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The scale of soft resummation in SCET vs perturbative QCD

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Abstract

We summarize and extend previous results on the comparison of threshold resummation, performed, using soft-collinear effective theory (SCET), in the Becher-Neubert approach, to the standard perturbative QCD formalism based on factorization and resummation of Mellin moments of partonic cross sections. We show that the logarithmic accuracy of this SCET result can be extended by half a logarithmic order, thereby bringing it in full agreement with the standard QCD result if a suitable choice is made for the soft scale μ_s which characterizes the SCET result. We provide a master formula relating the two approaches for other scale choices. We then show that with the Becher-Neubert scale choice the Landau pole, which in the perturbative QCD approach is usually removed through power- or exponentially suppressed terms, in the SCET approach is removed by logarithmically subleading terms which break factorization. Such terms may become leading for generic choices of parton distributions, and are always leading when resummation is used far enough from the hadronic threshold.

Keywords: QCD, soft-gluon, threshold, resummation, soft-collinear effective theory

1. Soft resummation and scale choices

Threshold resummation [1, 2] plays an important role in extending and stabilizing the accuracy of perturbative results, and it may be of some relevance even for hadronic processes which are quite far from threshold [3, 4], due to the fact that the underlying partonic process can be rather closer to threshold than the hadronic one [5]. All-order resummed results are known to lead to a divergent series when expanded out in powers of the strong coupling: this is physically due to the fact that resummation is obtained by choosing as a scale of the parton-level process the maximum energy of the radiated partons [6, 7], which tends to zero in the threshold limit. The divergence can be tamed by introducing suitable subleading contributions, such as exponentially suppressed terms outside the physical kinematic region [8], or power-suppressed terms [9, 10].

In Ref. [11] it was suggested, within the context of a SCET approach to threshold resummation, that the divergence can be tamed by making a hadronic choice of scale. In SCET this is possible because resummed results are characterized by a "soft scale" μ_s: the Becher-Neubert (BN) scale choice consists of expressing μ_s in terms of kinematic variables of the hadronic scattering process. The meaning of this choice is not obvious in the conventional QCD approach, where, because of perturbative factorization, the partonic cross section, which is being resummed, is independent of the hadronic kinematic variables.

In Ref. [13] we have clarified this issue by explicitly exhibiting a relation between the μ_s dependent resummed SCET result, and the standard (μ_s independent) QCD expression. Specializing to the BN scale choice (while taking the Drell-Yan process [12] as an example) we were able to show that in the SCET result, with the BN scale choice, the divergence is removed through

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QCD:	$A(\alpha_s)$	$D(\alpha_s)$	$\bar{g}_0(\alpha_s)$	accuracy: $\alpha_s^n \ln^k N$
SCET:	$\Gamma_{\rm cusp}(\alpha_s)$	$\gamma_W(\alpha_s)$	H, $\tilde{s}_{\rm DY}$	accuracy: $\alpha_s^n \ln^k(\mu_s/M)$
LL	1-loop	—	tree-level	$k = 2n$
NLL*	2-loop	1-loop	tree-level	$2n-1 \le k \le 2n$
NLL	2-loop	1-loop	1-loop	$2n-2 \le k \le 2n$
NNLL*	3-loop	2-loop	1-loop	$2n-3 \le k \le 2n$
NNLL	3-loop	2-loop	2-loop	$2n-4 \le k \le 2n$

Table 1: Orders of logarithmic approximations and accuracy of the predicted logarithms in perturbative QCD (first header) and SCET (second header). The last columns refers to the content of the coefficient function. In Ref. [12], only the N^{k}LL* counting is considered for the SCET resummation.

terms which are logarithmically subleading in comparison to the logarithmic accuracy of the SCET result of Ref. [12], which is by half a logarithmic order lower than that of the standard QCD result.

Here we show that the accuracy of the SCET result of Ref. [12] can actually be increased to the same level as that of the QCD result, and we rederive, within this higher accuracy, our master formula, which thus becomes particularly transparent. We then use this improved master formula to discuss various problems related to the BN scale choice.

2. Resummation of the Drell-Yan process in SCET and QCD

We consider for definiteness inclusive Drell-Yan production, but the same discussion applies to other processes, such as Higgs production in gluon-gluon fusion, with minimal modifications. The dimensionless invariant mass distribution $\sigma(\tau, M^2) = \frac{1}{\tau\sigma_0}\frac{d\sigma_{\rm DY}}{dM^2}$, with M the invariant mass of the pair and σ_0 the leading order partonic cross section, can be written schematically (omitting a sum over partons) in factorized form as

$$\sigma(\tau, M^2) = \int_\tau^1 \frac{dz}{z}\,\mathcal{L}\left(\frac{\tau}{z}\right) C(z, M^2), \tag{1}$$

where $\mathcal{L}$ is the parton luminosity, s the hadronic center-of-mass energy squared, and $\tau = \frac{M^2}{s}$, so that the hadronic threshold limit is $\tau \to 1$. The perturbatively computable coefficient function $C(z, M^2)$ is normalized so that $C(z, M^2) = \delta(1-z)$ at leading order in the strong coupling α_s. In the sequel, without significant loss of generality, we shall always choose the renormalization and factorization scales $\mu_F^2 = \mu_R^2 = M^2$.

Standard QCD resummation follows from a Mellin-space renormalization-group argument [6, 7]: indeed, at the resummed level, both the convolution Eq. (1) and the gluon radiation phase space factorize, so at the resummed level one may write

$$\sigma(N, M^2) = \int_0^1 d\tau\,\tau^{N-1}\sigma(\tau, M^2) = \mathcal{L}(N)\,C(N, M^2) \tag{2}$$

where the N-space resummed coefficient function has the form [1, 2]

$$C_{\rm QCD}(N, M^2) = \bar{g}_0\left(\alpha_s(M^2)\right)\exp\bar{\mathcal{S}}\left(M^2, \frac{M^2}{N^2}\right) \tag{3}$$

where

$$\bar{\mathcal{S}}\left(M^2, \frac{M^2}{N^2}\right) = \int_0^1 dz\,\frac{z^{N-1}-1}{1-z} \times\left[\int_{M^2}^{M^2(1-z)^2}\frac{d\mu^2}{\mu^2}2A\left(\alpha_s(\mu^2)\right) + D\left(\alpha_s([1-z]^2M^2)\right)\right]. \tag{4}$$

The functions $\bar{g}_0(\alpha_s)$, $A(\alpha_s)$ and $D(\alpha_s)$ are power series in α_s, with $\bar{g}_0(0) = 1$ and $A(0) = D(0) = 0$.

Because resummation is obtained through exponentiation, it might seem natural to also exponentiate the function $\bar{g}_0$. However, unlike $\bar{\mathcal{S}}(M^2, M^2/N^2)$, $\bar{g}_0$ is independent of N and only depends on $\alpha_s(M^2)$. As a consequence, it turns out that simply including an extra term in $\bar{g}_0$ at each order increases the logarithmic accuracy of the coefficient function by half a logarithmic order. This is summarized in Table 1, where the logarithmic accuracy obtained by including a given number of terms in $\bar{g}_0$, A, and D is summarized. A given accuracy means that all and only the logarithmically enhanced contributions to the coefficient function listed in the last column are correctly predicted. At leading logarithmic (LL) accuracy only the largest power of $\ln N$ at each order in α_s is predicted; adding one order in each of the functions A, D, and $\bar{g}_0$ one then obtains the next-to-leading logarithmic (NLL) accuracy, which correctly predicts two powers more, and so on to N^{k}LL accuracy. However, if

$\bar{g}_0$ is exponentiated and a power counting is performed at the level of exponents, it may seem more natural to include one less order in $\bar{g}_0$. This results in the N^{k}LL* accuracy, also shown in table, which is lower by one power of $\ln N$ at each order in α_s than the N^{k}LL accuracy.

The resummed SCET expression for Drell-Yan pair production is given by [12][1]

$$C_{\rm SCET}(z, M^2, \mu_s^2) = H(M^2) U(M^2, \mu_s^2) S(z, M^2, \mu_s^2) \quad (5)$$

where $H(M^2)$ (hard function) is a power series in $\alpha_s(M^2)$,

$$S(z, M^2, \mu_s^2) = \tilde{s}_{\rm DY}\left(\ln\frac{M^2}{\mu_s^2} + \frac{\partial}{\partial\eta}, \mu_s\right)\frac{(1-z)^{2\eta}}{1-z}\frac{e^{-2\gamma\eta}}{\Gamma(2\eta)} \quad (6)$$

(soft function) depends on

$$\eta = \int_{M^2}^{\mu_s^2}\frac{d\mu^2}{\mu^2}\,\Gamma_{\rm cusp}\left(\alpha_s(\mu^2)\right), \qquad \Gamma_{\rm cusp}(\alpha_s) = A(\alpha_s), \quad (7)$$

$\tilde{s}_{\rm DY}(L, \mu_s)$ is a series in $\alpha_s(\mu_s^2)$ with L-dependent coefficients, and

$$\ln U(M^2, \mu_s^2) = -\int_{M^2}^{\mu_s^2}\frac{d\mu^2}{\mu^2} \times\left[\Gamma_{\rm cusp}\left(\alpha_s(\mu^2)\right)\ln\frac{\mu^2}{M^2} - \gamma_W\left(\alpha_s(\mu^2)\right)\right], \quad (8)$$

where $\gamma_W(\alpha_s)$ is also a series in α_s, with $\gamma_W(0) = 0$. The scale μ_s is a soft matching scale of the effective theory, and $C_{\rm SCET}$ formally does not depend on it, up to subleading terms. However, the SCET result resums powers of $\ln\frac{\mu_s}{M}$, so this choice of scale determines what is being resummed.

Again, a given logarithmic accuracy is obtained by including a finite number of terms in the perturbative expansion of the functions which determine the resummed result, namely $\Gamma_{\rm cusp}$, γ_W, H and $\tilde{s}_{\rm DY}$, according to Table 1. In Ref. [12], only the N^{k}LL* accuracy was considered: in fact, the order called N^{k}LL* in Tab. 1 is actually referred to N^{k}LL Ref. [12], which might be the source of some confusion. Computations using either of these two definitions of the logarithmic accuracy have been presented in the past, either in the contex of QCD (see e.g. Ref [14], where N^{k}LL* is referred to as N^{k}LL$_{\ln R}$) or SCET (see e.g. Ref. [15], where N^{k}LL is referred to as N^{k}LL′).

[1]The resummed expression as given in Ref. [12] actually depends on several hard energy scales, which here for simplicity are all taken to be equal to the hard scale M^2.

Here we show that in fact the N^{k}LL* of Ref. [12] can be promoted to higher N^{k}LL accuracy, by inclusion of the terms listed in the table. This result is obtained in the next Section, by explicitly computing the relation between this improved version of the SCET result, and the QCD result.

3. Comparison at NNLL

An analytic comparison between the QCD and SCET resummation formalisms can be performed [13] in N space, where the QCD result is naturally constructed, and where it admits a convergent perturbative expansion in powers of α_s. Namely, we determine the ratio $C_r(N, M^2, \mu_s^2)$ between the QCD and SCET expressions, Eqs. (3) and (5):

$$C_{\rm QCD}(N, M^2) = C_r(N, M^2, \mu_s^2)\, C_{\rm SCET}(N, M^2, \mu_s^2). \quad (9)$$

In Ref. [13] we computed $C_r(N, M^2, \mu_s^2)$ to NNLL, using the definition of NNLL of Ref. [12], which in Tab. 1 we call NNLL*. Here we show that the accuracy of the SCET expression can be upgraded, and the comparison can be carried out at full NNLL according to the definition of Table 1.

We first rewrite the QCD result in the more convenient form [13]

$$C_{\rm QCD}(N, M^2) = \hat{g}_0\left(\alpha_s(M^2)\right)\exp\hat{\mathcal{S}}_{\rm QCD}\left(M^2, \frac{M^2}{\bar{N}^2}\right) \quad (10)$$

with $\bar{N} = Ne^\gamma$ and

$$\hat{g}_0(\alpha_s) = \bar{g}_0(\alpha_s)\exp\left[2\zeta_2 A(\alpha_s) + \frac{8}{3}\zeta_3\beta_0\frac{C_F}{\pi}\alpha_s^2\right], \quad (11)$$

$$\hat{\mathcal{S}}_{\rm QCD}\left(M^2, \frac{M^2}{\bar{N}^2}\right) = \int_{M^2}^{M^2/\bar{N}^2}\frac{d\mu^2}{\mu^2}\left[A\left(\alpha_s(\mu^2)\right)\ln\frac{M^2}{\mu^2\bar{N}^2} + \hat{D}\left(\alpha_s(\mu^2)\right)\right], \quad (12)$$

$$A(\alpha_s) = \frac{A_1}{4}\alpha_s + \frac{A_2}{16}\alpha_s^2 + \frac{A_3}{64}\alpha_s^3 + O(\alpha_s^4), \quad (13)$$

$$\hat{D}(\alpha_s) = \frac{1}{2}D(\alpha_s) - 2\zeta_2\frac{C_F}{\pi}\beta_0\alpha_s^2 = \hat{D}_2\alpha_s^2 + O(\alpha_s^3), \quad (14)$$

and $\beta_0 = (11C_A - 2n_f)/(12\pi)$.

The functions A, D and $\bar{g}_0$ are computed at NNLL order according to Table 1; the relevant coefficients can be found in Ref. [13], except the two-loop contribution to $\bar{g}_0$, which can be determined by matching the expansion of Eq. (3) to the NNLO Drell-Yan cross section in Ref. [16],

$$\bar{g}_0(\alpha_s) = 1 + \frac{\alpha_s}{\pi}C_F\,(2\zeta_2 - 4)$$

$$+\frac{\alpha_s^2}{\pi^2}\frac{C_F}{16}\Bigg[C_A\left(-\frac{12}{5}\zeta_2^2+\frac{592}{9}\zeta_2+28\zeta_3-\frac{1535}{12}\right)+C_F\left(\frac{72}{5}\zeta_2^2-70\zeta_2-60\zeta_3+\frac{511}{4}\right)+n_f\left(8\zeta_3-\frac{112}{9}\zeta_2+\frac{127}{6}\right)\Bigg], \quad (15)$$

and the coefficient A_2, which we give here for completeness

$$A_2=\frac{4C_F}{\pi^2}\left[\left(\frac{67}{9}-2\zeta_2\right)C_A-\frac{10}{9}n_f\right]. \quad (16)$$

The explicit expression of A_3 is not needed here.

On the other hand, at NNLL the Mellin transform of the SCET result, Eq. (5), can be written as [13]

$$C_{\rm SCET}(N,M^2,\mu_s^2)=\hat{H}(M^2)\,E\left(\frac{M^2}{\bar{N}^2},\mu_s^2\right)\times\exp\hat{\mathcal{S}}_{\rm SCET}(N,M^2,\mu_s^2), \quad (17)$$

with

$$\hat{H}(M^2)=H(M^2)\exp\left[\frac{\zeta_2}{2}\frac{C_F}{\pi}\alpha_s(M^2)\right], \quad (18)$$

$$E\left(\frac{M^2}{\bar{N}^2},\mu_s^2\right)=\tilde{s}_{\rm DY}\left(\ln\frac{M^2}{\mu_s^2\bar{N}^2},\mu_s^2\right)\exp\left[-\frac{\zeta_2}{2}\frac{C_F}{\pi}\alpha_s(\mu_s^2)\right], \quad (19)$$

$$\hat{\mathcal{S}}_{\rm SCET}(N,M^2,\mu_s^2)=\int_{M^2}^{\mu_s^2}\frac{d\mu^2}{\mu^2}\left[\Gamma_{\rm cusp}\left(\alpha_s(\mu^2)\right)\ln\frac{M^2}{\mu^2\bar{N}^2}+\hat{\gamma}_W\left(\alpha_s(\mu^2)\right)\right], \quad (20)$$

$$\hat{\gamma}_W(\alpha_s)=\gamma_W(\alpha_s)-\frac{\zeta_2}{2}\frac{C_F}{\pi}\beta_0\alpha_s^2, \quad (21)$$

and $\hat{\gamma}_W(\alpha_s)=\hat{D}(\alpha_s)$ at this order.

In comparison to Ref. [13], we now also include the two-loop contributions to the functions H and $\tilde{s}_{\rm DY}$, which were given explicitly in Ref. [12]. Note that, in order to be accurate to order α_s^2, the definition of the function E slightly differs from Ref. [13].

Putting everything together we find

$$C_r(N,M^2,\mu_s^2)=\frac{\hat{g}_0\left(\alpha_s(M^2)\right)}{\hat{H}(M^2)E\left(\frac{M^2}{\bar{N}^2},\mu_s^2\right)}\exp\hat{\mathcal{S}}\left(\mu_s^2,\frac{M^2}{\bar{N}^2}\right) \quad (22)$$

with

$$\hat{\mathcal{S}}\left(\mu_s^2,\frac{M^2}{\bar{N}^2}\right)=\int_{\mu_s^2}^{M^2/\bar{N}^2}\frac{d\mu^2}{\mu^2}\left[A\left(\alpha_s(\mu^2)\right)\ln\frac{M^2}{\mu^2\bar{N}^2}+\hat{D}\left(\alpha_s(\mu^2)\right)\right]. \quad (23)$$

It is easy to see that

$$\frac{\hat{g}_0\left(\alpha_s(M^2)\right)}{\hat{H}(M^2)E(M^2,M^2)}=1+O\left(\alpha_s^3(M^2)\right), \quad (24)$$

so to NNLL accuracy Eq. (22) can be written

$$C_r(N,M^2,\mu_s^2)=\frac{E(M^2,M^2)}{E\left(\frac{M^2}{\bar{N}^2},\mu_s^2\right)}\exp\hat{\mathcal{S}}\left(\mu_s^2,\frac{M^2}{\bar{N}^2}\right). \quad (25)$$

Using the 2-loop expression of $\tilde{s}_{\rm DY}$ from Ref. [12] in Eq. (19), we find

$$E\left(\frac{M^2}{\bar{N}^2},\mu_s^2\right)=1+E_1(L)\alpha_s(\mu_s^2)+E_2(L)\alpha_s^2(\mu_s^2)+O(\alpha_s^3) \quad (26)$$

where

$$E_1(L)=\frac{A_1}{8}L^2, \quad (27)$$

$$E_2(L)=\frac{A_1^2}{128}L^4-\frac{L^3}{3}\frac{A_1}{8}\beta_0+\frac{L^2}{2}\frac{A_2}{16}+L\hat{D}_2+\frac{C_AC_F}{\pi^2}\left[\frac{607}{324}+\frac{67}{144}\zeta_2-\frac{3}{4}\zeta_2^2-\frac{11}{72}\zeta_3\right]+\frac{C_Fn_f}{\pi^2}\left[-\frac{41}{162}-\frac{5}{72}\zeta_2+\frac{\zeta_3}{36}\right], \quad (28)$$

and

$$L\equiv\ln\frac{M^2}{\mu_s^2\bar{N}^2}. \quad (29)$$

Note that $L=0$ when the two arguments of E are equal to each other.

Eq. (25) establishes our first new result. Indeed, it is immediate to check that $C_r(N,M^2,\mu_s^2)=1$ for $\mu_s=M/\bar{N}$, up to subleading (NNNLL*) terms. This means that with this scale choice the SCET result now reproduces the QCD result to full NNLL accuracy, rather than to the lower NNLL* accuracy of Ref. [12].

We are however interested in studying C_r for generic scale choices, and in particular with the BN scale choice. The result becomes especially transparent by casting the ratio $E(M^2,M^2)/E\left(\frac{M^2}{\bar{N}^2},\mu_s^2\right)$ Eq. (25) in the form of an exponential of an integral, of the same kind as the form adopted in Eq. (23). This can be done at the price of including terms of order α_s^3 or higher in the ratio, which is allowed at NNLL. The ensuing expression of C_r is particularly simple and suitable for analytic comparisons. It should however be kept in mind that a numerical comparison of the SCET and QCD expressions should rather be performed using the exact expression Eq. (25), and possibly also retaining the subleading terms in Eq. (24).

We get

$$\ln \frac{E(M^2,M^2)}{E\left(\frac{M^2}{\bar{N}^2},\mu_s^2\right)} = -\alpha_s(\mu_s^2)\frac{A_1}{4}\frac{L^2}{2} + \alpha_s^2(\mu_s^2)\left[\beta_0\frac{A_1}{8}\frac{L^3}{3} - \frac{A_2}{16}\frac{L^2}{2} - \hat{D}_2 L\right] + O(\alpha_s^3). \tag{30}$$

Using

$$\frac{L^{k+1}}{k+1} = \int_{\mu_s^2}^{M^2/\bar{N}^2}\frac{d\mu^2}{\mu^2}\,\ln^k\frac{M^2}{\mu^2\bar{N}^2} \tag{31}$$

and taking the running of α_s into account, we finally obtain

$$C_r(N,M^2,\mu_s^2) = \exp\int_{\mu_s^2}^{M^2/\bar{N}^2}\frac{d\mu^2}{\mu^2}\,\ln\frac{M^2}{\mu^2\bar{N}^2} \times\left[A\left(\alpha_s(\mu^2)\right) - \frac{A_1\alpha_s(\mu^2)}{4} - \frac{A_2\alpha_s^2(\mu^2)}{16}\right], \tag{32}$$

which is our NNLL master QCD-SCET comparison formula. It generalizes to full NNLL the result of Ref. [13]. Its most notable feature, which determines the relative accuracy of the comparison, is that (recall the expansion Eq. (13)) the exponent in Eq. (32) is of order α_s^3. Note that this is however due to the exponentiation Eq. (30). If one does not exponentiate (as in the original SCET expression), when expanding C_r in powers of α_s, terms proportional to A_1 and A_2 only cancel up to $O(\alpha_s^2)$.

4. The Becher-Neubert scale choice

As briefly discussed in Sect. 1, the BN approach is based on the idea of choosing for μ_s a scale determined by hadronic, rather than partonic kinematics, namely $\mu_s = M(1-\tau)$. In Ref. [11, 12] a more general choice $\mu_s = M(1-\tau)g(\tau)$ was considered, with $g(\tau) = \text{const.} + O(1-\tau)$: the distinction may be relevant for phenomenology, but it is immaterial for our present goal, which is to determine the logarithmic accuracy of the SCET result with this scale choice.

Since the variable τ refers to hadron kinematics, the comparison can only be performed at the level of the physical cross section, Eq. (1). We therefore define

$$\sigma_{\text{QCD}}(\tau,M^2) = \int_\tau^1\frac{dz}{z}\,\mathcal{L}\left(\frac{\tau}{z}\right)C_{\text{QCD}}(z,M^2), \tag{33}$$

$$\sigma_{\text{SCET}}(\tau,M^2) = \int_\tau^1\frac{dz}{z}\,\mathcal{L}\left(\frac{\tau}{z}\right)C_{\text{SCET}}(z,M^2,M^2(1-\tau)^2), \tag{34}$$

where $C_{\text{QCD}}(z,M^2)$ and $C_{\text{SCET}}(z,M^2,\mu_s^2)$ are the inverse Mellin transforms of Eqs. (10) and (17), respectively, and in the SCET case after performing the inverse Mellin transform at fixed μ_s we have set $\mu_s = M(1-\tau)$. Of course, $C_{\text{QCD}}(z,M^2)$ is given by a divergent series in powers of $\alpha_s(M^2)$, so it should be understood as the order-by-order Mellin inversion up to arbitrarily high but finite order. Using Eq. (9) we find

$$\sigma_{\text{QCD}}(\tau,M^2) = \int_\tau^1\frac{dz}{z}\,\sigma_{\text{SCET}}\left(\frac{\tau}{z},M^2\right)C_r(z,M^2,M^2(1-\tau)^2) \tag{35}$$

where $C_r(z,M^2,M^2(1-\tau)^2)$ is the inverse Mellin transform of Eq. (25), performed at fixed μ_s and evaluated at $\mu_s = M(1-\tau)$.

In order to compute $C_r(z,M^2,M^2(1-\tau)^2)$ it is convenient to rewrite Eq. (25) as

$$C_r(N,M^2,\mu_s^2) = \frac{E(M^2,M^2)}{E(\mu_s^2,\mu_s^2)}\frac{E(\mu_s^2,\mu_s^2)}{E\left(\frac{M^2}{\bar{N}^2},\mu_s^2\right)}\exp\hat{\mathcal{S}}\left(\mu_s^2,\frac{M^2}{\bar{N}^2}\right). \tag{36}$$

The first ratio is just a function of $\alpha_s(M^2)$ and $\alpha_s(\mu_s^2)$, independent of N, while

$$\frac{E(\mu_s^2,\mu_s^2)}{E\left(\frac{M^2}{\bar{N}^2},\mu_s^2\right)}\exp\hat{\mathcal{S}}\left(\mu_s^2,\frac{M^2}{\bar{N}^2}\right) = 1 + F_r\left(\alpha_s(\mu_s^2),L\right), \tag{37}$$

where $F_r(\alpha_s,L)$ is of order α_s^3.

The inverse Mellin transform can be now computed using the results of Ref. [13]. We find

$$\sigma_{\text{QCD}}(\tau,M^2) = \frac{E(M^2,M^2)}{E(\mu_s^2,\mu_s^2)}\left[\sigma_{\text{SCET}}(\tau,M^2) + F_r\left(\alpha_s(\mu_s^2),2\frac{\partial}{\partial\xi}\right)\Sigma(\tau,M^2,\xi)\Big|_{\xi=0}\right], \tag{38}$$

where

$$\Sigma(\tau,M^2,\xi) = \frac{(1-\tau)^{-\xi}}{e^{\gamma\xi}\Gamma(\xi)}\int_\tau^1\frac{dz}{z}\,\sigma_{\text{SCET}}\left(\frac{\tau}{z},M^2\right)\ln^{\xi-1}\frac{1}{z} \tag{39}$$

for $\mu_s = M(1-\tau)$. We have shown in App. B of Ref. [13] that $\Sigma(\tau,M^2,\xi)$ can be expressed in terms of derivatives of $\sigma_{\text{SCET}}(\tau,M^2)$ with respect to $\ln(1-\tau)$, up to terms suppressed by positive powers of $(1-\tau)$:

$$\Sigma(\tau,M^2,\xi) = \sum_{k=0}^\infty c_k(\xi)\frac{d^k\sigma_{\text{SCET}}(\tau,M^2)}{d\ln^k(1-\tau)}\left[1+O(1-\tau)\right], \tag{40}$$

with coefficients c_k which do not depend on τ. It follows that the term proportional to F_r in Eq. (38) does not contain any extra logarithmic enhancement with respect to

$\sigma_{\rm SCET}(\tau, M^2)$. On the other hand

$$\frac{E(M^2, M^2)}{E(\mu_s^2, \mu_s^2)} = 1 + E_2(0)\left[\alpha_s^2(M^2) - \alpha_s^2(\mu_s^2)\right] + \dots \quad (41)$$
$$= 1 + 4E_2(0)\beta_0\alpha_s^3(M^2)\ln(1-\tau) + O(\alpha_s^4).$$

We conclude that the leading difference between the QCD and SCET expressions is

$$\sigma_{\rm QCD}(\tau, M^2) - \sigma_{\rm SCET}(\tau, M^2) = \\ = \sigma_{\rm SCET}(\tau, M^2)\left[\alpha_s^3 4\beta_0 E_2(0)\ln(1-\tau) + \dots\right], \quad (42)$$

where the ellipse denotes terms which are either of relative order $O(\alpha_s^3)$, but without any logarithmic enhancement, or $O(\alpha_s^4)$.

Because the log counting is now done at the level of hadronic cross sections, it is based on counting powers of $\ln(1-\tau)$. Also, because the SCET result violates standard QCD factorization (i.e., it does not factorize upon Mellin transformation), the difference between $\sigma_{\rm QCD}$ and $\sigma_{\rm SCET}$ depends on the parton luminosity (it is not universal) through $\sigma_{\rm SCET}$ itself. A generic leading-log term in $\sigma_{\rm SCET}$ has the form

$$\sigma_{\rm SCET} \sim \alpha_s^k \ln^{2k+p}(1-\tau), \quad (43)$$

where $\alpha_s^k \ln^{2k}(1-\tau)$ is due to the leading log behavior of the coefficient function, and $\ln^p(1-\tau)$ generally comes from the parton luminosity.

If we assume that the parton luminosity does not lead to any logarithmic enhancement, then

$$\sigma_{\rm QCD} - \sigma_{\rm SCET} \sim \alpha_s^{k+3}\ln^{2k+1}(1-\tau) = \alpha_s^h \ln^{2h-5}(1-\tau), \quad (44)$$

where we have set $h = k + 3$. This corresponds to a NNNLL* correction. It is interesting to observe that, had we used the exponentiated version Eq. (32) of C_r, a NNNLL, rather than NNNLL* correction, would have been obtained. The argument can be generalized to the case in which C_r is computed to all orders in α_s rather than just to order α_s^2. Indeed, no leading logarithmic enhancement arises from the factor $1 + F_r(\alpha_s(\mu_s^2), L)$; the only possible source of powers of $\ln(1-\tau)$ in C_r is the ratio $E(M^2, M^2)/E(\mu_s^2, \mu_s^2)$. It is easy to see, however, that all terms in the expansion Eq. (41) are at most of order $\alpha_s^k(M^2)\ln^{k-2}(1-\tau)$, with $k \geq 3$. Thus, the conclusion Eq. (44) holds to all orders in α_s.

In conclusion, we restate three observations which were already made in Ref. [13]. First, we note that the BN scale choice removes the divergence of the perturbative expansion at the cost of introducing logarithmically suppressed non-universal terms. This is to be contrasted with the commonly used Minimal prescription [8], which also introduces non-universal terms (with support outside the physically accessible kinematic region) but are more suppressed than any power, or with the Borel prescription [9, 10], which introduces power-suppressed but universal terms.

Second, we observe that quite in general we do expect PDFs to contain logarithmically enhanced terms. In this case the terms introduced by the BN scale choice to tame the perturbative divergence can become leading or even super-leading (i.e., more logarithmically enhanced than the leading log).

Finally, we remark that threshold resummation is often useful in situations where τ is far from threshold, but nevertheless the partonic subprocess is close to threshold [5, 3, 4]. In this case $M(1-\tau) \sim M$, and consequently C_r Eq. (25) is actually leading log.

The phenomenological implications of our results remain to be investigated. They are potentially of considerable interest, given the increasingly important role that threshold resummation, in its various implementations, is playing for LHC phenomenology.

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