DESY 15-196
HU-Mathematik-2015-12
HU-EP-15/50
MITP/15-092

Exact Bremsstrahlung and Effective Couplings

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Abstract

We calculate supersymmetric Wilson loops on the ellipsoid for a large class of $\mathcal{N}=2$ SCFT using the localization formula of Hama and Hosomichi. From them we extract the radiation emitted by an accelerating heavy probe quark as well as the entanglement entropy following the recent works of Lewkowycz-Maldacena and Fiol-Gerchkovitz-Komargodski. Comparing our results with the $\mathcal{N}=4$ SYM ones, we obtain interpolating functions $f(g^2)$ such that a given $\mathcal{N}=2$ SCFT observable is obtained by replacing in the corresponding $\mathcal{N}=4$ SYM result the coupling constant by $f(g^2)$. These "exact effective couplings" encode the finite, relative renormalization between the $\mathcal{N}=2$ and the $\mathcal{N}=4$ gluon propagator, they interpolate between the weak and the strong coupling. We discuss the range of their applicability.

Contents

1 Introduction

Wilson loops are very important observables in the study of gauge theories. Not only have they been used as probes of confinement [1], but they also capture a large piece of the high energy scattering data produced by Feynman diagrams [2, 3]. Among the possible Wilson loop geometries, a prominent place is occupied by the Wilson loop with a cusp [4], see figure 1. Its expectation value is divergent and from its degree of divergence, known as the cusp anomalous dimension Γ_{cusp}, one can extract the *light-like* cusp anomalous dimension K that governs the anomalous dimensions of high spin twist operators [5] (see also [6]).

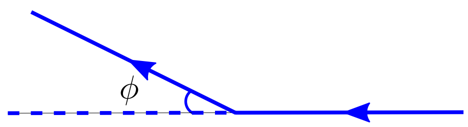


Figure 1: *The Wilson line with an Euclidean cusp angle ϕ.*

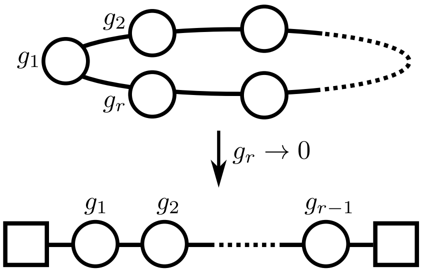


Figure 2: *The $\hat{A}_{r-1}$ $\mathcal{N}=2$ SCFT elliptic quivers with $SU(N)^r$ color group. Linear quiver theories can be obtained by taking a limit in which one of the gauge couplings $g_r \to 0$. This procedure produces the correct results only in the weak coupling expansion. The strong coupling limit and $g_r \to 0$ do not commute.*

The recent years have also seen the emergence of a plethora of exact results for $\mathcal{N} \geq 2$ supersymmetric gauge theories in four dimensions, see [7] for a review. These include the vacuum expectation values of supersymmetric Wilson loops and 't Hooft Loops [8–10] as well as other observables, not immediately given by localization, such as the cusp anomalous dimension, the quark anti-quark potential [11–13] and the entanglement entropy[1] [14]. [2]

In a parallel development [17], integrability was discovered in the spectral problem of planar $\mathcal{N}=2$ SCFTs, for a purely gluonic subset of local operators with $SU(2,1|2)$ symmetry that is closed under renormalization. Using diagrammatic arguments it was found that integrability in this sector of local operators is immediately inherited from $\mathcal{N}=4$ SYM. The Hamiltonian that acts on the spin chain states (local operators) and is equal to the mixing matrix of anomalous dimensions is given by the $\mathcal{N}=4$ SYM result after replacing the $\mathcal{N}=4$ SYM coupling constant by $g^2 \to f(g_i^2)$, a function of all the marginal couplings of the $\mathcal{N}=2$ SCFT. Thus, anomalous dimensions of $\mathcal{N}=2$ operators in that sector can be obtained from the $\mathcal{N}=4$ results (that can be obtained from integrability [18]) by replacing the $\mathcal{N}=4$ coupling g^2 by $g^2 \to f(g_i^2)$.

In the present article, we built on the the work of [14] and [19] who showed how to compute the entanglement entropy and Bremsstrahlung in $\mathcal{N} \geq 2$ theories in four dimensions by using the BPS Wilson loops on the ellipsoid [20]. We compute the large N limit of the b-deformed BPS Wilson loops of [20] for a large class of $\mathcal{N}=2$ SCFT, the $\hat{A}_{r-1}$, elliptic quivers with $SU(N)^r$ color group[3], the quiver diagram of which is depicted in figure 2. From them we extract following [14, 19] the entanglement entropy and the radiation emitted by an accelerating heavy probe quark. Our results interpolate between the weak and the strong coupling. From the weak coupling we can understand the first few terms in the expansion of the localization result using Feynman diagrams, while from the strong coupling the leading term using AdS/CFT.

[1]The traditional definition of entanglement entropy comes from thermal field theory where the antisymmetric boundary conditions for the fermions break supersymmetry completely.

[2]The Zamolodchikov metric, which is the metric on theory space, is another very interesting non-BPS observable that can be extracted from exact localization results [15, 16].

[3]Obtaining similar results for other $\mathcal{N}=2$ SCFT with a Lagrangian description is straightforward. For theories that do not have a Lagrangian description, localization is not applicable, and thus the road is not completely paved yet. However, we believe that such Wilson loops could be obtained relatively straightforward, by using the results of [21–24] as well as by combining with AGT intuition.

This paper is structured as follows. We begin in section 2 with a review of the Bremsstrahlung function, the integrability of the purely gluonic sector in $\mathcal{N}=2$ SCFTs. We then overview in section 3 the setup of the circular Wilson loops on the ellipsoid and their computation via localization. We follow up in section 4 with the saddle point approximation in the planar limit. We discuss the results on Bremsstrahlung function and the entanglement entropy in section 5. The technical aspects of the weak and strong coupling solutions to the saddle point equations are kept in the appendices C and D. We present an interpretation of some aspects of results, in particular the universality of the coupling substitution in section 6. Finally, we conclude and make some suggestions for future work in section 7.

2 Review

In this section, we present a short review of the main ingredients appearing in this paper: the cusp anomalous dimension, the Bremsstrahlung function and the integrability of the purely gluonic sector of $\mathcal{N}=2$ SCFTs. We end this section with a couple of brief arguments about the universality of the effective couplings.

2.1 The cusp anomalous dimension and the Bremsstrahlung function

The energy emitted by an uniformly accelerating probe quark is proportional to the Bremsstrahlung function B

$$\Delta E = 2\pi B \int dt \dot{v}^2 \,, \tag{1}$$

for small velocities v. It is well known, see for example [25] for an old review and [26] for a more recent presentation, that B can be obtained from a Wilson line that makes a sudden turn by an angle ϕ, see figure 1, at a single specific point that we refer to as the cusp. As discussed already in [4], the vacuum expectation value of such a Wilson loop is both UV and IR divergent with divergences of the form

$$\langle W_\varphi \rangle \sim e^{-\Gamma_{\text{cusp}}(\varphi) \log \frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}}} \,, \tag{2}$$

where Λ_{UV}, respectively Λ_{IR} is the UV, respectively IR cutoff. We can analytically continue to Minkowski signature by setting $\phi = i\varphi$.

The first important observation for this Wilson loop is the fact that, for large Euclidean angles $\phi \sim \pi$, its cusp anomalous dimension leads to the quark-antiquark potential, while for large values of the Minkowski angle φ, it grows linearly with φ with a slope equal to the light-like cusp anomalous dimension K, *i.e.*

$$\Gamma_{\text{cusp}}(\varphi) \sim K\varphi \,. \tag{3}$$

The number K, determines the leading logarithmic behavior of the anomalous dimensions of finite twist operators in the large spin limit [5] as

$$\Delta - S \sim K \log(S) \,. \tag{4}$$

The light-like cusp anomalous dimension has been calculated for $\mathcal{N}=4$ SYM to four loops [27–29], also using integrability [30], see [31] for a review, and lately even using resurgence techniques [32, 33].

The second important observation is that, for small φ, the divergence of the Wilson loop becomes quadratic in φ, namely

$$\Gamma_{\text{cusp}}(\varphi) = B\varphi^2 + \mathcal{O}(\varphi^4)\,. \tag{5}$$

Importantly, the Bremsstrahlung function can be obtained from other geometries that allow us to compute it via localization. In [26], it was argued that for $\mathcal{N}=4$ SYM, B can be obtained from the Wilson loop expectation value on the sphere as $B = \frac{\lambda}{2\pi}\partial_\lambda \log\langle W\rangle$.[4] Furthermore, according to a conjecture by [19], the Bremsstrahlung function of $\mathcal{N}=2$ theories should be given by

$$B = \pm\frac{1}{4\pi^2}\frac{d}{db}\log\langle W^{\pm}(b)\rangle_{|b=1}\,, \tag{6}$$

where $\langle W^{\pm}(b)\rangle$ are the Wilson loop expectation values of circular loops on the ellipsoid with parameter b, see (21). This formula is true for $\mathcal{N}=4$ SYM [14] but needs to be subjected to further checks or to be derived rigorously for $\mathcal{N}=2$ theories. One of the goals of our study is to precisely provide such non-trivial checks, both in the weak and in the strong coupling limit.

2.2 The integrable purely gluonic $SU(2,1|2)$ sector

Parallel to the developments discussed so far, in [17][5], integrability was discovered in the spectral problem of planar $\mathcal{N}=2$ SCFTs for a purely gluonic subset of local operators that is closed under renormalization.[6] Using the spin chain approach to the problem (see [18] for a recent review) it was found that integrability in this sector of local operators is immediately inherited from $\mathcal{N}=4$ SYM. The Hamiltonian acting on the spin chain states, or local operators, is equal to the mixing matrix of anomalous dimensions and is given by the $\mathcal{N}=4$ SYM result after substituting the $\mathcal{N}=4$ SYM coupling constant by $f(g_1^2,\ldots,g_r^2)$ a function of all the marginal couplings of the $\mathcal{N}=2$ SCFT. Thus, anomalous dimensions of $\mathcal{N}=2$ operators in that sector can be recovered from the $\mathcal{N}=4$ results obtained from integrability, by replacing the $\mathcal{N}=4$ coupling g^2 by the effective coupling $f(g_i^2)$.

In the spin chain picture, the chiral operators, members of the chiral ring $\text{Tr}(\phi^\ell)$ with ϕ being the adjoint scalar in the $\mathcal{N}=2$ vector multiplet, correspond to the BMN (feromagnetic) vacua of the spin chain. Following [42] and [38], the choice of the vacuum breaks the $SU(2,1|2)$ symmetry of the sector down to $SU(2|2)$. This is the leftover symmetry under which the elementary excitations, or magnons, $\lambda_+^{\mathcal{I}}, \mathcal{D}_{+\dot\alpha}$ of the sector transform, as shown in table 1. The index $\mathcal{I}=1,2$ is the $SU(2)_R$ symmetry index (the bosonic part of the $SU(2|2)$) while $\dot\alpha = \dot+, \dot-$ is the right-handed spinor index. The broken $SU(2,1|2)$ generators give rise to Goldstone excitations,

[4] Here and elsewhere, we use the following definition of the couplings $\lambda = Ng_{\text{YM}}^2 = (4\pi g)^2$.

[5] See [34–41] for work on which it was based.

[6] These set of operators is closed under the action of $SU(2,1|2)$. The spinor indices are chosen so that no mixing with quarks (hypermultiplets) can occur. All operators in it obey the condition $\Delta = 2j - r$, where Δ is the engineering dimension, $(j,\bar{\jmath})$ are the Lorenz spin labels spin and r is the $\text{U}(1)_r$ charge. See [17] for a precise description of the sector.

	$SU(2)_{\dot\alpha}$	$SU(2)_R$	$SU(2)_\alpha$
$SU(2)_{\dot\alpha}$	$\mathcal{L}^{\dot\alpha}{}_{\dot\beta}$	$\mathcal{Q}^{\dot\alpha}{}_{\mathcal{J}}$	$\mathcal{D}^{\dagger\dot\alpha}{}_{\beta}$
$SU(2)_R$	$\mathcal{S}^{\mathcal{I}}{}_{\dot\beta}$	$\mathcal{R}^{\mathcal{I}}{}_{\mathcal{J}}$	$\lambda^{\dagger\mathcal{I}}{}_{\beta}$
$SU(2)_\alpha$	$\mathcal{D}^{\alpha}{}_{\dot\beta}$	$\lambda^{\alpha}{}_{\mathcal{J}}$	$\mathcal{L}^{\alpha}{}_{\beta}$

Table 1: *In the spin chain description of the purely gluonic $SU(2,1|2)$ sector, after the choice of Tr(ϕ^ℓ) as the vacuum of the spin chain has been made, the $SU(2,1|2)$ symmetry of the sector break down to $SU(2|2)$. In this table, the we depict the broken generators of $SU(2,1|2)$ that become the elementary excitations $\lambda_+^{\mathcal{I}}, \mathcal{D}_{+\dot\alpha}$ as well as their transformation properties under the leftover $SU(2|2)$.*

i.e. gapless magnons, with their energy fixed by symmetry to [42]

$$E_{\lambda,\mathcal{D}} = \Delta - r = \sqrt{1 + 8f(g_i^2)\sin^2\left(\frac{p}{2}\right)}\,, \tag{7}$$

where the function $f(g_i^2)$ cannot be fixed by symmetry. For the $\mathcal{N}=4$ SYM magnons, loop calculations [43] as well as AdS/CFT and string theory calculations were needed in order to determine this unknown function, which is by now believed to be to simply equal to g^2. Moreover the scattering matrix of the elementary excitations $\lambda_+^{\mathcal{I}}, \mathcal{D}_{+\dot\alpha}$ is also fixed by symmetry again up to the function $f(g_i^2)$. Then from the spin chain picture it is clear that the $\mathcal{N}=2$ anomalous dimensions in that sector can be obtained from the $\mathcal{N}=4$ results using the substitution rule $g^2 \to f(g_i^2)$. Let us stress that the light-like cusp anomalous dimension K discussed above which is the leading logarithmic behavior of the anomalous dimensions of finite twist $\Delta - S$ operators in the large spin limit is also an observable in the sector because the twist $\Delta - S$ operators are in the purely gluonic $SU(2,1|2)$ sector. Thus for $\mathcal{N}=2$ SCFTs the light-like cusp anomalous dimension is simply given by the $\mathcal{N}=4$ results by the substitution

$$K_{\mathcal{N}=2}(g_i^2) = K_{\mathcal{N}=4}(f(g_i^2))\,. \tag{8}$$

To be more precise, there is a purely gluonic sector and its respective effective coupling $f_j(g_1^2,\dots,g_r^2)$ for each vector multiplet V_j of the theory, but the effective couplings are all related by permutations of the marginal couplings in them. From the Feynman diagrams (weak coupling) point of view this interpolating function $f_j(g_i^2)$, dubbed “effective coupling” [44], is the relative finite renormalization of the $\mathcal{N}=2$ gluon propagator

$$f_k(g_1^2,\dots,g_r^2) = g_k^2 + g_k^2\left[\left(\mathcal{Z}_{g_k}^{\mathcal{N}=2}\right)^2 - \left(\mathcal{Z}_{g_k}^{\mathcal{N}=4}\right)^2\right]. \tag{9}$$

and as such depends on all the marginal couplings g_i^2 of the $\mathcal{N}=2$ SCFT. From the strong coupling point of view and using AdS/CFT correspondence[7], the effective coupling computes the relation (via the AdS/CFT dictionary) between the effective tension of the string and the coupling constant of the $\mathcal{N}=2$ SCFT

$$T_{\text{eff}}^2 = \frac{R^4}{(2\pi\alpha')^2} = f(g_i^2)\,. \tag{10}$$

[7]Note that the quivers that we are considering have a gravity dual description.

Following [17] one can compute anomalous dimensions using the proposed $f(g_i^2)$ of [44] that were extracted from the circular Wilson loop expectation value on S^4, calculated thanks to localization [8]. In particular, this method allows us to compute the light-like cusp anomalous dimension in $\mathcal{N}=2$ theories (8), provided that we know the effective couplings.

In [17,44] some arguments and calculations were presented arguing that the coupling substitution procedure is universal, *i.e.* independent of the particular observable in the purely gluonic sector. In the string theory side the observables of the purely gluonic $SU(2,1|2)$ sector correspond to string states classical living in the $AdS_5 \times S^1$ factor[8] of the geometry with the S^1 corresponding to the $U(1)_r$ of the $\mathcal{N}=2$ theories. The chiral $\mathrm{Tr}(\phi^\ell)$ with $\Delta = r$ are charged under the $U(1)_r$ and correspond to sugra KK reduction modes on this S^1. See [36] and also [45] for a recent discussion. It is vital to study as many observables as possible in order to finally decide for the universal nature of $f(g^2)$. Moreover, it is quite important to find out whether the validity of the coupling substitution rule extends beyond the purely gluonic sector.

2.3 The universality of the coupling substitution

Finally, we wish to conclude this review section by connecting with our work a recent, independent result in favor of the existence of the universal coupling substitution rule. In [46,47], the cusp anomalous dimension Γ_{cusp} of QCD was compared with the one of $\mathcal{N}=4$ SYM and it was found that, upon replacing the coupling constant g by the light-like cusp anomalous dimension K as

$$\Gamma_{\mathrm{cusp}}\left(\varphi, g^2\right) = \Omega\left(\varphi, K(g^2)\right)\,, \tag{11}$$

the function Ω is *independent of the choice of the theory*, at least to three loops. All the dependence on the particular theory stands in K. It is very important to stress that this result is true for any φ. In (11), both Γ_{cusp} and K have to be computed in the same scheme. Expanding Ω in φ, we get

$$\Omega(\varphi, K(g^2)) = \varphi^2 \tilde{B}(K(g^2)) + \mathcal{O}(\varphi^3)\,, \tag{12}$$

and hence comparison with (5) leads to

$$B(g^2) = \tilde{B}(K(g^2))\,, \tag{13}$$

where $\tilde{B}$ is an *universal function*, at least to 3-loops. Since

$$B_{\mathcal{N}=4}(g^2) = \tilde{B}(K_{\mathcal{N}=4}(g^2))\,, \qquad B_{\mathcal{N}=2}(g^2) = \tilde{B}(K_{\mathcal{N}=2}(g^2)) \tag{14}$$

and since from [17] we have the effective coupling relation

$$K_{\mathcal{N}=2}(g^2) = K_{\mathcal{N}=4}(f(g^2))\,, \tag{15}$$

[8] Specifically, the geometry does not factorize, but has an $U(1)$ isometry.

we can write

$$B_{\mathcal{N}=2}(g^2) = \tilde{B}(K_{\mathcal{N}=4}\big(f(g^2)\big) = B_{\mathcal{N}=4}\big(f(g^2)\big)\,. \tag{16}$$

Following [46], this implies that, at least to 3-loops, the effective coupling $f(g^2)$ is universal, *i.e.* valid for B as well as for the anomalous dimensions.

A short comment about notation is due. Since we will often be comparing $\mathcal{N}=4$ quantities to $\mathcal{N}=2$ ones, we need to be painstakingly clear about denoting them properly. In general $\mathcal{N}=4$ quantities will be denoted as such, for example $\langle W_{\mathcal{N}=4}\rangle$ for the vacuum expectation value of the Wilson loop on the sphere. We will specifically be considering the $\mathbb{Z}_r$ cyclic quiver $\mathcal{N}=2$ theories, see figure 2. They have r gauge groups and we will designate the corresponding quantities, such as the Wilson loop expectation values or the Bremsstrahlung functions, simply by labeling them by an index $k \in \{1,\ldots,r\}$, for example B_k. Furthermore, for the sake of brevity, we shall often abbreviate the dependence of a function $f(g_1^2,\ldots,g_r^2)$ of all the couplings as $f(g_i^2)$.

3 Wilson loops on Ellipsoids

The 4D ellipsoid is defined, in embedding coordinates, via the equation

$$\frac{x_0^2}{r^2} + \frac{x_1^2+x_2^2}{\ell^2} + \frac{x_3^2+x_4^2}{\tilde{\ell}^2} = 1\,. \tag{17}$$

We are interested in the Maldacena-Wilson loops on the ellipsoid

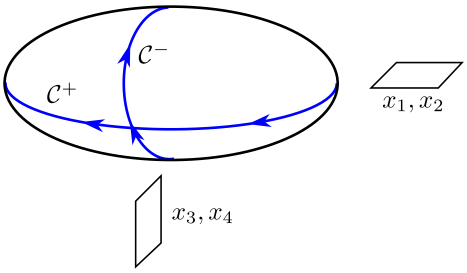


Figure 3: *We sketch here two circular Wilson loops $\mathcal{C}^\pm$ that can be computed on the ellipsoid via localization. The 4D ellipsoid itself should be understood as a fibration of the 3D ellipsoid over the interval $x_0 \in [-r, r]$.*

$$\langle W_k(\mathcal{C}^\pm)\rangle \equiv \langle W_k^\pm\rangle = \left\langle \frac{1}{N}\mathrm{Tr}_\Box \mathrm{Pexp}\oint_{\mathcal{C}^\pm} ds\left(iA_\mu^{(k)}(x)\dot{x}^\mu + \phi^{(k)}(x)|\dot{x}|\right)\right\rangle\,, \tag{18}$$

where $\Box$ denotes the fundamental representation and $\mathcal{C}^\pm$ are the two circular loop located depicted in figure 3. In our case, we have r gauge groups and the index k labels the adjoint scalar $\phi^{(k)}$ and the gauge field $A_\mu^{(k)}$ in the vector multiplet of the k-th gauge group.

We define the deformation parameter b as

$$b := \sqrt{\ell/\tilde{\ell}}\,. \tag{19}$$

The case $b = 1$ corresponds to the sphere S^4, in which case the localization result was already given by [8]. In [20], the following expression for the partition function (written for simplicity here for a single SU(N) gauge group) was given

$$Z = \int d\hat{a} e^{-\frac{8\pi^2}{g_{\rm YM}^2}\mathrm{Tr}(\hat{a}^2)} Z_{\text{1-loop}}(a,b)\left|Z_{\rm inst}(a,b)\right|^2 , \tag{20}$$

where $\hat{a} := \sqrt{\ell\tilde{\ell}}a$. The matrix $a = \mathrm{diag}(a_1, \ldots, a_N)$ is subject to the condition $\sum_{i=1}^N a_i = 0$ and is an element of the Cartan subalgebra of $\mathfrak{su}(N)$. The two Wilson loops that we are able to compute are drawn in figure 3 and are given by

$$\langle W^{\pm}(b)\rangle = \frac{1}{Z}\int d\hat{a}\mathrm{Tr}\left(e^{-2\pi b^{\pm 1}\hat{a}}\right) e^{-\frac{8\pi^2}{g_{\rm YM}^2}\mathrm{Tr}(\hat{a}^2)} Z_{\text{1-loop}}\left|Z_{\rm inst}\right|^2 \tag{21}$$

The perturbative part $Z_{\text{1-loop}}$ of (20) is given by a product over the vector multiplet and hypermultiplet contributions (here only in the bifundamental representation of two SU(N) gauge groups) with

$$\begin{aligned} Z^{\rm vect}_{\text{1-loop}} &= \prod_{i<j=1}^{N} \Upsilon\big(i\sqrt{\ell\tilde{\ell}}(a_i - a_j); b\big)\Upsilon\big(-i\sqrt{\ell\tilde{\ell}}(a_i - a_j); b\big) \\ Z^{\rm hyper}_{\text{1-loop}} &= \prod_{i,j=1}^{N} \Upsilon\big(i\sqrt{\ell\tilde{\ell}}(a_i^{(1)} - a_j^{(2)}) + Q/2; b\big)^{-1} , \end{aligned} \tag{22}$$

where $Q = b + b^{-1}$ and Υ was defined in appendix A.1. We remark that both $Z^{\rm vect}_{\text{1-loop}}$ and $Z^{\rm hyper}_{\text{1-loop}}$ are invariant under the transformation $b \leftrightarrow b^{-1}$.

For massless theories, we can rescale the integration variable a and get rid of the factor of $\sqrt{\ell\tilde{\ell}}$, whereas for massive one, the factor of $\sqrt{\ell\tilde{\ell}}$ is part of the ambiguity of the mass. Rearranging some factors in the special functions, leads us to the expression, proven in appendix B,

$$Z = \int da e^{-\frac{N}{2g^2}\sum_{i=1}^N a_i^2} Z_{\text{1-loop}}(a,b)\left|Z_{\rm inst}(a,b)\right|^2 , \tag{23}$$

with g given by $\lambda = N g_{\rm YM}^2 = (4\pi g)^2$ and where now the 1-loop part is given by

$$\begin{aligned} Z^{\rm vect}_{\text{1-loop}} &= \prod_{i<j=1}^{N} (a_i - a_j)^2 \prod_{i,j=1}^{N} H_v(a_i - a_j; b) , \\ Z^{\rm hyper}_{\text{1-loop}} &= \prod_{i,j=1}^{N} H_h(a_i^{(1)} - a_j^{(2)}; b)^{-1} , \end{aligned} \tag{24}$$

with the functions $H_v(x; b)$ and $H_h(x; b)$ defined in (107). These functions have the advantage of being simpler to work with for the weak coupling expansion since they are even in x, invariant under $b \leftrightarrow b^{-1}$ and normalized as $H_v(0; b) = H_h(0; b) = 1$.

4 Saddle point approximation

We only consider the A class of $\mathcal{N}=2$ SCFTs, *i.e.* the $\mathbb{Z}_r$ cyclic quivers or the linear quivers.[9] We shall concentrate on the cyclic, or class $\hat{A}_{r-1}$, of quiver theories, since in the weak coupling limit we can recover the results for the class A_{r-1} linear quivers by taking a limit, see figure 2. The partition function is can then be written as

$$\begin{aligned} Z &= \int \prod_{k=1}^{r} d^{N-1}a^{(k)} \prod_{i<j=1}^{N} \left(a_i^{(k)} - a_j^{(k)}\right)^2 e^{-\frac{N}{2g_k^2}\sum_{i=1}^{N}\left(a_i^{(k)}\right)^2} Z_{\text{1-loop}} \left|Z_{\text{inst}}\right|^2 \\ &= \int \prod_{k=1}^{r} d^{N-1}a^{(k)} e^{-N\mathcal{S}_{\text{eff}}} \,, \end{aligned} \tag{25}$$

where the instanton part Z_{inst} will be ignored, since we are going to perform a planar limit[10] computation. We remind that, since we are dealing with $SU(N)$ gauge groups, the Coulomb parameters satisfy

$$\sum_{i=1}^{N} a_i^{(k)} = 0 \,, \qquad \forall k = 1, \ldots, r \,. \tag{26}$$

In the large N limit that we are interested in we can safely ignore the instanton part [48], and the effective action is given by

$$\mathcal{S}_{\text{eff}} = \sum_{k=1}^{r} \left[\sum_{i=1}^{N} \frac{1}{2g_k^2} \left(a_i^{(k)}\right)^2 - \frac{1}{N} \sum_{i<j=1}^{N} \ln\left(a_i^{(k)} - a_j^{(k)}\right)^2 \right] - \frac{1}{N} \ln\left(Z_{\text{1-loop}}\right) . \tag{27}$$

For the $\hat{A}_{r-1}$ quivers, the perturbative part of the partition function can be written as

$$\ln\left(Z_{\text{1-loop}}\right) = \sum_{k,l=1}^{r} \sum_{i,j=1}^{N} \left[\delta_{kl} \log H_v(a_i^{(k)} - a_j^{(l)}) - \frac{\delta_{k,l+1} + \delta_{k,l-1}}{2} \log H_h(a_i^{(k)} - a_j^{(l)}) \right] . \tag{28}$$

Solving the matrix model in the planar limit is done by considering the saddle point approximation. Specifically, we are computing a vacuum expectation value of some quantitiy W in the large N limit

$$\langle\, W \,\rangle = \frac{\int \prod_{k=1}^{r} d^{N-1}a^{(k)} W(a^{(\ell)}) e^{-N\mathcal{S}_{\text{eff}}}}{\int \prod_{k=1}^{r} d^{N-1}a^{(k)} e^{-N\mathcal{S}_{\text{eff}}}} \,, \tag{29}$$

where by abuse of notation W is also a function of the eigenvalues $a^{(k)} = (a_1^{(k)}, \ldots, a_N^{(k)})$ of the k^{th} gauge group. Let the effective action have an extremum ($\frac{\partial \mathcal{S}_{\text{eff}}}{\partial a}\big|_{a=b} = 0$) at $a = (a^{(1)}, \ldots, a^{(r)}) \equiv b$ and expand the integral $a = b + \frac{1}{\sqrt{N}}x$, we get

$$\begin{aligned} \int da W(a) e^{-N\mathcal{S}_{\text{eff}}} =& \frac{e^{-N\mathcal{S}_{\text{eff}}(b)}}{N^{\text{power}}} \int dx e^{-\frac{1}{2}(\mathcal{S}_{\text{eff}})_{,ij}(b) x_i x_j} \Bigg[W(b) + \\ &+ \frac{1}{N^{\frac{1}{2}}} \left(W_{,i}(b) x_i - \frac{1}{6} W(b) (\mathcal{S}_{\text{eff}})_{,ijk} x_i x_j x_k \right) + \cdots \Bigg] , \end{aligned} \tag{30}$$

[9] Similar calculations can be found in [48–51], and for the $\mathcal{N}=2^*$ theory on the ellipsoid in [52].

[10] See [48] and [49] for a discussion on this subject.

where "power" is a number that will drop out in the end and we have used the shorthand $W_{,ijk\cdots} = \partial_i\partial_j\partial_k\cdots W$. It follows that the leading term in the planar limit is given by the function evaluated at the saddle point, *i.e.*

$$\langle\, W(a)\,\rangle = W(b) + \mathcal{O}(1/N)\,. \tag{31}$$

In our case, the saddle point equations $\partial\mathcal{S}_{\text{eff}}/\partial a_i^{(k)} = 0$ imply that for all $i = 1,\dots,N$ and all $k = 1,\dots,r$, we must have

$$\begin{aligned}\frac{a_i^{(k)}}{2g_k^2} =& \frac{1}{N}\sum_{j\neq i}\frac{1}{a_i^{(k)}-a_j^{(k)}} \\ &- \frac{1}{N}\sum_{l=1}^{r}\sum_{j=1}^{N}\left[\delta_{kl}K_v(a_i^{(k)}-a_j^{(l)}) - \frac{\delta_{k,l+1}+\delta_{k,l-1}}{2}K_h(a_i^{(k)}-a_j^{(l)})\right]\,.\end{aligned} \tag{32}$$

where K_v and K_h are defined in (108). In the $N\to\infty$ limit, we replace the eigenvalues $a_i^{(k)}$ by normalized densities that, due to (26), are localized on a symmetric interval $[-\mu_k,\mu_k]$

$$\rho_k(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x-a_i^{(k)}\right)\,,\qquad \int_{-\mu_k}^{\mu_k}\rho_k(x)dx = 1\,, \tag{33}$$

which transforms the saddle point equations (32) into integral equations. Specifically, we obtain the following system of coupled integral equations:

$$\frac{x}{2g_k^2} = ⨍_{-\mu_k}^{\mu_k} dy\frac{\rho_k(y)}{x-y} - \sum_{l=1}^{r}\int_{-\mu_l}^{\mu_l}\left[\delta_{kl}K_v(x-y) - \frac{\delta_{k,l+1}+\delta_{k,l-1}}{2}K_h(x-y)\right]\rho_l(y)dy \tag{34}$$

for $k = 1,\dots,r$. For numerical approximations at small values of the 't Hooft couplings, it is sometimes helpful to rewrite (34) by inverting the Hilbert kernel, *i.e.* by acting with $⨍_{-\mu_k}^{\mu_k}\frac{dx}{\sqrt{\mu_k^2-x^2}}\frac{1}{z-x}$ on both sides of the equation. Using (115) and (119) we get the set of equations

$$\begin{aligned}\rho_k(x) =& \frac{1}{2\pi g_k^2}\sqrt{\mu_k^2-x^2} - \frac{1}{\pi^2}⨍_{-\mu_k}^{\mu_k}\frac{dy}{x-y}\sqrt{\frac{\mu_k^2-x^2}{\mu_k^2-y^2}}\sum_{l=1}^{r}\int_{-\mu_l}^{\mu_l}\rho_l(z)dz\Big[\delta_{kl}K_v(y-z) \\ &- \frac{\delta_{k,l+1}+\delta_{k,l-1}}{2}K_h(y-z)\Big]\,,\end{aligned} \tag{35}$$

subject to the normalization condition for the densities

$$1 = \frac{\mu_k^2}{4g_k^2} + \frac{1}{\pi}\int_{-\mu_k}^{\mu_k}\frac{dy\,y}{\sqrt{\mu_k^2-y^2}}\sum_{l=1}^{r}\int_{-\mu_l}^{\mu_l}\rho_l(z)dz\Big[\delta_{kl}K_v(y-z) - \frac{\delta_{k,l+1}+\delta_{k,l-1}}{2}K_h(y-z)\Big]\,. \tag{36}$$

Thanks to (31), having obtained the densities by solving the saddle point equations (35), we can compute the Wilson loop expectation values for the k-th gauge group by plugging the densities

$$W_k^{\pm}(g_1,\dots,g_r;b) = \left\langle\,\frac{1}{N}\sum_{i=1}^{N}e^{-2\pi a_i^{(k)}b^{\pm1}}\,\right\rangle = \int_{-\mu_k}^{\mu_k}\rho_k(x)e^{-2\pi x b^{\pm1}}dx\,. \tag{37}$$

4.1 Weak coupling results

By *weak coupling*, we understand the regime for which *all* the couplings are small, *i.e.*

$$g_i^2 = t\kappa_i \, , \tag{38}$$

with the coefficients κ_i being order one constants and $t << 1$. Appendix C contains further details on the weak coupling expansion of the Wilson loops. From the vacuum expectation values of the Wilson loops on the ellipsoids, we define the "full" effective couplings $f_k(g_i^2; b)$ via

$$\langle W^+_{\mathcal{N}=4}(f_k(g_1^2, \ldots, g_r^2; b); b)\rangle = \langle W_k^+(g_1^2, \ldots, g_r^2; b)\rangle \, . \tag{39}$$

We could have just as easily used the other Wilson loop $\langle W^-\rangle$ in the above. The corresponding effective coupling is simply $f_k(g_i^2, b^{-1})$. It is useful to expand these effective couplings in power of $(b-1)$ around $b = 1$. We define the coefficients of this expansion as

$$f_k(g_1^2, \ldots, g_r^2; b) := \sum_{n=0}^{\infty} f_k^{(n)}(g_1^2, \ldots, g_r^2)(b-1)^n \, . \tag{40}$$

In appendix C, we write explicitly the linear equations that need to be solved to obtain the Wilson loop expectation values on the ellipsoids. From the result (143) for the $\mathbb{Z}_2$ quiver, we get

$$\begin{aligned} f_1^{(0)}(g_1^2, g_2^2) = g_1^2 + \left(g_2^2 - g_1^2\right) \Big\{ & 12\zeta(3) g_1^4 - 40\zeta(5) g_1^4 \left[3g_1^2 + g_2^2\right] \\ & - \frac{4}{3} g_1^4 \left[10\pi^2\zeta(5) g_1^4 + 108\zeta(3)^2 \left(2g_1^4 - g_2^2 g_1^2 + g_2^4\right) - 105\zeta(7)\left(8g_1^4 + 5g_2^2 g_1^2 + g_2^4\right)\right] + \\ & - \frac{8}{9} g_1^4 \big[8\pi^4 g_1^6 \zeta(5) + 21\pi^2\left(11g_1^2 + 5g_2^2\right) g_1^4 \zeta(7) + 27\big(5g_1^6(60\zeta(3)\zeta(5) - 91\zeta(9)) \\ & - g_2^2 g_1^4 (100\zeta(3)\zeta(5) + 371\zeta(9)) + g_2^4 g_1^2 (20\zeta(3)\zeta(5) - 161\zeta(9)) + g_2^6(100\zeta(3)\zeta(5) - 21\zeta(9))\big)\big] \Big\} \\ & + \mathcal{O}(g^{14}) \, , \end{aligned} \tag{41}$$

for the term constant in $b - 1$ and

$$f_1^{(1)}(g_1^2, g_2^2) = -\left(g_2^2 - g_1^2\right)\pi^2 \left\{ \frac{80}{3} g_1^8 \zeta(5) + \frac{16}{9} g_1^8 \left(16\pi^2 g_1^2 \zeta(5) + 21\left(11g_1^2 + 5g_2^2\right)\zeta(7)\right)\right\} + \mathcal{O}(g^{14}) \, , \tag{42}$$

for the linear piece. In order to not overload the reader with information, we refrain from presenting any additional orders in the $(b-1)$ expansion, since they can be easily taken from (143).

A short remark is in order. The terms in the expansions (41) and (42) are homogeneous polynomials in the two couplings g_1 and g_2 of a given degree. By $\mathcal{O}(g^n)$, we mean that the results exclude polynomials of homogeneous degree greater or equal to n. Lastly, the expression for the other effective coupling $f_2(g_1^2, g_2^2; b)$ can be obtained by using the $\mathbb{Z}_2$ cyclic symmetry of the theory

$$f_2(g_1^2, g_2^2; b) = f_1(g_2^2, g_1^2; b) \, . \tag{43}$$

For the general $\mathbb{Z}_r$ cyclic quivers, we have the results

$$\begin{aligned} f_k^{(0)}(g_i^2) &= g_k^2 + 6\zeta(3) g_k^4 \left[g_{k-1}^2 + g_{k+1}^2 - 2g_k^2\right] - 20\zeta(5) g_k^4 \left[g_{k-1}^4 + g_{k+1}^4 - 6g_k^4 + 2g_k^2\left(g_{k-1}^2 + g_{k+1}^2\right)\right] \\ &+ g_k^4 \Big[70\zeta(7)\Big(g_{k-1}^6 + g_{k+1}^6 - 16g_k^6 + 3g_k^4\left(g_{k-1}^2 + g_{k+1}^2\right) + 4g_k^2\left(g_{k-1}^4 + g_{k+1}^4\right)\Big) \\ &- 2\zeta(2)(20\zeta(5)) g_k^4\left(g_{k-1}^2 + g_{k+1}^2 - 2g_k^2\right) + (6\zeta(3))^2\Big(8g_k^6 - 2g_{k-1}^6 - 2g_{k+1}^6 + g_{k-1}^4 g_{k-2}^2 + g_{k+2}^2 g_{k+1}^4 \\ &- 6g_k^4\left(g_{k-1}^2 + g_{k+1}^2\right) + 2g_k^2\left(g_{k-1}^4 + g_{k-1}^2 g_{k+1}^2 + g_{k+1}^4\right)\Big)\Big] + \mathcal{O}(g^{12}) \end{aligned} \tag{44}$$

already present in [44] and

$$f_k^{(1)}(g_i^2) = 80 g_k^8 (2g_k^2 - g_{k+1}^2 - g_{k-1}^2)\zeta(2)\zeta(5) + \mathcal{O}(g^{12})\,, \tag{45}$$

for the first $b-1$ correction. The effective couplings for the general cyclic quivers are symmetric under $\mathbb{Z}_r$.

4.2 Strong coupling results

Similarly to the weak coupling approximation in subsection 4.1, we define the *strong coupling regime* to be the one in which all the couplings are large, *i.e.* we suppose that they all scale like (38) with $t >> 1$.

In appendix D, we present the strong coupling analysis that we omitted in [44]. We obtain namely that, at the leading order, the densities ρ_k behave like

$$\rho_k(x) \sim \frac{1}{2\pi g_k^2}\sqrt{\mu_k^2 - x^2} \tag{46}$$

with the widths

$$\mu_k = 2\sqrt{\frac{r g_1^2 \cdots g_r^2}{\sum_{i=1}^r \prod_{k\neq i} g_k^2}} \qquad \forall k\,. \tag{47}$$

This implies that the Wilson loops expectation values go like

$$\langle W_k^{\pm}\rangle = \frac{\sqrt{\mu_k}\, e^{2\pi b^{\pm 1}\mu_k}}{8\pi^2 b^{\pm 3/2} g_k^2} + \mathcal{O}(b-1)^2\,. \tag{48}$$

The leading piece in the couplings of the above can be written as

$$\log\langle W_k^{\pm}\rangle = 2\pi\mu_k \pm (b-1)\left(2\pi\mu_k - \frac{3}{2}\right) + O\left((b-1)^2\right)\,, \tag{49}$$

up to logarithmic corrections that are sub-leading and can be dropped. Due to the exponential term, the strong coupling limit of the effective couplings is simply given by comparing (176) with the width μ for $\mathcal{N}=4$. Since for $\mathcal{N}=4$ SYM we have $\mu^2 = 4g^2$, comparing with (176) leads to

$$f_k^{(0)}(g_1,\dots,g_r) = r\left(\sum_{j=1}^r \frac{1}{g_j^2}\right)^{-1} = \frac{r g_1^2\cdots g_r^2}{\sum_{i=1}^r \prod_{k\neq i} g_k^2}\,, \tag{50}$$

up to constant and logarithmic corrections. Furthermore, to that order of precision $f_k^{(1)}$ is zero. This can be seen by plugging (49) into (39), solving for f_k:

$$f_k(g_1,\dots,g_r;b) = \left(\frac{3}{8\pi b}\mathbf{W}_{-1}\left(-\frac{8\sqrt[3]{2}\pi^{2/3} b g_k^{4/3} e^{-\frac{4\pi}{3}b\mu_k}}{3\sqrt[3]{\mu_k}}\right)\right)^2\,, \tag{51}$$

and using the asymptotic expansion $\mathbf{W}_{-1}(x) = \log(-x) - \log(-\log(-x)) + \cdots$ for the appropriate branch of the product log function, defined as the inverse of the function xe^x.

5 The Bremsstrahlung function and the entanglement entropy

Having in section 4 derived the vacuum expectation value of the Wilson loops on the ellipsoid, it is now time to reap the fruits of our labor and investigate the quantities that we can easily obtain from them, namely the Bremsstrahlung function and the entanglement entropy.

5.1 The Bremsstrahlung function

For $\mathcal{N} = 4$, we can obtain the Wilson loop on the ellipsoid by simply making the substitution $g \to gb^{\pm 1}$, leading to the planar limit result

$$\langle W^{\pm}_{\mathcal{N}=4}(g^2; b)\rangle = \frac{I_1(4\pi g b^{\pm 1})}{2\pi g b^{\pm 1}} + \mathcal{O}((b-1)^2)\,, \tag{52}$$

where I_n are the modified Bessel functions of the first kind. It follows from (6) and (52) that we have in the planar limit the expression (see [26] for an earlier derivation of $B_{\mathcal{N}=4}$)

$$\begin{aligned} B_{\mathcal{N}=4}(g^2) =& \frac{g(I_0(4\pi g) + I_2(4\pi g))}{2\pi I_1(4\pi g)} - \frac{1}{4\pi^2} = \frac{g I_2(4g\pi)}{\pi I_1(4g\pi)} \\ =& g^2 - \frac{2\pi^2 g^4}{3} + \frac{2\pi^4 g^6}{3} - \frac{32\pi^6 g^8}{45} + \frac{104\pi^8 g^{10}}{135} - \frac{88\pi^{10} g^{12}}{105} + \frac{2588\pi^{12} g^{14}}{2835} + O\left(g^{15}\right) \end{aligned} \tag{53}$$

where we have used $\frac{d}{dx} I_1 = \frac{1}{2}(I_0 + I_2)$. One can check that for large g we have

$$B_{\mathcal{N}=4}(g) \sim \frac{g}{\pi} - \frac{3}{8\pi^2}\,. \tag{54}$$

In particular, $B_{\mathcal{N}=4}(g)$ is monotonically growing for all $g > 0$ and is hence invertible in that domain. It follows that the equation $B_{\mathcal{N}=4}(x) = y$ has an unique solution for y positive. We now define effective coupling $f_{B;k}$ for the $\mathbb{Z}_r$ quiver theories by demanding

$$B_{\mathcal{N}=4}\big(f_{B;k}(g_1^2, \ldots, g_r^2)\big) = B_k(g_1^2, \ldots, g_r^2)\,. \tag{55}$$

We find that, in the weak couping B_k goes like,

$$\begin{aligned} B_k =& g_k^2 - \frac{2\pi^2}{3} g_k^4 + \frac{2}{3} g_k^4 \Big[\pi^4 g_k^2 + 9\zeta(3)\left(-2g_k^2 + g_{k+1}^2 + g_{k-1}^2\right)\Big] \\ &+ \Big[8\pi^2\zeta(3)\left(2g_k^2 - g_{k+1}^2 - g_{k-1}^2\right) g_k^6 + 20\zeta(5)\left(6g_k^4 - 2\left(g_{k+1}^2 + g_{k-1}^2\right) g_k^2 - g_{k+1}^4 - g_{k-1}^4\right) g_k^4 - \frac{32}{45}\pi^6 g_k^8\Big] \\ &+ \Big[12\pi^4\left(-2g_k^2 + g_{k+1}^2 + g_{k-1}^2\right) g_k^8 \zeta(3) - \frac{40\pi^2}{3}\left(10g_k^4 - 3\left(g_{k+1}^2 + g_{k-1}^2\right) g_k^2 - 2\left(g_{k+1}^4 + g_{k-1}^4\right)\right) g_k^6 \zeta(5) \\ &+ 2g_k^4 \Big(16 g_k^6\left(9\zeta(3)^2 - 35\zeta(7)\right) - 3\left(g_{k+1}^2 + g_{k-1}^2\right) g_k^4\left(36\zeta(3)^2 - 35\zeta(7)\right) \\ &+ 4g_k^2\left(g_{k+1}^4\left(9\zeta(3)^2 + 35\zeta(7)\right) + 9g_{k-1}^2 g_{k+1}^2 \zeta(3)^2 + g_{k-1}^4\left(9\zeta(3)^2 + 35\zeta(7)\right)\right) \\ &- 36 g_{k-1}^6 \zeta(3)^2 + 18 g_{k-2}^2 g_{k-1}^4 \zeta(3)^2 + 18 g_{k+1}^4 g_{k+2}^2 \zeta(3)^2 + 35 g_{k-1}^6 \zeta(7) + g_{k+1}^6\left(35\zeta(7) - 36\zeta(3)^2\right)\Big) \\ &+ \frac{104}{135}\pi^8 g_k^{10}\Big] + \mathcal{O}\left(g^{12}\right)\,. \end{aligned} \tag{56}$$

For the $\mathbb{Z}_2$ quiver, the above can be checked from the explicit result (143) for the Wilson loop.

We now wish to discuss the relationships between all the different effective couplings. For the sake of clarity, we shall suppress the indices referring to the gauge groups. From the Wilson loops on the ellipsoids, we extract the effective coupling $f_k(g_i^2; b)$ via (39). This defines the b-dependent effective coupling $f_k(g_i^2; b)$. For $b = 1$, it reduces to the "Wilson loop" effective couplings

$$f_{W;k}(g_1^2, \dots, g_r^2) := f_k(g_1^2, \dots, g_r^2; b)_{\big|b=1}\,. \tag{57}$$

that we used in [44]. On the other hand, from the Bremsstrahlung function, we can extract $f_{B;k}(g_i^2)$ through (55). Let us see how the two are related. We have

$$\begin{aligned} B_k(g_i^2) &= \frac{1}{4\pi^2}\frac{d}{db}\log\langle W_k^+(g_i^2; b)\rangle_{\big|b=1} \\ &= \frac{1}{4\pi^2}\frac{d}{db}\log\langle W_{\mathcal{N}=4}(f_k(g_i^2; b); b)\rangle_{\big|b=1} \\ &= \frac{1}{4\pi^2}\frac{\partial}{\partial g^2}\log\langle W_{\mathcal{N}=4}(f_{W;k}(g_i^2))\rangle \frac{\partial f_k(g_i^2; b)}{\partial b}_{\big|b=1} + B_{\mathcal{N}=4}(f_{W;k}(g_i^2)) \end{aligned} \tag{58}$$

From [26], we take

$$B_{\mathcal{N}=4} = \frac{\lambda}{2\pi^2}\partial_\lambda \log\langle W_{\mathcal{N}=4}(\lambda)\rangle \Longrightarrow \frac{\partial}{\partial g^2}\log\langle W_{\mathcal{N}=4}(g^2)\rangle = \frac{2\pi^2}{g^2}B_{\mathcal{N}=4}(g^2)\,. \tag{59}$$

Since $B_{\mathcal{N}=4}(f_{B;k}(g_i^2)) \overset{!}{=} B_k(g_i^2)$, it follows that

$$B_{\mathcal{N}=4}(f_{B;k}(g_i^2)) = \left(1 + \frac{f_k^{(1)}(g_i^2)}{2 f_{W;k}(g_i^2)}\right) B_{\mathcal{N}=4}(f_{W;k}(g_i^2))\,, \tag{60}$$

where the first derivative $\partial_b f_k(g_i^2; b)_{\big|b=1} \equiv f_k^{(1)}(g_i^2)$ is given in (42) for the $\mathbb{Z}_2$ quiver and (45) in general. Hence, the discrepancy between $f_{B;k}$ and $f_{W;k}$ comes from the first derivative of the "full" effective coupling $f_k(g_i^2; b)$ at $b = 1$. To the order that we care to check, the discrepancies are always proportional to $\zeta(2n)$.

In the strong coupling limit, the result (48) for the Wilson loop expectation values implies that the Bremsstrahlung function for the k^{th} gauge group goes like

$$B_k \sim \frac{\mu_k}{2\pi} = \frac{1}{\pi}\sqrt{\frac{r g_1^2 \cdots g_r^2}{\sum_{i=1}^r \prod_{k\neq i} g_k^2}}\,, \tag{61}$$

ignoring constant and logarithmic contributions, where the widths μ_k of the densities are to be found in (47). Since the leading contribution to $f_k^{(1)}$ is zero at strong coupling, $f_{B;k} = f_{W;k}$ to the precision we have in that regime.

5.2 Entanglement entropy

Combining the results of [14, 19], for 4D $\mathcal{N} = 2$ SCFTs, the additional entanglement entropy of a spherical region due to the presence of a heavy probe located at its origin is given by

$$S = \log\langle W\rangle - 8\pi^2 h_W = \left(1 - \frac{2}{3}\partial_b\right)\langle \log W^+\rangle_{\big|b=1} = \log\langle W\rangle - \frac{8\pi^2}{3}B\,. \tag{62}$$

For $\mathcal{N}=4$, this result combined with expressions (52) and (53) gives

$$S_{\mathcal{N}=4}(g^2) = \log\left(\frac{I_1(4g\pi)}{2\pi g}\right) - \frac{8\pi g}{3}\frac{I_2(4g\pi)}{I_1(4g\pi)}\,, \tag{63}$$

which expanded for small values of the coupling is

$$S_{\mathcal{N}=4}(g^2) = -\frac{2\pi^2 g^2}{3} + \frac{10\pi^4 g^4}{9} - \frac{4\pi^6 g^6}{3} + \frac{208\pi^8 g^8}{135} - \frac{3536\pi^{10} g^{10}}{2025} + \frac{88\pi^{12} g^{12}}{45} + O\left(g^{13}\right)\,, \tag{64}$$

while its strong coupling asymptotic is given[11] by

$$S_{\mathcal{N}=4}(g^2) \sim 4\pi g + \frac{8\pi(32\pi g - 15)g}{9 - 96\pi g} - \log\left(\frac{\pi^2\sqrt{32 g^3}}{1 - \frac{3}{32\pi g}}\right)\,. \tag{65}$$

Unlike the Bremsstrahlung function $B_{\mathcal{N}=4}$ or the Wilson loop expectation $\log\langle W_{\mathcal{N}=4}\rangle$, the entanglement entropy $S_{\mathcal{N}=4}$ is not monotonically growing in g. Hence, we cannot in general find a single solution to the equation $S_{\mathcal{N}=4}(f_k(g_i^2)) = S_k(g_i^2)$. We restrict to simply stating the weak and strong coupling expansions of the entanglement entropies by using the results of the appendices C and D. We find

$$\begin{aligned} S_k =& -\frac{2\pi^2}{3}g_k^2 + \frac{10}{9}\pi^4 g_k^4 - \frac{4}{3}\Big[\pi^2 g_k^4\left(3\zeta(3)\left(-2g_k^2 + g_{k+1}^2 + g_{k-1}^2\right) + \pi^4 g_k^2\right)\Big] \\ &+ \frac{8\pi^2}{135}g_k^4\Big[-225\pi^2\zeta(3)\left(2g_k^2 - g_{k+1}^2 - g_{k-1}^2\right)g_k^2 - 225\zeta(5)\left(6g_k^4 - 2\left(g_{k+1}^2 + g_{k-1}^2\right)g_k^2 - g_{k+1}^4 - g_{k-1}^4\right) \\ &+ 26\pi^6 g_k^4\Big] + \mathcal{O}(g^{10})\,, \end{aligned} \tag{66}$$

in the weak coupling. Plugging (49) and (61) into (62), we get

$$S_k \sim \frac{2\pi}{3}\mu_k = \frac{4\pi}{3}\sqrt{\frac{r g_1^2\cdots g_r^2}{\sum_{i=1}^r\prod_{k\neq i} g_k^2}}\,. \tag{67}$$

up to constant and logarithmic contributions for the strong coupling limit, with μ_k taken from (47).

6 Universality of the coupling substitution rule

In this section we wish to argue that the effective couplings $f_k(g_i^2)$ that we have been calculating are universal, *i.e.* they are the same for any observable in the $SU(2,1|2)$ sector, up to some scheme dependence that is related to the way the theory is regulated in the infrared. Firstly, it is important to recall that in [17,44] we argued that in perturbation theory the functions $f_k(g_i^2)$ compute the finite renormalization of the $\mathcal{N}=2$ gluon propagator relative to the $\mathcal{N}=4$ one,

$$f_k(g_i^2) = g_k^2 + g_k^2\left[\left(\mathcal{Z}_{g_k}^{\mathcal{N}=2}\right)^2 - \left(\mathcal{Z}_{g_k}^{\mathcal{N}=4}\right)^2\right]\,. \tag{68}$$

We checked this proposal by a three loop calculation of the difference between the $\mathcal{N}=2$ and the $\mathcal{N}=4$ gluon propagator.

[11] We use $I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{4n^2-1}{8x} + \cdots\right)$.

In sections 4 and 5, we found that up to three loops[12] in the weak coupling expansion and at leading order in the strong coupling the effective couplings are universal, *i.e.* they same for the different observables that we studied, namely the Wilson loop and the Bremsstrahlung function. Hence, up to that order we see $f_W = f_B = f$ with

$$f_1(g_i^2) = \begin{cases} g_1^2 + 2\left(g_2^2 - g_1^2\right)\left[6\zeta(3)g_1^4 + 20\zeta(5)g_1^4\left(g_2^2 + 3g_1^2\right) + \cdots\right] , & g_1, g_2 \to 0 \\ 2\frac{g_1^2 g_2^2}{g_1^2 + g_2^2} + \cdots , & g_1, g_2 \to \infty \end{cases} , \tag{69}$$

for the $\mathbb{Z}_2$ quiver theory. These results are identical with the ones in [44] and thus in perturbation theory up to three loops the effective couplings are universal, and compute the relative finite renormalization of the gluon propagators (68). In the strong coupling the leading term matches the prediction of AdS/CFT [44], see also references therein.

Starting at four loops, we found, see equation (60), that there are discrepancies between the different effective couplings for the different observables that are always proportional to $\zeta(2) = \frac{\pi^2}{6}$. The first few terms read

$$\Delta f(g_i^2) = 80\zeta(2)g_1^8(g_1^2 - g_2^2)\zeta(5) - g_1^8(g_1^2 - g_2^2)\big(192 g_1^2\zeta(2)^2\zeta(5) + 112(11g_1^2 + 5g_2^2)\zeta(2)\zeta(7)\big) + \cdots . \tag{70}$$

Moreover, for the two observables B and $\langle W\rangle$, the difference between the two effective couplings is due to the dependence in b,

$$f_B - f_W = \Delta f \sim \frac{\partial f}{\partial b} , \tag{71}$$

see also equation (60). Beginning with this observation and stressing the fact that the parameter b determines how we cut off the low energy momenta for a given calculation, as it is related to the size and shape of the ellipsoid, we wish to argue that the way we extract $f_k(g_i^2)$ suffers from scheme dependence originating in the way the theory is regulated in the infrared.

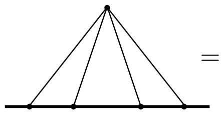
At the order g^{10}, where the discrepancies appear for the first time, the effective coupling reads

$$\begin{aligned} f_{W;1}(g_1, g_2) = \cdots + 2\left(g_2^2 - g_1^2\right) g_1^4 \Big[& 70\zeta(7)\Big(g_2^4 + 5g_1^2 g_2^2 + 8g_1^4\Big) \\ & - 40\zeta(2)\zeta(5)g_1^4 - 2(6\zeta(3))^2\Big(g_2^4 - g_1^2 g_2^2 + 2g_1^4\Big)\Big] . \end{aligned} \tag{72}$$

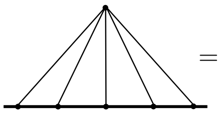
While it is very clear how to get from Feynman diagrams the $\zeta(7)$ and the $\zeta(3)^2$ pieces [44], it is *not possible in flat space and for the massless and finite theories that we are considering to produce a* $\zeta(2)\zeta(5)$ *term* by one or more Feynman diagrams. This can be understood by carefully looking at the classification of the massless four loop integrals [53]. The reader needs to keep in mind that the theories we are looking at are finite [54] and hence the poles always have to cancel. It would be very important to demonstrate this statement with an explicit calculation, but we leave this for future work.

Let us further stress that the way we have been computing the different $f_k(g_i^2)$ is through localization, which are always done *on a sphere [8] or on an ellipsoid* [20]. For those geometries, some of the fields couple conformally to the curvature *acquiring an effective mass term* $m^2 = \mu_R^2 \propto R \propto (\ell\tilde{\ell})^{-1}$, proportional to scale set

[12] Three loops for $f(g_i^2)$ is four loops for the observables, the Wilson loop, the Bremsstrahlung function and the entanglement entropy. An insertion of a tree level propagator creates an one loop correction for them and so on.



$= -5\zeta(5) + 12\zeta(2)\zeta(3)$



$= a\zeta(7) + b\zeta(5)\zeta(2) + c\zeta(3)\zeta(4)$

Figure 4: *The $\zeta(2)$s that we are looking for can be created by massive on-shell propagator diagrams. The thick lines indicate the massive propagators, while the thin lines stand for massless ones. The computation is done with the external massive particle momentum being on the mass shell. We expect the numbers a, b and c to be integers.*

by the Ricci scalar of the ellipsoid. Hence, if we want to reproduce the $\zeta(2)\zeta(5)$ term by Feynman diagrams, we would have to take into account that *some propagators in the integrals that we are computing become massive.* These mass terms will then be renormalized. For generic theories, this conformal coupling to the curvature usually begins to renormalize starting at two loops. For theories with supersymmetry, the conformal coupling to the curvature will start to renormalize one loop later, at three loops, and we believe that in our case with $\mathcal{N} = 2$ superconformal symmetry the effective mass term will start to renormalize at four loops. Moreover, the presence of massive modes in the loops forces us to specify a *mass renormalization scheme.* In the localization results, the size of the sphere or of the ellipsoid is a parameter independent of g. The mass term $m^2 = \mu_R^2 \propto R$ does not renormalize, but is instead kept fixed. This is a very particular scheme choice.

Our next step is to recall examples in 4D QFTs where terms proportional to $\zeta(2)$ are created. One instance that immediately comes to mind is the computation of bubbles with no external legs. A famous example is the Casimir effect, for which $\langle T_{00} \rangle = -\frac{\pi^2}{90L^4} = -\zeta(2)\frac{1}{15L^4}$. A second common way to create $\zeta(2)$s is through mass renormalization, when the mass counterterm is inserted in a bigger diagram and in particular in the large mass expansion [55, 56]. Feynman diagrams like the ones depicted in figure 4 with some on shell propagators are known to create $\zeta(2)\zeta(3)$ or $\zeta(2)\zeta(5)$ terms. On the left hand side of figure 4, taken from [56], a $\zeta(2)\zeta(3)$ finite term is created. The $\zeta(2)\zeta(5)$ term that we are looking for is going to come from a diagram like the one depicted on the right side of figure 4, at one loop higher than the $\zeta(2)\zeta(3)$ one.

Now that we have made a very particular choice for the mass renormalization scheme, we also have to UV regularize. Let's consider a UV cut off regularization with a UV scale Λ_{UV}. From an effective field theory point of view[13], we should be able to perform the calculation either with $\Lambda_{\text{UV}} >> \mu_R$, or with $\Lambda_{\text{UV}} << \mu_R$. In the latter case, we obtain an effective theory in which the massive fields have decoupled. The relation between the coupling constant in the effective theory without the heavy fields (EFT) and the full theory including the massive fields is given by the matching condition

$$g^2_{\text{EFT}} = z_{\text{match}}\, g^2_{\text{Full}}\,, \tag{73}$$

where z_{match} is a function that depends on the choice of the scheme and the mass of the heavy fields. For us, the matching condition translates to a relation between the finite coupling renormalization factor $\mathcal{Z}_g^{\mathcal{N}=2}$ for the

[13] We have in mind the well known relation between heavy quark QCD and the HQEF [57]. In particular, the decoupling theorem or the ramifications thereof, that is described in chapter 8 of [55].

theory in the large mass expansion and the full theory

$$\mathcal{Z}^{\mathcal{N}=2}_{g\,\mathrm{EFT}} = z_{\mathrm{match}}\, \mathcal{Z}^{\mathcal{N}=2}_{g\,\mathrm{full}}\,. \tag{74}$$

See [55, 57] for the general ideology. Now in this theory with very massive fields, it is easy to see how to create the terms with $\zeta(2)\zeta(5)$. We just need to look again at the figure 4. We see that $\zeta(2)\zeta(3)$ terms can be created when we replace the propagator of the massive fields by the on shell one. It would be very beautiful to understand why there are no $\zeta(2)\zeta(3)$ terms at order g^8, or at any order that we have checked - it is most certainly a cancellation due to supersymmetry. The $\zeta(2)\zeta(5)$ term that we are looking for is expected to come from the diagram depicted in the right of figure 4. These type of calculations can be done using [58] and we leave them for future work.

After all the arguments above, we come to the conclusion that the $\zeta(2)$s stem from the fact that on the sphere or on the ellipsoid some of the fields have a mass term $m^2 = \mu_R^2 \propto R$. As we we go up in loops, the mass is going to be renormalized and then we have to make the choice of a scheme. At the next loop order this mass renormalization scheme choice is going to interfere with our $f(g_i)$ that computes the finite coupling renormalization of the coupling constant. This point was also addressed by Lewkowycz and Maldacena [14] and by Fraser in [59]. Specifically, in [14] it was discussed that the entanglement entropy computed has a finite ambiguity related to the precise procedure for defining the entropy with the additional finite contributions arising due to the conformal coupling of the scalars to the curvature.

7 Conclusions and outlook

In this article, we calculated in the planar limit the vacuum expectation value of supersymmetric Wilson loops on ellipsoids for the $\mathcal{N}=2$ cyclic superconformal quivers, by using the localization formula of [20]. We provided explicit results, both in the weak and in the strong coupling limits. By comparing with $\mathcal{N}=4$ SYM, we obtained the effective couplings $f_W(g_i^2)$ such that $W_{\mathcal{N}=4}(f_W(g_i^2)) = W_{\mathcal{N}=2}(g_i^2)$.

Using the works of [14, 19], we extracted from the Wilson loops the Bremsstrahlung functions B and the entanglement entropy S. Thereby we learned an important lesson, the effective couplings are universal, *i.e.* the same for the different observables, up to four loops in the weak coupling and for the leading order in the strong coupling. For example, for the $\hat{A}_1$ or $\mathbb{Z}_2$ quiver theory, we find that the effective coupling of the first gauge groups is

$$f_1(g_i^2) = \begin{cases} g_1^2 + 2\left(g_2^2 - g_1^2\right)\left[6\zeta(3)g_1^4 + 20\zeta(5)g_1^4\left(g_2^2+3g_1^2\right)+\cdots\right]\,, & g_1, g_2 \to 0 \\ 2\frac{g_1^2 g_2^2}{g_1^2+g_2^2} + \cdots\,, & g_1, g_2 \to \infty \end{cases}\,. \tag{75}$$

Starting at five loops there are discrepancies between the effective couplings for the different observables that are always proportional to $\zeta(2)$. Specifically

$$\Delta f(g_i^2) = \zeta(2)\left(g_1^2 - g_2^2\right) g_1^8 \left(40\zeta(5) + \mathcal{O}(g^2)\right)\,. \tag{76}$$

The same exact observation was made by Fraser in [59] by comparing Wilson loops in different representations.

In section 6 we argued that the $\zeta(2)$ discrepancies are due to the artifact of the localization calculations being performed on the ellipsoid, imposing hard IR regulators and scheme dependence. Thus, we proposed that the effective couplings are universal up to the fact that one needs to properly take into account this scheme dependence. Following [17], any anomalous dimension in the purely gluonic, $SU(2,1|2)$, sectors of $\mathcal{N}=2$ superconformal gauge theories can be obtained from the $\mathcal{N}=4$ results by directly replacing the $\mathcal{N}=4$ coupling by the effective couplings. Moreover, again up to the scheme ambiguity, our results allow for the calculation of the cusp anomalous dimension $\Gamma_{\text{cusp}}(\varphi)$ in $\mathcal{N}=2$ theories. The light-like cusp anomalous dimension of $\mathcal{N}=4$ SYM is

$$K_{\mathcal{N}=4}(g^2) = 4g^2 - \frac{4\pi^2 g^4}{3} + \frac{44\pi^4 g^6}{45} - \left(32\zeta(3)^2 + \frac{292\pi^6}{315}\right) g^8 + O\left(g^{10}\right) \tag{77}$$

for small g and $K_{\mathcal{N}=4}(g^2) \sim 2g - \frac{3\log(2)}{2\pi} + \cdots$ for large g. Hence, inserting (44) and (50), we obtain the predictions for the $\mathbb{Z}_2$ quiver

$$\begin{aligned} K(g_i^2) =& 4g_1^2 - \frac{4\pi^2}{3} g_1^4 + \left[24\zeta(3)\left(-2g_1^2+g_2^2+g_5^2\right)g_1^4 + \frac{44}{45}\pi^4 g_1^6\right] - \Big[32\zeta(3)^2 g_1^8 \\ &+ 16\pi^2\zeta(3)\left(-2g_1^2+g_2^2+g_5^2\right)g_1^6 - 80\zeta(5)\left(6g_1^4 - 2\left(g_2^2+g_5^2\right)g_1^2 - g_2^4 - g_5^4\right)g_1^4 + \frac{292}{315}\pi^6 g_1^8\Big] + \mathcal{O}\left(g^{10}\right) \end{aligned} \tag{78}$$

in the weak coupling regime and

$$K(g_i^2) \sim 2\sqrt{\frac{2g_1^2 g_2^2}{g_1^2+g_2^2}} \tag{79}$$

in the strong coupling limit. Inserting K into the function Ω in equation (11) (whose explicit expression to three loops is found in equation (17) of [46]) provides us with the full $\Gamma_{\text{cusp}}(\varphi)$ function.

There are many interesting questions and problems left for future work. In our mind, the number one priority is to perform an explicit Feynman diagram calculation on the sphere where the scalars acquire a mass and to explicitly find and compute the diagrams responsible for the first $\zeta(2)\zeta(5)$ discrepancies given in (76).

Wilson loops compute a big part of the data needed to obtain the high-energy scattering of charged particles [2,3]. Our diagrammatic studies lead us to believe that light-like polygonal Wilson loops stand a good change to obey the substitution rule, perhaps even in theories with less supersymmetry. Checking whether the substitution rule works by explicit Feynman diagrams calculations is an important direction worth pursuing. In investigating this direction, it will be paramount to use the appropriate superspace formalism. Moreover, in $\mathcal{N}=4$ SYM, polygonal shaped Wilson loops are believed to be exactly dual to scattering amplitudes, see [60] and references therein for a recent review. This duality is due to the dual superconformal symmetry which is also believed to combine with the usual conformal symmetry to the Yangian of the full $\mathfrak{psu}(2,2|4)$ integrable model. If the substitution rule works for light-like polygonal Wilson loops and gives the correct $\mathcal{N}=2$ results, it implies that there exists a Yangian symmetry acting on them, something worth checking. However, in strong contrast with $\mathcal{N}=4$ SYM, the dual superconformal symmetry seems to break at two-loops[14] [61, 62] destroying the

[14]It is important to stress that the term responsible for breaking the dual superconformal symmetry is a finite constant $12\zeta(3)$

amplitude/Wilson loop duality. It would be very important to understand what this means for the integrability of the $\mathcal{N}=2$ SCFTs and to try to come up with ways to bypass this impasse.

In the case of $\mathcal{N}=4$ SYM, the cusp anomalous dimension can be obtained by studying a supersymmetric Wilson loops with L local fields inserted at the cusp [12, 13]. This setup is described by TBA equations very similar to the ones of the spectral problem [12, 13]. However, these TBA equations are simpler and can be recast in terms of a matrix model [63–65] with a spectral curve that can be mapped to the classical string algebraic curve. For $\mathcal{N}=2$ SCFTs, it is currently not clear what happens beyond the $SU(2,1|2)$ sector, but it is worth thinking whether it is possible to derive TBA equations for supersymmetric Wilson loops with local fields from the $SU(2,1|2)$ sector inserted at the cusp.

Another class of observables that is definitely worth studying is the correlation functions of chiral primary operators, studied in particular in [66–69]. The naive coupling substitution for them does not work, due to the fact that finite terms from the non-holomorphic part of the $\mathcal{N}=2$ effective action $\int d^8\theta\mathcal{H}(\mathcal{W},\bar{\mathcal{W}})$ contribute to the correlation functions. This is not the case for the anomalous dimensions in the $SU(2,1|2)$ sector where effective vertices from $\int d^8\theta\mathcal{H}(\mathcal{W},\bar{\mathcal{W}})$ cannot contribute [17]. Using the methods of [15, 16, 66–69], we can compute the exact Zamolodchikov metric, *i.e.* the metric in theory space, in the planar limit and from that recover the correlation functions of chiral primary operators. This is work in progress. The Zamolodchikov metric is another very interesting non-BPS observable. Investigating this direction is currently in progress.

While our results apply for the weak coupling of both the cyclic and the linear quivers, such as $\mathcal{N}=2$ SCQCD, see figure 2, they do not apply to the strong coupling of the linear quivers, since the limit $g_r \to 0$ does not commute with the strong coupling limit considered in section 4.2. As already discussed in [36], the strong coupling limit of $\mathcal{N}=2$ SCQCD is quite subtle. It would be important to understand more about the strong coupling limit of the linear quiver theories, in particular so as to improve our knowledge of their string duals.

Another theory in which a similar coupling substitution rule applies is ABJM with the interpolating function $h(\lambda)$ also appearing in the magnon dispersion relation and being computed by comparing with localization techniques. It would be very interesting to study the Kaluza-Klein reduction of $\mathcal{N}=4$ in the spirit of section 2 of [6], to see whether the interpolating function $h(\lambda)$ can be understood diagrammatically in a spirit similar to ours.

Last but not least, as we discussed in section 2.3, in [46, 47] another, "experimental" coupling substitution rule was discovered, in which the coupling $g_{\rm YM}$ was replaced by the light-like cusp anomalous dimension K and it was then found that, when so expressed, the full $\Gamma_{\rm cusp}(\varphi)$ is independent of the specific particle content of the gauge theory, at least up to three loops. It would be very interesting to try to understand this fact using a diagrammatic argument of the form of [17], and to try to decide whether there should be other observables

that comes from the non-holomorphic part of the $\mathcal{N}=2$ effective action $\int d^8\theta\mathcal{H}(\mathcal{W},\bar{\mathcal{W}})$ and is suppressed in the Regge limit. The amplitude/Wilson loop duality that is broken at two-loops is restored in the Regge limit. Finally this finite constant and all the other finite contributions at higher loops are all encoded in the exact Zamolodchikov metric that can be obtained from localization using the methods of [15].

that could be obtained in similar ways.

Note added

As we have been finishing writing up this note, a closely related paper [70] appeared in the arXiv.

Acknowdlegments

We thank James Drummond, Bartomeu Fiol, Johannes Henn, Zohar Komargodski, Sven Moch, Volker Schomerus, Matthias Steinhauser, Jörg Teschner, Konstantin Zarembo for useful discussions and feedback.

A Special functions

For the reader's convenience, we gather here the definitions and properties of all special functions used in the main text.

A.1 Barnes Γ_2 and associated functions

We begin with the function $\Upsilon(x;b)$ which is defined for $0 < \Re(x) < Q = b + b^{-1}$ as the integral

$$\log \Upsilon(x;b) := \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left[\left(\frac{Q}{2} - x \right) \frac{t}{2} \right]}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right] . \tag{80}$$

Unless necessary, we shall often suppress the b and just write $\Upsilon(x)$. It is clear from the definition that

$$\Upsilon(x) = \Upsilon(Q - x) , \qquad \Upsilon\left(Q/2\right) = 1 . \tag{81}$$

One can show from the alternative definition below that the following shift identities are obeyed

$$\Upsilon(x + b) = \gamma(xb) b^{1-2bx} \Upsilon(x) , \qquad \Upsilon(x + b^{-1}) = \gamma(xb^{-1}) b^{2xb^{-1}-1} \Upsilon(x) . \tag{82}$$

where

$$\gamma(x) := \frac{\Gamma(x)}{\Gamma(1 - x)} \Longrightarrow \gamma(-x) = -\frac{1}{x^2} \gamma(x)^{-1} . \tag{83}$$

It follows from (82) that Υ is an entire function with zeroes at

$$x = -n_1 b - n_2 b^{-1} , \quad \text{or} \quad x = (n_1 + 1)b + (n_2 + 1)b^{-1} , \tag{84}$$

where $n_i \in \mathbb{N}_0$.

The function Υ can be connected to the Barnes double Gamma function $\Gamma_2(x; \epsilon, \epsilon_2)$. Introducing the Barnes ζ_2 function

$$\zeta_2(x, s; \epsilon_1, \epsilon_2) := \sum_{n_1, n_2 \geq 0} \frac{1}{(x + n_1\epsilon_1 + n_2\epsilon_2)^s} , \tag{85}$$

convergent for $\mathrm{Re}(s) \geq 2$, we define $\Gamma_2(x;\epsilon_1,\epsilon_2)$ via the *analytic continuation*

$$\log \Gamma_2(x;\epsilon_1,\epsilon_2) = \left[\frac{\partial}{\partial t}\zeta_2(x,s;\epsilon_1,\epsilon_2)\right]_{s=0} . \tag{86}$$

From this definition, one can prove (see A.54 of [71]) the *difference property*

$$\frac{\Gamma_2(x+\epsilon_1;\epsilon_1,\epsilon_2)}{\Gamma_2(x;\epsilon_1,\epsilon_2)} = \frac{\sqrt{2\pi}}{\epsilon_2^{\frac{x}{\epsilon_2}-\frac{1}{2}}\Gamma\left(\frac{x}{\epsilon_2}\right)}, \qquad \frac{\Gamma_2(x+\epsilon_2;\epsilon_1,\epsilon_2)}{\Gamma_2(x;\epsilon_1,\epsilon_2)} = \frac{\sqrt{2\pi}}{\epsilon_1^{\frac{x}{\epsilon_1}-\frac{1}{2}}\Gamma\left(\frac{x}{\epsilon_1}\right)} . \tag{87}$$

In order to express the Υ function using the Barnes double Gamma function, we have to first define the *normalized* function

$$\Gamma_b(x) := \frac{\Gamma_2(x;b,b^{-1})}{\Gamma_2(\frac{Q}{2};b,b^{-1})} . \tag{88}$$

We remark that the log of the function $\Gamma_b(x)$ has also an integral representation as

$$\log\Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left(\frac{e^{-xt}-e^{-\frac{Qt}{2}}}{(1-e^{-tb})(1-e^{-tb^{-1}})} - \frac{\left(\frac{Q}{2}-x\right)^2}{2}e^{-t} - \frac{\frac{Q}{2}-x}{t}\right) . \tag{89}$$

Then, using (88) we can express the $\Upsilon(x;b)$ function as

$$\Upsilon(x;b) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)} . \tag{90}$$

This, together with the difference properties of Γ_2 proves the shift identities (82). In addition, using the shift identities leads us to the following expression

$$\Upsilon(x;b) = -x b^{(b-b^{-1})x}\frac{\Gamma(-bx)\Gamma(-b^{-1}x)\Gamma_2(Q/2;b,b^{-1})^2}{2\pi\Gamma_2(x;b,b^{-1})\Gamma_2(-x;b,b^{-1})} , \tag{91}$$

the advantage of which is to show which pieces of $\Upsilon(x;b)$ are symmetric under $x \to -x$.

One very often encounters a product formula for the function $\Gamma_2 = \prod_{n_1,n_2}(x+\epsilon_1 n_1+\epsilon_2 n_2)^{-1}$ that is unfortunately not quite correct. To get the product formula for $\Gamma_2(x)$ working, one has to use (A.62) of [71]. Specifically, we set for $\Re(s) > 2$

$$\chi(s;\epsilon_1,\epsilon_2) := \sum_{n_1,n_2\geq 0}{}' \frac{1}{(\epsilon_1 n_1+\epsilon_2 n_2)^s} , \tag{92}$$

where the prime removes the value $(n_1,n_2) = (0,0)$ from the sum. The function $\chi(s;\epsilon_1,\epsilon_2)$ can be analytically continued for all $s \in \mathbb{C}$ except for $s=1$ and $s=2$ where there are poles. We have the residues

$$\mathrm{Res}(\chi(s;\epsilon_1,\epsilon_2), s=1) = \frac{1}{2}\left(\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right), \qquad \mathrm{Res}(\chi(s;\epsilon_1,\epsilon_2), s=2) = \frac{1}{\epsilon_1\epsilon_2} \tag{93}$$

and the finite parts

$$\begin{aligned}
\text{Res}\Big(\frac{\chi(s;\epsilon_1,\epsilon_2)}{s-1}, s=1\Big) &= -\frac{\log\epsilon_1}{\epsilon_1} + \frac{1}{2}\left(\frac{1}{\epsilon_1}-\frac{1}{\epsilon_2}\right)\log\epsilon_2 + \frac{\gamma}{\epsilon_1} + \frac{\gamma}{2\epsilon_2} - \frac{1}{2\epsilon_1}\log 2\pi \\
&\quad -\frac{i}{b}\int_0^\infty \frac{\psi(i\frac{\epsilon_1}{\epsilon_2}y+1)-\psi(-i\frac{\epsilon_1}{\epsilon_2}y+1)}{e^{2\pi y}-1}dy \\
\text{Res}\Big(\frac{\chi(s;\epsilon_1,\epsilon_2)}{s-2}, s=2\Big) &= \frac{\zeta(2)}{\epsilon_1^2} + \frac{\zeta(2)}{2\epsilon_2^2} + \frac{1}{\epsilon_1\epsilon_2}(\gamma - 1 - \log\epsilon_2) \\
&\quad -\frac{i}{\epsilon_2}\int_0^\infty \frac{\zeta_H(2,i\frac{\epsilon_1}{\epsilon_2}y+1)-\zeta_H(2,-i\frac{\epsilon_1}{\epsilon_2}y+1)}{e^{2\pi y}-1}dy\,,
\end{aligned} \tag{94}$$

where ψ is the digamma function, γ is the Euler - Mascheroni constant and $\zeta_H(s,q)$ is the Hurwitz-ζ function with ($\Re(s)>1$ and $\Re(q)>0$)

$$\zeta_H(s,q) := \sum_{n=0}^{\infty}\frac{1}{(q+n)^s}\,. \tag{95}$$

Finally, using the shorthands

$$r_\chi := \text{Res}(\frac{\chi(s;\epsilon_1,\epsilon_2)}{s-1}, s=1)\,, \qquad s_\chi := \text{Res}(\frac{\chi(s;\epsilon_1,\epsilon_2)}{s-2}, s=2) + \text{Res}(\chi(s;\epsilon_1,\epsilon_2), s=2)\,, \tag{96}$$

we present the formula

$$\Gamma_2(x;\epsilon_1,\epsilon_2) = \frac{e^{-r_\chi x + \frac{s_\chi x^2}{2}}}{x}\prod_{n_1,n_2\geq 0}{}^{'}\frac{e^{\frac{x}{\epsilon_1 n_1+\epsilon_2 n_2}-\frac{x^2}{2(\epsilon_1 n_1+\epsilon_2 n_2)^2}}}{1+\frac{x}{\epsilon_1 n_1+\epsilon_2 n_2}}\,. \tag{97}$$

There is furthermore the following nice identity of which (89) is a special case

$$\log\frac{\Gamma_2(x;\epsilon_1,\epsilon_2)}{\Gamma_2(\epsilon_+/2;\epsilon_1,\epsilon_2)} = \int_0^\infty\frac{dt}{t}\left[\frac{e^{-tx}-e^{-\frac{t\epsilon_+}{2}}}{(1-e^{-t\epsilon_1})(1-e^{-t\epsilon_2})} - \frac{e^{-t}\left(\frac{\epsilon_+}{2}-x\right)^2}{2(\epsilon_1\epsilon_2)} - \frac{\frac{\epsilon_+}{2}-x}{t(\epsilon_1\epsilon_2)}\right]. \tag{98}$$

We have for the special case $\epsilon_1=\epsilon_2$ (beware that in the first equation we have $\epsilon^\star$, not $e^\star$!)

$$\begin{aligned}
\Gamma_2(x;\epsilon,\epsilon) &= \epsilon^{\frac{x}{\epsilon}-\frac{1}{2}\left(\frac{x}{\epsilon}\right)^2-1}\Gamma_2(x/\epsilon;1,1)\,, \\
\Gamma_2(x;1,1) &= \frac{e^{-\left(\gamma-\frac{1}{2}\right)x+\frac{1}{2}\left(\frac{\pi^2}{6}+\gamma+1\right)x^2}}{x}\prod_{n=1}^{\infty}\frac{e^{(n+1)\left(\frac{x}{n}-\frac{x^2}{2n^2}\right)}}{\left(1+\frac{x}{n}\right)^{n+1}}\,,
\end{aligned} \tag{99}$$

where we have used proposition 8.3 in [72] and the identities

$$\begin{aligned}
\int_0^\infty \frac{\psi(iy+1)-\psi(-iy+1)}{e^{2\pi y}-1}dy &= i\left(\frac{\log(2\pi)}{2}-\frac{\gamma}{2}-\frac{1}{2}\right), \\
\int_0^\infty \frac{\zeta_H(2,iy+1)-\zeta_H(2,-iy+1)}{e^{2\pi y}-1}dy &= i(\zeta(2)-1)\,.
\end{aligned} \tag{100}$$

Using the product formula for the usual Gamma function

$$\Gamma(x) = \frac{e^{-\gamma x}}{x}\prod_{n=1}^{\infty}\frac{e^{\frac{x}{n}}}{1+\frac{x}{n}}\,, \tag{101}$$

and the definition of the Barnes G function, (obeying $G(x+1) = \Gamma(x)G(x)$)

$$G(1+x) := (2\pi)^{\frac{x}{2}} e^{-\frac{x+x^2(1+\gamma)}{2}} \prod_{n=1}^{\infty} \left(1+\frac{x}{n}\right)^n e^{\frac{x^2}{2n}-x}\,, \tag{102}$$

we get

$$\Gamma_2(x;1,1) = (2\pi)^{\frac{x}{2}} \frac{\Gamma(x)}{G(1+x)}\,. \tag{103}$$

Since $G(2) = 1$, we have $\Gamma_2(1;1,1) = \sqrt{2\pi}$, so that (90) becomes for $b = 1$ the expression

$$\Upsilon(x;1) = \gamma(x)^{-1} G(1+x)G(1-x)\,, \tag{104}$$

where we have used the shift properties of the Barnes G function. It follows from (83) that

$$\Upsilon(x;1)\Upsilon(-x;1) = -x^2 e^{-2(1+\gamma)x^2} H(-ix)^2\,, \tag{105}$$

where

$$H(x) := e^{-(1+\gamma)x^2} G(1+ix)G(1-ix) = \prod_{n=1}^{\infty} \left(1+\frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}}\,. \tag{106}$$

Let us introduce two new special functions that will prove useful later on. Specifically, we need

$$\begin{aligned} H_v(x;b) :=& \prod_{m,n=0}^{\infty} \sqrt{\left(1+\frac{x^2}{(b(m+1)+b^{-1}n)^2}\right)\left(1+\frac{x^2}{(bm+b^{-1}(n+1))^2}\right)} \\ &\times e^{-\frac{x^2}{2(b(m+1)+b^{-1}n)^2}-\frac{x^2}{2(bm+b^{-1}(n+1))^2}}\,, \\ H_h(x;b) :=& e^{-\zeta(2)\frac{b^2+b^{-2}}{2}x^2} \prod_{m,n=0}^{\infty} \left(1+\frac{x^2}{\left(b(m+1/2)+b^{-1}(n+1/2)\right)^2}\right) e^{-\frac{x^2}{(b(m+1)+b^{-1}(n+1))^2}}\,. \end{aligned}$$

It is easy to see that both these functions are even in x, that $H_v(0;b) = H_h(0;b) = 1$ and that

$$\lim_{b\to 1} H_v(x;b) = \lim_{b\to 1} H_h(x;b) = H(x) = \prod_{n=1}^{n} \left(1+\frac{x^2}{n^2}\right)^n e^{-\frac{x^2}{n}}\,. \tag{107}$$

Furthermore, $H_a(x;b) = H_a(x;b^{-1})$, both for $a = v$ and for $a = h$.

We now need to consider the logarithms of the functions H_v and H_h defined in (107). Let us define

$$K_v(x;b) := -\frac{d}{dx}\log(H_v(x;b))\,, \qquad K_h(x;b) := -\log\frac{d}{dx}\log(H_h(x;b))\,. \tag{108}$$

We can easily compute the logarithms of H_v and H_h, finding that

$$\begin{aligned} K_v(x;b) &= -2\sum_{n=1}^{\infty}(-1)^n x^{2n+1} \frac{\zeta_2(b,2n+2;b,b^{-1})+\zeta_2(b^{-1},2n+2;b,b^{-1})}{2}\,, \\ K_h(x;b) &= -2\sum_{n=1}^{\infty}(-1)^n x^{2n+1} \zeta_2((b+b^{-1})/2, 2n+2; b, b^{-1})\,. \end{aligned} \tag{109}$$

We need the following special values

$$\begin{aligned} \zeta_2(1,2n+2;1,1) &= \zeta(2n+1)\,, \\ \frac{d}{db}\zeta_2(b,2n+2;b,b^{-1})_{|b=1} &= -(2n+2)\zeta(2n+2)\,, \\ \frac{d}{db}\zeta_2(b^{-1},2n+2;b,b^{-1})_{|b=1} &= (2n+2)\zeta(2n+2)\,, \\ \frac{d}{db}\zeta_2({}^{(b+b^{-1})}\!/_2,2n+2;b,b^{-1})_{|b=1} &= 0\,. \end{aligned} \tag{110}$$

Hence $K_v(x;1)=K_h(x;1)=K(x):=-2\sum_{n=1}^{\infty}(-1)^n\zeta(2n+1)x^{2n+1}$ and

$$\frac{d}{db}K_v(x;b)_{|b=1}=\frac{d}{db}K_h(x;b)_{|b=1}=0\,. \tag{111}$$

Similarly, expanding in b around $b=1$, replacing the summation variables as $n_1={}^{(r+s)}\!/_2$, $n_2={}^{(r-s)}\!/_2$ with $r\in\mathbb{N}_0$ and $s\in\{-r,-r+2,\dots,r\}$, we find the expansions

$$\begin{aligned} K_v(x;b) =& -2\sum_{n=1}^{\infty}(-1)^n x^{2n+1}\Big[\zeta(2n+1) \\ &+\frac{4}{3}(n+1)\big(n\zeta(2n+1)+(2n+3)\zeta(2n+3)\big)(b-1)^2 \\ &-4(n+1)\big(n\zeta(2n+1)+(2n+3)\zeta(2n+3)\big)(b-1)^3+\mathcal{O}(b-1)^4\Big]\,, \\ K_h(x;b) =& -2\sum_{n=1}^{\infty}(-1)^n x^{2n+1}\Big[\zeta(2n+1) \\ &+\frac{2}{3}(n+1)\big(2n\zeta(2n+1)-(2n+3)\zeta(2n+3)\big)(b-1)^2 \\ &-2(n+1)\big(2n\zeta(2n+1)-(2n+3)\zeta(2n+3)\big)(b-1)^3+\mathcal{O}(b-1)^4\Big]\,, \end{aligned} \tag{112}$$

where we have used ($B_k(1)$ is the value of the k^{th} Bernoulli polynomial at 1)

$$\sum_{\substack{s=-r\\ \text{step }2}}^{r} s^m=\delta_{(m \bmod 2),0}\left(\delta_{m,0}+\frac{2^{m+1}}{m+1}\sum_{k=0}^{m}B_k(1)\binom{m+1}{k}\left(\frac{r}{2}\right)^{m+1-k}\right)\,. \tag{113}$$

A.2 Chebyshev polynomials

In this appendix, we summarize a couple of useful formulae involving the Chebyshev polynomials T_l and U_l of the first and second kind respectively. Important for us are the integral identities

$$⨍_{-1}^{1}\sqrt{1-y^2}\frac{U_k(y)}{x-y}dy=\pi T_{k+1}(x),\qquad ⨍_{-1}^{1}\frac{1}{\sqrt{1-y^2}}\frac{T_k(y)}{x-y}dy=-\pi U_{k-1}(x)\,. \tag{114}$$

which are only valid if $x\in(-1,1)$. The second part of equation (114) implies in particular for $k=1$

$$⨍_{-\mu_k}^{\mu_k}\frac{dx}{\sqrt{\mu_k^2-x^2}}\frac{1}{z-x}x=-\pi\,. \tag{115}$$

We can generalize (114) for arbitrary x to to

$$\begin{aligned}
&\frac{1}{\pi}\fint_{-\mu}^{\mu}\sqrt{\mu^2-y^2}\frac{U_k\left(\frac{y}{\mu}\right)}{x-y}dy=\mu T_{k+1}\left(\frac{x}{\mu}\right)-\operatorname{sgn}(x)\Theta(|x|-\mu)\sqrt{x^2-\mu^2}U_k\left(\frac{x}{\mu}\right)\,,\\
&\frac{1}{\pi}\fint_{-\mu}^{\mu}\frac{1}{\sqrt{\mu^2-y^2}}\frac{T_k\left(\frac{y}{\mu}\right)}{x-y}dy=-\frac{1}{\mu}U_{k-1}\left(\frac{x}{\mu}\right)+\operatorname{sgn}(x)\Theta(|x|-\mu)\frac{T_k\left(\frac{x}{\mu}\right)}{\sqrt{x^2-\mu^2}}\,,
\end{aligned} \tag{116}$$

where $\Theta(x)=1$ if $x\geq 0$ and is zero otherwise is the Heaviside function. For $\mu>0$, one can prove for $x_1\neq x_2$

$$\begin{aligned}
&\frac{1}{\pi}\fint_{-\mu}^{\mu}\frac{dy}{\sqrt{\mu^2-y^2}}\frac{1}{(x_1-y)(x_2-y)}\\
&=\frac{\pi\delta(x_1-x_2)\Theta(\mu-|x_1|)}{\sqrt{\mu^2-x_1^2}}+\frac{1}{x_1-x_2}\left(\frac{\operatorname{sgn}(x_2)\Theta(|x_2|-\mu)}{\sqrt{x_2^2-\mu^2}}-\frac{\operatorname{sgn}(x_1)\Theta(|x_1|-\mu)}{\sqrt{x_1^2-\mu^2}}\right)\,.
\end{aligned} \tag{117}$$

The identity (117) follows from (116) as well as from the observation: the equations (114) imply that for a function $\rho(x)$, that we can under some assumptions expand as

$$\rho(x)=\sqrt{\mu^2-x^2}\sum_{n=0}^{\infty}c_nU_n\left(\frac{x}{\mu}\right)\,, \tag{118}$$

we can invert the finite Hilbert kernel and write a δ-function relation like

$$\rho(x)=-\frac{\sqrt{\mu^2-x^2}}{\pi^2}\fint_{-\mu}^{\mu}\frac{dy}{\sqrt{\mu^2-y^2}}\frac{1}{x-y}\fint_{-\mu}^{\mu}dz\frac{\rho(z)}{y-z}\,. \tag{119}$$

B Rewriting the partition functions

For conformal field theories, we can claim that we can rewrite (20) as

$$Z=\int da\, e^{-\frac{8\pi^2}{g_{\text{YM}}^2}\operatorname{Tr}(a^2)}Z_{\text{1-loop}}\left|Z_{\text{inst}}\right|^2\,, \tag{120}$$

with the vector multiplet contribution

$$Z^{\text{vect}}_{\text{1-loop}}=\prod_{i<j=1}^{N}(a_i-a_j)^2\prod_{i,j=1}^{N}H_v(a_i-a_j;b) \tag{121}$$

and the hyper multiplet contribution

$$Z^{\text{hyper}}_{\text{1-loop}}=\prod_{i,j=1}^{N}H_h(a_i^{(1)}-a_j^{(2)};b)^{-1}\,. \tag{122}$$

Proof. We rescale the integration variables in (20) by $\sqrt{\ell\tilde{\ell}}$. This does not change the results for conformal field theories. We now consider the partition functions (22) separately.

First, we consider the vector multiplet contributions. We use (91) in order to split away the Vandermonde determinant contribution. We get

$$\begin{aligned} Z^{\text{vect}}_{\text{1-loop}} &= \prod_{i<j=1}^{N} \Upsilon\big(i(a_i-a_j);b\big)\Upsilon\big(-i(a_i-a_j);b\big) \\ &= \prod_{i<j=1}^{N} (a_i-a_j)^2 \frac{\prod_{s=\pm}\left[\Gamma(sib(a_i-a_j))\Gamma(sib^{-1}(a_i-a_j))\right]\Gamma_2(Q/2)^4}{(2\pi)^2\Gamma_2(i(a_i-a_j))^2\Gamma_2(-i(a_i-a_j))^2} \end{aligned} \tag{123}$$

Since we are only interested in the computation of the Wilson loops (21), we should be able to rescale the partition function even by an b-dependent function, so that we can drop the $\Gamma_2(Q/2)^4/(2\pi)^2$ part. Hence, we can use instead

$$Z^{\text{vect}}_{\text{1-loop}} = \prod_{i<j=1}^{N}(a_i-a_j)^2 \prod_{i<j=1}^{N} \frac{\Gamma(ib(a_i-a_j))\Gamma(-ib(a_i-a_j))}{\Gamma_2(i(a_i-a_j))\Gamma_2(-i(a_i-a_j))} \prod_{i<j=1}^{N}(b\leftrightarrow b^{-1}) \tag{124}$$

Using (97) and (101), we find

$$\frac{\Gamma(ibx)\Gamma(-ibx)}{\Gamma_2(ix)\Gamma_2(-ix)} = \frac{e^{s_\chi x^2}}{b^2}\frac{\prod_{n=1}^{\infty}\left(1+\frac{x^2}{(b^{-1}n)^2}\right)}{\prod'_{m,n\geq 0}\frac{e^{\frac{x^2}{(bm+b^{-1}n)^2}}}{1+\frac{x^2}{(bm+b^{-1}n)^2}}}\,, \tag{125}$$

where s_χ was defined in (96). Hence it follows that we can write (124) as

$$Z^{\text{vect}}_{\text{1-loop}} = \prod_{i<j=1}^{N}(a_i-a_j)^2 \prod_{i,j=1}^{N}\prod_{i<j=1}^{N} e^{\left(s_\chi-\frac{b^2+b^{-2}}{2}\zeta(2)\right)(a_i-a_j)^2} H_v(a_i-a_j;b)\,. \tag{126}$$

Second, let us look at the hyper multiplet contribution. We write down explicitly

$$\Upsilon(ix+Q/2) = \frac{\Gamma_2(Q/2)^2}{\Gamma_2(Q/2+ix)\Gamma_2(Q/2-ix)} \tag{127}$$

and use (97) to get

$$\begin{aligned} \Upsilon(ix+Q/2) &= e^{x^2 s_\chi}\frac{(Q/2+ix)(Q/2-ix)}{\left(Q/2\right)^2} \\ &\times \prod_{m,n\geq 0}{}' e^{-\frac{x^2}{(bm+b^{-1}n)^2}} \frac{\left(1+\frac{Q/2+ix}{bm+b^{-1}n}\right)\left(1+\frac{Q/2-ix}{bm+b^{-1}n}\right)}{\left(1+\frac{Q/2}{bm+b^{-1}n}\right)^2} \\ &= e^{x^2 s_\chi}\left(1+\frac{4x^2}{Q^2}\right)\prod_{m,n\geq 0}{}' e^{-\frac{x^2}{(bm+b^{-1}n)^2}}\left(1+\frac{x^2}{\left(b\left(m+1/2\right)+b^{-1}\left(n+1/2\right)\right)^2}\right)\,. \end{aligned} \tag{128}$$

At this point, we can split the product for the exponential pieces as

$$\prod_{m,n=0}^{\infty}(\cdots) = \prod_{m,n=1}^{\infty}(\cdots)\prod_{m=1}^{\infty}(\cdots)\prod_{n=1}^{\infty}(\cdots)\,, \tag{129}$$

express the $\prod_{m=1}^{\infty}(\cdots)\prod_{n=1}^{\infty}(\cdots)$ piece as $e^{-x^2(b^2+b^{-2})\zeta(2)}$ and absorb the factor $\left(1+\frac{4x^2}{Q^2}\right)$ inside the remaining product to obtain

$$
\begin{aligned}
\Upsilon(ix+Q/2) &= e^{x^2(2s_\chi-(b^2+b^{-2})\zeta(2))} \\
&\times \prod_{m,n=0}^{\infty}\left(1+\frac{x^2}{\big(b(m+1/2)+b^{-1}(n+1/2)\big)^2}\right)e^{-\frac{x^2}{(b(m+1)+b^{-1}(n+1))^2}} \\
&= e^{x^2\left(s_\chi-\frac{b^2+b^{-2}}{2}\zeta(2)\right)}H_h(x;b)\,.
\end{aligned} \tag{130}
$$

Hence, feeding (130) into (22) leads to

$$
\begin{aligned}
Z^{\text{hyper}}_{\text{1-loop}} &= \prod_{i,j=1}^{N}\Upsilon\big(i(a_i^{(1)}-a_j^{(2)})+Q/2;b\big)^{-1} \\
&= \prod_{i,j=1}^{N} e^{-(a_i^{(1)}-a_j^{(2)})^2\left(s_\chi-\frac{b^2+b^{-2}}{2}\right)}H_h(a_i^{(1)}-a_j^{(2)};b)^{-1}\,.
\end{aligned} \tag{131}
$$

Putting (126) and (131) together, the exponential terms cancel for conformal field theories, leaving us with the desired result.

□

C The weak coupling expansion

In this appendix, we wish to take the set of linear integral equations (35) and find an approximate solution for small values of the couplings. Our computations follow the principles outlined in [48]. For our purposes, we fix an integer $P \geq 1$ and expand the kernels K_v and K_h as

$$
K_a(x) \approx -2\sum_{n=1}^{P}(-1)^n \mathsf{k}_a(n)x^{2n+1}\,, \tag{132}
$$

where $a \in \{v, h\}$, the coefficients $\mathsf{k}_a(n)$ can be extracted from equation (109) and we have suppressed the b dependence. This expansion is sufficient in order to obtain the results up to order $g^{2(P+1)}$ in the couplings. We have for a given eigenvalue density ρ_k the expression

$$
\int_{-\mu}^{\mu} dz\rho_k(z)K_a(y-z) = -2\sum_{n=1}^{P}(-1)^n\mathsf{k}_a(n)\sum_{s=0}^{n}\binom{2n+1}{2s}y^{2(n-s)+1}\mathfrak{m}_{2s}^{(k)}\,, \tag{133}
$$

where $\mathfrak{m}_i^{(k)}$ is the i-th moment of the density ρ_k, i.e.

$$
\int_{-\mu}^{\mu}\rho_k(x)x^i = \mathfrak{m}_i^{(k)}\,. \tag{134}
$$

Observe that the odd moments have to vanish due to the symmetry of the densities. Using Chebyshev polynomials, we can derive the integral formula

$$
⨍_{-\mu}^{\mu}\frac{dy}{x-y}\frac{y^n}{\sqrt{\mu^2-y^2}} = -\pi\sum_{t=0}^{\lfloor\frac{n-1}{2}\rfloor}\frac{(t+1)C_t}{4^t}\mu^{2r}x^{n-1-2t}\,. \tag{135}
$$

where $C_r = \frac{1}{r+1}\binom{2r}{r}$ is the r-th Catalan number. Plugging (133) and (135) into (35), we the following result for the k^{th} density:

$$\begin{aligned}\rho_k(x) =&\frac{1}{2\pi g_k^2}\sqrt{\mu_k^2-x^2} - \frac{2}{\pi}\sqrt{\mu_k^2-x^2}\sum_{n=1}^{P}(-1)^n\sum_{l=1}^{r}\Big[\delta_{kl}\mathsf{k}_v(n) - \frac{\delta_{k,l+1}+\delta_{k,l-1}}{2}\mathsf{k}_h(n)\Big] \\ &\times\sum_{j=0}^{n}\binom{2n+1}{2j}\mathfrak{m}_{2j}^{(l)}\sum_{t=0}^{n-j}\frac{(t+1)C_t}{4^t}\mu_k^{2t}x^{2(n-j-t)}\,.\end{aligned} \tag{136}$$

We thus have expressed each of the densities ρ_k as functions of its P first non-trivial moments $\mathfrak{m}_{2i}^{(k)}$, $i = 1, \ldots, P$. Computing the moments by plugging (136) into the definition (134), we obtain a set of $r \times P$ linear equations for the same number of variables $\mathfrak{m}_{2i}^{(k)}$:

$$\begin{aligned}&\sum_{j=1}^{P}\sum_{l=1}^{r}\Bigg[\delta_{ij}\delta_{kl} + \sum_{n=j}^{P}(-1)^n\Big[\delta_{kl}\mathsf{k}_v(n) - \frac{\delta_{k,l+1}+\delta_{k,l-1}}{2}\mathsf{k}_h(n)\Big]\binom{2n+1}{2j} \\ &\times\frac{\mu_k^{2(n+i-j+1)}}{4^{n+i-j}}\binom{2i}{i}\binom{2(n-j)}{n-j}\frac{2(n-j)+1}{n-j+i+1}\Bigg]\mathfrak{m}_{2j}^{(l)} = \\ &= \frac{1}{g_k^2}\frac{C_i}{4^{i+1}}\mu_k^{2(i+1)} \\ &- \sum_{n=1}^{P}(-1)^n\Big[\mathsf{k}_v(n) - \mathsf{k}_h(n)\Big]\frac{\mu_k^{2(n+i+1)}}{4^{n+i}}\binom{2i}{i}\binom{2n}{n}\frac{2n+1}{n+i+1}\,,\end{aligned} \tag{137}$$

for $i = 1, \ldots, P$ and $k = 1, \ldots, r$. In (137), we have used the integral formula

$$\int_{-\mu}^{\mu} dx\sqrt{\mu^2-x^2}x^{2s} = \frac{2\pi C_s}{4^{s+1}}\mu^{2(s+1)}\,, \tag{138}$$

the fact that $m_0^{(l)} = 1\ \forall l$ and the following formula for the Catalan numbers

$$\sum_{t=0}^{m}(t+1)C_tC_{m+i-t} = \binom{2i}{i}\binom{2m}{m}\frac{2m+1}{m+i+1}\,. \tag{139}$$

Observe that the last line of (137) vanishes for $b = 1$, since in that case $\mathsf{k}_v(n) = \mathsf{k}_h(n)$. The set of linear equations (137) allows us to solve for the moments as functions of the densities widths μ_k and of the couplings g_k^2. The μ_k are then expressed as functions of the coupling by normalizing the densities. Specifically, plugging (136) into the normalization condition for the densities (33), we arrive at

$$\begin{aligned}1 =&\frac{\mu_k^2}{4g_k^2} - \sum_{l=1}^{r}\sum_{n=1}^{P}(-1)^n\Big[\delta_{kl}\mathsf{k}_v(n) - \frac{\delta_{k,l+1}+\delta_{k,l-1}}{2}\mathsf{k}_h(n)\Big] \\ &\times\sum_{j=1}^{n}\binom{2n+1}{2j}\frac{\mu_k^{2(n-j+1)}\mathfrak{m}_{2j}^{(l)}}{4^{n-j}}\binom{2(n-j)}{n-j}\frac{2(n-j)+1}{n-j+1} \\ &-\sum_{n=1}^{P}(-1)^n\Big[\mathsf{k}_v(n) - \mathsf{k}_h(n)\Big]\frac{\mu_k^{2(n+1)}}{4^n}\binom{2n}{n}\frac{2n+1}{n+1}\,.\end{aligned} \tag{140}$$

Again, the last term of (140) vanishes for $b=1$. Inserting in (140) the expressions obtained from (137) for the moments allows us to solve for the μ_k as functions of the g_k^2.

In order to solve equations (137) and (140), it is numerically good to linearize in the couplings as

$$\mu_k = 2g_k \left(1 + \sum_{i=1}^{P+1} \alpha_i^{(k)} g_k^{2i}\right) , \qquad \mathfrak{m}_{2i}^{(k)} = C_i g_k^{2i} \left(1 + \sum_{j=1}^{P+1-i} \beta_{j,i}^{(k)} g_k^{2i}\right) . \tag{141}$$

We then expand the equations in powers of g_k up to the power $2(P+1)$. Once we have solved for the coefficients $\alpha_i^{(k)}$ and the $\beta_{j,i}^{(k)}$, we have obtained the widths and the moments and the Wilson loop expectation values (37) can be expressed as

$$\langle W_k^{\pm} \rangle = \sum_{n=0}^{\infty} \frac{(2\pi)^{2n}}{(2n)!} b^{\pm 2n} \mathfrak{m}_{2n}^{(k)} = 2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\zeta(2n)}{B_{2n}} b^{\pm 2n} \mathfrak{m}_{2n}^{(k)} , \tag{142}$$

where B_n are the Bernoulli numbers.

For the $\mathbb{Z}_2$ quiver, we obtain the expansion up to order $(b-1)^2$

$$\begin{aligned}
\langle W_1^+ \rangle = {}& 1 + 2\pi^2 g_1^2 + \frac{4}{3}\pi^4 g_1^4 + \frac{4}{9}\pi^2 g_1^4 \Big[54\zeta(3)\left(g_2^2 - g_1^2\right) + \pi^4 g_1^2\Big] \\
&+ \frac{4}{45}\pi^2 g_1^4 \Big[360\pi^2\zeta(3)\left(g_2^2 - g_1^2\right) g_1^2 + 900\zeta(5)(g_1 - g_2)(g_1 + g_2)\left(3g_1^2 + g_2^2\right) + \pi^6 g_1^4\Big] \\
&+ \frac{8}{675}\pi^2 g_1^4 \Big[- 1350\pi^4\zeta(3)\left(g_1^2 - g_2^2\right) g_1^4 + 2250\pi^2\zeta(5)\left(13g_1^4 - 9g_2^2 g_1^2 - 4g_2^4\right) g_1^2 \\
&+ 675\left(g_1^2 - g_2^2\right)\left(36\zeta(3)^2\left(2g_1^4 - g_2^2 g_1^2 + g_2^4\right) - 35\zeta(7)\left(8g_1^4 + 5g_2^2 g_1^2 + g_2^4\right)\right) + \pi^8 g_1^6\Big] + \cdots \\
&+ \Bigg[4\pi^2 g_1^2 + \frac{16}{3}\pi^4 g_1^4 + \frac{8}{3}\pi^2 g_1^4 \Big[18\zeta(3)\left(g_2^2 - g_1^2\right) + \pi^4 g_1^2\Big] \\
&+ \frac{32}{45}\pi^2 g_1^4 \Big[180\pi^2\zeta(3)\left(g_2^2 - g_1^2\right) g_1^2 + 225\zeta(5)(g_1 - g_2)(g_1 + g_2)\left(3g_1^2 + g_2^2\right) + \pi^6 g_1^4\Big] \\
&+ \frac{16}{135}\pi^2 g_1^4 \Big[810\pi^4\zeta(3)\left(g_2^2 - g_1^2\right) g_1^4 + 900\pi^2\zeta(5)\left(13g_1^4 - 9g_2^2 g_1^2 - 4g_2^4\right) g_1^2 \\
&+ 135\left(g_1^2 - g_2^2\right)\left(36\zeta(3)^2\left(2g_1^4 - g_2^2 g_1^2 + g_2^4\right) - 35\zeta(7)\left(8g_1^4 + 5g_2^2 g_1^2 + g_2^4\right)\right) + \pi^8 g_1^6\Big] + \cdots \Bigg](b-1) \\
&+ \Bigg[2\pi^2 g_1^2 + 8\pi^4 g_1^4 + \frac{4}{3}\pi^2 g_1^4 \Big[66\zeta(3)\left(g_2^2 - g_1^2\right) - 120\zeta(5)\left(4g_1^2 + g_2^2\right) + 5\pi^4 g_1^2\Big] \\
&+ \frac{16}{45}\pi^2 g_1^4 \Big[780\pi^2\zeta(3)\left(g_2^2 - g_1^2\right) g_1^2 - 75\zeta(5)\left(\left(34\pi^2 - 81\right) g_1^4 + 2\left(27 + 4\pi^2\right) g_2^2 g_1^2 + 27g_2^4\right) \\
&+ 1575\zeta(7)\left(15g_1^4 + 4g_2^2 g_1^2 + 2g_2^4\right) + 7\pi^6 g_1^4\Big] + \frac{8}{15}\pi^2 g_1^4 \Big[- 10\pi^4 g_1^4 \big((53\zeta(3) + 88\zeta(5))g_1^2 \\
&+ (20\zeta(5) - 53\zeta(3))g_2^2\big) + 140\pi^2 g_1^2 \big((65\zeta(5) + 166\zeta(7))g_1^4 + 45(\zeta(7) - \zeta(5))g_2^2 g_1^2 \\
&+ 20(\zeta(7) - \zeta(5))g_2^4\big) + 15\big(8\left(57\zeta(3)^2 + 480\zeta(5)\zeta(3) - 595\zeta(7) - 1806\zeta(9)\right) g_1^6 \\
&- 3\left(228\zeta(3)^2 + 720\zeta(5)\zeta(3) - 595\zeta(7) + 840\zeta(9)\right) g_2^2 g_1^4 + 4\big(6\zeta(3)(19\zeta(3) - 40\zeta(5)) + 595\zeta(7) \\
&- 840\zeta(9)\big)g_2^4 g_1^2 - \left(228\zeta(3)^2 + 720\zeta(5)\zeta(3) - 595\zeta(7) + 840\zeta(9)\right) g_2^6\big) + \pi^8 g_1^6\Big] + \ldots \Bigg](b-1)^2 \\
&+ \mathcal{O}(b-1)^3
\end{aligned} \tag{143}$$

From (143) one can extract the effective coupling (41)ff.

D The strong coupling limit

In this subsection, we want to make a strong coupling analysis of the saddle point equations (35) in the case in which $b = 1$, i.e. on the sphere. In so doing, we will follow the same procedure as explained in the appendices of [48], with some additional details and complications. Before we begin in earnest with the study of the strong coupling limit, it is necessary to derive some integral identities. Of particular importance are (116) and (117). We define the function

$$\theta(x) := x \coth(\pi x) \,. \tag{144}$$

We use the same conventions for the Fourier transform as [48], i.e.

$$\hat{f}(\omega) := \int_{-\infty}^{\infty} dx e^{ix\omega} f(x) \,, \tag{145}$$

so that in the sense of distributions we get

$$\hat{\theta}(\omega) = -\frac{1}{2\sinh^2(\frac{\omega}{2})} \,. \tag{146}$$

From equation (4.14) of [48], we take the following "formal" integral formula

$$K(x) = ⨍_{-\infty}^{\infty} \frac{\theta(w)}{x-w} dw \,. \tag{147}$$

To make (147) correct, one needs to shift the argument of $K(x)$ by z and average over z such that the first two moments vanish. Specifically, we observe that

$$\begin{aligned}\int_{-\mu}^{\mu} dy \rho(y)(K(x-y) - K(x)) &= \int_{-\mu}^{\mu} dy \rho(y) \left(-yK'(x) + \frac{1}{2} y^2 K''(x) + \cdots \right) \\ &= -\mathfrak{m}_1 K'(x) + \frac{1}{2}\mathfrak{m}_2 K''(x) + \cdots = \frac{1}{2}\mathfrak{m}_2 K''(x) + \cdots\end{aligned} \tag{148}$$

so that (147) is applicable, i.e.

$$\int_{-\mu}^{\mu} dy \rho(y)(K(x-y) - K(x)) = ⨍_{-\infty}^{\infty} \frac{dw}{x-w} \int_{-\mu}^{\mu} dy \rho(y) \left[\theta(w-y) - \theta(w)\right] \,, \tag{149}$$

which has been checked numerically. If we use (149) for the kernel K as well as equation (117), we are able to show[15]

$$\begin{aligned}&\frac{1}{\pi^2} ⨍_{-\mu}^{\mu} \frac{dy}{x-y} \sqrt{\frac{\mu^2 - x^2}{\mu^2 - y^2}} \int_{-\mu}^{\mu} dz \rho(z) \Big[K(y-z) - K(y)\Big] \\ &= -\frac{\sqrt{\mu^2 - x^2}}{\pi^2} ⨍_{-\infty}^{\infty} dw \int_{-\mu}^{\mu} dz \rho(z) \big(\theta(w-z) - \theta(w)\big) ⨍_{-\mu}^{\mu} \frac{dy}{(x-y)(w-y)} \frac{1}{\sqrt{\mu^2 - y^2}} \\ &= \theta(x) - (\rho \star \theta)(x) + \frac{1}{\pi} \int_{|w|>\mu} \frac{\mathrm{sgn(w)dw}}{w-x} \sqrt{\frac{\mu^2 - x^2}{w^2 - \mu^2}} \Big[(\rho \star \theta)(w) - \theta(w)\Big],\end{aligned} \tag{150}$$

[15] The "sgn" part of the equation is a convention that can be absorbed in the definition of the square roots.

where we have defined the convolution as

$$(\rho\star\theta)(x) := \int_{-\mu}^{\mu} dy\rho(y)\theta(x-y)\,. \tag{151}$$

Having set up the necessary additional identities, we can start our analysis of the strong coupling behavior of the saddle point equations. We take the system of r coupled integral equations for the cyclic quivers (35) for $b=1$ and using (119) we rewrite them as:

$$\rho_k(x) = \frac{8\pi\sqrt{\mu_k^2-x^2}}{\lambda_k} - \frac{1}{\pi^2}\int_{-\mu_k}^{\mu_k}\frac{dy}{x-y}\sqrt{\frac{\mu_k^2-x^2}{\mu_k^2-y^2}}\sum_{l=1}^{r}\frac{\hat{\mathbf{a}}_{kl}}{2}\int_{-\mu_l}^{\mu_l}\rho_l(z)\left[K(y-z)-K(z)\right]\,, \tag{152}$$

where $\hat{\mathbf{a}}_{kl} := 2\delta_{kl}-\delta_{k,l+1}-\delta_{k,l-1}$ is the SU$\,(r)$ Cartan matrix and we have used $\sum_{l=1}^{r}\hat{\mathbf{a}}_{kl}=0$. Using (150), we arrive at

$$\begin{aligned}\rho_k(x) - \sum_{l=1}^{r}\frac{\hat{\mathbf{a}}_{kl}}{2}\int_{-\mu_l}^{\mu_l}dy\rho_l(y)\left(\theta(x-y)-\theta(x)\right) &= \frac{\sqrt{\mu_k^2-x^2}}{2\pi g_k^2}\\ -\frac{1}{\pi}\sum_{l=1}^{r}\frac{\hat{\mathbf{a}}_{kl}}{2}\int_{|w|>\mu_k}\frac{\text{sgn}(w)dw}{w-x}\sqrt{\frac{\mu_k^2-x^2}{w^2-\mu_k^2}}\int_{-\mu_l}^{\mu_l}dy\rho_l(y)(\theta(w-y)-\theta(w)) &=: \mathbb{F}_k(x)\,.\end{aligned} \tag{153}$$

The functions $\mathbb{F}_k$ are the *driving terms* of the integral equations, i.e. our equations are written as

$$\rho_k(x) - \sum_{l=1}^{r}\frac{\hat{\mathbf{a}}_{kl}}{2}\int_{-\mu_l}^{\mu_l}dy\rho_l(y)\left(\theta(x-y)-\theta(x)\right) = \mathbb{F}_k(x)\,. \tag{154}$$

In the strong coupling limit, the $\mathbb{F}_k(x)$ are dominated by their first term. Fourier transforming (154), using (146) and $\sum_{l=1}^{r}\hat{\mathbf{a}}_{kl}=0$, we get the set of equations

$$\hat{\rho}_k(\omega) + \sum_{l=1}^{r}\frac{\hat{\mathbf{a}}_{kl}}{2}\frac{\hat{\rho}_l(\omega)}{2\sinh^2\frac{\omega}{2}} = \hat{\mathbb{F}}_k(\omega) + e^{-i\mu_k\omega}\hat{X}_k^-(\omega) + e^{i\mu_k\omega}\hat{X}_k^+(\omega), \tag{155}$$

where the functions $\hat{X}_k^{\pm}$ are analytic in the upper/lower half plane respectively. Their presence is due to the fact that (154) only holds for $x\in(-\mu_k,\mu_k)$. We can now rewrite (155) as

$$\sum_{l=1}^{r}\mathbb{A}_{kl}(\omega)\hat{\rho}_l(\omega) = \hat{\mathbb{F}}_k(\omega) + e^{-i\mu_k\omega}\hat{X}_k^-(\omega) + e^{i\mu_k\omega}\hat{X}_k^+(\omega) \tag{156}$$

where

$$\mathbb{A}_{kl}(\omega) := \frac{(e^{\omega}+e^{-\omega})\delta_{kl}-\delta_{k,l-1}-\delta_{k,l+1}}{e^{\omega}+e^{-\omega}-2} \tag{157}$$

with the indices k,l subject to the identification $k\equiv k+r$ and $l\equiv l+r$. The implicit dependence on the number of gauge groups r contained in many quantities, such as $\mathbb{A}_{kl}$, will not be explicitly noted in this appendix so as to not clutter the notation. Solving (156) for the densities by inverting the matrix $\mathbb{A}$, we get

$$\begin{aligned}\mathbb{K}(\omega)\hat{\rho}_k(\omega) = &\sum_{l=1}^{r}\left(\chi_{|k-l|-1}(e^{\omega})+\chi_{r-1-|k-l|}(e^{\omega})\right)\\ &\times\left(\hat{\mathbb{F}}_l(\omega)+e^{-i\mu_l\omega}\hat{X}_l^-(\omega)+e^{i\mu_l\omega}\hat{X}_l^+(\omega)\right)\,,\end{aligned} \tag{158}$$

where $\chi_j(u) := \sum_{m=0}^{j} u^{j-2m}$ are the characters of the $j+1$ dimensional representations of SU (2) and we have defined the kernel

$$\mathbb{K}(\omega) := \frac{e^{r\omega} - 2 + e^{-r\omega}}{e^{\omega} - 2 + e^{-\omega}} = \left(\frac{\sinh \frac{r\omega}{2}}{\sinh \frac{\omega}{2}}\right)^2 . \tag{159}$$

Using the well known formula $\sin(x) = \pi \left(\Gamma(\frac{x}{\pi})\Gamma(1-\frac{x}{\pi})\right)^{-1}$, we can express $\mathbb{K}(\omega)$ using the Γ function as

$$\mathbb{K}(\omega) = \frac{1}{\mathbb{G}_+(\omega)\mathbb{G}_-(\omega)}, \quad \text{where} \quad \mathbb{G}_\pm(\omega) := \frac{1}{r}\left(\frac{\Gamma\left(1 \mp \frac{ir\omega}{2\pi}\right)}{\Gamma\left(1 \mp \frac{i\omega}{2\pi}\right)}\right)^2 , \tag{160}$$

where the functions $\mathbb{G}_\pm(\omega)$ have second order poles at $\mp i\nu_n$ with

$$\nu_n = 2\pi \frac{n}{r}, \qquad n = 1, 2, 3, \ldots, \text{ with } n \notin r\mathbb{N}. \tag{161}$$

Expanding around the poles, we get

$$\mathbb{G}_\pm(\omega) = \frac{\alpha_n}{(\omega \pm i\nu_n)^2} \pm \frac{\beta_n}{\omega \pm i\nu_n} + \cdots \tag{162}$$

where

$$\alpha_n := -\frac{4\pi^2}{r^3\Gamma(n)^2\Gamma\left(1-\frac{n}{r}\right)^2}, \qquad \beta_n := \frac{4\pi i\left(r\psi(n) - \psi\left(1-\frac{n}{r}\right)\right)}{r^3\Gamma(n)^2\Gamma\left(1-\frac{n}{r}\right)^2}, \tag{163}$$

and $\psi(x) := \Gamma'(x)/\Gamma(x)$ is the digamma function. Observe that both α_n and β_n go to zero very rapidly and that the β_n are purely imaginary. We solve (158) by multiplying by $\mathbb{G}_+(\omega)e^{-i\mu_k\omega}$ and taking the negative frequency part. Ignoring to first approximation the $\hat{X}_k^\pm$, we get

$$\hat{\rho}_k(\omega) = \mathbb{G}_-(\omega)e^{i\mu_k\omega}\left[\mathbb{G}_+(\omega)e^{-i\mu_k\omega}\sum_{l=1}^{r}\mathbb{B}_{kl}(\omega)\hat{\mathbb{F}}_l(\omega)\right]_- , \tag{164}$$

where we used the matrix $\mathbb{B}$ with

$$\mathbb{B}_{kl}(\omega) := \chi_{|k-l|-1}(e^\omega) + \chi_{r-1-|k-l|}(e^\omega) \tag{165}$$

and the definition

$$\mathcal{F}_\pm(\omega) := \pm \lim_{\epsilon\to 0+}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi i}\frac{\mathcal{F}(\omega')}{\omega' - \omega \mp i\epsilon}. \tag{166}$$

The functions $\mathcal{F}_\pm(\omega)$ are analytic in the upper/lower half-planes and the integral contours are to be closed in the upper/lower half-plane. The contour integral of (164) will give us the residues of $\mathbb{G}_+(\omega)\mathbb{B}_{kl}(\omega)$ at $-i\nu_n$. Expanding, we find

$$\operatorname{Res}\left(\mathbb{G}_+(\omega)\mathbb{B}_{kl}(\omega), -i\nu_n\right) = \alpha_n(\partial_\omega\mathbb{B}_{kl})(-i\nu_n) + \beta_n\mathbb{B}_{kl}(-i\nu_n). \tag{167}$$

Through brute force, we discover that

$$\mathbb{B}_{kl}(-i\nu_n) = \begin{cases} r & \text{for } n \in r\mathbb{N} \\ (-1)^{k+l-1}r & \text{for } n \in \frac{r}{2}\mathbb{N} \\ 0 & \text{otherwise} \end{cases} . \tag{168}$$

as well as

$$(\partial_\omega \mathbb{B}_{kl})(-i\nu_n) = \begin{cases} 0 & \text{for } n \in r\mathbb{N} \\ 0 & \text{for } n \in \frac{r}{2}\mathbb{N} \\ \frac{ir\cos(|k-l|\nu_n)}{\sin(\nu_n)} & \text{otherwise} \end{cases} . \tag{169}$$

Using the above, we find the residue of the poles $-i\nu_n$:

$$\mathbb{R}_{kl}(n) := \operatorname{Res}\left(\mathbb{G}_+(\omega)\mathbb{B}_{kl}(\omega), -i\nu_n\right) = \begin{cases} 0 & \text{for } n \in r\mathbb{N}, \\ \beta_n(-1)^{k+l-1}r & \text{for } n \in \frac{r}{2}\mathbb{N}, \\ \alpha_n \frac{ir\cos(|k-l|\nu_n)}{\sin(\nu_n)} & \text{otherwise} \end{cases} \tag{170}$$

We observe experimentally that the matrices $\mathbb{R}_{kl}(n)$ all commute, that the vector $(1,1,\ldots,1)^t$ is the only common eigenvector of $\mathbb{R}_{kl}(n)$ and that it has eigenvalue zero. Plugging (166) into (164) we get

$$\hat{\rho}_k(\omega) = \frac{1}{\mathbb{K}(\omega)}\sum_{l=1}^{r}\mathbb{B}_{kl}(\omega)\hat{\mathbb{F}}_l(\omega) - \mathbb{G}_-(\omega)e^{i\mu_k\omega}\sum_{n=1}^{\infty}\frac{e^{-\mu_k\nu_n}}{\omega+i\nu_n}\sum_{l=1}^{r}\mathbb{R}_{kl}(n)\hat{\mathbb{F}}_l(-i\nu_n)\,, \tag{171}$$

where we have to use (170) for the residues. Observe that, contrary to what one might think at first glance, (171) is indeed analytic in the lower half plane since the poles at $-i\nu_n$ cancel.

We now need to normalize the densities. Since the Fourier transform of (171) will represent the density $\rho_k(x)$ well only for positive x, we use the fact that the densities should be symmetric and demand

$$1 = 2\int_0^{\mu_k} dx\rho_k(x) = \lim_{\epsilon\to 0+}\int_{-\infty}^{\infty}\frac{d\omega}{\pi i}\frac{\hat{\rho}_k(\omega)}{\omega - i\epsilon} = 2(\hat{\rho}_k)_+(0)\,, \tag{172}$$

where we used (166). Thus, we need to close the contour in the upper half plane.

Taking the residues and using $\frac{1}{\omega\mp i\epsilon} = \text{p.v.} \pm i\pi\delta$ as well as the symmetry under $\omega\to-\omega$ of $\mathbb{B}$, $\mathbb{K}$ and $\hat{\mathbb{F}}_k$ gives

$$\begin{aligned} 1 =& \frac{1}{\mathbb{K}(0)}\sum_{l=1}^{r}\mathbb{B}_{kl}(0)\hat{\mathbb{F}}_l(0) + 2i\mathbb{G}_-(0)\sum_{n=1}^{\infty}\frac{e^{-\mu_k\nu_n}}{\nu_n}\sum_{l=1}^{r}\mathbb{R}_{kl}(n)\hat{\mathbb{F}}_l(-i\nu_n) \\ &- 2\sum_{m,n=1}^{\infty}\frac{e^{-\mu_k(\nu_m+\nu_n)}}{\nu_m(\nu_m+\nu_n)}\left(\beta_m - i\alpha_m\frac{\nu_n+2\nu_m+\nu_n\nu_m\mu_k+\nu_m^2\mu_k}{\nu_m(\nu_m+\nu_n)}\right)\sum_{l=1}^{r}\mathbb{R}_{kl}(n)\hat{\mathbb{F}}_l(-i\nu_n) \end{aligned} \tag{173}$$

Using $\mathbb{K}(0) = r^2$, $\mathbb{B}_{kl}(0) = r$ and $\mathbb{G}_-(0) = \frac{1}{r}$, we get

$$\begin{aligned} 1 =& \frac{1}{r}\sum_{l=1}^{r}\hat{\mathbb{F}}_l(0) + \frac{2i}{r}\sum_{n=1}^{\infty}\frac{e^{-\mu_k\nu_n}}{\nu_n}\sum_{l=1}^{r}\mathbb{R}_{kl}(n)\hat{\mathbb{F}}_l(-i\nu_n) \\ &- 2\sum_{m,n=1}^{\infty}\frac{e^{-\mu_k(\nu_m+\nu_n)}}{\nu_m(\nu_m+\nu_n)}\left(\beta_m - i\alpha_m\frac{\nu_n+2\nu_m+\nu_n\nu_m\mu_k+\nu_m^2\mu_k}{\nu_m(\nu_m+\nu_n)}\right)\sum_{l=1}^{r}\mathbb{R}_{kl}(n)\hat{\mathbb{F}}_l(-i\nu_n)\,. \end{aligned} \tag{174}$$

The terms in the second line of (171) are exponentially suppressed. We will now make an assumption that is justified by the self-consistency of the results. As in [48], in the large coupling limit, the driving terms defined in (153) are dominated by their first term, so that

$$\mathbb{F}_k(x) = \frac{1}{2\pi g_k^2}\sqrt{\mu_k^2 - x^2} \Longrightarrow \hat{\mathbb{F}}_k(\omega) = \frac{\mu_k J_1(\mu_k\omega)}{2g_k^2\omega}, \quad \hat{\mathbb{F}}_k(0) = \frac{\mu_k^2}{4g_k^2}\,, \quad \hat{\mathbb{F}}_k(-i\nu_n) = \frac{e^{\mu_k\nu_n}}{g_k^2}\sqrt{\frac{\mu_k}{8\pi\nu_n^3}}\,, \tag{175}$$

where J_n are the Bessel functions of the first kind. Using this approximation for the driving terms and dropping the exponentially suppressed terms in (174), explicit numerical solutions show that for large values of the couplings, the widths of the densities μ_k are very close to being equal to each other (as long as the ratios of the couplings ${}^{g_k}/_{g_l}$ is roughly of order one) and are approximately given by

$$\mu_k = 2\sqrt{\frac{r g_1^2 \cdots g_r^2}{\sum_{i=1}^r \prod_{k\neq i} g_k^2}} \qquad \forall k\,. \tag{176}$$

It can be seen that this is a solution of (174) if we ignore the (numerically suppressed) pieces containing $\hat{\mathbb{F}}_k(-i\nu_n)$, which implies

$$1 = \frac{1}{\mathbb{K}(0)} \sum_{l=1}^{r} \mathbb{B}_{kl}(1)\hat{\mathbb{F}}_l(0) = \frac{1}{r^2}\sum_{l=1}^{r} r\frac{\mu_l^2}{4g_l^2} = \frac{1}{4r}\sum_{l=1}^{r}\frac{\mu_l^2}{g_l^2}\,. \tag{177}$$

An a posteriori justification for neglecting the $\hat{\mathbb{F}}_k(-i\nu_n)$ is that, for g_k that are not wildly different, see (38), the $\hat{\mathbb{F}}_k$ will be roughly equal, so that the term $\sum_{l=1}^r \mathbb{R}_{kl}(n)\hat{\mathbb{F}}_l(-i\nu_n)$ will be small since $(1,1,\ldots,1)$ is an eigenvector of the $\mathbb{R}$ matrices with eigenvalue zero.

The densities themselves have in the strong coupling limit the same shape as the $\mathcal{N}=4$ one.The leading behavior of the Wilson loop expectation values at large values of the coupling is hence (using the asymptotic expression $I_1(x) \sim {}^{e^x}/_{\sqrt{2\pi x}}$)

$$\langle W_k^{\pm}\rangle = \frac{\sqrt{\mu_k}e^{2\pi b^{\pm 1}\mu_k}}{8\pi^2 b^{\pm 3/2} g_k^2} + \mathcal{O}(b-1)^2\,. \tag{178}$$

Due to the exponential term, the strong coupling limit of the effective couplings is simply given by comparing (176) with the width μ for $\mathcal{N}=4$. Since for $\mathcal{N}=4$ SYM we have $\mu^2 = 4g^2$, comparing with (176) leads to equation (50) in the main text. In particular, the strong coupling limit of the Bremsstrahlung functions is

$$B_k = \frac{\mu_k}{2\pi} - \frac{3}{8\pi^2}\,. \tag{179}$$

A more detailed analysis of (174) should also allow one to extract the leading corrections to the relation (50).

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