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MONOPOLE-FERMION AND DYON-FERMION BOUND STATES IV: DYON-FERMION SYSTEM WITH $j = |q| - \frac{1}{2}$ AND LARGE $\kappa|q|$

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Abstract

In the first part of the paper, we give analytic, approximate results for dyon-fermion binding energies and wave functions, valid for large values of $A = \frac{1}{2}Z|eg|\kappa$, where κ is the extra magnetic moment. In the second part, more general results are obtained for the same problem that are valid when *either* A is large *or* the binding is weak. Numerical results for the binding energy are tabulated and compared. The case of very strong binding is also discussed.

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1. Introduction

In an earlier paper (paper I) [1], we investigated some properties of the dyon-fermion bound system, as described by the Hamiltonian [2]

$$H = \vec{\alpha}\cdot(\vec{p} - Ze\vec{A}) + \beta M - \frac{\varsigma}{r} - \frac{\kappa q \beta \vec{\sigma}\cdot\vec{r}}{2Mr^3}, \tag{1.1}$$

where the notation is that of refs. [1] and [2]. The numerical method given in [1] yields highly accurate results.

Since it is also useful to have formulas that are approximate but more explicit, the limit of weak binding

$$M - E \ll M, \tag{1.2}$$

has been investigated in paper II [3]. The results there were derived under the assumption that A is neither small nor large,

$$\left|A + \frac{1}{A}\right| = O(1), \tag{1.3}$$

where

$$A = \frac{1}{2}\kappa|q|. \tag{1.4}$$

For the monopole problem ($\varsigma = 0$), approximate, explicit results have also been obtained in paper III [4] for a different limiting case, namely, that of large A. In Part A of the present paper, we shall generalize these results of paper III to the case of the dyon. Furthermore, in Part B we construct a covering approximation [4] which is valid in both cases, i.e., when either (1.2) holds or when A is large. Indeed, the structure of this paper is very similar to that of paper III.

2. Eigenvalue Problem

With the Kazama-Yang decomposition [5] of the bound-state wave functions for states of minimum angular momentum $j = |q| - \frac{1}{2}$, the eigenvalue problem

$$H\psi = E\psi \tag{2.1}$$

leads to the following coupled differential equations for the radial functions

$$\begin{aligned} \frac{dG}{d\rho} &= \left(A - B - \frac{\tilde{\varsigma}}{\rho} - \frac{1}{\rho^2}\right)F, \\ \frac{dF}{d\rho} &= \left(A + B + \frac{\tilde{\varsigma}}{\rho} - \frac{1}{\rho^2}\right)G, \end{aligned} \tag{2.2}$$

where the notation is still that of ref. [1]. In particular, B is the eigenvalue parameter

$$B = AE/M. \tag{2.3}$$

These are the differential equations to be treated here.

Equation (2.2) is invariant under

$$F \leftrightarrow G, \qquad B \to -B, \quad \text{and} \quad \tilde{\varsigma} \to -\tilde{\varsigma}. \tag{2.4}$$

In this paper, unless explicitly stated otherwise, we shall assume $A > 0$, so that $\tilde{\varsigma} = \varsigma$.

Part A. WKB Approximation

3. Wave Function between Turning Points

The WKB treatment of the radial equations is somewhat different for the present dyon case as compared with the monopole case of paper III. There are two reasons for this difference. First, we have not been able to obtain for the

dyon a second-order differential equation with only singularities at $\rho = 0$ and $\rho = \infty$. Thus it is more natural to investigate directly the coupled first-order equations (2.2). Secondly, even for the monopole, the WKB approximations for F and G, as given by (III.3.9), are simpler than that of $T = F - G$, as given by (III.3.8). This is another indication that the second-order equation is less natural.

With the variable τ defined by

$$\tau = \rho A^{1/2}, \tag{3.1}$$

(2.2) are

$$\begin{aligned} \frac{dG}{d\tau} &= A^{1/2}\left(1 - y - \frac{\hat{\varsigma}}{\tau} - \frac{1}{\tau^2}\right)F, \\ \frac{dF}{d\tau} &= A^{1/2}\left(1 + y + \frac{\hat{\varsigma}}{\tau} - \frac{1}{\tau^2}\right)G, \end{aligned} \tag{3.2}$$

where

$$y = B/A \tag{3.3}$$

and

$$\hat{\varsigma} = \varsigma/\sqrt{A}. \tag{3.4}$$

From (3.2), the turning points are determined by the roots of the quadratic equations

$$1 - y - \frac{\hat{\varsigma}}{\tau} - \frac{1}{\tau^2} = 0 \tag{3.5}$$

and

$$1 + y + \frac{\hat{\varsigma}}{\tau} - \frac{1}{\tau^2} = 0. \tag{3.6}$$

Since $|y| < 1$, each of these two quadratic equations has one positive and one negative root. Explicitly, the roots of (3.5) are

$$a = 2\{-\hat{\varsigma} + [\hat{\varsigma}^2 + 4(1-y)]^{1/2}\}^{-1} > 0, \tag{3.7}$$

$$d = 2\{-\hat{\varsigma} - [\hat{\varsigma}^2 + 4(1-y)]^{1/2}\}^{-1} < 0, \tag{3.8}$$

whereas those of (3.6) are

$$b = 2\{\hat{\varsigma} + [\hat{\varsigma}^2 + 4(1+y)]^{1/2}\}^{-1} > 0, \tag{3.9}$$

$$c = 2\{\hat{\varsigma} - [\hat{\varsigma}^2 + 4(1+y)]^{1/2}\}^{-1} < 0. \tag{3.10}$$

Note that $a = b$ if and only if $y = -\hat{\varsigma}$, and $c = d$ if and only if $y = \hat{\varsigma}$.

By the symmetry (2.4), it is sufficient to consider the case

$$y \geq -\hat{\varsigma}. \tag{3.11}$$

When $y > -\hat{\varsigma}$, the ordering of the four roots is as follows:

$$a > b > 0 > c > d \qquad \text{if } y > \hat{\varsigma}; \tag{3.12}$$

$$a > b > 0 > d > c \qquad \text{if } y < \hat{\varsigma}. \tag{3.13}$$

In both cases, the wave functions F and G are oscillatory for

$$b < \tau < a. \tag{3.14}$$

In the range (3.14), the WKB approximation to F and G is

$$\begin{aligned} F &\simeq F_0(\tau)e^{iA^{1/2}\phi(\tau)} + \text{c.c.}, \\ G &\simeq G_0(\tau)e^{iA^{1/2}\phi(\tau)} + \text{c.c.}, \end{aligned} \tag{3.15}$$

where

$$\phi(\tau) = \int d\tau \left(-1 + y^2 + 2y\frac{\hat{\varsigma}}{\tau} + \frac{2+\hat{\varsigma}^2}{\tau^2} - \frac{1}{\tau^4}\right)^{1/2}. \tag{3.16}$$

The amplitudes $F_0(\tau)$ and $G_0(\tau)$ can be determined in a similar way as in paper III: We substitute (3.15) into (3.2) and demand that terms of order $A^{1/2}$ vanish. We are thus led to the equations

$$G_0(\tau) = i\gamma(\tau)F_0(\tau), \tag{3.17}$$

$$\gamma(\tau)\frac{d}{d\tau}F_0(\tau) + \frac{d}{d\tau}[\gamma(\tau)F_0(\tau)] = 0, \tag{3.18}$$

where

$$\gamma(\tau) = \left(\frac{-1 + y + \frac{\hat{\varsigma}}{\tau} + \frac{1}{\tau^2}}{1 + y + \frac{\hat{\varsigma}}{\tau} - \frac{1}{\tau^2}}\right)^{1/2}. \tag{3.19}$$

Solving (3.18) and substituting for $\gamma(\tau)$, we find

$$\begin{aligned} F_0(\tau) &= C\gamma(\tau)^{-1/2} \\ &= C\left(\frac{1 + y + \frac{\hat{\varsigma}}{\tau} - \frac{1}{\tau^2}}{-1 + y + \frac{\hat{\varsigma}}{\tau} + \frac{1}{\tau^2}}\right)^{1/4}. \end{aligned} \tag{3.20}$$

The phase of C (up to $n\pi$) is determined by the boundary condition that

$$F \to 0, \qquad G \to 0, \tag{3.21}$$

as $\tau \to \infty$. Approximation through the Airy integral then gives

$$F \simeq C\left[-\left(A - B - \frac{\varsigma}{\rho} - \frac{1}{\rho^2}\right)\right]^{-1/4}\left(A + B + \frac{\varsigma}{\rho} - \frac{1}{\rho^2}\right)^{1/4}$$

$$\cdot \exp\left[iA^{1/2}\int_a^{A^{1/2}\rho} d\tau\left(-1 + y^2 + \frac{2y\hat{\varsigma}}{\tau} + \frac{2+\hat{\varsigma}^2}{\tau^2} - \frac{1}{\tau^4}\right)^{1/2}\right] + \text{c.c.},$$

and (3.22)

$$G \simeq iC\left[-\left(A - B - \frac{\varsigma}{\rho} - \frac{1}{\rho^2}\right)\right]^{1/4}\left(A + B + \frac{\varsigma}{\rho} - \frac{1}{\rho^2}\right)^{-1/4}$$

$$\cdot \exp\left[iA^{1/2}\int_a^{A^{1/2}\rho} d\tau\left(-1 + y^2 + \frac{2y\hat{\varsigma}}{\tau} + \frac{2+\hat{\varsigma}^2}{\tau^2} - \frac{1}{\tau^4}\right)^{1/2}\right] + \text{c.c.},$$

with

$$C = |C|e^{i\pi/4}. \tag{3.23}$$

The normalization condition is

$$\int_0^\infty d\rho\,(|F|^2 + |G|^2) = M/A, \tag{3.24}$$

which we approximate as

$$\int_{A^{1/2}b}^{A^{1/2}a} d\rho\,(|F|^2 + |G|^2) = M/A. \tag{3.25}$$

Substituting (3.22) into (3.25), we find

$$2|C|^2\int_{A^{1/2}b}^{A^{1/2}a} d\rho\left\{\left[-\left(A - B - \frac{\varsigma}{\rho} - \frac{1}{\rho^2}\right)\right]^{-1/2}\left(A + B + \frac{\varsigma}{\rho} - \frac{1}{\rho^2}\right)^{1/2}\right.$$

$$\left. + \left[-\left(A - B - \frac{\varsigma}{\rho} - \frac{1}{\rho^2}\right)\right]^{1/2}\left(A + B + \frac{\varsigma}{\rho} - \frac{1}{\rho^2}\right)^{-1/2}\right\}$$

$$= M/A, \tag{3.26}$$

or

$$|C| = \frac{1}{2}\left[\frac{AB}{M}\int_{A^{1/2}b}^{A^{1/2}a}\frac{(\rho^2 + \varsigma\rho/B)\,d\rho}{\{[1 + \varsigma\rho - (A-B)\rho^2][(A+B)\rho^2 + \varsigma\rho - 1]\}^{1/2}}\right]^{-1/2}. \tag{3.27}$$

4. Wilson-Sommerfeld Quantization

Similar to the monopole case [4], when A is large, the binding energy can be determined approximately by the Wilson-Sommerfeld quantization condition [6]

$$A^{1/2}[\phi(a) - \phi(b)] = n\pi. \tag{4.1}$$

When A is positive, the right-hand side of (4.1) is $n\pi$ for the reason already discussed in paper III.

In the general dyon-fermion case, there are two distinct possibilities. Let $K_1(\rho)$ and $K_2(\rho)$ denote the two factors appearing on the right-hand sides of (2.2):

$$K_1(\rho) = A - B - \frac{\tilde{\varsigma}}{\rho} - \frac{1}{\rho^2}, \tag{4.2}$$

$$K_2(\rho) = A + B + \frac{\tilde{\varsigma}}{\rho} - \frac{1}{\rho^2}. \tag{4.3}$$

The two possibilities are:

(i) K_1 vanishes at one turning point, while K_2 vanishes at the other. Then the right-hand side of the Wilson-Sommerfeld quantization condition is $n\pi$, as given by (4.1).

(ii) K_1 or K_2 vanishes at both turning points. Then the right-hand side of the Wilson-Sommerfeld quantization condition is $(n + \frac{1}{2})\pi$. Since $\rho > 0$ at both turning points, case (ii) requires

$$A - B < 0 \tag{4.4}$$

or

$$A + B < 0. \tag{4.5}$$

Therefore,

$$A < 0 \qquad \text{for case (ii).} \tag{4.6}$$

Furthermore, in this case (ii), at both turning points $K_1(\rho) = 0$ if $\tilde{\varsigma} < 0$ (i.e., $\varsigma > 0$) and $K_2(\rho) = 0$ if $\tilde{\varsigma} > 0$ (i.e., $\varsigma < 0$). On the other hand, by a similar consideration,

$$A > 0 \qquad \text{for case (i).} \tag{4.7}$$

Here the zeroes of $K_1(\rho)$ and $K_2(\rho)$ are given by (3.7)–(3.10).

By (3.16), eq. (4.1) can be written as

$$A^{1/2} I(y, \hat{\varsigma}) = n\pi, \tag{4.8}$$

where

$$I(y,\hat{\varsigma}) = \int_b^a d\tau \left[-\left(1 - y - \frac{\hat{\varsigma}}{\tau} - \frac{1}{\tau^2}\right)\left(1 + y + \frac{\hat{\varsigma}}{\tau} - \frac{1}{\tau^2}\right)\right]^{1/2}. \tag{4.9}$$

This integral can be expressed in terms of elliptic integrals of the first and second kinds. Note the similarity between this integral and the corresponding one for the monopole case, (III.4.12). However, the present integrand is less symmetric, and the resulting elliptic integrals are not complete.

In order to recognize the integral in terms of elliptic integrals, we first factorize the integrand,

$$I(y,\hat{\varsigma}) = (1 - y^2)^{1/2} \int_b^a d\tau \left[-\left(1 - \frac{a}{\tau}\right)\left(1 - \frac{b}{\tau}\right)\left(1 - \frac{c}{\tau}\right)\left(1 - \frac{d}{\tau}\right)\right]^{1/2}. \tag{4.10}$$

We shall proceed to evaluate $I(y, \hat{\varsigma})$ under the assumption that $y > \hat{\varsigma}$ so that (3.12) holds, and subsequently show that the result is valid also for $y < \hat{\varsigma}$.

The first step is to take out a factor τ^{-2} so that the square root is that of a fourth-order polynomial in τ (rather than in τ^{-1}). Next we integrate $\int d\tau\, \tau^{-2}$ by parts to make the square root appear in the denominator,

$$I(y,\hat{\varsigma}) = (1 - y^2)^{1/2} \cdot \int_b^a d\tau \, \frac{-2\tau^2 + \frac{3}{2}(a+b+c+d)\tau - (ab+ac+ad+bc+bd-cd)}{[(a-\tau)(\tau-b)(\tau-c)(\tau-d)]^{1/2}}. \tag{4.11}$$

This integral may now be expressed in terms of the following three basic integrals [7]:

$$I_0 = \int_b^a \frac{d\tau}{[(a-\tau)(\tau-b)(\tau-c)(\tau-d)]^{1/2}} = gK(k), \tag{4.12}$$

$$\begin{aligned} I_1 &= \int_b^a \left[\frac{a-\tau}{(\tau-b)(\tau-c)(\tau-d)}\right]^{1/2} d\tau \\ &= (a-b) g \frac{1}{\alpha^2} [K(k) + (\alpha^2 - 1)\Pi(\alpha^2, k)], \end{aligned} \tag{4.13}$$

$$\begin{aligned} I_2 &= \int_b^a \left[\frac{(\tau-c)(\tau-d)}{(a-\tau)(\tau-b)}\right]^{1/2} d\tau \\ &= -\frac{1}{2} g \frac{(b-c)(b-d)}{\alpha^2(\alpha^2-1)} [\alpha^2 E(k) + (k^2 - \alpha^2) K(k) \\ &\qquad + (2\alpha^2 - \alpha^4 - k^2)\Pi(\alpha^2, k)], \end{aligned} \tag{4.14}$$

where $K(k)$, $E(k)$, and $\Pi(\alpha^2, k)$ are complete elliptic integrals of the first, second, and third kind, respectively. In these formulas,

$$k = \left[\frac{(a-b)(c-d)}{(a-c)(b-d)}\right]^{1/2}, \tag{4.15}$$

$$\alpha = \left(\frac{a-b}{a-c}\right)^{1/2}, \tag{4.16}$$

and

$$g = \frac{2}{[(a-c)(b-d)]^{1/2}}. \tag{4.17}$$

In terms of I_0, I_1, and I_2, we have

$$I(y,\hat{\varsigma}) = (1-y^2)^{1/2}\Big[-2I_2 - \frac{1}{2}(3a+3b-c-d)I_1 + \frac{1}{2}(3a^2+ab-3ac-3ad-2bc-2bd+2cd)I_0\Big], \tag{4.18}$$

or, in terms of the complete elliptic integrals K, E, and Π:

$$I(y,\hat{\varsigma}) = (1-y^2)^{1/2}[(a-c)(b-d)]^{-1/2} \cdot \{(-ac-2ad-bc-2bd+c^2+cd)K(k) - 2(a-c)(b-d)E(k) + (b-c)(a+b+c+d)\Pi(\frac{a-b}{a-c},k)\}. \tag{4.19}$$

The complete elliptic integral of the third kind can be expressed in terms of incomplete elliptic integrals $F(\theta,k')$ and $E(\theta,k')$ of the first and second kind [7]

$$\Pi(\alpha^2,k) = K(k) + \alpha[(\alpha^2-k^2)(1-\alpha^2)]^{-1/2} \cdot \left\{\frac{\pi}{2} - [E(k)-K(k)]F(\theta,k') - K(k)E(\theta,k')\right\}, \tag{4.20}$$

where

$$k' = (1-k^2)^{1/2} = \left[\frac{(b-c)(a-d)}{(a-c)(b-d)}\right]^{1/2}, \tag{4.21}$$

and

$$\sin\theta = \left(\frac{1-\alpha^2}{k'^2}\right)^{1/2} = \left(\frac{b-d}{a-d}\right)^{1/2}. \tag{4.22}$$

Using the relation (4.20), we find for the phase integral

$$I(y,\hat{\varsigma}) = (1-y^2)^{1/2}\Big([(a-c)(b-d)]^{-1/2} \cdot \{[-2a(c+d)+b(a+b-c-d)]K(k) - 2(a-c)(b-d)E(k)\} + (a+b+c+d)\Big\{\frac{\pi}{2} - [E(k)-K(k)]F(\theta,k') - K(k)E(\theta,k')\Big\}\Big). \tag{4.23}$$

This formula is discussed further in appendix A.

The normalization constant (3.27) can also be evaluated using the integrals (4.12)–(4.14). With $J_N(y,\hat{\varsigma})$ defined by

$$|C| = \frac{1}{2}[A^{1/2}y(1-y^2)^{-1/2}M^{-1}J_N(y,\hat{\varsigma})]^{-1/2}, \tag{4.24}$$

we find

$$J_N(y,\hat{\varsigma}) = [(a-c)(b-d)]^{1/2}E(k) + [(a-c)(b-d)]^{-1/2}\Big[\frac{2\hat{\varsigma}}{y(1-y^2)}b - ab - cd\Big]K(k) + \frac{2\hat{\varsigma}}{y(1-y^2)}\Big\{\frac{\pi}{2} - [E(k)-K(k)]F(\theta,k') - K(k)E(\theta,k')\Big\}. \tag{4.25}$$

We close this section with a discussion of the limit $\varsigma \to 0$. Then $c \to -a$, $d \to -b$, and

$$k|_{\varsigma=0} \equiv k_0 = \frac{a-b}{a+b}. \tag{4.26}$$

The phase integral then reduces to

$$I(y,0) = 2(1-y^2)^{1/2}(a+b)[K(k_0) - E(k_0)], \tag{4.27}$$

which we wish to compare with (III.4.21). Since the argument used for the elliptic integrals in paper III,

$$k_{\rm III} = \left(\frac{2y}{1+y}\right)^{1/2} = \left(1-\frac{b^2}{a^2}\right)^{1/2} \quad , \quad k'_{\rm III} = \frac{b}{a} \tag{4.28}$$

is related to k_0 by a Gauss transformation [7],

$$k_0 = \frac{1-k'_{\rm III}}{1+k'_{\rm III}}, \tag{4.29}$$

it follows that (4.27) above is equal to (III.4.21). Similarly, for $\varsigma = 0$ the normalization integral (4.25) reduces to $(1-y)^{-1/2}E(k_{\rm III})$ and $|C|$ reduces to (III.4.23).

Part B. Covering Approximation

5. Wave Function

In this Part B, we generalize the covering approximation of paper III, where $\varsigma = 0$, to the dyon case. This approximation is valid for both the weak-binding case of paper II and the WKB case of Part A. We are therefore forced to impose the restriction (II.8.21) on ς:

$$|\varsigma|/A^{1/2} \ll 1. \tag{5.1}$$

The underlying reason for this restriction is that, even for the special case $A = B$, (2.2) cannot be solved explicitly in terms of known functions.

With $\varsigma \neq 0$, the inversion symmetry [8] is lost, and thus it is necessary to consider separately the regions of small and large values of ρ. The interior and

the exterior regions are defined respectively by [eqs. (II.8.3) and (II.8.6)]:

$$\rho \ll \min[|\varsigma|^{-1}, (A-B)^{-1/2}] \tag{5.2}$$

and

$$\rho \gg A^{-1/2}. \tag{5.3}$$

In the limit of weak binding with A not too large,

$$\frac{A-B}{A} \ll 1, \qquad A = O(1), \tag{5.4}$$

(2.2) can, in the interior region, be approximated as

$$\frac{d\tilde{g}}{d\tilde{\eta}} = -\frac{2B}{A+B}\frac{1}{\tilde{\eta}^2}\tilde{f},$$
$$\frac{d\tilde{f}}{d\tilde{\eta}} = \left(A+B-\frac{1}{\tilde{\eta}^2}\right)\tilde{g}, \tag{5.5}$$

because of (5.1). The coefficient $2B/(A+B)$ in the first equation is determined from the requirement that, in the limit (5.1), it be the same as that of F in (2.2) at the turning point $\rho = A^{1/2}b$. In the weak-binding approximation

$$\rho = \tilde{\eta},$$
$$F(\rho) = \tilde{f}(\tilde{\eta}), \tag{5.6}$$
$$G(\rho) = \tilde{g}(\tilde{\eta}).$$

We shall modify (5.5) and (5.6) such that they are valid whenever (5.1) is satisfied, even if

$$A \gg 1. \tag{5.7}$$

Equation (5.5) makes no reference to ς. In fact, applying the inversion symmetry to (III.5.3), we obtain (5.5) of the present paper. Thus, for the dyon case the covering approximation for the interior region is the same as for the monopole case. Omitting an overall constant, we have in the interior region

$$F(\rho)=\frac{F_0(\rho)}{\tilde{f}_0(\tilde{\eta})}\tilde{f}(\tilde{\eta}),$$

$$G(\rho)=\frac{G_0(\rho)}{\tilde{g}_0(\tilde{\eta})}\tilde{g}(\tilde{\eta}), \tag{5.8}$$

where

$$F_0(\rho)=\left[\frac{(A+B)\rho^2+\varsigma\rho-1}{1+\varsigma\rho-(A-B)\rho^2}\right]^{1/4},$$

$$\tilde{f}_0(\tilde{\eta})=\left\{\frac{A+B}{2B}[(A+B)\tilde{\eta}^2-1]\right\}^{1/4},$$

$$G_0(\rho)=iF_0(\rho)^{-1},$$

$$\tilde{g}_0(\tilde{\eta})=i\tilde{f}_0(\tilde{\eta})^{-1}. \tag{5.9}$$

Here $\tilde{\eta}$ is related to ρ by

$$\tilde{I}_1(\rho)=\tilde{I}_2(\tilde{\eta}), \tag{5.10}$$

where $\tilde{I}_1$ and $\tilde{I}_2$ are "complementary" to the I_1 and I_2 of paper III in the sense that we integrate from the *lower* turning point:

$$\tilde{I}_1(\rho)=\int_{A^{1/2}b}^{\rho}d\rho'\,[(-A+B+\varsigma\rho'^{-1}+\rho'^{-2})(A+B+\varsigma\rho'^{-1}-\rho'^{-2})]^{1/2},$$

$$\tilde{I}_2(\tilde{\eta})=\int_{(A+B)^{-1/2}}^{\tilde{\eta}}d\tilde{\eta}'\left[\frac{2B}{A+B}\tilde{\eta}'^{-2}(A+B-\tilde{\eta}'^{-2})\right]^{1/2}. \tag{5.11}$$

Furthermore, the solution to (5.5) is given by

$$\tilde{g}(\tilde{\eta})=\tilde{z}^{1/2}K_{ip}(\tilde{z}),$$

$$\tilde{f}(\tilde{\eta})=\left(\frac{A+B}{2B}\right)^{1/2}\frac{d}{d\tilde{z}}[\tilde{z}^{1/2}K_{ip}(\tilde{z})], \tag{5.12}$$

with

$$\tilde{z}=\left(\frac{2B}{A+B}\right)^{1/2}\tilde{\eta}^{-1},$$

$$p=\left(2B-\frac{1}{4}\right)^{1/2}. \tag{5.13}$$

We note that

$$F(\rho)\simeq\tilde{f}(\tilde{\eta})\qquad\text{and}\qquad G(\rho)\simeq\tilde{g}(\tilde{\eta}) \tag{5.14}$$

at the lower turning point and that the approximation (5.8) reduces to (5.6) in the weak-binding limit, and to the WKB solution when A is large [but $|\varsigma|$ restricted by (5.1)].

We next consider the exterior region. In the limit (5.4), eq. (2.2) can then be approximated as

$$\frac{dg}{d\eta}=\left(A-B-\frac{\varsigma}{\eta}-\frac{1}{\eta^2}\right)f,$$

$$\frac{df}{d\eta}=2Bg. \tag{5.15}$$

In this exterior region we write the solution in the covering approximation as

$$F(\rho)=\frac{F_0(\rho)}{f_0(\eta)}f(\eta),$$

$$G(\rho)=\frac{G_0(\rho)}{g_0(\eta)}g(\eta), \tag{5.16}$$

with $F_0(\rho)$ and $G_0(\rho)$ given by (5.9), and

$$f_0(\eta) = \left[\frac{2B\eta^2}{1+\varsigma\eta-(A-B)\eta^2}\right]^{1/4},$$

$$g_0(\eta) = if_0(\eta)^{-1}. \tag{5.17}$$

The relation between ρ and η is now determined by

$$I_1(\rho) = I_2(\eta), \tag{5.18}$$

where

$$I_1(\rho) = \int_\rho^{A^{1/2}a} d\rho' \, [(-A+B+\varsigma\rho'^{-1}+\rho'^{-2})(A+B+\varsigma\rho'^{-1}-\rho'^{-2})]^{1/2},$$

$$I_2(\eta) = \int_\eta^{(A-B)^{-1/2}} d\eta' \, [(-A+B+\varsigma\eta'^{-1}+\eta'^{-2})(2B)]^{1/2}. \tag{5.19}$$

Finally, the solution to (5.15) is given by [compare paper II, sect. 3]

$$f(\eta) = W_{\lambda,ip}(z),$$

$$g(\eta) = \left(\frac{A-B}{2B}\right)^{1/2} \frac{d}{dz} W_{\lambda,ip}(z), \tag{5.20}$$

with $W_{\lambda,ip}(z)$ a Whittaker function,

$$z = 2[2B(A-B)]^{1/2}\eta,$$

$$\lambda = \frac{1}{2}\left(\frac{2B}{A-B}\right)^{1/2}\varsigma, \tag{5.21}$$

and p given by (5.13).

6. Whittaker Function

In paper III (sect. 6) we obtained an approximation for $K_{ip}(z)$ that holds when

$$z \ll 1, \tag{6.1}$$

and when

$$p > z \gg 1. \tag{6.2}$$

That formula will here be used for the interior region. For the exterior region we need a corresponding formula for the Whittaker function. Such a formula will be obtained in this section.

When (6.1) holds, then [9]

$$W_{\lambda,ip}(z) \simeq z^{1/2}\left[\frac{\Gamma(-2ip)z^{ip}}{\Gamma(\frac{1}{2}-\lambda-ip)} + \text{c.c.}\right], \tag{6.3}$$

whereas when (6.2) holds, the WKB approximation to the Whittaker equation [compare (5.15)] gives

$$W_{\lambda,ip}(z) \simeq \text{const.}\left(-\frac{1}{4}+\frac{\lambda}{z}+\frac{p'^2}{z^2}\right)^{-1/4} \cdot \sin\left[-\int\left(-\frac{1}{4}+\frac{\lambda}{z}+\frac{p'^2}{z^2}\right)^{1/2} dz\right], \tag{6.4}$$

with

$$p' = (2B)^{1/2} = \left(p^2+\frac{1}{4}\right)^{1/2}. \tag{6.5}$$

Evaluating the integral we find [10]

$$W_{\lambda,ip}(z) \simeq \text{const.}\left(-\frac{1}{4}+\frac{\lambda}{z}+\frac{p'^2}{z^2}\right)^{-1/4}$$

$$\cdot \sin\left[p' \ln \frac{2p'^2+\lambda z+2p'(p'^2+\lambda z-\frac{1}{4}z^2)^{1/2}}{z(p'^2+\lambda^2)^{1/2}}\right.$$

$$\left. -(p'^2+\lambda z-\tfrac{1}{4}z^2)^{1/2}+\lambda \sin^{-1}\frac{-\frac{1}{2}z+\lambda}{(p'^2+\lambda^2)^{1/2}}+\lambda\frac{\pi}{2}+\frac{1}{4}\pi\right], \quad (6.6)$$

where the constant of integration has been determined by demanding that in the Airy approximation $W_{\lambda,ip}$ has the correct behaviour outside the turning point. We rewrite the logarithm in terms of an arccosh, and define

$$I_W(p',z) = p' \cosh^{-1}\frac{2p'^2+\lambda z}{(p'^2+\lambda^2)^{1/2}z} - (p'^2+\lambda z-\tfrac{1}{4}z^2)^{1/2}$$

$$+\lambda \sin^{-1}\frac{-\frac{1}{2}z+\lambda}{(p'^2+\lambda^2)^{1/2}}+\lambda\frac{\pi}{2}. \quad (6.7)$$

Expression (6.6) can then be written as

$$W_{\lambda,ip}(z) \simeq \text{const.}\left(-\frac{1}{4}+\frac{\lambda}{z}+\frac{p'^2}{z^2}\right)^{-1/4} \sin[I_W(p',z)+\tfrac{1}{4}\pi]. \quad (6.8)$$

We note the similarity between the amplitude $(-\frac{1}{4}+\lambda/z+p'^2/z^2)^{-1/4}$ and $f_0(\eta)$ of eq. (5.17). The expression (6.8) for $W_{\lambda,ip}$ corresponds to the expression (6.6) of paper III for K_{ip}.

In the covering approximation, a formula is needed for the Whittaker function $W_{\lambda,ip}(z)$ that reduces to (6.3) when (6.1) is satisfied, and to (6.8) when (6.2) is satisfied. This is accomplished by the Ansatz

$$W_{\lambda,ip}(z) \simeq A_W\left(-\frac{1}{4}+\frac{\lambda}{z}+\frac{p'^2}{z^2}\right)^{-1/4} \sin[(p/p')I_W(p',z)+\Phi_W]. \quad (6.9)$$

We determine the amplitude A_W and the phase Φ_W by expanding (6.9) for $z \ll p$, and comparing with (6.3). Thus, we find

$$A_W = \frac{p'^{1/2}}{p}\left|\frac{\Gamma(1+2ip)}{\Gamma(\frac{1}{2}-\lambda+ip)}\right|, \quad (6.10)$$

$$\Phi_W = \arg\Gamma(1+2ip) - \arg\Gamma(\tfrac{1}{2}-\lambda+ip)$$

$$- p\ln\frac{4p'^2}{(p'^2+\lambda^2)^{1/2}} + p - \frac{p}{p'}\lambda\left[\tan^{-1}\frac{\lambda}{p'}+\frac{\pi}{2}\right]. \quad (6.11)$$

If we use the Legendre duplication formula, and let $\lambda \to 0$, we find that

$$\Phi_W \underset{\lambda\to 0}{\to} \Phi_K \qquad \text{of paper III}, \quad (6.12)$$

as one should expect. Moreover,

$$A_W \underset{\lambda\to 0}{\to} \left(\frac{p'}{p}\right)^{1/2} [\sinh(\pi p)]^{-1/2}, \quad (6.13)$$

and

$$I_W(p',z) \underset{\lambda\to 0}{\to} I_K(p',z/2) \qquad \text{of paper III}. \quad (6.14)$$

Thus, the relation [9]

$$W_{0,ip}(z) = \left(\frac{z}{\pi}\right)^{1/2} K_{ip}(z/2), \quad (6.15)$$

is satisfied by the appropriate limit of (6.9) of the present paper and (6.7) of paper III.

From (6.7)–(6.11), the desired approximation to the Whittaker function is

$$W_{\lambda,ip}(z) \simeq \frac{p'^{1/2}}{p}\left|\frac{\Gamma(1+2ip)}{\Gamma(\frac{1}{2}-\lambda+ip)}\right|\left(-\frac{1}{4}+\frac{\lambda}{z}+\frac{p'^2}{z^2}\right)^{-1/4}$$

$$\cdot \sin\Bigg[p\cosh^{-1}\frac{2p'^2+\lambda z}{(p'^2+\lambda^2)^{1/2}z}-\frac{p}{p'}(p'^2+\lambda z-\tfrac{1}{4}z^2)^{1/2}$$

$$+\lambda\frac{p}{p'}\sin^{-1}\frac{-\frac{1}{2}z+\lambda}{(p'^2+\lambda^2)^{1/2}}+\arg\Gamma(1+2ip)$$

$$-\arg\Gamma(\tfrac{1}{2}-\lambda+ip)-p\ln\frac{4p'^2}{(p'^2+\lambda^2)^{1/2}}$$

$$+p-\lambda\frac{p}{p'}\sin^{-1}\frac{\lambda}{(p'^2+\lambda^2)^{1/2}}\Bigg], \qquad (6.16)$$

which is valid when either (6.1) or (6.2) is satisfied.

7. Energy Levels

We determine the energy eigenvalues by matching the interior solution (5.8) with the exterior solution (5.16). Let us match $F(\rho)$,

$$C_1\frac{\tilde{f}(\tilde{\eta})}{\tilde{f}_0(\tilde{\eta})}=C_2\frac{f(\eta)}{f_0(\eta)}, \qquad (7.1)$$

where C_1 and C_2 are constants. The left-hand side is given by (5.9) and (5.12),

$$C_1\left\{\frac{A+B}{2B}[(A+B)\tilde{\eta}^2-1]\right\}^{-1/4}\left(\frac{A+B}{2B}\right)^{1/2}\frac{d}{d\tilde{z}}[\tilde{z}^{1/2}K_{ip}(\tilde{z})], \qquad (7.2)$$

with $K_{ip}(\tilde{z})$ given in the covering approximation by (III.6.15). In particular, the above derivative is given by (III.7.5)–(III.7.8) as

$$\tilde{z}^{-1/2}(p'^2-\tilde{z}^2)^{1/4}\sin[(p/p')I_K(p',\tilde{z})+\Phi_K-\psi_B]. \qquad (7.3)$$

We now invoke (5.13), and write the left-hand side of (7.1) as

$$C_1 A_K\left(\frac{A+B}{2B}\right)^{1/4}\sin[(p/p')I_K(p',\tilde{z})+\Phi_K-\psi_B]. \qquad (7.4)$$

Substituting for $\tilde{z}$ in terms of $\tilde{\eta}$ [eq. (5.13)], and comparing (III.6.8) with (B.3) of appendix B, we obtain

$$I_K(p',\tilde{z})=\tilde{I}_2(\tilde{\eta}). \qquad (7.5)$$

Using further (5.10) and (B.1), we can write (7.4) as

$$C_1 A_K\left(\frac{A+B}{2B}\right)^{1/4}\sin\{(p/p')\sqrt{A}[\phi(\sqrt{A}\rho)-\phi(b)]+\Phi_K-\psi_B\}. \qquad (7.6)$$

The right-hand side of (7.1) is given by (5.17), (5.20), and (6.9),

$$C_2 A_W\left[\frac{2B\eta^2}{1+\varsigma\eta-(A-B)\eta^2}\right]^{-1/4}\left(-\frac{1}{4}+\frac{\lambda}{z}+\frac{p'^2}{z^2}\right)^{-1/4}$$

$$\cdot\sin[(p/p')I_W(p',z)+\Phi_W]. \qquad (7.8)$$

Because of (5.21), we can rewrite this as

$$\sqrt{2}C_2 A_W\left(\frac{A-B}{2B}\right)^{1/4}\sin[(p/p')I_W(p',z)+\Phi_W], \qquad (7.9)$$

i.e., the amplitude factor is again independent of the radial variable. *The matching will thus be a matching of phase.*

We next relate $I_W(p',z)$ to $I_2(\eta)$, using (6.7) and (B.5),

$$I_W(p',z)=I_2(\eta), \qquad (7.10)$$

with [cf. (5.18) and (B.4)]

$$I_2(\eta)=I_1(\rho)=\sqrt{A}[\phi(a)-\phi(\sqrt{A}\rho)]. \qquad (7.11)$$

Thus, the right-hand side of (7.1) can be written

$$-\sqrt{2}C_2 A_W\left(\frac{A-B}{2B}\right)^{1/4}\sin\{\frac{p}{p'}\sqrt{A}[-\phi(a)+\phi(\sqrt{A}\rho)]-\Phi_W\}. \qquad (7.12)$$

The ρ-dependence is here the same as for the interior region [expression (7.6)], so the two regions can be matched provided the phases are the same,

$$-\frac{p}{p'}\sqrt{A}\,\phi(b)+\Phi_K-\psi_B=-\frac{p}{p'}\sqrt{A}\,\phi(a)-\Phi_W+n\pi, \qquad (7.13)$$

or, with $\Phi_{WKB}=\phi(a)-\phi(b)$,

$$\frac{p}{p'}\sqrt{A}\,\Phi_{WKB}+\Phi_W+\Phi_K-\psi_B=n\pi. \qquad (7.14)$$

This is the equation for the energy eigenvalue, as determined in the covering approximation.

8. Numerical Results

Numerical results for the binding energy $\epsilon=(A-B)/A=(M-E)/M$ are given in table 1 for $\varsigma=-\alpha$ and $+\alpha$, in table 2 for $\varsigma=0.1$ and 0.5, and in table 3 for $\varsigma=1.0$ and 5.0. For a set of A-values ranging from 0.5 to 100, we compare the accurate results of paper I [1] and those of the weak-binding approximation of paper II [3] with those of the WKB method (part A of this paper) and with those of the covering approximation (part B of this paper). Five levels are considered, $n=1$ to 5. [We do not include the $n=0$ state for which the binding is very strong.]

For large values of A, where the WKB method applies, it gives excellent results. The results of the covering approximation are excellent for practically all A and ς. When the binding is weak, they are comparable with, or even better than the results of the weak-binding approximation. When A is large,

they are comparable with the results of the WKB method. Even when (5.1) is violated, the covering approximation in many cases remains excellent. The only case where we have found it to break down is the almost trivial one, where p becomes imaginary [cf. (5.13)], i.e., for

$$B=A(1-\epsilon)\lesssim\frac{1}{8}. \qquad (8.1)$$

Likewise, the WKB approximation has some validity beyond the range of parameters for which it was derived. With $A=O(1)$, it is still good, provided the binding is strong (compare table 3).

We close with some remarks on the spectrum. For a fixed value of ς, and for A (or κ) sufficiently large, the binding energy increases with increasing A (or κ). However, as is seen from table 3, for sufficiently small values of A (or κ), the binding energy increases with *decreasing* A (or κ). This is further illustrated in fig. 1, where we have plotted E/M vs. $\log A$, for $\varsigma=1.0$, $n=0,1,2,3$, and A ranging from 10^{-3} to 100.

The maxima observed in fig. 1 can be roughly understood as follows. At large r the magnetic-moment interaction is just like the angular-momentum interaction for hydrogen-like atoms, with $-\kappa|q|(M+E)/2M$ corresponding to $l(l+1)$. Thus, at large r, a positive κ amounts to an attraction. On the other hand, at short distances the wave functions behave like $\exp[-(1/r)(|\kappa q|/2M)]$; in this sense the magnetic-moment interaction is repulsive at short distances. If now the wave function is concentrated at large r, where the magnetic-moment interaction is attractive, increasing κ will increase the binding. If, however, the wave function is concentrated at small r, where the magnetic-moment interaction is effectively repulsive, decreasing κ will increase the binding. It follows that there is some intermediate κ (which depends on ς and n) for which the energy has an extremum with respect to variations in κ.

For $\kappa<0$, these extrema appear to be absent. As follows from the argument of the last paragraph, the magnetic-moment interaction is then repulsive for small and large values of r. In fig. 2 we have plotted E/M vs. $\log(-A)$ for $\varsigma = 1.0$, $n = 1,2,3$, and A ranging from -10^{-3} to -100. It is seen to change monotonically with A.

9. Maximal Binding

As is clear from figs. 1 and 2, there are sets of parameters A and ς for which the binding becomes maximal,

$$E_B \equiv M - E = 2M, \quad \text{or} \quad B = -A. \tag{9.1}$$

For a given ς, we shall refer to a value of A for which this occurs as $A_{\rm crit}$. Beyond this point, which is similar to the case of $Z = 137$ for hydrogen-like atoms [11], the one-particle description presumably makes no sense. This critical value can be determined as the eigenvalue A of eq. (2.2) for $B = -A$,

$$\begin{aligned} \frac{dG}{d\rho} &= \left(2A - \frac{\tilde{\varsigma}}{\rho} - \frac{1}{\rho^2}\right)F, \\ \frac{dF}{d\rho} &= \left(\frac{\tilde{\varsigma}}{\rho} - \frac{1}{\rho^2}\right)G, \end{aligned} \tag{9.2}$$

by the method of paper I. Solutions are shown in fig. 3 for values of ς up to 10.

As $\varsigma \to 0$, the eigenvalue of (9.2), $A_{\rm crit}$, can be determined analytically by the method of paper II (a derivation of which is given in appendix C). For $A > 0$, we find

$$A_{\rm crit} \simeq \frac{1}{4\varsigma}\exp\left[-\frac{(n+\frac{1}{4})\pi}{\varsigma} - 3\gamma\right], \qquad \varsigma \ll 1, \tag{9.3}$$

for $n = 0,1,2,\cdots$, and with $\gamma = 0.577\cdots$ Euler's constant. Similarly, when $A < 0$, we have

$$A_{\rm crit} = -\frac{1}{4\varsigma}\exp\left[-\frac{(n-\frac{1}{4})\pi}{\varsigma} - 3\gamma\right], \qquad \varsigma \ll 1, \tag{9.4}$$

for $n = 1,2,3,\cdots$. Comparing now with (9.3), and considering the ground state, we can give the allowed regions of A for small ς as

$$A < -\frac{1}{4\varsigma}\exp\left[-\frac{3\pi}{4\varsigma} - 3\gamma\right], \tag{9.5a}$$

or

$$A > \frac{1}{4\varsigma}\exp\left[-\frac{\pi}{4\varsigma} - 3\gamma\right]. \tag{9.5b}$$

When $\varsigma \gg 1$ and $A > 0$, $A_{\rm crit}$ can be determined from the following equation, derived in appendix D,

$$A_{\rm crit} = \varsigma^2 \frac{(n+\bar{\varsigma})[n+\bar{\varsigma}-(n^2+2n\bar{\varsigma})^{1/2}]}{[n+\bar{\varsigma}+(n^2+2n\bar{\varsigma})^{1/2}]^2}, \qquad \varsigma \gg 1, \tag{9.6}$$

with

$$\bar{\varsigma} = \varsigma(1 + 8A_{\rm crit}/\varsigma^2)^{1/4}. \tag{9.7}$$

For $n = 0$, this is explicit, $A_{\rm crit} = \varsigma^2$. At $\varsigma = 10$, (9.6) is good to 3% and 9% for $n = 0$ and 1, respectively. We note that

$$A_{\rm crit} \underset{\varsigma\to\infty}{\to} \varsigma^2, \qquad A > 0. \tag{9.8}$$

When $\varsigma \gg 1$ and $A < 0$, the critical value of A is determined in appendix D as

$$A_{\rm crit} = -\frac{3}{8}\varsigma^4[-1 + 2n + (1+3\varsigma^2)^{1/2}]^{-2}, \qquad \varsigma \gg 1. \tag{9.9}$$

In contrast to (9.8), we note that

$$A_{\rm crit} \underset{\varsigma\to\infty}{\to} -\frac{1}{8}\varsigma^2, \qquad A < 0. \tag{9.10}$$

In summary, we have given analytic results for dyon-fermion binding energies and wave functions approximately for two cases. The WKB approximation in Part A applies when $A = \frac{1}{2}Z|eg|\kappa \gg 1$, and the covering approximation of Part B applies more generally when *either* A is large *or* the binding is weak. The formulas for the binding energies are given in the WKB approximation by (4.8) and (4.23), and in the covering approximation by (7.14). All the results apply only to the case of lowest angular momentum $j = Z|eg| - \frac{1}{2}$. Generalization to higher angular momentum states will be considered in paper VI.

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Appendix A. Symmetric Forms of the Elliptic Integrals

In evaluating the phase integral $I(y, \hat{\varsigma})$ [eq. (4.23)], we assumed $\hat{\varsigma} < y$ in order to secure $c > d$. We shall here show that the result is invariant under the interchange $c \leftrightarrow d$. Let us recall

$$k = \left[\frac{(a-b)(c-d)}{(a-c)(b-d)}\right]^{1/2}, \qquad k' = \left[\frac{(a-d)(b-c)}{(a-c)(b-d)}\right]^{1/2}, \tag{A.1}$$

and

$$\sin\theta = \left(\frac{b-d}{a-d}\right)^{1/2}. \tag{A.2}$$

The complete elliptic integrals $K(k)$ and $E(k)$ can be transformed into symmetric forms by a Gauss transformation [7]. Let

$$\begin{aligned} k_1 &= \frac{1-k'}{1+k'} \\ &= \frac{[(a-c)(b-d)]^{1/2} - [(a-d)(b-c)]^{1/2}}{[(a-c)(b-d)]^{1/2} + [(a-d)(b-c)]^{1/2}}, \end{aligned} \tag{A.3}$$

which is odd under $c \leftrightarrow d$. Then [7]

$$K(k_1) = \frac{1+k'}{2} K(k), \tag{A.4}$$

$$E(k_1) = \frac{1}{1+k'}[E(k) + k'K(k)]. \tag{A.5}$$

These are invariant under $c \leftrightarrow d$ since they only depend on the square of the modulus, k_1^2. Solving for $K(k)$ and $E(k)$, we find

$$K(k) = \frac{2}{1+k'}K(k_1)$$

$$= 2[(a-c)(b-d)]^{1/2}\{[(a-c)(b-d)]^{1/2}$$

$$+ [(a-d)(b-c)]^{1/2}\}^{-1}K(k_1), \tag{A.6}$$

$$E(k) = (1+k')E(k_1) - \frac{2k'}{1+k'}K(k_1)$$

$$= [(a-c)(b-d)]^{-1/2}\{[(a-c)(b-d)]^{1/2}$$

$$+ [(a-d)(b-c)]^{1/2}\}E(k_1) - 2[(a-d)(b-c)]^{1/2}$$

$$\cdot \{[(a-c)(b-d)]^{1/2} + [(a-d)(b-c)]^{1/2}\}^{-1}K(k_1). \tag{A.7}$$

In order to write the incomplete elliptic integrals $F(\theta, k')$ and $E(\theta, k')$ in terms of symmetric ones, we shall perform an imaginary argument transformation followed by a Landen transformation [7]. By the imaginary argument transformation [7],

$$F(\psi, k) = iF(\theta, k'), \tag{A.8}$$

$$E(\psi, k) = i\{F(\theta, k') - E(\theta, k') + \tan\theta[1 - k'^2 \sin^2\theta]^{1/2}\}, \tag{A.9}$$

with

$$\sin\theta = -i\tan\psi = \left(\frac{b-d}{a-d}\right)^{1/2}. \tag{A.10}$$

We next apply a Landen transformation [7], with k_1 given by eq. (A.3) and

$$\sin\phi = \frac{(1+k')\sin\psi\cos\psi}{[1-k^2\sin^2\psi]^{1/2}}$$

$$= \frac{i}{a-b}\{[(a-c)(b-d)]^{1/2} + [(a-d)(b-c)]^{1/2}\}, \tag{A.11}$$

i.e., the new argument ϕ is invariant under $c \leftrightarrow d$. Then $F(\psi, k)$ and $E(\psi, k)$ transform as

$$F(\phi, k_1) = (1+k')F(\psi, k), \tag{A.12}$$

$$E(\phi, k_1) = \frac{2}{1+k'}[E(\psi, k) + k'F(\psi, k)] - \frac{1-k'}{1+k'}\sin\phi, \tag{A.13}$$

and combining this with (A.8) and (A.9), we find

$$F(\theta, k') = -\frac{i}{1+k'}F(\phi, k_1), \tag{A.14}$$

$$E(\theta, k') = \left(\frac{b-d}{a-c}\right)^{1/2} + i\{-F(\phi, k_1) + \frac{1}{2}(1+k')E(\phi, k_1)$$

$$+ \frac{1}{2}(1-k')\sin\phi\}. \tag{A.15}$$

Collecting then everything, i.e., using (A.4), (A.5), (A.14) and (A.15), we find that the WKB phase of eq. (4.23) can be written as

$$I(y, \zeta) = (1-y^2)^{1/2}$$

$$\cdot\Bigg(\frac{[-3(a+b)+c+d](c+d) + 4[(a-c)(b-d)(a-d)(b-c)]^{1/2}}{[(a-c)(b-d)]^{1/2} + [(a-d)(b-c)]^{1/2}}K(k_1)$$

$$- 2\{[(a-c)(b-d)]^{1/2} + [(a-d)(b-c)]^{1/2}\}E(k_1) + \frac{1}{2}\pi(a+b+c+d)$$

$$+ i(a+b+c+d)[E(k_1)F(\phi, k_1) - K(k_1)E(\phi, k_1)]\Bigg), \tag{A.16}$$

which is seen to be symmetric under the interchange $c \leftrightarrow d$.

Appendix B. Phase Integrals

In this appendix we present a brief discussion of the phase integrals that appear in sect. 5.

First, we note that $\tilde{I}_1(\rho)$ of (5.11) can be written in terms of $\phi(\tau)$ of (3.9),

$$\tilde{I}_1(\rho) = \sqrt{A}\,[\phi(\sqrt{A}\,\rho) - \phi(b)], \tag{B.1}$$

where ϕ can be expressed by elliptic integrals.

Second, $\tilde{I}_2(\tilde{\eta})$ is related to the I_2 defined in paper III. With $x = 1/\tilde{\eta}'$, we have

$$\tilde{I}_2(\tilde{\eta}) = (2B)^{1/2} \int_{\tilde{\eta}^{-1}}^{(A+B)^{1/2}} dx\,[x^{-2} - (A+B)^{-1}]^{1/2}. \tag{B.2}$$

This integral is related to that of (III.5.7) by the substitutions $\eta \to \tilde{\eta}^{-1}$, $A - B \to (A+B)^{-1}$,

$$\tilde{I}_2(\tilde{\eta}) = (2B)^{1/2}\left\{\cosh^{-1}(\sqrt{A+B}\,\tilde{\eta}) - \left[1 - \frac{1}{(A+B)\tilde{\eta}^2}\right]^{1/2}\right\}. \tag{B.3}$$

For the phase integrals of the exterior region we proceed in a similar way. First, $I_1(\rho)$ can be expressed in terms of ϕ [cf. (3.9) and (5.19)],

$$I_1(\rho) = \sqrt{A}\,[\phi(a) - \phi(\sqrt{A}\,\rho)]; \tag{B.4}$$

and

$$\begin{aligned} I_2(\eta) &= (2B)^{1/2} \int_{\eta}^{\eta_t} \frac{d\eta'}{\eta'}\,[1 + \varsigma\eta' - (A-B)\eta'^2]^{1/2} \\ &= (2B)^{1/2}\left\{\cosh^{-1}\frac{2+\varsigma\eta}{[\varsigma^2 + 4(A-B)]^{1/2}\eta} - [1 + \varsigma\eta - (A-B)\eta^2]^{1/2} \right. \\ &\qquad \left. + \frac{1}{2}\frac{\varsigma}{(A-B)^{1/2}}\cos^{-1}\frac{2(A-B)\eta - \varsigma}{[\varsigma^2 + 4(A-B)]^{1/2}}\right\}. \end{aligned} \tag{B.5}$$

where

$$\eta_t = 2\{-\varsigma + [\varsigma^2 + 4(A-B)]^{1/2}\}^{-1}. \tag{B.6}$$

Appendix C. Critical Value of A for $\varsigma \ll 1$

When $\varsigma \ll 1$, we can find approximate, analytic expressions for the value $A = A_{\rm crit}$ that corresponds to maximal binding, $E_B = 2M$ or $B = -A$. This is achieved by solving (9.2) using the method of paper II. A can be either positive or negative.

Let us assume

$$|A| \ll \varsigma^2 \tag{C.1}$$

(this will subsequently be justified) and take

$$x = \frac{1}{\rho}. \tag{C.2}$$

Then eq. (9.2) can be written

$$\begin{aligned} \frac{dG}{dx} &= \left(1 + \frac{\varsigma}{x} - \frac{2A}{x^2}\right)F, \\ \frac{dF}{dx} &= \left(1 - \frac{\varsigma}{x}\right)G. \end{aligned} \tag{C.3}$$

Consider now

$$\text{Region I:} \qquad x \gg \frac{2|A|}{\varsigma} \tag{C.4}$$

(i.e., "small" ρ). Equation (C.3) is then approximated by

$$\begin{aligned} \frac{dG}{dx} &= \left(1 + \frac{\varsigma}{x}\right)F, \\ \frac{dF}{dx} &= \left(1 - \frac{\varsigma}{x}\right)G. \end{aligned} \tag{C.5}$$

This equation can be solved exactly. Let

$$\begin{aligned} S &= F + G, \\ T &= F - G, \end{aligned} \tag{C.6}$$

the equations for which become

$$\frac{d^2S}{dx^2}+\frac{1}{x}\frac{dS}{dx}-\left(1+\frac{1}{x}-\frac{\varsigma^2}{x^2}\right)S=0, \tag{C.7}$$

$$T=\frac{x}{\tilde{\varsigma}}\left(\frac{dS}{dx}-S\right). \tag{C.8}$$

The equation for S can be solved in terms of a Whittaker function (subject to the boundary condition $S\to 0$, $T\to 0$ as $x\to\infty$),

$$S(x)=N_1x^{-1/2}W_{-1/2,i\varsigma}(2x), \tag{C.9}$$

with N_1 a constant. Using (C.8), (C.2) and expanding for

$$\rho\gg 1, \tag{C.10}$$

we find

$$F(\rho)\simeq\frac{N_1}{\sqrt{2}}\left[\frac{\Gamma(-2i\varsigma)}{\Gamma(1-i\varsigma)}\left(1+\frac{iA}{|A|}\right)\left(\frac{2}{\rho}\right)^{i\varsigma}+\text{c.c.}\right]. \tag{C.11}$$

Consider next

$$\text{Region II:}\qquad \rho\gg\varsigma^{-1}, \tag{C.12}$$

where (9.2) simplifies to

$$\frac{dG}{d\rho}=\left(2A-\frac{\tilde{\varsigma}}{\rho}\right)F,$$
$$\frac{dF}{d\rho}=\frac{\tilde{\varsigma}}{\rho}G. \tag{C.13}$$

With

$$\rho=\frac{z^2}{8A\tilde{\varsigma}}=\frac{z^2}{8|A|\varsigma}, \tag{C.14}$$

we find that F must satisfy

$$\frac{d^2F}{dz^2}+\frac{1}{z}\frac{dF}{dz}-\left(1-\frac{4\varsigma^2}{z^2}\right)F=0. \tag{C.15}$$

The solution that satisfies the boundary condition $F\to 0$ as $z\to\infty$ is a modified Bessel function,

$$F(\rho)=N_2K_{2i\varsigma}\left(\sqrt{8|A|\varsigma\rho}\right). \tag{C.16}$$

For

$$\rho\ll\frac{1}{4|A|}, \tag{C.17}$$

we can expand

$$F(\rho)\simeq N_2'\left[\frac{(2|A|\varsigma\rho)^{-i\varsigma}}{\Gamma(1-2i\varsigma)}-\text{c.c.}\right] \tag{C.18}$$

with N_2' another constant.

It follows from (C.2), (C.4) and (C.12) that the two regions overlap when (C.1) is satisfied. In this region of overlap, the power expansions (C.11) and (C.18) also overlap because $\varsigma\ll 1$.

The matching of (C.11) and (C.18) is then straightforward. For $A>0$ the condition is

$$(4A\varsigma)^{2i\varsigma}e^{i\pi/2}\left[\frac{\Gamma(1-2i\varsigma)}{\Gamma(1+2i\varsigma)}\right]^2\frac{\Gamma(1+i\varsigma)}{\Gamma(1-i\varsigma)}=e^{2ki\pi}, \tag{C.19}$$

with k an integer. For $\varsigma\ll 1$ the above Γ-functions can be expanded as [12]

$$\Gamma(1+li\varsigma)\simeq 1-li\varsigma\gamma\simeq e^{-li\varsigma\gamma}, \tag{C.20}$$

where $\gamma=0.577\cdots$ is Euler's constant. Equation (C.19) can then be explicitly solved for $A=A_{\text{crit}}$,

$$A_{\text{crit}}\simeq\frac{1}{4\varsigma}\exp\left[-\frac{(n+\frac{1}{4})\pi}{\varsigma}-3\gamma\right], \tag{C.21}$$

where k has been identified as $-n$.

For $A < 0$, the matching condition similarly gives

$$(4|A|\varsigma)^{2i\varsigma} e^{-i\pi/2} \left[\frac{\Gamma(1-2i\varsigma)}{\Gamma(1+2i\varsigma)}\right]^2 \frac{\Gamma(1+i\varsigma)}{\Gamma(1-i\varsigma)} = e^{2mi\pi}, \qquad \text{(C.22)}$$

with m an integer. With the approximation (C.20) we now get

$$A_{\rm crit} = -\frac{1}{4\varsigma} \exp\left[-\frac{(n-\frac{1}{4})\pi}{\varsigma} - 3\gamma\right], \qquad \text{(C.23)}$$

where m has been identified as $-n$. We note that (C.21) and (C.23) satisfy (C.1) for $\varsigma \ll 1$.

Appendix D. Critical Value of A for $\varsigma \gg 1$

The asymptotic behaviour of $A_{\rm crit}$ for $\varsigma \gg 1$ can be determined analytically. With

$$x = \varsigma\rho, \qquad \text{(D.1)}$$

eq. (9.2) takes the form

$$\frac{dG}{dx} = \varsigma\left(\frac{2A}{\varsigma^2} - \frac{A}{|A|}\frac{1}{x} - \frac{1}{x^2}\right)F,$$
$$\frac{dF}{dx} = \varsigma\left(\frac{A}{|A|}\frac{1}{x} - \frac{1}{x^2}\right)G. \qquad \text{(D.2)}$$

We rewrite this as

$$\frac{dG}{dx} = -\varsigma\left(\frac{1}{x} - y\right)\left(\frac{1}{x} - y'\right)F, \qquad \text{(D.3a)}$$

$$\frac{dF}{dx} = -\varsigma\left(\frac{1}{x} - \frac{A}{|A|}\right)\frac{1}{x}G, \qquad \text{(D.3b)}$$

with

$$y = \frac{1}{2}\left[-\frac{A}{|A|} + (1+8\overline{A})^{1/2}\right],$$
$$y' = \frac{1}{2}\left[-\frac{A}{|A|} - (1+8\overline{A})^{1/2}\right], \qquad \text{(D.4)}$$

$$\overline{A} = \frac{A}{\varsigma^2}. \qquad \text{(D.5)}$$

Consider first $A > 0$. Then the right-hand sides of eqs. (D.3a) and (D.3b) vanish for $x = y^{-1}$ and $x = 1$, respectively. Linearizing in x^{-1}, we get

$$\frac{dG}{dx} = -\varsigma\left(\frac{1}{x} - y\right)(y - y')F,$$
$$\frac{dF}{dx} = -\varsigma\left(\frac{1}{x} - 1\right)G. \qquad \text{(D.6)}$$

We shall solve this equation exactly. The boundary conditions that F and G vanish at the origin and at infinity yield an implicit equation relating $A_{\rm crit}$ to ς.

The first step is to symmetrize eq. (D.6) by the rescaling

$$x_1 = \frac{1}{2}(1+y)x, \qquad F = (1+8\overline{A})^{-1/8}\overline{F}, \qquad G = (1+8\overline{A})^{1/8}\overline{G}, \tag{D.7}$$

which yields

$$\frac{d\overline{G}}{dx_1} = -\overline{\varsigma}\left(\frac{1}{x_1} - 1 + \epsilon\right)\overline{F}, \qquad \frac{d\overline{F}}{dx_1} = -\overline{\varsigma}\left(\frac{1}{x_1} - 1 - \epsilon\right)\overline{G}, \tag{D.8}$$

with

$$\overline{\varsigma} = \varsigma(1+8\overline{A})^{1/4}, \qquad \epsilon = (1-y)/(1+y). \tag{D.9}$$

In terms of the sum and difference,

$$\overline{S} = \overline{F} + \overline{G}, \qquad \overline{T} = \overline{F} - \overline{G}, \tag{D.10}$$

eq. (D.8) takes the form

$$\frac{d\overline{S}}{dx_1} = -\overline{\varsigma}\left[\left(\frac{1}{x_1} - 1\right)\overline{S} + \epsilon\overline{T}\right], \qquad \frac{d\overline{T}}{dx_1} = -\overline{\varsigma}\left[-\left(\frac{1}{x_1} - 1\right)\overline{T} - \epsilon\overline{S}\right]. \tag{D.11}$$

With $A > 0$, F and G have the same number of nodes, so that, for large ς, $\overline{S}$ tends to be small compared with $\overline{T}$ [1]. Elimination of $\overline{S}$ gives a Whittaker equation for $\overline{T}$,

$$\overline{T} = W_{\kappa\mu}(z), \tag{D.12}$$

with

$$\kappa = \overline{\varsigma}(1-\epsilon^2)^{-1/2}, \qquad \mu = \overline{\varsigma} - \frac{1}{2}, \tag{D.13}$$

$$z = 2\overline{\varsigma}(1-\epsilon^2)^{1/2}x_1. \tag{D.14}$$

This solution satisfies the boundary condition of $\overline{T} \to 0$ at the origin provided

$$\frac{1}{2} - \kappa + \mu \qquad \text{is a non-positive integer,} \tag{D.15}$$

or

$$\overline{\varsigma}[1 - (1-\epsilon^2)^{-1/2}] = -n, \tag{D.16}$$

where the right-hand side has been identified in terms of the quantum number n. The asymptotic behaviour $A_{\rm crit} = A_{\rm crit}(\varsigma, n)$ is given by this condition. Using (D.9) and (D.4), we can write the above condition as

$$A = A_{\rm crit} = \varsigma^2 \frac{(n+\overline{\varsigma})[n+\overline{\varsigma} - (n^2+2n\overline{\varsigma})^{1/2}]}{[n+\overline{\varsigma}+(n^2+2n\overline{\varsigma})^{1/2}]^2}, \tag{D.17}$$

with [cf. eq. (D.9)]

$$\overline{\varsigma} = \varsigma(1+8A_{\rm crit}/\varsigma^2)^{1/4}. \tag{D.18}$$

For $A < 0$ the right-hand side of (D.3a) vanishes for $x = y^{-1}$ and for $x = y'^{-1}$, whereas the right-hand side of (D.3b) has no zero for $x^{-1} > 0$. We therefore replace x^{-1} in the second equation by $\frac{1}{2}(y+y')$,

$$\frac{dG}{dx} = \varsigma\left(-2\hat{A} + \frac{1}{x} - \frac{1}{x^2}\right)F, \qquad \frac{dF}{dx} = -\frac{3}{4}\varsigma G, \tag{D.19}$$

with

$$\tilde{A} = |A|/\varsigma^2. \tag{D.20}$$

We now get a Whittaker equation for F,

$$\frac{d^2F}{dx^2} + \frac{3}{4}\varsigma^2\left(-2\tilde{A} + \frac{1}{x} - \frac{1}{x^2}\right)F = 0, \tag{D.21}$$

with the solution

$$F = W_{\kappa'\mu'}(z), \tag{D.22}$$

$$\kappa' = \frac{1}{4}\left(\frac{3}{2|A|}\right)^{1/2}\varsigma^2, \qquad \mu' = \frac{1}{2}(1+3\varsigma^2)^{1/2},$$
$$z = 2\left(\frac{3}{2}|A|\right)^{1/2} x. \tag{D.23}$$

Requiring an acceptable behaviour at the origin, we are again led to condition (D.15), now in terms of κ' and μ',

$$\frac{1}{2} - \frac{1}{4}\left(\frac{3}{2|A|}\right)^{1/2}\varsigma^2 + \frac{1}{2}(1+3\varsigma^2)^{1/2} = -(n-1). \tag{D.24}$$

Since $n \geq 1$ for $A < 0$, we have identified the non-positive integer as $-(n-1)$. This equation can be solved explicitly for $A_{\rm crit} = -|A|$,

$$A_{\rm crit} = -\frac{3}{8}\varsigma^4[-1 + 2n + (1+3\varsigma^2)^{1/2}]^{-2}. \tag{D.25}$$

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Table Captions

Table 1. Binding energies $(E - M)/M$ vs. $A = \frac{1}{2}|q|\kappa$ for $\varsigma = -\alpha$ and α (fine-structure constant).

Exact: Numerical results of paper I [1].

WBA: Weak-binding approximation of paper II [3].

WKB: Part A of this paper.

CA: Covering approximation, Part B of this paper.

Where no value is quoted, the state does not exist.

Table 2. Binding energies $(E - M)/M$ vs. $A = \frac{1}{2}|q|\kappa$ for $\varsigma = 0.1$ and 0.5.

Exact: Numerical results of paper I [1].

WBA: Weak-binding approximation of paper II [3].

WKB: Part A of this paper.

CA: Covering approximation, Part B of this paper.

(For $A = 2.0$, $\varsigma = 0.5$, and $n = 3$, the two entries in the table in paper II are inadvertently interchanged.)

Table 3. Binding energies $(E - M)/M$ vs. $A = \frac{1}{2}|q|\kappa$ for $\varsigma = 1.0$ and 5.0.

Exact: Numerical results of paper I [1].

WBA: Weak-binding approximation of paper II [3].

WKB: Part A of this paper.

CA: Covering approximation, Part B of this paper.

When all four entries are missing, the state does not exist. When the entry is missing for WBA or CA, the approximation fails. For $\varsigma = 5$ and $A \leq 5$, the one-particle description used here presumably makes no sense [11].

Figure Captions

Fig. 1. E/M vs. $A = \frac{1}{2}\kappa|q| > 0$ for dyon-fermion states of minimal angular momentum, $j = |q| - \frac{1}{2}$ and $\varsigma = 1.0$. The four lowest levels are shown, $n = 0$, 1, 2, and 3.

Fig. 2. E/M vs. $A < 0$ for dyon-fermion states of minimal angular momentum and $\varsigma = 1.0$. The three lowest levels are shown, $n = 1$, 2, and 3. (The quantum number n counts the maximum number of nodes of F or G, which is at least 1 for $A < 0$.)

Fig. 3. Critical values $A_{\rm crit}$ for which the binding becomes maximal, $E_B = M - E = 2M$, as a function of ς. (a) $A_{\rm crit} > 0$. (b) $A_{\rm crit} < 0$.

Table 1. Binding energies $(E-M)/M$ vs. $A=\frac{1}{2}|q|\kappa$ for $\varsigma=-\alpha$ and α

		$\varsigma=-\alpha$					$\varsigma=\alpha$				
A	Method	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	Units	10^{-4}					10^{-3}	10^{-5}	10^{-6}	10^{-6}	10^{-6}
	Exact	1.239					0.933	3.040	7.311	3.161	1.751
	WBA	1.238					0.933	3.040	7.311	3.161	1.751
	WKB	5.863					1.655	4.289	8.766	3.563	1.913
	CA	1.240					0.934	3.040	7.311	3.161	1.751
1.0	Units	10^{-3}					10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-6}
	Exact	4.604					6.403	1.464	1.740	5.523	2.628
	WBA	4.583					6.376	1.464	1.740	5.523	2.628
	WKB	5.948					7.837	1.964	2.092	6.188	2.846
	CA	4.613					6.417	1.464	1.740	5.523	2.628
2.0	Units	10^{-2}	10^{-4}				10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
	Exact	2.236	6.809				2.487	1.190	0.931	1.659	5.616
	WBA	2.187	6.802				2.434	1.189	0.931	1.659	5.616
	WKB	2.398	8.329				2.651	1.368	1.091	1.868	6.092
	CA	2.244	6.811				2.497	1.190	0.931	1.659	5.616
5.0	Units	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-6}	10^{-2}	10^{-2}	10^{-3}	10^{-4}	10^{-5}
	Exact	7.794	1.008	1.230	1.216	2.394	8.071	1.114	1.621	2.646	5.419
	WBA	7.253	0.996	1.228	1.216	2.394	7.515	1.101	1.618	2.645	5.418
	WKB	7.907	1.046	1.314	1.361	4.054	8.184	1.152	1.711	2.835	5.820
	CA	7.825	1.009	1.230	1.216	2.394	8.103	1.115	1.621	2.646	5.419
10.0	Units	10^{-1}	10^{-2}	10^{-3}	10^{-3}	10^{-4}	10^{-1}	10^{-2}	10^{-3}	10^{-3}	10^{-4}
	Exact	1.439	3.412	8.146	1.902	4.224	1.465	3.545	8.812	2.232	5.858
	WBA	1.271	3.291	8.063	1.896	4.221	1.294	3.419	8.722	2.226	5.854
	WKB	1.446	3.451	8.305	1.957	4.398	1.472	3.583	8.974	2.291	6.052
	CA	1.444	3.416	8.149	1.902	4.224	1.470	3.549	8.816	2.233	5.858
20.0	Units	10^{-1}	10^{-2}	10^{-2}	10^{-2}	10^{-3}	10^{-1}	10^{-2}	10^{-2}	10^{-2}	10^{-3}
	Exact	2.238	7.948	2.905	1.063	3.866	2.260	8.087	2.991	1.116	4.191
	WBA	1.863	7.368	2.814	1.049	3.845	1.883	7.498	2.898	1.102	4.168
	WKB	2.242	7.976	2.922	1.072	3.912	2.264	8.115	3.009	1.126	4.238
	CA	2.244	7.956	2.906	1.063	3.866	2.265	8.095	2.993	1.116	4.191
50.0	Units	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}	10^{-1}	10^{-1}	10^{-1}	10^{-2}	10^{-2}
	Exact	3.378	1.696	8.864	4.692	2.492	3.394	1.708	8.956	4.760	2.543
	WBA	2.595	1.460	8.152	4.469	2.423	2.609	1.477	8.238	4.535	2.473
	WKB	3.379	1.697	8.876	4.701	2.499	3.395	1.709	8.967	4.769	2.549
	CA	3.382	1.697	8.868	4.693	2.493	3.398	1.709	8.960	4.761	2.543
100.0	Units	10^{-1}	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-1}	10^{-1}	10^{-1}	10^{-2}	10^{-2}
	Exact	4.224	2.519	1.568	9.915	6.316	4.236	2.529	1.576	9.983	6.372
	WBA	4.530	2.054	1.368	9.041	5.932	4.542	2.063	1.376	9.105	5.985
	WKB	4.224	2.520	1.568	9.921	6.321	4.237	2.530	1.577	9.989	6.377
	CA	4.228	2.521	1.568	9.917	6.317	4.240	2.531	1.577	9.985	6.373

Table 2. Binding energies $(E-M)/M$ vs. $A=\frac{1}{2}|q|\kappa$ for $\varsigma=0.1$ and 0.5

		$\varsigma=0.1$					$\varsigma=0.5$				
A	Method	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.5	Units	10^{-2}	10^{-3}	10^{-4}	10^{-4}	10^{-4}	10^{-1}	10^{-2}	10^{-2}	10^{-3}	10^{-3}
	Exact	1.072	1.874	7.264	3.815	2.344	1.055	3.005	1.369	7.761	4.984
	WBA	1.070	1.872	7.255	3.811	2.342	0.988	2.863	1.319	7.531	4.860
	WKB	1.215	2.013	7.608	3.946	2.407	1.072	3.045	1.382	7.820	5.014
	CA	1.080	1.880	7.277	3.820	2.346	1.079	3.043	1.380	7.810	5.009
1.0	Units	10^{-2}	10^{-3}	10^{-4}	10^{-4}	10^{-4}	10^{-1}	10^{-2}	10^{-2}	10^{-3}	10^{-3}
	Exact	2.029	2.819	9.496	4.640	2.733	1.151	3.258	1.456	8.148	5.186
	WBA	2.013	2.811	9.479	4.634	2.730	1.086	3.127	1.411	7.943	5.077
	WKB	2.201	3.008	9.912	4.785	2.799	1.167	3.300	1.470	8.210	5.218
	CA	2.039	2.823	9.505	4.643	2.734	1.168	3.283	1.464	8.180	5.202
2.0	Units	10^{-2}	10^{-3}	10^{-3}	10^{-4}	10^{-4}	10^{-2}	10^{-2}	10^{-2}	10^{-3}	10^{-3}
	Exact	4.202	5.507	1.519	6.535	3.563	1.357	3.861	1.667	9.077	5.667
	WBA	4.105	5.485	1.516	6.526	3.559	1.281	3.731	1.624	8.889	5.568
	WKB	4.367	5.794	1.580	6.725	3.642	1.370	3.906	1.683	9.145	5.702
	CA	4.223	5.513	1.520	6.537	3.564	1.370	3.879	1.672	9.097	5.676
5.0	Units	10^{-2}	10^{-2}	10^{-3}	10^{-3}	10^{-4}	10^{-1}	10^{-2}	10^{-2}	10^{-2}	10^{-3}
	Exact	9.873	1.834	4.554	1.569	7.100	1.843	5.732	2.379	1.223	7.270
	WBA	9.208	1.809	4.537	1.566	7.092	1.700	5.546	2.330	1.203	7.174
	WKB	9.981	1.875	4.677	1.608	7.244	1.851	5.775	2.398	1.231	7.312
	CA	9.915	1.836	4.556	1.569	7.101	1.853	5.745	2.383	1.224	7.275
10.0	Units	10^{-1}	10^{-2}	10^{-2}	10^{-3}	10^{-3}	10^{-1}	10^{-2}	10^{-2}	10^{-2}	10^{-2}
	Exact	1.629	4.409	1.328	4.565	1.842	2.373	8.558	3.663	1.827	1.036
	WBA	1.444	4.253	1.313	4.547	1.839	2.107	8.175	3.579	1.799	1.024
	WKB	1.635	4.448	1.345	4.636	1.871	2.378	8.594	3.682	1.837	1.042
	CA	1.634	4.415	1.329	4.566	1.843	2.382	8.572	3.666	1.828	1.037
20.0	Units	10^{-1}	10^{-2}	10^{-2}	10^{-2}	10^{-3}	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}
	Exact	2.398	8.975	3.551	1.466	6.373	3.011	1.301	6.181	3.186	1.782
	WBA	2.005	8.331	3.440	1.446	6.335	2.540	1.206	5.954	3.121	1.758
	WKB	2.402	9.003	3.568	1.476	6.426	3.013	1.304	6.198	3.197	1.788
	CA	2.404	8.985	3.553	1.466	6.374	3.018	1.303	6.185	3.188	1.782
50.0	Units	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}	10^{-1}	10^{-1}	10^{-1}	10^{-2}	10^{-2}
	Exact	3.496	1.785	9.542	5.198	2.868	3.940	2.128	1.215	7.174	4.361
	WBA	2.697	1.547	8.785	4.954	2.789	3.077	1.853	1.120	6.829	4.230
	WKB	3.497	1.787	9.553	5.207	2.875	3.941	2.129	1.216	7.183	4.367
	CA	3.500	1.787	9.546	5.200	2.869	3.946	2.130	1.216	7.177	4.362
100.0	Units	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-2}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-2}
	Exact	4.314	2.594	1.629	1.042	6.726	4.651	2.875	1.862	1.233	8.296
	WBA	4.618	2.119	1.424	0.951	6.320	4.945	3.420	1.633	1.128	7.798
	WKB	4.314	2.595	1.630	1.042	6.731	4.651	2.875	1.862	1.234	8.300
	CA	4.318	2.595	1.630	1.042	6.727	4.655	2.876	1.862	1.233	8.297

Table 3. Binding energies $(E-M)/M$ vs. $A=\frac{1}{2}|q|\kappa$ for $\varsigma=1.0$ and 5.0

		$\varsigma=1.0$					$\varsigma=5.0$				
A	Method	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
	Units	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}					
	Exact	3.329	1.106	5.268	3.041	1.969					1.188
0.5	WBA										
	WKB	3.330	1.110	5.289	3.051	1.975					1.186
	CA	3.331	1.110	5.284	3.048	1.973					
	Units	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}					10^{-1}
	Exact	3.018	1.055	5.115	2.976	1.937			>1.8?	1.196	8.040
1.0	WBA	2.573	0.927	4.609	2.730	1.800					
	WKB	3.024	1.059	5.135	2.986	1.942				1.195	8.030
	CA	3.058	1.063	5.144	2.989	1.943					6.48
	Units	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}				10^{-1}	10^{-1}
	Exact	2.922	1.068	5.218	3.035	1.971			1.243	8.320	5.808
2.0	WBA	2.580	0.970	4.823	2.841	1.863					
	WKB	2.929	1.072	5.238	3.045	1.977			1.242	8.312	5.804
	CA	2.954	1.074	5.240	3.045	1.976				7.452	5.532
	Units	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}			10^{-1}	10^{-1}	10^{-1}
	Exact	3.077	1.215	5.979	3.435	2.198		1.222	8.194	5.734	4.165
5.0	WBA	2.750	1.132	5.659	3.281	2.113					
	WKB	3.083	1.219	5.999	3.446	2.204		1.221	8.189	5.731	4.164
	CA	3.098	1.219	5.992	3.441	2.201			7.948	5.679	4.143
	Units	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}		10^{-1}	10^{-1}	10^{-1}	10^{-1}
	Exact	3.381	1.461	7.375	4.208	2.646	1.436	9.425	6.512	4.682	3.482
10.0	WBA	2.967	1.365	7.038	4.056	2.565					
	WKB	3.384	1.464	7.394	4.220	2.652	1.436	9.422	6.511	4.681	3.481
	CA	3.395	1.464	7.384	4.212	2.648		9.152	6.496	4.678	3.480
	Units	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-2}		10^{-1}	10^{-1}	10^{-1}	10^{-1}
	Exact	3.811	1.849	9.917	5.764	3.600	1.155	7.807	5.548	4.085	3.097
20.0	WBA	3.220	1.697	9.421	5.564	3.503			6.010	4.356	3.254
	WKB	3.813	1.851	9.933	5.775	3.607	1.155	7.806	5.548	4.085	3.098
	CA	3.822	1.852	9.926	5.768	3.601		7.806	5.551	4.087	3.099
	Units	10^{-1}	10^{-1}	10^{-1}	10^{-2}	10^{-2}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}
	Exact	4.508	2.572	1.559	9.831	6.414	9.506	6.709	4.955	3.766	2.929
50.0	WBA	3.554	2.243	1.433	9.304	6.178			5.472	4.165	3.226
	WKB	4.508	2.573	1.560	9.839	6.420	9.505	6.709	4.955	3.767	2.929
	CA	4.515	2.574	1.560	9.834	6.416	9.504	6.716	4.959	3.768	2.930
	Units	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}
	Exact	5.077	3.233	2.160	1.481	1.035	8.689	6.364	4.856	3.794	3.018
100.0	WBA	5.353	2.674	1.899	1.354	0.971			5.186	4.118	3.302
	WKB	5.078	3.233	2.161	1.482	1.036	8.689	6.364	4.856	3.794	3.018
	CA	5.082	3.235	2.161	1.482	1.035	8.703	6.369	4.858	3.795	3.018

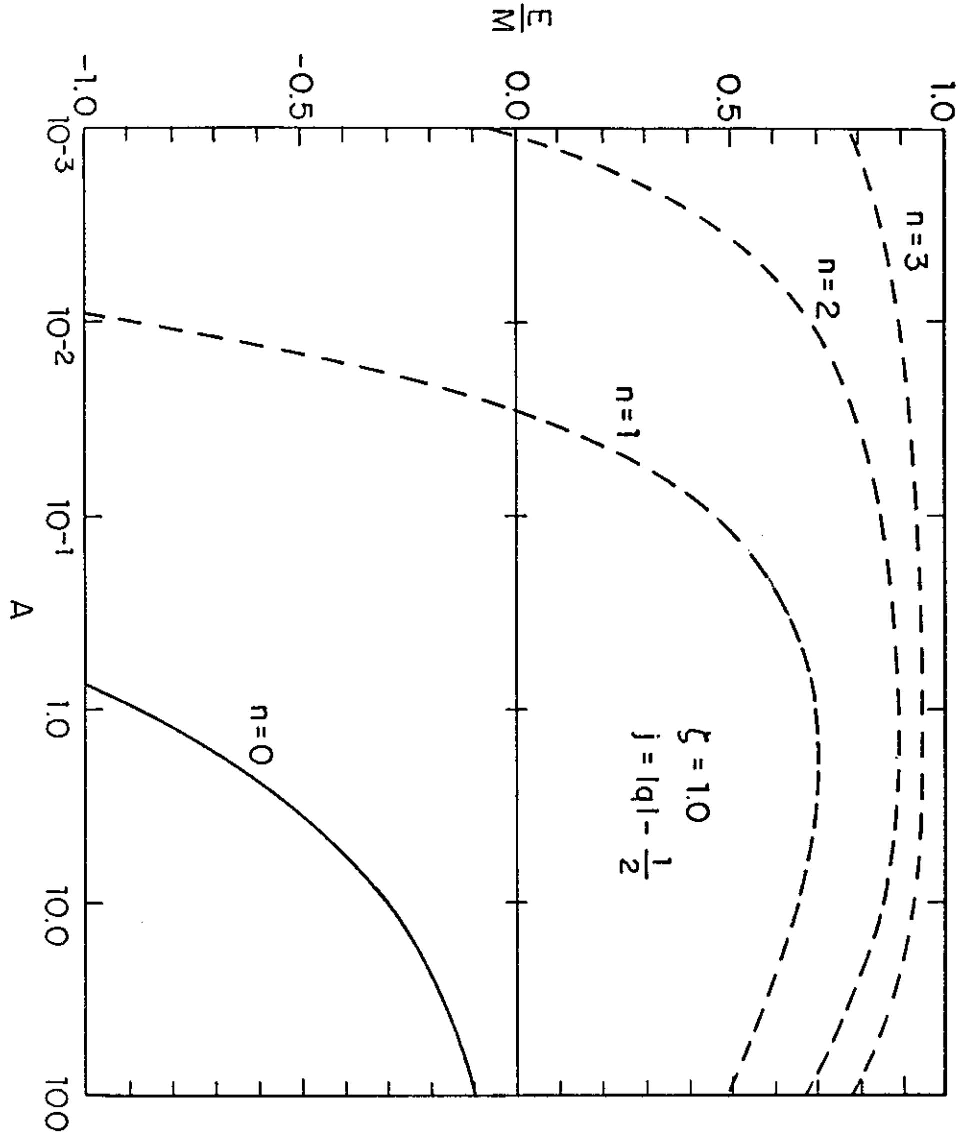
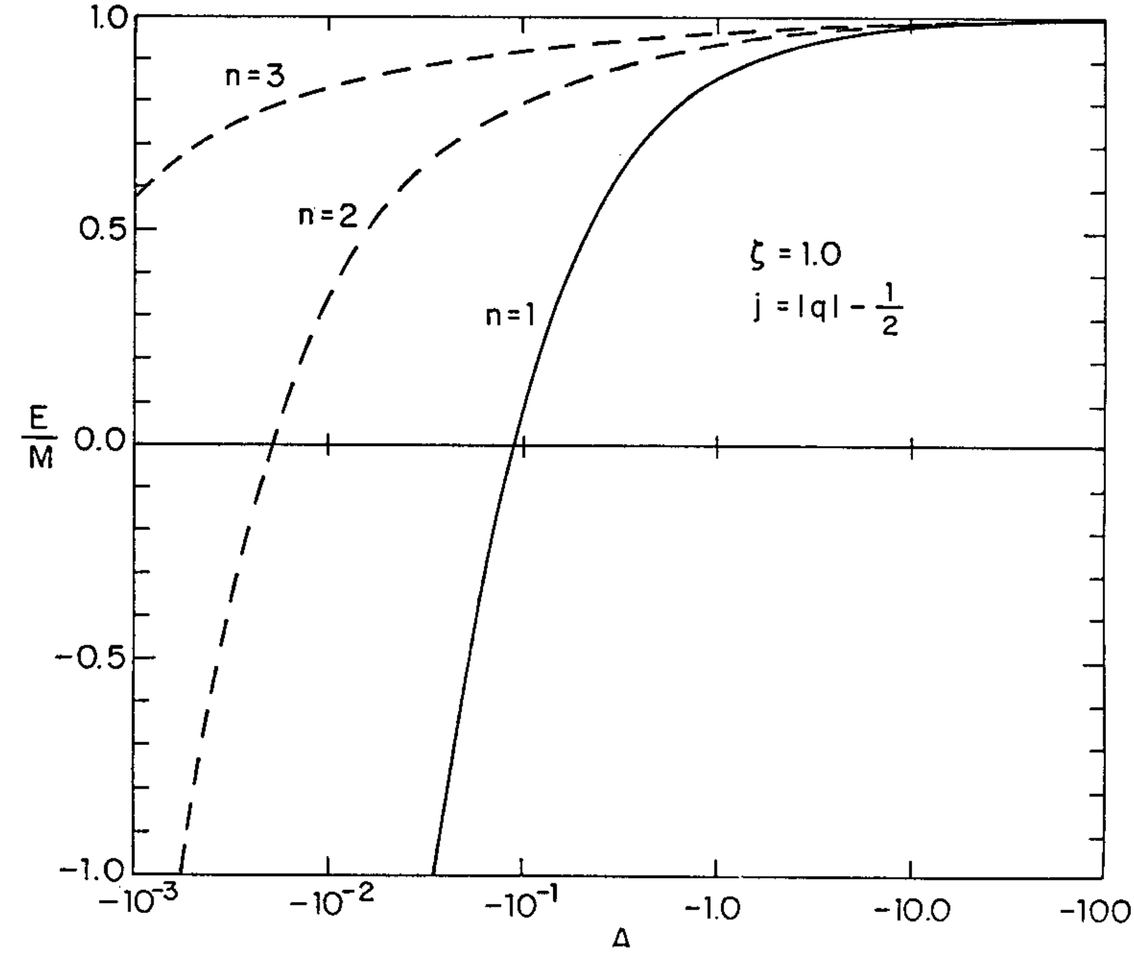
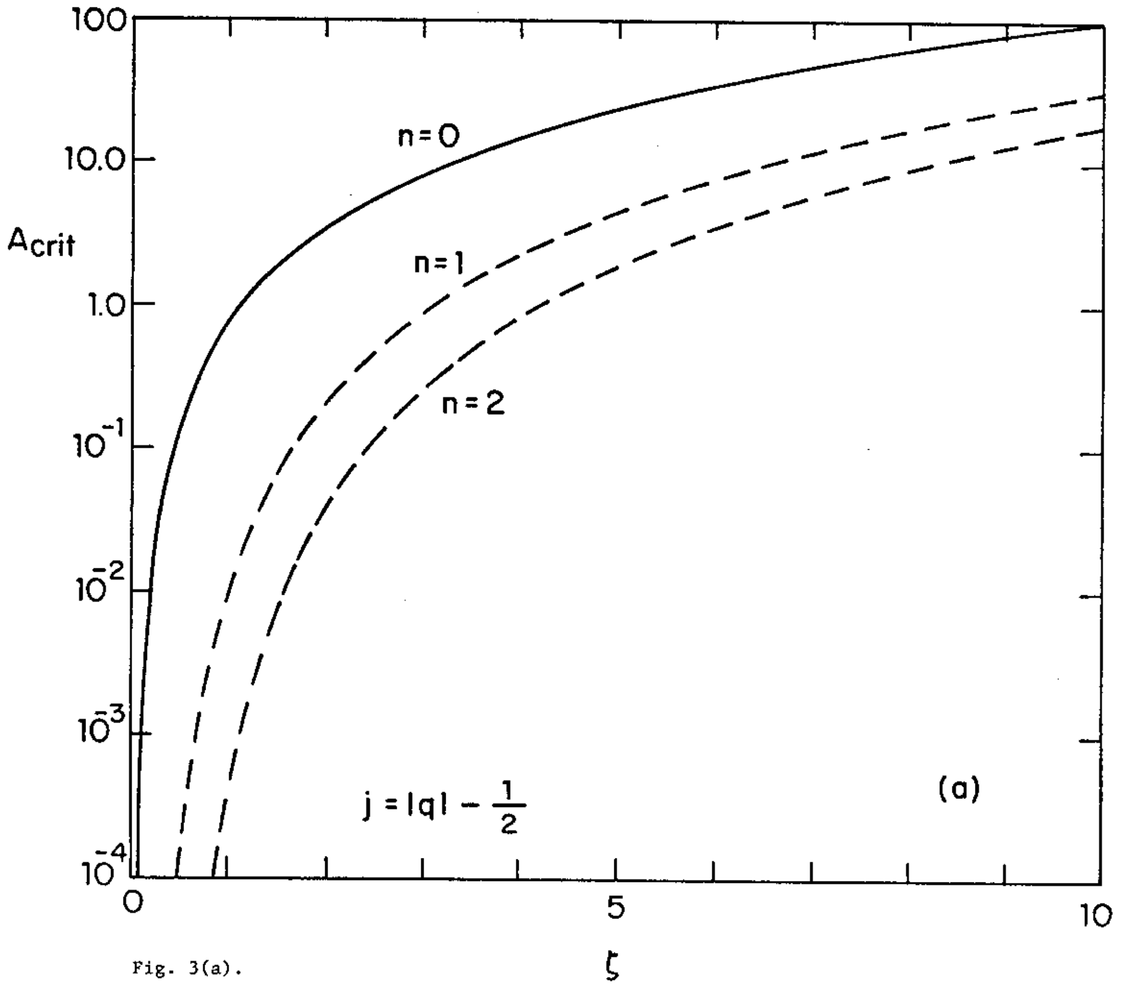


Fig. 1.



1.0
0.5
0.0
-0.5
-1.0
E/M
n=3
n=2
n=1
ζ = 1.0
j = |q| - 1/2
-10^{-3}
-10^{-2}
-10^{-1}
-1.0
-10.0
-100
Δ

Fig. 2.



100
10.0
1.0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
A_{crit}
n=0
n=1
n=2
j = |q| - 1/2
(a)
0
5
10
ζ

Fig. 3(a).

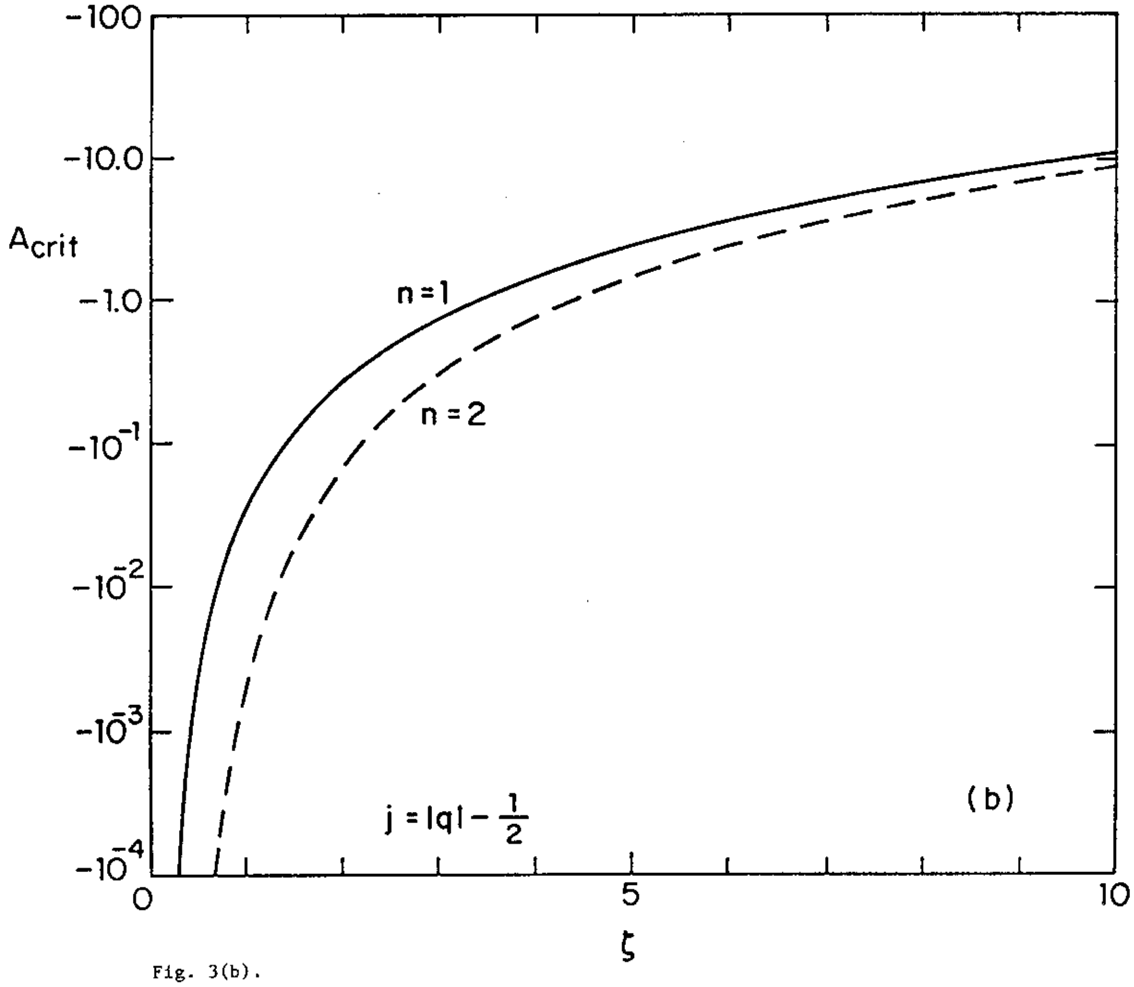


Fig. 3(b).