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Non-perturbative effects and the refined topological string

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Abstract: The partition function of ABJM theory on the three-sphere has non-perturbative corrections due to membrane instantons in the M-theory dual. We show that the full series of membrane instanton corrections is completely determined by the refined topological string on the Calabi–Yau manifold known as local $\mathbb{P}^1 \times \mathbb{P}^1$, in the Nekrasov–Shatashvili limit. Our result can be interpreted as a first-principles derivation of the full series of non-perturbative effects for the closed topological string on this Calabi–Yau background. Based on this, we make a proposal for the non-perturbative free energy of topological strings on general, local Calabi–Yau manifolds.

Contents

1. Introduction

Large N dualities relate gauge theories to string theories, and provide in principle a non-perturbative definition of string theory on certain backgrounds. The genus expansion of string theory amplitudes emerges then as an asymptotic, $1/N$ expansion of gauge theory amplitudes. Most of the work on large N dualities has focused on the large N or planar limit of the correspondence. One can also use these dualities to extract information about subleading $1/N$ corrections, although this is typically more difficult and it has been comparatively much less explored. In principle, large N dualities could be also used to study non-perturbative stringy effects, which correspond to corrections which are exponentially suppressed as N becomes large. Results along this direction have been even rarer.

In this paper we use large N dualities to completely determine the non-perturbative structure of the free energy of M-theory on $\mathrm{AdS}_4 \times \mathbb{S}^7/\mathbb{Z}_k$. As a bonus, we obtain as well the non-perturbative structure for the free energy of topological string theory on the Calabi–Yau manifold known as local $\mathbb{P}^1\times\mathbb{P}^1$, since both problems are formally identical. The non-perturbative structure we find turns out to be encoded by the *refined* topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$, in the so-called Nekrasov–Shatashvili (NS) limit [1].

The solution to this problem has been based on the convergence of many different results. First of all, a large N dual to M-theory on $\mathrm{AdS}_4 \times \mathbb{S}^7/\mathbb{Z}_k$ was proposed already in [2] in terms of the theory of N coincident M2 branes. In [3], based on previous work [4], this theory was constructed as an $\mathcal{N} = 6$ supersymmetric $U(N) \times U(N)$ Chern–Simons–matter theory known as ABJM theory. In this large N duality, the geometric parameter k in M-theory corresponds to the Chern–Simons coupling. The second ingredient was the localization computation of [5], where the partition function of ABJM theory on the three-sphere was reduced to a matrix integral which we will call the ABJM matrix model. This matrix model has been intensively studied from many points of view, and a variety of results have been found. The planar free energy, as well as the subleading $1/N$ corrections in the standard 't Hooft or genus expansion, were determined in [6]. This expansion makes contact with the type IIA reduction of M-theory and it captures all worldsheet instanton corrections to the partition function. However, in order to make contact with the M-theory regime, one should study the ABJM matrix model in the so-called M-theory expansion, where N is large but k is fixed. This was first done in [7], where the leading, large N limit was studied. In order to understand in more detail the M-theory expansion, and the corrections to the large N limit, a new method was introduced in [8], based on an equivalence with an ideal Fermi gas. In this approach, the Planck constant of the quantum gas is naturally identified with the inverse string coupling, and the semiclassical limit of the gas corresponds then to the strong string coupling limit in type IIA theory. One of the main virtues of the Fermi gas approach is that it makes it possible to calculate systematically non-perturbative stringy effects. These effects were anticipated in [9], where they were interpreted as membrane instanton effects in M-theory, or equivalently as D2-brane effects in type IIA theory. Thus, the Fermi gas approach opened the way for a quantitative determination of these effects in the M-theory dual to ABJM theory.

During the last year, the Fermi gas approach has led to many results on the partition function of ABJM theory. The equivalence between this method and the TBA system of [10, 11] has been particularly useful. We now have a lot of data, like for example WKB expansions at small k of the membrane instanton corrections [8, 12]. The calculation of the values of the partition function for various values of N and k [13, 14, 15], and their extrapolation to large N, have produced numerical results for the exponentially small corrections. In [15, 16], it was noticed that the corrections due to worldsheet instantons, which are known explicitly, are singular for integer values of k. Since the partition function is regular for all k, it was postulated that these singularities should be cancelled by membrane instanton corrections, as well as corrections coming from bound states of membranes and fundamental strings. This principle, which we will call the HMO cancellation mechanism, when combined with WKB expansions and numerical results, has led to conjectural exact results in k for the very first membrane instanton corrections [15, 12, 16] and to a conjecture for the structure of bound states [16]. According to this conjecture, the bound states are completely determined by the worldsheet instantons and the membrane instanton corrections. The remaining open problem is then to find an analytic description of the membrane instanton corrections in the M-theory regime, i.e. as an expansion at large N but exact in k.

In this paper we find precisely such a description. It turns out that the membrane instanton expansion at large N, which involves two independent generating functionals, is completely determined by the NS limit of the refined topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$. This limit is described by the two quantum periods of the mirror manifold [17, 18, 19], which are equal to the two generating functionals we were looking for. The Chern–Simons coupling k of ABJM theory corresponds to the quantum deformation parameter $\hbar$, and the standard large radius expansion of the periods corresponds precisely to the large N expansion in ABJM theory. Since the periods can be calculated exactly as a function of $\hbar$, this equivalence solves the problem of computing the non-perturbative corrections to the free energy of ABJM theory.

So far we are lacking a proof of this equivalence, which we have checked by comparing the existing results on membrane instantons in ABJM theory to the explicit results for the quantum periods, so our result here should be regarded as a conjecture. It can be stated quite precisely as an equivalence between the solution of the TBA system describing the ABJM partition function which is analytic at $k = 0$, and the problem of quantizing the periods of local $\mathbb{P}^1 \times \mathbb{P}^1$.

One of the first insights which made possible a precise quantitative understanding of the ABJM matrix model is its equivalence [20] to the matrix model describing Chern–Simons theory on $\mathbb{RP}^3$ [21], which is dual at large N to topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$ [22]. This implies, for example, that the worldsheet instanton corrections in ABJM theory are determined by the worldsheet instanton corrections in this topological string theory. We can then *define* the non-perturbative partition function of topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$ through the ABJM matrix model. With this non-perturbative definition, our computation of exponentially small corrections to this matrix model partition function can be also regarded as a derivation of the full structure of non-perturbative effects for topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$. The fact that the Fermi gas approach could be used to obtain a precise quantitative understanding of non-perturbative effects in this topological string model was pointed out in [8], and emphasized in [23].

The non-perturbative structure of topological strings has been the subject of much speculation in recent years, and there are by now various proposals on how it should look like. We would like to emphasize, however, that our derivation of the non-perturbative structure in this particular example is done from first principles, once we define it through the large N matrix model dual, and it fits a large amount of data on the large N asymptotics of the matrix model. Our result says that the non-perturbative part of the standard topological string free energy is determined by the refined topological string in the NS limit, on the same background. Inspired by this concrete result, we make a proposal for the non-perturbative structure of topological strings on arbitrary local CY manifolds, where the non-perturbative effects are encoded in the refined topological string. It turns out that our proposal (as well as our concrete, first-principles calculation for local $\mathbb{P}^1 \times \mathbb{P}^1$) is similar to a recent proposal by Lockhart and Vafa [24], which was inspired by localization in five-dimensional supersymmetric Yang–Mills theories, and we point out the resemblances as well as the differences between the two proposals.

The organization of this paper is as follows. In section 2 we review the known results on the grand potential of ABJM theory obtained in [8, 13, 14, 15, 12, 16]. In section 3 we show that these results are encoded in the NS limit of the refined topological string, and in particular in the quantum periods. In section 4 we point out that this leads to the determination of the non-perturbative structure of the topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$, and we make a proposal on how to extend this to arbitrary, local CY manifolds. We also discuss the relationship of our results and proposal to the work of [24]. Finally, in section 5 we conclude and discuss some avenues for further research. In Appendix A we explain how to calculate the quantum A-periods from the TBA system of the Fermi gas, and in Appendix B we make some comments on the quantum mirror map.

2. The partition function of ABJM theory

2.1 The grand potential

As it was shown in [5], the partition function of ABJM theory on the three-sphere, $Z(N,k)$, is given by the matrix integral

$$Z(N,k) = \frac{1}{N!^2}\int \frac{\mathrm{d}^N\mu}{(2\pi)^N}\frac{\mathrm{d}^N\nu}{(2\pi)^N}\frac{\prod_{i<j}\left[2\sinh\left(\frac{\mu_i-\mu_j}{2}\right)\right]^2\left[2\sinh\left(\frac{\nu_i-\nu_j}{2}\right)\right]^2}{\prod_{i,j}\left[2\cosh\left(\frac{\mu_i-\nu_j}{2}\right)\right]^2}\exp\left[\frac{\mathrm{i}k}{4\pi}\sum_{i=1}^N(\mu_i^2-\nu_i^2)\right]. \tag{2.1}$$

This matrix integral can be calculated in two different regimes. In the *'t Hooft expansion* one considers the limit

$$N\to\infty, \quad \lambda=\frac{N}{k} \text{ fixed}, \tag{2.2}$$

and the partition function has the standard $1/N$ expansion,

$$Z(N,k)=\exp\left[\sum_{g=0}^{\infty}N^{2-2g}F_g(\lambda)\right], \tag{2.3}$$

which corresponds to the genus expansion of type IIA superstring theory on $\mathrm{AdS}_4\times\mathbb{CP}^3$ [3]. The genus g free energies $F_g(\lambda)$ can be calculated exactly as a function of λ, and order by order in the genus expansion, by using matrix model techniques [6]. They contain non-perturbative information in α', since they involve exponentially small corrections of the form

$$\mathcal{O}\left(\mathrm{e}^{-2\pi\sqrt{2\lambda}}\right). \tag{2.4}$$

It was conjectured in [6] that these terms correspond to worldsheet instantons wrapping a two-cycle $\mathbb{CP}^1\subset\mathbb{CP}^3$, which were first considered in [27].

In the *M-theory expansion*, one computes the partition function in the regime

$$N\to\infty, \quad k \text{ fixed}. \tag{2.5}$$

This is the regime which is suitable for the dual description in terms of M-theory on $\mathrm{AdS}_4\times\mathbb{S}^7/\mathbb{Z}_k$. In this regime, one expects to find as well non-perturbative effects in the string coupling constant, which in type IIA theory correspond to Euclidean D2-brane instantons wrapping three-cycles in the target space. In [9] an appropriate, explicit family of generalized Lagrangian submanifolds with the topology of $\mathbb{RP}^3\subset\mathbb{CP}^3$ was proposed as an explicit candidate for this type of cycles, leading to exponentially small corrections of the form

$$\exp\left(-k\pi\sqrt{2\lambda}\right). \tag{2.6}$$

In order to understand the M-theory expansion of the ABJM matrix integral, one needs a suitable approach, different from the standard $1/N$ expansion of matrix integrals. A first step in this direction was taken in [7], where the leading contribution to the partition function at large N and fixed k was determined for various $\mathcal{N}=3$ Chern–Simons–matter theories. A more systematic approach to the problem was introduced in [8], and it is based on an analogy to a quantum, ideal Fermi gas. One first notices (see also [28]) that the matrix integral (2.1) can be written as

$$Z(N,k) = \frac{1}{N!}\sum_{\sigma\in S_N}(-1)^{\epsilon(\sigma)}\int \frac{\mathrm{d}^N x}{(2\pi k)^N}\frac{1}{\prod_i 2\cosh\left(\frac{x_i}{2}\right)2\cosh\left(\frac{x_i - x_{\sigma(i)}}{2k}\right)}. \tag{2.7}$$

This in turn can be interpreted as the canonical partition function of a one-dimensional Fermi gas with a non-trivial one-particle density matrix

$$\rho(x_1,x_2) = \frac{1}{2\pi k}\frac{1}{\left(2\cosh\frac{x_1}{2}\right)^{1/2}}\frac{1}{\left(2\cosh\frac{x_2}{2}\right)^{1/2}}\frac{1}{2\cosh\left(\frac{x_1-x_2}{2k}\right)}. \tag{2.8}$$

The one-particle Hamiltonian $\hat{H}$ of this system is then defined as

$$\hat{\rho} = \mathrm{e}^{-\hat{H}}, \qquad \langle x_1|\hat{\rho}|x_2\rangle = \rho(x_1,x_2), \tag{2.9}$$

and the Planck constant of the Fermi gas is

$$\hbar_{\mathrm{FG}} = 2\pi k. \tag{2.10}$$

The semiclassical or WKB expansion is then around $k = 0$, and it corresponds to the strong string coupling expansion in the type IIA dual. The Fermi gas approach makes it possible to determine both the subleading $1/N$ corrections and non-perturbative corrections due to D2-brane instantons. Various aspects of this approach have been developed in [13, 14, 32, 13, 16, 33] and we will review some of them in this section.

The Fermi gas approach suggests to look instead to the grand partition function (see also [29])

$$\Xi(\mu,k) = 1 + \sum_{N=1}^{\infty} Z(N,k)z^N, \tag{2.11}$$

where

$$z = \mathrm{e}^{\mu} \tag{2.12}$$

plays the rôle of the fugacity and μ is the chemical potential. The grand potential is then defined as

$$J(\mu,k) = \log\Xi(\mu,k). \tag{2.13}$$

The canonical partition function is recovered from the grand-canonical potential as

$$Z(N,k) = \oint \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{\Xi(\mu,k)}{z^{N+1}}. \tag{2.14}$$

As explained in [15], the grand potential has a "naive" part, which is the one obtained with the standard techniques in Statistical Mechanics, and an oscillatory part which restores the $2\pi\mathrm{i}$ periodicity in μ. It turns out that the contour in (2.14) can be deformed to the imaginary axis if one replaces the grand potential by its "naive" part, which will be the only one we will consider in this paper. Therefore, we can write

$$Z(N,k) = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\mathrm{d}\mu\,\exp\left[J(\mu,k)-\mu N\right], \tag{2.15}$$