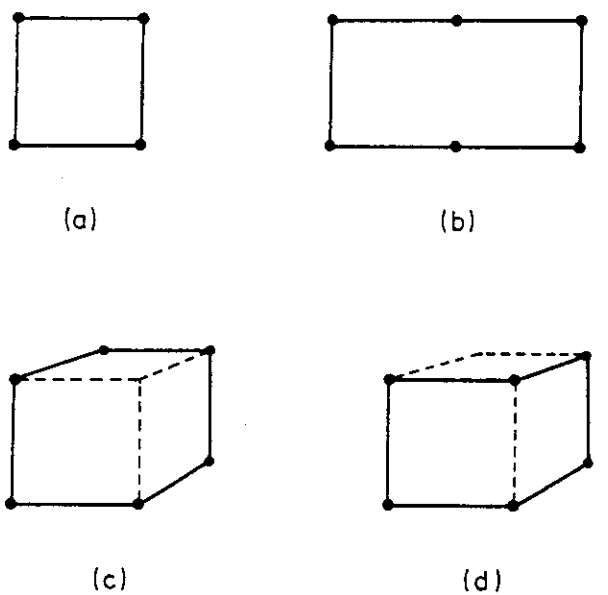

FIG.1

Figure caption

Fig. 1: Elementary loops on the lattice. The set $\mathcal{S}_0$ contains all plaquette loops (a), $\mathcal{S}_1$ all planar rectangles (b), $\mathcal{S}_2$ all parallelograms (c) and $\mathcal{S}_3$ all bent rectangles (d). Loops $\mathcal{C}$ that differ by orientation only are considered equal and are hence counted only once in eq. (1).