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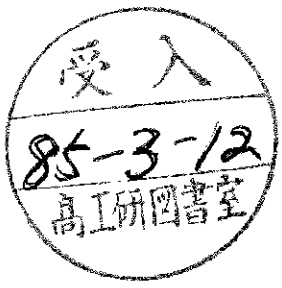

EFFECTIVE TRANSFER MATRIX FOR LOW-LYING GLUEBALL STATES IN LATTICE GAUGE THEORY

by

G. Münster

II. Institut f. Theoretische Physik, Universität Hamburg

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Effective Transfer Matrix for low-lying Glueball states in Lattice Gauge Theory

Gernot Münster
II. Institut für Theoretische Physik der Universität Hamburg, Germany

Abstract:

In four-dimensional lattice gauge theory without quarks the lowest particle states in the strong coupling region correspond to a spin zero and a spin two glueball. They are degenerate in the lowest order of the strong coupling expansion. There exists an effective transfer matrix, acting in the lowest order eigenspace, whose eigenvalues are identical with the corresponding ones of the full transfer matrix to all orders in the strong coupling expansion. It is constructed here up to the 8^{th} nontrivial order for different gauge groups. Diagonalization yields the energy-momentum dispersion relations for glueball states.

1. Introduction

The low-lying spectrum of lattice gauge theories has been studied by different methods [1]. One of them is the strong coupling expansion. For several low lying particle states (glueballs) strong coupling expansions of their masses have been derived for different gauge groups and lattice actions [2-8]. In the following we limit ourselves to the consideration of Euclidean lattice gauge theory with standard Wilson action. For zero-momentum states the strong coupling calculation of their energies simplifies in comparison to the case of nonzero momentum. This is due to the fact that zero-momentum states transform irreducibly under some representation of the cubic lattice symmetry group. This allows the explicit construction of operators which create such states out of the vacuum [2,9]. On the other hand, for states with generic nonzero momentum, the lattice symmetry does not give any restrictions on the operators creating them. If one likes to calculate their energy one has to determine the states properly. In particular they contain parts belonging to different irreducible representations of the cubic group and the relative amplitudes of these contributions must be calculated.

In another article [10] we considered a triple glueball vertex function in the framework of the strong coupling expansion. The calculation required knowledge of the energy-momentum dispersion relation for low-lying glueballs and of the corresponding eigenstates. This information can be obtained from the results of the present article.

The essential ingredient of our method is an effective transfer matrix whose eigenvalues coincide with the eigenvalues of the full transfer matrix belonging to the lowest glueball states. We calculate the effective transfer matrix up to 12th order in the strong coupling expansion. Since the expansion starts at fourth order this amounts to a calculation of eight orders. Diagonalizing the matrix the energy-momentum dispersion relations for the lowest spin zero and the lowest spin two glueballs are obtained.

For gauge groups SU(2) and Z_2 Kimura and Ukawa [11] already studied the energy-momentum dispersion relation of the scalar glueball. In their calculation, however, they do not use the proper eigenstate but a state which behaves like the zero-momentum state with respect to rotations. As a result the two-plaquette correlation function failed to exponentiate. Their results for the energy are nevertheless correct up to terms quadratic in the momentum.

Our notations are the following. The models are defined on a hypercubical lattice in 4 dimensions. The gauge field $U(b) \in G$ is attached to the links b of the lattice and takes values in the gauge group G. The ordered product of variables U(b) on the boundary of an elementary plaquette p is called U(p). The action is

$$S = - \frac{2}{g^2} \sum_p Re\, tr\, U(p) \tag{1}$$

The sum extends over all unoriented plaquettes p and g is the bare coupling constant. Expectation values of functions $\mathcal{O}$ of the link variables are evaluated by

$$\langle \mathcal{O} \rangle = Z^{-1} \int \prod_b dU(b)\ \mathcal{O}\ \exp(-S) \tag{2}$$

where

$$Z = \int \prod_b dU(b)\ \exp(-S) \tag{3}$$

is the partition function.

The quantity which corresponds to the Euclidean time evolution operator exp(-tH), where H is the Hamiltonian, is the transfer matrix T [12]. It is known to exist for the lattice gauge theories under consideration [13,14]. The lattice Hamiltonian is defined by

$$H = - a^{-1} \log T \tag{4}$$

where a is the lattice spacing. In the following we set a = 1. Physical states with energy E are eigenvectors of the transfer matrix with eigenvalue

$$\lambda = \exp(-E) \tag{5}$$

The transfer matrix commutes with spatial lattice translation. Consequently physical states carry definite momenta $\vec{p}$ and we write

$$T\, \Psi(\vec{p}) = \exp(-E(\vec{p}))\, \Psi(\vec{p}) \tag{6}$$

$$\vec{p} = (p_1, p_2, p_3), \quad |p_i| \leq \pi \tag{7}$$

For a particular particle the corresponding function $E(\vec{p})$ is its energy-momentum dispersion relation, and its mass is

$$m = E(\vec{0}) \tag{8}$$

Concerning the strong coupling expansion we use the notations and definitions of [3]. In particular the expansion parameter is denoted u.

2. Effective Transfer-matrix

In the framework of degenerate perturbation theory in quantum mechanics the concept of an effective Hamiltonian is well-known [15]. Consider a Hamiltonian of the form

$$H = H_0 + V \tag{9}$$

where H_0 is a solvable unperturbed Hamiltonian and V is a perturbation. Suppose H_0 has a n-fold degenerate eigenvalue E_0 with eigenspace $\mathcal{H}_0$. In higher orders of perturbation theory the degeneracy is lifted giving rise to n distinct eigenvalues $E^{(i)}$, i = 1, ..., n of the full Hamiltonian H. An effective Hamiltonian $\hat{H}$ is an operator acting only in the n-dimensional space $\mathcal{H}_0$, whose eigenvalues are identical with the $E^{(i)}$. There are different ways to construct $\hat{H}$ order by order in perturbation theory.

In the following we apply this concept to the strong coupling expansion of lattice gauge theory. The transfer matrix T is normalized such that its largest eigenvalue, belonging to the vacuum Ψ_0, equals one.

$$T\, \Psi_0 = \Psi_0 \tag{10}$$

In the strong coupling region the three next largest eigenvalues correspond to particle states [4,5,16]

$$T\, \Psi_i(\vec{p}) = \lambda_i(\vec{p})\, \Psi_i(\vec{p}), \quad i = 1,2,3 \tag{11}$$

For zero momentum Ψ_1 belongs to the trivial representation A_1 of the cubic lattice symmetry group and is called 0^{++}-glueball, whereas Ψ_2 and Ψ_3 are degenerate and transform according to the representation E of the cubic group, being called 2^{++}-glueball [1]. Both types of states have even spatial parity and even C-parity [2,5].

In the strong coupling expansion one obtains

$$\lambda_i(\vec{p}) = u^4 + O(u^5) \quad , \quad i = 1,2,3 \tag{12}$$

Thus in the leading (fourth) order of the expansion the three states are degenerate. Furthermore there are no other states with even C-parity degenerate with the states above to leading order.

For some gauge groups, in particular for SU(3), there is a C-odd triplet, transforming under the representation T_1 of the cubic group, which is degenerate with the states above in the leading order of the expansion. It is called 1^{+-}-glueball. For simplicity we restrict ourselves to the C-even states in the following. Of course, everything can also be applied to the C-odd states.

It is well known how to construct states which generate the states $\Psi_i(\vec{p})$ in the leading order of the strong coupling expansion. Let

$$p_i(\vec{x}) \quad , \qquad i \in \{1,2,3\} \tag{13}$$

be a space-like unoriented plaquette at time t = 0 with center $\vec{x}$ and being perpendicular to the i-axis. We call i the orientation of this plaquette. Define

$$\chi_i(\vec{x}) = \left\{ \mathrm{tr}\, U(p_i(\vec{x})) - \langle \mathrm{tr}\, U(p) \rangle \right\} \Psi_0 \tag{14}$$

where trU(p) acts as a multiplication operator on the vacuum state Ψ_0. Since the momenta commute with T it is convenient to work with momentum eigenstates from the beginning

$$\varphi_i(\vec{p}) = N \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \chi_i(\vec{x}) \tag{15}$$

The sum extends over all possible centers of plaquettes and N is an irrelevant normalization factor. These states satisfy

$$T \varphi_i(\vec{p}) = u^4 \varphi_i(\vec{p}) + O(u^5) \tag{16}$$

and they are orthogonal to the vacuum. In the strong coupling expansion as well as in Monte Carlo calculations they are used to create the lowest glueball states from the vacuum. The matrix

$$C_{ik}(\vec{p}) = \langle \Psi_i(\vec{p}) \mid \varphi_k(\vec{p}) \rangle \tag{17}$$

is nonsingular in the strong coupling expansion.

Now let

$$P = \sum_{i=1}^{3} | \Psi_i(\vec{p}) \rangle \langle \Psi_i(\vec{p}) | \tag{18}$$

be the projector on the space of glueballs with momentum $\vec{p}$ and consider the matrix

$$\varrho_{ik}(n,\vec{p}) = \langle \varphi_i(\vec{p}) \mid P\, T^n\, P \mid \varphi_k(\vec{p}) \rangle \tag{19}$$

Obviously we have

$$\varrho(n,\vec{p}) = C^+ C\, \hat{T}^n \tag{20}$$

where

$$\hat{T} = C^{-1} D\, C \tag{21}$$

and

$$D = \mathrm{diag}(\lambda_1(\vec{p}), \lambda_2(\vec{p}), \lambda_3(\vec{p})) \tag{22}$$

This means that knowledge of the matrices ϱ would allow to obtain an effective transfer matrix $\hat{T}$, whose eigenvalues are the desired exact eigenvalues $\lambda_i(\vec{p})$ of the glueball states.

In fact it turns out that the matrices ϱ can be calculated in the strong coupling expansion. The essential observation is that the moment-cumulant-transformation [17,3] provides a method to obtain the projection operator P exactly to all orders of the expansion. I shall dispense with showing this in detail but merely indicate the line of reasoning. The starting point are the correlation functions

$$\tau_{ik}(n,\vec{p}) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \langle \mathrm{tr}\, U(p_i(\vec{x}_o+\vec{x},n))\ \mathrm{tr}\, U(p_k(\vec{x}_o,0))\rangle_c \qquad (23)$$

The second argument of the plaquettes are the time coordinates. We write τ in the form

$$\tau_{ik}(n,\vec{p}) = \langle \varphi_i(\vec{p}) \mid T^n \mid \varphi_k(\vec{p})\rangle$$

$$= \varrho_{ik}(n,\vec{p}) + \langle \varphi_i(\vec{p}) \mid (1-P)T^n(1-P) \mid \varphi_k(\vec{p})\rangle \qquad (24)$$

The second term in (24) is of order u^{6n} in the strong coupling expansion. The formulation of the strong coupling expansion for correlation functions in terms of lattice graphs [3] can now be generalized to the case of correlation matrices like τ. The contribution of a graph is then given by a certain matrix. Moreover it can be shown that the moment-cumulant-transformation applied to correlation functions [3,17] also works here in the sense of matrix algebra. It yields an expansion for $\frac{1}{n}\log\tau$ in terms of connected graphs. As usual it allows to isolate those contributions which exponentiate and represent corrections to the leading term $4\log u$. In this way the piece ϱ is projected out from τ.

Using these methods I calculated the effective transfer matrix $\hat{T}$ up to order u^{12}, which are 8 nontrivial orders of the expansion. From it the effective Hamiltonian

$$\hat{H} = -\mathrm{Log}\,\hat{T} \qquad (25)$$

can be derived up to order u^8.

3. Strong-coupling expansion

In the following we present the results for the effective Hamiltonian $\hat{H}$. Define

$$\hat{p}_j = 2\sin\tfrac{1}{2}p_j$$
$$t_j = 2\cos\tfrac{1}{2}p_j \quad , \quad j = 1,2,3 \qquad (26)$$
$$A_{ik} = t_i t_k - t_i^2\,\delta_{ik}$$
$$B_{ik} = (t_i^2 - 2)\,\delta_{ik} \qquad (27)$$

In terms of the matrices A, B and $\mathbb{1}$ the expansions are as follows.

For SU(2):

$$\hat{H} = (-4\log u + 2u^2)\mathbb{1} - u^4\{\left[\tfrac{50}{3} - \hat{p}^2\right]\mathbb{1} + 2A\}$$
$$- u^6\{\left[-\tfrac{10606}{405} + \hat{p}^2\right]\mathbb{1} + 7A + 11B\}$$
$$- u^8\{\left[\tfrac{43442}{243} - \tfrac{35}{3}\hat{p}^2 + \tfrac{1}{2}(\hat{p}^2)^2\right]\mathbb{1} + \left[\tfrac{67}{3} + \hat{p}^2\right]A$$
$$+ \left[-1 + 3\hat{p}^2\right]B + 3A^2 + 3B^2 + 2(AB + BA)\} \qquad (28)$$

For SU(3):

$$\hat{H} = (-4\log u - 3u + 9u^2 - \tfrac{27}{2}u^3)\mathbb{1}$$
$$- u^4\{\left[-9 - \hat{p}^2\right]\mathbb{1} + 2A\}$$
$$- u^5\{\left[\tfrac{201}{2} - 3\hat{p}^2\right]\mathbb{1} + 6A\}$$
$$- u^6\{\left[-\tfrac{1247947}{10240} + \tfrac{7}{2}\hat{p}^2\right]\mathbb{1} + 2A + 11B\}$$
$$- u^7\{\left[-\tfrac{36\,028\,989}{71\,680} + 12\,\hat{p}^2\right]\mathbb{1} - 21A + 3B\}$$

–

$$- u^8\{ \left[\frac{113\,631\,493}{40\,960} - \frac{459}{8}\hat{p}^2 + \frac{1}{2}(\hat{p}^2)^2\right]\mathbb{1} + \left[\frac{673}{4} + \hat{p}^2\right]A + \left[\frac{83}{2} + 3\hat{p}^2\right]B + 3A^2 + 3B^2 + 2(AB + BA)\} \quad (29)$$

For SU(∞):

$$\hat{H} = (- 4\log u)\mathbb{1} - u^4\{ [18 - \hat{p}^2]\mathbb{1} + 2A\} - u^6\{ [54 - \hat{p}^2]\mathbb{1} + 11A + 11B\} - u^8\{ \left[124 - 9\hat{p}^2 + \frac{1}{2}(\hat{p}^2)^2\right]\mathbb{1} + \left[19 + \hat{p}^2\right]A + \left[1 + 3\hat{p}^2\right]B + 3A^2 + 3B^2 + 2(AB + BA)\} \quad (30)$$

For Z_2:

$$\hat{H} = (- 4\log u + u^2)\mathbb{1} - u^4\{ \left[\frac{35}{2} - \hat{p}^2\right]\mathbb{1} + 2A\} - u^6\{ \frac{5}{3}\mathbb{1} + 9A + 11B\} - u^8\{ \left[\frac{567}{4} - 10\hat{p}^2 + \frac{1}{2}(\hat{p}^2)^2\right]\mathbb{1} + \left[20 + \hat{p}^2\right]A + 3\hat{p}^2B + 3A^2 + 3B^2 + 2(AB + BA)\} \quad (31)$$

For Z_3:

$$\hat{H} = (- 4\log u - u + \frac{5}{2}u^2 - \frac{7}{3}u^3)\mathbb{1} - u^4\{ \left[\frac{55}{4} - \hat{p}^2\right]\mathbb{1} + 2A\} - u^5\{ \left[\frac{131}{5} - \hat{p}^2\right]\mathbb{1} + 2A\} - u^6\{ \left[-\frac{197}{6} + \hat{p}^2\right]\mathbb{1} + 7A + 11B\} - u^7\{ \left[\frac{1}{7} + \hat{p}^2\right]\mathbb{1} - A + B\} - u^8\{ \left[\frac{2479}{8} - 18\hat{p}^2 + \frac{1}{2}(\hat{p}^2)^2\right]\mathbb{1} + \left[56 + p^2\right]A + \left[16 + 3\hat{p}^2\right]B + 3A^2 + 3B^2 + 2(AB + BA)\} \quad (32)$$

For U(1), Wilson action:

$$\hat{H} = (- 4\log u + \frac{3}{2}u^2)\mathbb{1} - u^4\{ \left[\frac{409}{24} - \hat{p}^2\right]\mathbb{1} + 2A\} - u^6\{ \left[-\frac{281}{24} + \frac{1}{2}\hat{p}^2\right]\mathbb{1} + 8A + 11B\} - u^8\{ \left[\frac{411349}{2880} - \frac{32}{3}\hat{p}^2 + \frac{1}{2}(\hat{p}^2)^2\right]\mathbb{1} + \left[\frac{125}{6} + \hat{p}^2\right]A + \left[-\frac{1}{2} + 3\hat{p}^2\right]B + 3A^2 + 3B^2 + 2(AB + BA)\} \quad (33)$$

For U(1), Villain action:

$$\hat{H} = (-4\log u + 2u^2)\mathbb{1} - u^4\{[17 - \hat{p}^2]\mathbb{1} + 2A\}$$
$$- u^6\{\left[-\frac{62}{3} + \hat{p}^2\right]\mathbb{1} + 7A + 11B\}$$
$$- u^8\{\left[\frac{339}{2} - 12\,\hat{p}^2 + \frac{1}{2}(\hat{p}^2)^2\right]\mathbb{1} + \left[23 + \hat{p}^2\right]A$$
$$+ \left[-1 + 3\hat{p}^2\right]B + 3A^2 + 3B^2 + 2(AB + BA)\} \qquad (34)$$

Diagonalization of $\hat{H}$ yields the energy momentum dispersions $E_i(\vec{p})$ for the glueball states. Here we only write down the results for the energy $E_1(\vec{p})$ of the 0^{++}-glueball and leave the other two as an exercise for the reader. Because the momentum dependence is rather complicated we present the energy in the form

$$E_1(\vec{p}) = m + f\cdot\hat{p}^2 + g\cdot\left((\hat{p}^2)^2 - 3\sum_i \hat{p}_i^4\right) + h\cdot(\hat{p}^2)^2 + O(\hat{p}^6) \qquad (35)$$

with

$$m = -4\log u + \sum_{k=1}^{\infty} m_k u^k \qquad (36)$$

$$f = \sum_{k=4}^{\infty} f_k u^k\,,\quad g = \sum_{k=4}^{\infty} g_k u^k\,,\quad h = \sum_{k=8}^{\infty} h_k u^k \qquad (37)$$

The coefficients are contained in table 1. The eigenvector $v^{(1)}$ belonging to the eigenvalue E_1 of $\hat{H}$ is to lowest order in u given by

$$v_k^{(1)} = t_k(\mu_1 + t_k^2)^{-1} \qquad (38)$$

where μ_1 is a root of the equation

$$\mu^3 - (t_1^2t_2^2 + t_2^2t_3^2 + t_3^2t_1^2)\mu - 2t_1^2t_2^2t_3^2 = 0 \qquad (39)$$

What are these results good for? First of all the energy $E_1(\vec{p})$ as well as the corresponding eigenvector $v^{(1)}$ for a particular momentum with

$$\hat{p}_1 = O(u^{-1}) \qquad (40)$$

were needed in [10] in the course of a calculation of a triple glueball coupling constant. In this case $\hat{H}$ can be determined up to order u^4 from the results above.

Secondly the energy momentum dispersion $E(\vec{p})$ allows to consider the question of restoration of Euclidean invariance [11]. In the continuum limit Euclidean invariance should hold and $E(\vec{p})$ should approach its relativistic form

$$E = (m^2 + \vec{p}^2)^{1/2} = m + \vec{p}^2/2m + O(p^4) \qquad (41)$$

In particular the quantity

$$C_1 = 2mf \qquad (42)$$

should approach the value 1. For gauge group SU(2) it has been considered in [11]. For SU(3) the strong coupling expansion of C_1 is

$$C_1 = \tfrac{1}{3}u^4\{-\log u\,[56 + 168u + 36u^2 - 600u^3 + 5313u^4]$$
$$-42u + 162u^3 - 134u^4 + O(u^5)\} \qquad (43)$$

It is plotted in fig. 1 as a function of β together with the diagonal Padé approximant applied to the power series in (43). The behaviour is similar to the SU(2) case. In the crossover region around $\beta \approx 5$ the function C_1 reaches the value 1 and continues to increase for larger values of β. As the strong coupling expansion ceases to be reliable beyond the crossover, the true function C_1 may well approach its limit value 1 there, as indicated by Monte Carlo calculations [18].

Finally I would like to point out that the correlation matrix ς can also be determined in a Monte Carlo calculation. If the masses of other glueballs in the parity-sector under consideration are sufficiently higher and if n is sufficiently large, the measurable correlations $\tau_{ik}(n,\vec{p})$ are approximately equal to the ς_{ik}. Numerical diagonalization of ς would give the proper energies and eigenstates. Such an analysis could be done by extending the calculations of [19].

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Table caption:

Tab. 1: Coefficients of the strong coupling expansions (36), (37) for different gauge groups. For U(1) both cases of Wilson's and Villain's action are listed.

Figure caption:

Fig. 1: The quantity C_1, defined in (42), as a function of $\beta = 6/g^2$ for gauge group SU(3). The upper curve shows the expansion (43), the lower curve is a diagonal Padé approximant.

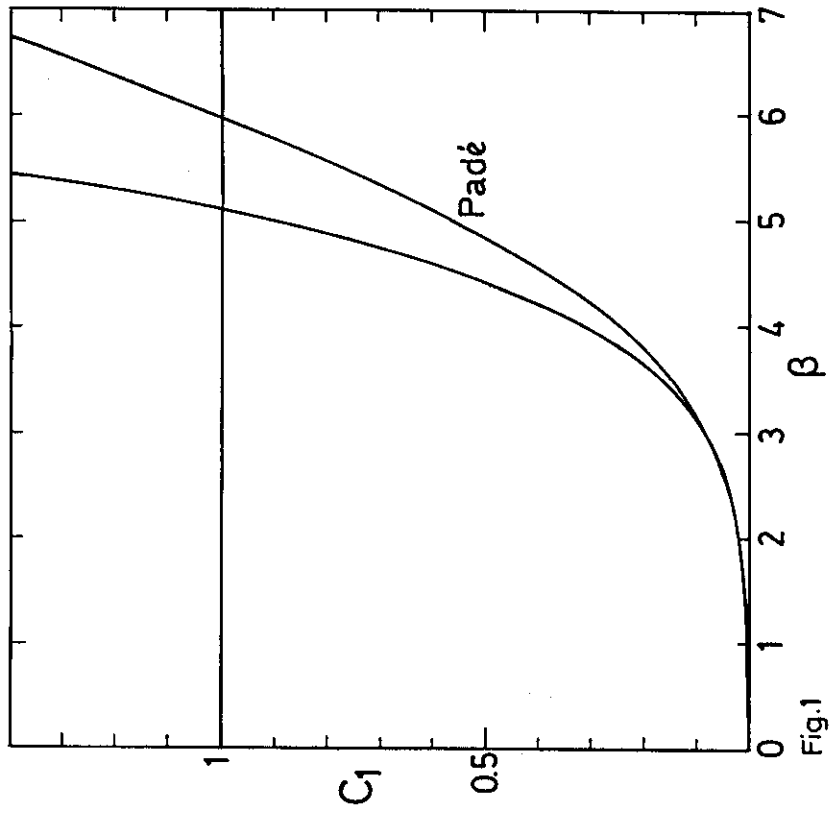

Fig.1

Table 1

	SU(2)	SU(∞)	Z_2	U(1), Wilson	U(1), Villain	SU(3)	Z_3
m_1						-3	-1
m_2	2	0	1	$\frac{3}{2}$	2	9	$\frac{5}{2}$
m_3						$-\frac{27}{2}$	$-\frac{7}{3}$
m_4	$-\frac{98}{3}$	-34	$-\frac{67}{2}$	$-\frac{793}{24}$	-33	-7	$-\frac{119}{4}$
m_5						$-\frac{297}{2}$	$-\frac{211}{5}$
m_6	$-\frac{20984}{405}$	-164	$-\frac{287}{3}$	$-\frac{1783}{24}$	$-\frac{172}{3}$	$\frac{858\ 827}{10\ 240}$	$-\frac{271}{6}$
m_7						$\frac{47\ 641\ 149}{71\ 680}$	$\frac{41}{7}$
m_8	$-\frac{151\ 496}{243}$	-564	$-\frac{2279}{4}$	$-\frac{1660309}{2880}$	$-\frac{1239}{2}$	$-\frac{183\ 140\ 613}{40960}$	$-\frac{8463}{8}$
f_4	$\frac{7}{3}$	$\frac{7}{3}$	$\frac{7}{3}$	$\frac{7}{3}$	$\frac{7}{3}$	$\frac{7}{3}$	$\frac{7}{3}$
f_5						7	$\frac{7}{3}$
f_6	$\frac{22}{3}$	12	$\frac{29}{3}$	$\frac{17}{2}$	$\frac{22}{3}$	$\frac{3}{2}$	$\frac{22}{3}$
f_7						-25	$-\frac{4}{3}$
f_8	$\frac{578}{9}$	60	$\frac{184}{3}$	$\frac{1123}{18}$	65	$\frac{1771}{8}$	$\frac{296}{3}$
g_4	$-\frac{1}{27}$	$-\frac{1}{27}$	$-\frac{1}{27}$	$-\frac{1}{27}$	$-\frac{1}{27}$	$-\frac{1}{27}$	$-\frac{1}{27}$
g_5						$-\frac{1}{9}$	$-\frac{1}{27}$
g_6	$-\frac{1}{36}$	$-\frac{11}{108}$	$-\frac{7}{108}$	$-\frac{5}{108}$	$-\frac{1}{36}$	$\frac{7}{108}$	$-\frac{1}{36}$
g_7						$\frac{5}{12}$	$\frac{1}{36}$
g_8	$-\frac{103}{648}$	$-\frac{17}{216}$	$-\frac{23}{216}$	$-\frac{41}{324}$	$-\frac{37}{216}$	$-\frac{533}{216}$	$-\frac{5}{8}$
h_8	$-\frac{25}{18}$	$-\frac{25}{18}$	$-\frac{25}{18}$	$-\frac{25}{18}$	$-\frac{25}{18}$	$-\frac{25}{18}$	$-\frac{25}{18}$