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Nonperturbative Renormalization of Quantum Gauge Theories +

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Contents:

Abstract

A systematic nonperturbative formulation of quantum gauge theories is given, which defines the renormalized theory as an appropriate limit of a lattice-regularized one. First some features of the regularization in particular with respect to fermions and to gauge invariance are discussed. The conditions for the existence of the limit, which are those of general renormalizability, are worked out in detail. A fundamental parameter singularity and basic equivalence relations are important features. Mass scales and RG invariants are defined in a general way. Directly measurable quantities and order parameters are considered. Various types of β functions are studied. The present knowledge about fundamental singularities in specific theories and the urgent need to investigate ones with matter fields are discussed. The consequent manner of applying KW transformations to QFT and the occurring restrictions are pointed out. An additional condition arising for correlation functions and its implications for particular transformation types are considered. Using an appropriate auxiliary-parameter representation only long-distance properties of the action enter QFT and a characterization of its parameters becomes possible. The need of investigating suitable classes of KW transformations is stressed.

1. Introduction

It is rather clear by now that particle physics is to be described by quantum gauge theories. For this purpose, however, perturbative quantum field theory is actually not sufficient. In addition to the obvious defects that masses cannot be calculated and that confinement cannot be explained, increasingly also the analysis of amplitudes suffers from scheme-dependence ambiguities and from unknown nonperturbative effects. A severe conceptual drawback is that the very definition of the theory rests on the perturbative expansion. Thus, a truely nonperturbative framework is urgently needed. This would also allow to attack unification problems from the dynamical side.

Renormalization is crucial for the definition of a quantum field theory (QFT). Historically it means the removal of divergences from the terms of the perturbative expansion. The more appropriate view is to consider it as the definition of the theory by a particular limit, which is a definite mathematical concept. Any limit is based on a sequence of certain quantities. In the present context the respective quantities are so-called regularized functions. The regularization then depends on the numbering of the sequence.

A nonperturbative formulation clearly must be based on a nonperturbative regularization. Then for gauge theories in four dimensions so far the only possibility is lattice regularization. The lattice will be considered strictly in this sense here (which is to be contrasted to the applications where it provides approximations or models). Therefore, one should keep in mind that not the lattice quantities but only their limits are of physical interest.

The lattice formulation of gauge theories [1,2] has the additional virtue of allowing gauge-invariant quantization. Thus, gauge fixing with its unpleasant features such as ambiguities in certain gauges [3] and operator ordering problems [4] can be avoided. In correlations of variant fields gauge fixing can be shown to produce implicitly a highly artificial average of invariant fields [5]. Instead of this in the gauge-invariant formulation one has to specify appropriate invariant fields [6]. Such choices are expected to be restricted by the limit.

A difficulty which appears in lattice regularization is the so-called fermion degeneracy [7]. There are two ways to handle the additional degrees of freedom: to suppress them in the limit [7] or to interpret them (after some reduction) in a phenomenological manner as flavors [8]. Because for the latter view a consequent justification is missing, here the first way is used. The suppression mechanism will later be seen to fit rather naturally into the general representation of the limit. A hint where to look for a deeper understanding comes from the direct relation to the axial-vector anomaly [9,10].

Since the lattice provides the only nonperturbative regularization, it is important to be able to include neutrinos too. For this purpose the most general form of the suppression term for ordinary fermions [11] can be extended to include handed fermions as limiting cases [12], which can be cast into a simple representation with one additional parameter [13].

The envisaged general limit [6,14] which defines QFT in a nonperturbative way, due to the lattice regularization used is a particular type of continuum limit. Its ultimate check can only occur by comparison with experiment. In its formulation the aim is to use a minimum of ingredients and to start as much as possible from first principles. Particular features will be that parameter singularities play a fundamental role and that the procedure is based on renormalization group (RG) invariants. The existence of this limit means general (nonperturbative) renormalizability.

Some guide for the lattice formulation is that it should also lead to the usual classical and perturbative results. The classsical continuum limit is essentially that of the action alone. The perturbative one gives perturbative QFT with the lattice providing a particular regularization. Because it occurs for the terms of an expansion, which poses a problem of interchange of limits, and because the expansion is at best asymptotic, the mathematical nature of the perturbative limit for any regularization is rather unclear. Furthermore, it does not refer to parameter singularities and (being based on normalization conditions for correlation functions) is not RG invariant.

It appears worthwhile to remember here that the RG of QFT is that of transformations between different fixings of the scheme. The concept has been introduced by Stueckelberg and Peterman [15] for continuous transformations in the perturbative framework. The RG of Gell-Mann and Low [16] is the dilatation subgroup of this continuous one. In the nonperturbative formulation a precise definition of the general RG (also including discrete transformations) will be given [6,14].

Because of some formal similarity to the dilatation subgroup of the RG, the Kadanoff-Wilson (KW) transformations [17] in statistical mechanics are called RG transformations, too. This will not be done here since both concepts will be used and must not be mixed up. The consequent use of KW transformations in QFT will basically involve a rescaling within the correlation functions which occur in the definition of the QFT limit. Therefore a study of (gauge invariant) transformations of correlation functions becomes necessary which goes beyond the usual considerations of transformations of partition functions.

In the following a systematic formulation of the indicated nonperturbative framework will be given. The knowledge presently available about the respective structure of theories of physical interest will be discussed. The way in which KW transformations become useful for QFT will be pointed out.

In particular, after specifying the lattice formulation in sect.2, the general limit will be considered in sect.3. Then in sect.4 the information available about particular theories will be discussed and in sect.5 the aspects of using KW transformations in QFT will be treated. Finally in sect.6 some general conclusions will be collected.

2. Lattice formulation

2.1. Preliminaries

2.1.1. Definitions and remarks

A hypercubic lattice is used since more complicated lattices so far appear to have no advantage. Euclidean space of four dimensions is considered, the results in which are to give those in Minkowski space properly by analytic continuation. The quantities to start from are correlation functions

$$\langle \sigma \rangle = \frac{\int e^S \sigma}{\int e^S} \tag{2.1}$$

where S is the action and σ some function of fields. The abbreviation $\int$ stands for the product of all integrations, i.e. gauge-field group integrations $\int_U$, fermion Grassmann integrations $\int_\psi$ and, if present, scalar-field integrations $\int_\phi$.

A typical example for S is the QCD-type action

$$S = -\sum_{n',n} \bar\psi_{n'} \Big(\sum_\lambda (\gamma_\lambda D_\lambda - \eta W_\lambda) + \tilde m \Big)_{n'n} \psi_n + \frac{1}{g^2} \sum_p \mathrm{Tr}\,(U_p + U_p^\dagger - 2) \tag{2.2}$$

where

$$\begin{aligned} U_p &= U^\dagger_{\sigma n} U^\dagger_{\lambda, n+\hat\sigma} U_{\sigma, n+\hat\lambda} U_{\lambda n} \\ D_{\lambda n' n} &= \big(U^\dagger_{\lambda n'} \delta_{n'+\hat\lambda, n} - U_{\lambda n} \delta_{n', n+\hat\lambda} \big)/2 \\ W_{\lambda n' n} &= \big(U^\dagger_{\lambda n'} \delta_{n'+\hat\lambda, n} + U_{\lambda n} \delta_{n', n+\hat\lambda} - 2\delta_{n'n} \big)/2 \end{aligned} \tag{2.3}$$

and

$$\eta = r \exp(i\gamma_5 \theta) \quad \text{with} \quad r > 0 . \tag{2.4}$$

The term $\sum_\lambda \eta W_\lambda$ in (2.2) serves to suppress unwanted fermion degrees of freedom. It has been introduced for r=1 and θ=0 by Wilson [7] and first given in its general form by the present author [11]. It breaks the chiral symmetry which the classical theory otherwise would have for $\tilde m$=0 (which, however, in the quantized theory anyway does not persist). It is interesting that (2.2) for $\tilde m$=0 is invariant under

$$\psi \to e^{i\alpha\gamma_5}\psi\,,\quad \bar\psi \to \bar\psi\, e^{i\alpha\gamma_5}\,,\quad \theta \to \theta - 2\alpha \tag{2.5}$$

i.e. under a generalized chiral transformation.

It is important to realize that the lattice formulation does not involve a length. The latter is only introduced by the limit prescription which is imposed. Thus, for the classical limit (i.e. for that of S alone) one uses the correspondences

$$\begin{aligned} &x_\lambda \mathrel{\hat=} a\, n_\lambda\,, \quad k_\lambda \mathrel{\hat=} a^{-1}\alpha_\lambda\,, \quad m \mathrel{\hat=} a^{-1}\tilde m \\ &\psi(x) \mathrel{\hat=} a^{-\frac{3}{2}}\psi_n\,, \quad \phi(x) \mathrel{\hat=} a^{-1}\phi_n \\ &g A_\lambda(x) \mathrel{\hat=} a^{-1} B_{\lambda n} \quad \text{where} \quad U_{\lambda n} = e^{iB_{\lambda n}} \end{aligned} \tag{2.6}$$

to introduce the lattice spacing a. Then for $a \to 0$, one can e.g. check that (2.2) gives the usual continuum action.

2.1.2. Schwinger-Dyson equations and Ward Takahashi identities

The integrations in (2.1) have invariance properties which are not only of conceptual importance but also give rise to relations [5,11] useful in practice (and needed in the following).

Schwinger-Dyson equations for fermions follow from the fact that $(\partial/\partial\psi_{n\beta})^2=0$ such that because of $\int_\psi = \prod_{n,\beta} \frac{\partial}{\partial\psi_{n\beta}}\frac{\partial}{\partial\bar\psi_{n\beta}}$ one has the general relation

$$\int_\psi \frac{\partial}{\partial\psi_{n\beta}} F = 0\,. \tag{2.7}$$

Inserting $F = e^S\sigma$ (with S depending on products $\bar\psi_{n'\beta'}\,\psi_{n\beta}$) one gets the equation

$$\int_\psi e^S\left(\frac{\partial S}{\partial\psi_{n\beta}}\sigma + \frac{\partial\sigma}{\partial\psi_{n\beta}}\right) = 0 \tag{2.8}$$

and analogous relations follow by using right derivatives instead of left derivatives.

To obtain Schwinger-Dyson equations for gauge fields as well as all kinds of Ward-Takahashi identities one has to start from transformations of the fields which leave $\int$ invariant. This allows to define general derivatives

$$\partial^J F = \lim_{\varepsilon\to 0}\Big(F(\psi',\bar\psi',U') - F(\psi,\bar\psi,U)\Big)/\varepsilon\,, \tag{2.9}$$

where ε parametrizes the transformation and J denotes the particular one considered. These derivatives have the property

$$\int \partial^J F = 0\,. \tag{2.10}$$

Inserting again $F = e^S\sigma$ one obtains

$$\int e^S\Big((\partial^J S)\sigma + \partial^J\sigma\Big) = 0\,. \tag{2.11}$$

For example, the transformations of the gauge field at a certain link exploiting the right invariance or the left invariance of the Haar measure lead to "right" or "left" Schwinger-Dyson equations, respectively. Gauge type transformations and chiral transformations lead to various kinds of Ward-Takahashi identities. It is to be noted that from these relations at the same time the proper definition of currents arises.

2.2. Fermion considerations

2.2.1. Propagator limits

In order to get definite criteria for an adequate fermion treatment one has to see where precisely the naive description goes wrong. For this purpose first the free propagator

$$G_{n'n} = \frac{\int_\psi e^S \psi_{n'}\bar\psi_n}{\int_\psi e^S} \tag{2.12}$$

(with all $U_{\lambda n}\equiv 1$ in (2.2)) is considered for $\eta = 0$. Its Fourier representation on an infinite lattice is

$$G_{n'n} = -\frac{1}{(2\pi)^4}\int_{-\pi}^{\pi}\!\!\cdots\!\int_{-\pi}^{\pi} d^4\alpha\; e^{i\alpha\cdot(n'-n)}\, \frac{i\sum_\lambda \gamma_\lambda \sin\alpha_\lambda - \tilde m}{\sum_\lambda \sin^2\alpha_\lambda + \tilde m^2} \tag{2.13}$$

which with integrations over the full period of $\sin\alpha_\lambda$ is not yet in a reasonable form for the limit. However, by subdividing and shifting of integration intervals one gets the equivalent representation

$$G_{n'n} = -\frac{1}{(2\pi)^4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdots\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d^4\alpha\, e^{i\alpha\cdot(n'-n)}\, \frac{i\sum_\lambda \varphi_\lambda(n'-n)\gamma_\lambda \sin\alpha_\lambda - \varphi(n'-n)\tilde{m}}{\sum_\lambda \sin^2\alpha_\lambda + \tilde{m}^2} \tag{2.14}$$

where

$$\varphi_\lambda(n) = \begin{cases} 16 & \text{if } n_\lambda \text{ odd},\ n_\mu \text{ even } (\mu \neq \lambda) \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

$$\varphi(n) = \begin{cases} 16 & \text{if all } n_\mu \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{2.16}$$

which is appropriate. Realizing that the limit is to be taken in the sense of distributions and using (2.6) one readily sees that $\alpha^{-3} G_{n'n}$ has the correct limit.

The averaging over the 16 corners of an elementary cube, which in the free case is due to the application to a test function, in perturbation theory still works in diagrams without multiplication of fermion propagators. However, wrong results arise for fermion loops. Thus, at this point one gets a true criterion [11].

It is to be stressed that considerations of the spectrum on the lattice do not help within this respect. Actually, it is not too difficult to construct modified actions which do not show a spectrum degeneracy, which however, still run into trouble with fermion loops. Thus, the simple example given is typical for the general situation.

The way out is to use $\eta \neq 0$ in which case one obtains from (2.12)

$$G_{n'n} = -\frac{1}{(2\pi)^4}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} d^4\alpha\, e^{i\alpha\cdot(n'-n)}\, \frac{i\sum_\lambda \gamma_\lambda \sin\alpha_\lambda - \tilde{m} - w e^{-i\gamma_5\theta}}{\sum_\lambda \sin^2\alpha_\lambda + \tilde{m}^2 + w(w + 2\tilde{m}\cos\theta)} \tag{2.17}$$

where $w = r\sum_\lambda (1 - \cos\alpha_\lambda)$. Again by decomposing and shifting integration intervals one achieves integrations from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. Then 15 of the 16 arising terms are suppressed in the limit due to the sign change from $1 - \cos\alpha_\lambda$ to $1 + \cos\alpha_\lambda$ caused by the indicated shift. Then the correct results are obtained also for fermion loops in perturbation theory. Therefore, the formulation with $\eta \neq 0$ appears to be appropriate for the nonperturbative case, too.

2.2.2. Axial-vector anomaly

A second criterion for the fermion treatment is that one must get the correct anomaly. To check this it is important to start from the properly defined Ward-Takahashi identity in which currents and other quantities are unambiguously determined. For the transformation $\psi_n \to e^{i\alpha_n\gamma_5}\psi_n$, $\bar{\psi}_n \to \bar{\psi}_n e^{i\alpha_n\gamma_5}$, (2.9) applied to (2.2) gives

$$\partial_n^5 S = -\sum_\lambda \left(J^5_{\lambda n} - J^5_{\lambda, n-\hat{\lambda}}\right) + 2i\tilde{m}\bar{\psi}_n\gamma_5\psi_n + \Delta \tag{2.18}$$

where

$$J^5_{\lambda n} = \frac{i}{2}\left(\bar{\psi}_n\gamma_\lambda\gamma_5 U^+_{\lambda n}\psi_{n+\hat{\lambda}} + \bar{\psi}_{n+\hat{\lambda}}\gamma_\lambda\gamma_5 U_{\lambda n}\psi_n\right) \tag{2.19}$$

and

$$\Delta = -i\sum_{\lambda, n'}\left(\bar{\psi}_{n'}\eta\gamma_5 W_{\lambda n' n}\psi_n + \bar{\psi}_n\eta\gamma_5 W_{\lambda n n'}\psi_{n'}\right). \tag{2.20}$$

In passing it is to be noted that "point-splitting" forms of currents arise here automatically while in conventional approaches they must be introduced by hand.

For simplicity now σ in (2.11) is specialized to $\sigma = 1$, though general σ can straightforwardly be treated too [10,11]. One then has

$$\int_\psi e^S\left(\sum_\lambda\left(J^5_{\lambda n} - J^5_{\lambda, n-\hat{\lambda}}\right) - 2i\tilde{m}\bar{\psi}_n\gamma_5\psi_n\right) = \int_\psi e^S\Delta\ . \tag{2.21}$$

By using the Schwinger-Dyson equation (2.8) the r.h.s. of (2.21) gets the form

$$\int_\psi e^S\Delta = i\,\mathrm{tr}\left[\gamma_5(GX + XG)_{nn}\right]\int_\psi e^S \tag{2.22}$$

where

$$X = \eta\sum_\lambda W_\lambda\ , \quad G = \left(\sum_\lambda(\gamma_\lambda D_\lambda - \eta W_\lambda) + \tilde{m}\right)^{-1}. \tag{2.23}$$

From (2.21) and (2.22) it is seen that the correct anomaly occurs if one has [10,11]

$$\frac{1}{a^4}\, tr\left[\gamma_5 (GX+XG)_{nn}\right] \rightarrow \frac{g^2}{16\pi^2}\, Tr \sum_{\mu,\nu,\lambda,\rho} \epsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x)\ . \tag{2.24}$$

To establish the connection between degeneracy suppression and anomaly it suffices to prove (2.24) for an external field $U_{\lambda n}$ using (2.6) and keeping g^2 fixed as has been done [10,18].

It is to be noted that at the quantized level in any formulation an additional regularization is needed which breaks chiral symmetry and gives rise to the anomaly. The particular feature here is that the respective regularization at the same time is related to the degeneracy phenomenon.

2.2.3. Neutrinos

With the replacements

$$\gamma_\lambda \longrightarrow \gamma_\lambda \frac{1+\gamma_5 \sin\chi}{\cos\chi} \quad \text{where } -\frac{\pi}{2} < \chi < \frac{\pi}{2} \tag{2.25}$$

and

$$\psi \rightarrow \sqrt{\frac{2}{\cos\chi}}\, \psi \ ,\quad \bar\psi \rightarrow \sqrt{\frac{2}{\cos\chi}}\, \bar\psi \tag{2.26}$$

in (2.2), instead of (2.17) one gets the free propagator

$$G_{n'n} = -\frac{1}{(2\pi)^4}\int_{-\pi}^{\pi}\!\!\cdots\!\int_{-\pi}^{\pi} d^4\alpha\, e^{i\alpha\cdot(n'-n)}\ \frac{i\sum_\lambda \gamma_\lambda \frac{1+\gamma_5 \sin\chi}{2}\sin\alpha_\lambda - \frac{\cos\chi}{2}(\tilde m + w e^{-i\gamma_5\theta})}{\sum_\lambda \sin^2\alpha_\lambda + \tilde m^2 + w(w+2\tilde m\cos\theta)} \tag{2.27}$$

Obviously for (2.27) the limit can be performed in the same way as indicated for (2.17). Thus letting in addition $\chi \rightarrow \pm\frac{\pi}{2}$ corresponding to the handedness and $m \rightarrow 0$, the correct results in perturbation theory are to be expected for neutrinos, too. Therefore, this formulation should also be appropriate for the nonperturbative case.

2.3. Gauge-invariant quantization

2.3.1. Effective fields in variant formulation

In conventional approaches, in addition to allowing quantization at all, gauge-fixing leads to nontrivial results for correlations of gauge-variant fields. It is useful to analyse in more detail what happens within this respect before looking for an alternative to such correlations in the invariant formulation. On the lattice this can be done in a well defined way.

After chosing a gauge-fixing function F(U) one defines $\phi(U)$ by

$$\phi(U)\int_V F(U') = 1 \tag{2.28}$$

where V denotes gauge transformations. Then the relation

$$\int_U e^S = \int_V\int_U e^S \phi F \tag{2.29}$$

leads to the definition of correlation functions with gauge fixing

$$\langle \mathcal{O} \rangle_F = \frac{\int e^S \phi F \mathcal{O}}{\int e^S \phi F}\ . \tag{2.30}$$

Conversely now, multiplying numerator and denominator in (2.30) by $\int_V$ and doing steps of type (2.29) backwards one arrives at [5,11]

$$\langle \mathcal{O} \rangle_F = \langle \mathcal{O}_{eff} \rangle \tag{2.31}$$

where

$$\mathcal{O}_{eff}(U,\psi,\bar\psi) = \frac{\int_V F(U')\, \mathcal{O}(U',\psi',\bar\psi')}{\int_V F(U')}\ . \tag{2.32}$$

Thus, a correlation function with gauge fixing is equivalent to one without it of an effective field combination $\mathcal{O}_{eff}$. For a gauge-variant $\mathcal{O}$, $\mathcal{O}_{eff}$ is a highly artificial average involving different pieces of F in the numerator and in the denominator of (2.32).

On the lattice it is seen that the choice of F actually needs some care [5]. In order that (2.32) can be sensible, appropriate forms of F [5,6] are to be used.

It is to be noted here that the Ward-Takahashi identities which play a major role in conventional renormalization are nontrivial solely due to gauge fixing [11]. In fact, for the derivative (2.9) associated to the gauge transformation V one has $\partial^V_{n\alpha} S = 0$. What then remains of (2.11) is $\int e^S \partial^V_{n\alpha}\sigma = 0$. Using this and the definition (2.30) one obtaines $\langle F^{-1}\partial^V_{n\alpha}(F\sigma)\rangle_F = 0$ or

$$\langle \sigma\, \partial^V_{n\alpha} \ln F \rangle_F + \langle \partial^V_{n\alpha}\sigma \rangle_F = 0 \tag{2.33}$$

which is the general form of the mentioned identities.

2.3.2. Choice of appropriate invariant fields

Invariant fields occuring in σ, like

$$\bar\psi_{n\beta'}\, \psi_{n\beta} \tag{2.34}$$

$$\frac{1}{N!} \sum_{\alpha_1,\dots,\alpha_N} \epsilon_{\alpha_1,\dots,\alpha_N}\, \psi_{n\beta_1\alpha_1} \cdots \psi_{n\beta_N\alpha_N} \tag{2.35}$$

$$\frac{1}{2}\, \mathrm{Tr}\, (U_p + U_p^\dagger) \tag{2.36}$$

for mesons, baryons or glue, respectively, are, of course, to be used in any formulation. However, e.g. for $\bar\psi_{n'}\psi_n$ in the gauge-variant formulation one needs an alternative in the invariant one. One observes that for $\bar\psi_{n'}\psi_n$ the numerator of (2.32) contains terms of type

$$\bar\psi_{n'\beta'}\, S^{(i)}_{n'n}\, \psi_{n\beta} = \psi_{n'\beta'}\, U(\mathcal{P}^{(i)}_{n'n})\, \psi_{n\beta} \tag{2.37}$$

where $\mathcal{P}^{(i)}_{n'n}$ is a path from n to n' and $U(\mathcal{P}^{(i)}_{n'n})$ denotes the ordered product of gauge-field factors along it. The coefficients of these terms as well as the denominator of (2.32) involve expressions of type

$$L^{(j)} = \mathrm{Tr}\ U(\mathcal{L}^{(j)}) \tag{2.38}$$

where $\mathcal{L}^{(i)}$ is a loop and $U(\mathcal{L}^{(i)})$ again the ordered product. Thus, in the invariant formulation matters are considerably simplified by directly choosing one invariant bilocal field of type (2.37).

To find a replacement for gauge-field correlations of the variant formulation one has to look for appropriate field-strength quantities on the lattice. For this purpose one observes that in the Schwinger-Dyson equations, i.e. in the equations (2.11) related to the invariance of the gauge-group measure, such quantities are given by [5,11]

$$\mathcal{F}^{[\alpha]}_{\sigma\lambda,n} = (U^\dagger_{p\alpha} - U_{p\alpha})/(2i) \tag{2.39}$$

where U_{p_1} is just U_p of (2.3) and $\alpha = 2,3,4$ denote products around the plaquette starting at the other corners. Then, similarly as for the matter field in (2.37), one can define bilocal field-strength combinations

$$\mathrm{Tr}\ \left(\mathcal{F}_{\sigma\lambda,n'}\, U(\mathcal{P}^{(j)}_{n'n})\, \mathcal{F}_{\mu\nu,n}\, U(\mathcal{P}^{(i)}_{nn'})\right). \tag{2.40}$$

Furthermore, more general multilocal fields arise by additional insertions of (2.39) into (2.40) or (2.37).

Clearly, physics must not depend on the choice of paths in (2.37) and (2.40). In the variant formulation the respective freedom is contained in the choice of the gauge too, changes of which may be absorbed by the wave-function renormalization. Thus one expects that the required independence can be established at the level of the renormalized theory.

In addition to the field combinations suggested by particle physics order parameters, testing phases of the regularized theory, and closer in spirit to statistical mechanics are of interest. Best known within this respect are loops [1,2], i.e. combinations (2.38). There have also been efforts [19] to use certain quantities of type (2.37).

3. General limit

3.1. Definitions, conditions, properties

3.1.1. Implications of variable dependences

A particular lattice theory by (2.1) gives a set of correlation functions of form

$$G(n_1, \ldots, n_r; g_1, \ldots, g_\ell) \tag{3.1}$$

which are to be used to define the limit. In (3.1) the n_ϱ are the integer variables of the particular function and the g_i are the essential parameters of the theory. For the moment one may think of the g_i as of the usual bare parameters; more precise criteria for essential parameters will be given later.

The first step now is to relate the n_ϱ to physical lengths x_ϱ. If ν numbers the sequence which is to define the limit, this may be done by putting $n_\varrho(\nu) = \mathrm{int}\,(\nu x_\varrho/b)$ where b is the length unit. Actually it is sufficient to require only

$$n_\varrho(\nu) \cong \nu \frac{x_\varrho}{b} \tag{3.2}$$

where the equivalence sign $\cong$ is defined by

$$A(\nu) \cong B(\nu) \quad \text{if} \quad A(\nu)/B(\nu) \to 1 \quad \text{for} \quad \nu \to \infty . \tag{3.3}$$

Obviously continuous ν can be used.

A basic requirement now is that in physical quantities the dependences on the x_ϱ must not be wiped out for $\nu \to \infty$. This means that only dependences on the lattice variables of the types

$$n_\varrho / n_{\varrho'} \tag{3.4}$$

and

$$n_\varrho f(g_1, \ldots, g_\ell) \tag{3.5}$$

are allowed to occur, where

$$f(g_1, \ldots, g_\ell) \to 0 \qquad \text{for} \quad g_i \to g_i^c \tag{3.6}$$

holds for the approach from some region of the parameter space. Then one gets $n_\varrho(\nu)/n_{\varrho'}(\nu) \to x_\varrho/x_{\varrho'}$ and $n_\varrho(\nu) f(g_1(\nu), \ldots, g_\ell(\nu)) \to x_\varrho C$ with finite C for suitable dependences $g_i(\nu)$ as will be explained in the following.

In general the functions (3.1) themselves are not expected to have only the allowed dependences (3.4) and (3.5) but certain combinations

$$P_\sigma(n_1, \ldots, n_s; g_1, \ldots, g_\ell) \tag{3.7}$$

of them. In (3.7) σ denotes the particular combination. The subclasses of such functions which are of main interest will be discussed later. It suffices to perform the infinite-volume limit for the combinations (3.7) considered. This is henceforth understood having been done before letting $\nu \to \infty$, which in view of (3.2) is necessary.

3.1.2. Necessary structure of functions

There may be up to ℓ independent "taming" functions, i.e. functions of the type occurring in (3.5). Thus in general one has

$$f_i(g_1, \ldots, g_\ell) \quad \text{with} \quad i = 1, \ldots, k \tag{3.8}$$

where $k \leq \ell$, with

$$f_i(g_1, \ldots, g_\ell) \to 0 \qquad \text{for} \quad g_j \to g_j^c \tag{3.9}$$

for the approach from some region of parameter space. The value of k depends on the particular theory and on the region of parameter space considered. If $k < \ell$ there are further ℓ-k independent functions

$$f_i(g_1, \ldots, g_\ell) \quad \text{with} \quad i = k+1, \ldots, \ell \tag{3.10}$$

which are not subject to (3.9).(For $k<\ell$, if the functions (3.8) have suitable properties, (3.9) can possibly be extended to the approach of a ℓ-k dimensional manifold.)

According to the allowed dependences (3.4) and (3.5) and to (3.8) - (3.10), the structure of the functions (3.7) must be such that

$$\begin{aligned} &P_\sigma(n_1(\nu), \ldots, n_r(\nu); g_1, \ldots, g_\ell) \\ &\cong F_{\sigma,x}(\nu f_1, \ldots, \nu f_k; f_{k+1}, \ldots, f_\ell) \quad \text{for} \quad g_i \to g_i^c \end{aligned} \tag{3.11}$$

with $f_j \equiv f_j(g_1,\dots,g_\ell)$ and $x \equiv (x_1,\dots,x_r)$, where linear combinations of $\nu f_1,\dots,\nu f_k$ multiply some or all of the x_ρ. This implies the property

$$F_{\sigma,sx}(v_1,\dots,v_\ell) = F_{\sigma,x}(sv_1,\dots,sv_k; v_{k+1},\dots,v_\ell) \tag{3.12}$$

of the functions $F_{\sigma,x}$.

3.1.3. Basic equivalence relations

The task now is to determine the dependences $g_i(\nu)$ such that $g_i(\nu) \to g_i^c$ for $\nu \to \infty$ in an appropriate way. Then the limit of interest

$$\begin{aligned} Q_{\sigma,x} &= \lim_{\nu\to\infty} P_\sigma(n_1(\nu),\dots,n_r(\nu); g_1(\nu),\dots,g_\ell(\nu)) \\ &= \lim_{\nu\to\infty} F_{\sigma,x}(\nu f_1,\dots,\nu f_k; f_{k+1},\dots,f_\ell), \end{aligned} \tag{3.13}$$

where now $f_j \equiv f_j(g_1(\nu),\dots,g_\ell(\nu))$, can be calculated.

For the determination of $g_i(\nu)$ one has to prescribe ℓ defining values $Q_{(\bar\sigma,\bar x)(j)}$ with $j = 1,\dots,\ell$ and to consider the corresponding ℓ relations (3.13) with the $g_i(\nu)$ as unknown. In other words, one has to solve the system of equivalence relations

$$F_{(\bar\sigma,\bar x)(j)}(\nu f_1,\dots,\nu f_k; f_{k+1},\dots,f_\ell) \cong Q_{(\bar\sigma,\bar x)(j)} \tag{3.14}$$

where $j = 1,\dots,\ell$ and $f_i \equiv f_i(g_1(\nu),\dots,g_\ell(\nu))$.

In order that (3.14) has a solution, conditions of the type known from the inverse function theorem must hold. The inversion here occurs in two steps. For the first one the inverse set of the functions $F_{(\bar\sigma,\bar x)(j)}$ is needed, i.e. the existence of

$$H_{(\bar\sigma,\bar x)(i)}(u_1,\dots,u_\ell) = v_i \quad , \quad u_j = F_{(\bar\sigma,\bar x)(j)}(v_1,\dots,v_\ell), \tag{3.15}$$

with $i,j = 1,\dots,\ell$, in a suitable interval (which depends on the experimental values of basic physical constants). For the second step the inverse set of the functions f_i, i.e. the existence of

$$h_j(\phi_1,\dots,\phi_\ell) = g_j \quad , \quad \phi_i = f_i(g_1,\dots,g_\ell), \tag{3.16}$$

with $i,j = 1,\dots,\ell$, is to be required in a suitable region (which for the ϕ_i with $i = 1,\dots,k$ is in the vicinity of zero).

3.1.4. General properties of solutions

Assuming now that the described conditions are satisfied, the general solution of (3.14) can be written down and the resulting properties can be studied.

By applying (3.15) to (3.14) one obtains

$$\nu f_i(g_1(\nu),\dots,g_\ell(\nu)) \cong R_i \quad \text{for } i = 1,\dots,k \tag{3.17}$$

$$f_i(g_1(\nu),\dots,g_\ell(\nu)) \cong R_i \quad \text{for } i = k+1,\dots,\ell \tag{3.18}$$

where

$$R_i = H_{(\bar\sigma,\bar x)(i)}(Q_{(\bar\sigma,\bar x)(1)},\dots,Q_{(\bar\sigma,\bar x)(\ell)}), \quad i = 1,\dots,\ell . \tag{3.19}$$

Then using (3.16), from (3.17) and (3.18) one gets

$$g_j(\nu) \cong h_j\left(\frac{R_1}{\nu},\dots,\frac{R_k}{\nu}; R_{k+1},\dots,R_\ell\right), \quad j = 1,\dots,\ell. \tag{3.20}$$

Now inserting (3.20) into (3.13) one obtains

$$Q_{\sigma,x} = F_{\sigma,x}(R_1,\dots,R_\ell) \tag{3.21}$$

for the basic renormalized quantities.

Clearly, the determination of $g_j(\nu)$ up to equivalences is sufficent for obtaining $Q_{\sigma,x}$. The r.h.s. of (3.20) describes a definite curve in parameter space. The particular direction of approach of the singularity, implied by this is what guarantees simultaneous "taming" by the independent functions f_i with $i = 1,\dots,k$ (which can be read off from (3.17)).

The quantities (3.21) behave under dilatations $x_\rho \to sx_\rho$ as

$$Q_{\sigma,sx} = F_{\sigma,x}(sR_1,\dots,sR_k; R_{k+1},\dots,R_\ell), \tag{3.22}$$

which follows from (3.12). Among the constants $R_1,\dots,R_\ell$ of the renormalized theory, $R_1,\dots,R_k$ have the nature of mass scales, for which on thus gets a precise definition (masses in general occur as linear combinations of these scales with coefficients depending on the particular function under consideration). Then the number of independent "taming" functions of a theory is just that of its mass scales. The Λ parameter of QCD is an example of such a scale. The investigation of these scales beyond pure (one-parameter) gauge theory appears important with respect to the mass problems in particle physics.

3.1.5. Consistency and RG invariance

In order to check the consistency of the outlined procedure one uses (3.21) to select ℓ particular values

$$Q_{(\tilde{\sigma},\tilde{x})(j)} = F_{(\tilde{\sigma},\tilde{x})(j)}(R_1,\dots,R_\ell) \quad , \quad j=1,\dots,\ell, \tag{3.23}$$

requiring that for these values invertibility of type (3.15) holds, i.e. that

$$H_{(\tilde{\sigma},\tilde{x})(i)}(u_1,\dots,u_\ell) = v_i \quad , \quad u_j = F_{(\tilde{\sigma},\tilde{x})(j)}(v_1,\dots,v_\ell) \tag{3.24}$$

with $i,j = 1,\dots,\ell$ exists. Then one uses the set of the $Q_{(\hat{\sigma},\hat{x})(j)}$ for a new solution of the equivalence relations (3.14), which leads to new constants

$$\tilde{R}_i = H_{(\tilde{\sigma},\tilde{x})(i)}(\hat{Q}_{(\tilde{\sigma},\hat{x})(1)},\dots,Q_{(\tilde{\sigma},\hat{x})(\ell)}) \quad , \quad i=1,\dots,\ell. \tag{3.25}$$

On the other hand, one can use (3.24) to invert (3.23). Comparing the result of this with (3.25) it is obvious that

$$\tilde{R}_i = R_i \qquad , \qquad i = 1,\dots,\ell \tag{3.26}$$

i.e. that the R_i are universal constants. From (3.26) one gets $\tilde{g}_j(\nu) \cong g_j(\nu)$ and, therefore, again (3.21). Thus one has, in fact, consistency.

The transformation from one defining set to another one,

$$\{ Q_{(\bar{\sigma},\bar{x})(i)} \ , \ i=1,\dots,\ell \} \longrightarrow \{ Q_{(\hat{\sigma},\hat{x})(j)} \ , \ j=1,\dots,\ell \} \tag{3.27}$$

is nothing else but the general RG transformation of QFT. Its general form can be explicitly written down by inserting (3.19) into (3.23), which gives

$$\begin{aligned} Q_{(\tilde{\sigma},\tilde{x})(j)} = F_{(\tilde{\sigma},\tilde{x})(j)}\Big(& H_{(\bar{\sigma},\bar{x})(1)}\big(Q_{(\bar{\sigma},\bar{x})(1)},\dots,Q_{(\bar{\sigma},\bar{x})(\ell)}\big), \\ & \dots, H_{(\bar{\sigma},\bar{x})(\ell)}\big(Q_{(\bar{\sigma},\bar{x})(1)},\dots,Q_{(\bar{\sigma},\bar{x})(\ell)}\big)\Big). \end{aligned} \tag{3.28}$$

Due to the invertibility properties expressed by (3.15) and (3.24) a RG transformation is invertible, too. By (3.27) (or (3.28)) the concept of RG transformations is now precisely and generally defined. It is to be noted that also discrete transformations occur.

Having the general definition of the RG it is seen that the consistency shown before means that the quantities $Q_{\sigma x}$ given by (3.21) as well as the constants R of the theory are general RG invariants.

The necessity of using asymptotic equivalence instead of equality now becomes also transparent. For one fixing of the scheme by (3.14) one could, of course, use equality. The solution of this, however, after a RG transformation in general does only satisfy equivalence relations.

3.2. Function combinations of interest

3.2.1. Class of functions related to physics

The most important class of function combinations P_σ among those introduced by (3.7) is the one related to S-matrix elements and physical masses. In order to see how this class of combinations is to be constructed a look on the conventional formulation is useful. Starting from the general set of correlation functions, the first step there is to form connected functions because these are the ones of physical interest (in additon they have more reasonable mathematical properties). Similarly, in a next step one even restricts to one-particle-irreducible functions. Traditionally renormalized functions

$$\Gamma_{ren}^{(u_1,\dots,u_r)} = Z_1^{u_1/2} \dots Z_r^{u_r/2} \, \Gamma^{(u_1,\dots,u_r)} \tag{3.29}$$

and renormalized parameters

$$g_i^{ren} = g_i^{ren}(g_1,\dots,g_\ell; \nu/\mu) \tag{3.30}$$

are introduced. Quantities which are invariant under the dilatation subgroup of the RG satisfy

$$\left(\mu \frac{\partial}{\partial\mu} + \sum_{i=1}^{\ell} \beta_i^{ren} \frac{\partial}{\partial g_i^{ren}}\right) Q = 0 \tag{3.31}$$

where

$$\beta_i^{ren} = \mu \frac{d g_i^{ren}}{d\mu} \tag{3.32}$$

are the renormalized β functions.

A simple rule for the construction of quantities which satisfy (3.31) from functions (3.29) is that in the respective combinations all wave-function renormalization factors Z_ς must cancel out. This obviously holds for the class of functions

$$\frac{\Gamma_{ren}^{(u_1,\dots,u_r)}}{\left(\Gamma_{ren}^{(u_{\alpha_1}=2)} \cdots \Gamma_{ren}^{(u_{\alpha_M}=2)}\right)^{1/2}} \tag{3.33}$$

(with $M = u_1 + \dots + u_r$) which determines physical quantities as will be pointed out now.

The mass m_α follows from the special case

$$\frac{\Gamma_{ren}^{(u_\alpha=2)}(x,x')}{\left(\Gamma_{ren}^{(u_\alpha=2)}(y,y')\,\Gamma_{ren}^{(u_\alpha=2)}(x,x')\right)^{1/2}} \tag{3.34}$$

of (3.33) because m_α^2 is a simple zero with coefficient one of

$$\left(\left(\frac{d}{dp^2}\, \frac{\hat{\Gamma}_{ren}^{(u_\alpha=2)}(p,-p)}{\hat{\Gamma}_{ren}^{(u_\alpha=2)}(q,-q)}\right)\Bigg|_{q=p}\right)^{-1} \tag{3.35}$$

where $\hat{\Gamma}_{ren}^{(u_\alpha=2)}$ is the momentum-space transform of $\Gamma_{ren}^{(u_\alpha=2)}$.

The S matrix is given by the on-shell values of the function

$$(p_1^2 - m_{\alpha_1}^2)^{\frac{1}{2}} \cdots (p_M^2 - m_{\alpha_M}^2)^{\frac{1}{2}} \frac{\hat{\Gamma}_{ren}^{(u_1,\dots,u_r)}}{\left(\hat{\Gamma}_{ren}^{(u_{\alpha_1}=2)} \cdots \hat{\Gamma}_{ren}^{(u_{\alpha_M}=2)}\right)^{1/2}} \tag{3.36}$$

(where to simplify the notation a possible spin structure has been suppressed). (3.36) is a straightforward transform of (3.33). It is to be noted that in standard formulations the product of factors $((p^2 - m_\alpha^2)/\hat{\Gamma}^{(u_\alpha=2)})^{1/2}$ does not occur because of the restriction to rather special normalization conditions.

RG-invariant coupling strengths (i.e. physical ones) can also be defined by particular values of functions (3.36). For example, the definition of the electric charge in the Thomson limit is of this type. Furthermore, the so-called invariant charge of ϕ^4 theory is a special case of (3.36).

3.2.2. General starting point

The crucial observation now is that wave-function renormalization effects cancel out from (3.33) such that one can as well start from

$$\frac{\Gamma^{(u_1,\dots,u_r)}}{\left(\Gamma^{(u_{\alpha_1}=2)} \cdots \Gamma^{(u_{\alpha_M}=2)}\right)^{1/2}} \tag{3.37}$$

formed by bare functions. If (3.37) is constructed from lattice functions, one gets nothing else but particular combinations P_σ as introduced by (3.7). The requirement of dropping out of the wave-function factors of the perturbative framework, in the general case is replaced by the condition that only the dependences (3.4) and (3.5) are allowed to occur. Then with the structure (3.11) and the properties (3.15) and (3.16) not only invariance under the dilatation subgroup but under the general RG is guaranteed for the result. S-matrix elements, physical masses and RG-invariant coupling strengths are now in an obvious way given by the $Q_{\sigma,\chi}$ related by (3.13) to the particular P_σ of the form (3.37).

Obviously the traditional renormalized functions and renormalized constants do not occur in the general formulation; their introduction appears rather as an unnecessary complication from the general point of view. It is also seen that the general conditions are weaker than the conventional ones by avoiding to fix wave-function features and by replacing equalities by asymptotic equivalences (which actually turned out to be a necessity).

3.2.3. Nonlocal fields and order parameters

The considerations in sect. 3.2.1. started from local fields. However, also the nonlocal objects discussed in sect. 2.3.2. are of interest. A hint for the construction of appropriate combinations P_σ in that case comes from the renormalization properties of loops in continuum perturbation theory [20],

$$W_{ren} = e^{Zp} Z_c(\gamma_1) \dots Z_c(\gamma_r) W \,, \qquad W = \langle L \rangle \,, \tag{3.38}$$

where p is the perimeter and γ_ρ the angle of the ρ-th cusp (for a loop with a cross point, in addition mixing with loops in contact at this point is to be taken into account). Using again the rule of dropping out of wave-function renormalization, for rectangular loops with extensions u_i, v_i the combinations

$$\frac{W(u_1,v_1)\dots W(u_s,v_s)}{W(\tilde u_1,\tilde v_1)\dots W(\tilde u_s,\tilde v_s)} \qquad \text{with} \quad \sum_{i=1}^{s}(u_i+v_i-\tilde u_i-\tilde v_i)=0\,, \tag{3.39}$$

where $W(u,v) = \langle L_{u,v}\rangle$, are appropriate. On the other hand, constructing (3.39) from lattice functions one expects to obtain a new class of P_σ (which has, of course, to be checked using the nonperturbative criteria). This class is just that of general Creutz ratios [21,22].

W(u,v) is primarily an order parameter [1]. With respect to particle physics it provides a criterion for confinement in pure gauge theories [2]. The ratios (3.39), though not directly related to S-matrix elements, are useful to determine properties of the theory. They also give clearer signals from the order-parameter point of view.

For $\langle \bar\psi_{n'\beta'} S_{n'n} \psi_{n\beta}\rangle$ similar renormalization properties as for $\langle L \rangle$ are to be expected (with a relation of type (3.38) supplemented by factors for the string ends with matter fields). This again leads to combinations which are candidates for functions P_σ. The usefulness of correlation functions of type $\langle \phi_{n'} S_{n'n} \phi_n \rangle$ as order parameters in the presence of matter fields needs further investigation [19]. It appears that also in this case combinations P_σ should be better order parameters.

For a systematic treatment of the multilocal fields of sect. 2.3.2 in particle physics, an appropriate choice of the paths $\mathcal{P}^{(i)}_{n'n}$ related to the $S^{(i)}_{n'n}$ is crucial. According to the available hints they all should have the same length, the same number of corners and, for convenience, no crossings (the crossings at the $\mathcal{F}_{\sigma\lambda,n}$ are, however, to be accounted for). In order that it can be used in all functions, the length must be the maximal one, which in the infinite-volume limit gives paths running to infinity. Then the construction of P_σ related to S-matrix elements and masses should be possible similarly as discussed for local fields before.

3.3. Some β-function relations

3.3.1. Functions in renormalized theory

In view of the almost exclusive use of the dilatation subgroup of the RG in conventional approaches it appears worthwhile to have a brief look on the features of this subgroup in the general formulation, too. This means to specialize (3.27) to

$$\tilde\sigma = \bar\sigma \,, \qquad \tilde x_g = s\,\bar x_g \tag{3.40}$$

by which (3.25) with (3.26) becomes

$$R_i = H_{(\bar\sigma, s\bar x)(i)}\left(Q_{(\bar\sigma, s\bar x)(1)},\dots,Q_{(\bar\sigma, s\bar x)(\ell)}\right)\,, \quad i=1,\dots,\ell\,. \tag{3.41}$$

By using (3.12) and (3.15), (3.41) can be cast into the form

$$R_i = \begin{cases} s^{-1} H_{(\bar\sigma,\bar x)(i)}\left(Q_{(\bar\sigma, s\bar x)(1)},\dots,Q_{(\bar\sigma, s\bar x)(\ell)}\right) & \text{for } i=1,\dots,k, \\ H_{(\bar\sigma,\bar x)(i)}\left(Q_{(\bar\sigma, s\bar x)(1)},\dots,Q_{(\bar\sigma, s\bar x)(\ell)}\right) & \text{for } i=k+1,\dots,\ell. \end{cases} \tag{3.42}$$

Now, considering (3.42) being inserted into (3.21), one gets the RG equation

$$\left(s^{-1}\frac{\partial}{\partial s^{-1}} + \sum_{i=1}^{\ell} B_i \frac{\partial}{\partial Q_{(\bar{\sigma},s\bar{x})(i)}}\right) Q_{\sigma,x} = 0 \qquad (3.43)$$

where

$$B_i = s^{-1}\frac{d\,Q_{(\bar{\sigma},s\bar{x})(i)}}{d s^{-1}} \qquad (3.44)$$

is the general β function of the renormalized theory.

Clearly (3.43) and (3.44) are the analogues of (3.31) and (3.32), respectively. It is, however, to be noted that the $Q_{(\bar{\sigma},s\bar{x})(i)}$ and the $g_i^{ren}(\mu)$ are in general different within two respects. Firstly, a $Q_{(\bar{\sigma},s\bar{x})(i)}$ is a particular value of a RG-invariant quantity which a $g_i^{ren}(\mu)$ needs not to be (and usually is not). Secondly, the $Q_{(\bar{\sigma},s\bar{x})(i)}$ are neither required to be small nor to allow any expansion while the $g_i^{ren}(\mu)$, of course, must do.

Inserting (3.42) into $\frac{dR_i}{ds^{-1}} = 0$ it is seen that the functions B_i satisfy

$$\begin{aligned} H_{(\bar{\sigma},\bar{x})(i)} + \sum_j B_j \frac{\partial H_{(\bar{\sigma},\bar{x})(i)}}{\partial Q_{(\bar{\sigma},s\bar{x})(j)}} &= 0 \quad \text{for } i=1,\dots,k \\ \sum_j B_j \frac{\partial H_{(\bar{\sigma},\bar{x})(i)}}{\partial Q_{(\bar{\sigma},s\bar{x})(j)}} &= 0 \quad \text{for } i=k+1,\dots,\ell \end{aligned} \qquad (3.45)$$

where $H_{(\bar{\sigma},\bar{x})(i)} \equiv H_{(\bar{\sigma},\bar{x})(i)}(Q_{(\bar{\sigma},s\bar{x})(1)},\dots,Q_{(\bar{\sigma},s\bar{x})(\ell)})$. A consequence of (3.45) is that

$$B_i = B_i\left(Q_{(\bar{\sigma},s\bar{x})(1)},\dots,Q_{(\bar{\sigma},s\bar{x})(\ell)}\right) \qquad (3.46)$$

is the functional dependence of the B_i, which considering $\ell n\, s^{-1}$ as "time" allows the picture of "stationary flows".

3.3.2. Bare β functions

Because (3.2) allows continuous ν, one can define bare β functions by

$$\beta_i = \nu\frac{d g_i(\nu)}{d\nu} \qquad (3.47)$$

By differentiating (3.17) and (3.18) with respect to ν one obtains

$$\begin{aligned} f_i + \sum_{j=1}^{\ell} \beta_j \frac{\partial f_i}{\partial g_j} &\cong 0 \quad \text{for } i=1,\dots,k \\ \sum_{j=1}^{\ell} \beta_j \frac{\partial f_i}{\partial g_j} &\cong 0 \quad \text{for } i=k+1,\dots,\ell \end{aligned} \qquad (3.48)$$

where $f_i \equiv f_i(g_1(\nu),\dots,g_\ell(\nu))$ and $g_j \equiv g_j(\nu)$. From (3.48) it follows that

$$\beta_j \cong \beta_j(g_1(\nu),\dots,g_\ell(\nu)). \qquad (3.49)$$

Formally (3.48) and (3.49) are similar to (3.45) and (3.46), respectively. However, apart from the fact that the nature of the $g_i(\nu)$ and of the $Q_{(\bar{\sigma},s\bar{x})(i)}$ is entirely different, mathematically the relations involving the β_i are only asymptotic equivalences while those of the B_i are equations.

By inserting (3.17) and (3.18) into (3.23) with (3.40) and using (3.12) one arrives at the equivalence

$$Q_{(\bar{\sigma},s\bar{x})(i)} \cong F_{(\bar{\sigma},\bar{x})(i)}(s\nu f_1,\dots,s\nu f_k; f_{k+1},\dots,f_\ell) \qquad (3.50)$$

with $i = 1,\dots,\ell$ and $f_j \equiv f_j(g_1(\nu),\dots,g_\ell(\nu))$, which connects the $Q_{(\bar{\sigma},s\bar{x})}$ and the $g_j(\nu)$. Differentiation of (3.50) with respect to ν gives

$$0 \cong \sum_{j=1}^{\ell} \beta_j \frac{\partial F_{(\bar{\sigma},\bar{x})(i)}}{\partial g_j} + \nu\frac{\partial F_{(\bar{\sigma},\bar{x})(i)}}{\partial \nu}. \qquad (3.51)$$

Because of $\nu\frac{\partial F_{(\bar{\sigma},\bar{x})(i)}}{\partial\nu} = s\frac{\partial F_{(\bar{\sigma},\bar{x})(i)}}{\partial s} = -B_i$ from (3.51) it follows that

$$B_i \cong \sum_{j=1}^{\ell} \beta_i \frac{\partial F_{(\bar{\sigma},\bar{x})(i)}(s\nu f_1,\dots,s\nu f_k; f_{k+1},\dots,f_\ell)}{\partial g_j(\nu)} \qquad (3.52)$$

with $i = 1,\dots,\ell$, $f_j \equiv f_j(g_1(\nu),\dots,g_\ell(\nu))$ and $\beta_j \equiv \beta_j(g_1(\nu),\dots,g_\ell(\nu))$. It is to be noted that due to (3.15) and (3.16) the derivatives $\partial F_{(\bar{\sigma},\bar{x})(i)}(v_1,\dots,v_\ell)/\partial v_j$ and $\partial f_i(g_1,\dots,g_\ell)/\partial g_j$ are finite matrices. The equivalence (3.52) is the precise general relation between the two types of β functions.

4. Situation in specific theories

4.1. Properties of pure gauge theories

4.1.1. Knowledge from Monte-Carlo simulation

Though proofs remain to be given by analytical methods, Monte-Carlo simulations are valuable for exploring the situation. Here information from such simulations about the parameter singularities in specific theories are considered which are of interest in the present context.

Results which test (3.11) directly have been obtained [22-24] for SU(N), where N = 1,...,4, and U(1) using a special case of (3.39), namely

$$\frac{W(2\nu,2\nu)\,W(\nu,\nu)}{(W(2\nu,\nu))^2} \cong F(\nu f(g)) \tag{4.1}$$

with ν = 1,2. Their qualitative behaviour is shown in Fig.1. It is seen that the steepness of the curves increases with ν for $g^2 > g_o^2$, where $g_o^2 = 0$ for SU(N) and $g_o^2 \approx 0.99$ for U(1). This is in accordance with

$$\frac{dF(\nu f(g))}{dg} \approx F' f' \nu \tag{4.2}$$

where $F' \neq 0$ and $f' \neq 0$ due to (3.15) and (3.16), respectively. For SU(N) the interpretation is consistent with $k = 1$ and $f(g) \to 0$ for $g^2 \searrow g_o^2$. For U(1) the same is true in the confinement region $g^2 > g_o^2$, while in the Coulomb region $g^2 < g_o^2$ one has the case $k = 0$ without a mass scale. Thus one gets reasonable results though $\nu = 2$ is actually far from $\nu \to \infty$ in any sense.

If one is willing to make assumptions about the form of W(u,v), one can also extract information about $f(g)$ from Monte-Carlo data. In particular, assuming that for u,v large enough one has

$$\ln W(u,v) \approx c_o - c_1(u+v) - c_2 f^2 u v \quad , \tag{4.3}$$

for special cases of (3.39) like

$$\frac{W(u+1,v+1)\,W(u,v)}{W(u+1,v)\,W(u,v+1)} \quad \text{or} \quad \frac{W(u+1,u+1)}{W(u+2,u)} \tag{4.4}$$

one gets

$$P \approx \exp(-c_2 f^2) \quad . \tag{4.5}$$

Using this, the data of the simulations for SU(N) with N = 2,3 [21] and for U(1) [23] allow to obtain the string tension $c_2 f^2$. For SU(N) there is agreement at large g^2 with

$$f^2 \sim \ln g^2 + \ln(N - \delta_{N2}) \quad , \tag{4.6}$$

following from the strong-coupling expansion, and at small g^2 with

$$f^2 \sim e^{-\frac{1}{\beta_o g^2}} (\beta_o g^2)^{-\frac{\beta_1}{\beta_o^2}} \quad , \tag{4.7}$$

to be discussed below. The crossover between the behaviours (4.6) and (4.7) is fast (occurring roughly near $g^2 \approx 2$ and $g^2 \approx 1$ for N = 2 and N = 3, respectively). For U(1) the fit

$$f \sim \left(\frac{1}{0.99} - \frac{1}{g^2}\right)^{0.33} \quad \text{for} \quad g^2 > 0.99 \tag{4.8}$$

and

$$f \approx 0 \quad \text{for} \quad g^2 < 0.99 \tag{4.9}$$

holds within the accuracy of the data.

4.1.2. Use of bare β function

For ℓ = k = 1 integration of (3.48) gives

$$f(g(\nu)) \cong f(g(\nu_o)) \exp\left(-\int_{g(\nu_o)}^{g(\nu)} \frac{dg}{\beta(g)}\right) . \tag{4.10}$$

The approximation which replaces $\cong$ by = in (4.10) is essentially that of scaling [25], which is what is to be improved by Symanzik's program [26], and which so far is only under control by reference to perturbation theory. From the considerations in sect.3.1 it follows that this approximation in general is not RG invariant.

Of course, (4.10) is only of interest if the β function is given. By (3.47) it is actually simply $f(g(\nu)) \cong f(g(\nu_o))\nu_o/\nu$ as one also gets directly from (3.17), i.e. from $f(g(\nu)) \cong R\nu^{-1}$.

For SU(N) from perturbation theory one obtains the representation

$$\beta(g(\nu)) = -\beta_0\, g^3(\nu) - \beta_1\, g^5(\nu) - \mathcal{O}(g^7(\nu)) . \qquad (4.11)$$

Neglecting $\mathcal{O}(g^7)$ and inserting (4.11) into the scaling approximation of (4.10) one arrives at the asymptotic-scaling approximation (4.7). Thus, though the perturbative expressions from which (4.11) is derived by far do not show the dependence (3.5), via $\beta(g)$ one obtains an approximate function $f(g)$.

Some comments on the nature of (4.11) appear in order at this point. It is well known that the lowest-order coefficients β_0 and β_1 [27] are invariant with respect to perturbative transformations. Actually one starts from the relation analogous to (4.11) which holds for β^{ren} and $g^{ren}(\mu)$, in which case the transformation is between $g^{ren}(\mu)$ and some $\tilde{g}^{ren}(\tilde{\mu})$. Keeping ν finite, a transformation from $g^{ren}(\mu)$ to $g(\nu)$ then leads to (4.11). Thus, in addition to its perturbative nature the finiteness of ν is important for (4.11) to hold approximately. From (3.50) and (3.52) it is seen that the general relations are rather subtle and allow, in fact, at best an approximation for finite ν.

4.2. Gauge theories with matter fields

4.2.1. Masses in QCD

According to the structure of (3.11) the hadron masses in QCD must arise from lattice expressions of form

$$\tilde{M}_\tau(g,\tilde{m}_1,\ldots,\tilde{m}_f) = \sum_{j=1}^{k} c_{\tau j}\, f_j(g,\tilde{m}_1,\ldots,\tilde{m}_f) \qquad (4.12)$$

where τ denotes the particular hadron. They are obtained from

$$M_\tau = \lim_{\nu\to\infty} \frac{\nu}{b}\tilde{M}_\tau(g(\nu),\tilde{m}_1(\nu),\ldots,\tilde{m}_f(\nu)) = \frac{1}{b}\sum_{j=1}^{k} c_{\tau j} R_j \qquad (4.13)$$

after determining by (3.14) the dependences $g(\nu)$, $\tilde{m}_i(\nu)$, i.e. the curve (3.20) in $(g,\tilde{m}_1,\ldots,\tilde{m}_f)$ space along which one has to approach the fundamental singularity at g^c, $\tilde{m}_i^c$.

Having $g^c = 0$ with the taming function $f(g)$ discussed before in the pure gauge-field case and $\tilde{m}_i^c = 0$ with taming functions $f_i(\tilde{m}_i) = \tilde{m}_i$ in the free-fermion case, naive combination leads to expect the singularity at the point $g^c = 0$, $\tilde{m}_i^c = 0$ in $(g,\tilde{m}_1,\ldots,\tilde{m}_f)$ space and to envisage the possibility $k = \ell$ where $\ell = f + 1$. This picture is supported by considerations of β functions of the asymptotic free theory to be presented later.

It is to be realized that by (3.9) the functions (4.12) for any τ have to go to zero if $g^c,\tilde{m}_i^c$ is approached. For an approximate evaluation one should at least be close to this critical region. Unfortunately this is not the case in the respective Monte-Carlo simulations [28 and references given there] and strong-coupling computations [7,29] though they allow to some extent a fair description of spectra. For example, for pseudoscalar and vector mesons the strong-coupling results

$$\begin{aligned} 2(\cosh \tilde{M}_{ps} - 1) &= \frac{((\tilde{m}+4)^2-4)((\tilde{m}+4)^2-1)}{(\tilde{m}+4)^2 - 3/2} \\ 2(\cosh \tilde{M}_{v} - 1) &= \frac{((\tilde{m}+4)^2-3)((\tilde{m}+4)^2-2)}{(\tilde{m}+4)^2 - 3/2} \end{aligned} \qquad (4.14)$$

exhibit singularities only at different values of $\tilde{m}$ (the ones closest to $\tilde{m} = 0$ at $\tilde{m} = -2$ and at $\tilde{m} \approx -2.3$, respectively). Of course, one can blame this to the fact that $g^2 = \infty$ is really far from $g^2 = 0$. However, for $g^2 \approx 1$, which can be reached by Monte-Carlo simulations, the situation is also far from the ideal one. This is shown qualitatively in Fig.2 for the meson example.

4.2.2. Use of β functions in QCD

Requiring f small enough such that asymptotic freedom persists, one can again try to get information by using β functions from perturbation theory. Similarly as (4.11) for $g(\nu)$, there are bare functions associated to the $\tilde{m}_i(\nu)$. At the one-loop level one gets for large ν

$$\beta_i \approx -\beta_m \, \tilde{m}_i(\nu) \, g^2(\nu) \tag{4.15}$$

where $\beta_m = (2\pi^2)^{-1}$ for SU(3) [30]. From (4.11) and (4.15) it is seen that there is an ultraviolet-stable fixed point at $g=0$, $\tilde{m}_i=0$, indicating that one has, in fact, $g^c=0$, $\tilde{m}_i^c=0$.

The flow curves, which are obtained by integrating (3.47) with the one-loop β functions, are given by

$$\frac{\tilde{m}_i(\nu)}{\tilde{m}_i(\nu_0)} = \left(\frac{g^2(\nu)}{g^2(\nu_0)}\right)^{\frac{\beta_m}{2\beta_0}} \tag{4.16}$$

where $\beta_0 = (11-2f/3)(16\pi^2)^{-1}$ for SU(3). The flow pattern described by (4.16) indicates that one has, in fact, the case $k=\ell$. It, furthermore, suggests the possibility of a phase-transition line along the g^2 axis.

4.2.3. Situation in other cases

While the information about the singularity structure in QCD is restricted to indications on the location in parameter space and on the value of k, even these features are not clear in other theories of particle physics. This means that one is still far from a truely nonperturbative description.

For QED obviously the region with $g^2 < 0.99$ of the U(1) gauge field is to be considered. Again the question is which structure arises upon combination of gauge field and matter field. Now, however, due to the lack of asymptotic freedom, no hints from perturbation theory are available.

In the standard model of electroweak interactions again the perturbative knowledge does not help. Ideas about possible features may, however, be obtained from the Higgs models related to its sectors. For the fundamental representation of the Higgs field the confinement phase and the Higgs phase are continuously connected [31], i.e. not actually different. In the abelian case there is in addition a separate Coulomb phase. At the present stage of affaires [32,33], however, conclusions about a possible singularity structure of an electroweak theory appear premature.

4.3. Actions with auxiliary parameters

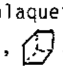
The lattice action leading to a particular continuum theory is by far not unique. For the description of more general action forms, representations involving auxiliary parameters are convenient. This is illustrated in the present sect. by typical examples.

The motivations for using more general forms are firstly to get a faster approach of the limit and secondly to study nearby phase structures in the larger parameter space. In addition, the fermion description discussed in sect.2.2 apparently involves auxiliary parameters.

The dependence of the auxiliary parameters on the essential ones is restricted by the fact that one must get the proper classical limit and possibly by further conditions related to the particular purpose under consideration.

Instead of the gauge-field part of (2.2) one may consider the generalization

$$S_U = \sum_i u_i(g) \sum_{\mathcal{C}_i} 2\,\mathrm{Re}\cdot\mathrm{Tr}\left(U(\mathcal{C}_i) - 1\right), \tag{4.17}$$

which in addition to the product of gauge-field factors along the path $\mathcal{C}_0$ around a plaquette uses products along more general loops $\mathcal{C}_i$, in particular , , , for i = 1,2,3, respectively. There are no unique criteria for improving the convergence by the choice of the auxiliary parameters $u_i(g)$. In Symanzik's program at the tree level one gets [34] $u_0 = \frac{5}{3} g^{-2}$, $u_1 = -\frac{1}{12} g^{-2}$, $u_i = 0$ for $i > 1$, while Wilson proposes [35] $u_0 = 4.376\, g^{-2}$, $u_1 = -0.252\, g^{-2}$, $u_2 = 0$, $u_3 = -0.17\, g^{-2}$, $u_i = 0$ for $i > 3$ according to block-spin considerations.

Another possibility of generalizing the gauge-field action is to use the character expansion

$$S_U = \sum_r v_r(g) \sum_p 2\,\mathrm{Re}\, \chi_r(U_p)\,. \tag{4.18}$$

Examples of choices of the auxiliary parameters v_r are the fundamental-adjoint action for SU(2) [36], with $v_0 = -2v_{1/2} - 3v_1$, $v_{1/2} + 4v_1 = g^{-2}$, $v_r = 0$ for $r > 1$, and Manton's action [37] where for SU(2) [38] $v_0 = (\frac{1}{2} - \frac{\pi^2}{3}) g^{-2}$, $v_r = \frac{1}{2}(-1)^{2r+1}(2r+1) r^{-2}(r+1)^{-2} g^{-2}$

(more general forms than $v_r = \tilde{v}_r/g^2$ occur, e.g., for the heat-kernel action [38]). An illustration of the effect of nearby singularities in the larger parameter space is provided by the fundamental-adjoint example for SU(2), which has the phase structure shown in Fig.3 as found by Monte-Carlo simulation [36]. The first-order-transition line points just to the region of the $v_{1/2}$ axis where the crossover between the behaviours (4.6) and (4.7) occurs.

For the fermion part of (2.2) a more general form, no longer restricted to nearest neighbors, is

$$S_\psi = -\sum_{n',n} \bar{\psi}_{n'} \Big(\sum_\lambda \sum_j (c_j \gamma_\lambda D_\lambda^{(j)} - \eta_j W_\lambda^{(j)}) + \tilde{m} \Big)_{n'n} \psi_n \qquad (4.19)$$

where

$$D^{(j)}_{\lambda n' n} = \Big(U(P_{n',n'+j\hat{\lambda}})\,\delta_{n'+j\hat{\lambda},n} - U(P_{n+j\hat{\lambda},n})\,\delta_{n',n+j\hat{\lambda}} \Big)/2 \;,$$
$$W^{(j)}_{\lambda n' n} = \Big(U(P_{n',n'+j\hat{\lambda}})\,\delta_{n'+j\hat{\lambda},n} + U(P_{n+j\hat{\lambda},n})\,\delta_{n',n+j\hat{\lambda}} - 2\delta_{n'n} \Big)/2 \;. \qquad (4.20)$$

The choice of coefficients in (4.19) corresponding to tree level improvement is [39] $c_1 = \frac{4}{3}$, $c_2 = -\frac{1}{6}$, $\eta_1 = \frac{4}{3}r$, $\eta_2 = -\frac{1}{3}r$ and $c_j = \eta_j = 0$ for $j > 2$. A choice with a continuum-like spectrum, however no longer satisfying the criteria of sect. 2.2, is [40] $c_j = (-1)^{j+1}/j$, $\eta_j = 0$.

In (2.2) r and Θ are auxiliary parameters, the (r,Θ) space being restricted to the region $r > 0$. Thus the actual nature of the fermion description is that of an auxiliary-parameter representation. That there is no dependence of auxiliary parameters on essential ones is due to the special parametrization introduced. Instead of the form of the fermion part in (2.2) one could e.g. use

$$S_\psi = \sum_n \Big(\frac{1}{8\varrho} \sum_{\pm\lambda} \bar{\psi}_{n+\hat{\lambda}} \,(\gamma_\lambda + r e^{i\gamma_5 \varphi})\, U(P_{n+\hat{\lambda},n})\psi_n - \bar{\psi}_n \psi_n \Big) \qquad (4.21)$$

where $\lambda = \pm 1, \ldots, \pm 4$ and $\gamma_{-\lambda} = -\gamma_\lambda$. Then in general one has the dependence $\varrho(\tilde{m})$, $r(\tilde{m})$, $\varphi(\tilde{m})$ describing a suitable curve in (ϱ, r, φ) space (restricted to $r > 0$).

With the particular choice

$$\varrho = (\tilde{m}^2 + 16r^2 + 8\tilde{m}r\cos\Theta)^{1/2}$$
$$\varphi = \arctan \frac{\tilde{m}\sin\Theta}{\tilde{m}\cos\Theta + 4r} \qquad (4.22)$$
$$r = r$$

after appropriate rescaling of $\psi, \bar{\psi}$ one arrives at the form (2.2).

5. Kadanoff-Wilson transformations

5.1. Transformation of the partition function

5.1.1. Definitions and remarks

A KW transformation amounts to integrate out short distance degrees of freedom keeping the partition function fixed, which leads to a new (effective) action. In statistical mechanics it in general allows an easier and more appropriate evaluation in the critical region.

Given an action, which may depend on several types of fields and on ℓ essential parameters,

$$S(U, \psi, \bar{\psi}, \phi; g_1, \ldots, g_\ell; 1) \,, \qquad (5.1)$$

after performing a KW transformation, which reduces the degrees of freedom by a factor λ^d, one gets an effective action denoted by

$$S(U, \psi, \bar{\psi}, \phi; g_1, \ldots, g_\ell; \lambda) \,. \qquad (5.2)$$

By a further transformation, reducing by $(\tilde{\lambda}/\lambda)^d$, one arrives at an action with $\tilde{\lambda}$ instead of λ in (5.2), and so on. To condense the notation, in the following mostly $S(U,g,\lambda)$ will be used, which may be considered as a shorthand for (5.2) or as a special case of it.

A general formulation of the transformations is

$$e^{S(\tilde{U},g,\tilde{\lambda})} = \int_U \mathcal{T}_{\tilde{\lambda}\lambda}(\tilde{U},U)\, e^{S(U,g,\lambda)} \tag{5.3}$$

with the composition rule

$$\int_{U'} \mathcal{T}_{\tilde{\lambda}\lambda'}(\tilde{U},U')\,\mathcal{T}_{\lambda'\lambda}(U',U) = \mathcal{T}_{\tilde{\lambda}\lambda}(\tilde{U},U). \tag{5.4}$$

The requirement to keep the partition function fixed

$$\int_{\tilde{U}} e^{S(\tilde{U},g,\tilde{\lambda})} = \int_U e^{S(U,g,\lambda)} \tag{5.5}$$

leads to the condition

$$\int_{\tilde{U}} \mathcal{T}_{\tilde{\lambda}\lambda}(\tilde{U},U) = 1 . \tag{5.6}$$

In statistical mechanics (5.5) means to keep physics fixed, while in QFT it is a convenient technical condition.

There is obviously considerable freedom in the choice of KW transformations. Here gauge invariance is an additional requirement. Invariant block variables for pure gauge fields have been constructed with scale factors (minimal ratios $\tilde{\lambda}/\lambda$) 2 [41] and, in particular for d=4, with $\sqrt{3}$ [42] and $\sqrt{2}$ [43].

A central issue in the applications of KW transformations is the existence of a reasonable fixed point. The rescaling of fields crucial for linear transformations is not needed for nonlinear ones [44]. Therefore, for gauge theories no tuning problems of a similar type are to be expected. Nevertheless, since the fixed point on one hand depends on the particular transformation and on the other hand is responsible for fundamental properties of the theory, it is an important task to get the transformations under control.

Further conditions on the transformations, which are related to correlation functions and to the application in QFT, will be discussed later.

5.1.2. Example for use of auxiliary parameters

The form of the action changes in general under the transformation (5.3). A straightforward representation is then in terms of the types of contributions which can be generated, as may e.g. be expressed by the expansion

$$S(U,g,\lambda) = \sum_i c_i(g,\lambda)\, S_i(U). \tag{5.7}$$

This is seen to be a particular auxiliary-parameter representation, the λ dependence describing the effect of the mappings (5.3). The points generated by subsequent KW transformations are on a curve in auxiliary-parameter space which starts at λ=1 from a prescribed point depending on g.

In the vicinity of a fixed point of the transformation (5.3), which is given by the property

$$c_i(g,\tilde{\lambda}) = c_i(g,\lambda) = c_i^* , \tag{5.8}$$

an overview is obtained by using the linearization

$$c_i(g,\tilde{\lambda}) - c_i^* \approx \sum_j T_{ij}\,(c_j(g,\lambda) - c_j^*) \tag{5.9}$$

where

$$T_{ij} = \left.\frac{\partial c_i(g,\tilde{\lambda})}{\partial c_j(g,\lambda)}\right|_{c_k(g,\lambda)=c_k^*} . \tag{5.10}$$

Assuming that (5.10) can be diagonalized, one gets

$$T_{ij} = \sum_\ell u_{i\ell}\left(\frac{\tilde{\lambda}}{\lambda}\right)^{y_\ell} v_{\ell j} \tag{5.11}$$

with $\sum_j v_{\ell' j} u_{j\ell} = \delta_{\ell'\ell}$ and $\sum_\ell u_{i\ell} v_{\ell j} = \delta_{ij}$. The form $\left(\frac{\tilde{\lambda}}{\lambda}\right)^{y_\ell}$ of the (real) eigenvalues is dictated by (5.4). With

$$\bar{S}_\ell = \sum_i S_i u_{i\ell} \quad , \quad \rho_\ell = \sum_j v_{\ell j} c_j \quad , \tag{5.12}$$

i.e. transforming to the eigenbasis, one obtains from (5.7)

$$S(U,g,\tilde{\lambda}) = \sum_\ell \rho_\ell(g,\tilde{\lambda})\,\bar{S}_\ell(U) , \tag{5.13}$$

where according to (5.9)

$$\rho_\ell(g,\tilde{\lambda}) \approx \rho_\ell^* + \left(\frac{\tilde{\lambda}}{\lambda}\right)^{y_\ell}(\rho_\ell(g,\lambda) - \rho_\ell^*) . \tag{5.14}$$

The behaviour of (5.14) for increasing $\tilde{\lambda}$ is related to calling an eigendirection relevant, irrelevant or marginal if $y_i > 0$, < 0 or $= 0$, respectively. It is to be stressed that this here refers only to the linear approximation; in particular the marginal directions need to be checked beyond this.

With respect to a more general characterization to be given later it is useful to state the geometrical description in the present case:

a) For r marginal eigenvalues one has a r dimensional hyperplane of fixed points. After projecting to an appropriate subspace the situation with one fixed point remains to be considered.

b) Points which for increasing $\tilde{\lambda}$ are driven to the fixed point are said to belong to the "critical" hyperplane".

c) Points not belonging to the "critical" hyperplane in the case of k' relevant eigenvalues are driven to the k' dimensional "renormalized" hyperplane and away from the fixed point.

5.1.3. General auxiliary-parameter representations

Instead of (5.7) one could, for example, as well use an expansion of e^S. Then, there are various possibilities for the type of expansion. Also, one is not restricted to series representations. Even in case of a series the auxiliary parameters c_i need not be the coefficients (e.g. for pure SU(N) theory because of $g^c = 0$ a representation of S of form $(S_0 + \sum_{i=1}^{\infty} c_i S_i)/c_0$ is convenient). Thus, there are many auxiliary-parameter representations out of which a suitable one can be selected.

The general representation can be written in the form

$$S(U,g,\lambda) = \check{S}(U;c_1,c_2,\ldots) \tag{5.15}$$

where $c_i \equiv c_i(g,\lambda)$, or if $S(U,g,\lambda)$ is considered as a shorthand of (5.2) more explicitly $c_i \equiv c_i(g_1,\ldots,g_l;\lambda)$. The notation (5.15) straightforwardly extends to other functions, e.g. for $e^S = E$ one gets $E(U,g,\lambda) = \check{E}(U,c_1,c_2,\ldots)$.

In order to generalize the geometric description of the preceding linearized example, one has to note that by an appropriate nonlinear map hyperplanes are replaced by manifolds. This suggests a generalization in terms of manifolds and the use of the related mathematical tools for working out precise conditions. In this way also fixed-point structures of more general nature can be reached.

Restricting for simplicity to structures which do not involve additional complications, the generalized geometrical description is:

a) There may be a r dimensional manifold of fixed points. After mapping to an appropriate manifold in parameter space the situation with one fixed point remains to be considered.

b) Points driven to the fixed point for increasing λ are said to belong to the "critical" manifold.

c) Points not belonging to the "critical" manifold are driven to the k' dimensional "renormalized" manifold and away from the fixed point.

This is illustrated in a simple case in Fig.4.

According to the geometrical description, the generalization of (5.13) is of form

$$S(U,g,\lambda) = \bar{S}(U;\zeta_1,\zeta_2,\ldots)\,, \tag{5.16}$$

where a component $\zeta_i(g,\lambda)$ is called relevant, irrelevant or marginal if it is driven away from the fixed point, towards it or nowhere at all, respectively (now understood generally, i.e. beyond the linear approximation).

5.1.4. Associated type of β function

If in (5.16) the special case is considered where one component is relevant and all others are irrelevant, for large λ one may use

$$S(U,g,\lambda) \approx \bar{S}(U,\zeta(g,\lambda)). \tag{5.17}$$

With the functions $c_i(g,1)$ prescribed, $\zeta(g,\lambda)$ is determined via the transformations (5.3) for the respective discrete values of λ. It is understood here that there is one relevant parameter g.

Starting from a point closer to the fixed point may be compensated for by doing more transformation steps. The condition for arriving at the same result is

$$\zeta(g',\lambda') = \zeta(g,\lambda). \tag{5.18}$$

Obviously (5.18) defines a dependence of g' on λ' which can be expressed by

$$\zeta(g^{KW}(\lambda'),\lambda') = C\,. \tag{5.19}$$

It is important to keep in mind that $g^{KW}(\lambda')$ is the value related to the starting point of the KW transformations. The factor λ' is the one reached after the particular number of transformation steps. In (5.19) C is given by C $= \zeta(g,\lambda)$ where $\zeta(g,\lambda)$ is determined as described before.

Though originally derived only for the discrete values λ' occuring in (5.3), (5.19) may more generally be considered for continuous λ'. With suitable invertibility properties of the function ζ one then gets $g^{KW}(\lambda)$ for continuous values λ, too. Then one can define the β function

$$\beta^{KW} = \lambda \frac{d g^{KW}(\lambda)}{d\lambda} . \tag{5.20}$$

The relation of $g^{KW}(\lambda)$ to $g(v)$ and of β^{KW} to β will be established later.

5.2. Transformation of correlation functions

5.2.1. General formulation

Now the formulation of KW transformations introduced in sect.5.1.1. for partition functions is extended to correlation functions which is important for the application to QFT.

Inserting (5.3) into the general form

$$\int_{\tilde{U}} e^{S(\tilde{U},g,\tilde{\lambda})} \sigma(\tilde{U},\tilde{\lambda}) = \int_{U} e^{S(U,g,\lambda)} \sigma(U,\lambda) \tag{5.21}$$

of the transformation, it is seen that

$$\int_{\tilde{U}} \sigma(\tilde{U},\tilde{\lambda}) \mathcal{T}_{\tilde{\lambda}\lambda}(\tilde{U},U) = \sigma(U,\lambda) \tag{5.22}$$

is the condition which $\sigma(\tilde{U},\tilde{\lambda})$ has to satisfy. For the special case $\sigma(\tilde{U},\tilde{\lambda}) = 1$, with $\sigma(U,\lambda) = 1$ (5.22) becomes just (5.6). The crucial feature of (5.22) is that it maps from $\sigma(\tilde{U},\tilde{\lambda})$ to $\sigma(U,\lambda)$. In practice this means that for some desired final $\sigma(\tilde{U},\tilde{\lambda})$ to be obtained by a KW transformation, (5.22) gives the necessary initial $\sigma(U,\lambda)$.

5.2.2. Particular transformation types

To study the implications of (5.22) in the case of blocking transformations for definiteness the one of Swendsen [41] for pure gauge fields shown in Fig.5 is considered. As illustrated in Fig.6, for a given $\sigma(\tilde{U},\tilde{\lambda})$ already by one transformation step a large number of contributions to $\sigma(U,\lambda)$ arises (which increases with dimension). For more steps the number of terms grows very rapidly. In this way $\sigma(U,\lambda)$ becomes a definite linear combination of form

$$\sigma(U,\lambda) = \sigma_0(U,\lambda) + \sum_{i\neq 0} \sigma_i(U,\lambda,\tilde{\lambda}) . \tag{5.23}$$

Here σ_0 is the contribution involving only the first term of the transformation in Fig.5, with gauge-field factors along a path $\mathcal{P}_0$ (consisting of one or several loops in the present case). While the path $\mathcal{P}_0$ is the same as occurs for $\sigma(\tilde{U},\tilde{\lambda})$, the $\mathcal{P}_i$ with $i \neq 0$ according to the construction are ones distributed around $\mathcal{P}_0$. Since for $\sigma_0(U,\lambda)$ the subdivision along $\mathcal{P}_0$ is finer by a factor $\tilde{\lambda}/\lambda$ than for $\sigma(\tilde{U},\tilde{\lambda})$, the effect in lattice units is an expansion of all extensions occurring in $\sigma_0(U,\lambda)$ by a factor $\tilde{\lambda}/\lambda$ as compared to those in $\sigma(\tilde{U},\tilde{\lambda})$.

It is to be stressed that it is necessary to choose $\sigma(U,\lambda)$ out of the image set of (5.22) because otherwise $\sigma(\tilde{U},\tilde{\lambda})$ and then (5.21) would simply not be defined. In particular, one is not allowed to replace the "thick path" consisting of the distribution of $\mathcal{P}_i$ by a "thin path" made up of $\mathcal{P}_0$ alone.

Another possibility is to use KW transformations of the thinning type which preserve the overall properties of σ. This in particular means that one has a path $\mathcal{P}_0$ common to $\sigma(\tilde{U},\tilde{\lambda})$ and to $\sigma(U,\lambda)$ (instead of to $\sigma_0(U,\lambda)$ only in (5.23)). The effect of (5.22) then is solely the expansion of the extensions in $\sigma(\tilde{U},\tilde{\lambda})$ by a factor $\tilde{\lambda}/\lambda$ related to the change of subdivisions. A simple example is shown in Fig.7.

Getting straightforward properties of σ under thinning transformations, however, the problem is actually shifted to S. At least in higher dimensions this may show up by drastic changes of the fixed-point structure. On the other hand, the blocking transformations with all old variables involved in new ones tend to prevent such changes.

Of course, also for usual blocking transformations [41-43] and their obvious generalizations it remains to be checked what the fixed-point structure actually is. Further, in view of the effects on σ it is important to know more about the thinning type of transformations. It is to be noted here that Kadanoff's reformulation [45] of Migdal's transformation [46] realizes a thinning transformation of the indicated type, which, however, is exact only in 2 dimensions.

5.3. Application to quantum field theory

5.3.1. Rescaling of QFT functions

In order to define the limit as pointed out in sect.3, one has to construct functions $P_\sigma(n_1(\nu),\dots,n_r(\nu);g_1,\dots,g_\ell)$ which satisfy (3.11). For this purpose one needs the correlation functions $G(n_1(\nu),\dots,n_r(\nu);g_1,\dots,g_\ell)$ with $n_\varsigma(\nu)$ given by (3.2) and with g_i still independent of ν. Thus, G is to be calculated for large n_ς and close to the singularity at g_i^c, for which an appropriate method is to be found.

A reformulation by KW transformations becomes possible if one restricts the values of x_ς and ν such that

$$n_\varsigma(\nu) = \nu\,\frac{x_\varsigma}{b}\,, \quad \frac{x_\varsigma}{b}\ \text{integer}\,, \quad \nu = 1,2,3,\dots \tag{5.24}$$

replaces the equivalence (3.2). Then extending the notation $\sigma(U,\lambda)$ of sect.5.2 to $\sigma(\nu,U,\lambda)$, where ν results from the dependence on $n_\varsigma(\nu)$, the functions of interest read

$$G(n_1(\nu),\dots,n_r(\nu);g) = \frac{\int_U e^{S(U,g,1)}\,\sigma(\nu,U,1)}{\int_U e^{S(U,g,1)}} \tag{5.25}$$

(again with g possibly being a shorthand for $g_1,\dots,g_\ell$ and U for all fields). By a KW transformation according to (5.21) and (5.5) one gets

$$G(n_1(\nu),\dots,n_r(\nu);g) = \frac{\int_{\tilde U} e^{S(\tilde U,g,\tilde\lambda)}\,\sigma(\nu,U,\tilde\lambda)}{\int_{\tilde U} e^{S(\tilde U,g,\tilde\lambda)}} \tag{5.26}$$

as the transform of (5.25) provided the condition (5.22) is satisfied.

5.3.2. Rescaling for particular transformations

In the case of a thinning transformation (in the sense of sect.5.2.2) one has

$$\sigma(\nu,U,\lambda) = \sigma(\nu/\tau,U,\lambda/\tau) \tag{5.27}$$

if ν/τ and λ/τ are integers, because $\nu > 1$ means to expand all extensions given for $\nu = 1$ by a factor ν, while (for the transformations considered) λ describes a shrinking of them by a factor λ. Then for $\tilde\lambda = \nu$ (5.26) using (5.27) becomes

$$G(n_1(\nu),\dots,n_r(\nu);g) = \frac{\int_{\tilde U} e^{S(\tilde U,g,\nu)}\,\sigma(1,\tilde U,1)}{\int_{\tilde U} e^{S(\tilde U,g,\nu)}} \tag{5.28}$$

with the remarkable property that the ν dependence is transformed completely to the action. Thus the ultraviolet limit in σ is replaced by an infrared limit in S.

For a blocking transformation, however, it turns out that (5.22) cannot be rigorously satisfied. The reason is that the form (5.23) for $\sigma(\nu,U,1)$ is in contradiction to the fact that $\sigma(\nu,U,1)$ must arise from $\sigma(1,U,1)$ by an expansion of all extensions by a factor ν. If this holds for σ_0 it cannot be achieved for the σ_i with $i \neq 0$ as is illustrated by a simple example in Fig.8.

At this point it is instructive to satisfy (5.22) in an approximate way, requiring $\sigma(\nu,U,1)$ to be properly given by $\sigma(1,U,1)$. For this purpose one introduces an associate quantity $\bar\sigma$ for which $\bar\sigma(U,\nu) = \sigma(1,U,1)$ holds. Then $\bar\sigma(U,1)$ is to be calculated according to (5.22). The approximation to be made when doing this is for one step illustrated in Fig.9.

For definiteness the transformation of Fig.5 is used. The result depicted in Fig.9 arises from an original loop of perimeter L_j. The values β_j are adjustable to optimize the approximation of smoothing the loops with resulting perimeter $2L_j$. The factor γ_j is obtained by summing the contributions, which gives

$$\gamma_j = \alpha + \beta_j\left[\left(1+2(d-1)\right)^{L_j} - 1\right] \tag{5.29}$$

where $L_j = 2L_{j-1} = 2^{j-1}L_1$. For $\nu = 2^n$ one then has

$$\bar{\sigma}(U,1) \approx \gamma_1 \cdots \gamma_n\, \sigma(\nu, U, 1)\,. \tag{5.30}$$

Thus it turns out that replacing $\sigma(\nu,U,1)$ in (5.25) by (5.30) the transformation leads to (5.28).
The factor $\gamma_1\cdots\gamma_n$ in (5.30) is nothing else but the nonperturbative analogue of the perimeter-divergence factor in (3.38). Evaluating the leading contribution one obtains

$$\gamma_1 \cdots \gamma_n \approx \beta_1 \cdots \beta_n \left(1+2(d-1)\right)^{(\nu-1)L_1} \tag{5.31}$$

It is obvious that for a blocking transformation involving more paths the expression 1+2(d-1) in (5.31) would be replaced by a larger one.

5.3.3. Implications for QFT limit

In the combinations $P_{\bar\sigma}$ needed for QFT, the factors (5.31) are to cancel out. Thus, for the blocking as well as for the thinning type of transformations one arrives at the form of $P_{\bar\sigma}$ which is made up by functions of type (5.28). The restriction (5.24) implies that $n_s(1)$ is to be chosen large enough (i.e. b small enough) for a sufficiently accurate description. This means that in a way one has now a two-stage limit.

For one essential parameter g by (3.14) with (3.11) the defining relation for $g(\nu)$ is

$$P_{\bar\sigma}\left(\bar n_1(\nu),\dots,\bar n_r(\nu); g(\nu)\right) \cong Q_{\bar\sigma,\bar x}\,. \tag{5.32}$$

For the transformed $P_{\bar\sigma}$ (made up of functions of form (5.28)), assuming that (5.16) holds with one relevant component ζ, with respect to the dependences on ζ, g and ν (5.32) is of type

$$\mathcal{P}\left(\bar n_1(1),\dots,\bar n_r(1); \zeta(g(\nu),\nu)\right) \cong Q_{\bar\sigma,\bar x}\,. \tag{5.33}$$

By using the inverse function $\mathcal{R}$ of $\mathcal{P}$, (5.33) becomes

$$\zeta(g(\nu),\nu) \cong \mathcal{R}\left(Q_{\bar\sigma,\bar x}, \bar n(1)\right). \tag{5.34}$$

Comparing (5.34) with (5.19) it is seen that $g(\nu) \cong g^{KW}(\nu)$ provided that $C = \mathcal{R}(Q_{\bar\sigma,\bar x}, \bar n(1))$ and that the above assumptions hold. Then also β and β^{KW} defined by (3.47) and (5.20) become equivalent.

In contrast to the application to critical phenomena in statistical physics, the use of KW transformations in QFT turns out to be subject to substantial restrictions. This is apparent from the following overview:

1) For the partition function in (5.5) $\tilde\lambda$ can increase freely, which can be fully exploited for any value of g.
2) For correlation functions in (5.21) the condition (5.22) on σ also implies that a maximal value $\tilde\lambda = \lambda_{max}$ occurs where a characteristic extension in σ reaches the length of one link.
3) For QFT functions in (5.26) only $\tilde\lambda \leq \nu$ is generally reasonable and $\tilde\lambda = \nu$ is the appropriate choice.
4) As is transparent from (5.34), the basic equivalence relations of QFT, (3.14) with (3.11), require that for increasing $\tilde\lambda = \nu$ simultaneously g is forced to approach g^c.

It should be noticed that according to (5.34) the magnitude of $\mathcal{R}(Q_{\bar\sigma,\bar x}, \bar n(1))$ also decides about possible methods for the further evaluation.

The assumption that (5.16) can be used to get (5.34) implies that only the long-distance properties of S enter the definition of QFT. On the other hand, (5.34) obviously keeps ζ at a fixed value. Therefore, irrelevant components must be either negligible in the corresponding region or not there at all due to an appropriate choice of the form of S (for example in Fig.4 on the "renormalized" trajectory).

5.3.4. Criteria for essential parameters

The assumption that the action (5.1) can be chosen such that (5.16) becomes

$$S(U,\dots; g_1,\dots,g_\ell; \lambda) = \bar S(U,\dots; \zeta_1,\dots,\zeta_{\ell'})\,, \tag{5.35}$$

with k' relevant components $\zeta_i \equiv \zeta_i(g_1,\dots,g_\ell;\lambda)$ and ℓ'-k' marginal ones $\zeta_j \equiv \zeta_j(g_1,\dots,g_\ell)$, in the language of sect.5.1.3 means that on the "renormalized" manifold S is determined by ℓ' parameters.

If one has $\ell' = \ell$ in (5.35) this confirms that there are, in fact, ℓ essential parameters as anticipated. If $\ell' < \ell$ more parameters than necessary have been introduced and their number can be reduced to ℓ'. On the other hand, if $\ell' > \ell$ the original choice is not complete and may be supplemented. In all this appropriate invertibility properties of the ζ_i are assumed.

Now in the case $\ell' = \ell$, which one achieves as indicated, k' is studied. In the transformed P_σ (constructed from functions of form (5.28)), the ν dependence enters via the k' relevant components $\zeta_i(g_1,\ldots,g_\ell;\nu)$ of (5.35). It is wiped out if by the choice of the g_j one shifts ζ_i to the fixed-point value ζ_i^*. On the other hand, in (3.11) with (3.9) the ν dependence is wiped out if g_i tends to g_i^c. This relates $\zeta_j \to \zeta_j^*$ to $g_i \to g_i^c$ and leads to k' = k.

Since the fixed-point structure depends on the particular KW transformation used, it is important to investigate which class allows the present characterization in a universal way.

6. Conclusions

For a truely nonperturbative description of particle physics at present only lattice regularization is available. A suitable lattice formulation can be given. In particular, the problems with fermions can be overcome by techniques a deeper understanding of which should result from the further investigation of nonperturbative features. The gauge-invariant quantization which becomes possible not only allows to avoid the difficulties of gauge-fixing but also reveals important details of the theory.

General renormalizability means the existence of the limit which defines the quantized theory in a nonperturbative way. Two types of conditions for this occur. The first type is related to the allowed variable dependences while the second one concerns invertibility properties of the functions involved. The singularity in the space of essential parameters plays a fundamental role. The dependence of the parameters on the numbering of the sequence needed for the limit is determined by a system of equivalence relations.

From the given framework certain general features of the nonperturbatively renormalized theory follow. The notions of mass scales and constants become precise. RG invariance can be defined in a general way. Clear relations between various types of β functions arise.

As compared to the perturbative approach rather different renormalizability criteria are obtained. The reference to the fundamental parameter singularity, the mathematically sound definition of the limit, the (general) RG invariance of the formulation and the straightforward inclusion of gauge fields are features which contrast favorably to those of the perturbative case.

The knowledge about the critical singularities of interest is (qualitatively) fair in pure gauge theories, however, very unsatisfactory in the presence of matter fields. While in QCD due to asymptotic freedom there are indications on the singularity structure, almost nothing is known in other cases. To make progress within this respect is rather urgent because a truely nonperturbative description is a prerequisite for the settling of any mass and unification problems.

The consequent application of KW transformations to QFT basically consists in a rescaling within the correlation functions which occur in the definition of the QFT limit. In contrast to the application in statistical mechanics their use here is subject to particular restrictions.

For the transformation of correlation function a nontrivial condition exists which leads to features not there when merely transforming the partition function. Specific behaviours arise for (gauge invariant) transformations of the blocking and thinning types. For QFT functions the nonperturbative mechanism of multiplicative renormalization effects is related to the blocking nature.

Only the long-distance properties of the action enter the definition of QFT if in an appropriate auxiliary-parameter representation one goes to the "renormalized" manifold. This allows a characterization of the essential parameters of QFT in terms of relevant and marginal components.

Since KW transformations turn out to be of interest for nonperturbative QFT from the conceptual as well as from the technical point of view, it appears urgent to investigate detailed properties of suitable classes of them.

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Figure captions

Fig.1. Qualitative behavior of (4.1) for a) SU(N) and b) U(1)

Fig.2. Qualitative behavior of $\tilde{M}$ for mesons,
a) for $g^2 \approx 1$ in Monte-Carlo simulation,
b) for $g^2 = 0$ as needed theoretically

Fig.3. Phase diagram for SU(2) fundamental-adjoint action

Fig.4. Example of fixed point FP, "critical" surface CS, "renormalized" trajectory RT, and general trajectory T

Fig.5. Example of invariant blocking transformation

Fig.6. Illustration of effect of one blocking step on σ,
a) some links with gauge-field factors of $\sigma(\tilde{U},\tilde{\lambda})$,
b) a few typical contributions arising to $\sigma(U,\lambda)$

Fig.7. Example of effect of thinning transformation on σ, a) $\sigma(\tilde{U},\tilde{\lambda})$, b) $\sigma(U,\lambda)$

Fig.8. Example for behavior of σ,
a) for increasing ν of QFT definition
b) for increasing λ of blocking transformation

Fig.9. Approximation after one step of blocking transformation of σ

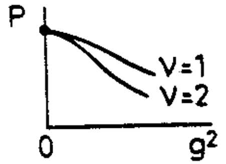

Fig. 1a

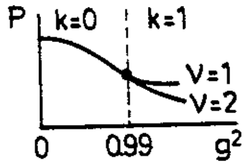

Fig. 1b

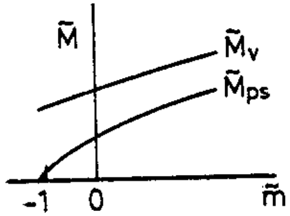

Fig. 2a

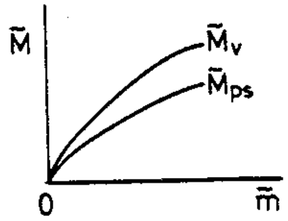

Fig. 2b

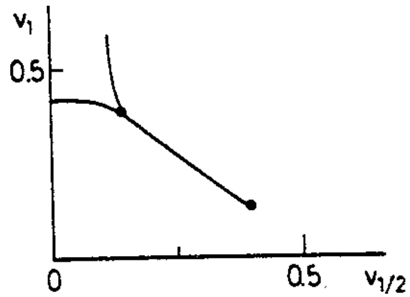

Fig. 3

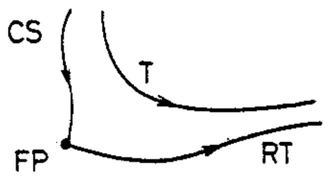

Fig. 4

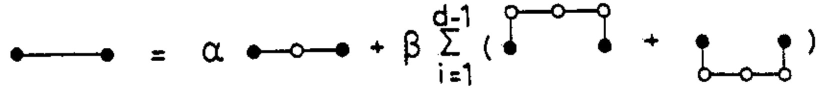

Fig. 5

Fig. 6a

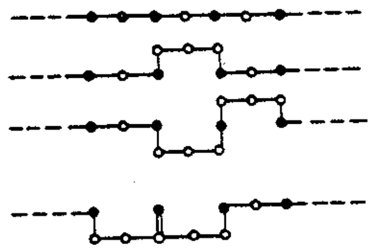

Fig. 6b

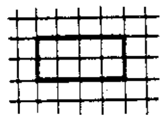

Fig. 7a

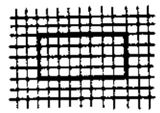

Fig. 7b

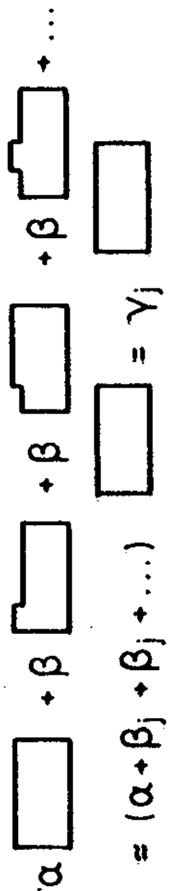

Fig.9

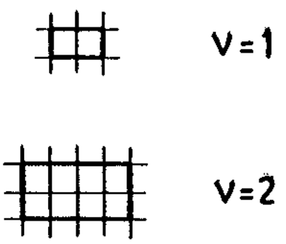

Fig. 8a

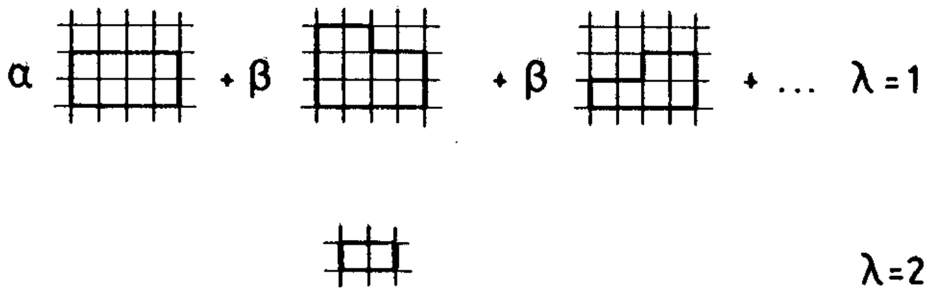

Fig.8b