

On the relation between the modular double of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ and the quantum Teichmüller theory

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Abstract

We exhibit direct relations between the modular double of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ and the quantum Teichmüller theory. Explicit representations for the fusion- and braiding operations of the quantum Teichmüller theory are immediate consequences. Our results include a simplified derivation of the Clebsch-Gordan decomposition for the principal series of representation of the modular double of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$.

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1. Introduction

The modular double of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ is a non-compact quantum group closely related to the quantum deformation of the universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{R})$. It has some interesting features that are responsible for its relevance to conformal field theory [PT1, T01], integrable models [BT2], and quantum Teichmüller theory. The generators of the modular double are represented by *positive* self-adjoint operators, which was shown in [BT1] to be responsible for the remarkable self-duality of the modular double: It is simultaneously the modular double of $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$, with deformation parameter \tilde{q} given as $\tilde{q} = e^{\pi i b^{-2}}$ if $e^{\pi i b^2}$. This self-duality was pointed out independently in [F99] and in [PT1], and it has profound consequences in the applications of this mathematical structure. One may, for example, use it to explain the quantum-field theoretical self-dualities of the Liouville theory [T01] and of the Sinh-Gordon model [BT2].

There are various hints that there must be close connections between the quantization of the Teichmüller spaces constructed in [F97, CF1, Ka1] on the one hand, and the modular double of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ on the other hand. First hints came from the observations made in [T03] that the fusion move in the quantum Teichmüller theory gets represented in terms of the 6j-symbols of the modular double [PT1, PT2]. One may also observe [FK] that the quantum Teichmüller theory is essentially build from the basic data of the modular double of the quantum $(ax + b)$ -group, the so-called multiplicative unitary. As the $(ax + b)$ -group is nothing but the Borel half of $SL(2, \mathbb{R})$, one may expect relations between the quantum Teichmüller theory and the modular double of $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$ to follow by combining the quantum double construction of $\text{Fun}_q(SL(2, \mathbb{R}))$ from the quantum $(ax + b)$ -group [Ip] with the duality between the modular doubles of $\text{Fun}_q(SL(2, \mathbb{R}))$ and $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ described in [PT1], and proven in [Ip].

However, all these hints are somewhat indirect. We'll here exhibit a direct link by establishing a relation between the Casimir operator of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ and the geodesic length operators of the quantum Teichmüller theory. The key observation is that the co-product of the modular double of $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$ gets represented by an operator in the quantum Teichmüller theory that has a simple geometrical interpretation in terms of changes of triangulation of the underlying Riemann surfaces. The combinatorial structure of the quantum Teichmüller theory can be used to find an explicit expression for the Clebsch-Gordan operator that describes the decomposition of the tensor product of two irreducible representations of the modular double into irreducible

representations.

An immediate consequence is the direct relation between the kernel representing the fusion operation from quantum Teichmüller theory and the b - $6j$ symbols of the modular double [PT1, PT2]. We also find, not surprisingly, that the R -operator of the modular double [F99, BT1] is directly related to the braiding operation of the quantum Teichmüller theory.

The Clebsch-Gordan maps of the modular double have previously been constructed in [PT2] as an integral operator with an explicit kernel. However, especially the proof of the completeness for the Clebsch-Gordan decomposition given in [PT2] was quite complicated. The construction of the Clebsch-Gordan operator given in this paper will allow us to re-derive the main results of [PT2] on the Clebsch-Gordan decomposition in a simpler, and hopefully more transparent way. The explicit construction of the Clebsch-Gordan operator presented below reduces the proof of completeness to the results of [Ka3, Ka4] on the spectral decomposition of the geodesic length operators in Teichmüller theory. The proof of this result given in [Ka4] is much simpler than the proof of the corresponding result on the Casimir operators of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ given in [PT2].

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2. Some notations and conventions

The special function $e_b(U)$ can be defined in the strip $|\Im z| < |\Im c_b|$, $c_b \equiv i(b + b^{-1})/2$ by means of the integral representation

$$\log e_b(z) \equiv \frac{1}{4} \int_{i0-\infty}^{i0+\infty} \frac{dw}{w} \frac{e^{-2izw}}{\sinh(bw) \sinh(b^{-1}w)}. \quad (2.1)$$

Closely related is the function $w_b(x)$ defined via

$$e_b(x) = e^{-\frac{\pi i}{12}(1+2c_b^2)} e^{\frac{\pi i}{2}x^2} (w_b(x))^{-1}. \quad (2.2)$$

Another useful combination is the function $D_\alpha(x)$, defined as

$$D_\alpha(x) := \frac{w_b(x + \alpha)}{w_b(x - \alpha)}. \quad (2.3)$$

For tensor products we will be using the following leg-numbering notation. Let us first define, as usual,

$$\mathbb{X}_r := 1 \otimes \dots \otimes \underset{r\text{-th}}{\mathbb{X}} \otimes \dots \otimes 1. \quad (2.4)$$

We are using the slightly unusual convention to label tensor factors from the right to the left, as, for example, in $\mathcal{H}_2 \otimes \mathcal{H}_1$.

3. Modular double

3.1 Principal series representations of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$

We will be considering the Hopf-algebra $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ which has generators E, F and K subject to the relations,

$$\begin{aligned} KE &= qEK, & [E, F] &= -\frac{K^2 - K^{-2}}{q - q^{-1}}. \\ KE &= qEK, \end{aligned} \quad (3.1)$$

The algebra $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ has the central element

$$Q = (q - q^{-1})^2 FE - qK^2 - qK^{-2}. \quad (3.2)$$

The co-product is given as

$$\begin{aligned} \Delta(K) &= K \otimes K, & \Delta(E) &= E \otimes K + K^{-1} \otimes E, \\ \Delta(F) &= F \otimes K + K^{-1} \otimes F. \end{aligned} \quad (3.3)$$

This implies

$$\begin{aligned} \Delta(Q) &= K^{-1}F \otimes EK + K^{-1}E \otimes FK \\ &+ Q \otimes K^2 + K^{-2} \otimes Q + (q + q^{-1})K^{-2} \otimes K^2. \end{aligned} \quad (3.4)$$

This algebra has a one-parameter family of representations \mathcal{P}'_α

$$\begin{aligned} E'_s &\equiv \pi_s(E) := e^{+\pi b \mathfrak{q}} \frac{\cosh \pi b(\mathfrak{p} - s)}{\sin \pi b^2} e^{+\pi b \mathfrak{q}}, & K'_s &\equiv \pi_s(K) := e^{-\pi b \mathfrak{p}}, \\ F'_s &\equiv \pi_s(F) := e^{-\pi b \mathfrak{q}} \frac{\cosh \pi b(\mathfrak{p} + s)}{\sin \pi b^2} e^{-\pi b \mathfrak{q}}, \end{aligned} \quad (3.5)$$

where \mathfrak{p} and \mathfrak{q} are operators acting on functions $f(q)$ as $\mathfrak{p}f(q) = (2\pi i)^{-1} \frac{\partial}{\partial q} f(q)$ and $\mathfrak{q}f(q) = qf(q)$, respectively. In the definitions (3.5) we are parameterizing q as $q = e^{\pi i b^2}$. There is a maximal dense subspace $\mathcal{P}'_s \subset L^2(\mathbb{R})$ on which all polynomials formed out of E'_s, F'_s and K'_s are well-defined [BT2, Appendix B].

3.2 Modular duality

These representations are distinguished by a remarkable self-duality property: It is automatically a representation of the quantum group $\mathcal{U}_{\tilde{q}}(\mathfrak{sl}(2, \mathbb{R}))$, where $\tilde{q} = e^{\pi i/b^2}$ if $q = e^{\pi i b^2}$. This representation is generated from operators $\tilde{E}_\alpha, \tilde{F}_\alpha$ and \tilde{K}_α which are defined by formulae obtained from those in (3.5) by replacing $b \rightarrow b^{-1}$. The subspace \mathcal{P}_α is simultaneously a maximal domain for the polynomial functions of $\tilde{E}_\alpha, \tilde{F}_\alpha$ and \tilde{K}_α [BT2, Appendix B].

This phenomenon was observed independently in [PT1] and in [F99]. It is closely related to the fact that E_α , F_α and K_α are *positive* self-adjoint generators which allows one to construct \tilde{E}_α , \tilde{F}_α and \tilde{K}_α via [BT1]

$$\tilde{e} = e^{1/b^2}, \quad \tilde{f} = f^{1/b^2}, \quad \tilde{K} = K^{1/b^2} \quad (3.6)$$

using the notations

$$\begin{aligned} e &:= 2 \sin(\pi b^2) E, & f &:= 2 \sin(\pi b^2) F, \\ \tilde{e} &:= 2 \sin(\pi b^{-2}) \tilde{E}, & \tilde{f} &:= 2 \sin(\pi b^{-2}) \tilde{F}. \end{aligned} \quad (3.7)$$

It was proposed in [PT1, BT1] to construct a noncompact quantum group which has as *complete* set of tempered representations the self-dual representations \mathcal{P}_α . It's gradually becoming clear how to realize this suggestion precisely. Relevant steps in this direction were taken in [BT1] by defining co-product, R-operator and Haar-measure of such a quantum group. Further important progress in this direction was recently made in [Ip]. Following [F99], we will in the following call this noncompact quantum group the modular double of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$.

3.3 The Whittaker model for $\mathcal{DU}_q(\mathfrak{sl}_2)$

A unitarily equivalent family of representations of the modular double is

$$2 \sin \pi b^2 E_s = e^{\pi b(2q-p)}, \quad K_s = e^{-\pi b p}, \quad (3.8a)$$

$$2 \sin \pi b^2 F_s = e^{\pi b(q-p/2)} (2 \cosh(2\pi b s) + 2 \cosh(2\pi b p)) e^{\pi b(q-p/2)}, \quad (3.8b)$$

A joint domain of definition is the space \mathcal{P} of entire functions which decay faster than any polynomial when going to infinity along the real axis. It is easy to see that this representation is unitarily equivalent to the one defined in (3.5), $X'_s = U_s \cdot X \cdot U_s^{-1}$, with

$$U_s := e^{-\frac{\pi i}{2} p^2} w_b(p - s). \quad (3.9)$$

In any representation in which E_r are invertible we may represent the action of the Casimir on the tensor product of two representations, defined as,

$$Q_{2,1} \equiv (\pi_{s_2} \otimes \pi_{s_1})(Q), \quad (3.10)$$

by the formula

$$\begin{aligned} Q_{2,1} &= K_2^{-1} (qK_2^2 + q^{-1}K_2^{-2} + Q_2) E_2^{-1} E_1 K_1 + K_2^{-1} E_2 (qK_1^2 + q^{-1}K_1^{-2} + Q_1) E_1^{-1} K_1 \\ &\quad + Q_2 K_1^2 + K_2^{-2} Q_1 + (q + q^{-1}) K_2^{-2} K_1^2. \end{aligned} \quad (3.11)$$

Our main task is to diagonalize this operator.

3.4 The model space

It will be useful for us to introduce a space which contains all the irreducible representations of the modular double with multiplicity one.

Let us consider the space $\mathcal{M} := \mathcal{P} \otimes L^2(\mathbb{R}_+, d\mu)$. We'll choose the measure $d\mu \equiv d\mu(s)$ as

$$d\mu(s) := ds \, 4 \sinh(2\pi b s) \sinh(2\pi b^{-1} s). \quad (3.12)$$

This space may be identified with the space of functions of two variables, taken to be functions of $f(p_2, s_1)$. We will consider an operator $Q \equiv \pi_{\mathcal{M}}(Q)$ which will represent the action of the Casimir Q on \mathcal{M} . Its action is

$$Q \cdot f(p_2, s_1) := 2 \cosh(2\pi b s_1) f(p_2, s_1). \quad (3.13)$$

The space \mathcal{M} becomes a representation of the modular double generated by the operators $E \equiv \pi_{\mathcal{M}}(E)$, $F \equiv \pi_{\mathcal{M}}(F)$ and $K \equiv \pi_{\mathcal{M}}(K)$ which are defined as

$$2 \sin \pi b^2 E = e^{\pi b(2q-p)}, \quad K = e^{-\pi b p}, \quad (3.14a)$$

$$2 \sin \pi b^2 F = e^{\pi b(q-p/2)} (Q + 2 \cosh(2\pi b p)) e^{\pi b(q-p/2)}, \quad (3.14b)$$

It is clear by definition that \mathcal{M} decomposes into irreducible representations of the modular double as

$$\mathcal{M} \simeq \int^{\oplus} d\mu(s) \mathcal{P}_s, \quad (3.15)$$

with action of the generators defined above.

3.5 The R-operator

Let us introduce the rescaled generators e and f via

$$e := 2 \sin \pi b^2 E, \quad f := 2 \sin \pi b^2 F. \quad (3.16)$$

Let us furthermore introduce an anti-self-adjoint element h such that $K = q^h$. We will then define the following operator on $\mathcal{M} \otimes \mathcal{M}$:

$$R = q^{h \otimes h} E_b(e \otimes f) q^{h \otimes h}. \quad (3.17)$$

R coincides with the R-operator proposed by L. Faddeev in [F99]. Notice that $|g_b(x)| = 1$ for $x \in \mathbb{R}^+$. This implies that R is manifestly unitary.

Theorem 1. *The operator R has the following properties:*

$$(i) \quad R \Delta(X) = \Delta'(X) R, \quad (3.18)$$

$$(ii) \quad (\text{id} \otimes \Delta)R = R_{13}R_{12}, \quad (\Delta \otimes \text{id})R = R_{13}R_{23}, \quad (3.19)$$

$$(iii) \quad (\sigma \otimes \text{id})R = R^{-1}, \quad (\text{id} \otimes \sigma)R = R^{-1}. \quad (3.20)$$

Lemma 1. *Let U and V be positive self-adjoint operators such that $UV = q^2VU$ where $q = e^{i\pi b^2}$. The function $E_b(x)$ satisfies the identities*

$$E_b(U) E_b(V) = E_b(U + V), \quad (3.21)$$

$$E_b(V) E_b(U) = E_b(U) E_b(q^{-1}UV) E_b(V). \quad (3.22)$$

Furthermore, (3.21) \Leftrightarrow (3.22).

In the literature, eqs. (3.21) and (3.22) are often referred to as the quantum exponential and the quantum pentagon relations.

To prove the first formula in (3.19), we use the quantum exponential relation (3.21) from Lemma 1 with identification $U = e_1 K_2^{-1} f_3$ and $V = e_1 f_2 K_3$,

$$\begin{aligned} (\text{id} \otimes \Delta)R &\stackrel{(3.17)}{=} (\text{id} \otimes \Delta)(q^{h_1 h_2} g_b(e_1 f_2) q^{h_1 h_2}) \\ &\stackrel{(3.3)}{=} q^{h_1 h_2 + h_1 h_3} E_b(e_1 f_2 K_3 + e_1 K_2^{-1} f_3) q^{h_1 h_2 + h_1 h_3} \\ &\stackrel{(3.21)}{=} q^{h_1 h_2 + h_1 h_3} E_b(e_1 K_2^{-1} f_3) E_b(e_1 f_2 K_3) q^{h_1 h_2 + h_1 h_3} \\ &= q^{h_1 h_3} E_b(e_1 f_3) q^{h_1 h_3} \cdot q^{h_1 h_2} E_b(e_1 f_2) q^{h_1 h_2} = R_{13} R_{12}. \end{aligned}$$

The second formula in (3.19) is proved in the same way.

The R-operator allows us to introduce the braiding of tensor products of the representations \mathcal{P}_s . Specifically, let the operator $B : \mathcal{P}_{s_2} \otimes \mathcal{P}_{s_1} \rightarrow \mathcal{P}_{s_1} \otimes \mathcal{P}_{s_2}$ be defined by $B_{s_2, s_1} \equiv PR_{s_2, s_1}$, where P is the operator that permutes the two tensor factors. Property (i) from Theorem 1 implies as usual that $B_{s_2, s_1} \circ \Delta(X) = \Delta(X) \circ B_{s_2, s_1}$.

4. The Clebsch-Gordan maps of the modular double

In this section we are going to re-derive the main results of [PT2] on the Clebsch-Gordan decomposition of tensor products of representations of $\mathcal{DU}_q(\mathfrak{sl}(2, \mathbb{R}))$ in a completely new way. The most difficult part in [PT2] was to prove the completeness of the eigenfunctions of the Casimir operator Q_{21} acting on the tensor product of two representations. This result will now be obtained by first constructing an explicit unitary operator which maps Q_{21} to a simple standard form Q''_1 , and then applying the result of Kashaev [Ka4] on the completeness of the eigenfunctions of Q''_1 . The resulting proof is much shorter than the one given in [PT2].

4.1 Definition of Clebsch-Gordan maps

The goal is to construct the Clebsch-Gordan projection maps,

$$C_{s_2 s_1}^{s_3} : \mathcal{P}_{s_2} \otimes \mathcal{P}_{s_1} \rightarrow \mathcal{P}_{s_3}, \quad (4.1)$$

that satisfy

$$C_{s_2 s_1}^{s_3} \cdot (\pi_{s_2} \otimes \pi_{s_1})(X) = \pi_{s_3}(X) \cdot C_{s_2 s_1}^{s_3}. \quad (4.2)$$

It will be convenient to consider the unitary operators

$$C_{s_2 s_1} : \mathcal{P}_{s_2} \otimes \mathcal{P}_{s_1} \rightarrow \mathcal{M} \simeq \int^{\oplus} d\mu(s_3) \mathcal{P}_{s_3}, \quad (4.3)$$

related to $C_{s_2 s_1}^{s_3}$ as

$$C_{s_2 s_1} = \int^{\oplus} d\mu(s_3) C_{s_2 s_1}^{s_3}. \quad (4.4)$$

We note that $C_{s_2 s_1}$ is characterized by the properties

$$(C_{s_2 s_1})^{-1} \cdot E \cdot C_{s_2 s_1} = E_2 K_1 + K_2^{-1} E_1, \quad (4.5a)$$

$$(C_{s_2 s_1})^{-1} \cdot K \cdot C_{s_2 s_1} = K_2 K_1, \quad (4.5b)$$

$$(C_{s_2 s_1})^{-1} \cdot Q \cdot C_{s_2 s_1} = Q_{21}. \quad (4.5c)$$

The “missing” property

$$(C_{s_2 s_1})^{-1} \cdot F \cdot C_{s_2 s_1} = F_2 K_1 + K_2^{-1} F_1,$$

is an easy consequence of (4.5), since invertibility of E allows us to express F in terms of E , K and Q in our representations.

4.2 Factorization of Clebsch-Gordan maps

We will construct the Clebsch-Gordan maps in the following factorized form:

$$C_{s_2 s_1} := \nu_{s_2 s_1}^{s_{21}} \cdot S_1 \cdot C_1 \cdot (T_{12})^{-1}, \quad (4.6)$$

where

- The operator T_{12} satisfies

$$T_{12} \cdot E_2 \cdot (T_{12})^{-1} = E_2 K_1 + K_2^{-1} E_1, \quad (4.7a)$$

$$T_{12} \cdot K_2 \cdot (T_{12})^{-1} = K_2 K_1, \quad (4.7b)$$

$$T_{12} \cdot Q'_{21} \cdot (T_{12})^{-1} = Q_{21}, \quad (4.7c)$$

where

$$Q'_1 := 2 \cosh 2\pi b(q_1 - p_1) + e^{-2\pi b q_1} Q_1 + e^{-2\pi b p_1} Q_2 + e^{-2\pi b(p_1 + q_1)}. \quad (4.8)$$

This means that T_{12} generates the representation of the co-product in the representation of the Borel-subalgebra generated by E and K on $\mathcal{P}_{s_2} \otimes \mathcal{P}_{s_1}$, and it simplifies Q_{21} to an operator that acts nontrivially only on one tensor factor.

- The operator C_1 maps $L^2(\mathbb{R}^2)$ to itself, commutes with K_2 and E_2 and maps Q'_1 to a simple form,

$$(C_1)^{-1} \cdot Q''_1 \cdot C_1 = Q'_1, \quad (4.9)$$

with Q''_1 being defined as

$$Q''_1 = 2 \cosh 2\pi b p_1 + e^{-2\pi b q_1}. \quad (4.10)$$

- S_1 maps $L^2(\mathbb{R}^2)$ to \mathcal{M} in such a way that Q''_1 is mapped to the multiplication operator Q ,

$$S_1^{-1} \cdot E \cdot S_1 = E_2, \quad (4.11a)$$

$$S_1^{-1} \cdot K \cdot S_1 = K_2, \quad (4.11b)$$

$$S_1^{-1} \cdot Q \cdot S_1 = Q''_1. \quad (4.11c)$$

- $\nu_{s_2 s_1}^{s_{21}}$ is a normalization factor that may depend on the positive self-adjoint operator s_{21} defined by $Q_{21} = 2 \cosh(2\pi b s_{21})$. A convenient choice for $\nu_{s_2 s_1}^{s_{21}}$ will be defined later.

It follows easily that the operator defined in (4.6) satisfies (4.5).

4.3 Construction of the Clebsch-Gordan maps

The operators T_{12} and C_1 in (4.6) can be constructed explicitly as

$$T_{12} := e_b(q_1 + p_2 - q_2) e^{-2\pi i p_1 q_2}, \quad (4.12a)$$

$$C_1^{-1} := e_b(q_1 - s_2) e^{2\pi i s_1 p_1} \frac{e_b(s_1 - p_1)}{e_b(s_1 + p_1)} e^{2\pi i s_2 q_1}. \quad (4.12b)$$

The operator S_1 essentially coincides with the operator that maps L_1 to diagonal form. This operator can be represented by the integral kernel

$$\langle p_1 | s_1 \rangle = \frac{e_b(s_1 + p_1 + c_b - i0)}{e_b(s_1 - p_1 - c_b + i0)} e^{-2\pi i s_1 (p_1 + c_b)}. \quad (4.13)$$

The functions $\phi_{s_1}(p_1) := \langle p_1 | s_1 \rangle$ are nothing but the eigenfunctions of the operator L_1 in the representation where p_1 is diagonal. It was shown in [Ka4] that the eigenfunctions $\phi_{s_1}(p_1)$ are delta-function orthogonalized and complete in $L^2(\mathbb{R})$,

$$\int_{\mathbb{R}_+} dp_1 \langle s_1 | p_1 \rangle \langle p_1 | s'_1 \rangle = \delta(s_1 - s'_1). \quad (4.14a)$$

$$\int_{\mathbb{R}_+} d\mu(s_1) \langle p_1 | s_1 \rangle \langle s_1 | p'_1 \rangle = \delta(p_1 - p'_1). \quad (4.14b)$$

This is equivalent to unitarity of the operator S_1 .

4.4 Verification of intertwining property

We want to demonstrate that the operator T_{12} satisfies (4.7). In order to see this, let us calculate

$$\begin{aligned}
T_{12} \cdot E_2 \cdot (T_{12})^{-1} &= e_b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2) \cdot e^{\pi b(2\mathbf{q}_2 - (\mathbf{p}_2 + \mathbf{p}_1))} \cdot (e_b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2))^{-1} \\
&= e^{\frac{\pi b}{2}(2\mathbf{q}_2 - (\mathbf{p}_2 + \mathbf{p}_1))} \cdot \frac{e_b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2 - ib/2)}{e_b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2 + ib/2)} \cdot e^{\frac{\pi b}{2}(2\mathbf{q}_2 - (\mathbf{p}_2 + \mathbf{p}_1))} \\
&= e^{\frac{\pi b}{2}(2\mathbf{q}_2 - (\mathbf{p}_2 + \mathbf{p}_1))} \cdot (1 + e^{2\pi b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2)}) \cdot e^{\frac{\pi b}{2}(2\mathbf{q}_2 - (\mathbf{p}_2 + \mathbf{p}_1))} \\
&= E_2 K_1 + (K_2)^{-1} E_1.
\end{aligned}$$

Equation (4.7c) is verified as follows: Let us write $T_{12} = t_{12} e^{-2\pi i p_1 q_2}$, and calculate

$$\begin{aligned}
T_{12} \cdot \tilde{L}_1 \cdot (T_{12})^{-1} &= \tag{4.15} \\
&= t_{12} \cdot (2 \cosh 2\pi b(\mathbf{q}_1 - \mathbf{p}_1 - \mathbf{q}_2) + e^{-2\pi b(\mathbf{q}_1 - \mathbf{q}_2)} Q_1 + e^{-2\pi b p_1} Q_2 + e^{-2\pi b(\mathbf{q}_1 + \mathbf{p}_1 - \mathbf{q}_2)}) \cdot t_{12}^{-1} \\
&= 2 \cosh 2\pi b(\mathbf{q}_1 - \mathbf{p}_1 - \mathbf{q}_2) \\
&\quad + Q_1 e^{-\pi b(\mathbf{q}_1 - \mathbf{q}_2)} (1 + e^{2\pi b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2)}) e^{-\pi b(\mathbf{q}_1 - \mathbf{q}_2)} \\
&\quad + Q_2 e^{-\pi b p_1} (1 + e^{2\pi b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2)}) e^{-\pi b p_1} \\
&\quad + e^{-\pi b(\mathbf{q}_1 + \mathbf{p}_1 - \mathbf{q}_2)} (1 + e^{2\pi b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2 + ib/2)}) (1 + e^{2\pi b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2 - ib/2)}) e^{-\pi b(\mathbf{q}_1 + \mathbf{p}_1 - \mathbf{q}_2)}
\end{aligned}$$

Comparison of this expression with (3.11) proves (4.7c).

The calculations needed to verify (4.9) are very similar.

4.5 The b-Clebsch-Gordan coefficients

The b-Clebsch-Gordan coefficients are defined as the matrix elements of the Clebsch-Gordan operator,

$$\left(\begin{smallmatrix} s_3 & s_2 & s_1 \\ p_3 & p_2 & p_1 \end{smallmatrix} \middle| \begin{smallmatrix} s_2 & s_1 \\ p_2 & p_1 \end{smallmatrix} \right)_b := \langle s_3, p_3 \mid C_{s_2 s_1} \mid p_2, p_1 \rangle. \tag{4.16}$$

We have

Proposition 1.

There exists a choice of coefficients $\nu_{s_2 s_1}^{s_2 s_1}$ such that the following statements are true:

(a) The b-Clebsch-Gordan coefficients are explicitly given by the formula

$$\begin{aligned}
\left(\begin{smallmatrix} s_3 & s_2 & s_1 \\ p_3 & p_2 & p_1 \end{smallmatrix} \middle| \begin{smallmatrix} s_2 & s_1 \\ p_2 & p_1 \end{smallmatrix} \right)_b &= \delta(p_{21} - p_2 - p_1) \left(\frac{w_b(s_1 + s_2 - s_3) w_b(s_1 + s_3 - s_2) w_b(s_2 + s_3 - s_1)}{w_b(s_1 + s_2 + s_3)} \right)^{\frac{1}{2}} \\
&\quad \times e^{\frac{\pi i}{2}(p_3^2 - p_1^2 - p_2^2)} \frac{w_b(p_1 - s_1) w_b(p_2 - s_2)}{w_b(p_{21} - s_{21})} e^{\pi i(p_2(s_1 + c_b) - p_1(s_2 + c_b))} \\
&\quad \times \int_{\mathbb{R}} dp e^{\pi i p(s_1 + s_2 - s_{21} + c_b)} D_{\frac{1}{2}(s_2 - s_1 - s_{21} - c_b)}(p + p_2) D_{\frac{1}{2}(s_1 - s_2 - s_{21} - c_b)}(p - p_1) \\
&\quad \times D_{\frac{1}{2}(s_1 + s_2 + s_{21} - c_b)}(p). \tag{4.17}
\end{aligned}$$

(b) The following Weyl-symmetries hold:

$$\begin{pmatrix} s_3 & s_2 & s_1 \\ p_3 & p_2 & p_1 \end{pmatrix}_b = \begin{pmatrix} s_3 & s_2 & -s_1 \\ p_3 & p_2 & p_1 \end{pmatrix}_b = \begin{pmatrix} s_3 & -s_2 & s_1 \\ p_3 & p_2 & p_1 \end{pmatrix}_b = \begin{pmatrix} -s_3 & s_2 & s_1 \\ p_3 & p_2 & p_1 \end{pmatrix}_b. \quad (4.18)$$

(c) The b -Clebsch-Gordan coefficients are real,

$$\left[\begin{pmatrix} s_3 & s_2 & s_1 \\ p_3 & p_2 & p_1 \end{pmatrix}_b \right]^* = \begin{pmatrix} s_3 & s_2 & s_1 \\ p_3 & p_2 & p_1 \end{pmatrix}_b. \quad (4.19)$$

The proof can be found in Appendix A.1.

The unitarity of the Clebsch-Gordan maps $C_{s_2 s_1}$ is equivalent to the following orthogonality and completeness relations for the Clebsch-Gordan coefficients,

$$\int_{\mathbb{R}^2} dp_2 dp_1 \left[\begin{pmatrix} s_{21} & s_2 & s_1 \\ p_{21} & p_2 & p_1 \end{pmatrix} \right]^* \begin{pmatrix} s'_{21} & s_2 & s_1 \\ p'_{21} & p_2 & p_1 \end{pmatrix} = \delta(p_{21} - p'_{21}) \delta(s_{21} - s'_{21}) \quad (4.20)$$

$$\int_{\mathbb{R}_+} d\mu(s_{21}) \int_{\mathbb{R}} dp_{21} \left[\begin{pmatrix} s_{21} & s_2 & s_1 \\ p_{21} & p_2 & p_1 \end{pmatrix} \right]^* \begin{pmatrix} s_{21} & s_2 & s_1 \\ p'_{21} & p_2 & p'_1 \end{pmatrix} = \delta(p_1 - p'_1) \delta(p_2 - p'_2). \quad (4.21)$$

We finally want to compare our results with those of [PT2]. In this reference the authors constructed Clebsch-Gordan maps $C'^{s_3}_{s_2, s_1} : \mathcal{P}'_{s_2} \otimes \mathcal{P}'_{s_1} \rightarrow \mathcal{P}'_{s_3}$ as integral operators of the form

$$(C'^{s_3}_{s_2, s_1} \psi)(x_3) = \int_{\mathbb{R}^2} dx_1 dx_2 \begin{pmatrix} s_3 & s_2 & s_1 \\ x_3 & x_2 & x_1 \end{pmatrix}_b \psi(x_2, x_1), \quad (4.22)$$

where

$$\begin{aligned} \begin{pmatrix} s_3 & s_2 & s_1 \\ x_3 & x_2 & x_1 \end{pmatrix}_b &= N(s_3, s_2, s_1) D_{-\frac{1}{2}(s_1+s_2+s_3+c_b)} \left(x_2 - x_1 - \frac{s_3+c_b}{2} \right) \\ &\times D_{-\frac{1}{2}(s_2-s_3-s_1+c_b)} \left(x_2 - x_3 - \frac{s_1+c_b}{2} \right) D_{-\frac{1}{2}(s_1-s_3-s_2+c_b)} \left(x_3 - x_1 - \frac{s_2+c_b}{2} \right). \end{aligned} \quad (4.23)$$

The normalization factor will be chosen as

$$N(s_3, s_2, s_1) = \left(\frac{w_b(s_1 + s_2 + s_3) w_b(s_1 + s_2 - s_3)}{w_b(s_1 + s_3 - s_2) w_b(s_2 + s_3 - s_1)} \right)^{\frac{1}{2}}. \quad (4.24)$$

Proposition 2. *We have*

$$C'^{s_3}_{s_2, s_1} = U_{s_3} \cdot C^{s_3}_{s_2, s_1} \cdot (U_{s_2}^{-1} \otimes U_{s_1}^{-1}). \quad (4.25)$$

The proof is given in Appendix A.2.

4.6 The fusion operation

Let us now consider tensor products of three representations. There are two natural ways to construct unitary operators

$$C_{s_3 s_2 s_1} : \mathcal{P}_{s_3} \otimes \mathcal{P}_{s_2} \otimes \mathcal{P}_{s_1} \rightarrow \mathcal{M} \otimes \int^{\oplus} d\mu(s) e_s, \quad (4.26)$$

that satisfy

$$C_{s_3 s_2 s_1} \cdot (\pi_{s_3} \otimes \pi_{s_2} \otimes \pi_{s_1})(X) = (\pi_{\mathcal{M}}(X) \otimes 1) \cdot C_{s_3 s_2 s_1}. \quad (4.27)$$

In (4.26) we used the notation e_s for the one-dimensional module of the algebra of functions $f : \mathbb{S} \rightarrow \mathbb{C}$ with action given as $f \cdot e_s = f(s)e_s$. The variable s represents the multiplicity with which the representation \mathcal{M} appears in the triple tensor product $\mathcal{P}_{s_3} \otimes \mathcal{P}_{s_2} \otimes \mathcal{P}_{s_1}$. Two such operators can be constructed as

$$C_{s_3(s_2 s_1)} := \int^{\oplus} d\mu(s_{21}) C_{s_3 s_{21}} \cdot (1 \otimes C_{s_2 s_1}^{s_{21}}), \quad (4.28)$$

$$C_{(s_3 s_2)s_1} := \int^{\oplus} d\mu(s_{32}) C_{s_3 s_1} \cdot (C_{s_3 s_2}^{s_{32}} \otimes 1). \quad (4.29)$$

The fusion operator $A_{s_3 s_2 s_1} : \int^{\oplus} d\mu(s_{32}) \int^{\oplus} d\mu(s_4) \mathcal{P}_{s_4} \rightarrow \int^{\oplus} d\mu(s_{21}) \int^{\oplus} d\mu(s_4) \mathcal{P}_{s_4}$ is defined as

$$A_{s_3 s_2 s_1} := C_{s_3(s_2 s_1)} \cdot [C_{(s_3 s_2)s_1}]^{\dagger}. \quad (4.30)$$

This operator commutes with π_{s_4} and is therefore of the form

$$A_{s_3 s_2 s_1} = \int^{\oplus} d\mu(s_4) A_{s_3 s_2 s_1}^{s_4}, \quad (4.31)$$

where $A_{s_3 s_2 s_1}^{s_4}$ is a unitary operator $A_{s_3 s_2 s_1}^{s_4} : \int^{\oplus} d\mu(s_{32}) e_{s_{32}} \longrightarrow \int^{\oplus} d\mu(s_{21}) e_{s_{21}} \simeq L^2(\mathbb{S}, d\mu)$.

4.7 The b-j symbols

The b-6j symbols are defined as the matrix elements of the operator $A_{s_3 s_2 s_1}^{s_4}$,

$$\left\{ \begin{matrix} s_1 & s_2 & s_{21} \\ s_3 & s_4 & s_{32} \end{matrix} \right\}_b := \langle s_{21} | A_{s_3 s_2 s_1}^{s_4} | s_{32} \rangle. \quad (4.32)$$

Proposition 2 allows us to use the results from [PT2, TV] for the calculation of these matrix elements. The result is

$$\begin{aligned} \left\{ \begin{matrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \end{matrix} \right\}_b &= \Delta(\alpha_3, \alpha_2, \alpha_1) \Delta(\alpha_5, \alpha_4, \alpha_3) \Delta(\alpha_6, \alpha_4, \alpha_2) \Delta(\alpha_6, \alpha_5, \alpha_1) \\ &\times \int_c du S_b(u - \alpha_{321}) S_b(u - \alpha_{543}) S_b(u - \alpha_{642}) S_b(u - \alpha_{651}) \\ &\times S_b(\alpha_{4321} - u) S_b(\alpha_{6431} - u) S_b(\alpha_{6532} - u) S_b(2Q - u). \end{aligned} \quad (4.33)$$

The expression involves the following ingredients:

- We have used the notations $\alpha_i = \frac{Q}{2} + is_i$, as well as $\alpha_{ijk} = \alpha_i + \alpha_j + \alpha_k$, $\alpha_{ijkl} = \alpha_i + \alpha_j + \alpha_k + \alpha_l$ for $i, j, k, l \in \{1, 2, 3, 4, 5, 6\}$.

- $\Delta(\alpha_3, \alpha_2, \alpha_1)$ is defined as

$$\Delta(\alpha_3, \alpha_2, \alpha_1) = \left(\frac{S_b(\alpha_1 + \alpha_2 + \alpha_s - Q)}{S_b(\alpha_1 + \alpha_2 - \alpha_s)S_b(\alpha_1 + \alpha_s - \alpha_2)S_b(\alpha_2 + \alpha_s - \alpha_1)} \right)^{\frac{1}{2}}.$$

- The integral is defined in the cases that $\alpha_k \in Q/2 + i\mathbb{R}$ by a contour \mathcal{C} which approaches $2Q + i\mathbb{R}$ near infinity, and passes the real axis in the interval $(3Q/2, 2Q)$. For other values of the variables α_k it is defined by analytic continuation.

5. Quantum Teichmüller theory

This section presents the definitions and results from the quantum Teichmüller theory that will be needed in this paper. We will use the formulation introduced by R. Kashaev [Ka1], see also [T05] for a more detailed exposition and a discussion of its relation to the framework of Fock [F97] and Chekhov and Fock [CF1]. The formulation from [Ka1] starts from the quantization of a somewhat enlarged space $\hat{\mathcal{T}}(C)$. The usual Teichmüller space $\mathcal{T}(C)$ can then be characterized as subspace of $\hat{\mathcal{T}}(C)$ using certain linear constraints. This is motivated by the observation that the spaces $\hat{\mathcal{T}}(C)$ have natural polarizations, which is not obvious in the formulation of [F97, CF1].

5.1 Algebra of operators and its representations

For a given surface C with constant negative curvature metric and at least one puncture one considers ideal triangulations τ . Such ideal triangulations are defined by maximal collection of non-intersecting open geodesics which start and end at the punctures of C . We will assume that the triangulations are decorated, which means that a distinguished corner is chosen in each triangle.

We will find it convenient to parameterize triangulations by their dual graphs which are called fat graphs φ_τ . The vertices of φ_τ are in one-to-one correspondence with the triangles of τ , and the edges of φ_τ are in one-to-one correspondence with the edges of τ . The relation between a triangle t in τ and the fat graph φ_τ is depicted in Figure 1. φ_τ inherits a natural decoration of its vertices from τ , as is also indicated in Figure 1.

The quantum theory associated to the Teichmüller space $\mathcal{T}(C)$ is defined on the kinematical level by associating to each vertex $v \in \varphi_0$, $\varphi_0 = \{\text{vertices of } \varphi\}$, of φ a pair of generators p_v, q_v which are supposed to satisfy the relations

$$[p_v, q_{v'}] = \frac{\delta_{vv'}}{2\pi i}. \quad (5.1)$$

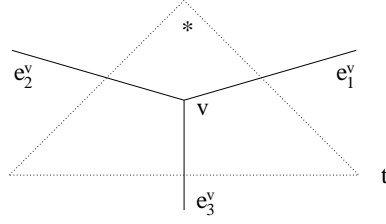


Figure 1: *Graphical representation of the vertex v dual to a triangle t . The marked corner defines a corresponding numbering of the edges that emanate at v .*

There is a natural representation of this algebra on the Schwarz space $\hat{\mathcal{S}}(C)$ of rapidly decaying smooth functions $\psi(q)$, $q : \varphi_0 \ni v \rightarrow q_v$, generated from $\pi_\varphi(q_v) := \mathbf{q}_v$, $\pi_\varphi(p_v) := \mathbf{p}_v$, where

$$\mathbf{q}_v \psi(q) := q_v \psi(q), \quad \mathbf{p}_v \psi(q) := \frac{1}{2\pi i} \frac{\partial}{\partial q_v} \psi(q). \quad (5.2)$$

For each surface C we have thereby defined an algebra $\hat{\mathcal{A}}(C)$ together with a family of representations π_φ of $\hat{\mathcal{A}}(C)$ on the Schwarz spaces $\hat{\mathcal{S}}_\varphi(C)$ which are dense subspaces of the Hilbert space $\mathcal{K}(\varphi) \simeq L^2(\mathbb{R}^{4g-4+2n})$. The next step is to show that the choice of fat graph φ is inessential by constructing unitary operators $\pi_{\varphi_2 \varphi_1} : \mathcal{K}(\varphi_1) \rightarrow \mathcal{K}(\varphi_2)$ intertwining the representations π_{φ_1} and π_{φ_2} .

5.2 The projective representation of the Ptolemy groupoid on $\mathcal{K}(\varphi)$

The groupoid generated by the changes from one fat graph to another is called the Ptolemy groupoid. It can be described in terms of generators and relations, see e.g. [T05, Section 3] for a summary of the relevant results and further references.

Following [Ka3] closely we shall define a projective unitary representation of the Ptolemy groupoid in terms of the following set of unitary operators

$$\begin{aligned} \mathbf{A}_v &\equiv e^{\frac{\pi i}{3}} e^{-\pi i(\mathbf{p}_v + \mathbf{q}_v)^2} e^{-3\pi i \mathbf{q}_v^2} \\ \mathbf{T}_{vw} &\equiv e_b(\mathbf{q}_v + \mathbf{p}_w - \mathbf{q}_w) e^{-2\pi i \mathbf{p}_v \mathbf{q}_w}, \end{aligned} \quad \text{where } v, w \in \varphi_\circ. \quad (5.3)$$

The special function $e_b(U)$ can be defined in the strip $|\Im z| < |\Im c_b|$, $c_b \equiv i(b + b^{-1})/2$ by means of the integral representation

$$\log e_b(z) \equiv \frac{1}{4} \int_{i0-\infty}^{i0+\infty} \frac{dw}{w} \frac{e^{-2izw}}{\sinh(bw) \sinh(b^{-1}w)}. \quad (5.4)$$

These operators are unitary for $(1 - |b|)\Im b = 0$. They satisfy the following relations [Ka3]

$$(i) \quad T_{vw} T_{uw} T_{uv} = T_{uv} T_{vw}, \quad (5.5a)$$

$$(ii) \quad A_v T_{uv} A_u = A_u T_{vu} A_v, \quad (5.5b)$$

$$(iii) \quad T_{vu} A_u T_{uv} = \zeta A_u A_v P_{uv}, \quad (5.5c)$$

$$(iv) \quad A_u^3 = \text{id}, \quad (5.5d)$$

where $\zeta = e^{\pi i c_b^2/3}$, $c_b \equiv \frac{i}{2}(b + b^{-1})$. The relations (5.5a) to (5.5d) allow us to define a projective representation of the Ptolemy groupoid as follows.

- Assume that $\omega_{uv} \in [\varphi', \varphi]$. To ω_{uv} let us associate the operator

$$u(\omega_{uv}) \equiv T_{uv} : \mathcal{K}(\varphi) \ni \mathfrak{v} \rightarrow T_{uv} \mathfrak{v} \in \mathcal{K}(\varphi').$$

- For each fat graph φ and vertices $u, v \in \varphi_0$ let us define the following operators

$$A_u^\varphi : \mathcal{K}(\varphi) \ni \mathfrak{v} \rightarrow A_u \mathfrak{v} \in \mathcal{K}(\rho_u \circ \varphi).$$

$$P_{uv}^\varphi : \mathcal{K}(\varphi) \ni \mathfrak{v} \rightarrow P_{uv} \mathfrak{v} \in \mathcal{K}((uv) \circ \varphi).$$

It follows immediately from (5.5a)-(5.5d) that the operators T_{uv} , A_u and P_{uv} can be used to generate a unitary projective representation of the Ptolemy groupoid.

The corresponding automorphisms of the algebra $\mathcal{A}(C)$ are

$$a_{\varphi_2 \varphi_1}(\mathcal{O}) := \text{ad}[U_{\varphi_2 \varphi_1}](\mathcal{O}) := U_{\varphi_2 \varphi_1} \cdot \mathcal{O} \cdot U_{\varphi_2 \varphi_1}. \quad (5.6)$$

The automorphism $a_{\varphi_2 \varphi_1}$ generate the canonical quantization of the changes of coordinates for $\hat{\mathcal{T}}(C)$ from one fat graph to another [Ka1].

5.3 The reduction to the Teichmüller spaces

Recall that the quantum theory defined in this way is not quite the one we are interested in. It is the quantum theory of an enlarged space $\hat{\mathcal{T}}(C)$ which is the product of the Teichmüller space with the first homology of C , both considered as real vector spaces [Ka1, T05]. The embedding of the Teichmüller space $\mathcal{T}(C)$ into $\hat{\mathcal{T}}(C)$ can be described classically in terms of a certain set of constraints $z_c = 0$ which characterize the locus of $\mathcal{T}(C)$ within $\hat{\mathcal{T}}(C)$.

To define the quantum representatives of the constraints let us introduce an embedding of the first homology $H_1(\Sigma, \mathbb{R})$ into $\hat{\mathcal{T}}(C)$ as follows. Each graph geodesic g_γ which represents an element $\gamma \in H_1(\Sigma, \mathbb{R})$ may be described by an ordered sequence of vertices $v_i \in \varphi_0$, and edges $e_i \in \varphi_1$, $i = 0, \dots, n$, where $v_0 = v_n$, $e_0 = e_n$, and we assume that v_{i-1} , v_i are connected

by the single edge e_i . We will define $\omega_i = 1$ if the arcs connecting e_i and e_{i+1} turn around the vertex v_i in the counterclockwise sense, $\omega_i = -1$ otherwise. The edges emanating from v_i will be numbered e_j^i , $j = 1, 2, 3$ according to the convention introduced in Figure 1. To each $c \in H_1(\Sigma, \mathbb{R})$ we will assign

$$z_c \equiv \sum_{i=1}^n u_i, \quad u_i := \omega_i \begin{cases} -q_{v_i} & \text{if } \{e_i, e_{i+1}\} = \{e_3^i, e_1^i\}, \\ p_{v_i} & \text{if } \{e_i, e_{i+1}\} = \{e_2^i, e_3^i\}, \\ q_{v_i} - p_{v_i} & \text{if } \{e_i, e_{i+1}\} = \{e_1^i, e_2^i\}. \end{cases} \quad (5.7)$$

Let C_φ be the subspace in $\hat{\mathcal{T}}(C)$ that is spanned by the z_c , $c \in H_1(\Sigma, \mathbb{R})$.

Lemma 2. — [Kal] *The mapping $H_1(\Sigma, \mathbb{R}) \ni c \mapsto z_c \in C_\varphi$ is an isomorphism of Poisson vector spaces.*

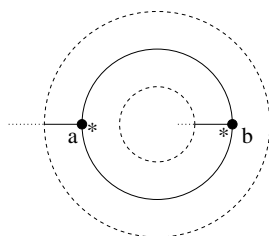
Replacing q_v by q_v and p_v by p_v in the definition above gives the definition of the operators $z_c \equiv z_{\varphi, c}$ which represent the constraints in the quantum theory. Let us note that the constraints transform under a change of fat graph as $a_{\varphi_2 \varphi_1}(z_{\varphi_1, c}) = z_{\varphi_2, c}$.

5.4 Length operators

A particularly important class of coordinate functions on the Teichmüller spaces are the geodesic length functions. The quantization of these observables was studied in [CF1, CF2, T05].

Such length operators can be constructed in general as follows [T05]. We will first define the length operators for a case in which the choice of fat graph φ simplifies the representation of the curve c . We then explain how to generalize the definition to all other cases.

Let A_c be an annulus embedded in the surface C containing the curve c , and let φ be a fat graph which looks inside of A_c as depicted in Figure 5.8.



Annulus A_c : Region bounded by the two dashed circles, and part of φ_σ contained in A_c . (5.8)

Let

$$L_{\varphi, c} := 2 \cosh 2\pi b p_c + e^{-2\pi b q_c}, \quad (5.9)$$

where $p_c := \frac{1}{2}(p_a - q_a - p_b)$, $q_c := \frac{1}{2}(q_a + p_a + p_b - 2q_b)$.

In all remaining cases we will define the length operator $L_{\varphi,c}$ as follows: There always exists a fat graph φ_0 for which the definition above can be used to define $L_{\varphi_0,c}$. Let then

$$L_{\varphi,c} := a_{\varphi,\varphi_0}(L_{\varphi_0,c}). \quad (5.10)$$

It can be shown that the length operators $L_{\varphi,c}$ are unambiguously defined in this way [T05].

The length operators satisfy the following properties:

- (a) **Spectrum:** $L_{\varphi,c}$ is self-adjoint. The spectrum of $L_{\varphi,c}$ is simple and equal to $[2, \infty)$. This is necessary and sufficient for the existence of an operator $l_{\varphi,c}$ - the *geodesic length operator* - such that $L_{\varphi,c} = 2 \cosh \frac{1}{2} l_{\varphi,c}$.

- (b) **Commutativity:**

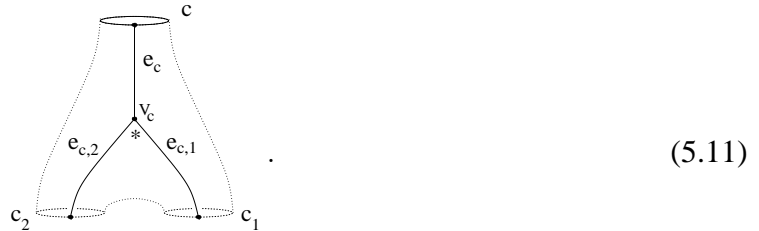
$$[L_{\varphi,c}, L_{\varphi,c'}] = 0 \quad \text{if } c \cap c' = \emptyset.$$

- (c) **Mapping class group invariance:**

$$a_{\mu}(L_{\varphi,c}) = L_{\mu.\varphi,c}, \quad a_{\mu} \equiv a_{[\mu.\varphi,\varphi]}, \quad \text{for all } \mu \in \text{MC}(\Sigma).$$

It can furthermore be shown that this definition reproduces the classical geodesic length functions on $\mathcal{T}(C)$ in the classical limit.

As an example for the use of (5.10) that will be important for the following let us assume that the curve c is the boundary component of a trinion P_c embedded in C within which the fat graph φ' looks as follows:



Let c_{ϵ} , $\epsilon = 1, 2$ be the curves which represent the other boundary components of P_c as indicated in Figure 5.11.

Proposition 3. L_c is given by

$$L_{\varphi',c} = 2 \cosh(y_c^2 + y_c^1) + e^{-y_c^2} L_{c_1} + e^{y_c^1} L_{c_2} + e^{y_c^1 - y_c^2}, \quad (5.12)$$

where y_c^{ϵ} , $\epsilon = 1, 2$ are defined as $y_c^2 = 2\pi b(q_c + z_{c_2})$, $y_c^1 = -2\pi b(p_c - z_{c_1})$.

The proof of (5.12) can be found in Appendix B.

5.5 The annulus

As a basic building block let us develop the quantum Teichmüller theory of an annulus in some detail. To the simple closed curve c that can be embedded into A we associate

- the constraint

$$z := \frac{1}{2}(p_a - q_a + p_b), \tag{5.13}$$

- the length operator L is defined as in (5.9).

The operator L is positive-self-adjoint, and its spectral decomposition [Ka4] was recalled in the above.

For later use let us construct the change of representation from the representation in which p_a and p_b are diagonal to a representation where z and L are diagonal. To this aim let us introduce $d := \frac{1}{2}(q_a + p_a - p_b + 2q_b)$. We have

$$\begin{aligned} [z, d] &= (2\pi i)^{-1}, & [z, p] &= 0, & [z, q] &= 0, \\ [p, q] &= (2\pi i)^{-1}, & [d, p] &= 0, & [d, q] &= 0. \end{aligned}$$

Let $\langle p, z |$ be an eigenvector of p and z with eigenvalues p and z , respectively, and $|p_a, p_b\rangle$ an eigenvector of p_a and p_b with eigenvalues p_a and p_b , respectively. It follows easily that

$$\langle p, z | p_a, p_b \rangle = \delta(p_b - z + p) e^{\pi i(p+z-p_a)^2}. \tag{5.14}$$

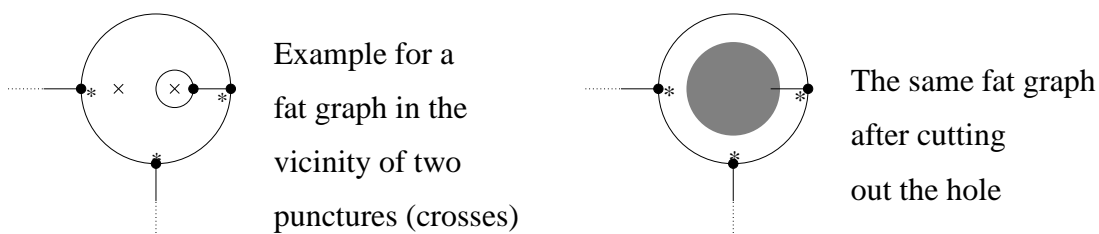
The transformation

$$\psi(s, z) = \int_{\mathbb{R}^2} dp dp_a \frac{w_b(s - p + c_b - i0)}{w_b(s + p - c_b + i0)} e^{\pi i(p+z-p_a)^2} \Psi(p_a, z - p), \tag{5.15}$$

will then map a wave function $\Psi(p_a, p_b)$ in the representation which diagonalizes p_a, p_b to the corresponding wave function $\psi(s, z)$ in the representation which diagonalizes L and z .

5.6 Teichmüller theory for surfaces with holes

The formulation of quantum Teichmüller theory introduced above has only punctures (holes with vanishing geodesic circumference) as boundary components. In order to generalize to holes of non-vanishing geodesic circumference one may represent each hole as the result of cutting along a geodesic surrounding a pair of punctures.



On a surface C with n holes one may choose φ to have the following simple standard form near at most $n - 1$ of the holes:



The price to pay is a fairly complicated representation of the closed curves which surround the remaining holes.

The simple form of the fat graph near the incoming boundary components allows us to use the transformation (5.15) to pass to a representation where the length operators and constraints associated to these holes are diagonal. In order to describe the resulting hybrid representation let us denote by s_b and z_b the assignments of values s_h and z_h to each incoming hole h , while p assigns real numbers p_v to all vertices v of φ which do not coincide with any vertex \hat{h} or h' associated to an incoming hole h . The states will then be described by wave-functions $\psi(p; s_b, z_b)$ on which the operators L_h and z_h act as operators of multiplication by $2 \cosh 2\pi b s_h$ and z_h , respectively.

For a given hole h one may define a projection $\mathcal{H}(C_{h(s,z)})$ of $\mathcal{H}(C)$ to the eigenspace with fixed eigenvalues $2 \cosh 2\pi b s$ and c of L_h and z_h . States in $\mathcal{H}(C_{h(s,z)})$ are represented by wave-functions $\psi_h(p_h)$, where p_h assigns real values to all vertices in $\varphi_0 \setminus \{\hat{h}, h'\}$. The mapping class action on $\mathcal{H}(C)$ commutes with L_h and z_h . It follows that the operators $M(\mu)$ representing the mapping class group action on $\mathcal{H}(C)$ project to operators $M_{s,z}(\mu)$ generating an action of $\text{MCG}(C)$ on $\mathcal{H}(C_{h(s,z)})$.

5.7 The cutting operation

Cutting C along the curve c embedded in an annulus as considered above will produce two surfaces C'' and C' with boundary containing copies of the curve c . We may regard C'' and C' as subsurfaces of C . The mapping class groups $\text{MCG}(C'')$ and $\text{MCG}(C')$ thereby get embedded as subgroups into $\text{MCG}(C)$. The images of $\text{MCG}(C'')$ and $\text{MCG}(C')$ are generated by the Dehn twist along c together with diffeomorphisms of C'' and C' which act trivially on A , respectively.

The spectral decomposition of L_c and z_c defines a natural counterpart of the operation to cut C into C'' and C' within the quantum Teichmüller theory. It produces an isomorphism

$$S_c : \mathcal{H}(C) \mapsto \int_{\mathbb{R}_+}^{\oplus} ds \int_{\mathbb{R}}^{\oplus} dc \mathcal{H}(C''_{h''(s,z)}) \otimes \mathcal{H}(C'_{h'(s,z)}). \quad (5.17)$$

The explicit form of the operator S_c is easily found with the help of the integral transformation (5.15). To this aim it is sufficient to split the set φ_0 of vertices of φ as $\varphi_0 = \varphi''_0 \cup \{a, b\} \cup \varphi'_0$,

where a and b are the vertices lying inside A , and the set φ'_0 contains the vertices in $\varphi_0 \setminus \{a, b\}$ located in C' . Writing accordingly $\Psi(p) = \Psi(p'', p_a, p_b, p')$, with $p'' : \varphi''_0 \mapsto \mathbb{R}$ and $p' : \varphi'_0 \mapsto \mathbb{R}$, we may use the integral transformation (5.15) to map $\Psi(p) = \Psi(p'', p_a, p_b, p')$ to a function $\psi(p'', s, z, p')$ which represents an element of the Hilbert space on the right of (5.17).

6. Relation between the modular double and quantum Teichmüller theory

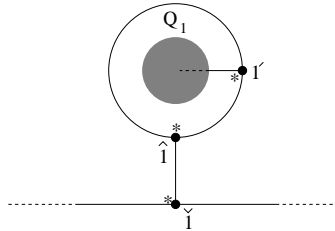
We are now ready to address our main aim. Recall that the modular double is characterized by the following main objects: The operators C_{s_2, s_1} which generate the co-product, and the R-operator R . We are going to show that these operators have very natural interpretations in within the quantum Teichmüller theory.

6.1 The hole algebra

Recall that the representation $\pi_{\mathcal{M}}$ of the modular double $\mathcal{DU}_q(\mathfrak{sl}_2)$ has positive self-adjoint generators E, K, F . It will again be convenient to replace the generator F by the Casimir Q

$$F = (q - q^{-1})^{-2} (Q + qK^2 + q^{-1}K^{-2}) E^{-1}. \quad (6.1)$$

We will identify the representation $\pi_{\mathcal{M}}$ of the algebra $\mathcal{DU}_q(\mathfrak{sl}_2)$ with the hole algebra which is associated to the following subgraph of a fat graph σ :

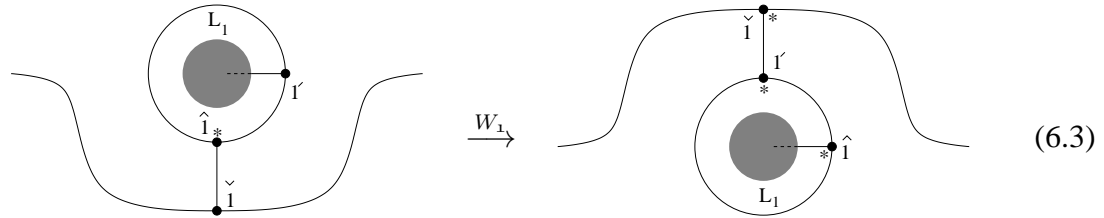


The identification is such that

$$\begin{aligned} E &\mapsto e^{\pi b(2q_1 - p_1)}, \\ K &\mapsto e^{-\pi b p_1}, \end{aligned} \quad Q \mapsto L. \quad (6.2)$$

We furthermore note that local changes of the fat graph are naturally mapped to unitary equivalence transformations of the representation $\pi_{\mathcal{M}}$. A particularly important one is the equivalence transformation corresponding to the automorphism w . We have

Proposition 4. *The automorphism w coincides with the automorphism associated to the following move W_1*

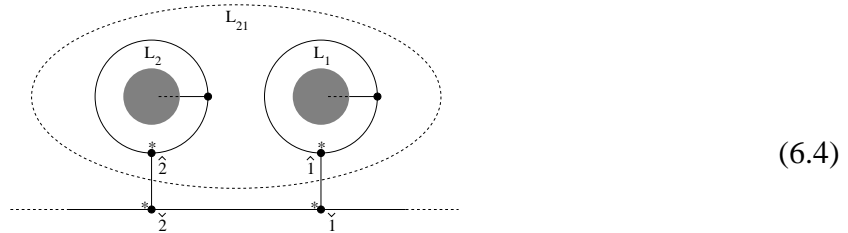


after setting the constraint z_1 to zero.

The proof is given in Appendix B.

6.2 Tensor products of representations

It is clearly natural to identify the tensor product of two representations with the following subgraph



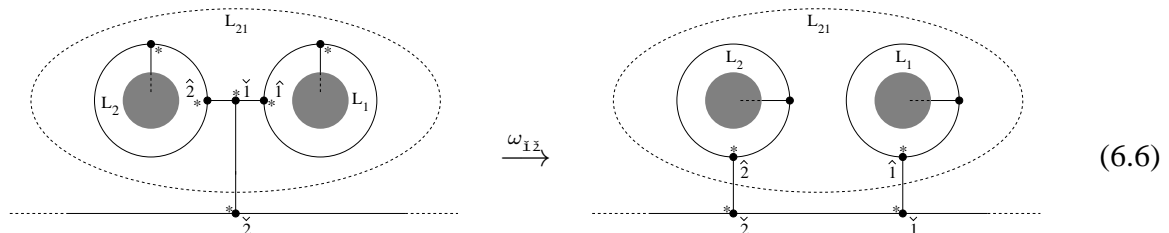
Let L_{21} be the operator which represents the geodesic length observable in the representation corresponding to the fat graph above.

The key observation to be made is formulated in the following proposition:

Proposition 5. *The projection of the length operator L_{21} onto the subspace of vanishing constraints becomes equal to the Casimir Q_{21} ,*

$$L_{21} \mapsto Q_{21} . \tag{6.5}$$

Proof. In order to calculate the explicit form of the length operator in the representation associated to the fat graph (6.4), we may take Proposition 3 as a starting point. It remains to calculate the change of representation induced by the move $\omega_{\hat{1}\hat{2}}$ which is diagrammatically represented as



This move is represented by the operator $T_{\bar{1}\bar{2}}$. This calculation is obtained from the one described above in (4.15) by simple substitutions, resulting in the expression

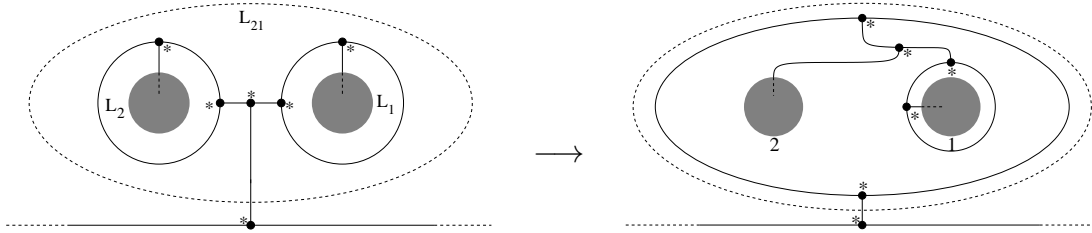
$$\begin{aligned} L_{21} = & e^{+2\pi b z_1} : e^{2\pi b(q_1 - q_2 + p_2 - p_1)} (2 \cosh 2\pi b(p_2 - z_2) + L_2) : \\ & + e^{-2\pi b z_2} : e^{-\pi b(q_1 - q_2)} (2 \cosh 2\pi b(p_1 - z_1) + L_1) : \\ & + e^{2\pi b(p_2 - z_2)} L_1 + e^{-2\pi b(p_1 - z_1)} L_2 + 2 \cos \pi b^2 e^{2\pi b(p_2 - z_2 - p_1 + z_1)}. \end{aligned} \quad (6.7)$$

Setting the constraints to zero and comparing with (3.11) yields the claimed result. \square

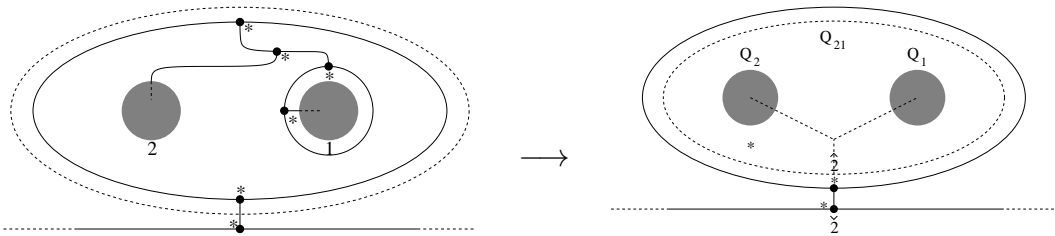
6.3 The Clebsch-Gordan maps

Note that the operator Q'_1 defined in (4.8) essentially coincides with the particular representation of a length operator given in (5.12) after setting the constraints to zero. It follows immediately from this observation that one may without loss of generality assume that the projection of the operator C_v^T defined in (7.2) to $z_v^i = 0$, $i = 1, 2$ coincides with the operator $\nu_{s_2 s_1}^{s_{21}} \cdot S_1 \cdot C_1$ which appears as a building block in the construction of the Clebsch-Gordan maps $C_{s_2 s_1}$ given in equation (4.6). More precisely:

- The operator C_1 corresponds to the following move:



- The operator S_1 corresponds to the cutting operation:



It remains to notice that the operator $(T_{12})^{-1}$ which appears in the factorized representation of the full Clebsch-Gordan maps,

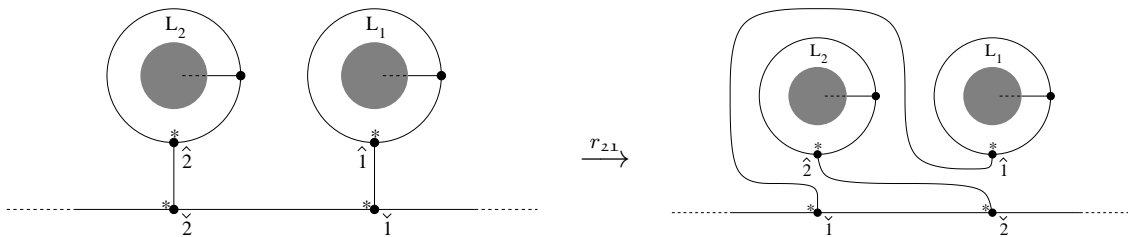
$$C_{s_2 s_1} := S_1 \cdot C_1 \cdot (T_{12})^{-1}, \quad (6.8)$$

corresponds to the move $\omega_{\bar{1}\bar{2}}$ depicted in (6.6).

These observations may be summarized by saying that the Clebsch-Gordan maps of the modular double represent the quantum cutting operation associated to the curve c_{21} surrounding holes h_2 and h_1 within the hybrid representation assigned to the graph in (6.4).

6.4 The R-operator

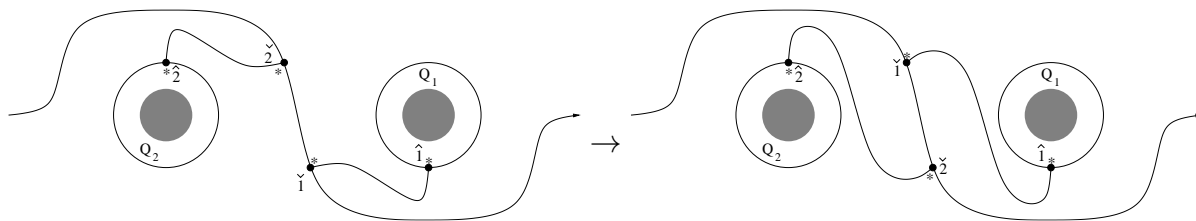
Let us consider the following move r_{21}



When this move is composed with the operation of exchanging all indices 1 and 2 one gets the braid move representing the clockwise rotation of holes 2 and 1 around each other until the positions have been exchanged. The operator r_{21} which represents this move within the quantum Teichmüller theory is easily found to be

$$r_{21} = W_2^{-1} \cdot A_{\hat{z}} T_{\hat{1}\hat{2}}^{-1} A_{\hat{z}}^{-1} \cdot W_2. \tag{6.9}$$

This is easily seen by noting that the operator $A_{\hat{z}}(T_{\hat{1}\hat{2}})^{-1}A_{\hat{z}}^{-1}$ represents the following move:



Proposition 6. *The operator r_{21} gets mapped to the R-operator R.*

Proof. We have

$$A_{\hat{z}} T_{\hat{1}\hat{2}}^{-1} A_{\hat{z}}^{-1} = e^{-\pi i p_1 p_2} e_b \left(q_{\hat{1}} - \frac{1}{2} p_{\hat{1}} + q_{\hat{2}} - \frac{1}{2} p_{\hat{2}} \right) e^{-\pi i p_1 p_2}.$$

This is identified as the operator

$$q^{-h \otimes h} E_b(e \otimes e) q^{-h \otimes h}.$$

Using $\text{ad}[W_2](e) = f$ yields R, as claimed. □

7. A representation of the Moore-Seiberg groupoid in genus 0

We will now present an important application of the results above. It was shown in [T05] that the quantum Teichmüller theory defines a representation of the Moore-Seiberg groupoid, which is important for understanding relations to conformal field theory. The results of this paper will allow us to calculate the operators which generate the representation of the Moore-Seiberg groupoid explicitly.

7.1 Pants decompositions

Let us consider hyperbolic surfaces C of genus 0 with n holes. We will assume that the holes are represented by geodesics in the hyperbolic metric. A pants decomposition of a hyperbolic surface C is defined by a cut system which in this context may be represented by a collection $\mathcal{C} = \{\gamma_1, \dots, \gamma_{n-3}\}$ of non-intersecting simple closed geodesics on C . The complement $C \setminus \mathcal{C}$ is a disjoint union $\bigsqcup_v C_{0,3}^v$ of three-holed spheres (trinions). One may reconstruct C from the resulting collection of trinions by pairwise gluing of boundary components.

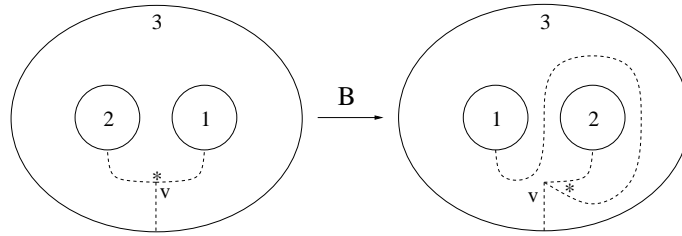
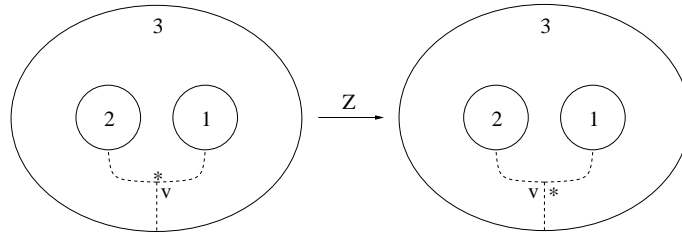
For given lengths of the three boundary geodesics there is a unique hyperbolic metric on each trinion $C_{0,3}^v$. Introducing a numbering of the boundary geodesics $\gamma_i(v)$, $i = 1, 2, 3$, one gets three distinguished geodesic arcs $\gamma_{ij}(v)$, $i, j = 1, 2, 3$ which connect the boundary components pairwise. Up to homotopy there are exactly two tri-valent graphs Γ_{\pm}^v on $C_{0,3}^v$ that do not intersect any $\gamma_{ij}(v)$. We may assume that these graphs glue to two connected graphs Γ_{\pm} on C . The pair of data $\sigma = (C_{\sigma}, \Gamma_{\sigma})$, where Γ_{σ} is one of the MS graphs Γ_{\pm} associated to a hyperbolic pants decomposition, can be used to distinguish different pants decompositions in hyperbolic geometry. The role of the graph Γ_{σ} is to distinguish pants decompositions obtained from each other by means of Dehn twists, rotations of one boundary component by 2π before gluing.

7.2 The Moore-Seiberg groupoid

Let us note [MS, BK] that any two different pants decompositions σ_2, σ_1 can be connected by a sequence of elementary moves localized in subsurfaces of $C_{g,n}$ of type $C_{0,3}, C_{0,4}$. These will be called the B, S and F , respectively. Graphical representations for the elementary moves B, Z , and F are given in Figures 2, 3 and 4, respectively.

One may formalize the resulting structure by introducing a two-dimensional CW complex $\mathcal{M}(C)$ with set of vertices $\mathcal{M}_0(C)$ given by the pants decompositions σ , and a set of edges $\mathcal{M}_1(C)$ associated to the elementary moves.

The Moore-Seiberg groupoid is defined to be the path groupoid of $\mathcal{M}(C)$. It can be described in terms of generators and relations, the generators being associated with the edges in $\mathcal{M}_1(C)$,

Figure 2: The move $B_v : \sigma \rightarrow B_v\sigma$ Figure 3: The move $Z_v : \sigma \rightarrow Z_v\sigma$

and the relations associated with the faces of $\mathcal{M}(C)$. The classification of the relations was first presented in [MS], and rigorous mathematical proofs have been presented in [FG, BK]. The relations are all represented by sequences of moves localized in subsurfaces $C_{g,n}$ with genus $g = 0$ and $n = 3, 4, 5$ punctures. Graphical representations of the relations can be found in [MS, FG, BK].

7.3 Representation of the Moore-Seiberg groupoid

A representation of the Moore-Seiberg groupoid can be obtained from the quantum Teichmüller theory as follows [T05].

7.3.1

The starting point is a construction which produces a fat graph φ_σ associated to pants decompositions σ . This construction depends on a choice of decoration for a pants decomposition σ which is the choice of a distinguished boundary component for each trinion. The distinguished boundary component will be called outgoing, the other boundary component incoming. The decoration is indicated by an asterisk in the Figures 2, 3 and 4. We identify the Z-move as the elementary change of decorations. In the following we will use the notation σ for *decorated* pants decompositions.

The construction described in [T05] can be applied for a subset of decorated pants decompositions which is defined by the condition that outgoing boundary components are never glued to another. Such pants decompositions will be called admissible.

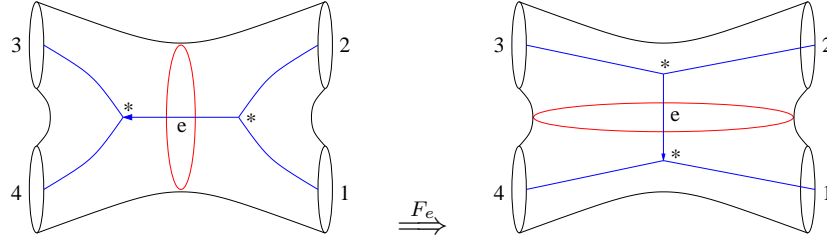


Figure 4: The move $F_e : \sigma_s \rightarrow \sigma_t \equiv F_e \sigma_s$

There is a natural fat graph φ_σ associated to σ which is defined by gluing the following pieces:

- For each curve c separating two incoming boundary component let us insert an annulus A_c , with fat graph locally of the form depicted in Figure (5.8).
- Trinions: See Figure (5.11).
- Holes: See Figure (5.16).

Gluing these pieces in the obvious way will produce the connected graph φ_σ associated to the Moore-Seiberg graph σ we started from.

7.3.2

Following [T05], we will now describe how to map a maximal commuting family of length operators to diagonal form. We will start from the hybrid representation described above in which the length operators and constraints associated to the incoming holes are diagonal. Recall that states are represented by wave-functions $\psi(p; s_b, z_b)$ in such a representation, where $p : \tilde{\varphi}_0 \mapsto \mathbb{R}$, and $\tilde{\varphi}_0$ is the subset of φ_0 that does not contain \hat{h} nor h' for any incoming hole h . A maximal commuting family of length operators is associated to the cut system \mathcal{C}_σ of a pants decomposition.

To each vertex v of Γ_σ assign the length operator L_v^2 and L_v^1 to the incoming and L_v to the outgoing boundary components of the pair of pants P_v containing v . The main tool is the operator C_v^T which maps L_v to a simple standard form,

$$C_v^T \cdot L_v \cdot (C_v^T)^{-1} = 2 \cosh 2\pi b p_v + e^{-2\pi b q_v}. \quad (7.1)$$

Such an operator can be constructed explicitly as

$$C_v^T(s_v^2, s_v^1) := e^{-2\pi i s_2 q_v} \frac{e_b(s_v^1 + p_v)}{e_b(s_v^1 - p_v)} e^{-2\pi i s_v^1 p_v} (e_b(q_v - s_v^2))^{-1} e^{-2\pi i (z_v^2 p_v + z_v^1 q_v)}, \quad (7.2)$$

where s_v^ι , $\iota = 1, 2$ are the positive self-adjoint operators defined by $L_v^\iota = 2 \cosh 2\pi b s_v^\iota$, and z_v^2 , z_v^1 are the constraints associated to the incoming boundary components of P_v . The operator C_v

is clearly related to the operator C_1 that appeared as a key ingredient of the Clebsch-Gordan maps for the modular double in the previous part.

The map to the length representation is then constructed as the ordered product over the operators C_v , $v \in \varphi_{\sigma,0}$. The resulting operator may be represented as the following explicit integral transformation: Let s be the assignment $v : \tilde{\varphi}_0 \mapsto \mathbb{R}_+$. Define

$$\Phi(s, z_b) = \int_{\mathbb{R}^{n-3}} \left(\prod_{v \in \tilde{\varphi}_0} dp_v K_{s_b^2 s_b^1}^{z_v^2 z_v^1}(s_v, p_v) \right) \psi(p; s_b, z_b). \quad (7.3)$$

The kernel $K_{s_2 s_1}^{z_2 z_1}(s, p)$ has the explicit form

$$\begin{aligned} K_{s_2 s_1}^{z_2 z_1}(s, p) &= \langle s | C^T | p \rangle \quad (7.4) \\ &= \langle s | e^{-2\pi i s_2 \mathbf{q}} \frac{e_b(s_1 + \mathbf{p})}{e_b(s_1 - \mathbf{p})} e^{-2\pi i s_1 \mathbf{p}} (e_b(\mathbf{q} - s_2))^{-1} e^{-2\pi i(z_2 \mathbf{p} + z_1 \mathbf{q})} | p \rangle \\ &= \int_{\mathbb{R}} dp' \langle s | p' \rangle \frac{w_b(s_1 - p' - s_2)}{w_b(s_1 + p' + s_2)} \langle p' | e^{-2\pi i s_2 \mathbf{q}_1} (e_b(\mathbf{q} - s_2))^{-1} e^{-2\pi i(z_2 \mathbf{p} + z_1 \mathbf{q})} | p \rangle \\ &= \zeta_0 \int_{\mathbb{R}} dp' e^{-2\pi i(s_2 - z_b)(s_2 + p' - p + z_1)} e_b(p - z_1 - s_2 - p' + z_b) \\ &\quad \times \frac{w_b(s_1 - p' - s_2)}{w_b(s_1 + p' + s_2)} \frac{w_b(s + p' - z_b)}{w_b(s - p' + z_b)} e^{-2\pi i z_2(2p - z_1)}. \end{aligned}$$

In the last step we have used the complex conjugate of equation (A.11) in Appendix A below.

7.3.3

The construction above canonically defines operators $U_{\sigma_2 \sigma_1}$ intertwining between the representations π_{σ_1} and π_{σ_2} as

$$U_{\sigma_2 \sigma_1} := C_{\sigma_2} \cdot W_{\varphi_{\sigma_2} \varphi_{\sigma_1}} \cdot C_{\sigma_1}^{-1}, \quad (7.5)$$

where $W_{\varphi_{\sigma_2} \varphi_{\sigma_1}}$ is any operator representing the move $[\varphi_{\sigma_2}, \varphi_{\sigma_1}]$ between the fat graph associated to σ_1 and σ_2 , respectively. In this way one defines operators \hat{B}_v , \hat{A}_e , and \hat{Z}_v associated to the elementary moves B_v , F_e and Z_v between different MS-graphs, respectively. These operators satisfy operatorial versions of the Moore-Seiberg consistency conditions [T05], which follow immediately from the relations of the Ptolemy groupoid using (7.5).

One should note that the definition (7.5) can be applied only if the decorated pants decompositions σ_2 and σ_1 are both admissible. However, this restriction will quickly turn out to be inessential. To begin with, let us note that the definition (7.5) can indeed be applied to all operators that appear in the relations of the Moore-Seiberg groupoid. A quick inspection of the relations listed in [BK, T05] shows that all the decorated pants decompositions appearing therein are admissible.

The wave-function $\phi(s) := \Phi(s, z_b)|_{z_b=0}$ represents the projection of the wave-function Φ to the subspace defined by vanishing constraints $z_{\varphi, c}$. The operators $U_{\sigma_2 \sigma_1}$ commute with the constraints $z_{\varphi, c}$, in the sense that

$$U_{\sigma_2 \sigma_1} \cdot z_{\varphi_{\sigma_1}, c} = z_{\varphi_{\sigma_2}, c} \cdot U_{\sigma_2 \sigma_1}. \quad (7.6)$$

The projections of the operators \hat{B}_v , \hat{A}_e , and \hat{Z}_v define operators B_v , A_e , and Z_v which satisfy the relations of the Moore-Seiberg groupoid up to possible projective phases.

7.3.4

The representation of the Moore-Seiberg groupoid defined via (7.5) has nice locality properties in the sense that the operator representing a move localized in a subsurface C' of C will only act on the variables s_e associated to the edges e of Γ_σ that have nontrivial intersection with C' . In order to make this precise and easily visible in the notations, let us introduce the one-dimensional Hilbert space $\mathcal{H}_{s_2 s_1}^{s_3}$ associated to a three-holed sphere $C_{0,3}$ with parameters s_i , $i = 1, 2, 3$ associated to the boundary components according the numbering convention indicated on the left of Figure 2. Note that edges e of the MS graph determine curves c_e in the cut system. The eigenvalues L_e of the operators L_{c_e} will be parameterized, as before, in terms of real numbers s_e such that $L_e = 2 \cosh 2\pi b s_e$. To a pants decomposition we may then associate the direct integral of Hilbert spaces

$$\mathcal{H}_\sigma \simeq \int_{\mathbb{R}_+^h}^{\oplus} \prod_{e \in \sigma_1} d\mu(s_e) \bigotimes_{v \in \sigma_0} \mathcal{H}_{s_2(v) s_1(v)}^{s_3(v)}. \quad (7.7)$$

We denoted the set of internal edges of the MS graph σ by σ_1 , and the set of vertices by σ_0 .

As a useful notation let us introduce ‘‘basis vectors’’ $\langle s |$ for \mathcal{H}_σ , more precisely distributions on dense subspaces of \mathcal{H}_σ such that the wave-function $\psi(s)$ of a state $|\psi\rangle$ is represented as $\psi(s) = \langle s | \psi \rangle$. Representing \mathcal{H}_σ as in (7.7) one may identify

$$\langle s | \simeq \bigotimes_{v \in \sigma_0} v_{s_2(v) s_1(v)}^{s_3(v)}, \quad (7.8)$$

where $v_{s_2 s_1}^{s_3}$ is understood as an element of the dual $(\mathcal{H}_{s_2, s_1}^{s_3})^t$ of the Hilbert space $\mathcal{H}_{s_2 s_1}^{s_3}$.

The operators B_v , Z_v and A_e will be represented in the following form: Given functions $B_{s_2 s_1}^{s_3}$ and $Z_{s_2 s_1}^{s_3}$ of three variables one may define multiplication operators B and Z as

$$B \cdot v_{s_2 s_1}^{s_3} = B_{s_2 s_1}^{s_3} v_{s_2 s_1}^{s_3}, \quad (7.9a)$$

$$Z \cdot v_{s_2 s_1}^{s_3} = Z_{s_2 s_1}^{s_3} v_{s_1 s_3}^{s_2}. \quad (7.9b)$$

For each vertex v of Γ_σ the representation (7.7) suggests an obvious way to lift B and Z to operators B_v and Z_v mapping \mathcal{H}_σ to $\mathcal{H}_{B_v\sigma}$ and $\mathcal{H}_{Z_v\sigma}$, respectively.

Let furthermore σ_s and σ_t be the pants decompositions of $C_{0,4}$ depicted on the left and right of Figure 4 respectively. The operators $A : \mathcal{H}_{\sigma_s} \rightarrow \mathcal{H}_{\sigma_t}$ can be represented as

$$A \cdot v_{s_3 s_2 s_1}^{s_4} \otimes v_{s_2 s_1}^{s_{21}} = \int_{\mathbb{S}}^{\oplus} d\mu(s_{32}) F_{s_{21} s_{32}} \begin{bmatrix} s_3 & s_2 \\ s_4 & s_1 \end{bmatrix} v_{s_{32} s_1}^{s_4} \otimes v_{s_3 s_2}^{s_{32}}. \quad (7.9c)$$

For each pants decomposition σ and each edge e of Γ_σ one may then use A to define operators $A_e : \mathcal{H}_\sigma \rightarrow \mathcal{H}_{F_e\sigma}$.

7.3.5

Indeed, it is easy to see that the operators B_v , A_e , and Z_v defined via (7.5) are of the form described in Subsection 7.3.4. The fact that B_v and Z_v act as multiplication operators on \mathcal{H}_σ follows from the observation that these operators, as can easily be checked, commute with the length operators associated to the boundary components of a trinion. The form claimed for A_e follows from the fact that this operator commutes with length operators associated to the boundary components of the four-holed sphere $C_{0,4}$ containing e .

It is now clear how the representation of the Moore-Seiberg groupoid is extended from pairs (σ_2, σ_1) of admissible pants decompositions to all pairs (σ_2, σ_1) of pants decompositions.

7.4 Explicit form of the generators

We now come to one of the main applications of the connection between the modular double and the quantum Teichmüller theory: It will allow us to calculate the explicit representation of the operators B_v , A_e , and Z_v . As explained above, it suffices to find the corresponding operators B , Z and A which take the form specified in equations (7.9) above. The result will be

$$B_{s_2 s_1}^{s_3} = e^{\pi i (s_3^2 - s_2^2 - s_1^2 + c_b^2)}, \quad (7.10a)$$

$$F_{s_{21} s_{32}} \begin{bmatrix} s_3 & s_2 \\ s_4 & s_1 \end{bmatrix} = \left\{ \begin{matrix} s_1 & s_2 & s_{21} \\ s_3 & s_4 & s_{32} \end{matrix} \right\}_b, \quad (7.10b)$$

$$Z_{s_2 s_1}^{s_3} = 1. \quad (7.10c)$$

This result will be a rather easy consequence of the the relations between the modular double and quantum Teichmüller theory observed above.

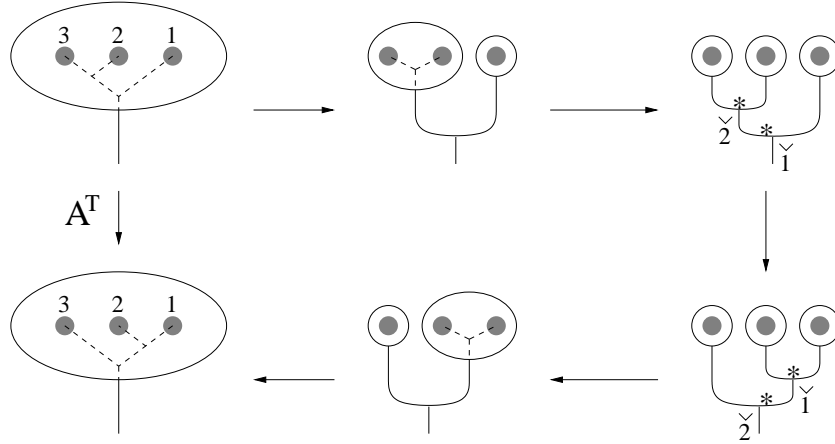


Figure 5: Graphical representation for Equation 7.11.

7.4.1 The operator A

It follows from the relation between Clebsch-Gordan maps and operators C_v observed above that the operators A can be expressed in terms of the operators $A^T(s_3, s_2, s_1)$ defined as

$$A^T(s_3, s_2, s_1) = C_2(s_3, s_{21})C_1(s_2, s_1) \cdot T_{12}^{-1} \cdot [C_1(s_{32}, s_1)C_2(s_3, s_2)]^{-1}. \quad (7.11)$$

We have given a diagrammatic representation for the the operator A^T in Figure 5.

By projecting $A^T(s_3, s_2, s_1)$ to vanishing constraints one gets an operator $A_{s_3 s_2 s_1}^T : \mathcal{H}_{\sigma_s} \rightarrow \mathcal{H}_{\sigma_t}$. It is not hard to see that we have $A_{s_3 s_2 s_1}^T = A_{s_3 s_2 s_1}$, where $A_{s_3 s_2 s_1}$ is the operator defined in (4.30). Indeed, we may express $A_{s_3 s_2 s_1}$ in the following form

$$A_{s_3 s_2 s_1} = C_2(s_3, s_{21})C_1(s_2, s_1) \cdot T_{23}^{-1} T_{12}^{-1} T_{23} T_{13} \cdot [C_1(s_{32}, s_1)C_2(s_3, s_2)]^{-1}.$$

By using (5.5a) one easily simplifies this expression to the form given in (7.11). This result allows us to conclude that the matrix elements of the fusion operator A are given by the b-6j symbols $\left\{ \begin{matrix} s_1 & s_2 & s_{21} \\ s_3 & s_4 & s_{32} \end{matrix} \right\}_b$.

7.4.2 The operator B

It follows from our main result in Subsection 6.4 that the operator B_v is represented by $P_{21}r_{21}$, where P_{21} is the permutation operator. It was shown in [BT1] that the Clebsch-Gordan maps diagonalize this operator, with eigenvalue being $e^{\pi i(s_3^2 - s_2^2 - s_1^2 + c_b^2)}$.

7.4.3 The operator Z

Let us, on the one hand, consider the relation in the Moore-Seiberg groupoid drawn in Figure 6.

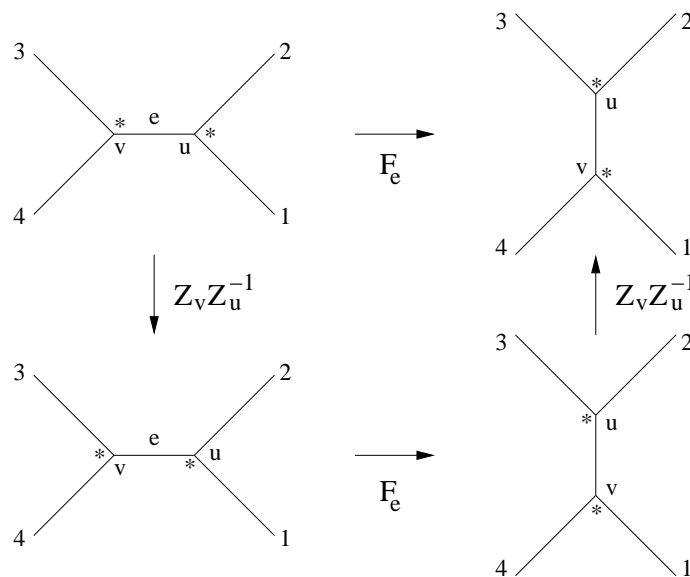


Figure 6: A relation in the Moore-Seiberg groupoid.

This relation implies the symmetry relation

$$F_{s_{21}s_{32}} \begin{bmatrix} s_3 & s_2 \\ s_4 & s_1 \end{bmatrix} = (Z_{s_2s_1}^{s_{21}})^{-1} Z_{s_3s_{21}}^{s_4} F_{s_{21}s_{32}} \begin{bmatrix} s_1 & s_4 \\ s_2 & s_3 \end{bmatrix} (Z_{s_{32}s_3}^{s_2})^{-1} Z_{s_4s_1}^{s_{32}}. \quad (7.12)$$

Note, on the other hand, that the coefficients $\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \alpha_4 & \alpha_t \end{smallmatrix} \right\}_b$ satisfy the tetrahedral symmetries

$$\left\{ \begin{smallmatrix} \alpha_1 & \alpha_2 & \alpha_s \\ \alpha_3 & \alpha_4 & \alpha_t \end{smallmatrix} \right\}_b = \left\{ \begin{smallmatrix} \alpha_2 & \alpha_1 & \alpha_s \\ \alpha_4 & \alpha_3 & \alpha_t \end{smallmatrix} \right\}_b = \left\{ \begin{smallmatrix} \alpha_2 & \alpha_s & \alpha_1 \\ \alpha_4 & \alpha_t & \alpha_3 \end{smallmatrix} \right\}_b = \left\{ \begin{smallmatrix} \alpha_3 & \alpha_4 & \alpha_s \\ \alpha_1 & \alpha_2 & \alpha_t \end{smallmatrix} \right\}_b, \quad (7.13)$$

as follows easily from the integral representation (4.33). From the comparison it is easy to see that we must have $Z_{s_2s_1}^{s_3} = 1$, as claimed.

A. Calculation of the Clebsch-Gordan coefficients

A.1 Proof of Proposition 1

As the most tedious step let us calculate the matrix elements of the operator $\mathbb{T}_{12} C_1^{-1}$, which is defined as

$$C \left(\begin{smallmatrix} s_{21} & s_2 & s_1 \\ p_{21} & p_2 & p_1 \end{smallmatrix} \right) := \langle p_2, p_1 | \mathbb{T}_{12} C_1^{-1} | p_{21}, s_{21} \rangle. \quad (\text{A.1})$$

Proposition 7. *The matrix elements of $\mathbb{T}_{12} C_1^{-1}$ are explicitly given by the formula*

$$\begin{aligned} C \left(\begin{smallmatrix} s_{21} & s_2 & s_1 \\ p_{21} & p_2 & p_1 \end{smallmatrix} \right) &= \delta(p_{21} - p_2 - p_1) e^{\frac{\pi i}{2}(\Delta_{s_1} + \Delta_{s_2} - \Delta_{s_{21}})} w_b(s_1 + s_2 - s_{21}) w_b(s_{21} + s_2 - s_1) \\ &\times e^{\frac{\pi i}{2}(p_3^2 - p_1^2 - p_2^2)} \frac{w_b(p_1 - s_1) w_b(p_2 - s_2)}{w_b(p_{21} - s_{21})} e^{\pi i(p_2(s_1 + c_b) - p_1(s_2 + c_b))} \\ &\times \int_{\mathbb{R}} dp e^{\pi i p(s_1 + s_2 - s_{21} + c_b)} D_{\frac{1}{2}(s_2 - s_1 - s_{21} - c_b)}(p + p_2) D_{\frac{1}{2}(s_1 - s_2 - s_{21} - c_b)}(p - p_1) \\ &\times D_{\frac{1}{2}(s_1 + s_2 + s_{21} - c_b)}(p). \end{aligned} \quad (\text{A.2})$$

Proof. By using

$$e_b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{q}_2) e^{-2\pi i \mathbf{p}_1 \mathbf{q}_2} = e^{-2\pi i \mathbf{p}_1 \mathbf{q}_2} e_b(\mathbf{q}_1 + \mathbf{p}_2 - \mathbf{p}_1), \quad (\text{A.3})$$

it is easy to see that

$$C \left(\begin{smallmatrix} s_{21} & s_2 & s_1 \\ p_{21} & p_2 & p_1 \end{smallmatrix} \right) = \delta(p_{21} - p_2 - p_1) C_{s_2 s_1}^{s_{21}}(p_1, p_{21}), \quad (\text{A.4})$$

where

$$C_{s_2 s_1}^{s_{21}}(p_1, p_{21}) = \langle p_1 | e_b(\mathbf{q}_1 - \mathbf{p}_1 + p_{21}) C_1^{-1} | s_{21} \rangle. \quad (\text{A.5})$$

As a preparation it will be convenient to rewrite C_1^{-1} using (2.2) in the form

$$C_1^{-1} := e_b(\mathbf{q}_1 - s_2) e^{2\pi i s_2 \mathbf{q}_1} \frac{w_b(s_1 + \mathbf{p}_1 + s_2)}{w_b(s_1 - \mathbf{p}_1 - s_2)}. \quad (\text{A.6})$$

The matrix element (A.4) may then be calculated by inserting two resolutions of the identity as follows,

$$\begin{aligned} C_{s_2 s_1}^{s_{21}}(p_1, p_{21}) &= \int_{\mathbb{R}^2} dp' dp'' \langle p_1 | e_b(\mathbf{q}_1 - \mathbf{p}_1 + p_{21}) | p' \rangle \times \\ &\times \langle p' | e_b(\mathbf{q}_1 - s_2) e^{2\pi i s_2 \mathbf{q}_1} | p'' \rangle \frac{w_b(s_1 + p'' + s_2)}{w_b(s_1 - p'' - s_2)} \langle p'' | s_{21} \rangle. \end{aligned} \quad (\text{A.7})$$

The ingredients of the kernel are the following:

1. *The matrix element $\langle p_1 | e_b(\mathbf{q}_1 - \mathbf{p}_1 + p_{21}) | p' \rangle$:*

We may use the integral identity

$$e_b(x) = \zeta_\circ \int_{\mathbb{R}-i0} dy e^{-2\pi ixy} e^{-\pi iy^2} e_b(y + c_b), \quad (\text{A.8})$$

where $\zeta_\circ = e^{\frac{\pi i}{12}(1-4c_b^2)}$, in order to represent the function e_b in this matrix element. We get

$$\begin{aligned} \langle p_1 | e_b(\mathbf{q}_1 - \mathbf{p}_1 + p_{21}) | p' \rangle &= \zeta_\circ \int_{\mathbb{R}-i0} dy e^{-\pi iy^2} e_b(y + c_b) \langle p_1 | e^{-2\pi i(\mathbf{q}_1 - \mathbf{p}_1 + p_{21})y} | p' \rangle \\ &= \zeta_\circ \int_{\mathbb{R}-i0} dy e^{-\pi iy^2} e_b(y + c_b) e^{\pi iy(p_1 + p' - 2p_{21})} \delta(p_1 + y - p') \\ &= \zeta_\circ e^{2\pi i(p' - p_1)(p_1 - p_{21})} e_b(p' - p_1 + c_b). \end{aligned} \quad (\text{A.9})$$

2. *The matrix element $\langle p' | e_b(\mathbf{q}_1 - s_2) e^{2\pi i s_2 \mathbf{q}_1} | p'' \rangle$:*

We now use a variant of the integral identity (A.8) which takes the form

$$e_b(x) = \zeta_\circ^{-1} \int_{\mathbb{R}-i0} dy e^{-2\pi ixy} \frac{e^{2\pi ic_b y}}{e_b(-y - c_b)}. \quad (\text{A.10})$$

A calculation similar to the one leading to (A.9) gives now

$$\begin{aligned} \langle p' | e_b(\mathbf{q}_1 - s_2) e^{2\pi i s_2 \mathbf{q}_1} | p'' \rangle &= \zeta_\circ^{-1} \int_{\mathbb{R}-i0} dy \frac{e^{2\pi i(s_2 + c_b)y}}{e_b(-y - c_b)} \langle p' + y - s_2 | p'' \rangle \\ &= \zeta_\circ^{-1} \frac{e^{2\pi i(s_2 + c_b)(s_2 + p'' - p')}}{e_b(p' - s_2 - p'' - c_b)}. \end{aligned} \quad (\text{A.11})$$

3. *The integral over p' :*

Let us focus on the integral over p' appearing in (A.7):

$$\mathcal{I}' := \int_{\mathbb{R}} dp' \langle p_1 | e_b(\mathbf{q}_1 - \mathbf{p}_1 + p_{21}) | p' \rangle \langle p' | e_b(\mathbf{q}_1 - s_2) e^{2\pi i s_2 \mathbf{q}_1} | p'' \rangle. \quad (\text{A.12})$$

Inserting (A.9) and (A.11) yields

$$\begin{aligned} \mathcal{I}' &= e^{2\pi i(s_2 + c_b)(s_2 + p'')} e^{2\pi i p_1(p_{21} - p_1)} \times \\ &\quad \times \int_{\mathbb{R}+i0} dp' e^{-2\pi i p'(p_{21} + s_2 + c_b - p_1)} \frac{e_b(p' - p_1 + c_b)}{e_b(p' - s_2 - p'' - c_b)}. \end{aligned} \quad (\text{A.13})$$

By using

$$\int_{\mathbb{R}} dz e^{-2\pi iz(u+c_b)} \frac{e_b(z+c_b)}{e_b(z-x-c_b)} = \zeta_\circ^{-1} \frac{e_b(u-x)}{e_b(-x-c_b)e_b(u)}, \quad (\text{A.14})$$

we may calculate

$$\mathcal{I}' = \zeta_\circ^{-1} \frac{e^{2\pi i(s_2 + c_b)(s_2 + p'' - p_1)} e_b(p_{21} - p'')}{e_b(p_1 - s_2 - p'' - c_b) e_b(p_{21} + s_2 - p_1)}. \quad (\text{A.15})$$

It is now convenient to rewrite the resulting expression in terms of the function $w_b(x)$ related to $e_b(x)$ via (2.2). We find using $p_{21} = p_2 + p_1$

$$\mathcal{I}' = e^{\pi i(s_2+c_b)(s_2-p_1)} e^{-\pi i p_2 s_2} e^{\frac{\pi i}{2}(p_3^2-p_1^2-p_2^2)} \frac{w_b(p_2+s_2)w_b(p''-p_{21})}{w_b(p''+s_2-p_1+c_b)} e^{\pi i p''(s_2-p_2+c_b)}. \quad (\text{A.16})$$

Taking into account that

$$\langle p'' | s_{21} \rangle = \frac{w_b(s_{21}-p''-c_b)}{w_b(s_{21}+p''+c_b)}, \quad (\text{A.17})$$

we may use (A.15) to get a single integral representation for $C_{s_2 s_1}^{s_{21}}(p_1, p_{21})$, which takes the form

$$C_{s_2 s_1}^{s_{21}}(p_1, p_{21}) = e^{\frac{\pi i}{2}(p_3^2-p_1^2-p_2^2)} e^{\pi i(s_2+c_b)(s_2-p_1)} e^{-\pi i p_2 s_2} w_b(p_2+s_2) \mathcal{I}'', \quad (\text{A.18})$$

where the integral \mathcal{I}'' is defined as

$$\mathcal{I}'' := \int_{\mathbb{R}-i0} dp \frac{w_b(p-p_{21})}{w_b(p+s_2-p_1+c_b)} \frac{w_b(p+s_1+s_2)}{w_b(p+s_{21}+c_b)} \frac{w_b(p+s_2-s_1)}{w_b(p-s_{21}+c_b)} e^{\pi i p(s_2-p_2+c_b)}. \quad (\text{A.19})$$

We'll need to rewrite this integral further. Let us first introduce the combination

$$D_a(x) := \frac{w_b(x+a)}{w_b(x-a)}. \quad (\text{A.20})$$

In terms of this function we may write \mathcal{I}'' as

$$\begin{aligned} \mathcal{I}'' := \int_{\mathbb{R}} dp e^{\pi i p(s_2-p_2+c_b)} & D_{-\frac{1}{2}(s_2+p_2+c_b)}\left(p + \frac{1}{2}(s_2-p_1-p_{21}+c_b)\right) \\ & \times D_{\frac{1}{2}(s_1+s_2-s_{21}-c_b)}\left(p + \frac{1}{2}(s_1+s_2+s_{21}+c_b)\right) \\ & \times D_{\frac{1}{2}(s_2-s_1+s_{21}-c_b)}\left(p + \frac{1}{2}(s_2-s_1-s_{21}+c_b)\right). \end{aligned} \quad (\text{A.21})$$

By using the identities [BT2, Equation (A.34)]

$$\begin{aligned} \int dx D_\alpha(x+u) D_\beta(x+v) D_\gamma(x+w) e^{-2\pi i x \delta} &= \\ = A_{\alpha\beta\gamma\delta} D_{\alpha+\beta+c_b}(u-v) e^{-2\pi i(\alpha+\beta+c_b)w} &\int dx e^{-2\pi i x \gamma^*} D_{\alpha^*}(x+v) D_{\beta^*}(x+u) D_{\delta^*}(x+w), \end{aligned} \quad (\text{A.22})$$

and

$$D_a(x) D_b(y) = D_{\frac{1}{2}(a+b+x-y)}\left(\frac{1}{2}(a-b+x+y)\right) D_{\frac{1}{2}(a+b+y-x)}\left(\frac{1}{2}(b-a+x+y)\right), \quad (\text{A.23})$$

we find that $C_{p_{21} | p_2 p_1}^{s_{21} | s_2 s_1}$ is indeed represented by the formula (A.2), a claimed. \square

With the help of the integral identity (A.22) it is straightforward to check that the expression given in (4.17) has the Weyl-symmetries (4.18). The reality (4.19) follows immediately since

$$\left[\left(\begin{smallmatrix} s_3 & s_2 & s_1 \\ p_3 & p_2 & p_1 \end{smallmatrix} \middle| b \right) \right]^* = \left(\begin{smallmatrix} -s_3 & -s_2 & -s_1 \\ p_3 & p_2 & p_1 \end{smallmatrix} \middle| b \right). \quad (\text{A.24})$$

Keeping in mind that

$$\left[\langle s_3, p_3 | C_{s_2 s_1} | p_2, p_1 \rangle \right]^* = \langle p_2, p_1 | T_{12} C_1^{-1} | s_3, p_3 \rangle, \quad (\text{A.25})$$

one may complete the proof of Proposition 1 by comparing the expressions (4.17) and (A.2).

A.2 Proof of Proposition 2

As the main technical step let us calculate the Fourier-transformation of the b-Clebsch-Gordan coefficients $\left(\begin{smallmatrix} s_3 & s_2 & s_1 \\ x_3 & x_2 & x_1 \end{smallmatrix} \middle| b\right)_b$ defined in (4.23). We need to calculate the following integral:

$$\left(\begin{smallmatrix} s_3 & s_2 & s_1 \\ k_3 & k_2 & k_1 \end{smallmatrix} \middle| b\right)_b^{\text{PT}} = \int_{\mathbb{R}} dx_3 e^{-2\pi i k_3 x_3} \int_{\mathbb{R}^2} dx_2 dx_1 e^{2\pi i (k_2 x_2 + k_1 x_1)} \left(\begin{smallmatrix} s_3 & s_2 & s_1 \\ x_3 & x_2 & x_1 \end{smallmatrix} \middle| b\right)_b. \quad (\text{A.26})$$

Proposition 8. *We have*

$$\begin{aligned} \left(\begin{smallmatrix} s_3 & s_2 & s_1 \\ k_3 & k_2 & k_1 \end{smallmatrix} \middle| b\right)_b^{\text{PT}} &= \delta(k_3 - k_2 - k_1) e^{\pi i (k_2 (s_1 + c_b) - k_1 (s_2 + c_b))} \\ &\times N(s_3, s_2, s_1) w_b(-s_1 - s_2 - s_3) w_b(s_1 + s_3 - s_2) w_b(s_2 + s_3 - s_1) \\ &\times \int_{\mathbb{R}} dy e^{-\pi i (s_3 - s_2 - s_1 - c_b) y} D_{\frac{1}{2}(s_1 + s_2 + s_3 - c_b)}(y) \\ &\times D_{\frac{1}{2}(s_2 - s_3 - s_1 - c_b)}(y + k_2) D_{\frac{1}{2}(s_1 - s_3 - s_2 - c_b)}(y - k_1). \end{aligned} \quad (\text{A.27})$$

Proof. After using the integral transformation

$$D_a(x) = w_b(2a + c_b) \int_{\mathbb{R}} dy e^{-2\pi i x y} D_{-a - c_b}(y) \quad (\text{A.28})$$

in order to express the function $D_a(x)$ which appears in the first line of (4.23), we get the integral

$$\begin{aligned} \left(\begin{smallmatrix} s_3 & s_2 & s_1 \\ k_3 & k_2 & k_1 \end{smallmatrix} \middle| b\right)_b^{\text{PT}} &= N(s_3, s_2, s_1) w_b(-s_1 - s_2 - s_3) \\ &\times \int_{\mathbb{R}} dy e^{\pi i (s_3 + c_b) y} D_{\frac{1}{2}(s_1 + s_2 + s_3 - c_b)}(y) \int_{\mathbb{R}} dx_3 e^{-2\pi i k_3 x_3} \int_{\mathbb{R}^2} dx_2 dx_1 e^{2\pi i (k_2 x_2 + k_1 x_1)} e^{-2\pi i (x_2 - x_1) y} \\ &\times D_{-\frac{1}{2}(s_2 - s_3 - s_1 + c_b)}^{\text{PT}}(x_2 - x_3 - \frac{s_1 + c_b}{2}) D_{-\frac{1}{2}(s_1 - s_3 - s_2 + c_b)}(x_3 - x_1 - \frac{s_2 + c_b}{2}). \end{aligned} \quad (\text{A.29})$$

Substituting the variables of integration as $x_2 = y_2 + x_3 + (s_1 + c_b)/2$, $x_1 = y_1 + x_3 - (s_2 + c_b)/2$ yields

$$\begin{aligned} \left(\begin{smallmatrix} s_3 & s_2 & s_1 \\ k_3 & k_2 & k_1 \end{smallmatrix} \middle| b\right)_b^{\text{PT}} &= N(s_3, s_2, s_1) w_b(-s_1 - s_2 - s_3) e^{\pi i (k_2 (s_1 + c_b) - k_1 (s_2 + c_b))} \\ &\times \int_{\mathbb{R}} dy e^{\pi i (s_3 - s_2 - s_1 - c_b) y} D_{\frac{1}{2}(s_1 + s_2 + s_3 - c_b)}(y) \int_{\mathbb{R}} dx_3 e^{-2\pi i (k_3 - k_2 - k_1) x_3} \\ &\times \int_{\mathbb{R}} dy_2 e^{-2\pi i y_2 (y - k_2)} D_{-\frac{1}{2}(s_2 - s_3 - s_1 + c_b)}(y_2) \int_{\mathbb{R}} dy_1 e^{2\pi i y_1 (y + k_1)} D_{-\frac{1}{2}(s_1 - s_3 - s_2 + c_b)}(y_1). \end{aligned} \quad (\text{A.30})$$

The integrals over y_2 and y_1 may be carried out using (A.28), while the integral over x_3 yields a delta-distribution $\delta(k_3 - k_2 - k_1)$. We arrive at the formula

$$\begin{aligned} \left(\begin{smallmatrix} s_3 & s_2 & s_1 \\ k_3 & k_2 & k_1 \end{smallmatrix} \middle| b\right)_b^{\text{PT}} &= \delta(k_3 - k_2 - k_1) N(s_3, s_2, s_1) e^{\pi i (k_2 (s_1 + c_b) - k_1 (s_2 + c_b))} \\ &\times w_b(-s_1 - s_2 - s_3) w_b(s_1 + s_3 - s_2) w_b(s_2 + s_3 - s_1) \\ &\times \int_{\mathbb{R}} dy e^{\pi i (s_3 - s_2 - s_1 - c_b) y} D_{\frac{1}{2}(s_1 + s_2 + s_3 - c_b)}(y) \\ &\times D_{\frac{1}{2}(s_2 - s_3 - s_1 - c_b)}(y - k_2) D_{\frac{1}{2}(s_1 - s_3 - s_2 - c_b)}(y + k_1). \end{aligned} \quad (\text{A.31})$$

Substituting $y \rightarrow -y$ and using that $D_a(x) = D_a(-x)$ completes the proof. \square

It remains to compare the resulting expression (A.27) with (4.17). We find that

$$\begin{pmatrix} s_3 & s_2 & s_1 \\ p_3 & p_2 & p_1 \end{pmatrix} = N(s_3, s_2, s_1) \frac{w_b(p_1 - s_1)w_b(p_2 - s_2)}{w_b(p_3 - s_3)} e^{\frac{\pi i}{2}(p_3^2 - p_1^2 - p_2^2)} \begin{pmatrix} s_3 & s_2 & s_1 \\ p_3 & p_2 & p_1 \end{pmatrix}_b^{\text{PT}}. \quad (\text{A.32})$$

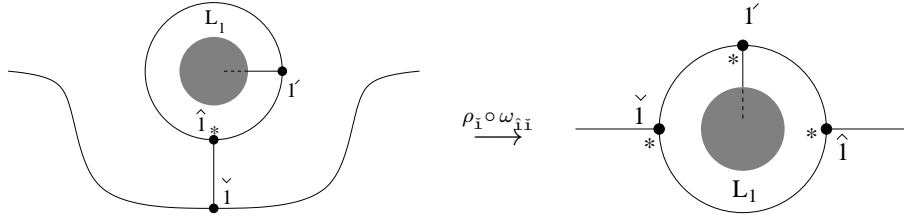
We observe that the terms in the first line of (A.32) represent the unitary transformation between the representation (3.5) and the Whittaker model (3.8). The prefactor in the second line depends only on the triple of Casimir eigenvalues and represents a change of normalization of the Clebsch-Gordan maps.

B. Proofs of some technical results

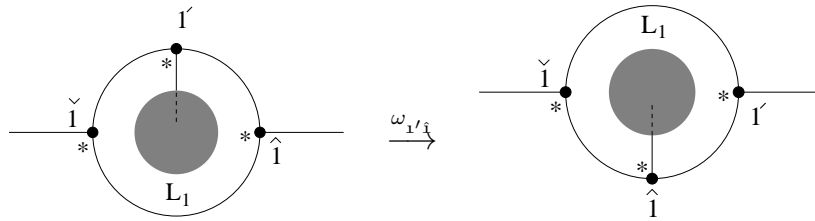
B.1 Proof of Proposition 4

The move W_1 defined in (6.3) may be factorized into the following three simple moves:

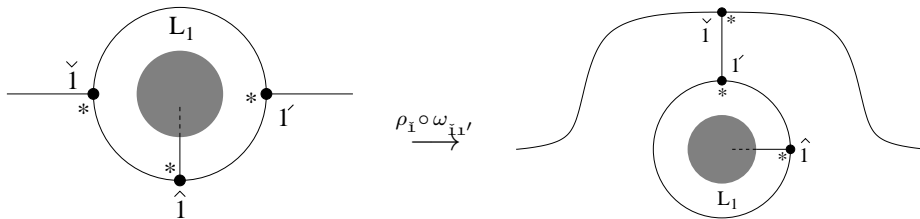
First move: The move $\rho_{\hat{1}} \circ \omega_{\hat{1}\hat{1}}$, diagrammatically represented as follows



Second move: The move $\omega_{\hat{1}'\hat{1}}$, diagrammatically represented as follows



Third move: The move $\rho_{\hat{1}} \circ \omega_{\hat{1}\hat{1}'}$, diagrammatically represented as follows



We have

$$\text{ad}[W_1](e^{-\pi b p_{\hat{1}}}) = e^{-\pi b(-p_{\hat{1}} + 2z_1)}, \quad (\text{B.1})$$

where $z_1 := \frac{1}{2}(p_0 - q_0 + p_1)$. This is equivalent to $\text{ad}[W_1](K_1) = K_1^{-1}$.

Furthermore

$$\text{ad}[W_1](e^{\pi b(2q_i - p_i)}) = e^{-\frac{\pi b}{2}(2q_i - p_i)} (2 \cosh 2\pi b(p_i - z_1) + L_1) e^{-\frac{\pi b}{2}(2q_i - p_i)} \quad (\text{B.2})$$

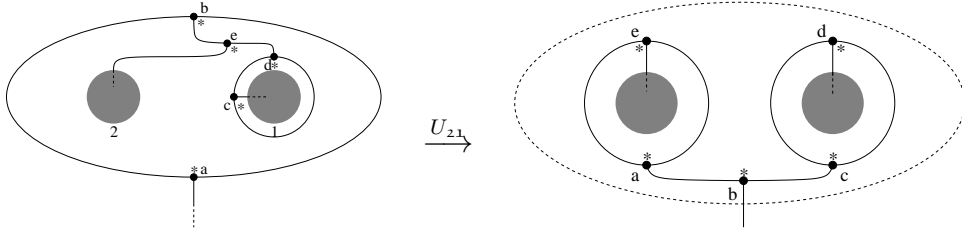
This is equivalent to $\text{ad}[W_1](E_1) = F_1$.

B.2 Proof of Proposition 3

We need to calculate $\text{ad}[U_{21}](L)$, where

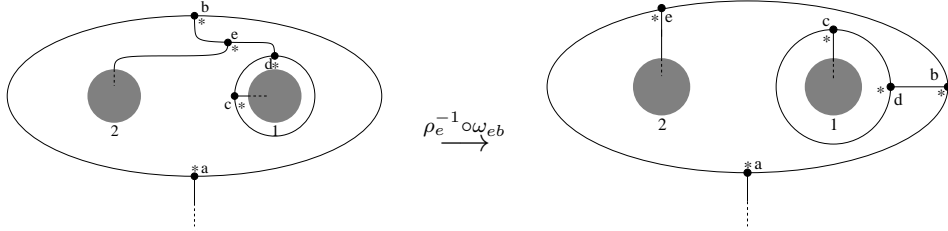
$$L := 2 \cosh \pi b(p_b + q_a - p_a) + e^{\pi b(2q_b - p_b - (q_a + p_a))}, \quad (\text{B.3})$$

and U_{21} is the operator representing the move U_{21} which is diagrammatically represented as:



The calculation may be performed in three steps.

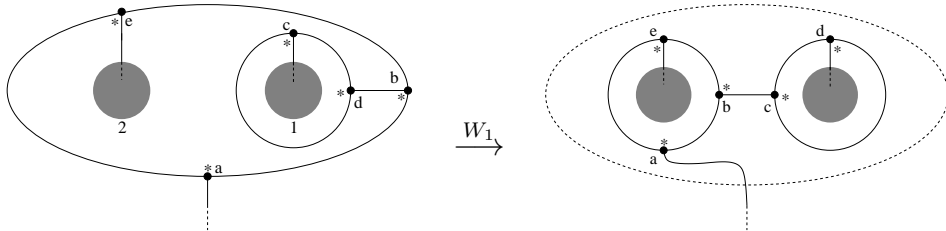
First step: The move $\rho_e^{-1} \circ \omega_{eb}$, diagrammatically represented as follows



Calculation of $\text{ad}[A_e^{-1}T_{eb}](L)$:

$$L' := \text{ad}[A_e^{-1}T_{eb}](L) = 2 \cosh \pi b(p_b - q_e + q_a - p_a) + e^{\pi b(p_b + (2p_e - q_e) - (q_a + p_a))} + e^{\pi b(2q_b + q_e - p_b - (q_a + p_a))}. \quad (\text{B.4})$$

Second step: The move W_1 , diagrammatically represented as follows

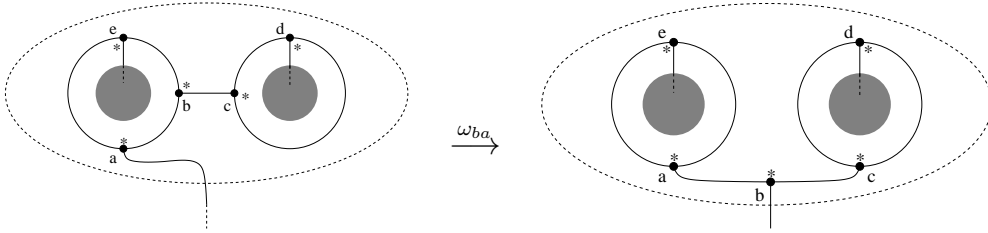


Calculation of $\text{ad}[W_1](L')$:

$$\begin{aligned} L'' &:= \text{ad}[W_1](L') & (B.5) \\ &= 2 \cosh \pi b (-p_b + 2z_1 - q_e + q_a - p_a) + e^{\pi b (-p_b + z_1 + (2p_e - q_e) - (q_a + p_a))} \\ &\quad + : e^{-\pi b (2q_b + q_e - p_b + (q_a + p_a))} (2 \cosh 2\pi b (p_b - z_1) + L_1) :, \end{aligned}$$

where $z_1 := \frac{1}{2}(p_c + p_d - q_c)$.

Third step: The move ω_{ba} , diagrammatically represented as follows



Calculation of $\text{ad}[T_{ba}](L'')$: We factorize $T_{ba} = T'_{ba} e^{-2\pi i p_b q_a}$ and collect the terms with equal weight with respect to the adjoint action of the argument $q_b - q_a + p_a$ of $T'_{ba} = e_b(q_b - q_a + p_a)$:

$$\begin{aligned} L''' &:= \text{ad}[T'_{ba}] \left(: e^{2\pi b (p_b - z_1)} (e^{-\pi b (2q_b + q_e - q_a + p_a)} + e^{-\pi b (q_a - p_a)}) : \right. \\ &\quad \left. + e^{\pi b (-p_b + z_1 + (2p_e - q_e) - (q_a + p_a))} + e^{-\pi b (2q_b + q_e + p_a - q_a)} L_1 \right. \\ &\quad \left. + : e^{-2\pi b (p_b - z_1)} (e^{-\pi b (2q_b + q_e + p_a - q_a)} + e^{\pi b (q_a - p_a)}) : \right) \\ &= : e^{2\pi b (p_b - z_1)} (e^{-\pi b (2q_b + q_e - q_a + p_a)} + e^{-\pi b (q_a - p_a)}) (1 + e^{2\pi b (q_b - q_a + p_a)})^{-1} : \\ &\quad + e^{\pi b (-p_b + z_1 + (2p_e - q_e) - (q_a + p_a))} + e^{-\pi b (2q_b + q_e + p_a - q_a)} L_1 \\ &\quad + : e^{-2\pi b (p_b - z_1)} (e^{-\pi b (2q_b + q_e + p_a - q_a)} + e^{\pi b (q_a - p_a)}) (1 + e^{2\pi b (q_b - q_a + p_a)}) :, \end{aligned}$$

where $z_2 := \frac{1}{2}(p_a - q_a + q_e)$. Collecting the terms yields

$$\begin{aligned} L''' &= e^{-2\pi b (p_b - z_1 + q_b + z_2)} + e^{-2\pi b (p_b - z_1)} L_2 + e^{-2\pi b (q_b + z_2)} L_1 \\ &\quad + 2 \cosh 2\pi b (p_b - q_b - z_1 - z_2). \end{aligned} \quad (B.6)$$

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