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MONOPOLE-FERMION AND DYON-FERMION BOUND STATES (V). WEAKLY BOUND STATES FOR THE MONOPOLE-FERMION SYSTEM

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ABSTRACT

We present explicit, approximate, remarkably precise results for the Kazama–Yang hamiltonian, which describes a Dirac monopole interacting with a spin-$\frac{1}{2}$ fermion that has an extra magnetic moment. The results are valid for bound states of angular momentum $j \geq Z|eg| + \frac{1}{2}$, where the radial wave functions are determined by four coupled differential equations. These equations have been solved analytically for $M - E \ll M$, which is a limit of considerable practical interest. Binding energies and wave functions are given.

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1. Introduction

An analysis of the monopole-fermion states of higher angular momenta is very complicated because one has to study four coupled differential equations. One way to attack this problem is through a Sturm-Liouville angle analysis as shown by Yang [1]; another is through asymptotic series expansions as discussed in paper I [2], where accurate numerical results were obtained for binding energies and wave functions, using a method that is valid for arbitrary parameters.

The monopole-fermion spectrum has bound states of zero energy [3] for all nonzero values of κ, the extra magnetic moment. Such states, for which the binding energy is equal to the fermion mass, exist for all angular momenta if $\kappa > 0$, and for all except the lowest angular momentum if $\kappa < 0$. Also, if $|\kappa|$ is not too small, additional states will exist. Most of these states will be very loosely bound. Unless $Z|eg\kappa|$ is fairly large, all excited states are in fact very loosely bound.

The present paper deals with that limit of weak binding, where explicit expressions for the binding energies can be obtained. These explicit results turn out to be highly accurate.

The hamiltonian we study is [3]

$$H = \vec{\alpha} \cdot (\vec{p} - Ze\vec{A}) + \beta M - \kappa q \beta \vec{\sigma} \cdot \vec{r}/(2Mr^3), \tag{1.1}$$

i.e., we study an infinitely heavy Dirac monopole of magnetic charge g interacting with a fermion that has an extra magnetic moment κ. In (1.1), M denotes the fermion mass, and

$$q = Zeg = Z(\pm\tfrac{1}{2}, \pm 1, \pm\tfrac{3}{2}, \dots), \tag{1.2}$$

with Ze the electric charge of the fermion.

Such loosely bound monopole-fermion states have been studied previously [3]-[5] for the case of lowest angular momentum, $j = |q| - \frac{1}{2}$. That case is much simpler than the present one, since there are then only two coupled ordinary differential equations.

2. Radial equations

For the bound-state eigensections, we make the following decomposition* [3]

$$\psi(\vec{r}) = \frac{1}{r}\begin{bmatrix} h_1(r)\xi^{(1)}_{jj_z} + \frac{q}{|q|}h_2(r)\xi^{(2)}_{jj_z} \\ -i\frac{\kappa}{|\kappa|}\left[h_3(r)\xi^{(2)}_{jj_z} + \frac{q}{|q|}h_4(r)\xi^{(1)}_{jj_z}\right]\end{bmatrix}, \tag{2.1}$$

where $j \geq |q| + \frac{1}{2}$, and $\xi^{(1)}_{jj_z}$ and $\xi^{(2)}_{jj_z}$ are eigensections of $\vec{J}^2$ and J_z, with eigenvalues $j(j+1)$ and j_z. Using lemma 1 of ref. [6], and changing the radial scale,

$$r = |\kappa q|\rho/(2M), \tag{2.2}$$

the eigenvalue equation

$$H\psi(\vec{r}) = E\psi(\vec{r}) \tag{2.3}$$

leads to the following set of four coupled radial equations,

$$\left(\frac{d}{d\rho} - \frac{\mu}{\rho}\right)h_1 + (A+B)h_3 + \frac{1}{\rho^2}h_4 = 0, \tag{2.4a}$$

$$\left(\frac{d}{d\rho} + \frac{\mu}{\rho}\right)h_2 + \frac{1}{\rho^2}h_3 + (A+B)h_4 = 0, \tag{2.4b}$$

$$(A-B)h_1 + \frac{1}{\rho^2}h_2 + \left(\frac{d}{d\rho} + \frac{\mu}{\rho}\right)h_3 = 0, \tag{2.4c}$$

$$\frac{1}{\rho^2}h_1 + (A-B)h_2 + \left(\frac{d}{d\rho} - \frac{\mu}{\rho}\right)h_4 = 0. \tag{2.4d}$$

In these equations, we have used the abbreviations [6]

$$\mu = [(j+\tfrac{1}{2})^2 - q^2]^{1/2}, \tag{2.5}$$

$$A = \tfrac{1}{2}\kappa|q|, \tag{2.6a}$$

$$B = \tfrac{1}{2}\kappa|q|\frac{E}{M}. \tag{2.6b}$$

* The present notation is that of paper I [2], which is slightly different from that of refs. [1], [3] and [6].

The boundary conditions imposed on h_i are [3]

$$\lim_{\rho\to 0} h_i(\rho) = \lim_{\rho\to\infty} h_i(\rho) = 0, \qquad i = 1,2,3,4. \tag{2.7}$$

In the limit of weak binding,

$$(M-E)/M \ll 1, \tag{2.8}$$

the radial equations (2.4) can be solved approximately as follows, at least when

$$A = O(1), \tag{2.9}$$

which will be assumed throughout this paper. In both of the two cases

$$\rho \ll |A-B|^{-1/2} \tag{2.10}$$

and

$$\rho \gg |A+B|^{-1/2}, \tag{2.11}$$

(2.4) can be approximated by equations that can be solved exactly. Clearly, when the condition (2.8) is fulfilled, these two regions (2.10) [the interior region] and (2.11) [the exterior region] overlap. Furthermore, when (2.9) holds, the exact solutions found in the two regions can be matched to determine the binding energy.

It turns out that the exterior region (2.11) is much easier to treat than the interior region (2.10). They are studied in sects. 3 and 4.

3. Exterior-region wave functions

In the exterior region,

$$\rho \gg |A+B|^{-1/2}, \tag{3.1}$$

the terms $\rho^{-2}h_4$ and $\rho^{-2}h_3$ may be neglected in eqs. (2.4a) and (2.4b). The equations to be solved are thus

$$\left(\frac{d}{d\rho} - \frac{\mu}{\rho}\right)h_1 + 2Ah_3 = 0, \tag{3.2a}$$

$$\left(\frac{d}{d\rho} + \frac{\mu}{\rho}\right)h_2 + 2Ah_4 = 0, \tag{3.2b}$$

$$\left(\frac{d}{d\rho}+\frac{\mu}{\rho}\right)h_3+(A-B)h_1+\frac{1}{\rho^2}h_2=0, \tag{3.2c}$$

$$\left(\frac{d}{d\rho}-\frac{\mu}{\rho}\right)h_4+\frac{1}{\rho^2}h_1+(A-B)h_2=0. \tag{3.2d}$$

We combine these to yield two coupled second-order equations for h_1 and h_2:

$$\left[\frac{1}{2A}\left(\frac{d}{d\rho}-\frac{\mu}{\rho}\right)\left(\frac{d}{d\rho}+\frac{\mu}{\rho}\right)-(A-B)\right]h_2-\frac{1}{\rho^2}h_1=0, \tag{3.3a}$$

$$\left[\frac{1}{2A}\left(\frac{d}{d\rho}+\frac{\mu}{\rho}\right)\left(\frac{d}{d\rho}-\frac{\mu}{\rho}\right)-(A-B)\right]h_1-\frac{1}{\rho^2}h_2=0. \tag{3.3b}$$

With

$$\mathcal{D}=\rho^2\left[\frac{d^2}{d\rho^2}-2A(A-B)\right], \tag{3.4}$$

the equations can be written as

$$[\mathcal{D}-\mu(\mu+1)]h_2-2Ah_1=0, \tag{3.5a}$$

$$[\mathcal{D}-\mu(\mu-1)]h_1-2Ah_2=0. \tag{3.5b}$$

Eliminating here h_1 or h_2, we find that the differential operator factorizes,

$$(\mathcal{D}-\bar{\alpha}_+)(\mathcal{D}-\bar{\alpha}_-)h_1=0,$$
$$(\mathcal{D}-\bar{\alpha}_+)(\mathcal{D}-\bar{\alpha}_-)h_2=0, \tag{3.6}$$

with

$$\bar{\alpha}_\pm=\mu^2\pm(\mu^2+4A^2)^{1/2}. \tag{3.7}$$

We shall work with h_1. Clearly, h_1 can be expressed in terms of the two solutions $h_1^{(+)}$ and $h_1^{(-)}$ which satisfy

$$(\mathcal{D}-\bar{\alpha}_+)h_1^{(+)}=0, \tag{3.8a}$$

$$(\mathcal{D}-\bar{\alpha}_-)h_1^{(-)}=0. \tag{3.8b}$$

Consider the first of these equations, (3.8a),

$$\left[\frac{d^2}{d\rho^2}-2A(A-B)-\frac{\bar{\alpha}_+}{\rho^2}\right]h_1^{(+)}=0. \tag{3.9}$$

The differential operator is a Bessel operator, and the solution $h_1^{(+)}$ that vanishes as $\rho\to\infty$ can be expressed in terms of a modified Bessel function. We thus find

$$h_1^{(-)}=\rho^{1/2}K_{\nu_-}\left(\sqrt{2A(A-B)}\,\rho\right),$$
$$h_1^{(+)}=\rho^{1/2}K_{\nu_+}\left(\sqrt{2A(A-B)}\,\rho\right), \tag{3.10}$$

where

$$\nu_\pm=\left[\tfrac{1}{4}+\mu^2\pm(\mu^2+4A^2)^{1/2}\right]^{1/2}. \tag{3.11}$$

The general solution h_1 that satisfies the boundary condition (2.7) as $\rho\to\infty$ is

$$h_1=\rho^{1/2}\left[N_-K_{\nu_-}\left(\sqrt{2A(A-B)}\,\rho\right)+N_+K_{\nu_+}\left(\sqrt{2A(A-B)}\,\rho\right)\right], \tag{3.12a}$$

where N_- and N_+ are two constants to be determined by the matching with the interior region, and by the normalization.

The other radial wave functions can be determined from the differential equations (3.2). We find

$$h_2=\frac{1}{2A}\rho^{1/2}\Big\{N_-[\mu-(\mu^2+4A^2)^{1/2}]K_{\nu_-}\left(\sqrt{2A(A-B)}\,\rho\right)$$
$$+N_+[\mu+(\mu^2+4A^2)^{1/2}]K_{\nu_+}\left(\sqrt{2A(A-B)}\,\rho\right)\Big\}, \tag{3.12b}$$

$$h_3=\frac{1}{2A}\rho^{-1/2}\Big\{\left(\mu-\tfrac{1}{2}\right)\left[N_-K_{\nu_-}\left(\sqrt{2A(A-B)}\,\rho\right)+N_+K_{\nu_+}\left(\sqrt{2A(A-B)}\,\rho\right)\right]$$
$$-[2A(A-B)]^{1/2}\rho\left[N_-K'_{\nu_-}\left(\sqrt{2A(A-B)}\,\rho\right)+N_+K'_{\nu_+}\left(\sqrt{2A(A-B)}\,\rho\right)\right]\Big\}, \tag{3.12c}$$

$$h_4=-\frac{1}{4A^2}\rho^{-1/2}\Big[\left(\mu+\tfrac{1}{2}\right)\Big\{N_-[\mu-(\mu^2+4A^2)^{1/2}]K_{\nu_-}\left(\sqrt{2A(A-B)}\,\rho\right)$$
$$+N_+[\mu+(\mu^2+4A^2)^{1/2}]K_{\nu_+}\left(\sqrt{2A(A-B)}\,\rho\right)\Big\}$$
$$+[2A(A-B)]^{1/2}\rho\Big\{N_-[\mu-(\mu^2+4A^2)^{1/2}]K'_{\nu_-}\left(\sqrt{2A(A-B)}\,\rho\right)$$
$$+N_+[\mu+(\mu^2+4A^2)^{1/2}]K'_{\nu_+}\left(\sqrt{2A(A-B)}\,\rho\right)\Big\}\Big], \tag{3.12d}$$

where primes denote differentiation with respect to the argument.

4. Interior-region wave functions

In the interior region,

$$\rho \ll |A-B|^{-1/2}, \tag{4.1}$$

the terms $(A-B)h_1$ and $(A-B)h_2$ may be neglected in eqs. (2.4c) and (2.4d). It is furthermore convenient to introduce the variable

$$x = \frac{1}{\rho}. \tag{4.2}$$

Equations (2.4) then reduce to

$$\left(\frac{d}{dx}+\frac{\mu}{x}\right)h_1 - \frac{2A}{x^2}h_3 - h_4 = 0, \tag{4.3a}$$

$$\left(\frac{d}{dx}-\frac{\mu}{x}\right)h_2 - h_3 - \frac{2A}{x^2}h_4 = 0, \tag{4.3b}$$

$$h_2 = \left(\frac{d}{dx}-\frac{\mu}{x}\right)h_3, \tag{4.3c}$$

$$h_1 = \left(\frac{d}{dx}+\frac{\mu}{x}\right)h_4. \tag{4.3d}$$

These can be combined to give two coupled second-order equations,

$$\left[\left(\frac{d}{dx}+\frac{\mu}{x}\right)^2 - 1\right]h_4 - \frac{2A}{x^2}h_3 = 0, \tag{4.4a}$$

$$\left[\left(\frac{d}{dx}-\frac{\mu}{x}\right)^2 - 1\right]h_3 - \frac{2A}{x^2}h_4 = 0. \tag{4.4b}$$

Eliminating also h_3, we obtain a fourth-order ordinary differential equation for h_4,

$$\Big\{x^4\frac{d^4}{dx^4} + 4x^3\frac{d^3}{dx^3} - 2x^2(x^2+\mu^2-1)\frac{d^2}{dx^2} - 4x^3\frac{d}{dx} + x^4 - 2(\mu-1)^2x^2 + [\mu^2(\mu^2-1) - 4A^2]\Big\}h_4 = 0. \tag{4.5}$$

This is the equation we have to solve, subject to the boundary condition,

$$h_4 \to 0 \quad \text{as} \quad x\to\infty \quad (\text{or } \rho \to 0). \tag{4.6}$$

It turns out that the solution to (4.5) can be expressed as a Fourier–Bessel transform of a solution to a second-order equation. We find that we can rearrange (4.5) so that a factor of x^2 can be cancelled, enabling us to write the equation in terms of a Bessel operator and quadratic powers of x. The method has previously been used in studying the Corben–Schwinger problem [7] of the scattering of a charged vector meson [8]. Appendix A of ref. [8] also contains discussions of some of the more subtle points of the Fourier–Bessel transform.

Let

$$f = x^{-1/2}h_4, \tag{4.7}$$

then the substitution into (4.5) gives the differential equation for f:

$$\Big\{\Big[\Big(x\frac{d}{dx}\Big)^2 - \big(\tfrac14+\mu^2\big)\Big]^2 - (\mu^2+4A^2) - 2x\Big[\Big(x\frac{d}{dx}\Big)^2 - \tfrac14 + (\mu-1)^2\Big]x + x^4\Big\}f = 0. \tag{4.8}$$

The point here is that $x\,(d/dx)$ appears only in the form of a square. It is therefore natural to introduce the Bessel operator

$$T_\nu = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{\nu^2}{x^2}, \tag{4.9}$$

where the constant ν is chosen so that the constant term in (4.8) is cancelled:

$$[\nu^2 - (\tfrac14+\mu^2)]^2 - (\mu^2+4A^2) = 0. \tag{4.10}$$

Therefore,

$$\nu^2 = \tfrac14 + \mu^2 \pm (\mu^2+4A^2)^{1/2}. \tag{4.11}$$

Comparing with (3.11), we see that

$$\nu = \pm\nu_- \quad \text{or} \quad \nu = \pm\nu_+. \tag{4.12}$$

It will turn out that all four values of ν are needed, and that

$$\nu_-^2 < 0 \tag{4.13}$$

when bound states exist.

Since

$$2x T_\nu x = T_\nu x^2 + x^2 T_\nu - 2, \tag{4.14}$$

eq. (4.8) can be written in the form

$$\begin{aligned}\{T_\nu x^2 T_\nu - [x^2 + 2\mu^2 - 2(\nu^2 - \tfrac{1}{4})] T_\nu - T_\nu x^2 + x^2 \\ - 2(\mu^2 - 2\mu + \nu^2 - \tfrac{1}{4})\} f = 0.\end{aligned} \tag{4.15}$$

This holds when ν is given by (4.12).

Since T_ν is a Bessel operator, we have

$$T_\nu Z_\nu(xy) = -y^2 Z_\nu(xy), \tag{4.16}$$

with Z_ν a cylindrical Bessel function of order $\pm\nu$. Because (4.15) is linear in x^2, after a Fourier–Bessel transform we are left with a differential equation that is linear in the Bessel operator, i.e., *of second order*.

We make the Ansatz

$$f(x) = \int_C dy\, Z_\nu(xy) g(y), \tag{4.17}$$

where C is a contour to be specified. Then (4.15) is satisfied provided g satisfies

$$\begin{aligned}\left\{\frac{d^2}{dz^2} + \frac{1}{z}\frac{d}{dz} - \frac{2}{1-z}\frac{d}{dz} - \frac{\nu^2}{4z^2} - \frac{1}{z(1-z)} - \frac{\mu^2 - \nu^2 + \frac{1}{4}}{2(1-z)^2}\right. \\ \left. - \frac{\mu^2 - 2\mu + \nu^2 - \frac{1}{4}}{2z(1-z)^2}\right\} g = 0,\end{aligned} \tag{4.18}$$

with

$$z = -y^2, \tag{4.19}$$

and with the contour C chosen such that boundary terms vanish. After appropriate factors are extracted, we recognize (4.18) to be the hypergeometric equation. The final solution is

$$f(x) = \int_C dy\, y^{1+\nu} Z_\nu(xy)(1+y^2)^p\, u(-y^2), \tag{4.20}$$

where $u(z)$ is a solution of the hypergeometric equation. We shall be using the following solutions [9]:

$$\begin{aligned}u_1(z) &= F(a, b; c; z), \\ u_2(z) &= F(a, b; a+b+1-c; 1-z), \\ u_5(z) &= z^{1-c} F(a+1-c, b+1-c; 2-c; z), \\ u_6(z) &= (1-z)^{c-a-b} F(c-a, c-b; c+1-a-b; 1-z),\end{aligned} \tag{4.21}$$

where F is the hypergeometric function. Of these four u's, of course, only two can be linearly independent. In (4.20) and (4.21),

$$p = \mu - 1 \quad \text{or} \quad p = -\mu, \tag{4.22}$$

and

$$\begin{aligned}\left.\begin{matrix} a \\ b \end{matrix}\right\} &= 1 + p + \tfrac{1}{2}(\nu \pm \bar{\nu}), \\ c &= 1 + \nu.\end{aligned} \tag{4.23}$$

Here, ν and $\bar{\nu}$ are used generically,

$$\nu^2 = \nu_+^2 \quad \text{and} \quad \bar{\nu}^2 = \nu_-^2, \tag{4.24a}$$

or

$$\nu^2 = \nu_-^2 \quad \text{and} \quad \bar{\nu}^2 = \nu_+^2, \tag{4.24b}$$

whereas ν_+ and ν_- refer to the specific values defined in (3.11).

The fourth-order differential equation (4.5) has four linearly independent solutions. It is readily seen from the leading terms in (4.5) that, apart from powers, two of those are exponentially small, whereas two are exponentially large, as $x \to \infty$. There are thus two linearly independent solutions h_4 that satisfy the boundary condition (4.6).

It is clear that a large number of different solutions $f(x)$ can be written down. First of all, we may choose different Bessel functions, and different hypergeometric functions. Further, we have two choices for p, four choices for $(\nu, \bar{\nu})$, and finally, there are several possible contours. In appendix A we give a survey of these different solutions, some of which can be related using identities of the Bessel or hypergeometric functions. It turns out that this set contains *all* the four linearly independent solutions.

Two convenient bases for the functions that vanish as $x \to \infty$ are $(f^{(1)}, f^{(5)})$ and $(f^{(2)}, f^{(6)})$, where

$$f^{(j)}(x) = \int_{C_1} dy\, y^{\nu} H_{\bar{\nu}}^{(1)}(xy)(1+y^2)^{-\mu} u_j(-y^2), \tag{4.25}$$

and u_j is one of the functions (4.21). (These are referred to as solutions of Class I in appendix A, and denoted by $f_{\rm I}^{(j)}(x)$ there.) The contour C_1 is shown in fig. 1. The combination of the contour C_1 and the Hankel function $H_{\bar{\nu}}^{(1)}(xy)$ makes all boundary terms vanish; $f^{(j)}(x)$ thus satisfies the differential equation. Further, as shown in appendix B, these solutions are well-behaved (i.e., decreasing) for $x \to \infty$, e.g.,

$$\left.\begin{matrix} f^{(2)}(x) \\ f^{(6)}(x) \end{matrix}\right\} \underset{x\to\infty}{\sim} \left(\frac{2}{\pi x}\right)^{1/2} \sin(\pi\mu) e^{-x} \left\{\begin{matrix} \Gamma(1-\mu)(2/x)^{-\mu+1} \\ \Gamma(\mu)(2/x)^{\mu} \end{matrix}\right. . \tag{4.26}$$

In order to match the interior-region wave function with the exterior-region wave function of sect. 3, we need to understand both the large-x and small-x behaviour of our solutions. In appendix C we present convergent series expansions that satisfy the differential equation (4.15), and in appendix D we relate those series to the functions (4.25) that are well-behaved at large x.

For the purpose of determining binding energies, we have chosen to work with the basis $f^{(1)}(x)$ and $f^{(5)}(x)$, whose behaviours for small values of x are given by eqs. (E.5) and (E.6), respectively. Thus, in the interior region,

$$h_4 = x^{1/2}[N_1 f^{(1)}(x) + N_5 f^{(5)}(x)], \tag{4.27}$$

where N_1 and N_5 are two constants to be determined from the matching with the exterior region and the normalization. The other radial wave functions can be determined from the differential equations (4.3). With the abbreviation

$$F(x) \equiv N_1 f^{(1)}(x) + N_5 f^{(5)}(x), \tag{4.28}$$

the interior-region wave functions can be written as

$$h_1 = x^{-1/2}[(\mu + \tfrac{1}{2})F(x) + xF'(x)], \tag{4.29a}$$

$$\begin{aligned} h_2 = \frac{1}{2A} x^{-1/2} \{ & [-(\mu - \tfrac{1}{2})^2(\mu + \tfrac{1}{2}) + (\mu - \tfrac{5}{2})x^2] F(x) \\ & + [(-\mu^2 + 2\mu + \tfrac{5}{4})x - x^3] F'(x) + (\mu + \tfrac{7}{2}) x^2 F''(x) + x^3 F'''(x) \}, \end{aligned} \tag{4.29b}$$

$$h_3 = \frac{1}{2A} x^{1/2} \{ [(\mu^2 - \tfrac{1}{4}) - x^2] F(x) + (2\mu + 1) x F'(x) + x^2 F''(x) \}, \tag{4.29c}$$

$$h_4 = x^{1/2} F(x), \tag{4.29d}$$

where primes denote differentiation with respect to x, and $x = 1/\rho$.

A summary of the properties of the functions $f^{(1)}(x)$ and $f^{(5)}(x)$ is given in appendix E.

5. Matching and eigenvalue condition

The interior- and exterior-region wave functions given by eqs. (4.29) and (3.12), respectively, have to match for

$$|A + B|^{-1/2} \ll \rho \ll |A - B|^{-1/2}. \tag{5.1}$$

The approximations that we have made in solving the differential equations are consistent in this region of overlap. Therefore, it is sufficient to match one of the four radial functions. We shall consider h_4. From eqs. (3.12d) and (4.29d) we then get the condition

$$\rho^{-1/2}\Big[N_1 f^{(1)}\Big(\frac{1}{\rho}\Big) + N_5 f^{(5)}\Big(\frac{1}{\rho}\Big)\Big]$$

$$\begin{aligned}
&= -\frac{1}{4A^2}\rho^{-1/2}\big[(\mu+\tfrac{1}{2})\{N_-[\mu-(\mu^2+4A^2)^{1/2}]K_{\nu_-}(\sqrt{2A(A-B)}\,\rho) \\
&\qquad + N_+[\mu+(\mu^2+4A^2)^{1/2}]K_{\nu_+}(\sqrt{2A(A-B)}\,\rho)\} \\
&\quad + [2A(A-B)]^{1/2}\rho\{N_-[\mu-(\mu^2+4A^2)^{1/2}]K'_{\nu_-}(\sqrt{2A(A-B)}\,\rho) \\
&\qquad + N_+[\mu+(\mu^2+4A^2)^{1/2}]K'_{\nu_+}(\sqrt{2A(A-B)}\,\rho)\}\big].
\end{aligned} \tag{5.2}$$

In the region (5.1) of overlap, the functions appearing in eq. (5.2) can be expanded in power series. For $\rho \gg 1$, the functions $f^{(1)}(1/\rho)$ and $f^{(5)}(1/\rho)$ can be expanded as given in appendices D and E. We have from (E.5) and (E.6) that

$$\rho^{1/2}[\text{L.H.S. of } (5.2)]$$

$$\begin{aligned}
\underset{\rho\gg 1}{\simeq} N_1\Big[&\Big(\frac{1}{2\rho}\Big)^{\nu_+}\frac{\sin(\pi\mu)}{\pi}\Gamma(-\alpha_-)\Gamma(-\alpha_+) \\
&-\Big(\frac{1}{2\rho}\Big)^{\nu_-}\frac{\sin(\pi\alpha_+)}{\pi}\frac{\Gamma(-\alpha_+)\Gamma(\alpha_-)\Gamma(1+\nu_+)\Gamma(-\nu_-)}{\Gamma(1-\mu+\alpha_-)\Gamma(\mu+\alpha_-)} \\
&-\Big(\frac{1}{2\rho}\Big)^{-\nu_-}\frac{\sin(\pi\alpha_-)}{\pi}\frac{\Gamma(-\alpha_-)\Gamma(\alpha_+)\Gamma(1+\nu_+)\Gamma(\nu_-)}{\Gamma(1-\mu+\alpha_+)\Gamma(\mu+\alpha_+)}\Big] \\
+N_5\Big[&\Big(\frac{1}{2\rho}\Big)^{-\nu_+}\frac{\sin(\pi\mu)}{\pi}\Gamma(\alpha_+)\Gamma(\alpha_-) \\
&+\Big(\frac{1}{2\rho}\Big)^{\nu_-}\frac{\sin(\pi\alpha_-)}{\pi}\frac{\Gamma(\alpha_-)\Gamma(-\alpha_+)\Gamma(1-\nu_+)\Gamma(-\nu_-)}{\Gamma(1-\mu-\alpha_+)\Gamma(\mu-\alpha_+)} \\
&+\Big(\frac{1}{2\rho}\Big)^{-\nu_-}\frac{\sin(\pi\alpha_+)}{\pi}\frac{\Gamma(\alpha_+)\Gamma(-\alpha_-)\Gamma(1-\nu_+)\Gamma(\nu_-)}{\Gamma(1-\mu-\alpha_-)\Gamma(\mu-\alpha_-)}\Big],
\end{aligned} \tag{5.3}$$

where we have made use of the abbreviations [compare eq. (E.1)]

$$\alpha_\pm = \tfrac{1}{2}(\nu_+ \pm \nu_-). \tag{5.4}$$

For the modified Bessel functions we have [9]

$$K_\nu(z) \simeq \frac{\pi}{2\sin(\pi\nu)}\left[\frac{(\frac{1}{2}z)^{-\nu}}{\Gamma(1-\nu)} - \frac{(\frac{1}{2}z)^{\nu}}{\Gamma(1+\nu)}\right], \tag{5.5}$$

valid for $|z| \ll 1$. Therefore,

$$\begin{aligned}
\rho^{1/2}[\text{R.H.S. of } (5.2)] \underset{\sqrt{2A(A-B)}\,\rho\ll 1}{\simeq} & -\frac{1}{4A^2}\Big\{N_-[\mu-(\mu^2+4A^2)^{1/2}]\frac{\pi}{2\sin(\pi\nu_-)} \\
&\cdot\Big[\frac{\mu+\frac{1}{2}-\nu_-}{\Gamma(1-\nu_-)}(\tfrac{1}{2}\sqrt{2A(A-B)}\,\rho)^{-\nu_-} - \frac{\mu+\frac{1}{2}+\nu_-}{\Gamma(1+\nu_-)}(\tfrac{1}{2}\sqrt{2A(A-B)}\,\rho)^{\nu_-}\Big] \\
&+ N_+[\mu+(\mu^2+4A^2)^{1/2}]\frac{\pi}{2\sin(\pi\nu_+)} \\
&\cdot\Big[\frac{\mu+\frac{1}{2}-\nu_+}{\Gamma(1-\nu_+)}(\tfrac{1}{2}\sqrt{2A(A-B)}\,\rho)^{-\nu_+} - \frac{\mu+\frac{1}{2}+\nu_+}{\Gamma(1+\nu_+)}(\tfrac{1}{2}\sqrt{2A(A-B)}\,\rho)^{\nu_+}\Big]\Big\}.
\end{aligned} \tag{5.6}$$

Comparing now the coefficients of the four different powers of ρ, we get equations that have the structure,

$$\rho^{-\nu_+}: \qquad N_+ = W_1 N_1, \tag{5.7a}$$

$$\rho^{\nu_+}: \qquad N_+ = W_2 N_5, \tag{5.7b}$$

$$\rho^{-\nu_-}: \qquad N_- = W_3 N_1 + W_4 N_5, \tag{5.7c}$$

$$\rho^{\nu_-}: \qquad N_- = W_5 N_1 + W_6 N_5. \tag{5.7d}$$

The coefficients $W_1, \ldots, W_6$ are given by

$$\begin{aligned}
W_1 = -8A^2[\mu+(\mu^2+4A^2)^{1/2}]^{-1}(\mu+\tfrac{1}{2}-\nu_+)^{-1}\frac{\sin(\pi\mu)}{\pi} \\
\cdot\frac{\Gamma(-\alpha_-)\Gamma(-\alpha_+)}{\Gamma(\nu_+)}(\tfrac{1}{4}\sqrt{2A(A-B)})^{\nu_+},
\end{aligned} \tag{5.8a}$$

$$\begin{aligned}
W_2 = -8A^2[\mu+(\mu^2+4A^2)^{1/2}]^{-1}(\mu+\tfrac{1}{2}+\nu_+)^{-1}\frac{\sin(\pi\mu)}{\pi} \\
\cdot\frac{\Gamma(\alpha_+)\Gamma(\alpha_-)}{\Gamma(-\nu_+)}(\tfrac{1}{4}\sqrt{2A(A-B)})^{-\nu_+},
\end{aligned} \tag{5.8b}$$

$$\begin{aligned}
W_3 = 8A^2[\mu-(\mu^2+4A^2)^{1/2}]^{-1}(\mu+\tfrac{1}{2}-\nu_-)^{-1}\frac{\sin(\pi\alpha_+)}{\pi} \\
\cdot\frac{\Gamma(-\alpha_+)\Gamma(\alpha_-)\Gamma(1+\nu_+)\Gamma(-\nu_-)}{\Gamma(1-\mu+\alpha_-)\Gamma(\mu+\alpha_-)\Gamma(\nu_-)}(\tfrac{1}{4}\sqrt{2A(A-B)})^{\nu_-},
\end{aligned} \tag{5.9a}$$

$$W_4 = -8A^2[\mu - (\mu^2 + 4A^2)^{1/2}]^{-1}(\mu + \tfrac{1}{2} - \nu_-)^{-1} \frac{\sin(\pi\alpha_-)}{\pi}$$
$$\cdot \frac{\Gamma(\alpha_-)\Gamma(-\alpha_+)\Gamma(1-\nu_+)\Gamma(-\nu_-)}{\Gamma(1-\mu-\alpha_+)\Gamma(\mu-\alpha_+)\Gamma(\nu_-)} \left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{\nu_-}, \tag{5.9b}$$

$$W_5 = 8A^2[\mu - (\mu^2 + 4A^2)^{1/2}]^{-1}(\mu + \tfrac{1}{2} + \nu_-)^{-1} \frac{\sin(\pi\alpha_-)}{\pi}$$
$$\cdot \frac{\Gamma(-\alpha_-)\Gamma(\alpha_+)\Gamma(1+\nu_+)\Gamma(\nu_-)}{\Gamma(1-\mu+\alpha_+)\Gamma(\mu+\alpha_+)\Gamma(-\nu_-)} \left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{-\nu_-}, \tag{5.10a}$$

$$W_6 = -8A^2[\mu - (\mu^2 + 4A^2)^{1/2}]^{-1}(\mu + \tfrac{1}{2} + \nu_-)^{-1} \frac{\sin(\pi\alpha_+)}{\pi}$$
$$\cdot \frac{\Gamma(\alpha_+)\Gamma(-\alpha_-)\Gamma(1-\nu_+)\Gamma(\nu_-)}{\Gamma(1-\mu-\alpha_-)\Gamma(\mu-\alpha_-)\Gamma(-\nu_-)} \left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{-\nu_-}. \tag{5.10b}$$

These coefficients have the following symmetry properties, as also follows from the properties of $f^{(1)}$, $f^{(5)}$, and K_ν,

$$W_2 = W_1(\nu_+ \to -\nu_+),$$
$$W_4 = W_3(\nu_+ \to -\nu_+), \qquad W_6 = W_5(\nu_+ \to -\nu_+),$$
$$W_5 = W_3(\nu_- \to -\nu_-), \qquad W_6 = W_4(\nu_- \to -\nu_-). \tag{5.11}$$

For the purpose of determining the binding energy, we do not need the constants N_1, N_5, N_-, and N_+. Therefore, we eliminate these from eqs. (5.7), and find the following condition on the W's:

$$W_2(W_3 - W_5) = W_1(W_6 - W_4). \tag{5.12}$$

This is the equation to be solved for the eigenvalue B.

6. Binding energies

The binding energies are given implicitly by eq. (5.12), which is to be solved for B. Inserting for the coefficients $W_1, \ldots, W_6$ of eqs. (5.8)–(5.10), the equation becomes rather involved. It has the following structure,

$$X_1\left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{-\nu_+ + \nu_-} + X_2\left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{-\nu_+ - \nu_-}$$
$$= X_3\left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{\nu_+ - \nu_-} + X_4\left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{\nu_+ + \nu_-}. \tag{6.1}$$

If we inserted the actual values for $X_1, \ldots, X_4$, we would see that (6.1) is symmetric under $\nu_+ \leftrightarrow \nu_-$. That this has to be the case is also clear from the symmetries of the K_{ν_-} and K_{ν_+} in the exterior region and the $f_{\pm\nu_-}$ and $f_{\pm\nu_+}$ in the interior region [see (C.1) and (D.1); an interchange $\nu_+ \leftrightarrow \nu_-$ would merely interchange them].

It was found by Yang [1] that excited states of non-minimal angular momentum only exist for

$$|A| > A_0 = \tfrac{1}{2}(\mu^2 - \tfrac{1}{4}). \tag{6.2}$$

This condition is equivalent to stating that

$$\nu_- \equiv i\beta \quad \text{is imaginary.} \tag{6.3}$$

On the other hand, ν_+ is always real,

$$\nu_+ > (\tfrac{1}{2} + 2\mu^2)^{1/2}, \tag{6.4}$$

c.f. eq. (3.11). That the condition (6.2) has to be satisfied can also be seen from the present analysis: Without the condition (6.3), the radial wave function would not be oscillatory, and the eigenvalue B (or the energy) would not be real.

We shall make use of (6.3) and (6.4) to simplify (6.1). Having made the assumption that the binding is weak,

$$\epsilon \equiv \frac{A-B}{A} \ll 1, \tag{6.5}$$

we can neglect the two terms proportional to $\left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{\nu_+}$ in eq. (6.1), and are left with $W_3 = W_5$, or

$$(\mu + \tfrac{1}{2} - \nu_-)^{-1} \frac{\sin(\pi\alpha_+)}{\pi} \left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{\nu_-} \frac{\Gamma(-\alpha_+)\Gamma(\alpha_-)\Gamma(1+\nu_+)\Gamma(-\nu_-)}{\Gamma(1-\mu+\alpha_-)\Gamma(\mu+\alpha_-)\Gamma(\nu_-)}$$
$$= (\mu + \tfrac{1}{2} + \nu_-)^{-1} \frac{\sin(\pi\alpha_-)}{\pi} \left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{-\nu_-} \frac{\Gamma(-\alpha_-)\Gamma(\alpha_+)\Gamma(1+\nu_+)\Gamma(\nu_-)}{\Gamma(1-\mu+\alpha_+)\Gamma(\mu+\alpha_+)\Gamma(-\nu_-)}. \tag{6.6}$$

We rewrite this as

$$\left(\tfrac{1}{4}\sqrt{2A(A-B)}\right)^{2\nu_-} = \frac{\mu + \frac{1}{2} - \nu_-}{\mu + \frac{1}{2} + \nu_-} \, \frac{\alpha_-}{\alpha_+} \, \frac{\Gamma(1-\mu+\alpha_-)\Gamma(\mu+\alpha_-)}{\Gamma(1-\mu+\alpha_+)\Gamma(\mu+\alpha_+)}$$
$$\cdot \left[\frac{\Gamma(\alpha_+)}{\Gamma(\alpha_-)}\right]^2 \left[\frac{\Gamma(\nu_-)}{\Gamma(-\nu_-)}\right]^2. \tag{6.7}$$

It is now convenient to make use of eq. (6.3), and define phase angles ψ and ϕ as follows:

$$e^{-2i\psi} = \frac{\mu + \frac{1}{2} - i\beta}{\mu + \frac{1}{2} + i\beta}, \tag{6.8}$$

$$e^{2i\phi} = \frac{\Gamma(1 + i\beta)}{\Gamma(1 - i\beta)}. \tag{6.9}$$

Thus,

$$\left[\frac{\Gamma(\nu_-)}{\Gamma(-\nu_-)}\right]^2 = e^{4i\phi}. \tag{6.10}$$

The above angles ψ and ϕ are generalizations of those introduced in ref. [4]. As $\mu \to 0$, with A positive, they reduce to those given there, since

$$\beta = [(\mu^2 + 4A^2)^{1/2} - \mu^2 - \tfrac{1}{4}]^{1/2} \tag{6.11}$$

in that limit reduces to the familiar $(2A - \frac{1}{4})^{1/2}$ [4].

The remaining factors on the right-hand side of (6.7) can also be written as a phase factor. We define the angle χ,

$$e^{2i\chi} = \frac{\nu_+ + i\beta}{\nu_+ - i\beta}\left[\frac{\Gamma(\frac{1}{2}\nu_+ + \frac{1}{2}i\beta)}{\Gamma(\frac{1}{2}\nu_+ - \frac{1}{2}i\beta)}\right]^2 \frac{\Gamma(1 - \mu + \frac{1}{2}\nu_+ - \frac{1}{2}i\beta)\Gamma(\mu + \frac{1}{2}\nu_+ - \frac{1}{2}i\beta)}{\Gamma(1 - \mu + \frac{1}{2}\nu_+ + \frac{1}{2}i\beta)\Gamma(\mu + \frac{1}{2}\nu_+ + \frac{1}{2}i\beta)}, \tag{6.12}$$

where we have substituted for α_+ and α_- according to (5.4) and (6.3). In the limit $\mu \to 0$,* χ is seen to vanish.

In terms of β, A, and the angles defined above, the fractional binding energy may be written as

$$\epsilon_{jn} = \frac{8}{A^2}\exp\left[\frac{-2}{\beta}(n\pi - 2\phi + \psi - \chi)\right], \tag{6.13}$$

for $n = 1, 2, \ldots$. As follows from the definitions of β, ϕ, ψ, and χ, if one formally takes the limit $\mu \to 0$, one recovers the result valid for the states of minimal angular momentum.

* With angular momentum quantized, μ is also quantized. For $|q| = \frac{1}{2}$, $\mu = \sqrt{j(j+1)}$, and the lowest values are: $\mu = 0, \sqrt{2}, \sqrt{6}, \ldots$.

With $|q| = \frac{1}{2}$, we show in fig. 2 a plot of the binding energies for the lowest states ($n = 1$, 2, and 3) of angular momentum $j = 1$, 2, and 3 versus A. For a few values of A, these binding energies are also given in table 1. There we compare them with the exact ones, determined by the method of paper I [2]. The agreement is seen to be excellent.

7. Fine structure

The binding energy depends in general in a rather complicated way on the angular momentum j. While this dependence is dramatic when $|A|$ is close to the critical value A_0 of (6.2), it becomes much weaker for $|A| \gg A_0$. As will be shown in sect. 10, the assumed range of validity (2.9) can be extended to $|A| \gg 1$, provided that we require ϵ to be sufficiently small. In that limit of large $|A|$ the energy depends only weakly on j. We shall here discuss that limit.

Let

$$\beta_0 \equiv (2|A| - \tfrac{1}{4})^{1/2}. \tag{7.1}$$

In the limit

$$\frac{\mu}{\beta_0} \ll 1, \qquad \text{or} \qquad j(j+1) - q^2 \ll 2|A|, \tag{7.2}$$

which for small $|q|$ corresponds to

$$j^2 \ll 2|A|, \tag{7.3}$$

the variation of the phase angles ϕ, ψ, and χ with μ (or j) can be determined explicitly by expanding in the small quantity μ/β_0.

We find

$$\beta = \beta_0\left[1 - \frac{\mu^2}{2\beta_0^2} + O\left(\frac{\mu^4}{\beta_0^4}\right)\right], \tag{7.4}$$

$$\phi = \phi_0 - \frac{\mu^2}{2\beta_0}\ln\beta_0 + O\left(\frac{\mu^4}{\beta_0^3}\right), \tag{7.5}$$

$$\psi = \psi_0 - \frac{\mu}{\beta_0} + O\left(\frac{\mu^3}{\beta_0^3}\right), \tag{7.6}$$

$$\chi = -\frac{\mu}{\beta_0}(1 - \mu) + O\left(\frac{\mu^4}{\beta_0^3}\right), \tag{7.7}$$

where ϕ_0 and ψ_0 are given as

$$\phi_0 = \mathrm{Im}\,\log\Gamma(1+i\beta_0), \tag{7.8}$$

$$\psi_0 = \frac{\pi}{2} - \frac{1}{2\beta_0}. \tag{7.9}$$

Here ϕ_0 corresponds exactly to the $\mu = 0$ case, whereas ψ_0 differs from the $\mu = 0$ value by terms $\sim O(1/\beta_0^3)$.

Consistent with (7.2), we must take $\beta_0 \gg 1$, and thus find

$$\phi_0 \simeq \beta_0(\ln\beta_0 - 1) + \frac{\pi}{4},$$
$$\psi_0 \simeq \frac{\pi}{2}. \tag{7.10}$$

The expression for the binding energy, eq. (6.13), then factorizes,

$$\epsilon_{jn} \simeq \epsilon_n \exp\left(-\frac{\mu^2}{\beta_0^3} n\pi\right), \tag{7.11}$$

with ϵ_n the binding energy for the states of minimal angular momentum, $\mu = 0$ [4]:

$$\epsilon_n = \frac{8}{A^2} \exp\left[\frac{-2}{\beta_0}(n\pi - 2\phi_0 + \psi_0)\right]. \tag{7.12}$$

For $|q| \ll j \ll |A|^{1/2}$, the "fine structure" is thus essentially gaussian in the angular momentum j.

The largest value of A considered in table 1 is $A = 50$ with $\beta_0 \simeq 10$. In that case, the binding energies predicted by (7.11) differ from those of (6.13) by 0.1–1.5%. They are most accurate for states of low n and low angular momentum.

8. Normalized eigensections

The radial wave functions for the external and internal regions are given in sects. 3 and 4, respectively. The relative normalization constants and the overall normalization shall here be determined.

We first express all normalization constants in terms of N_-. Solving (5.7c) and (5.7d), we find

$$N_1 = N_-(W_6 - W_4)/(W_3W_6 - W_4W_5). \tag{8.1}$$

We here use the matching condition, $W_3 \simeq W_5$, to reduce this to

$$N_1 = N_-/W_3. \tag{8.2}$$

Equations (5.7a) and (5.7b) then give

$$N_5 = N_1W_1/W_2 = N_-W_1/(W_2W_3), \tag{8.3}$$

$$N_+ = N_1W_1 = N_-W_1/W_3. \tag{8.4}$$

Using (6.7), we find

$$W_3^{-1} = \frac{(-1)^{n+1}}{4\sqrt{2}\,A}\frac{1}{\Gamma(1+\nu_+)}[\mu^2 + 4A^2 - \mu(\mu^2+4A^2)^{1/2}]^{1/2}$$
$$\cdot\left|\Gamma\left(1-\mu+\tfrac{1}{2}\nu_+ + \tfrac{1}{2}i\beta\right)\Gamma\left(\mu+\tfrac{1}{2}\nu_+ + \tfrac{1}{2}i\beta\right)\right|, \tag{8.5}$$

and hence

$$N_1 = N_-W_3^{-1}, \tag{8.6}$$

$$N_5 = N_-W_3^{-1}\frac{\mu+\frac{1}{2}+\nu_+}{\mu+\frac{1}{2}-\nu_+}\frac{\Gamma(-\nu_+)}{\Gamma(\nu_+)}\left|\frac{\Gamma(-\alpha_+)}{\Gamma(\alpha_+)}\right|^2\left(\frac{A^2}{8}\epsilon\right)^{\nu_+}, \tag{8.7}$$

$$N_+ = -N_-W_3^{-1}\frac{8A^2}{\mu+(\mu^2+4A^2)^{1/2}}\frac{1}{\mu+\frac{1}{2}-\nu_+}\frac{\sin(\pi\mu)}{\pi}$$
$$\cdot\frac{|\Gamma(-\alpha_+)|^2}{\Gamma(\nu_+)}\left(\frac{A^2}{8}\epsilon\right)^{\nu_+/2}. \tag{8.8}$$

For small arguments, one of the exterior-region solutions, $K_{\nu_+}(\sqrt{2A(A-B)}\,\rho)$, becomes large, $\sim \epsilon^{-\nu_+/2}\rho^{-\nu_+}$, but we note that these potentially large terms are multiplied by a small coefficient, $N_+ \sim \epsilon^{\nu_+/2}$.

The normalization constant N_- is determined by

$$\int d^3r\,\psi^\dagger(\vec{r})\psi(\vec{r}) = 1. \tag{8.9}$$

With [6]

$$\int d\Omega\, \xi^{(i)\dagger}_{jj_z} \xi^{(k)}_{jj_z} = \delta_{ik}, \tag{8.10}$$

and $r = \rho A/M$, the decomposition (2.1) then leads to

$$\frac{A}{M} \int_0^\infty d\rho\, \{h_1^2 + h_2^2 + h_3^2 + h_4^2\} = 1. \tag{8.11}$$

We consider separately the contributions from the interior and exterior regions,

$$I_{\rm int} = \frac{A}{M} \int_0^{\rho_0} d\rho\, \{h_1^2 + h_2^2 + h_3^2 + h_4^2\}_{\rm int}, \tag{8.12}$$

$$I_{\rm ext} = \frac{A}{M} \int_{\rho_0}^\infty d\rho\, \{h_1^2 + h_2^2 + h_3^2 + h_4^2\}_{\rm ext}, \tag{8.13}$$

with

$$I_{\rm int} + I_{\rm ext} = 1. \tag{8.14}$$

As a transition point between the interior and the exterior regions we take the geometric mean of the boundaries of the range of overlap,

$$\rho_0 = (A^2 - B^2)^{-1/4} \gg 1. \tag{8.15}$$

To leading order in the binding energy ϵ, the contribution of the interior region can be neglected. This can be seen by noting that

$$\frac{N_1}{N_-} f^{(1)} \sim O(1),$$
$$\frac{N_5}{N_-} f^{(5)} \sim O(\epsilon^{3\nu_+/4}) \ll 1. \tag{8.16}$$

Hence,

$$N_-^{-2} I_{\rm int} \sim O\Big(\int^{\rho_0} \rho\, d\rho\Big) = O(\rho_0^2) = O(\epsilon^{-1/2}). \tag{8.17}$$

In contrast, as will be evaluated explicitly,

$$N_-^{-2} I_{\rm ext} = O(\epsilon^{-1}). \tag{8.18}$$

For the purpose of determining the normalization, in the exterior region we may drop terms $\sim N_+$, since $N_+ \sim \epsilon^{\nu_+/2} N_-$, and terms with an explicit factor $[2A(A-B)]^{1/2}$. Among the remaining terms, the h_1- and h_2-contributions will dominate, because of an extra power of ρ. To leading order in ϵ, the normalization condition is thus

$$I_{\rm ext} \simeq \frac{A}{M} \int_{\rho_0}^\infty d\rho\, \{h_1^2 + h_2^2\}_{\rm ext} \simeq 1, \tag{8.19}$$

or

$$\frac{A}{M} N_-^2 \left\{ \frac{1}{4A^2} [\mu - (\mu^2 + 4A^2)^{1/2}]^2 + 1 \right\} \int_{\rho_0}^\infty \rho\, d\rho\, K^2_{i\beta}(\sqrt{2A(A-B)}\,\rho) = 1. \tag{8.20}$$

The lower limit may here be replaced by zero, and using (see, for example, eq. 6.576.4 of ref. [11])

$$\int_0^\infty x\, dx\, K^2_{i\beta}(x) = \frac{\pi\beta}{2\sinh(\pi\beta)}, \tag{8.21}$$

we finally get

$$N_-^2 = 8A^3 M\epsilon \frac{\sinh(\pi\beta)}{\pi\beta} [\mu^2 + 4A^2 - \mu(\mu^2 + 4A^2)^{1/2}]^{-1}. \tag{8.22}$$

For $j = 1$, and $A = 2$, we show in fig. 3 the approximate radial density distributions, h_1^2, h_2^2, h_3^2, and h_4^2. These are calculated from our analytical expressions (3.12) and (4.29) and are valid in the weak binding approximation. (The actual binding energies for the states considered are given in table 1.) Within the accuracy of the plots, the more accurate wave functions calculated numerically by the method of ref. [2] would not be distinguishable from the present ones. Also shown, in fig. 4, are plots for the case $A = 3$, $j = 2$.

9. Minimal angular momentum as a special case

For $\mu = 0$, or $j = |q| - \frac{1}{2}$, there is only one eigensection of angular momentum [6]

$$\eta_{jj_z} = \phi^{(2)}_{jj_z}. \tag{9.1}$$

With $\mu = 0$, we can formally express this in terms of $\xi^{(1)}_{jj_z}$ and $\xi^{(2)}_{jj_z}$ [6] as

$$\eta_{jj_z} = \frac{1}{\sqrt{2}}\left(-\xi^{(1)}_{jj_z} + \frac{q}{|q|}\xi^{(2)}_{jj_z}\right). \tag{9.2}$$

If we decompose the $j = |q| - \frac{1}{2}$ bound-state eigensection as

$$\psi(\vec{r}) = \frac{1}{r}\begin{bmatrix} \frac{\kappa q}{|\kappa q|}F(r)\eta_{jj_z} \\ -iG(r)\eta_{jj_z} \end{bmatrix}, \tag{9.3}$$

make use of eq. (9.2), and compare with the decomposition of eq. (2.1), we find the correspondence

$$\left.\begin{matrix} F \\ G \end{matrix}\right\} \underset{\mu=0}{\leftrightarrow} \frac{1}{\sqrt{2}}\frac{\kappa q}{|\kappa q|}\left\{\begin{matrix}(-h_1 + h_2) \\ (h_3 - h_4)\end{matrix}\right. \tag{9.4}$$

with

$$h_1 = -h_2 \quad \text{and} \quad h_3 = -h_4. \tag{9.5}$$

In appendix A, eq. (A.14), we have found for $\mu = 0$ in the interior region a solution

$$h_4 = \sqrt{x}\, f^{(1)}_{\mathrm{IV}} = C\sqrt{x}\, K_{\bar{\nu}}(x), \tag{9.6}$$

where $x = 1/\rho$, C is a constant, and $\bar{\nu}$ must be taken to be

$$\nu_-(\mu = 0) \equiv i\beta_0 \equiv i\beta(\mu = 0) = i\left(2|A| - \tfrac{1}{4}\right)^{1/2}. \tag{9.7}$$

The differential equations (4.3) furnish the other three radial wave functions,

$$-\frac{A}{|A|}h_3 = h_4 = C\sqrt{x}\, K_{i\beta_0}(x),$$
$$-\frac{A}{|A|}h_2 = h_1 = C\frac{d}{dx}\left[\sqrt{x}\, K_{i\beta_0}(x)\right]. \tag{9.8}$$

It follows from (9.4) and (9.5) that, for $A > 0$, the present h_i give precisely the solutions of the interior region found in ref. [4].

In the exterior region, we note that $N_+ \to 0$ as $\mu \to 0$ [cf. eq. (8.8)], and thus find from (3.12),

$$-\frac{A}{|A|}h_2 = h_1 = N_-\rho^{1/2}K_{i\beta_0}\left(\sqrt{2A(A-B)}\,\rho\right),$$
$$-h_3 = \frac{A}{|A|}h_4 = \frac{1}{2A}N_-\frac{d}{d\rho}\left[\rho^{1/2}K_{i\beta_0}\left(\sqrt{2A(A-B)}\,\rho\right)\right], \tag{9.9}$$

again in agreement with (9.4) and the solutions of ref. [4].

10. Range of validity

For clarity of presentation, we have assumed that $|A|$ is not large [see eq. (2.9)]. We here show that this condition can be relaxed provided that the binding is sufficiently weak.

Let

$$\bar{\beta}_0 = 1 + \beta, \tag{10.1}$$

with β defined in terms of A and μ by (6.11). The condition for the validity of the power expansion used in the exterior region is thus [compare (II.8.2) and (5.5)]

$$(A^2 - B^2)\rho^2 \ll \bar{\beta}_0. \tag{10.2}$$

Together with (3.1) we get

$$|A + B|^{-1/2} \ll \rho \ll \bar{\beta}_0^{1/2}(A^2 - B^2)^{-1/2}. \tag{10.3}$$

From (C.4) with $\nu = i\beta$ we find the following condition for the validity of the power expansion used in the interior region,

$$\rho^2 \gg \left|\frac{b_1}{b_0}\right| = \frac{1}{4}\frac{1}{|1 + i\beta|}\,\frac{\frac{9}{8} + \nu_+ + \frac{3}{2}\mu^2 - \mu}{\frac{9}{8} + \nu_+ + \frac{1}{2}\mu^2}. \tag{10.4}$$

Since the last factor is $O(1)$, we can rewrite this simply as

$$\rho \gg \bar{\beta}_0^{-1/2}. \tag{10.5}$$

Together with (4.1) we thus have

$$\tilde{\beta}_0^{-1/2} \ll \rho \ll |A-B|^{-1/2}. \qquad (10.6)$$

In order for the procedure of this paper to be valid, the ranges (10.3) and (10.6) must overlap. This requires

$$\tilde{\beta}_0^{-1/2} \ll \tilde{\beta}_0^{1/2}(A^2-B^2)^{-1/2}, \qquad (10.7)$$

and

$$|A+B|^{-1/2} \ll |A-B|^{-1/2}. \qquad (10.8)$$

The inequality (10.7) implies

$$\epsilon \ll \frac{1}{A^2}(1+\beta^2), \qquad (10.9)$$

whereas (10.8) implies

$$\epsilon \ll 1, \qquad (10.10)$$

which is the same as the original (2.8). Of these two, (10.9) is the more restrictive. It is somewhat more explicit to rewrite (10.9) as allowing the following two cases:

(i) When $\beta = O(1)$ [or $|A| \gtrsim A_0$], then the condition is

$$\epsilon \ll |A|^{-2}. \qquad (10.11)$$

(ii) When $\beta \gg 1$ [or $|A| \gg A_0$], then the condition is

$$\epsilon \ll |A|^{-1}. \qquad (10.12)$$

These are consistent with (II.8.20) and (II.8.22).

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Appendix A. Some Solutions to the Differential Equation (4.15)

This appendix gives a brief survey of some of the solutions to the fourth-order differential equation (4.15). We organize these according to the paths of integration shown in fig. 5.

Class I. Contour C_1

Since this contour starts and ends at $|y| \to \infty$, above the real axis, we choose a Hankel function $H_\nu^{(1)}$ in the integrand. The boundary terms therefore vanish, and eq. (4.15) is satisfied by

$$f_{\rm I}^{(j)}(x) = \int_{C_1} dy\, y^{1+\nu} H_\nu^{(1)}(xy)(1+y^2)^p\, u_j(-y^2). \qquad (A.1)$$

Here ν can take on four values [see eq. (4.24)], $p = -\mu$ or $\mu - 1$, and u_j is a hypergeometric function given by eq. (4.21). It will be shown in appendix B that the functions defined by eq. (A.1) all vanish exponentially as $x \to \infty$. At most two of these can be linearly independent, since eq. (4.15) also has two solutions that become exponentially large as $x \to \infty$.

For large values of x, the asymptotic behaviour of the integral (A.1) is controlled by that of $u_j(-y^2)$ in the neighborhood of $y = i$. In this respect, the simplest hypergeometric functions are u_2 and u_6. A natural choice of functions which are well-behaved as $x \to \infty$ is thus

$$f_{\rm I}^{(2)}(x) = \int_{C_1} dy\, y^{1+\nu} H_\nu^{(1)}(xy)(1+y^2)^{-\mu}\, u_2(-y^2), \qquad (A.2)$$

$$f_{\rm I}^{(6)}(x) = \int_{C_1} dy\, y^{1+\nu} H_\nu^{(1)}(xy)(1+y^2)^{-\mu}\, u_6(-y^2). \qquad (A.3)$$

We have here picked a value for p, $p = -\mu$. Taking the other value, $p = \mu - 1$, we would merely interchange u_2 and u_6 (and the two solutions).

For large x, we find these two solutions to be given by (see appendix B)

$$f_{\rm I}^{(2)}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin(\pi\mu)\Gamma(1-\mu)e^{-x}\left(\frac{2}{x}\right)^{-\mu+1}\left[1 + O\left(\frac{1}{x}\right)\right], \qquad (A.4)$$

$$f_{\rm I}^{(6)}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin(\pi\mu)\Gamma(\mu)e^{-x}\left(\frac{2}{x}\right)^{\mu}\left[1+O\left(\frac{1}{x}\right)\right]. \tag{A.5}$$

We note that the leading powers are different. The functions $f_{\rm I}^{(2)}(x)$ and $f_{\rm I}^{(6)}(x)$ are therefore linearly independent.

Class II. Contour C_2

The contours C_1 and C_2 are identical for large $|y|$. Therefore, the Hankel function $H_\nu^{(1)}$ must again be used, and the integrand is identical to that of (A.1):

$$f_{\rm II}^{(j)}(x) = \int_{C_2} dy\, y^{1+\nu} H_\nu^{(1)}(xy)(1+y^2)^p u_j(-y^2). \tag{A.6}$$

For large x, the region around $y \simeq -i$ dominates the integral, and the simplest functions are again those for which $j = 2$ and 6. A computation analogous to the one given in appendix B gives (with $p = -\mu$)

$$f_{\rm II}^{(2)}(x) = e^{i(\mu-1-2\nu)\pi/2}\left(\frac{2}{\pi x}\right)^{1/2} \sin(\pi\mu)\Gamma(1-\mu)e^{x}\left(\frac{2}{x}\right)^{-\mu+1}\left[1+O\left(\frac{1}{x}\right)\right], \tag{A.7}$$

$$f_{\rm II}^{(6)}(x) = e^{i(-\mu-2\nu)\pi/2}\left(\frac{2}{\pi x}\right)^{1/2} \sin(\pi\mu)\Gamma(\mu)e^{x}\left(\frac{2}{x}\right)^{\mu}\left[1+O\left(\frac{1}{x}\right)\right], \tag{A.8}$$

valid as $x \to \infty$. Again, taking the other value of p is equivalent to an interchange of $f_{\rm II}^{(2)}(x)$ and $f_{\rm II}^{(6)}(x)$.

Class III. Contour C_3

The functions

$$f_{\rm III}^{(j)}(x) = \int_{C_3} dy\, y^{1+\nu} H_\nu^{(1)}(xy)(1+y^2)^p u_j(-y^2) \tag{A.9}$$

are very similar to those of Class II. In fact, with $p = -\mu$, we find asymptotically, as $x \to \infty$,

$$f_{\rm III}^{(2)}(x) = -e^{i(3\mu-3-2\nu)\pi/2}\left(\frac{2}{\pi x}\right)^{1/2} \sin(\pi\mu)\Gamma(1-\mu)e^{x}\left(\frac{2}{x}\right)^{-\mu+1}\left[1+O\left(\frac{1}{x}\right)\right], \tag{A.10}$$

$$f_{\rm III}^{(6)}(x) = -e^{i(-3\mu-2\nu)\pi/2}\left(\frac{2}{\pi x}\right)^{1/2} \sin(\pi\mu)\Gamma(\mu)e^{x}\left(\frac{2}{x}\right)^{\mu}\left[1+O\left(\frac{1}{x}\right)\right]. \tag{A.11}$$

Up to a phase factor, as $x \to \infty$, these are thus equal to the Class II functions. Presumably, the Class II and Class III functions differ by some function which is exponentially small as $x \to \infty$ (related to the Class I functions), and which therefore cannot be determined by an asymptotic evaluation of the kind given in appendix B.

Class IV. Contour C_4

The contour C_4, which is the positive real axis, is especially useful for small values of x. We therefore choose the Bessel function in the integrand to be $J_\nu(xy)$, which has the simplest behaviour for small x. Furthermore, for the present problem, the only relevant function of this Class IV is the one with $j = 1$. Therefore, we study the function

$$f_{\rm IV}^{(1)}(x) = \int_0^\infty dy\, y^{1+\nu} J_\nu(xy)(1+y^2)^p u_1(-y^2). \tag{A.12}$$

This integral (A.12) is convergent at the lower limit when

$$\mathrm{Re}\,\nu > -1. \tag{A.13}$$

At the upper limit, it is absolutely integrable if $|\mathrm{Re}\,\bar\nu| < \frac{1}{2}$; it is however summable for all finite values of $\bar\nu$ because of the oscillatory nature of $J_\nu(xy)$. It is thus convenient to think of the right-hand side of (A.12) as including an extra factor of $\exp(-\epsilon y)$ with $\epsilon \to 0^+$ after integration. This is precisely Abel summability. The $f_{\rm IV}^{(1)}(x)$ interpreted in this way is studied further in appendix C.

We comment briefly on the special case $p = -\mu = 0$. In this case, the integration can be carried out explicitly to yield (see p. 855 of ref. [11])

$$f_{\rm IV}^{(1)}(x) = \frac{\Gamma(1+\nu)}{\Gamma(1+\frac{1}{2}\nu+\frac{1}{2}\bar\nu)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\bar\nu)} K_{\bar\nu}(x). \tag{A.14}$$

Apart from a normalization constant, this is a radial wave function for $j = |q| - \frac{1}{2}$. The connection with that case is discussed in sect. 9.

Appendix B. Asymptotic Expansion of $f_{\rm I}^{(2)}(x)$ and $f_{\rm I}^{(6)}(x)$

As $x \to \infty$, relatively simple asymptotic expansions can be obtained for $f_{\rm I}^{(2)}(x)$ and $f_{\rm I}^{(6)}(x)$. These functions are defined by (A.1). For large x, the main contribution to the integrals (A.2) and (A.3) comes from the neighborhood of $y = +i$. It is therefore convenient to shift the variable of integration,

$$y = e^{i\pi/2}(1+t^2)^{1/2}. \tag{B.1}$$

The Hankel function can be expressed in terms of a modified Bessel function [10],

$$H_\nu^{(1)}(xy) = \frac{2}{i\pi}\, e^{-i\pi\nu/2} K_\nu\big(x\sqrt{1+t^2}\,\big), \tag{B.2}$$

and, using $y\,dy = -t\,dt$, we find

$$f_{\rm I}^{(j)}(x) = -\frac{2}{i\pi}\int_0^\infty dt\, t(1+t^2)^{\nu/2} K_\nu\big(x\sqrt{1+t^2}\,\big) t^{2p}\Delta_j. \tag{B.3}$$

Here $t^{2p}\Delta_j$ is the discontinuity of $(1+y^2)^p u_j(-y^2)$ across the cut (taken along the real axis in the t-plane),

$$\Delta_j = \lim_{\epsilon\to 0^+}\left[e^{i\pi p} u_j(1+t^2-i\epsilon) - e^{-i\pi p} u_j(1+t^2+i\epsilon)\right]. \tag{B.4}$$

Since $u_2(1+t^2)$ and $t^{-4p-2}u_6(1+t^2)$ have no branch point at $t=0$, we find the following simple results

$$t^{2p}\Delta_2 = 2i\sin(\pi p)\, t^{2p} F(1+p+\alpha_+, 1+p+\alpha_-; 2+2p; -t^2), \tag{B.5}$$

$$t^{2p}\Delta_6 = 2i\sin(\pi p)\, t^{-2p-2} F(-p+\alpha_+, -p+\alpha_-; -2p; -t^2), \tag{B.6}$$

where we have introduced the abbreviations

$$\alpha_\pm = \tfrac{1}{2}(\nu \pm \bar{\nu}). \tag{B.7}$$

These expressions show explicitly that interchanging the two possible values of p, namely, $-\mu$ and $\mu - 1$, i.e., interchanging p and $-(1+p)$, is equivalent to interchanging $f_{\rm I}^{(2)}(x)$ and $f_{\rm I}^{(6)}(x)$.

We now expand the hypergeometric function,

$$f_{\rm I}^{(2)}(x) = -\frac{4}{\pi}\sin(\pi p)\int_0^\infty dt\, t^{1+2p}(1+t^2)^{\nu/2} K_\nu\big(x\sqrt{1+t^2}\,\big) \cdot \sum_{n=0} \frac{\Gamma(1+p+\alpha_+ + n)}{\Gamma(1+p+\alpha_+)}\,\frac{\Gamma(1+p+\alpha_- + n)}{\Gamma(1+p+\alpha_-)}\,\frac{\Gamma(2+2p)}{\Gamma(2+2p+n)}\,\frac{(-1)^n}{n!}\, t^{2n}. \tag{B.8}$$

This integral can be evaluated in closed form (see p. 95 of ref. [10]). With $p = -\mu$, and using $K_\nu(z) = K_{-\nu}(z)$, we find

$$f_{\rm I}^{(2)}(x) = \frac{2}{\pi}\sin(\pi\mu)\sum_{n=0}\frac{\Gamma(1-\mu+\alpha_+ + n)}{\Gamma(1-\mu+\alpha_+)}\,\frac{\Gamma(1-\mu+\alpha_- + n)}{\Gamma(1-\mu+\alpha_-)}\,\frac{\Gamma(2-2\mu)}{\Gamma(2-2\mu+n)} \cdot \frac{(-1)^n}{n!}\Gamma(-\mu+1+n)\left(\frac{2}{x}\right)^{-\mu+1+n} K_{-\nu+\mu-1-n}(x). \tag{B.9}$$

The other solution, $f_{\rm I}^{(6)}(x)$, can be obtained by repeating this procedure with $p = \mu - 1$. This amounts to replacing $-\mu$ by $\mu - 1$ in the above expression, keeping ν, α_+ and α_- unchanged.

The present expansion is an asymptotic one, useful as $x \to \infty$. We may then also expand the modified Bessel functions in an asymptotic series [10],

$$K_\gamma(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\sum_{m=0}\frac{\Gamma(\frac{1}{2}+\gamma+m)}{\Gamma(\frac{1}{2}+\gamma-m)}\,\frac{1}{m!}(2x)^{-m}, \tag{B.10}$$

and express $f_{\rm I}^{(2)}(x)$ as an asymptotic series where each coefficient is given by a sum,

$$f_{\rm I}^{(2)}(x) = \left(\frac{2}{\pi x}\right)^{1/2} e^{-x}\sin(\pi\mu)\sum_{n=0}\sum_{m=0}\frac{\Gamma(1-\mu+\alpha_+ + n)}{\Gamma(1-\mu+\alpha_+)} \cdot \frac{\Gamma(1-\mu+\alpha_- + n)}{\Gamma(1-\mu+\alpha_-)}\,\frac{\Gamma(2-2\mu)}{\Gamma(2-2\mu+n)}\,\frac{(-1)^n}{n!}\Gamma(-\mu+1+n) \cdot \frac{\Gamma(-\frac{1}{2}+\mu-\nu-n+m)}{\Gamma(-\frac{1}{2}+\mu-\nu-n-m)}\,\frac{4^{-m}}{m!}\left(\frac{2}{x}\right)^{-\mu+1+n+m}. \tag{B.11}$$

By an explicit evaluation, we find the first few terms to be given by

$$f_{\mathrm{I}}^{(2)}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sin(\pi\mu) e^{-x} \Gamma(-\mu+1) \left(\frac{2}{x}\right)^{-\mu+1}$$
$$\cdot \Big\{ 1 + \tfrac{1}{2}[-(-\mu+1)^2 - \tfrac{1}{4} + \tfrac{1}{2}(\nu^2+\bar{\nu}^2)]x^{-1}$$
$$+ \tfrac{1}{4}(3-2\mu)^{-1}[-\mu^5 + \tfrac{15}{2}\mu^4 - \tfrac{45}{2}\mu^3 + \tfrac{135}{4}\mu^2 - \tfrac{409}{16}\mu + \tfrac{259}{32}$$
$$- (\nu^2+\bar{\nu}^2)(-\mu^3 + \tfrac{9}{2}\mu^2 - \tfrac{29}{4}\mu + \tfrac{35}{8})$$
$$+ \tfrac{1}{4}(\nu^4+\bar{\nu}^4)(2-\mu) + \tfrac{1}{2}\nu^2\bar{\nu}^2(1-\mu)]x^{-2} + O(x^{-3}) \Big\}. \qquad \text{(B.12)}$$

The solution $f_{\mathrm{I}}^{(6)}(x)$ is obtained from this expression by the substitution $-\mu \to \mu - 1$, with ν and $\bar{\nu}$ kept unchanged.

Appendix C. Series Expansions for the Differential Equation of Sect. 4

We choose as the starting point the fourth-order ordinary differential equation (4.8). Let

$$f_\nu(x) = \sum_{k=0}^{\infty} b_k x^{\nu+2k} \qquad \text{(C.1)}$$

be a solution of (4.8), where ν is given by (4.12). The substitution of (C.1) into (4.8) gives the recurrence formula

$$\{[(\nu+2k)^2 - (\tfrac{1}{4}+\mu^2)]^2 - (\mu^2+4A^2)\}b_k$$
$$- 2[(\nu+2k-1)^2 - \tfrac{1}{4} + (\mu-1)^2]b_{k-1} + b_{k-2} = 0. \qquad \text{(C.2)}$$

In terms of the $\alpha_\pm$ of (B.7) with the ν and $\bar{\nu}$ of (4.24), (C.2) can be written in the form

$$4k(k+\nu)(k+\alpha_+)(k+\alpha_-)b_k$$
$$- [k(k+\nu) + (k-1+\alpha_+)(k-1+\alpha_-) + \mu(\mu-1)]b_{k-1}$$
$$+ \tfrac{1}{4}b_{k-2} = 0. \qquad \text{(C.3)}$$

Equation (C.3) is solved by

$$b_k = (-1)^k \left(\frac{1}{2}\right)^{1+\nu+2k} \frac{1}{k!} \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+k)} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l}$$
$$\cdot \frac{\Gamma(-\alpha_- - l)\Gamma(-\alpha_+ - l)}{\Gamma(1-\mu-l)\Gamma(\mu-l)}, \qquad \text{(C.4)}$$

as can be verified by considering separately each term in the sum over l. These coefficients are symmetric under interchange of α_+ and α_-, or $\bar{\nu} \to -\bar{\nu}$.

An alternative form of this solution is

$$a_k = (-1)^k \left(\frac{1}{2}\right)^{1+\nu+2k} \frac{1}{k!} \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+k)} \sum_{m=0}^{k} \binom{k}{m} \frac{\Gamma(1+p+\alpha_-+m)}{\Gamma(1+p+\alpha_-)}$$
$$\cdot \frac{\Gamma(-p+\alpha_++k-m)}{\Gamma(-p+\alpha_+)} \frac{\Gamma(-\alpha_- - k)\Gamma(-\alpha_+ - k + m)}{\Gamma(-p)\Gamma(1+p-k+m)}, \qquad \text{(C.5)}$$

where p is given by (4.22). The coefficients a_k and b_k are in fact equal, as will be shown presently. Both forms are useful.

Historically, the coefficients b_k and a_k were obtained by formal expansions of the integrand in (A.12).

In order to show that

$$a_k = b_k, \tag{C.6}$$

we consider

$$A_k = a_k/C_k, \qquad B_k = b_k/C_k, \tag{C.7}$$

where C_k is some k-dependent factor. If the two generating functions

$$G^A(z) = \sum_{k=0}^{\infty} A_k z^k, \qquad G^B(z) = \sum_{k=0}^{\infty} B_k z^k, \tag{C.8}$$

are identical for all z, then the two original coefficients a_k and b_k must be identical for all k. The factor C_k is chosen such that the expressions (C.8) can be summed in closed form.

Let us first consider

$$\begin{aligned} A_k = \frac{1}{k!} \sum_{m=0}^{k} \binom{k}{m} \frac{\Gamma(1+p+\alpha_-+m)}{\Gamma(1+p+\alpha_-)} \frac{\Gamma(-p+\alpha_+ + k - m)}{\Gamma(-p+\alpha_+)} \\ \cdot \frac{\Gamma(-\alpha_+ - k + m)}{\Gamma(-p)\Gamma(1+p-k+m)}, \end{aligned} \tag{C.9}$$

which implies a constant of proportionality

$$C_k = (-1)^k \left(\frac{1}{2}\right)^{1+\nu+2k} \frac{\Gamma(1+\nu)\Gamma(-\alpha_- - k)}{\Gamma(1+\nu+k)}. \tag{C.10}$$

With $k = j + m$, the generating function is seen to factorize,

$$\begin{aligned} G^A(z) &= \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{z^{m+j}}{m!\,j!} \frac{\Gamma(1+p+\alpha_-+m)\Gamma(-p+\alpha_++j)\Gamma(-\alpha_+-j)}{\Gamma(1+p+\alpha_-)\Gamma(-p+\alpha_+)\Gamma(-p)\Gamma(1+p-j)} \\ &= [\Gamma(1+p+\alpha_-)\Gamma(-p+\alpha_+)\Gamma(-p)]^{-1} G_1^A(z) G_2^A(z), \end{aligned} \tag{C.11}$$

where

$$\begin{aligned} G_1^A(z) &= \sum_{m=0}^{\infty} \frac{z^m}{m!} \Gamma(1+p+\alpha_-+m) \\ &= \Gamma(1+p+\alpha_-)(1-z)^{-p-\alpha_--1} \end{aligned} \tag{C.12}$$

and

$$\begin{aligned} G_2^A(z) &= \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\Gamma(-p+\alpha_++j)\Gamma(-\alpha_+-j)}{\Gamma(1+p-j)} \\ &= \frac{\sin(\pi p)}{\sin(\pi\alpha_+)} \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\Gamma(-p+j)\Gamma(-p+\alpha_++j)}{\Gamma(1+\alpha_++j)} \\ &= \frac{\sin(\pi p)}{\sin(\pi\alpha_+)} \frac{\Gamma(-p)\Gamma(-p+\alpha_+)}{\Gamma(1+\alpha_+)} F(-p,-p+\alpha_+;1+\alpha_+;z). \end{aligned} \tag{C.13}$$

In the last step we have used the reflection formula, and F is a hypergeometric function. Thus,

$$G^A(z) = \frac{\sin(\pi p)}{\sin(\pi\alpha_+)} \frac{1}{\Gamma(1+\alpha_+)} (1-z)^{-p-\alpha_--1} F(-p,-p+\alpha_+;1+\alpha_+;z). \tag{C.14}$$

The coefficients B_k are now given by b_k and C_k [eqs. (C.4) and (C.10)],

$$B_k = \frac{1}{k!} \frac{1}{\Gamma(-\alpha_- - k)} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \frac{\Gamma(-\alpha_- - l)\Gamma(-\alpha_+ - l)}{\Gamma(1+p-l)\Gamma(-p-l)}, \tag{C.15}$$

where p can be either $-\mu$ or $\mu - 1$. With $k = j + l$, the generating function can be written as

$$\begin{aligned} G^B(z) &= -\frac{\sin(\pi\alpha_-)}{\pi} \sum_{l=0}^{\infty} (-1)^l \frac{z^l}{l!} \frac{\Gamma(-\alpha_- - l)\Gamma(-\alpha_+ - l)}{\Gamma(1+p-l)\Gamma(-p-l)} \\ &\quad \cdot \sum_{j=0}^{\infty} \frac{z^j}{j!} \Gamma(1+\alpha_-+l+j) \\ &= -\frac{\sin(\pi\alpha_-)}{\pi} \sum_{l=0}^{\infty} (-1)^l \frac{z^l}{l!} \frac{\Gamma(-\alpha_- - l)\Gamma(-\alpha_+ - l)}{\Gamma(1+p-l)\Gamma(-p-l)} \\ &\quad \cdot \Gamma(1+\alpha_-+l)(1-z)^{-1-\alpha_--l} \\ &= (1-z)^{-1-\alpha_-} \frac{\Gamma(-\alpha_+)}{\Gamma(1+p)\Gamma(-p)} F(-p,1+p;1+\alpha_+;-z/(1-z)). \end{aligned} \tag{C.16}$$

By reexpressing the hypergeometric function of argument $-z/(1-z)$ in terms of one of argument z [9], and using the reflection formula for the Γ-functions, we arrive at the form (C.14). We have thus proved (C.6).

These coefficients are especially simple in the limit $\mu \to 0$. Let us consider the quantity

$$\bar{b}_k \equiv \lim_{\mu\to 0} \frac{2}{\mu\Gamma(-\alpha_-)\Gamma(-\alpha_+)\Gamma(1+\nu)} b_k. \tag{C.17}$$

Only the $l=0$ term contributes in this limit, and we obtain the series expansion for the modified Bessel function I_ν [10],

$$I_\nu(x) = \sum_{k=0}^{\infty} \bar{b}_k x^{\nu+2k}. \tag{C.18}$$

Note that this is exponentially increasing for large x.

In the remainder of this appendix we discuss in the general case the asymptotic behaviour of $f_\nu(x)$ as $x \to \infty$. We shall see that for all four values of ν, the functions $f_\nu(x)$ become exponentially large as $x \to \infty$.

The behaviour of $f_\nu(x)$ for large x is determined by that of a_k (or b_k) for large k. This latter problem is most naturally studied through the generating function (C.14). By (C.8) and Cauchy's theorem,

$$A_k = \frac{1}{2\pi i} \oint_C \frac{G^A(z)\, dz}{z^{k+1}}, \tag{C.19}$$

with the contour of integration encircling the origin (see fig. 6). The integrand has a branch point at $z=1$. We shall deform the contour of integration as indicated by the dashed curve labelled C' in fig. 6.

For large k, the dominant contribution to the integral will come from the region near $z=1$. We can then approximate

$$z^{-k-1} = (1+z-1)^{-k-1} \simeq e^{-(k+1)(z-1)}. \tag{C.20}$$

We next use an identity of hypergeometric functions [9],

$$\begin{aligned} F(-p,-p+\alpha_+;1+\alpha_+;z) &= \frac{\Gamma(1+\alpha_+)\Gamma(1+2p)}{\Gamma(1+p+\alpha_+)\Gamma(1+p)} F(-p,-p+\alpha_+;-2p;1-z) \\ &+ \frac{\Gamma(1+\alpha_+)\Gamma(-1-2p)}{\Gamma(-p+\alpha_+)\Gamma(-p)} (1-z)^{1+2p} F(1+p+\alpha_+,1+p;2+2p;1-z), \end{aligned} \tag{C.21}$$

and replace the $F(\alpha,\beta;\gamma;1-z)$ by 1, their limiting values as $z \to 1$. The integral (C.19) can then be approximated as

$$\begin{aligned} A_k \simeq \frac{1}{2\pi i} \frac{\sin(\pi p)}{\sin(\pi\alpha_+)} \int_{C'} dz\, e^{-(k+1)(z-1)} \Bigg[&\frac{\Gamma(1+2p)}{\Gamma(1+p+\alpha_+)\Gamma(1+p)} (1-z)^{-p-1-\alpha_-} \\ &+ \frac{\Gamma(-1-2p)}{\Gamma(-p+\alpha_+)\Gamma(-p)} (1-z)^{p-\alpha_-} \Bigg]. \end{aligned} \tag{C.22}$$

The contribution along the arc at $|z| \to \infty$ vanishes, and each of the remaining integrals is just the Hankel representation of the Γ-function. Thus,

$$\begin{aligned} A_k \simeq \frac{\sin(\pi p)}{\sin(\pi\alpha_+)} \Bigg\{ &\frac{\Gamma(1+2p)}{\Gamma(1+p)} [\Gamma(1+p+\alpha_-)\Gamma(1+p+\alpha_+)]^{-1} (k+1)^{p+\alpha_-} \\ &+ (p \leftrightarrow -p-1) \Bigg\}, \qquad k \gg 1. \end{aligned} \tag{C.23}$$

The coefficients of interest, a_k, may now be obtained from eqs. (C.7) and (C.10). Using also the Stirling formula for the k-dependent Γ-functions, we find that, as $k \to \infty$:

$$\begin{aligned} a_k \simeq -\left(\frac{1}{2}\right)^{2+\nu+2k} &\Gamma(1+\nu) \frac{\sin(\pi p)}{\sin(\pi\alpha_-)\sin(\pi\alpha_+)} \frac{\Gamma(1+2p)}{\Gamma(1+p)} \\ &\cdot [\Gamma(1+p+\alpha_-)\Gamma(1+p+\alpha_+)]^{-1} e^{2k} k^{-2k-1-\nu+p} + (p \leftrightarrow -p-1). \end{aligned} \tag{C.24}$$

The asymptotic behaviour of $f_\nu(x)$ as $x \to \infty$ can now be determined by replacing the sum over k by an integral, and evaluating the integral by the method of steepest descent.

Let us introduce the abbreviation

$$Z_\nu = -\left(\frac{1}{2}\right)^{2+\nu}\Gamma(1+\nu)\,\frac{\sin(\pi p)}{\sin(\pi\alpha_-)\sin(\pi\alpha_+)}\,\frac{\Gamma(1+2p)}{\Gamma(1+p)} \cdot [\Gamma(1+p+\alpha_-)\Gamma(1+p+\alpha_+)]^{-1}. \tag{C.25}$$

Then

$$f_\nu(x) \underset{x\to\infty}{\simeq} Z_\nu x^\nu \sum_k \left(\frac{1}{2}\right)^{2k} e^{2k} k^{-2k-1-\nu+p} x^{2k} + (p \leftrightarrow -p-1)$$
$$\simeq Z_\nu x^\nu \int dk\, e^{\chi(k)} + (p \leftrightarrow -p-1), \tag{C.26}$$

with

$$\chi(k) = 2k + 2k\ln(x/2) - (2k+1+\nu-p)\ln k. \tag{C.27}$$

The saddle point is determined by

$$\frac{d}{dk}\chi(k)\Big|_{k=k_s} = 0, \tag{C.28}$$

which gives [neglecting terms of $O(1/k)$ and $O(1/x)$]

$$k_s = \frac{x}{2}. \tag{C.29}$$

Further,

$$\frac{d^2}{dk^2}\chi(k)\Big|_{k=k_s} = -\frac{4}{x} + O\left(\frac{1}{x^2}\right), \tag{C.30}$$

and

$$f_\nu(x) \underset{x\to\infty}{\simeq} Z_\nu x^\nu e^{\chi(k_s)} \int dk\, e^{-(2/x)(k-k_s)^2}. \tag{C.31}$$

This leads to the asymptotic behaviour,

$$f_\nu(x) \underset{x\to\infty}{\simeq} f_\nu^{as}(x), \tag{C.32}$$

with

$$f_\nu^{as}(x) = \frac{1}{2}\sqrt{\frac{\pi}{2x}}\,\Gamma(1+\nu)\,\frac{\sin(\pi\mu)}{\sin(\pi\alpha_-)\sin(\pi\alpha_+)}\,e^x \cdot \left\{\frac{\Gamma(1-2\mu)}{\Gamma(1-\mu)}[\Gamma(1-\mu+\alpha_-)\Gamma(1-\mu+\alpha_+)]^{-1}\left(\frac{x}{2}\right)^{-\mu} + \frac{\Gamma(-1+2\mu)}{\Gamma(\mu)}[\Gamma(\mu+\alpha_-)\Gamma(\mu+\alpha_+)]^{-1}\left(\frac{x}{2}\right)^{\mu-1}\right\}. \tag{C.33}$$

Appendix D. Series Expansions of $f_I^{(1)}(x)$ and $f_I^{(5)}(x)$ for small x

In appendix B we have found that the solutions $f_I^{(j)}(x)$ of the differential equations (4.15), which were given in terms of integrals, are bounded as $x \to \infty$. In this appendix, we give the corresponding series expansions for small x. Since (4.15) is of fourth order, it follows from (C.1) that

$$f_I^{(j)}(x) = c_1^{(j)} f_{\nu_+}(x) + c_2^{(j)} f_{-\nu_+}(x) + c_3^{(j)} f_{\nu_-}(x) + c_4^{(j)} f_{-\nu_-}(x). \tag{D.1}$$

The coefficients $c_i^{(j)}$ depend on the choice of parameters ν, $\bar{\nu}$, and p adopted for $f_I^{(j)}(x)$. These relations will be analogous to the relationship between Bessel functions [10],

$$K_\nu(x) = \frac{\pi}{2\sin(\pi\nu)} [I_{-\nu}(x) - I_\nu(x)]. \tag{D.2}$$

The $f_{\pm\nu}(x)$ and $I_{\pm\nu}(x)$ all grow exponentially for large x, whereas $f_I^{(j)}(x)$ and $K_\nu(x)$ become exponentially small as $x \to \infty$.

We need the formulas for two linearly independent $f_I^{(j)}(x)$. Let us fix

$$p = -\mu, \tag{D.3}$$

and consider $f_I^{(1)}(x)$ and $f_I^{(5)}(x)$. Our approach is to evaluate

$$f_I^{(1)}(x) = \int_{C_1} dy\, y^{1+\nu} H_\nu^{(1)}(xy)(1+y^2)^{-\mu} F(a,b;c;-y^2) \tag{D.4}$$

to leading order for small x, in order to determine the coefficients $c_i^{(1)}$ by recognizing the various leading powers of $f_\nu(x)$ at small x. By symmetry arguments we shall later obtain the coefficients $c_i^{(5)}$ from $c_i^{(1)}$.

The following evaluation is only valid for ν and $\bar{\nu}$ both pure imaginary. While this is not the case for the physical values (4.24), we assume that the validity of our formulas can be extended by analytic continuation.

For small $|xy|$, we can approximate

$$H_\nu^{(1)}(xy) \simeq \bar{H}_\nu, \tag{D.5}$$

$$\bar{H}_\nu = A_1 (xy)^\nu + A_2 (xy)^{-\nu}, \tag{D.6}$$

with [10]

$$A_1 = -[i\sin(\pi\nu)]^{-1} \frac{e^{-i\pi\nu}}{\Gamma(1+\nu)} \left(\frac{1}{2}\right)^\nu, \tag{D.7a}$$

$$A_2 = [i\sin(\pi\nu)]^{-1} \frac{1}{\Gamma(1-\nu)} \left(\frac{1}{2}\right)^{-\nu}. \tag{D.7b}$$

On the other hand, for large y,

$$(1+y^2)^{-\mu} F(a,b;c;-y^2) \simeq \bar{F}, \tag{D.8}$$

$$\bar{F} = A_3 y^{-2(\mu+a)} + A_4 y^{-2(\mu+b)}, \tag{D.9}$$

with [9]

$$A_3 = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} = \frac{\Gamma(1+\nu)\Gamma(-\bar{\nu})}{\Gamma(1-\mu+\alpha_-)\Gamma(\mu+\alpha_-)}, \tag{D.10a}$$

$$A_4 = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} = \frac{\Gamma(1+\nu)\Gamma(\bar{\nu})}{\Gamma(1-\mu+\alpha_+)\Gamma(\mu+\alpha_+)}, \tag{D.10b}$$

provided

$$|\arg(y^2)| < \pi. \tag{D.11}$$

We will thus use the following approximation

$$\int_{C_1} dy\, y^{1+\nu} [H_\nu^{(1)}(xy) - \bar{H}_\nu][(1+y^2)^{-\mu} F(a,b;c;-y^2) - \bar{F}] \simeq 0. \tag{D.12}$$

At small $|y|$, the first and/or the second factor is small, while at large $|y|$, the last factor is small. It follows that

$$f_I^{(1)}(x) \simeq I_1 + I_2 - I_3, \tag{D.13}$$

where

$$I_1 = \int_{C_1} dy\, y^{1+\nu} H_\nu^{(1)}(xy) \bar{F}, \tag{D.14}$$

$$I_2 = \int_{C_1} dy\, y^{1+\nu} \bar{H}_\nu (1+y^2)^{-\mu} F(a,b;c;-y^2), \tag{D.15}$$

$$I_3 = \int_{C_1} dy\, y^{1+\nu} \bar{H}_\nu \bar{F}. \tag{D.16}$$

These three integrals can all be evaluated analytically.

We first consider I_1, and divide the contour of integration into the two parts C_a and C_b, as shown in fig. 7. On C_a the condition (D.11) is not satisfied. We shall therefore make use of the fact that the left-hand side of eq. (D.8) is symmetric under $y \leftrightarrow -y$, and evaluate $\bar{F}$ on C_a' instead of on C_a. We thus have

On C_a' ($y = Re^{-i\pi/4}$):

$$\bar{F} = A_3 e^{i(2+\nu+\bar{\nu})\pi/4} R^{-2-\nu-\bar{\nu}} + A_4 e^{i(2+\nu-\bar{\nu})\pi/4} R^{-2-\nu+\bar{\nu}}, \tag{D.17a}$$

On C_b ($y = Re^{i\pi/4}$):

$$\bar{F} = A_3 e^{-i(2+\nu+\bar{\nu})\pi/4} R^{-2-\nu-\bar{\nu}} + A_4 e^{-i(2+\nu-\bar{\nu})\pi/4} R^{-2-\nu+\bar{\nu}}, \tag{D.17b}$$

having inserted for a and b the values (4.23).

Further, we note that for small $|y|$ the integrand behaves like $|y|$ raised to an imaginary power (ν and $\bar{\nu}$ are here assumed to be imaginary). The contours C_a' and C_b can therefore be extended down to the origin,

$$I_1 = \int_0^\infty dR\, R^{-1-\bar{\nu}} A_3 [e^{-i\pi\bar{\nu}/4} H_\nu^{(1)}(xRe^{i\pi/4}) - e^{i\pi(\nu+\bar{\nu}/4)} H_\nu^{(1)}(xRe^{3i\pi/4})] + (\bar{\nu} \leftrightarrow -\bar{\nu}). \tag{D.18}$$

With $t = xR$, we obtain

$$I_1 = x^{\bar{\nu}} A_3 \int_0^\infty dt\, t^{-1-\bar{\nu}} [e^{-i\pi\bar{\nu}/4} H_\nu^{(1)}(te^{i\pi/4}) - e^{i\pi(\nu+\bar{\nu}/4)} H_\nu^{(1)}(te^{3i\pi/4})] + (\bar{\nu} \leftrightarrow -\bar{\nu}). \tag{D.19}$$

Next, we express the $H_\nu^{(1)}$ in terms of modified Bessel functions [10],

$$K_\nu(z) = \tfrac{1}{2} i\pi e^{i\pi\nu/2} H_\nu^{(1)}(ze^{i\pi/2}), \tag{D.20}$$

and use (see eq. 6.561.16 of ref. [11])

$$\int_0^\infty x^\rho K_\nu(ax)\, dx = 2^{\rho-1} a^{-\rho-1} \Gamma\Big(\frac{1+\rho+\nu}{2}\Big) \Gamma\Big(\frac{1+\rho-\nu}{2}\Big) \tag{D.21}$$

to get

$$I_1 = -\Big(\frac{x}{2}\Big)^{\bar{\nu}} \frac{\sin(\pi\alpha_+)}{\pi} \frac{\Gamma(-\alpha_+)\Gamma(\alpha_-)\Gamma(1+\nu)\Gamma(-\bar{\nu})}{\Gamma(1-\mu+\alpha_-)\Gamma(\mu+\alpha_-)} - \Big(\frac{x}{2}\Big)^{-\bar{\nu}} \frac{\sin(\pi\alpha_-)}{\pi} \frac{\Gamma(-\alpha_-)\Gamma(\alpha_+)\Gamma(1+\nu)\Gamma(\bar{\nu})}{\Gamma(1-\mu+\alpha_+)\Gamma(\mu+\alpha_+)}. \tag{D.22}$$

The remaining two integrals we do simultaneously, in order to exploit the cancellations at large $|y|$. In the variable $w = y^2$, the contour of integration is indicated by the solid line in fig. 8. We deform the contour as indicated and note that, since $\bar{F}$ is the large-argument limit of F, there are no contributions to $I_2 - I_3$ along the arcs AB and EF. Also, because ν and $\bar{\nu}$ are imaginary, there are no contributions near the origin (along CD). We are left then with

$$I_2 - I_3 = \tfrac{1}{2} \int_{\mathrm{BC+DE}} dw\, w^{\nu/2} [A_1 x^\nu w^{\nu/2} + A_2 x^{-\nu} w^{-\nu/2}] \cdot [(1+w)^{-\mu} F(a,b;c;-w) - A_3 w^{-\mu-a} - A_4 w^{-\mu-b}]. \tag{D.23}$$

The terms in this expression with coefficients A_3 and A_4 give integrals of the type (with $t = |w|$)

$$\lim_{\substack{\epsilon\to 0\\ R\to\infty}} \int_\epsilon^R dt\, t^{-1+\omega} = \lim_{\substack{\epsilon\to 0\\ R\to\infty}} \frac{1}{\omega}(R^\omega - \epsilon^\omega). \tag{D.24}$$

Since ω is pure imaginary, these vanish.

In the y plane, there are three branch points at $\pm i$ and 0. We take the first two corresponding branch cuts to be along the imaginary axis extending to infinity, and the third one to be the negative real axis. Thus in the $w = y^2$ plane, there are two branch cuts, one from $-\infty$ to -1, and the other from 0 to $+\infty$, both along the real axis. The phase of w is accordingly 0 along DE and 2π along BC:

$$I_2 - I_3 = \tfrac{1}{2}\int_0^\infty dt\,\{[A_1x^\nu t^\nu + A_2x^{-\nu}] - [A_1x^\nu e^{2\pi i\nu}t^\nu + A_2x^{-\nu}]\} \cdot (1+t)^{-\mu}F(a,b;c;-t). \tag{D.25}$$

The A_2 contributions cancel, and we are left with

$$I_2 - I_3 = \tfrac{1}{2}x^\nu A_1(1-e^{2\pi i\nu})\int_0^\infty dt\,t^\nu(1+t)^{-\mu}F(a,b;1+\nu;-t). \tag{D.26}$$

This integral gives (see eq. 7.512.10 of ref. [11])

$$I_2 - I_3 = \left(\frac{x}{2}\right)^\nu \frac{\sin(\pi\mu)}{\pi}\Gamma(-\alpha_-)\Gamma(-\alpha_+), \tag{D.27}$$

where we have inserted the values for a, b, and A_1.

To summarize, for small x, to leading order, we then have

$$\begin{aligned} f_{\rm I}^{(1)}(x) \simeq{}& \left(\frac{x}{2}\right)^\nu \frac{\sin(\pi\mu)}{\pi}\Gamma(-\alpha_-)\Gamma(-\alpha_+) \\ &- \left(\frac{x}{2}\right)^{\bar\nu} \frac{\sin(\pi\alpha_+)}{\pi}\,\frac{\Gamma(-\alpha_+)\Gamma(\alpha_-)\Gamma(1+\nu)\Gamma(-\bar\nu)}{\Gamma(1-\mu+\alpha_-)\Gamma(\mu+\alpha_-)} \\ &- \left(\frac{x}{2}\right)^{-\bar\nu} \frac{\sin(\pi\alpha_-)}{\pi}\,\frac{\Gamma(-\alpha_-)\Gamma(\alpha_+)\Gamma(1+\nu)\Gamma(\bar\nu)}{\Gamma(1-\mu+\alpha_+)\Gamma(\mu+\alpha_+)}, \end{aligned} \tag{D.28}$$

valid for ν and $\bar\nu$ both pure imaginary.

We need these formulas for two linearly independent functions. Let us now turn to $j = 5$ [cf. eq. (D.4)]:

$$\begin{aligned} f_{\rm I}^{(5)}(x) &= \int_{C_1} dy\,y^{1+\nu}H_\nu^{(1)}(xy)(1+y^2)^{-\mu}u_5(-y^2) \\ &= \int_{C_1} dy\,y^{1+\nu}(-y^2)^{-\nu}H_\nu^{(1)}(xy)(1+y^2)^{-\mu} \\ &\quad\cdot F(1-\mu-\alpha_-,1-\mu-\alpha_+;1-\nu;-y^2) \end{aligned} \tag{D.29}$$

where we have taken u_5 from eq. (4.21). The phase factor must be chosen to allow for a cut along the negative imaginary axis (starting at the origin), i.e.,

$$(-y^2)^{-\nu} = e^{i\pi\nu}(y^2)^{-\nu}. \tag{D.30}$$

Since [10]

$$H_\nu^{(1)}(xy) = e^{-i\pi\nu}H_{-\nu}^{(1)}(xy), \tag{D.31}$$

the phases cancel and it follows that

$$f_{\rm I}^{(5)}(x) = \int_{C_1} dy\,y^{1-\nu}H_{-\nu}^{(1)}(xy)(1+y^2)^{-\mu}F(1-\mu-\alpha_-,1-\mu-\alpha_+;1-\nu;-y^2). \tag{D.32}$$

We note that the integrand is the same as for $f_{\rm I}^{(1)}(x)$, with the replacement $(\nu \leftrightarrow -\nu)$. Hence,

$$f_{\rm I}^{(5)}(x) = f_{\rm I}^{(1)}(x;\nu\to-\nu). \tag{D.33}$$

An expansion of $f_{\rm I}^{(5)}(x)$ in powers of x will thus yield as the first terms,

$$\begin{aligned} f_{\rm I}^{(5)}(x) ={}& \left(\frac{x}{2}\right)^{-\nu} \frac{\sin(\pi\mu)}{\pi}\Gamma(\alpha_+)\Gamma(\alpha_-) \\ &+ \left(\frac{x}{2}\right)^{\bar\nu} \frac{\sin(\pi\alpha_-)}{\pi}\,\frac{\Gamma(\alpha_-)\Gamma(-\alpha_+)\Gamma(1-\nu)\Gamma(-\bar\nu)}{\Gamma(1-\mu-\alpha_+)\Gamma(\mu-\alpha_+)} \\ &+ \left(\frac{x}{2}\right)^{-\bar\nu} \frac{\sin(\pi\alpha_+)}{\pi}\,\frac{\Gamma(\alpha_+)\Gamma(-\alpha_-)\Gamma(1-\nu)\Gamma(\bar\nu)}{\Gamma(1-\mu-\alpha_-)\Gamma(\mu-\alpha_-)}. \end{aligned} \tag{D.34}$$

The above analysis was aimed at determining the leading powers. Since we know on general grounds that relations of the type (D.1) exist, we may replace the various leading powers by the complete series. Thus, we obtain from eqs. (C.1) and (C.4):

$$f_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \frac{\sin(\pi\mu)}{\pi}\Gamma(-\alpha_-)\Gamma(-\alpha_+) + O(x^{\nu+2}). \tag{D.35}$$

By comparing the leading terms, we are led to the following results:

$$f_{\rm I}^{(1)}(x) = 2f_\nu(x) - 2\frac{\sin(\pi\alpha_+)}{\sin(\pi\mu)}\,\frac{\Gamma(1+\nu)\Gamma(-\bar\nu)}{\Gamma(1-\mu+\alpha_-)\Gamma(\mu+\alpha_-)}\,f_{\bar\nu}(x)$$

$$-2\frac{\sin(\pi\alpha_-)}{\sin(\pi\mu)}\frac{\Gamma(1+\nu)\Gamma(\bar\nu)}{\Gamma(1-\mu+\alpha_+)\Gamma(\mu+\alpha_+)}f_{-\bar\nu}(x), \tag{D.36}$$

$$f_{\rm I}^{(5)}(x) = 2f_{-\nu}(x) + 2\frac{\sin(\pi\alpha_-)}{\sin(\pi\mu)}\frac{\Gamma(1-\nu)\Gamma(-\bar\nu)}{\Gamma(1-\mu-\alpha_+)\Gamma(\mu-\alpha_+)}f_{\bar\nu}(x)$$

$$+2\frac{\sin(\pi\alpha_+)}{\sin(\pi\mu)}\frac{\Gamma(1-\nu)\Gamma(\bar\nu)}{\Gamma(1-\mu-\alpha_-)\Gamma(\mu-\alpha_-)}f_{-\bar\nu}(x). \tag{D.37}$$

The fact that these combinations of series are actually free of terms that behave like e^x as $x \to \infty$ can be checked using the asymptotic form, eq. (C.33).

Appendix E. Summary on the Basis $f^{(1)}(x)$ and $f^{(5)}(x)$

In this appendix we shall give explicitly the functions $f^{(1)}$ and $f^{(5)}$ which we have used as a basis in our description of the interior region, together with some of their properties. These functions are actually the functions $f_{\rm I}^{(1)}$ and $f_{\rm I}^{(5)}$ of Class I, described in appendix A, with particular choices for the parameters: $p = -\mu$, $\nu = \nu_+$, and $\bar\nu = \nu_- = i\beta$.

In order to get more compact formulas, we shall here use the abbreviations*

$$\alpha_\pm = \tfrac{1}{2}(\nu_+ \pm i\beta), \tag{E.1}$$

where

$$\nu_+ = [(\mu^2+4A^2)^{1/2} + \mu^2 + \tfrac{1}{4}]^{1/2}, \tag{E.2a}$$

$$\beta = [(\mu^2+4A^2)^{1/2} - \mu^2 - \tfrac{1}{4}]^{1/2}, \tag{E.2b}$$

and with [6]

$$\mu = [(j+\tfrac{1}{2})^2 - q^2]^{1/2}. \tag{E.3}$$

The quantity β is real, i.e.,

$$|A| > \tfrac{1}{2}(\mu^2 - \tfrac{1}{4}) = \tfrac{1}{2}[j(j+1) - q^2]. \tag{E.4}$$

For small x we have the convergent expansions [cf. eqs. (D.36) and (D.37)]

$$f^{(1)}(x) = 2S(\nu_+, i\beta; x) - 2\frac{\sin(\pi\alpha_+)}{\sin(\pi\mu)}\frac{\Gamma(1+\nu_+)\Gamma(-i\beta)}{\Gamma(1-\mu+\alpha_-)\Gamma(\mu+\alpha_-)}S(i\beta,\nu_+;x)$$

$$-2\frac{\sin(\pi\alpha_-)}{\sin(\pi\mu)}\frac{\Gamma(1+\nu_+)\Gamma(i\beta)}{\Gamma(1-\mu+\alpha_+)\Gamma(\mu+\alpha_+)}S(-i\beta,\nu_+;x), \tag{E.5}$$

$$f^{(5)}(x) = 2S(-\nu_+, i\beta; x) + 2\frac{\sin(\pi\alpha_-)}{\sin(\pi\mu)}\frac{\Gamma(1-\nu_+)\Gamma(-i\beta)}{\Gamma(1-\mu-\alpha_+)\Gamma(\mu-\alpha_+)}S(i\beta,\nu_+;x)$$

$$+2\frac{\sin(\pi\alpha_+)}{\sin(\pi\mu)}\frac{\Gamma(1-\nu_+)\Gamma(i\beta)}{\Gamma(1-\mu-\alpha_-)\Gamma(\mu-\alpha_-)}S(-i\beta,\nu_+;x), \tag{E.6}$$

* This notation has been used in a more generic sense in the other appendices.

with (cf. appendix C)

$$S(\nu,\bar{\nu};x) = S(\nu,-\bar{\nu};x) = f_\nu(x) = \sum_{k=0}^{\infty} b_k x^{\nu+2k}, \tag{E.7}$$

$$b_k = (-1)^k \left(\frac{1}{2}\right)^{1+\nu+2k} \frac{1}{k!} \frac{\Gamma(1+\nu)}{\Gamma(1+\nu+k)}$$

$$\cdot \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \frac{\Gamma(-\frac{1}{2}\nu+\frac{1}{2}\bar{\nu}-l)\Gamma(-\frac{1}{2}\nu-\frac{1}{2}\bar{\nu}-l)}{\Gamma(1-\mu-l)\Gamma(\mu-l)}. \tag{E.8}$$

At large x, it is convenient to make use of the asymptotic expansions found in appendix B. If we use two Kummer relations [9], we can express $f^{(1)}$ and $f^{(5)}$ in terms of $f^{(2)}$ and $f^{(6)}$,

$$f^{(1)}(x) = \frac{\Gamma(1+\nu_+)\Gamma(-1+2\mu)}{\Gamma(\mu+\alpha_-)\Gamma(\mu+\alpha_+)} f^{(2)}(x) + \frac{\Gamma(1+\nu_+)\Gamma(1-2\mu)}{\Gamma(1-\mu+\alpha_+)\Gamma(1-\mu+\alpha_-)} f^{(6)}(x), \tag{E.9}$$

$$f^{(5)}(x) = \frac{\Gamma(1-\nu_+)\Gamma(-1+2\mu)}{\Gamma(\mu-\alpha_+)\Gamma(\mu-\alpha_-)} f^{(2)}(x) + \frac{\Gamma(1-\nu_+)\Gamma(1-2\mu)}{\Gamma(1-\mu-\alpha_-)\Gamma(1-\mu-\alpha_+)} f^{(6)}(x), \tag{E.10}$$

where (cf. appendix B)

$$f^{(2)}(x) = \left(\frac{2}{\pi x}\right)^{1/2} e^{-x}\sin(\pi\mu) \cdot \sum_{n=0}\sum_{m=0} \frac{\Gamma(1-\mu+\alpha_++n)}{\Gamma(1-\mu+\alpha_+)} \frac{\Gamma(1-\mu+\alpha_-+n)}{\Gamma(1-\mu+\alpha_-)} \frac{\Gamma(2-2\mu)}{\Gamma(2-2\mu+n)} \cdot \frac{(-1)^n}{n!}\Gamma(1-\mu+n) \frac{\Gamma(-\frac{1}{2}+\mu-\nu_+-n+m)}{\Gamma(-\frac{1}{2}+\mu-\nu_+-n-m)} \frac{4^{-m}}{m!} \left(\frac{2}{x}\right)^{1-\mu+n+m}, \tag{E.11}$$

$$f^{(6)}(x) = \left(\frac{2}{\pi x}\right)^{1/2} e^{-x}\sin(\pi\mu) \sum_{n=0}\sum_{m=0} \frac{\Gamma(\mu+\alpha_++n)}{\Gamma(\mu+\alpha_+)} \frac{\Gamma(\mu+\alpha_-+n)}{\Gamma(\mu+\alpha_-)} \frac{\Gamma(2\mu)}{\Gamma(2\mu+n)} \cdot \frac{(-1)^n}{n!}\Gamma(\mu+n) \frac{\Gamma(\frac{1}{2}-\mu-\nu_+-n+m)}{\Gamma(\frac{1}{2}-\mu-\nu_+-n-m)} \frac{4^{-m}}{m!} \left(\frac{2}{x}\right)^{\mu+n+m}. \tag{E.12}$$

As pointed out in appendix B, $f^{(2)}$ and $f^{(6)}$ are related. If we let $\mu \leftrightarrow 1-\mu$ (but leave ν_+ and β unchanged), then we have

$$f^{(2)}(x) \leftrightarrow f^{(6)}(x). \tag{E.13}$$

Since ν_+ and β are real, $\alpha_- = \alpha_+^*$, and the basis is real,

$$f^{(1)}(x) = f^{(1)}(x)^*,$$
$$f^{(5)}(x) = f^{(5)}(x)^*. \tag{E.14}$$

Also, the two functions are related by

$$\nu_+ \leftrightarrow -\nu_+: \qquad f^{(1)}(x) \leftrightarrow f^{(5)}(x). \tag{E.15}$$

The functions $f^{(1)}(x)$ and $f^{(5)}(x)$ may be thought of as generalizations of modified Bessel functions. In order to illustrate this relationship, we shall multiply $f^{(1)}(x)$ and $f^{(5)}(x)$ by suitable constants, and consider the limit $\mu \to 0$. We first note that $f_\nu(x)$ is a generalization of the modified Bessel function $I_\nu(x)$,

$$\lim_{\mu\to 0} 2[\mu\Gamma(-\tfrac{1}{2}\nu+\tfrac{1}{2}\bar{\nu})\Gamma(-\tfrac{1}{2}\nu-\tfrac{1}{2}\bar{\nu})\Gamma(1+\nu)]^{-1} f_\nu(x) = I_\nu(x), \tag{E.16}$$

as follows from (C.17) and (C.18). In (E.16), we have $(\nu,\bar{\nu}) = (\nu_+, i\beta)$, $(-\nu_+, i\beta)$, $(i\beta,\nu_+)$, or $(-i\beta,\nu_+)$.

Let us then define the real functions

$$k^{(1)}(x) = \frac{\Gamma(1+\alpha_-)\Gamma(1+\alpha_+)}{2\Gamma(1+\nu_+)} f^{(1)}(x), \tag{E.17}$$

$$k^{(5)}(x) = \frac{\Gamma(1-\alpha_+)\Gamma(1-\alpha_-)}{2\Gamma(1-\nu_+)} f^{(5)}(x). \tag{E.18}$$

This is just a convenient rescaling of $f^{(1)}(x)$ and $f^{(5)}(x)$. These functions (E.17) and (E.18) have simple limits as $\mu \to 0$,

$$\lim_{\mu\to 0} k^{(1)}(x) = \lim_{\mu\to 0} k^{(5)}(x) = K_{i\beta_0}(x), \tag{E.19}$$

with $K_{i\beta_0}$ a modified Bessel function and

$$\beta_0 = (2|A| - \tfrac{1}{4})^{1/2}. \qquad \text{(E.20)}$$

In fig. 9 we compare plots of the functions $k^{(1)}(x)$, $k^{(5)}(x)$ and $K_{i\beta}(x)$ for $A = 3$, $\mu = 10^{-4}$ and $\mu = \sqrt{2}$. (The latter case corresponds to $|q| = \frac{1}{2}$ and $j = 1$.)

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TABLE HEADING

Table 1. Binding energies $\epsilon_{jn} = (M - E_{jn})/M$ for a few values of $A = \frac{1}{4}\kappa$, j, and n, with $|q| = \frac{1}{2}$.

Exact: Numerical results of paper I [2],

WBA: Weak-binding approximation, eq. (6.13).

(The values for $j = 0$ are from table 1 of paper III [5].) Where both entries are missing, the state does not exist.

FIGURE CAPTIONS

Fig. 1. The contour of integration C_1 that defines the functions $f^{(j)}(x)$ of eq. (4.25). Other contours are discussed in appendix A.

Fig. 2. Plot of binding energies $(M - E_{jn})/M$ versus $A = \frac{1}{2}\kappa|q| = \frac{1}{4}\kappa$, where κ is the extra magnetic moment. The binding energies are as given by (6.13), with j the angular momentum. There are also bound states at $E_{j0} = 0$ [3].

Fig. 3. Squares of normalized radial wave functions h_1^2, h_2^2, h_3^2, and h_4^2 versus ρ. These are defined by the decomposition (2.1). For large ρ they are given by (3.12), whereas for small ρ they are given by (4.29). The parameters considered are $|q| = \frac{1}{2}$, $j = 1$, and $A = 2$. Results for three values of n are given. The minima are actually zeroes of the h_i.

Fig. 4. Same as fig. 3 but with $j = 2$, $A = 3$, and for $n = 1$ and 2.

Fig. 5. Contours of integration in the complex y-plane. The branch singularities are indicated by crosses.

Fig. 6. Contours of integration used for the evaluation of A_k.

Fig. 7. The contour C_1 divided into the two parts C_a and C_b, which make angles of $\pm 45°$ with the real y-axis. Note that the condition (D.11) is not satisfied on C_a.

Fig. 8. The contour C_1 as it appears in the $w = y^2$ plane (solid, ACDF). We deform the contour as indicated by the dashed curves (ABCDEF). The phase of $w = y^2$ at E and B is 0 and 2π, respectively.

Fig. 9. Comparison of the functions $k^{(1)}(x)$ and $k^{(6)}(x)$ of (E.17) and (E.18) with the modified Bessel function $K_{i\beta}(x)$ for $A = 3$ and a) $\mu = 10^{-4}$ and b) $\mu = \sqrt{2}$ ($|q| = \frac{1}{2}$ and $j = 1$). Solid: $K_{i\beta}(x)$; dashed: $k^{(1)}(x)$; dash-dotted: $k^{(6)}(x)$. In a) $K_{i\beta}(x)$ and $k^{(1)}(x)$ are indistinguishable.

Table 1

Binding energies $\epsilon_{jn} = (M - E_{jn})/M$ for a few values of $A = \frac{1}{4}\kappa$, j, and n, with $|q| = \frac{1}{2}$
Exact: Numerical results of paper I [2],
WBA: Weak-binding approximation, eq. (6.13).
(The values for $j = 0$ are from table 1 of paper III [5].) Where both entries are missing, the state does not exist.

j	n	Method	$A = \frac{1}{4}\kappa$					
			1	2	5	10	20	50
3	3	Exact				$7.246 \cdot 10^{-4}$	$1.622 \cdot 10^{-2}$	$7.834 \cdot 10^{-2}$
		WBA				$7.243 \cdot 10^{-4}$	$1.625 \cdot 10^{-2}$	$7.826 \cdot 10^{-2}$
	2	Exact				$6.656 \cdot 10^{-3}$	$5.343 \cdot 10^{-2}$	$1.557 \cdot 10^{-1}$
		WBA				$6.634 \cdot 10^{-3}$	$5.339 \cdot 10^{-2}$	$1.530 \cdot 10^{-1}$
	1	Exact				$6.238 \cdot 10^{-2}$	$1.810 \cdot 10^{-1}$	$3.222 \cdot 10^{-1}$
		WBA				$6.077 \cdot 10^{-2}$	$1.754 \cdot 10^{-1}$	$2.992 \cdot 10^{-1}$
2	3	Exact			$4.819 \cdot 10^{-5}$	$3.701 \cdot 10^{-3}$	$2.284 \cdot 10^{-2}$	$8.387 \cdot 10^{-2}$
		WBA			$4.819 \cdot 10^{-5}$	$3.707 \cdot 10^{-3}$	$2.293 \cdot 10^{-2}$	$8.376 \cdot 10^{-2}$
	2	Exact			$1.095 \cdot 10^{-3}$	$1.993 \cdot 10^{-2}$	$6.747 \cdot 10^{-2}$	$1.634 \cdot 10^{-1}$
		WBA			$1.095 \cdot 10^{-3}$	$1.999 \cdot 10^{-2}$	$6.755 \cdot 10^{-2}$	$1.603 \cdot 10^{-1}$
	1	Exact			$2.499 \cdot 10^{-2}$	$1.087 \cdot 10^{-1}$	$2.051 \cdot 10^{-1}$	$3.308 \cdot 10^{-1}$
		WBA			$2.490 \cdot 10^{-2}$	$1.078 \cdot 10^{-1}$	$1.990 \cdot 10^{-1}$	$3.066 \cdot 10^{-1}$
1	3	Exact		$9.915 \cdot 10^{-7}$	$7.155 \cdot 10^{-4}$	$6.769 \cdot 10^{-3}$	$2.730 \cdot 10^{-2}$	$8.739 \cdot 10^{-2}$
		WBA	$3.242 \cdot 10^{-19}$	$9.915 \cdot 10^{-7}$	$7.159 \cdot 10^{-4}$	$6.786 \cdot 10^{-3}$	$2.741 \cdot 10^{-2}$	$8.725 \cdot 10^{-2}$
	2	Exact		$8.498 \cdot 10^{-5}$	$6.721 \cdot 10^{-3}$	$2.994 \cdot 10^{-2}$	$7.612 \cdot 10^{-2}$	$1.679 \cdot 10^{-1}$
		WBA	$4.171 \cdot 10^{-13}$	$8.499 \cdot 10^{-5}$	$6.742 \cdot 10^{-3}$	$3.009 \cdot 10^{-2}$	$7.619 \cdot 10^{-2}$	$1.647 \cdot 10^{-1}$
	1	Exact	$5.367 \cdot 10^{-7}$	$7.238 \cdot 10^{-3}$	$6.306 \cdot 10^{-2}$	$1.344 \cdot 10^{-1}$	$2.188 \cdot 10^{-1}$	$3.361 \cdot 10^{-1}$
		WBA	$5.367 \cdot 10^{-7}$	$7.285 \cdot 10^{-3}$	$6.350 \cdot 10^{-2}$	$1.334 \cdot 10^{-1}$	$2.118 \cdot 10^{-1}$	$3.110 \cdot 10^{-1}$
0	3	Exact	$4.134 \cdot 10^{-7}$	$3.621 \cdot 10^{-5}$	$1.423 \cdot 10^{-3}$	$8.478 \cdot 10^{-3}$	$2.948 \cdot 10^{-2}$	$8.910 \cdot 10^{-2}$
		WBA	$4.134 \cdot 10^{-7}$	$3.622 \cdot 10^{-5}$	$1.424 \cdot 10^{-3}$	$8.503 \cdot 10^{-3}$	$2.961 \cdot 10^{-2}$	$8.893 \cdot 10^{-2}$
	2	Exact	$4.776 \cdot 10^{-5}$	$9.281 \cdot 10^{-4}$	$1.061 \cdot 10^{-2}$	$3.479 \cdot 10^{-2}$	$8.017 \cdot 10^{-2}$	$1.702 \cdot 10^{-1}$
		WBA	$4.776 \cdot 10^{-5}$	$9.290 \cdot 10^{-4}$	$1.065 \cdot 10^{-2}$	$3.496 \cdot 10^{-2}$	$8.020 \cdot 10^{-2}$	$1.668 \cdot 10^{-1}$
	1	Exact	$5.488 \cdot 10^{-3}$	$2.361 \cdot 10^{-2}$	$7.932 \cdot 10^{-2}$	$1.452 \cdot 10^{-1}$	$2.249 \cdot 10^{-2}$	$3.386 \cdot 10^{-1}$
		WBA	$5.519 \cdot 10^{-3}$	$2.383 \cdot 10^{-2}$	$7.968 \cdot 10^{-2}$	$1.437 \cdot 10^{-1}$	$2.173 \cdot 10^{-2}$	$3.130 \cdot 10^{-1}$

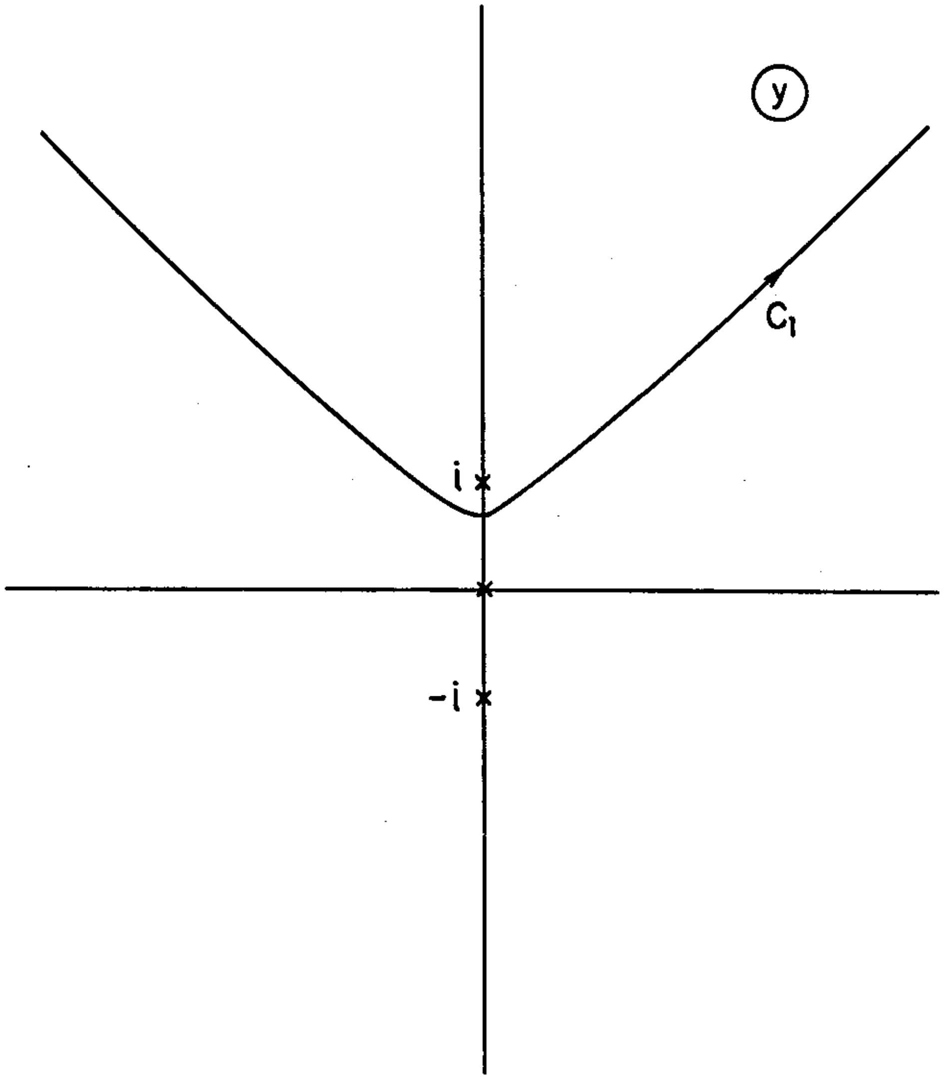

Figure 1.

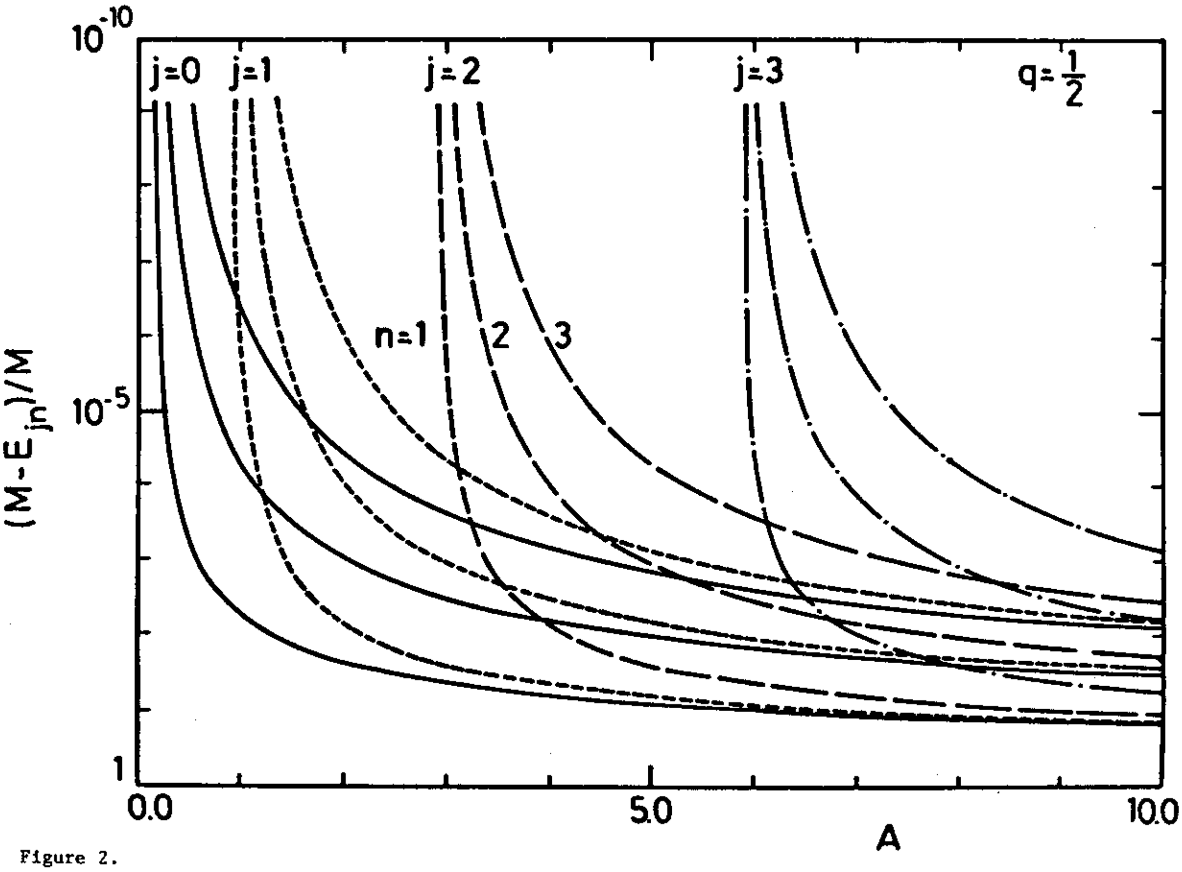

Figure 2.

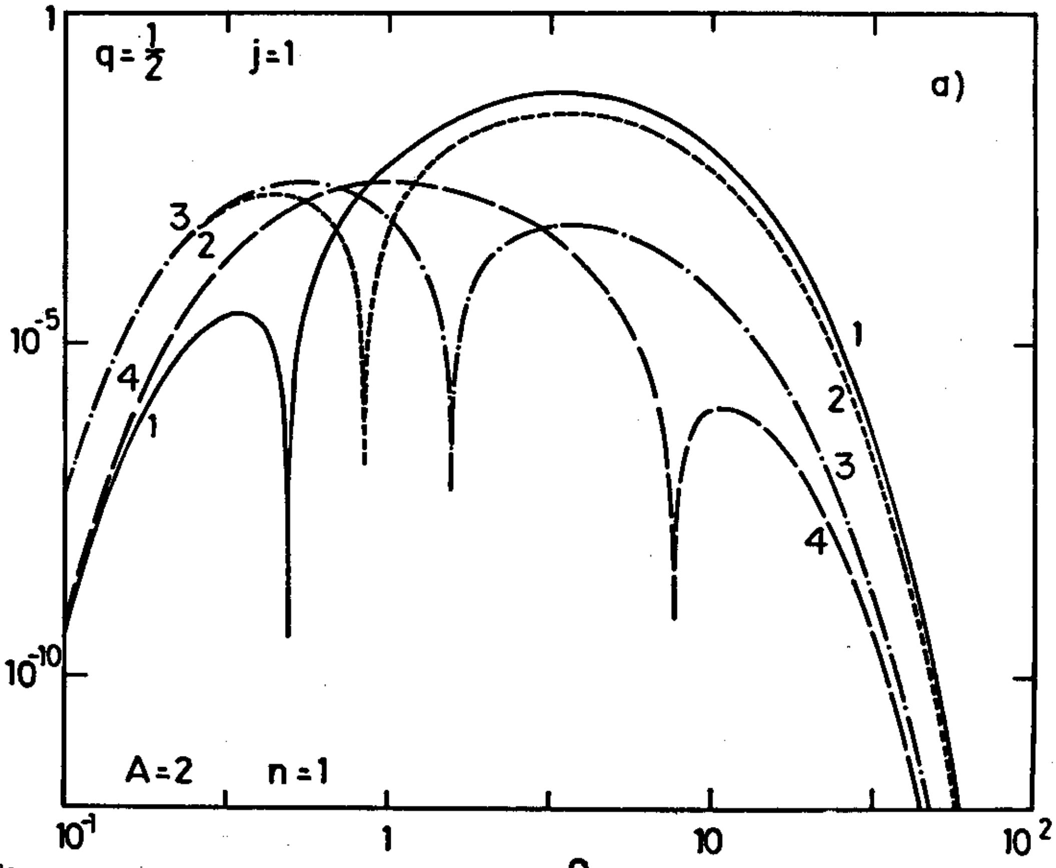

Figure 3a.

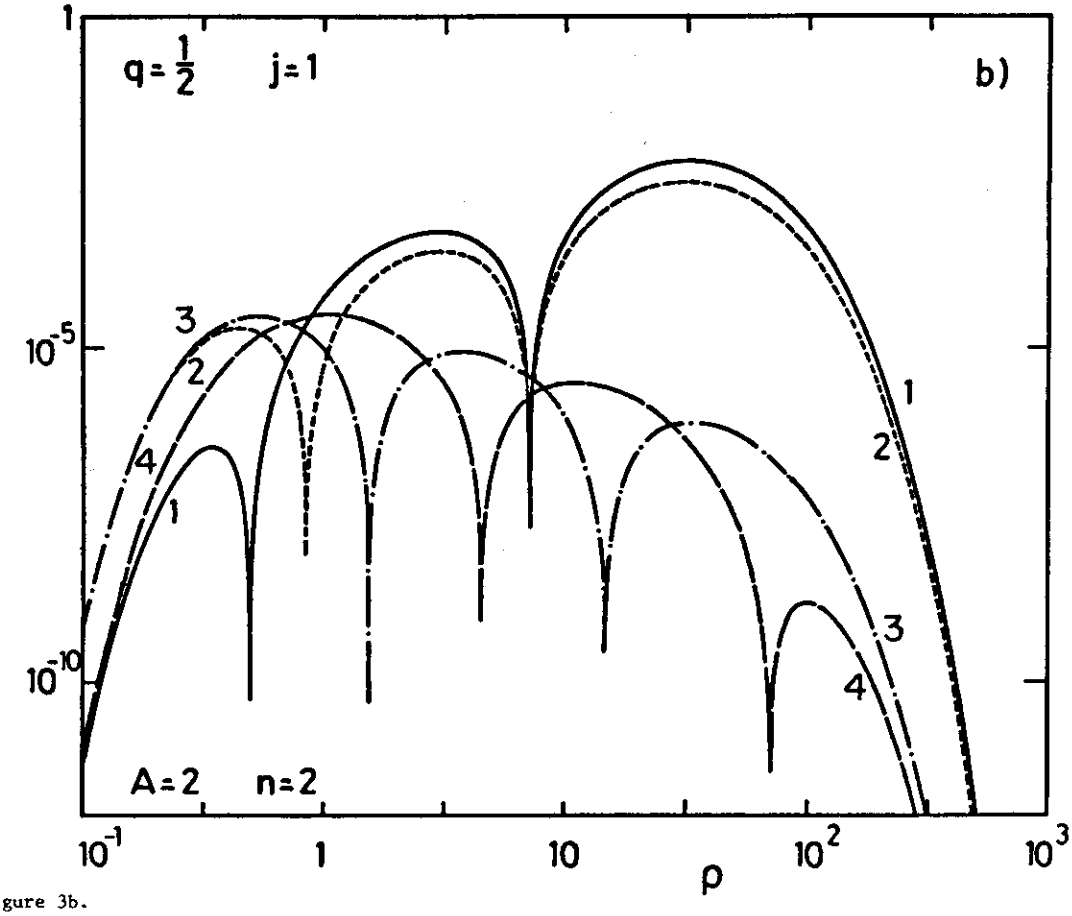

Figure 3b.

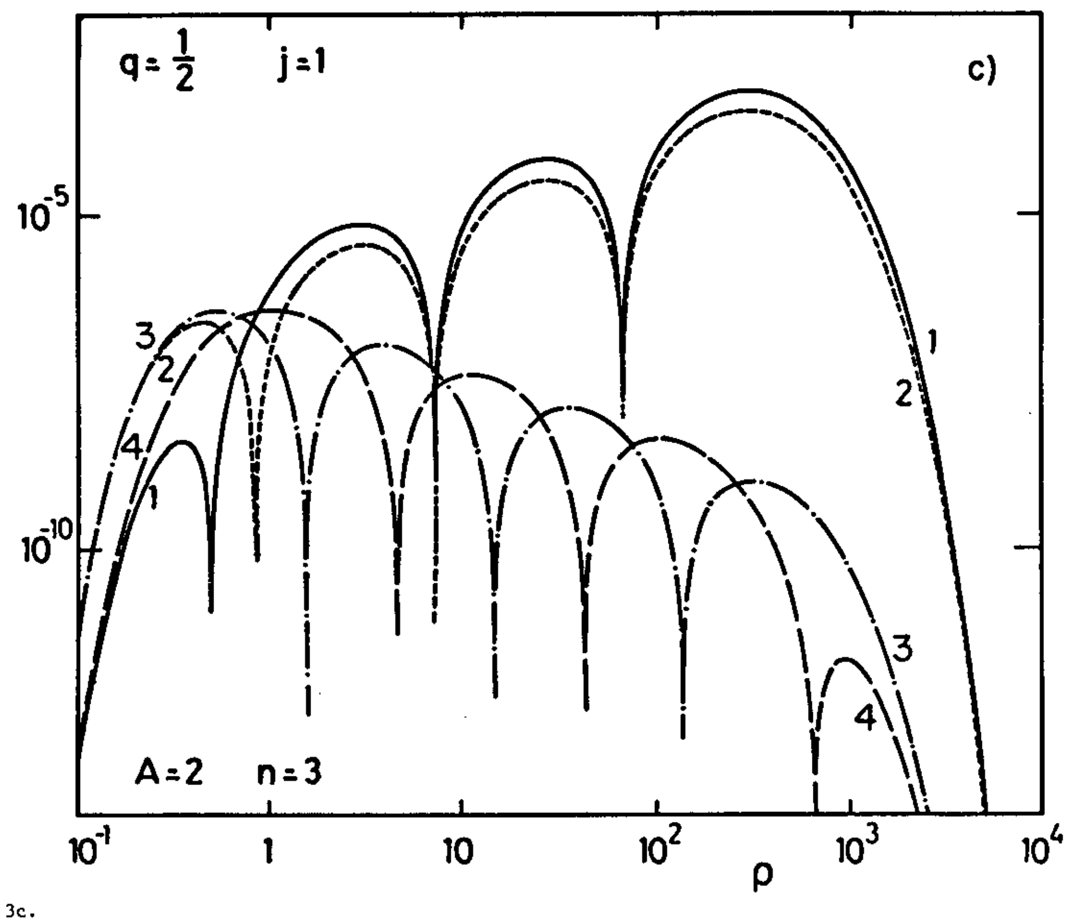

Figure 3c.

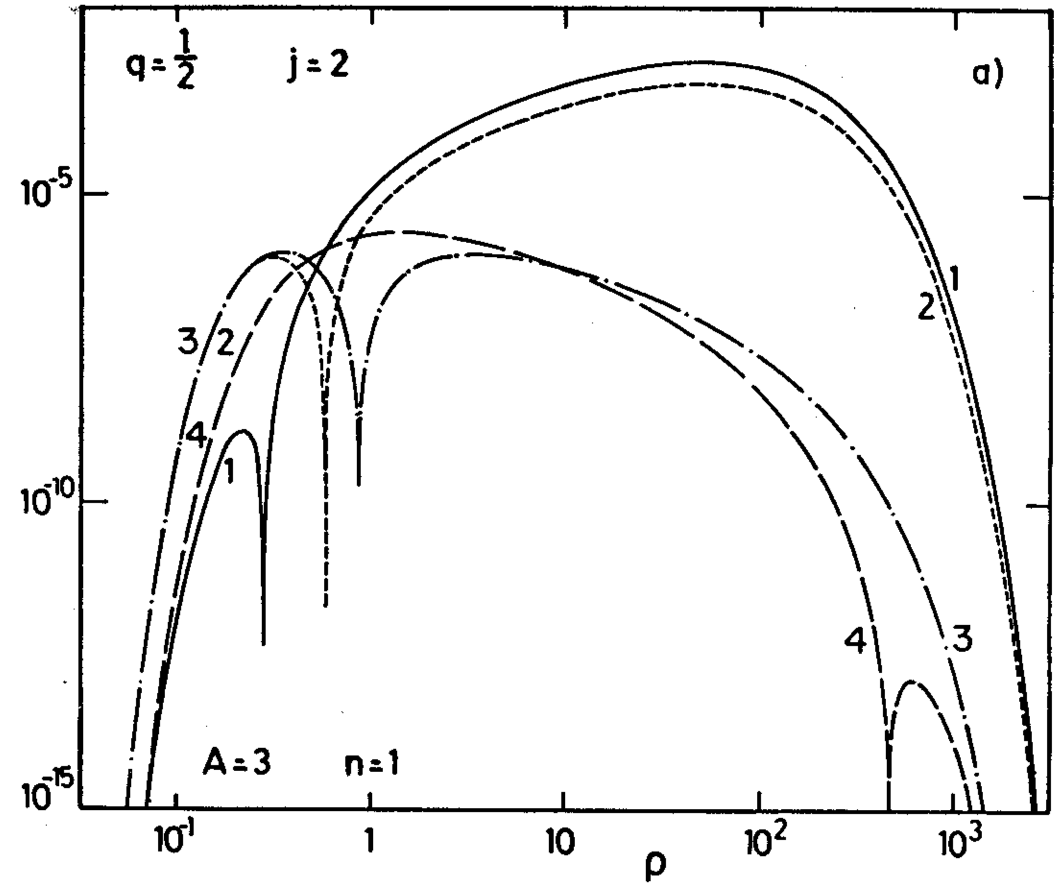

Figure 4a.

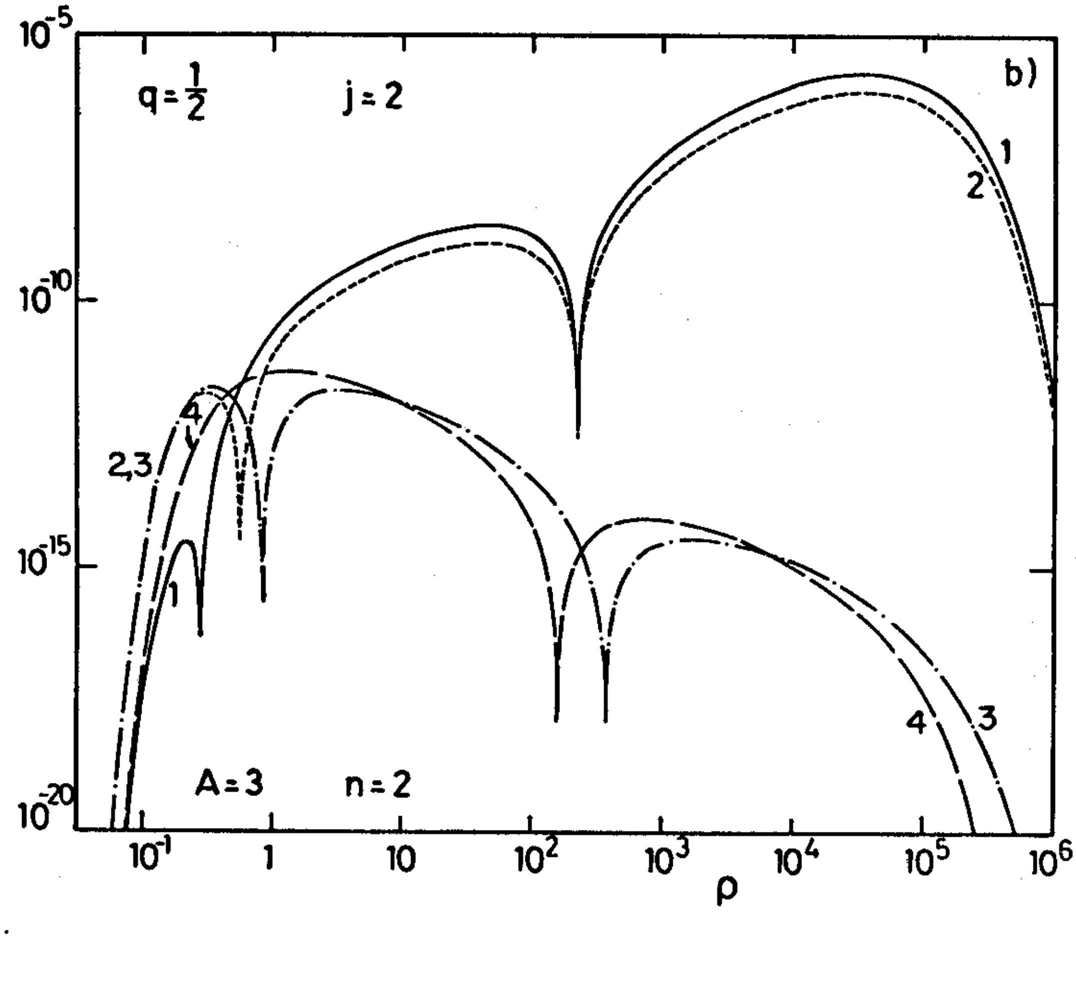

Figure 4b.

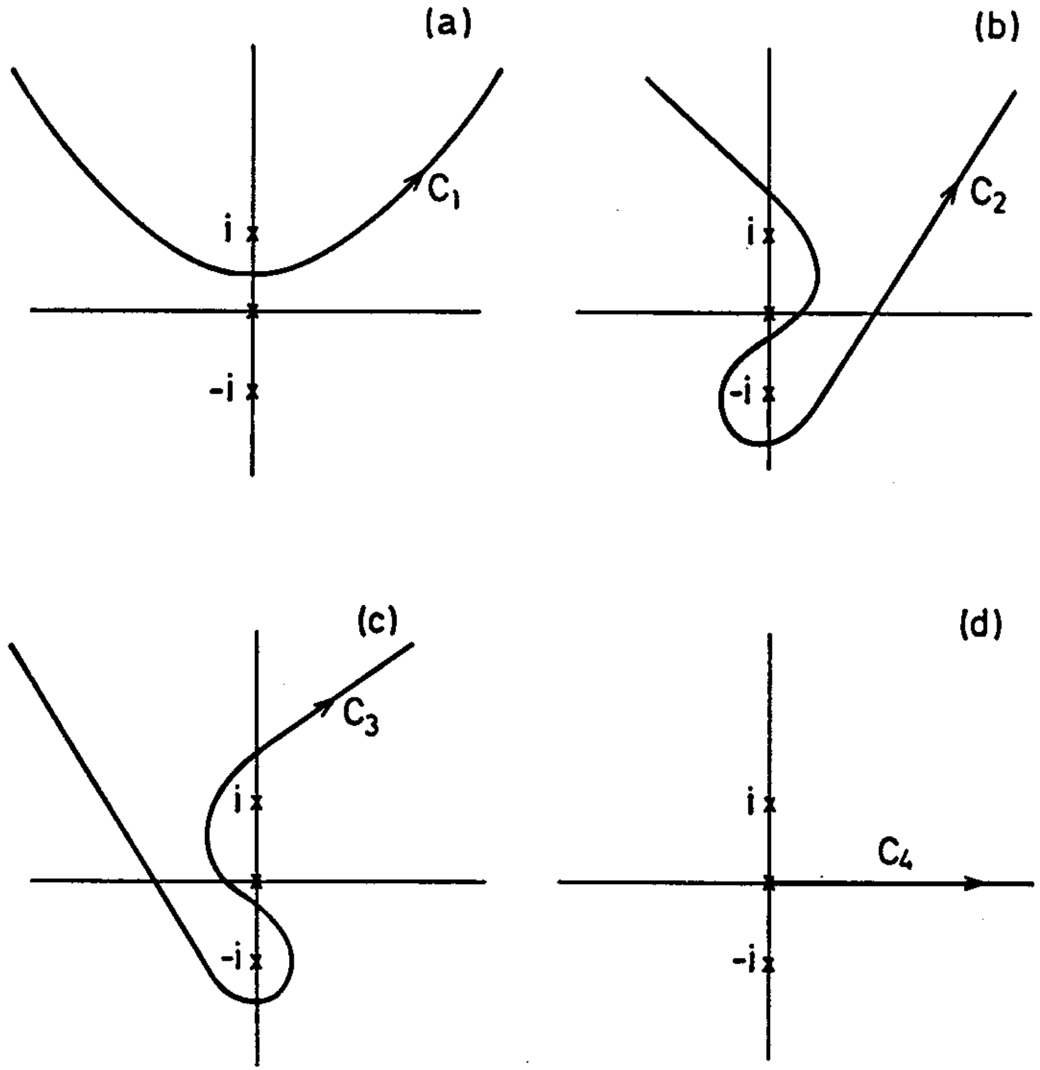

Figure 5.

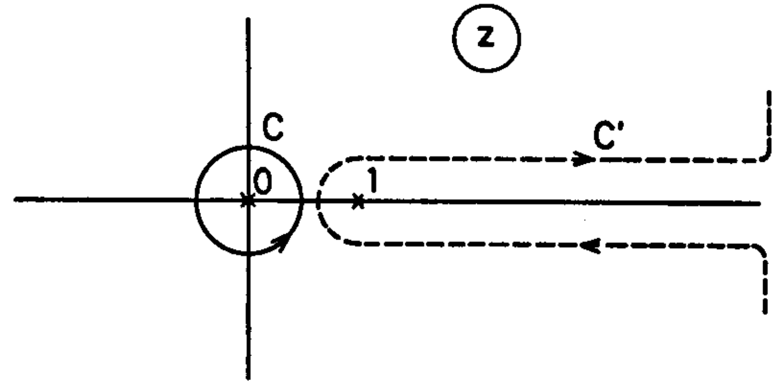

Figure 6.

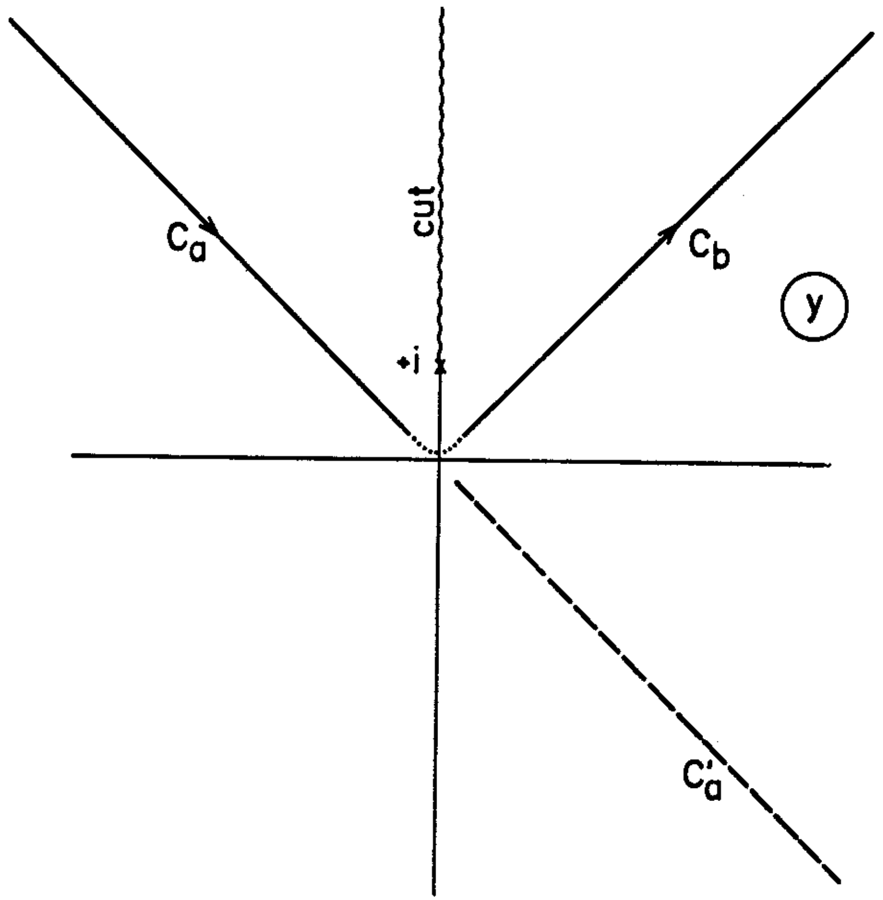

Figure 7.

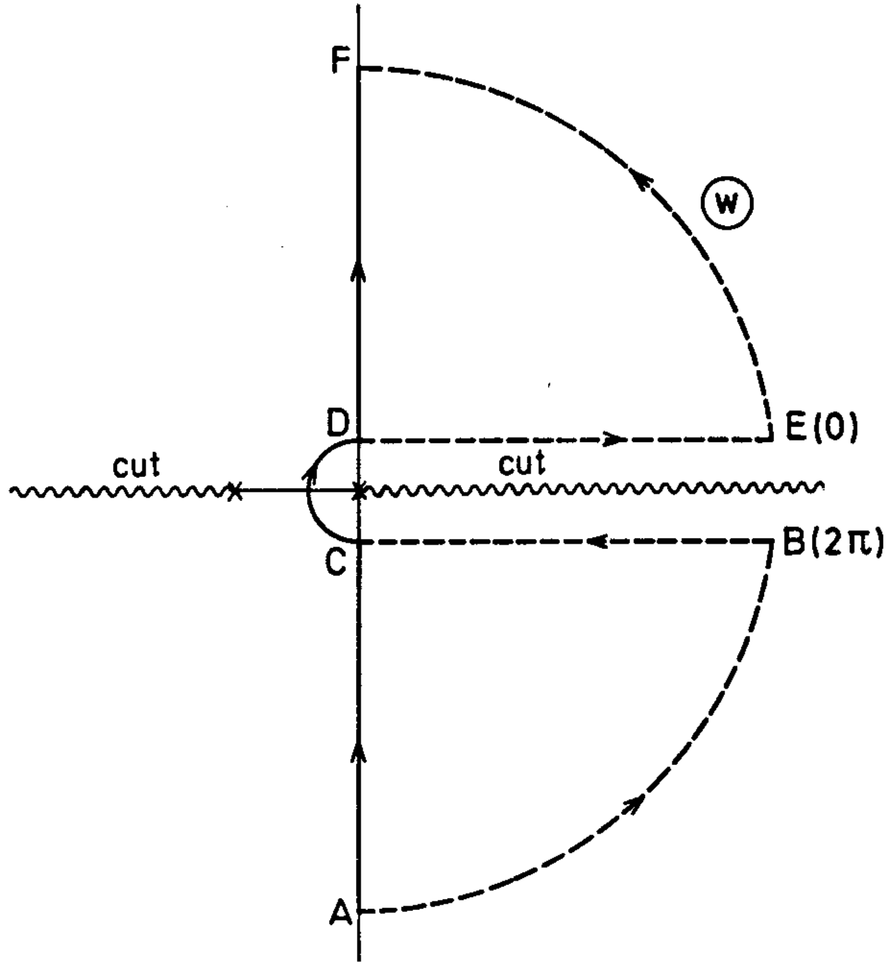

Figure 8.

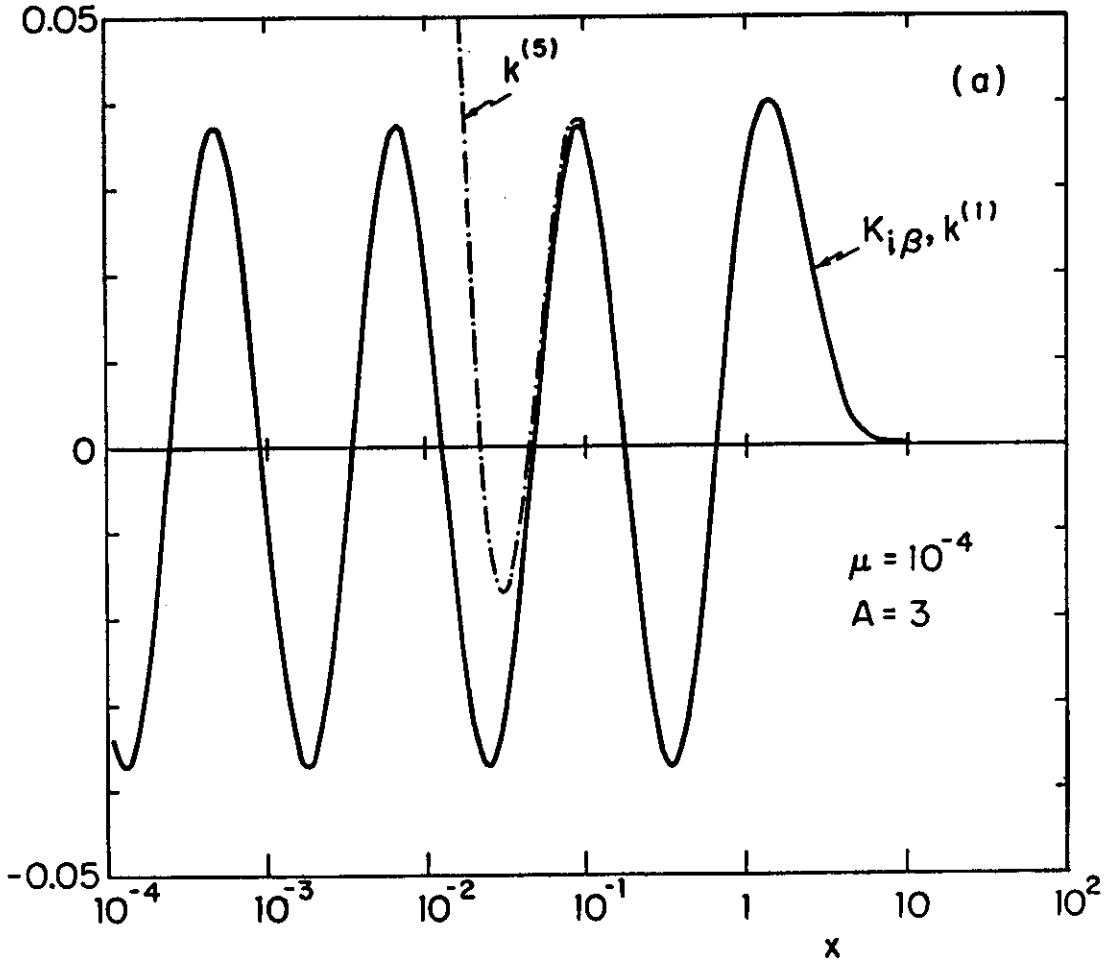

Figure 9a.

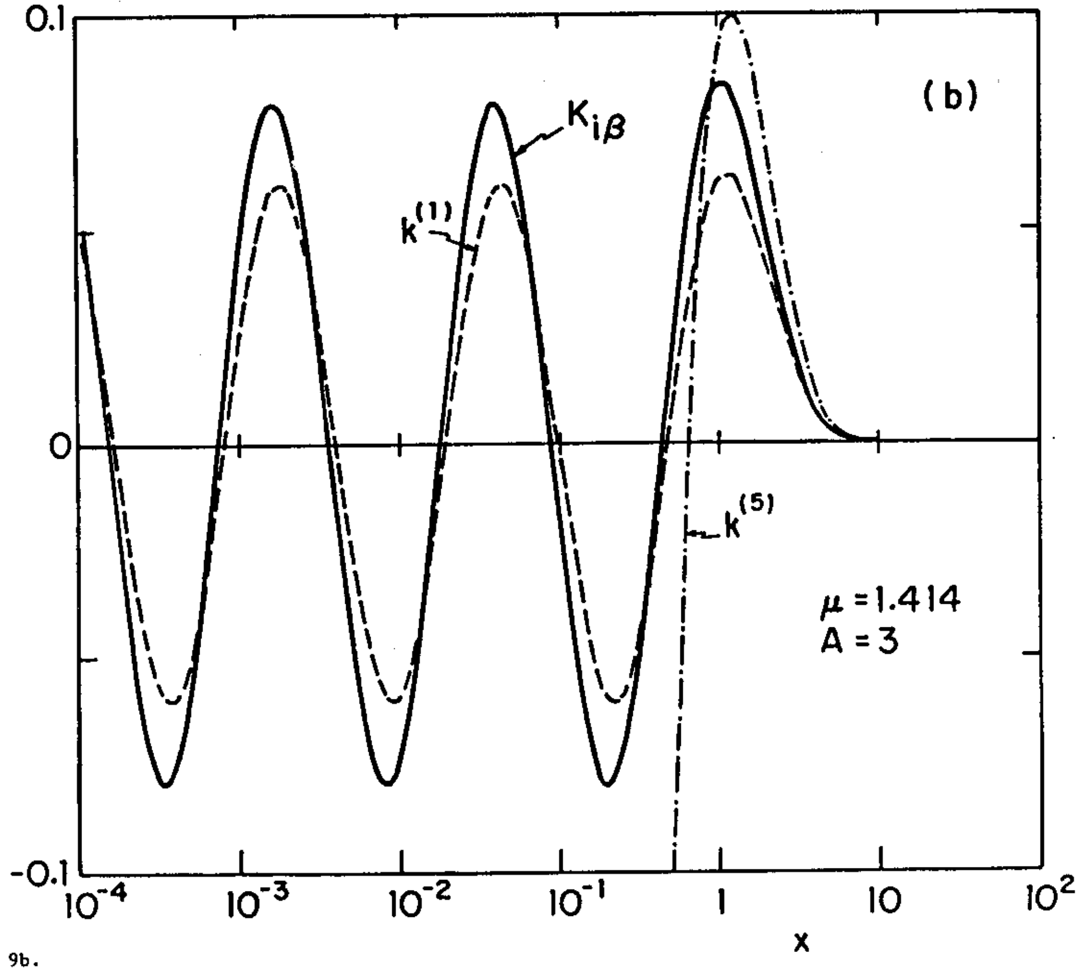

Figure 9b.