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Higgs production in gluon fusion beyond NNLO

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Abstract:

We construct an approximate expression for the cross section for Higgs production in gluon fusion at next-to-next-to-next-to-leading order (N^{3}LO) in α_s with finite top mass. We argue that an accurate approximation can be constructed by exploiting the analyticity of the Mellin space cross section, and the information on its singularity structure coming from large N (soft gluon, Sudakov) and small N (high energy, BFKL) all order resummation. We support our argument with an explicit comparison of the approximate and the exact expressions up to the highest (NNLO) order at which the latter are available. We find that the approximate N^{3}LO result amounts to a correction of 17% to the NNLO QCD cross section for production of a 125 GeV Higgs at the LHC (8 TeV), larger than previously estimated, and it significantly reduces the scale dependence of the NNLO result.

1 Introduction

The dominant Higgs production mechanism at the LHC is gluon fusion via a heavy fermion loop (mainly a top quark) [1], and indeed the recent announcement of the discovery of a Higgs-like particle [2, 3] is largely based on events in this channel. In view of this, an accurate determination of the cross section in this channel is of great interest. Next-to-leading order (NLO) corrections to the inclusive cross section, originally computed in Refs. [4, 5] in the large top mass ($m_t \to \infty$) approximation, and in Ref. [6] for general m_t are known to be as large as the leading order, and the NNLO corrections (first computed in Refs. [7–9] in the $m_t \to \infty$ limit and for finite top mass in Refs. [10–15]) about half as large as the leading order. The significant scale dependence of the NNLO result suggests that corrections at yet higher orders are not negligible: in fact they currently account for half or more of the uncertainty on the theory prediction for the cross section [16] (the other half being due to parton distributions and the strong coupling).

While computations of the full N^{3}LO correction to the cross section are in progress [17–19], it is interesting to derive approximate expressions for it. Several of us have argued (see e.g. [20–22]) that accurate approximations to partonic cross sections may be obtained from knowledge of their N space singularity structure, both at finite perturbative order, and at the resummed level. Because the $N \to \infty$ singularity and the rightmost singularity at finite N are known to all orders in α_s respectively from threshold (Sudakov) and high energy (BFKL) resummation, if this is indeed the case it is possible to construct reliable approximations even to very high orders in α_s. The possibility of constructing approximations based on the combination of results from large and small N resummation has also been considered in [23, 24].

In this paper, we will pursue this idea in the context of Higgs production in gluon fusion: we will determine the dominant small N and large N singularities up to N^{3}LO from resummation arguments, and, after testing our methodology against known results up to NNLO, we will use them to construct a N^{3}LO approximation.

2 The partonic cross section and its singularities

The factorized Higgs production cross section is

$$\sigma(\tau, m_H^2) = \tau \sum_{ij} \int_\tau^1 \frac{dz}{z} \mathscr{L}_{ij}\left(\frac{\tau}{z}, \mu_F^2\right) \frac{1}{z} \hat{\sigma}_{ij}\left(z, m_H^2, \alpha_s(\mu_R^2), \frac{m_H^2}{\mu_F^2}, \frac{m_H^2}{\mu_R^2}\right), \qquad \tau = \frac{m_H^2}{s}, \tag{2.1}$$

where $\mathscr{L}_{ij}(z, \mu^2)$ are the parton luminosities

$$\mathscr{L}_{ij}(z, \mu^2) = \int_z^1 \frac{dx}{x} f_i\left(\frac{z}{x}, \mu^2\right) f_j(x, \mu^2). \tag{2.2}$$

We introduce coefficient functions C_{ij}, defined as

$$\hat{\sigma}_{ij}\left(z, m_H^2, \alpha_s(\mu_R^2), \frac{m_H^2}{\mu_F^2}, \frac{m_H^2}{\mu_R^2}\right) = z\, \sigma_0\big(m_H^2, \alpha_s(\mu_R^2)\big)\, C_{ij}\left(z, \alpha_s(\mu_R^2), \frac{m_H^2}{\mu_F^2}, \frac{m_H^2}{\mu_R^2}\right), \tag{2.3}$$

where σ_0 is the leading order (LO) partonic cross section, so that the coefficient function is normalized to $\delta(1-z)$ at leading order:

$$C_{ij}(z, \alpha_s) = \delta(1-z)\delta_{ig}\delta_{jg} + \alpha_s C_{ij}^{(1)}(z) + \alpha_s^2 C_{ij}^{(2)}(z) + \alpha_s^3 C_{ij}^{(3)}(z) + \mathcal{O}(\alpha_s^4), \tag{2.4}$$

and for simplicity, we have suppressed the dependence on renormalization and factorization scales μ_F, μ_R. In the sequel, we will concentrate on the gluon fusion subprocess, while the contribution from other subprocesses will be only briefly discussed in Section 4, so in most of the discussion below we will drop the parton indices ij, and assume that both the coefficient function and luminosity refer to the gluon channel.

Because the cross section Eq. (2.1) is a convolution, its Mellin transform

$$\sigma(N, m_H^2) \equiv \int_0^1 d\tau\, \tau^{N-2} \sigma(\tau, m_H^2) \tag{2.5}$$

factorizes in terms of the Mellin space luminosity and coefficient function, respectively defined as

$$\begin{aligned} \mathscr{L}(N) &\equiv \int_0^1 dz\, z^{N-1} \mathscr{L}(z) \\ C(N, \alpha_s) &\equiv \int_0^1 dz\, z^{N-1} C(z, \alpha_s), \end{aligned} \tag{2.6}$$

according to

$$\sigma(N, m_H^2) = \sigma_0\big(m_H^2, \alpha_s\big) \mathscr{L}(N) C(N, \alpha_s). \tag{2.7}$$

While in momentum space the coefficient functions are distributions, if the Mellin transform integral has a finite convergence abscissa, the N space coefficient function is an analytic function of the complex variable N, given by the integral representation Eq. (2.6) to the right of the convergence abscissa, and by analytic continuation elsewhere. Therefore, it is fully determined by knowledge of its singularities.

The singularity structure of the perturbative expansion of $C(N, \alpha_s)$ is relatively simple. At any perturbative order, the rightmost singularity is a multiple pole located at $N = 1$ [25], with further multiple poles along the real axis at $N = 0, -1, -2, \ldots$, with residues of order one (this is also what is found in all known fixed order calculations); $\mathrm{Re}\, N = 1$ is the convergence abscissa of the Mellin transform, and as $N \to \infty$, $C(N, \alpha_s)$ grows as a power of $\ln N$. While knowledge of the residues of all poles is required in order to fully determine the function $C(N, \alpha_s)$, its behavior in the physical region $1 \le \mathrm{Re}\, N < \infty$ is mostly controlled by the residues of the leading (rightmost) pole at $N = 1$, together with that of the singularity at infinity. Both are known from resummation: Sudakov (soft gluon) resummation determines to all orders in the strong coupling the coefficients of the $\ln^m N$ terms which control the behavior as $N \to \infty$, while BFKL (high energy) resummation determines the residues of the leading $\frac{1}{(N-1)^n}$ multiple poles.

This suggests that an approximation of the coefficient function Eq. (2.6) may be constructed by simply combining the large N (soft) and small N (high energy) terms,

$$C_{\text{approx}}(N, \alpha_s) = C_{\text{soft}}(N, \alpha_s) + C_{\text{h.e.}}(N, \alpha_s), \tag{2.8}$$

where C_{soft} contains terms predicted by Sudakov resummation and $C_{\text{h.e.}}$ terms predicted by BFKL resummation. It is clear, however, that this is only correct if the small N singularities, controlled by $C_{\text{h.e.}}$, are unaffected by C_{soft}, while the large N logarithms, controlled by C_{soft}, are unaffected by $C_{\text{h.e.}}$. This is clearly nontrivial: for example, a term proportional to $\ln^m N$ has a cut at $N = 0$, while at each fixed order the expected behavior of the coefficient function is a pole, rather that a cut. So the approximate expressions for C_{soft} and $C_{\text{h.e.}}$ should reproduce this behavior, with no spurious singularities.

We will show in the sequel that an approximate expression of the form of Eq. (2.8) is possible, but both C_{soft} and $C_{\text{h.e.}}$ will have to be carefully constructed. Indeed we will now show that constructing C_{soft} in such a way that the small N singularity structure is preserved, the agreement at large N is considerably improved. This result may seem surprising, but it is in fact a consequence of analyticity.

2.1 Large N

We first discuss the computation of the large N (soft) part of the coefficient function. All contributions to $C(N, \alpha_s)$ which do not vanish as $N \to \infty$ may be computed from Sudakov resummation, using techniques summarized long ago in Ref. [26]. The resummed coefficient function has the form

$$C_{\rm res}(N,\alpha_s) = g_0(\alpha_s)\exp\left[\frac{1}{\alpha_s}g_1(\alpha_s\ln N) + g_2(\alpha_s\ln N) + \alpha_s g_3(\alpha_s\ln N) + \dots\right], \tag{2.9}$$

with

$$g_0(\alpha_s) = 1 + \alpha_s g_{0,1} + \alpha_s^2 g_{0,2} + \mathcal{O}(\alpha_s^3), \tag{2.10}$$

$$g_i(\lambda) = \sum_{k=k_{0,i}}^{\infty} g_{i,k}\lambda^k, \quad \text{for } i\geq 1, \qquad k_{0,1}=2, \quad k_{0,i\geq 2}=1. \tag{2.11}$$

Inclusion of all g_i with $1 \leq i \leq k+1$ and of g_0 up to order α_s^{k-1} gives the next^k-to-leading log approximation to $\ln C_{\rm res}(N,\alpha_s)$; it determines the coefficient of all contributions to the coefficient function of the form $\alpha_s^n \ln^m N$ with $2(n-k)+1 \leq m \leq 2n$. This can be extended to $2(n-k) \leq m \leq 2n$ by also including the order α_s^k contribution to g_0 . The functions g_1, g_2 and g_3 are known exactly, while g_0 is known up to $\mathcal{O}(\alpha_s^2)$. The function g_4 is only known in part [27,28], but the missing information (the 4-loop cusp anomalous dimension) only enters at $\mathcal{O}(\alpha_s^4)$. We can thus determine all large N non-vanishing contributions to $C(N,\alpha_s)$ up to $\mathcal{O}(\alpha_s^2)$, and all logarithmically enhanced contributions (but not the constant) to $\mathcal{O}(\alpha_s^3)$.

The accuracy of an approximation to the Higgs production cross section at the LHC based on the dominance of threshold terms can be studied [29] by using the saddle point method to determine which is the region in N space that gives the bulk of the contribution to the cross section. It turns out that, despite the fact that Higgs production at the LHC is far from the kinematic threshold, partly because of the underlying partonic kinematics and partly because of the shape of the cross section, at the LHC with 8 TeV center-of-mass energy, logarithmically enhanced terms are still providing most of the cross section, though the situation gradually changes as the center-of-mass energy increases.

However, our goal here is to construct an approximation to the coefficient function which holds for all N in the physical region. Now, it has been observed long ago [30] that the the quality of the soft approximation to the full coefficient function significantly depends on the choice of subleading terms which are included in the resummed result: indeed, while resummation uniquely determines the coefficients of logarithmically enhanced terms, there is a certain latitude in defining how the soft approximation is constructed, by making choices which differ by terms which vanish as $N \to \infty$. A similar situation has been observed recently in Drell-Yan production at the LHC [21], for which the threshold approximation is generally expected to be less good than for Higgs production. By comparing results which differ by terms which vanish as $N \to \infty$, we will now show that several preferred choices for such subleading terms are favored by the requirement that some aspects of the known small N singularity structure of the exact result be reproduced.

In order to outline our strategy, let us work with the simplest example. Let us first suppose that we know the N space resummed coefficient function and that we want to extract from Eq. (2.9) an approximate expression for the $\mathcal{O}(\alpha_s)$ coefficient $C^{(1)}(z)$, which is given by [6,31]

$$C^{(1)}(z) = 4A_g(z)\,\mathcal{D}_1(z) + d\,\delta(1-z) - 2A_g(z)\frac{\ln z}{1-z} + \mathcal{R}_{gg}(z), \tag{2.12}$$

$$\mathcal{D}_k(z) \equiv \left(\frac{\ln^k(1-z)}{1-z}\right)_+, \tag{2.13}$$

$$A_g(z) \equiv \frac{C_A}{\pi}\frac{1-2z+3z^2-2z^3+z^4}{z}. \tag{2.14}$$

The constant d and the function $\mathcal{R}_{gg}(z)$ are known functions of m_H/m_t; in particular $\mathcal{R}_{gg}(z)$ is an ordinary function, regular in $z=1$, so its Mellin transform vanishes as $N \to \infty$ and therefore its specific form is of no relevance for the large N behavior.

Expanding Eq. (2.9) to $\mathcal{O}(\alpha_s)$, and keeping NLL terms, we find

$$C_{\rm res}(N,\alpha_s) = 1 + \alpha_s C^{(1)}_{\rm res}(N) + \mathcal{O}(\alpha_s^2), \tag{2.15}$$
$$C^{(1)}_{\rm res}(N) = g_{1,2}\ln^2 N + g_{2,1}\ln N + g_{0,1}, \tag{2.16}$$

with

$$g_{1,2} = \frac{2C_A}{\pi}, \qquad g_{2,1} = \frac{4C_A}{\pi}\gamma_E, \tag{2.17}$$

where γ_E is the Euler-Mascheroni constant. The asymptotic behavior of the $\mathcal{O}(\alpha_s)$ coefficient as $N \to \infty$ is correctly reproduced by this expression, in that

$$\lim_{N\to\infty}\Big[C^{(1)}_{\rm res}(N) - C^{(1)}(N)\Big] = 0, \tag{2.18}$$

where $C^{(1)}(N)$ is the Mellin transform of Eq. (2.12); the constant $g_{0,1}$ is fixed by this condition.

On the other hand, the behavior of Eq. (2.16) at small values of N is incompatible with the known singularity structure. In particular, there is a logarithmic branch cut starting at $N = 0$ which is definitely unphysical, as the exact coefficient function has poles and not cuts at small N. This cut is a subleading singularity, given that the leading singularity is located at $N = 1$, but close enough to the leading one that the behavior of the coefficient function can be significantly affected. Even if we plan to eventually improve this expression by introducing the correct singularity at $N = 1$ according to Eq. (2.8), the logarithmic singularity will interfere with it and spoil the accuracy of the approximation.

This problem, however, is an artifact of the large N approximation, since powers of $\ln N$ are the large N approximation of powers of the digamma function $\psi_0(N)$ appearing in fixed order computations. Indeed, the inverse Mellin transform of Eq. (2.16) (using Eq. (A.6b) of Appendix A.1) is seen to be

$$\begin{aligned} C^{(1)}_{\rm res}(z,\alpha_s) &= g_{0,1}\delta(1-z) + 2g_{1,2}\mathcal{D}^{\log}_1(z) + (2\gamma_E g_{1,2} - g_{2,1})\mathcal{D}^{\log}_0(z), \\ &= g_{0,1}\delta(1-z) + \frac{4C_A}{\pi}\mathcal{D}^{\log}_1(z), \end{aligned} \tag{2.19}$$

where

$$\mathcal{D}^{\log}_k(z) \equiv \left(\frac{\ln^k \ln\frac{1}{z}}{\ln\frac{1}{z}}\right)_+, \tag{2.20}$$

which is seen to differ from the soft contribution Eq. (2.13) to the exact result Eq. (2.12).

This can be understood noting that singular terms as $z \to 1$ arise from integration of the real emission diagrams over the transverse momentum of the gluon, which has the form

$$p_{gg}(z)\int_\Lambda^{\frac{M(1-z)}{\sqrt{z}}}\frac{dk_T}{k_T} = \frac{A_g(z)}{1-z}\left(\ln\frac{1-z}{\sqrt{z}} + \ln\frac{M}{\Lambda}\right), \tag{2.21}$$

where Λ is a collinear cut-off and $p_{gg}(z)$ is the LO gluon-gluon Altarelli-Parisi splitting function for $z < 1$,

$$p_{gg}(z) = \frac{A_g(z)}{1-z}, \tag{2.22}$$

with $A_g(z)$ given by Eq. (2.14).

Indeed, Eq. (2.21) shows that logarithmically enhanced soft terms, rather than being proportional to $\frac{\ln\ln\frac{1}{z}}{\ln\frac{1}{z}}$, are of the form

$$\frac{1}{1-z}\ln\frac{1-z}{\sqrt{z}} = \frac{1}{1-z}\Big[\ln(1-z) + \mathcal{O}(1-z)\Big], \tag{2.23}$$

and they appear with a coefficient proportional to the Altarelli-Parisi splitting function. Explicitly, the latter in the $z \to 1$ limit may be expanded as

$$A_g(z) = \frac{C_A}{\pi}\Big[1 - (1-z) + 2(1-z)^2 + \mathcal{O}\big[(1-z)^3\big]\Big]. \tag{2.24}$$

Logarithmically enhanced contributions to the coefficient function are generated by the first terms in both expansions Eqs. (2.23) and (2.24), namely $\ln(1-z)$ and $A_g(1)$ respectively.

We will now argue that an optimal choice of the soft approximation, differing from Eq. (2.19) by subleading terms, is obtained by writing the large soft logs as powers of $\ln \frac{1-z}{\sqrt{z}}$, so in particular retaining the $\sqrt{z}$ in the denominator despite the fact that it is subleading, and furthermore, by retaining at least the first correction on the right-hand side of Eq. (2.24), also subleading. Therefore in this case our suggestion consists in the simple replacement

$$A_g(1)\,\mathcal{D}_1^{\log}(z) \to A_{g,m}(z)\,\hat{\mathcal{D}}_1(z) \tag{2.25}$$

in Eq. (2.19), where $A_{g,m}(z)$ is a finite m-th order expansion of $A_g(z)$ about $z = 1$, Eq. (2.24), and

$$\hat{\mathcal{D}}_1(z) \equiv \left(\frac{\ln(1-z)}{1-z}\right)_+ - \frac{\ln\sqrt{z}}{1-z}. \tag{2.26}$$

Note that we have chosen to apply the plus prescription only to the first term, singular in $z = 1$, which is the natural choice in fixed order calculations. In this way, $\hat{\mathcal{D}}_1(N)$ differs from $\mathcal{D}_1(N)$ only by terms vanishing at large N. Since $\mathcal{D}_1^{\log}(N)$ and $\hat{\mathcal{D}}_1(N)$ differ at large N by a constant, the coefficient $g_{0,1}$ must be modified accordingly, in order that the requirement Eq. (2.18) be satisfied. These technical details are discussed in Appendix A.1.

Our conclusion Eq. (2.25) relies on the following arguments:

- The replacement of $\mathcal{D}_1^{\log}(z)$, whose Mellin transform is

$$\mathcal{D}_1^{\log}(N) = \frac{1}{2}\big[\ln^2 N + 2\gamma_E \ln N\big], \tag{2.27}$$

  with $\hat{\mathcal{D}}_1(z)$, whose Mellin transform is

$$\hat{\mathcal{D}}_1(N) = \frac{1}{2}\big[\psi_0^2(N) + 2\gamma_E\psi_0(N) + \zeta_2 + \gamma_E^2\big] \tag{2.28}$$

  removes the logarithmic branch cut of $\mathcal{D}_1^{\log}(N)$, which is incompatible with the known analytic structure of the coefficient function. The only singularities are now isolated poles, as in the exact expression.

- The same features are shared by the Mellin transform of $\mathcal{D}_1(z)$, that is

$$\mathcal{D}_1(N) = \frac{1}{2}\big[\psi_0^2(N) - \psi_1(N) + 2\gamma_E\psi_0(N) + \zeta_2 + \gamma_E^2\big]. \tag{2.29}$$

  However, the presence of $\psi_1(N)$ exactly cancels the double poles of $\psi_0^2(N)$ in $N = 0, -1, -2, \ldots$, which are there in the exact result. Therefore, the choice of $\hat{\mathcal{D}}_1(N)$ is preferred over $\mathcal{D}_1(N)$.

- In the replacement Eq. (2.25) the factor $A_g(z)$ is expanded up to a finite order $m > 0$ about $z = 1$. This is because the inclusion of the full $A_g(z)$ would introduce a spurious singularity in $N = 1$. Indeed, the Mellin transform of $A_g(z)\,\hat{\mathcal{D}}_1(z)$ is given by

$$\int_0^1 dz\, z^{N-1} A_g(z)\,\hat{\mathcal{D}}_1(z) = \frac{C_A}{\pi}\Big[\hat{\mathcal{D}}_1(N-1) - 2\hat{\mathcal{D}}_1(N) + 3\hat{\mathcal{D}}_1(N+1) - 2\hat{\mathcal{D}}_1(N+2) + \hat{\mathcal{D}}_1(N+3)\Big]. \tag{2.30}$$

The first term, due to $1/z$ in $A_g(z)$, has a double and a simple pole in $N = 1$, while the exact singularity is a simple pole, with a (m_H/m_t)-dependent coefficient controlled by small z resummation. The expansion of $A_g(z)$ in powers of $1 - z$ to any finite order is not singular in $z = 0$, and therefore does not affect the singularity structure around $N = 1$.

We turn now to the general case. Each of the above arguments can be generalized to all orders, where N space resummed results contain powers of $\ln^k N$, whose inverse Mellin transform is a linear combination of distributions $\mathcal{D}_j^{\log}(z)$ Eq. (2.20) with $j \leq k-1$. The fact that the NLO result in z space depends on powers of $\ln \frac{1-z}{\sqrt{z}}$ rather than $\ln\ln\frac{1}{z}$ is of kinematical origin and ultimately comes from the upper bound for the transverse momentum of emitted gluons, Eq. (2.21), and therefore it persists to all orders. It follows that the exact result to all orders is expressed in terms of distributions $\hat{\mathcal{D}}_k(z)$, defined in Eq. (A.2c) of Appendix A.1 in analogy with Eq. (2.26). The Mellin transform of such distributions, $\hat{\mathcal{D}}_k(N)$, first, has poles rather than cuts as small N singularities, and also, in comparison to the distributions $\mathcal{D}_k(N)$, lacks contributions proportional to powers of $\psi_k(N)$ with k odd, which would change the pole structure (see Appendix A.1).

It has been shown in Refs. [30, 32] that the factor $A_g(z)$, Eq. (2.14), is present to all orders, because the full leading order anomalous dimension exponentiates. However, terms beyond the first in its expansion Eq. (2.24) generate contributions $\alpha_s^n(1-z)^j \ln^{2n-1}(1-z)$ with $j \geq 0$ to the coefficient functions, which are generally of the same order as other terms which we do not control. However, it can be shown [33] that the inclusion of the $\mathcal{O}\big[(1-z)^1\big]$ term in the expansion Eq. (2.24) correctly predicts, after exponentiation, the subdominant contributions of the form $\alpha_s^n \ln^{2n-1}(1-z)$ (i.e., in N space, terms behaving as $\alpha_s^n N^{-1}\ln^{2n-1} N$ at large N) to all orders, so the inclusion of this term rests on firm ground.

Including the $\mathcal{O}\big[(1-z)^1\big]$ from Eq. (2.24) we get

$$A_{g,1}(z) = \frac{C_A}{\pi}[1-(1-z)] = z\,A_g(1), \tag{2.31}$$

which is easily implemented to all orders by the replacement

$$\mathcal{D}_k^{\log}(z) \to z\,\hat{\mathcal{D}}_k(z); \qquad \mathcal{D}_k^{\log}(N) \to \hat{\mathcal{D}}_k(N+1). \tag{2.32}$$

Including also the next order gives

$$A_{g,2}(z) = \frac{C_A}{\pi}\big[1-(1-z)+2(1-z)^2\big] = \big[2-3z+2z^2\big]A_g(1), \tag{2.33}$$

which amounts to replacing

$$\mathcal{D}_k^{\log}(N) \to 2\hat{\mathcal{D}}_k(N) - 3\hat{\mathcal{D}}_k(N+1) + 2\hat{\mathcal{D}}_k(N+2), \tag{2.34}$$

in the N space expressions. The third order term of the expansion of $A_g(z)$ is accidentally zero, so $A_{g,2}(z) = A_{g,3}(z)$. We have checked that the inclusion of terms of order $(1-z)^4$ and higher in the expansion of $A_g(z)$ does not affect our results significantly. We will consider both the expansions to first and second order, and use their difference as a means to estimate the uncertainty on the result. Specifically, we will take the mid-point between them as our best prediction, with the first- and second-order expansion result giving the edges of the uncertainty band.

In summary, our soft approximation (to be combined with small N terms determined in the next Section) is constructed in the following way. The resummed expression Eq. (2.9) can be rewritten

$$C_{\rm res}(N,\alpha_s) = g_0(\alpha_s)\,\exp\sum_{n=1}^{\infty}\alpha_s^n\sum_{k=0}^{n} b_{n,k}\,\mathcal{D}_k^{\log}(N), \tag{2.35}$$

where the coefficients $b_{n,k}$ are obtained from the functions g_i, Eq. (2.11), and have been determined up to $n = 3$ [28]. The function $g_0(\alpha_s)$ is known only up to $\mathcal{O}(\alpha_s^2)$; the uncertainty associated to $g_{0,3}$ will be discussed in Sect. 3.

The replacements Eq. (2.32) or (2.34) are then applied to Eq. (2.35). We obtain, respectively,

$$C_{\mathrm{soft}_1}(N,\alpha_s) = \bar{g}_0(\alpha_s)\,\exp\sum_{n=1}^{\infty}\alpha_s^n\sum_{k=0}^{n} b_{n,k}\,\hat{\mathcal{D}}_k(N+1), \tag{2.36a}$$

$$C_{\mathrm{soft}_2}(N,\alpha_s) = \bar{g}_0(\alpha_s)\,\exp\sum_{n=1}^{\infty}\alpha_s^n\sum_{k=0}^{n} b_{n,k}\left[2\hat{\mathcal{D}}_k(N) - 3\hat{\mathcal{D}}_k(N+1) + 2\hat{\mathcal{D}}_k(N+2)\right], \tag{2.36b}$$

where we have defined

$$\bar{g}_0(\alpha_s) = g_0(\alpha_s)\exp\left[-\sum_{n=1}^{\infty}\alpha_s^n\sum_{k=0}^{n} b_{n,k} d_k\right] \tag{2.37}$$

$$d_k = \lim_{N\to\infty}\left[\hat{\mathcal{D}}_k(N) - \mathcal{D}_k^{\log}(N)\right] \tag{2.38}$$

so that the condition Eq. (2.18) is satisfied to all orders after the replacement. Explicit expressions for the coefficients $b_{n,k}$ and d_k are given in Appendix A.1.

Equations (2.36) can be cast in the form

$$C_{\mathrm{soft}}(N,\alpha_s) = \bar{g}_0(\alpha_s)\,\exp\sum_{n=1}^{\infty}\alpha_s^n\,S_n(N), \tag{2.39}$$

which is now expanded in powers of α_s:

$$C_{\mathrm{soft}}(N,\alpha_s) = 1 + \alpha_s C^{(1)}_{\mathrm{soft}}(N) + \alpha_s^2 C^{(2)}_{\mathrm{soft}}(N) + \alpha_s^3 C^{(3)}_{\mathrm{soft}}(N) + \mathcal{O}(\alpha_s^4). \tag{2.40}$$

We obtain

$$C^{(1)}_{\mathrm{soft}}(N) = S_1(N) + \bar{g}_{0,1} \tag{2.41a}$$

$$C^{(2)}_{\mathrm{soft}}(N) = \frac{1}{2}S_1^2(N) + S_2(N) + \bar{g}_{0,1}S_1(N) + \bar{g}_{0,2} \tag{2.41b}$$

$$C^{(3)}_{\mathrm{soft}}(N) = \frac{1}{6}S_1^3(N) + S_1(N)S_2(N) + S_3(N) + \bar{g}_{0,1}\left(\frac{1}{2}S_1^2(N) + S_2(N)\right) + \bar{g}_{0,2}S_1(N) + \bar{g}_{0,3}. \tag{2.41c}$$

As a test of our procedure we now compare the first two orders of our soft approximations Eq. (2.36) to the full result. Note that in the sequel when comparing to known results, and also when constructing our $\mathcal{O}(\alpha_s^3)$ approximation, we will always be retaining the exact m_t dependence.

As terms of comparison, at NLO we use the finite-m_t result of Ref. [6] (using the numerical implementation of Ref. [31]), while at NNLO we use the approximate finite-m_t result obtained by matching the double expansion in powers of $1-z$ and m_H/m_t of Refs. [11, 12] to the known small z terms computed in Ref. [10] according to Ref. [13] (see Refs. [14, 15] for further approximate finite-m_t results). Note that the soft limit only depends on m_t through the function $g_0(\alpha_s)$ of Eq. (2.9).

Results are shown, as functions of N along the real N axis, in Fig. 1. We find the comparison in N space to be most instructive, because the coefficient function is then an ordinary function, rather than a distribution as in z space. Furthermore, the saddle point which dominates the Mellin inversion is on the real axis [29]. All this said, it should be kept in mind that the physical cross section is obtained by Mellin inversion of the product of the N space coefficient function and luminosity: therefore, agreement on the real axis is certainly necessary, but in general not sufficient for agreement of the physical results. In particular, spurious singularities (and in particular spurious cuts) may substantially modify the behavior of the coefficient function in the complex plane.

In order to understand the role of various subleading terms, we also show in Fig. 1 the results obtained expanding the resummed expression, Eq. (2.35), which is built up from the distributions