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HYPERDIRE
HYPERgeometric functions DIfferential REduction: Mathematica-based packages for the differential reduction of generalized hypergeometric functions: Horn-type hypergeometric functions of two variables

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Abstract

HYPERDIRE is a project devoted to the creation of a set of Mathematica-based programs for the differential reduction of hypergeometric functions. The current version allows for manipulations involving the full set of Horn-type hypergeometric functions of two variables, including 30 functions.

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PROGRAM SUMMARY

Title of program: HYPERDIRE

Version: 1.0.0

Release: 1.0.0

Catalogue number:

Program obtained from: `https://sites.google.com/site/loopcalculations/home`

E-mail: `bvv@jinr.ru`

Licensing terms: GNU General Public Licence

Computers: all computers running Mathematica

Operating systems: operation systems running Mathematica

Programming language: Mathematica

Keywords: generalized hypergeometric functions, Feynman integrals

Nature of the problem: reduction of Horn-type hypergeometric functions of two variables to a set of basis functions

Method of solution: differential reduction

Restriction on the complexity of the problem: none

Typical running time: depending on the complexity of the problem

1 Introduction

This paper describes a major extension of HYPERDIRE [1,2], which is a set of Mathematica-based [3] program packages for manipulations involving Horn-type hypergeometric functions [4–7] on the basis of differential equations [8]. The creation of these program packages is motivated by the importance of Horn-type hypergeometric functions for analytical evaluations of Feynman diagrams, especially at the one-loop level [9]. Possible applications of the differential-reduction algorithm to Feynman diagrams beyond the one-loop level were discussed in Ref. [10].

The aim of this paper is to present a new version of HYPERDIRE, which includes the full set of Horn-type hypergeometric functions of two variables. Specifically, these are the 30 functions listed in Table 1. For completeness, we present also the full list of inverse operators for the differential reduction, in Appendix A.

2 Differential reduction

Let us consider the hypergeometric function $H(\vec{J};\vec{z})$ depending on a set of *contiguous* variables, $z_1, \cdots, z_k$, and a set of *discrete* variables, $J_1, \cdots, J_r$, and satisfying the following two additional conditions:

- There are linear differential operators which shift the values of the integer parameters J_a by ± 1 (step-up and step-down operators):

$$R_{a,\vec{K}} \frac{\partial^{\vec{K}}}{\partial \vec{z}} H(J_1, \cdots, J_{a-1}, J_a, J_{a+1}, \cdots, J_r; \vec{z}) = H(J_1, \cdots, J_{a-1}, J_a \pm 1, J_{a+1}, \cdots, J_r; \vec{z}) \;, \tag{1}$$

  where $\frac{\partial^{\vec{L}}}{\partial \vec{z}} = \frac{\partial^{l_1+\cdots+l_k}}{\partial z_1^{l_1} \cdots \partial z_k^{l_k}}$ and $R_{a,\vec{K}}(\vec{z})$ are rational functions.

- The function $H(\vec{J};\vec{z})$ satisfies the following homogeneous linear system of partial differential equations (PDEs):

$$P_{\vec{L}} \frac{\partial^{\vec{L}}}{\partial \vec{z}} H(\vec{J};\vec{z}) = 0 \;, \tag{2}$$

  where $P_{\vec{L}}(\vec{z})$ are polynomial functions.

We assume that Eq. (2) may be converted to the Pfaff form,

$$\sum_{J,k} P_{\vec{J};k}(\vec{a};\vec{z}) \frac{\partial}{\partial z_k} F(\vec{a};\vec{z}) = 0 \quad \Rightarrow \quad \left\{ d_k \omega_i(\vec{z}) = \Omega^k_{ij}(\vec{z}) \omega_j(\vec{z}) dz_k \;, \quad d_r \left[d_k \omega_i(\vec{z}) \right] = 0 \right\} . \tag{3}$$

Then, the differential operators inverse with respect to those defined by Eq. (1) may be constructed according to Ref. [8].

For certain values of the parameters, the coefficients entering the differential operators may be equal to zero or infinity. In this case, the result of the differential reduction may be expressed in terms of a simpler hypergeometric function. In Table 2, the sets of exceptional parameters are listed for all the Horn-type hypergeometric functions considered here.

Applying the direct and inverse differential operators to the hypergeometric function $H(\vec{J};\vec{z})$, the values of the parameters $\vec{J}$ can be shifted by arbitrary integers as

$$Q(\vec{z})H(\vec{J}+\vec{m};\vec{z}) = \sum_{\vec{k}=0}^{r} Q_{\vec{k}}(\vec{z}) \frac{\partial^{\vec{k}}}{\partial \vec{z}} H(\vec{J};\vec{z}) \ , \tag{4}$$

where $\vec{m}$ is a set of integers, $Q(\vec{z})$ and $Q_{\vec{J}}(\vec{z})$ are polynomials, and r is the holonomic rank of the homogeneous linear system of PDEs in Eq. (2).

Let us recall that, for a Horn-type hypergeometric function, the homogeneous linear system of PDEs can be derived from the coefficients of the series expansion about $\vec{z}=0$,[3]

$$H = \sum_{\vec{m}} C(\vec{m}) \vec{z}^{\vec{m}}.$$

In this case, the ratio of two coefficients may be represented as a ratio of two polynomials,

$$\frac{C(\vec{m}+\vec{e}_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})} \ , \tag{5}$$

where $\vec{e}_j = (0,\cdots,0,1,0,\cdots,0)$ denotes the unit vector with unity in its $j^{\rm th}$ component, so that the Horn-type hypergeometric function satisfies the following homogeneous linear system of PDEs:

$$\left[Q_j\left(\sum_{k=1}^{r} z_k \frac{\partial}{\partial z_k} \right) \frac{1}{z_j} - P_j\left(\sum_{k=1}^{r} z_k \frac{\partial}{\partial z_k} \right) \right] H(\vec{J};\vec{z}) = 0 \ , \tag{6}$$

where $j=1,\ldots,r$.

3 Horn-type hypergeometric functions of two variables

The Horn-type hypergeometric function $H(\vec{J};z_1,z_2)$ of two variables has the following series representation about $z_1=z_2=0$:

$$H(\vec{\gamma};\vec{\sigma};z_1,z_2) = \sum_{m_1,m_2=0}^{\infty} \left(\frac{\Pi_{a=0}^{K} \Gamma\left(\mu_1^{(a)} m_1 + \mu_2^{(a)} m_2 + \gamma_a\right) \Gamma^{-1}(\gamma_a)}{\Pi_{b=0}^{L} \Gamma\left(\nu_1^{(b)} m_1 + \nu_2^{(b)} m_2 + \sigma_b\right) \Gamma^{-1}(\sigma_b)} \right) z_1^{m_1} z_2^{m_2} \ , \tag{7}$$

[3]Under special conditions depending on the values of the parameters, also the Mellin-Barnes integral may be used for obtaining the homogeneous linear system of PDEs [11, 12].

where $\mu_j^{(a)}, \nu_k^{(b)} \in \mathbb{Z}$ and $\gamma_j, \sigma_k \in \mathbb{C}$. The sequences $\vec{\gamma} = (\gamma_1, \cdots, \gamma_K)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_L)$ are called *upper* and *lower* parameters, respectively. The function $H(\vec{J}; z_1, z_2)$ satisfies the following homogeneous linear system of PDEs of second order:

$$\begin{aligned} \theta_{11} H(\vec{J}; \vec{z}) &= \left\{ P_0(\vec{z})\theta_{12} + P_1(\vec{z})\theta_1 + P_2(\vec{z})\theta_2 + P_3(\vec{z}) \right\} H(\vec{J}; \vec{z}) \;, \\ \theta_{22} H(\vec{J}; \vec{z}) &= \left\{ R_0(\vec{z})\theta_{12} + R_1(\vec{z})\theta_1 + R_2(\vec{z})\theta_2 + R_3(\vec{z}) \right\} H(\vec{J}; \vec{z}) \;, \end{aligned} \tag{8}$$

where $\vec{z} = (z_1, z_2)$ accommodates the two variables, $\{P_j, R_j\}$ are rational functions, $\theta_j = z_j \partial_{z_j}$ with $j = 1, 2$, and $\theta_{i_1 \cdots i_k} = \theta_{i_i} \cdots \theta_{i_k}$. It is well known [4] that, under the condition $1 - P_0(\vec{z})R_0(\vec{z}) = 0$, Eq. (8) can be reduced to the Pfaff system in Eq. (3) of three PDEs

$$d\vec{f} = R\vec{f} \;, \tag{9}$$

where $\vec{f} = \Big(H(\vec{J}; \vec{z}), \theta_1 H(\vec{J}; \vec{z}), \theta_2 H(\vec{J}; \vec{z}) \Big)$. In this case, Eq. (8) has three solutions, and Eq. (4) takes the following form:

$$Q(\vec{z}) H(\vec{J} + \vec{m}; \vec{z}) = Q_0(\vec{z}) H(\vec{J}; \vec{z}) + Q_1(\vec{z}) \theta_1 H(\vec{J}; \vec{z}) + Q_2(\vec{z}) \theta_2 H(\vec{J}; \vec{z}) \;, \tag{10}$$

where $\vec{m}$ is a set of integers and $Q(\vec{z})$, $Q_0(\vec{z})$, $Q_1(\vec{z})$, and $Q_2(\vec{z})$ are polynomials.

In the case $1 - P_0(\vec{z})R_0(\vec{z}) = 0$, Eq. (8) has four independent solutions and may be reduced to the Pfaff system of four PDEs in Eq. (9) with $\vec{f} = \Big(H(\vec{J}; \vec{z}), \theta_1 H(\vec{J}; \vec{z}), \theta_2 H(\vec{J}; \vec{z}), \theta_{12} H(\vec{J}; \vec{z}) \Big)$. In this case, Eq. (4) has the following form:

$$Q(\vec{z}) H(\vec{J} + \vec{m}; \vec{z}) = Q_0(\vec{z}) H(\vec{J}; \vec{z}) + Q_1(\vec{z}) \theta_1 H(\vec{J}; \vec{z}) + Q_2(\vec{z}) \theta_2 H(\vec{J}; \vec{z}) + Q_{12}(\vec{z}) \theta_{12} H(\vec{J}; \vec{z}) \;, \tag{11}$$

where $\vec{m}$ is a set of integers and $Q(\vec{z})$, $Q_0(\vec{z})$, $Q_1(\vec{z})$, $Q_2(\vec{z})$, and $Q_{12}(\vec{z})$ are polynomials. In both cases, Eqs. (10) and (11), the construction of the inverse differential operators reduces to the construction of some inverse matrices, of the 3×3 and 4×4 types, respectively, with non-zero determinants. However, as was shown in Ref. [13], for some Horn-type hypergeometric functions, one of the four particular solutions under the condition $1 - P_0(\vec{z})R_0(\vec{z}) = 0$ is a Puiseux monomial in the neighborhood of the point $z_1 = z_2 = 0$. Examples include the functions G_3, H_3, H_6, H_6 (confluent), and H_8 (confluent). More details may be found in Ref. [5]. In applications to Feynman diagrams, such solutions correspond to diagrams which are exactly expressible in terms of Gamma functions and are typically associated with tadpoles [12]. In this case, the determinant of the corresponding matrix is equal to zero, and the differential reduction has the form of Eq. (10). To complete the differential reduction in this case, it is necessary to generate one PDE in addition to Eq. (8). A detailed analysis of such systems of PDEs for Horn-type hypergeometric functions of two variables was performed in Ref. [14]. The most systematic analysis of the criteria for the existence of such types of solutions for A-hypergeometric systems [7] was presented in Ref. [15].

In Table 3, the locus of singularities of the homogeneous linear system of PDEs of second order with two variables defined by Eq. (8) is specified for each of the Horn-type hypergeometric functions considered here.

4 HornFunctions — Mathematica-based program for the differential reduction of 30 Horn-type hypergeometric functions

In this section, we present the Mathematica-based[4] program package **HornFunctions** for the differential reduction of the 30 Horn-type hypergeometric functions of two variables. They are listed in Table 1. The differential reduction of the Appell functions, namely F_1, F_2, F_3, and F_4, was implemented in the program package **AppellF1F4** [2].

For the Horn-type hypergeometric functions defined in Eq. (7), the direct differential operators for the upper and lower parameters were constructed in Ref. [16]. For the upper parameters, they have the following form:

$$H(\vec{\gamma}+\vec{e_a};\vec{\sigma};\vec{z}) = \frac{1}{\gamma_a}\left(\mu_1^{(a)}\theta_1+\mu_2^{(a)}\theta_2+\gamma_a\right)H(\vec{\gamma};\vec{\sigma};\vec{z}) \,. \quad (12)$$

Similar relations also exist for the lower parameters:

$$H(\vec{\gamma};\vec{\sigma}-\vec{e}_b;\vec{z}) \;=\; \frac{1}{\sigma_b-1}\left(\nu_1^{(b)}\theta_1+\nu_2^{(b)}\theta_2+\sigma_b-1\right)H(\vec{\gamma};\vec{\sigma};\vec{z}) \,. \quad (13)$$

The program package **HornFunctions** allows one to automatically perform the differential reduction in accordance with Eq. (4). It is freely available from Ref. [17]. Its current version, 1.0, only handles non-exceptional values of the parameters.

4.1 Input format

The program may be loaded in the standard way:

<< "HornFunctions.m"

It includes the following basic routine for each Horn-type hypergeometric function:

$$\{\textbf{HornName}\}\textbf{IndexChange}[\text{changingVector}, \text{parameterVector}], \quad (14)$$

where {**HornName**}**IndexChange** defines the name of the Horn-type hypergeometric function to be modified, e.g. **H1IndexChange** for the function $H_1(\alpha,\beta,\gamma,\delta,z_1,z_2)$, "parameterVector" defines the list of parameters of that function, and "changingVector" defines the set of integers by which the values of these parameters are to be shifted, i.e. the vector $\vec{m}$ in Eq. (4). For example, the operator

$$\textbf{H1IndexChange}[\{1,-1,0,0\},\{\alpha,\beta,\gamma,\delta,z_1,z_2\}] \quad (15)$$

shifts the arguments of the function $H_1(\alpha,\beta,\gamma,\delta,z_1,z_2)$ so as to generate $H_1(\alpha+1,\beta-1,\gamma,\delta,z_1,z_2)$.

[4]It was tested using Mathematica 8.0.

4.2 Output format

The output structure of all the operators of the program package **HornFunctions** in Eq. (14) is as follows:

$$\{\{Q_0, Q_1, Q_2, Q_{12}\}, \{\text{parameterVectorNew}\}\}, \tag{16}$$

where "parameterVectorNew" is the new set of parameters, $\vec{J}+\vec{m}$, of the function **HornName** and Q_0, Q_1, Q_2, and Q_{12} are the rational coefficient functions of the differential operator in Eq. (11),

$$\textbf{HornFunct}(oldparam) = (Q_0 + Q_1\theta_1 + Q_2\theta_2 + Q_{12}\theta_1\theta_2)\textbf{HornFunct}(newparam) \tag{17}$$

In the case of three PDEs, which corresponds to Eq. (10), the last coefficient, Q_{12}, is identically zero.

4.3 Examples

Example 1:[5] Reduction of the Horn-type hypergeometric function $G_1(a, b_1, b_2, z_1, z_2)$.

G1IndexChange[{−1,−1,0}, {a,b1,b2,z1,z2}]

$$\left\{\left\{\frac{\mathrm{b1}(\mathrm{z1}+1)+(\mathrm{b2}-1)\mathrm{z1}-1}{(\mathrm{b1}-1)(\mathrm{z1}+\mathrm{z2}+1)}, \frac{-a+\mathrm{z1}(\mathrm{b1}+\mathrm{b2}-1)+1}{(a-1)(\mathrm{b1}-1)(\mathrm{z1}+\mathrm{z2}+1)}, \frac{a+\mathrm{b1z1}+\mathrm{b1}+\mathrm{b2z1}+\mathrm{b2}-\mathrm{z1}-2}{(a-1)(\mathrm{b1}-1)(\mathrm{z1}+\mathrm{z2}+1)}, 0\right\}, \{a-1, \mathrm{b1}-1, \mathrm{b2}, \mathrm{z1}, \mathrm{z2}\}\right\}$$

As the function $G_1(a, b_1, b_2, z_1, z_2)$ only satisfies three independent PDEs, it may be written without the mixed derivative $\theta_1\theta_2$, as:

$$\begin{aligned} G_1\left(a, b_1, b_2; z_1, z_2\right) = &\left[\frac{(b_2-1)\,z_1 + b\,(z_1+1) - 1}{(b-1)\,(z_1+z_2+1)} + \frac{-a + (b+b_2-1)\,z_1 + 1}{(a-1)(b-1)\,(z_1+z_2+1)}\theta_1 \right. \\ &\left. + \frac{a + bz_1 + b_2z_1 + b + b_2 - z_1 - 2}{(a-1)(b-1)\,(z_1+z_2+1)}\theta_2\right] G_1(a-1, b_1-1, b_2; z_1, z_2). \end{aligned} \tag{18}$$

Using Ref. [17], Eq. (18) may be checked numerically in specific examples.

Example 2: Reduction of the Horn-type hypergeometric function $H_1(a, b, c, d, z_1, z_2)$.

H1IndexChange[{−1,0,0,1}, {a,b,c,d,z1,z2}]

[5] All functions in the program package HYPERDIRE generate output without additional simplification for maximum efficiency of the algorithm. To get the output in a simpler form, we recommend to use the command **Simplify** in addition.

$$\left\{\left\{\frac{(a-1)d(\mathrm{z1}-1)(\mathrm{z2}+1)-b\mathrm{z1}(a\mathrm{z2}+a+c\mathrm{z2}-\mathrm{z2}-1)}{(a-1)d(\mathrm{z1}-1)(\mathrm{z2}+1)},\frac{-a(\mathrm{z2}+1)-b\mathrm{z1}(\mathrm{z2}+1)-c\mathrm{z1z2}+d\mathrm{z1z2}+d\mathrm{z1}+\mathrm{z2}+1}{(a-1)d(\mathrm{z1}-1)(\mathrm{z2}+1)},\right.\right.$$
$$\left.\left.\frac{\mathrm{z1}(b-\mathrm{z2}(a+c-1))-d(\mathrm{z1}-1)(\mathrm{z2}+1)}{(a-1)d(\mathrm{z1}-1)(\mathrm{z2}+1)},\frac{-2\mathrm{z1z2}+\mathrm{z2}+1}{(a-1)d(\mathrm{z1}-1)(\mathrm{z2}+1)}\right\},\{a-1,b,c,d+1,\mathrm{z1},\mathrm{z2}\}\right\}$$

This corresponds to the following mathematical formula:

$$\begin{aligned} H_1(a,b,c,d;z_1,z_2) \;=\; & \\ & \Bigg[\frac{(a-1)d\,(z_1-1)\,(z_2+1)-bz_1\,(az_2+a+cz_2-z_2-1)}{(a-1)d\,(z_1-1)\,(z_2+1)} \\ & +\frac{-a\,(z_2+1)-b\,(z_2+1)\,z_1-cz_2z_1+dz_1+dz_2z_1+z_2+1}{(a-1)d\,(z_1-1)\,(z_2+1)}\theta_1 \\ & +\frac{z_1\,(b-z_2(a+c-1))-d\,(z_1-1)\,(z_2+1)}{(a-1)d\,(z_1-1)\,(z_2+1)}\theta_2 \\ & +\frac{-2z_1z_2+z_2+1}{(a-1)d\,(z_1-1)\,(z_2+1)}\theta_1\theta_2\Bigg] H_1(a-1,b,c+1,d;z_1,z_2). \end{aligned} \tag{19}$$

Example 3: Reduction of the confluent Horn-type hypergeometric function $H_1(a,b,c,z_1,z_2)$.
H1cIndexChange[{0,1,1}, {a,b,c,z1,z2}]

$$\left\{\left\{\frac{a^2(-\mathrm{z1})-ab\mathrm{z1}+ac+bc+2b\mathrm{z1z2}-c\mathrm{z2}+2\mathrm{z1z2}}{ac+bc},\frac{a(-\mathrm{z1})+a-b\mathrm{z1}+b+2\mathrm{z1z2}-\mathrm{z2}}{ac+bc},\frac{\mathrm{z1}(a+b+2\mathrm{z2})-c}{c(a+b)},-\frac{1}{ac+bc}\right\},\{a,b+1,c+1,\mathrm{z1},\mathrm{z2}\}\right\}$$

This corresponds to the following mathematical formula:

$$\begin{aligned} H_1(a,b,c;z_1,z_2) = \Bigg[\quad & \frac{a^2\,(-z_1)-abz_1+ac+bc+2bz_1z_2-cz_2+2z_1z_2}{ac+bc}, \\ & \frac{a\,(-z_1)+a-bz_1+b+2z_1z_2-z_2}{ac+bc},\frac{z_1\,(a+b+2z_2)-c}{c(a+b)}, \\ & -\frac{1}{ac+bc}\Bigg] H_1(a,b+1,c+1;z_1,z_2). \end{aligned} \tag{20}$$

5 Conclusions

The differential-reduction algorithm [8] allows one to relate Horn-type hypergeometric functions with parameters whose values differ by integers. In this paper, we presented an extended version of the Mathematica-based program package HYPERDIRE [1, 2] for the differential reduction of generalized hypergeometric functions of Horn type with two variables to a set of basis functions.

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A Inverse differential operators

In this appendix, we present the full list of differential operators inverse to those defined by Eqs. (12) and (13), which shift the upper and lower parameters of the Horn-type hypergeometric functions, respectively. The corresponding results for the Appell hypergeometric functions, F_1, F_2, F_3, and F_4, were presented in Ref. [2]. The sets of upper and lower parameters are uniquely defined by the series representation of the Horn-type hypergeometric function in Eq. (7). In the remainder of this paper, we adopt the following notations. For each parameter a_i of the Horn-type hypergeometric function F_k, we denote the four coefficient functions appearing in the differential operators inverse to those defined by Eqs. (12) and (13) by A_{a_i,F_k}, B_{a_i,F_k}, C_{a_i,F_k}, and D_{a_i,F_k}. Specifically, we have

$$F_k(\ldots, a_i, \ldots) = (A_{a_i,F_k} + B_{a_i,F_k}\theta_1 + C_{a_i,F_k}\theta_2 + D_{a_i,F_k}\theta_1\theta_2)F_k(\ldots, a_i \pm 1, \ldots) \tag{21}$$

for upper and lower parameters, respectively. When only three non-trivial solutions exist, we put explicitly $D_{a_i,F_k} = 0$. In the special cases when one of the four solutions is a Puiseux monomial, we also present the extra PDE.

A.1 Non-confluent Horn-type hypergeometric functions

A.1.1 Function $G_1(a, b_1, b_2, z_1, z_2)$

$$G_1(\mathrm{a}, b_1, b_2, z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1+n_2}\,(b_1)_{n_2-n_1}\,(b_2)_{n_1-n_2}}{n_1!n_2!} z_1^{n_1} z_2^{n_2}. \tag{22}$$

The additional PDE reads:

$$\frac{(z_1+z_2+1)(4z_1z_2-1)}{z_1z_2}\theta_1\theta_2G_1(\vec{\gamma};z_1,z_2) =$$
$$\big(\big(-a^2z_1-a^2z_2-a^2+ab_2z_1+ab_1z_2+2ab_1z_1z_2+2ab_2z_1z_2-az_1-az_2-a\big)$$
$$+(-az_1-3az_2-2a+b_2z_1-b_1z_2+2b_1z_1z_2+2b_2z_1z_2-b_1-z_1-z_2-1)\,\theta_1$$
$$+(-3az_1-az_2-2a-b_2z_1+b_1z_2+2b_1z_1z_2$$
$$+2b_2z_1z_2-b_2-z_1-z_2-1)\,\theta_2\big)G_1(\vec{\gamma};z_1,z_2). \tag{23}$$

$$A_{a,G_1} = \frac{(a+b_2)(a-b_2z_1+b_1)+z_2(2(a+1)z_1(2a+b_1+b_2)-b_1(a+b_1))}{(a+b_1)(a+b_2)}, \tag{24}$$

$$B_{a,G_1} = \frac{z_2(2z_1(2a+b_1+b_2)+a+b_1)-(z_1+1)(a+b_2)}{(a+b_1)(a+b_2)}, \tag{25}$$

$$C_{a,G_1} = \frac{z_1(2z_2(2a+b_2)+a+b_2)-a(z_2+1)+b_1((2z_1-1)z_2-1)}{(a+b_1)(a+b_2)}, \tag{26}$$

$$D_{a,G_1} = 0, \tag{27}$$

$$A_{b_1,G_1} = \frac{b_1(-2az_2+a+b_1+b_2)}{(b_1+b_2)(a+b_1)}, \tag{28}$$

$$B_{b_1,G_1} = \frac{b_1(z_1(1-2z_2)+1)}{(b_1+b_2)z_1(a+b_1)}, \tag{29}$$

$$C_{b_1,G_1} = -\frac{b_1(2z_2+1)}{(b_1+b_2)(a+b_1)}, \tag{30}$$

$$D_{b_1,G_1} = 0, \tag{31}$$

$$A_{b_2,G_1} = A_{b_1,G_1}(z_1\leftrightarrow z_2, b_1\leftrightarrow b_2), \tag{32}$$

$$B_{b_2,G_1} = C_{b_1,G_1}(z_1\leftrightarrow z_2, b_1\leftrightarrow b_2), \tag{33}$$

$$C_{b_2,G_1} = B_{b_1,G_1}(z_1\leftrightarrow z_2, b_1\leftrightarrow b_2), \tag{34}$$

$$D_{b_2,G_1} = 0. \tag{35}$$

A.1.2 Function $G_2(a_1,a_2,b_1,b_2,z_1,z_2)$

$$G_2(a_1,a_2,b_1,b_2,z_1,z_2) = \sum_{n_1,n_2}\frac{(a_1)_{n_1}(a_2)_{n_2}(b_1)_{n_2-n_1}(b_2)_{n_1-n_2}}{n_1!n_2!}z_1^{n_1}z_2^{n_2}. \tag{36}$$

The additional PDE reads:

$$\theta_1\theta_2G_2(\vec{\gamma};\vec{\sigma};z_1,z_2) = \Big(-\frac{a_1a_2z_1z_2}{z_1z_2-1}-\frac{a_2z_1z_2}{z_1z_2-1}\theta_1-\frac{a_1z_1z_2}{z_1z_2-1}\theta_2\Big)G_2(\vec{\gamma};\vec{\sigma};z_1,z_2). \tag{37}$$

$$A_{a_1,G_2} = \frac{z_1(a_2z_2 - b_2)}{a_1+b_1} + 1, \tag{38}$$
$$B_{a_1,G_2} = -\frac{z_1+1}{a_1+b_1}, \tag{39}$$
$$C_{a_1,G_2} = \frac{z_1(z_2+1)}{a_1+b_1}, \tag{40}$$
$$D_{a_1,G_2} = 0, \tag{41}$$

$$A_{a_2,G_2} = A_{a_1,G_2}(z_1 \leftrightarrow z_2, a_1 \leftrightarrow a_2, b_1 \leftrightarrow b_2), \tag{42}$$
$$B_{a_2,G_2} = C_{a_1,G_2}(z_1 \leftrightarrow z_2, a_1 \leftrightarrow a_2, b_1 \leftrightarrow b_2), \tag{43}$$
$$C_{a_2,G_2} = B_{a_1,G_2}(z_1 \leftrightarrow z_2, a_1 \leftrightarrow a_2, b_1 \leftrightarrow b_2), \tag{44}$$
$$D_{a_2,G_2} = 0, \tag{45}$$

$$A_{b_1,G_2} = \frac{b_1(-a_2z_2 + a_1 + b_1 + b_2)}{(b_1+b_2)(a_1+b_1)}, \tag{46}$$
$$B_{b_1,G_2} = \frac{b_1(z_1+1)}{(b_1+b_2)z_1(a_1+b_1)}, \tag{47}$$
$$C_{b_1,G_2} = -\frac{b_1(z_2+1)}{(b_1+b_2)(a_1+b_1)}, \tag{48}$$
$$D_{b_1,G_2} = 0, \tag{49}$$

$$A_{b_2,G_2} = A_{b_1,G_2}(z_1 \leftrightarrow z_2, a_1 \leftrightarrow a_2, b_1 \leftrightarrow b_2), \tag{50}$$
$$B_{b_2,G_2} = C_{b_1,G_2}(z_1 \leftrightarrow z_2, a_1 \leftrightarrow a_2, b_1 \leftrightarrow b_2), \tag{51}$$
$$C_{b_2,G_2} = B_{b_1,G_2}(z_1 \leftrightarrow z_2, a_1 \leftrightarrow a_2, b_1 \leftrightarrow b_2), \tag{52}$$
$$D_{b_2,G_2} = 0. \tag{53}$$

A.1.3 Function $G_3(a_1, a_2, z_1, z_2)$

$$G_3(a_1, a_2, z_1, z_2) = \sum_{n_1,n_2} \frac{(a_1)_{2n_2-n_1}(a_2)_{2n_1-n_2}}{n_1!n_2!} z_1^{n_1} z_2^{n_2}. \tag{54}$$

The additional PDE reads:

$$\begin{aligned}\Big(& - a_1a_2z_1z_2 + (a_2z_1z_2 - 2a_1z_1z_2)\theta_1 + (2z_1z_2)\theta_1{}^2 \\ & + (a_1z_1z_2 - 2a_2z_1z_2)\theta_2 + (2z_1z_2)\theta_2{}^2 + (1 - 5z_1z_2)\theta_1\theta_2\Big)G_3(\vec{\gamma}; z_1, z_2) = 0.\end{aligned} \tag{55}$$

$$
\begin{aligned}
A_{a_1,G_3} &= \frac{2a_1\left(2a_1+2a_2+1\right)}{\left(2a_1+a_2\right)\left(2a_1+a_2+1\right)} \\
&\quad -\frac{3a_1z_2\left(\left(a_1+1\right)\left(5a_1+4a_2+2\right)-3a_2\left(2a_1+a_2+1\right)z_1\right)}{\left(2a_1+a_2\right)\left(2a_1+a_2+1\right)\left(a_1+2a_2\right)}, \qquad (56) \\
B_{a_1,G_3} &= \frac{a_1\left(\left(a_1+2a_2\right)\left(4z_1+1\right)+3z_1z_2\left(6\left(2a_1+a_2+1\right)z_1+5a_1+4a_2+2\right)\right)}{\left(2a_1+a_2\right)\left(2a_1+a_2+1\right)\left(a_1+2a_2\right)z_1}, \qquad (57) \\
C_{a_1,G_3} &= \frac{a_1\left(-3z_2\left(3\left(2a_1+a_2+1\right)z_1+10a_1+8a_2+4\right)-8a_1-7a_2-3\right)}{\left(2a_1+a_2\right)\left(2a_1+a_2+1\right)\left(a_1+2a_2\right)}, \qquad (58) \\
D_{a_1,G_3} &= 0, \qquad (59)
\end{aligned}
$$

$$
\begin{aligned}
A_{a_2,G_3} &= A_{a_1,G_3}(z_1 \leftrightarrow z_2, a_1 \leftrightarrow a_2), \qquad (60) \\
B_{a_2,G_3} &= B_{a_1,G_3}(z_1 \leftrightarrow z_2, a_1 \leftrightarrow a_2), \qquad (61) \\
C_{a_2,G_3} &= C_{a_1,G_3}(z_1 \leftrightarrow z_2, a_1 \leftrightarrow a_2), \qquad (62) \\
D_{a_2,G_3} &= 0. \qquad (63)
\end{aligned}
$$

A.1.4 Function $H_1(a,b,c,d,z_1,z_2)$

$$
H_1(\text{a,b,c,d},z_1,z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1-n_2}(b)_{n_1+n_2}(c)_{n_2}}{n_1!n_2!(d)_{n_1}} z_1^{n_1} z_2^{n_2}, \qquad (64)
$$

$$
\begin{aligned}
A_{a,H_1} &= \frac{abz_1\left(a^2+2a(b+3c-d+1)+b^2+2b(c-d+1)+4c(c-d+1)\right)}{\left(z_2+1\right)(a+b)(a+c)(a+b-2d+2)(a+c-d+1)} \\
&\quad +\frac{abz_1z_2}{\left(z_2+1\right)(a+c)(a+c-d+1)}+\frac{a(a+b+c)}{(a+b)(a+c)}, \qquad (65) \\
B_{a,H_1} &= -\frac{a\left(a^2-2d(a+b+c)+2ab+3ac+2a+b^2+bc+2(b+c)+2c^2\right)}{(a+b)(a+c)(a+b-2d+2)(a+c-d+1)} \\
&\quad +\frac{az_1\left(a^2+2a(b+3c-d+1)+b^2+2b(c-d+1)+4c(c-d+1)\right)}{\left(z_2+1\right)(a+b)(a+c)(a+b-2d+2)(a+c-d+1)} \\
&\quad +\frac{az_1z_2}{\left(z_2+1\right)(a+c)(a+c-d+1)}, \qquad (66) \\
C_{a,H_1} &= \frac{az_1\left(7a^2+2a(5b+5c-3d+6)+3b^2+b(6c-6d+8)+4(c+1)(c-d+1)\right)}{\left(z_2+1\right)(a+b)(a+c)(a+b-2d+2)(a+c-d+1)} \\
&\quad +\frac{a\left(z_2+1\right)}{z_2(a+b)(a+c)}+\frac{az_2z_1}{\left(z_2+1\right)(a+c)(a+c-d+1)}, \qquad (67) \\
D_{a,H_1} &= -\frac{a\left(z_2\left(-4z_1+z_2+2\right)+1\right)(3a+b+2c-2d+2)}{z_2\left(z_2+1\right)(a+b)(a+c)(a+b-2d+2)(a+c-d+1)}, \qquad (68)
\end{aligned}
$$

$$
\begin{aligned}
A_{b,H_1} &= \frac{z_1 z_2 \left(a^3 + a(b(b-2(c+d-1)) - 2c) - 2(b+1)c(3b-2d+2)\right)}{(z_2+1)\,(a+b)(b-d+1)(a+b-2d+2)} \\
&\quad + \frac{2a^2 z_1 z_2}{(z_2+1)\,(a+b)(a+b-2d+2)} - \frac{c z_2}{a+b} + \frac{a z_1}{(z_2+1)\,(b-d+1)} + 1, \qquad (69)\\
B_{b,H_1} &= \frac{z_2 z_1 \left(a^2 + 2a(b-c-d+1) + b^2 - 2b(3c+d-1) + 4c(d-1)\right)}{(z_2+1)\,(a+b)(b-d+1)(a+b-2d+2)} \\
&\quad + \frac{c z_2 (a+3b-2d+2)}{(a+b)(b-d+1)(a+b-2d+2)} + \frac{z_1}{(z_2+1)\,(b-d+1)} - \frac{1}{b-d+1}, \qquad (70)\\
C_{b,H_1} &= -\frac{z_2 z_1 \left(3a^2 + 2a(5b+c-3d+4) + 7b^2 + 6b(c-d+2) - 4(c+1)(d-1)\right)}{(z_2+1)\,(a+b)(b-d+1)(a+b-2d+2)} \\
&\quad - \frac{z_2+1}{a+b} - \frac{z_1}{(z_2+1)\,(b-d+1)}, \qquad (71)\\
D_{b,H_1} &= \frac{\left(z_2\left(-4z_1 + z_2 + 2\right) + 1\right)(a+3b-2d+2)}{(z_2+1)\,(a+b)(b-d+1)(a+b-2d+2)}, \qquad (72)
\end{aligned}
$$

$$
\begin{aligned}
A_{c,H_1} &= \frac{z_2 \left((a+c-d+1)\,(a - b z_2 - b + c) + b z_1 \left(z_2(a-b+2c+1) + a - b - 1\right)\right)}{(z_2+1)\,(a+c)(a+c-d+1)} \\
&\quad + \frac{1}{z_2+1}, \qquad (73)\\
B_{c,H_1} &= \frac{z_2 \left(z_1 \left(z_2(a-b+2c+1) + a - b - 1\right) - (z_2+1)\,(a-b+c)\right)}{(z_2+1)\,(a+c)(a+c-d+1)}, \qquad (74)\\
C_{c,H_1} &= \frac{z_1 z_2 \left(z_2(a-b+2c+1) - a - 3b - 1\right)}{(z_2+1)\,(a+c)(a+c-d+1)} - \frac{z_2+1}{a+c}, \qquad (75)\\
D_{c,H_1} &= \frac{z_2\left(-4z_1 + z_2 + 2\right) + 1}{(z_2+1)\,(a+c)(a+c-d+1)}, \qquad (76)
\end{aligned}
$$

$$
\begin{aligned}
A_{d,H_1} &= -\frac{(d-1)\left(-4d(b+c)+b(b+4c+5)+6c+4d^2-10d+6\right)}{(z_2+1)\,(a+b-2d+2)(a+b-2d+3)(a+c-d+1)} \\
&\quad -\frac{(d-1)}{(z_2+1)\,(b-d+1)(a+b-2d+2)(a+b-2d+3)(a+c-d+1)} \\
&\quad \times \left(a^3+a^2(3b+c-5d+6)+z_2\left(a^3+a^2(3b+c-5d+6)+a\left(-2d(5b+2c)\right.\right.\right. \\
&\quad \left.+b(3b+c+12)+5c+8d^2-19d+11\right)+b^3+b^2(2c-5d+6) \\
&\quad +b\left(-4cd-2(c-3)c+8d^2-19d+11\right)+2(d-1)(2d-3)(c-d+1)\big) \\
&\quad \left.+a\left(-2d(5b+2c)+3b(b+c+4)+5c+8d^2-19d+11\right)\right), \qquad (77)
\end{aligned}
$$

$$
\begin{aligned}
B_{d,H_1} &= -\frac{(d-1)}{(z_2+1)\,(b-d+1)(a+b-2d+2)(a+b-2d+3)(a+c-d+1)} \\
&\quad \times \left(z_2\left(a^2+a(2b-c-4d+5)+b^2+b(c-4d+5)-2c^2+c+4d^2-10d+6\right)\right. \\
&\quad \left.+a^2+a(2b+c-4d+5)-4d(b+c)+b(b+3c+5)+5c+4d^2-10d+6\right) \\
&\quad +\frac{(d-1)\left(a^2+a(2b+c-4d+5)-d(4b+3c+10)+(b+2)(b+2c+3)+4d^2\right)}{z_1(b-d+1)(a+b-2d+2)(a+b-2d+3)(a+c-d+1)} \\
&\quad -\frac{c(d-1)z_2}{z_1(b-d+1)(a+b-2d+2)(a+b-2d+3)}, \qquad (78)
\end{aligned}
$$

$$
\begin{aligned}
C_{d,H_1} &= \frac{(d-1)z_2\left(3a^2+a(2b+5c-4d+5)-b^2+b(3c-1)+c(2c-4d+5)\right)}{(z_2+1)\,(b-d+1)(a+b-2d+2)(a+b-2d+3)(a+c-d+1)} \\
&\quad +\frac{(d-1)\left(a^2+a(-2b+c+1)-b(3b+c-4d+5)+c\right)}{(z_2+1)\,(b-d+1)(a+b-2d+2)(a+b-2d+3)(a+c-d+1)}, \qquad (79)
\end{aligned}
$$

$$
D_{d,H_1} = \frac{(d-1)\left(z_2\left(-4z_1+z_2+2\right)+1\right)\left(-z_2(a+c-d+1)+b-d+1\right)}{z_1z_2\,(z_2+1)\,(b-d+1)(a+b-2d+2)(a+b-2d+3)(a+c-d+1)}. \qquad (80)
$$

A.1.5 Function $H_2(a,b,c,d,e,z_1,z_2)$

$$
H_2(\mathrm{a,b,c,d,e},z_1,z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1-n_2}(b)_{n_1}(c)_{n_2}(d)_{n_2}}{n_1!n_2!(e)_{n_1}} z_1^{n_1} z_2^{n_2}, \qquad (81)
$$

$$A_{a,H_2} = \frac{abz_1\left(a^2-e(a+c+d)+2ac+2ad+a+c^2+cd+c+d^2+d\right)}{(a+c)(a+d)(a+c-e+1)(a+d-e+1)} + \frac{a(a+c+d)}{(a+c)(a+d)}, \tag{82}$$

$$B_{a,H_2} = \frac{a\left(z_1-1\right)\left(a^2-e(a+c+d)+2ac+2ad+a+c^2+cd+c+d^2+d\right)}{(a+c)(a+d)(a+c-e+1)(a+d-e+1)}, \tag{83}$$

$$C_{a,H_2} = \frac{abz_1(2a+c+d-e+1)}{(a+c)(a+d)(a+c-e+1)(a+d-e+1)} + \frac{a\left(z_2+1\right)}{z_2(a+c)(a+d)}, \tag{84}$$

$$D_{a,H_2} = \frac{a\left(\left(z_1-1\right)z_2-1\right)\left(2a+c+d-e+1\right)}{z_2(a+c)(a+d)(a+c-e+1)(a+d-e+1)}, \tag{85}$$

$$A_{b,H_2} = \frac{az_1}{b-e+1}+1, \tag{86}$$

$$B_{b,H_2} = \frac{z_1-1}{b-e+1}, \tag{87}$$

$$C_{b,H_2} = -\frac{z_1}{b-e+1}, \tag{88}$$

$$D_{b,H_2} = 0, \tag{89}$$

$$A_{c,H_2} = 1-\frac{dz_2\left(a+bz_1+c-e+1\right)}{(a+c)(a+c-e+1)}, \tag{90}$$

$$B_{c,H_2} = -\frac{d\left(z_1-1\right)z_2}{(a+c)(a+c-e+1)}, \tag{91}$$

$$C_{c,H_2} = -\frac{z_2\left(a+bz_1+c-e+1\right)+a+c-e+1}{(a+c)(a+c-e+1)}, \tag{92}$$

$$D_{c,H_2} = \frac{1-\left(z_1-1\right)z_2}{(a+c)(a+c-e+1)}, \tag{93}$$

$$A_{d,H_2} = A_{c,H_2}(c\leftrightarrow d), \tag{94}$$

$$B_{d,H_2} = B_{c,H_2}(c\leftrightarrow d), \tag{95}$$

$$C_{d,H_2} = C_{c,H_2}(c\leftrightarrow d), \tag{96}$$

$$D_{d,H_2} = D_{c,H_2}(c\leftrightarrow d), \tag{97}$$

$$
\begin{aligned}
A_{e,H_2} &= -\frac{(e-1)}{(b-e+1)(a+c-e+1)(a+d-e+1)} \\
&\quad \times \left(a^2+a(b+c+d-2e+2)+b(c+d-e+1)+(c-e+1)(d-e+1)\right), \quad (98) \\
B_{e,H_2} &= -\frac{(e-1)(z_1-1)(a+c+d-e+1)}{z_1(b-e+1)(a+c-e+1)(a+d-e+1)}, \quad (99) \\
C_{e,H_2} &= \frac{b-be}{(b-e+1)(a+c-e+1)(a+d-e+1)}, \quad (100) \\
D_{e,H_2} &= -\frac{(e-1)((z_1-1)z_2-1)}{z_1 z_2(b-e+1)(a+c-e+1)(a+d-e+1)}. \quad (101)
\end{aligned}
$$

A.1.6 Function $H_3(a,b,c,z_1,z_2)$

$$
H_3(a,b,c,z_1,z_2) = \sum_{n_1,n_2} \frac{(a)_{2n_1+n_2}(b)_{n_2}}{n_1!n_2!(c)_{n_1+n_2}} z_1^{n_1} z_2^{n_2}. \quad (102)
$$

The additional PDE reads:

$$
(bz_2\theta_1 - az_1\theta_2 - z_1\theta_2^2 + (z_2-2z_1)\theta_1\theta_2)H_3(\vec{\gamma};\vec{\sigma};z_1,z_2) = 0. \quad (103)
$$

$$
\begin{aligned}
A_{a,H_3} &= \frac{2(a+1)z_1(2a+b-2c+2)+(a+b-2c+2)(a+bz_2-c+1)}{(a-c+1)(a+b-2c+2)}, \quad (104) \\
B_{a,H_3} &= \frac{(4z_1-1)(2a+b-2c+2)}{(a-c+1)(a+b-2c+2)}, \quad (105) \\
C_{a,H_3} &= \frac{(z_2-1)z_2(a+b-2c+2)+z_1(2z_2(2a+b-2c+2)-a)}{z_2(a-c+1)(a+b-2c+2)}, \quad (106) \\
D_{a,H_3} &= 0, \quad (107)
\end{aligned}
$$

$$
\begin{aligned}
A_{b,H_3} &= \frac{2az_2}{a+b-2c+2}+1, \quad (108) \\
B_{b,H_3} &= \frac{(4z_1-1)z_2}{z_1(a+b-2c+2)}, \quad (109) \\
C_{b,H_3} &= \frac{2z_2-1}{a+b-2c+2}, \quad (110) \\
D_{b,H_3} &= 0, \quad (111)
\end{aligned}
$$

$$A_{c,H_3} = \frac{(c-1)\left(-z_2\left(4a^2+a(3b-8c+10)+(b-2c+2)(b-2c+3)\right)-2abz_1\right)}{z_2(a-c+1)(a+b-2c+2)(a+b-2c+3)}, \quad (112)$$

$$B_{c,H_3} = -\frac{(c-1)(4z_1-1)(z_2(a-c+1)+bz_1)}{z_1z_2(a-c+1)(a+b-2c+2)(a+b-2c+3)}, \quad (113)$$

$$C_{c,H_3} = \frac{(c-1)}{z_2^2(a-c+1)(a+b-2c+2)(a+b-2c+3)} \times \left(z_2\left(-z_2(3a+b-4c+5)+2a+b-3c+4\right)-z_1\left(a+2bz_2-2c+3\right)\right), \quad (114)$$

$$D_{c,H_3} = 0. \quad (115)$$

A.1.7 Function $H_4(a,b,c,d,z_1,z_2)$

$$H_4(\text{a,b,c,d},z_1,z_2) = \sum_{n_1,n_2} \frac{(a)_{2n_1+n_2}(b)_{n_2}}{n_1!n_2!(c)_{n_1}(d)_{n_2}} z_1^{n_1} z_2^{n_2}, \quad (116)$$

$$A_{a,H_4} = \frac{4(a+1)z_2z_1\left(a^2+2a(b-c-d+2)-(b-d+1)(2c+d-3)\right)}{(z_2-1)(a-2c+2)(a-d+1)(a-2c-d+3)} + \frac{bz_2}{a-d+1} - \frac{4(a+1)z_1}{(z_2-1)(a-2c+2)} + 1, \quad (117)$$

$$B_{a,H_4} = \frac{2z_2\left(4z_1\left(a^2+2a(b-c-d+2)-(b-d+1)(2c+d-3)\right)-ab(z_2-1)\right)}{(z_2-1)(a-2c+2)(a-d+1)(a-2c-d+3)} - \frac{2bz_2}{(a-2c+2)(a-d+1)} - \frac{8z_1}{(z_2-1)(a-2c+2)} - \frac{2}{a-2c+2}, \quad (118)$$

$$C_{a,H_4} = \frac{4z_1\left(z_2(a+2b-2d+2)+a+2c-d+1\right)}{(z_2-1)(a-d+1)(a-2c-d+3)} + \frac{4z_1(2c-d-1)\left(z_2(b-d+1)+2c-1\right)}{(z_2-1)(a-2c+2)(a-d+1)(a-2c-d+3)} + \frac{z_2-1}{a-d+1}, \quad (119)$$

$$D_{a,H_4} = -\frac{2\left((z_2-1)^2-4z_1\right)(2a-2c-d+3)}{(z_2-1)(a-2c+2)(a-d+1)(a-2c-d+3)}, \quad (120)$$

$$A_{b,H_4} = \frac{az_2}{b-d+1} + 1, \quad (121)$$

$$B_{b,H_4} = \frac{2z_2}{b-d+1}, \quad (122)$$

$$C_{b,H_4} = \frac{z_2-1}{b-d+1}, \quad (123)$$

$$D_{b,H_4} = 0, \quad (124)$$

$$A_{c,H_4} = \frac{2(c-1)z_2}{(z_2-1)(a-2c+2)(a-2c+3)(a-2c-d+3)(a-2c-d+4)} \times \left(-2a^3 - a^2(b-10c-4d+17) + a(2(b(c-1)-(2c+d)(4c+d)) + 54c + 20d - 45) + abz_2(-b+d-1) + (2c-3)(2c+d-4)(2c+d-3)\right) + \frac{2(c-1)(2a-2c+3)}{(z_2-1)(a-2c+2)(a-2c+3)}, \quad (125)$$

$$B_{c,H_4} = -\frac{4(c-1)z_2(a+b-2c-d+4)}{(z_2-1)(a-2c+2)(a-2c+3)(a-2c-d+4)} + \frac{b(c-1)z_2(z_2(-a+b+2c-3)+3a-2(3c+d-5))}{z_1(a-2c+2)(a-2c+3)(a-2c-d+3)(a-2c-d+4)} - \frac{4b(c-1)z_2(z_2(b-d+1)+d-1)}{(z_2-1)(a-2c+2)(a-2c+3)(a-2c-d+3)(a-2c-d+4)} + \frac{c-1}{z_1(a-2c+2)(a-2c+3)} + \frac{4(c-1)}{(z_2-1)(a-2c+2)(a-2c+3)}, \quad (126)$$

$$C_{c,H_4} = \frac{2(c-1)(z_2(a-2b+1)+2d)}{(z_2-1)(a-2c+2)(a-2c-d+3)(a-2c-d+4)} + \frac{2(c-1)(d-bz_2)(z_2(b-d+1)+2c-3)}{(z_2-1)(a-2c+2)(a-2c+3)(a-2c-d+3)(a-2c-d+4)} - \frac{4(a+1)(c-1)}{(z_2-1)(a-2c+2)(a-2c-d+3)(a-2c-d+4)}, \quad (127)$$

$$D_{c,H_4} = \frac{(c-1)((z_2-1)^2-4z_1)(z_2(-a+b+2c-3)+2a-4c-d+6)}{z_1(z_2-1)(a-2c+2)(a-2c+3)(a-2c-d+3)(a-2c-d+4)}, \quad (128)$$

$$A_{d,H_4} = \frac{(d-1)(-(z_2-1)(a+b-d+1)(a-2c-d+3)-4abz_1)}{(z_2-1)(a-d+1)(b-d+1)(a-2c-d+3)}, \quad (129)$$

$$B_{d,H_4} = \frac{2b(d-1)(4z_1-z_2+1)}{(z_2-1)(a-d+1)(-b+d-1)(a-2c-d+3)}, \quad (130)$$

$$C_{d,H_4} = \frac{(d-1)(4z_1(a+bz_2-d+2)+(z_2-1)^2(a-2c-d+3))}{(z_2-1)z_2(a-d+1)(-b+d-1)(a-2c-d+3)}, \quad (131)$$

$$D_{d,H_4} = -\frac{2(d-1)((z_2-1)^2-4z_1)}{(z_2-1)z_2(a-d+1)(-b+d-1)(a-2c-d+3)}. \quad (132)$$

A.1.8 Function $H_5(a,b,c,z_1,z_2)$

$$H_5(\text{a,b,c},z_1,z_2) = \sum_{n_1,n_2} \frac{(a)_{2n_1+n_2}(b)_{n_2-n_1}}{n_1!n_2!(c)_{n_2}} z_1^{n_1} z_2^{n_2}, \quad (133)$$

$$
\begin{aligned}
A_{a,H_5} &= \frac{1}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a-c+1)(a+2b-3c+3)} \\
&\times 2z_1\left(4(a+1)z_1\left(2(a-c+1)(a+2b-3c+3)\right.\right. \\
&-z_2\left.\left(7a^2+a(16b-15c+23)+(2b-3c+3)(2b-3c+5)\right)\right) \\
&+z_2\left(13a^3+a^2(4b-13c+47)\right. \\
&-3z_2\left(4a^3-3a^2(b+c-5)+a(-2b(3b-3c+2)-9c+17)\right. \\
&-\left.\left(2b^2+b-2\right)(2b-3c+3)\right)+a(b(20-14c)+c(3c-25)+46) \\
&-2(2b-3c+3)(b(2b+3c-2)-c-2))) \\
&+\frac{\left(z_2-1\right)\left(a+bz_2-c+1\right)}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a-c+1)} \\
&-\frac{4(2b-1)z_1}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)}, \qquad (134) \\
B_{a,H_5} &= -\frac{2z_1z_2}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a-c+1)(a+2b-3c+3)} \\
&\times\left(8z_1\left(7a^2+a(16b-15c+23)+(2b-3c+3)(2b-3c+5)\right)\right. \\
&+3z_2\left(21a^2+2a(15b-9c+23)+(6b+7)(2b-3c+3)\right)-48a^2 \\
&+8a(-8b+c-11)-2(2b-3c+3)(8b+5c+5)) \\
&-\frac{\left(z_2-1\right)\left(z_2(a+2b)+2(a-c+1)\right)}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a-c+1)} \\
&+\frac{16z_1\left(2z_1+1\right)}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)}, \qquad (135) \\
C_{a,H_5} &= \frac{1}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a-c+1)(a+2b-3c+3)} \\
&\times(-8z_1^2\left(z_2\left(7a^2+a(16b-15c+23)+(2b-3c+3)(2b-3c+5)\right)\right. \\
&+2a(3a-3c+7)+4b-6c+6) \\
&-2z_1\left(z_2\left(3z_2\left(3a^2+a(8-6b)-(3b-2)(2b-3c+3)\right)-17a^2\right.\right. \\
&+a(20b+7c-43)+(2b-3c+3)(12b+c-10))+4a^2-8a(b-1) \\
&-2(2b-1)(2b-3c+3))) \\
&+\frac{\left(z_2-1\right)^2}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a-c+1)}, \qquad (136) \\
D_{a,H_5} &= -\frac{2\left(-z_2+z_1\left(16z_1+9z_2\left(3z_2-4\right)+8\right)+1\right)\left(4a+2b-3c+3\right)}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a-c+1)(a+2b-3c+3)}, \qquad (137)
\end{aligned}
$$

$$
\begin{aligned}
A_{b,H_5} &= \frac{b}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a+2b+1)(a+2b-3c+3)(a+2b-3c+4)} \\
&\times \Big(2z_1\Big(6z_2\left(a^3+2a^2(3b-3c+5)+a\left(12b^2-24(b+2)c+36b+18c^2+31\right)\right. \\
&\left.+(2b+1)(2b-3c+3)(2b-3c+4)\right)+27az_2^2(a+2b+1)(a+2b-3c+4) \\
&-4(2a+2b+1)(a+2b-3c+3)(a+2b-3c+4)\Big) \\
&+z_2\Big(-23a^3+9az_2\left(3a^2-3c(2a+b+1)+6ab+9a+6b+4\right) \\
&+a^2(-40b+30c-57)+2a\left(10b^2+b(7-18c)+6c(3c-5)+7\right) \\
&+2(2b+1)(2b-3c+3)(2b-3c+4)\Big)\Big) \\
&-\frac{2b(2a+2b+1)}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a+2b+1)}, \qquad (138) \\
B_{b,H_5} &= \frac{-3b(3c-2)z_2\left(3z_2(6b-15c+7)-2(8b-21c+9)-24(c-1)z_1\right)}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a+2b+1)(a+2b-3c+3)(a+2b-3c+4)} \\
&+\frac{108bz_1z_2^2}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a+2b-3c+3)} \\
&+\frac{bz_2\left(27z_2(4a+4b+5c+2)-4(23a+22b+39c+5)+48z_1\right)}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a+2b+1)(a+2b-3c+4)} \\
&-\frac{16bz_1^2+8bz_1}{z_1\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a+2b+1)} \\
&+\frac{b\left(z_2-1\right)}{z_1\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a+2b+1)}, \qquad (139) \\
C_{b,H_5} &= \frac{b}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a+2b+1)(a+2b-3c+3)(a+2b-3c+4)} \\
&\times \Big(-z_2\Big(12z_1\left(3a^2+a(12b-9c+13)+6(2-3b)c+2b(6b+13)-9c^2\right)+49a^2 \\
&-3c(35a+28b+25)+112ab+157a+28b^2+146b+9c^2+84\Big) \\
&+9z_2^2\left(3a^2-3c(2a+b+1)+6ab+9a+6b+4\right) \\
&+2\left(4z_1+1\right)\left(8a^2+2a(10b-9c+13)+(2b+1)(4b-9c+12)\right)\Big) \\
&+\frac{54bz_1z_2^2}{\left(z_2+z_1\left(6z_2-4\right)-1\right)(a+2b)(a+2b-3c+3)}, \qquad (140) \\
D_{b,H_5} &= \frac{3b\left(-z_2+z_1\left(16z_1+9z_2\left(3z_2-4\right)+8\right)+1\right)\left(2a+4b-3c+4\right)}{z_1(a+2b)(a+2b+1)(a+2b-3c+3)(a+2b-3c+4)} \\
&\times\frac{1}{\left(z_2+z_1\left(6z_2-4\right)-1\right)}, \qquad (141)
\end{aligned}
$$

$$
\begin{aligned}
A_{c,H_5} &= -\frac{a(c-1)}{(a-c+1)(a+2b-3c+3)(a+2b-3c+4)(a+2b-3c+5)} \\
&\times \frac{1}{(z_2+z_1(6z_2-4)-1)}\big(z_2\big(9a^2-9c(4a+7b+17)+17ab+54a+20b^2 \\
&+87b+54c^2+105\big)+2z_1\big(3z_2\big(4a^2+a(18b-27c+38)+6(2b-3c)^2 \\
&+94b-144c+94)-2\big(2a^2+a(14b-23c+31)+24b^2-70(b+2)c \\
&+90b+54c^2+89\big)\big)-9a^2-18a(b-2c+3)+64bc-4b(5b+22) \\
&-54c^2+3(51c-35)\big) \\
&+\frac{16a(c-1)z_1^2(a+2b+1)}{(z_2+z_1(6z_2-4)-1)(a-c+1)(a+2b-3c+3)(a+2b-3c+5)} \\
&-\frac{(c-1)(2b-3c+3)(2b-3c+4)(2b-3c+5)}{(a-c+1)(a+2b-3c+3)(a+2b-3c+4)(a+2b-3c+5)}, \qquad (142) \\
B_{c,H_5} &= \frac{(c-1)}{(a-c+1)(a+2b-3c+3)(a+2b-3c+4)(a+2b-3c+5)} \\
&\times \frac{1}{(z_2+z_1(6z_2-4)-1)}\big(2\big(9a^2+4z_1(4z_1(a+2b+1)(a+2b-3c+4) \\
&+c(16a+26b+13)-2(a+2b+1)(3a+4b+9))+a(8b-13c+19) \\
&-2(b+1)(2b+1)\big)+z_2\big(12z_1\big(3a^2+a(12b-9c+13) \\
&+(2b+1)(6b-9c+13)\big)-19a^2 \\
&+a(-16b+27c-39)+4(b+1)(2b+1)\big)\big), \qquad (143) \\
C_{c,H_5} &= \frac{(c-1)}{(a-c+1)(a+2b-3c+3)(a+2b-3c+4)(a+2b-3c+5)} \\
&\times \frac{1}{z_2(z_2+z_1(6z_2-4)-1)}\big(z_2^2\big(-\big(6z_1\big(3a^2+2a(6b-9c+13) \\
&+3(2b-3c)^2+46b-72c+47\big)+8a^2-27c(a+b+3)+11ab+42a \\
&+8b^2+39b+27c^2+58\big)\big) \\
&+z_2\big(4z_1\big(4a^2+4z_1(a+2b+1)(a+2b-3c+4) \\
&+a(20b-32c+46)-70bc+6b(4b+15)+54c^2-143c+93\big) \\
&+15a^2+a(24b-53c+81)+2\big(-28bc+8b(b+5)+27c^2-80c+57\big)\big) \\
&+2(4z_1+1)\big(-3a^2+4z_1(a-c+2)(a+2b-3c+4)-6a(b-2c+3) \\
&+14bc-4b(b+5)-13c^2+38c-27\big)\big), \qquad (144) \\
D_{c,H_5} &= \frac{(c-1)(-z_2+z_1(16z_1+9z_2(3z_2-4)+8)+1)}{(a-c+1)(a+2b-3c+3)(a+2b-3c+4)(a+2b-3c+5)} \\
&\times \frac{(4z_1(a+2b-3c+4)-a+c-1)}{z_1z_2(z_2+z_1(6z_2-4)-1)}. \qquad (145)
\end{aligned}
$$

A.1.9 Function $H_6(a, b, c, z_1, z_2)$

$$H_6\left(a,b,c,z_1,z_2\right)=\sum_{n_1,n_2}\frac{(a)_{2n_1-n_2}(b)_{n_2-n_1}(c)_{n_2}}{n_1!n_2!}z_1^{n_1}z_2^{n_2}. \tag{146}$$

The additional PDE reads:

$$(acz_1z_2+2cz_1z_2\theta_1+(az_1z_2-cz_1z_2)\theta_2-z_1z_2{\theta_2}^2+(2z_1z_2-1)\theta_1\theta_2)H_6(\vec{\gamma};z_1,z_2)=0. \tag{147}$$

$$A_{a,H_6} = \frac{a\left(z_1\left(cz_2(a+c)-2(a+1)(2(a+b)+c)\right)+(a+2b)(a+b+c)\right)}{(a+b)(a+2b)(a+c)}, \tag{148}$$

$$B_{a,H_6} = -\frac{a\left(4z_1+1\right)\left(2(a+b)+c\right)}{(a+b)(a+2b)(a+c)}, \tag{149}$$

$$C_{a,H_6} = \frac{a\left(z_2\left(z_1\left(4(a+b)+z_2(a+c)+2c\right)+a+2b\right)+a+2b\right)}{z_2(a+b)(a+2b)(a+c)}, \tag{150}$$

$$D_{a,H_6} = 0, \tag{151}$$

$$A_{b,H_6} = \frac{bcz_2\left(z_1\left(z_2(a+2b-c+1)+2a\right)-2a-3b-1\right)}{(a+b)(a+2b)(a+2b+1)} + \frac{2b(2a+2b+1)}{(a+2b)(a+2b+1)}, \tag{152}$$

$$B_{b,H_6} = \frac{b\left(4z_1+1\right)\left(a+b+cz_1z_2\right)}{z_1(a+b)(a+2b)(a+2b+1)}, \tag{153}$$

$$C_{b,H_6} = \frac{b\left(z_2\left(z_1\left(z_2(a+2b-c+1)-2c\right)-2a-3b-1\right)-3a-4b-1\right)}{(a+b)(a+2b)(a+2b+1)}, \tag{154}$$

$$D_{b,H_6} = 0, \tag{155}$$

$$A_{c,H_6} = \frac{z_2\left(2az_1-b\right)}{a+c}+1, \tag{156}$$

$$B_{c,H_6} = \frac{\left(4z_1+1\right)z_2}{a+c}, \tag{157}$$

$$C_{c,H_6} = -\frac{2z_1z_2+z_2+1}{a+c}, \tag{158}$$

$$D_{c,H_6} = 0. \tag{159}$$

A.1.10 Function $H_7(a, b, c, d, z_1, z_2)$

$$H_7(\text{a,b,c,d},z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{2n_1-n_2}(b)_{n_2}(c)_{n_2}}{n_1!n_2!(d)_{n_1}} z_1^{n_1} z_2^{n_2}, \tag{160}$$

$$\begin{aligned}
A_{a,H_7} &= \frac{a(a+b+c)}{(a+b)(a+c)} + \frac{4az_1}{(z_2+1)(a+b)(a+c)(a+b-2d+2)(a+c-2d+2)} \\
&\times \Big((a+1)\left(a^2+2a(b+c-d+1)+b^2+b(c-2d+2)+c(c-2d+2)\right) \\
&+z_2\left(a^3+a^2(2b+2c-2d+3)\right. \\
&+a\left(b^2-2d(b+c+1)+3bc+4b+c^2+4c+2\right)+b^2(c+1) \\
&+b(c+1)(c-2d+2)+c(c-2d+2))\Big), \qquad (161) \\
B_{a,H_7} &= \frac{2a(4z_1-1)(a^2+2a(b+c-d+1)+b^2+(b+c)(c-2d+2))}{(a+b)(a+c)(a+b-2d+2)(a+c-2d+2)}, \qquad (162) \\
C_{a,H_7} &= \frac{a}{(z_2+1)(a+b)(a+c)(a+b-2d+2)(a+c-2d+2)} \\
&\times (4z_1z_2(c(a+b)+a(a+b+2)+b+c-2d+2) \\
&-4z_1\left(a^2+2a(b+c-d+1)+b^2+b(c-2d+2)+c(c-2d+2)\right)\Big) \\
&+\frac{a(z_2+2)}{(z_2+1)(a+b)(a+c)} + \frac{a}{(z_2^2+z_2)(a+b)(a+c)}, \qquad (163) \\
D_{a,H_7} &= \frac{2a(z_2((4z_1-1)z_2-2)-1)(2a+b+c-2d+2)}{z_2(z_2+1)(a+b)(a+c)(a+b-2d+2)(a+c-2d+2)}, \qquad (164)
\end{aligned}$$

$$\begin{aligned}
A_{b,H_7} &= -\frac{4cz_1z_2(z_2(a+b+1)+a)}{(z_2+1)(a+b)(a+b-2d+2)} + \frac{z_2(a+b-cz_2-c)}{(z_2+1)(a+b)} + \frac{1}{z_2+1}, \qquad (165) \\
B_{b,H_7} &= \frac{2c(1-4z_1)z_2}{(a+b)(a+b-2d+2)}, \qquad (166) \\
C_{b,H_7} &= -\frac{4z_1z_2(z_2(a+b+1)-c)+a+b-2d+2}{(z_2+1)(a+b)(a+b-2d+2)} - \frac{z_2(z_2+2)}{(z_2+1)(a+b)}, \qquad (167) \\
D_{b,H_7} &= \frac{(2-8z_1)z_2^2+4z_2+2}{(z_2+1)(a+b)(a+b-2d+2)}, \qquad (168)
\end{aligned}$$

$$\begin{aligned}
A_{c,H_7} &= A_{b,H_7}(b \leftrightarrow c), \qquad (169) \\
B_{c,H_7} &= B_{b,H_7}(b \leftrightarrow c), \qquad (170) \\
C_{c,H_7} &= C_{b,H_7}(b \leftrightarrow c), \qquad (171) \\
D_{c,H_7} &= D_{b,H_7}(b \leftrightarrow c), \qquad (172)
\end{aligned}$$

$$
\begin{aligned}
A_{d,H_7} &= -\frac{2(d-1)}{z_2(z_2+1)(a+b-2d+2)(a+b-2d+3)(a+c-2d+2)(a+c-2d+3)} \\
&\times \left(-abc + z_2(z_2+1)\left(2a^3 + a\left(2\left(b^2 - 6d(b+c) + 2bc + 8b + c^2 + 8d^2\right)\right.\right.\right. \\
&\left.\left.\left. +16c - 42d + 27\right) + b^2(c-2d+3) + b\left(-8cd + c(c+11) + 8d^2 - 22d + 15\right)\right)\right) \\
&- \frac{2a^2(d-1)(4b+4c-10d+13)}{(a+b-2d+2)(a+b-2d+3)(a+c-2d+2)(a+c-2d+3)} \\
&- \frac{2bc(d-1)z_2(2a+b+c-2d+2)}{(z_2+1)(a+b-2d+2)(a+b-2d+3)(a+c-2d+2)(a+c-2d+3)} \\
&+ \frac{2(d-1)(2d-3)\left(c^2 + c(5-4d) + 4d^2 - 10d + 6\right)}{(a+b-2d+2)(a+b-2d+3)(a+c-2d+2)(a+c-2d+3)}, \qquad (173) \\
B_{d,H_7} &= -\frac{(d-1)(4z_1-1)}{z_1 z_2(a+b-2d+2)(a+b-2d+3)(a+c-2d+2)(a+c-2d+3)} \\
&\times \left(z_2\left(a^2 + a(2b+2c-4d+5) + b^2 + b(c-4d+5)\right.\right. \\
&\left.\left. +(c-2d+2)(c-2d+3)\right) - bc\right), \qquad (174) \\
C_{d,H_7} &= \frac{2(d-1)}{z_2(z_2+1)(a+b-2d+2)(a+b-2d+3)(a+c-2d+2)(a+c-2d+3)} \\
&\times \left(-bc - z_2\left(a^2 - 2(a+1)d + 4a - b^2 + bc + b - c^2 + c + 3\right)\right. \\
&\left. +z_2^2\left(-\left(2a^2 + 2a(b+c-2d+3) - 2d(b+c+1) + 2bc + 3b + 3c + 3\right)\right)\right), \qquad (175) \\
D_{d,H_7} &= -\frac{(d-1)\left(z_2\left((4z_1-1)z_2 - 2\right) - 1\right)}{(a+b-2d+3)(a+c-2d+2)(a+c-2d+3)} \\
&\times \frac{(z_2(2a+b+c-4d+5) + a - 2d + 3)}{z_1 z_2^2 (z_2+1)(a+b-2d+2)}. \qquad (176)
\end{aligned}
$$

A.2 Confluent Horn-type hypergeometric functions

A.2.1 Function $\Phi_1(a, b, c, z_1, z_2)$

$$
\Phi_1(\text{a,b,c},z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1+n_2}(b)_{n_1}}{n_1!n_2!(c)_{n_1+n_2}} z_1^{n_1} z_2^{n_2}. \qquad (177)
$$

The additional PDE reads:

$$
\theta_1\theta_2\Phi_1(\vec{\gamma};\vec{\sigma}; z_1, z_2) = (\frac{z_2}{z_1}\theta_1 - b\theta_2)\Phi_1(\vec{\gamma};\vec{\sigma}; z_1, z_2). \qquad (178)
$$

$$
\begin{aligned}
A_{a,\Phi_1} &= \frac{bz_1+z_2}{a-c+1}+1, && (179)\\
B_{a,\Phi_1} &= \frac{z_1-1}{a-c+1}, && (180)\\
C_{a,\Phi_1} &= \frac{1}{-a+c-1}, && (181)\\
D_{a,\Phi_1} &= 0, && (182)
\end{aligned}
$$

$$
\begin{aligned}
A_{b,\Phi_1} &= 1, && (183)\\
B_{b,\Phi_1} &= 0, && (184)\\
C_{b,\Phi_1} &= -\frac{z_1}{z_2}, && (185)\\
D_{b,\Phi_1} &= 0, && (186)
\end{aligned}
$$

$$
\begin{aligned}
A_{c,\Phi_1} &= \frac{a}{-a+c-1}+1, && (187)\\
B_{c,\Phi_1} &= 0, && (188)\\
C_{c,\Phi_1} &= \frac{c-1}{z_2(a-c+1)}, && (189)\\
D_{c,\Phi_1} &= 0. && (190)
\end{aligned}
$$

A.2.2 Function $\Phi_2(b_1, b_2, c, z_1, z_2)$

$$
\Phi_2\left(b_1, b_2, \text{c}, z_1, z_2\right) = \sum_{n_1,n_2} \frac{(b_1)_{n_1}(b_2)_{n_2}}{n_1!n_2!(c)_{n_1+n_2}} z_1^{n_1} z_2^{n_2}. \quad (191)
$$

The additional PDE reads:

$$
\theta_1\theta_2\Phi_2(\vec{\gamma};\vec{\sigma};z_1,z_2) = (\frac{b_2 z_2}{z_1-z_2}\theta_1 - \frac{b_1 z_1}{z_1-z_2}\theta_2)\Phi_2(\vec{\gamma};\vec{\sigma};z_1,z_2). \quad (192)
$$

$$
\begin{aligned}
A_{b_1,\Phi_2} &= \frac{z_1}{b_1+b_2-c+1}+1, && (193)\\
B_{b_1,\Phi_2} &= -\frac{1}{b_1+b_2-c+1}, && (194)\\
C_{b_1,\Phi_2} &= -\frac{z_1}{z_2\left(b_1+b_2-c+1\right)}, && (195)\\
D_{b_1,\Phi_2} &= 0, && (196)
\end{aligned}
$$

$$
\begin{aligned}
A_{b_2,\Phi_2} &= A_{b_1,\Phi_2}(b_1 \leftrightarrow b_2, z_1 \leftrightarrow z_2), && (197)\\
B_{b_2,\Phi_2} &= C_{b_1,\Phi_2}(b_1 \leftrightarrow b_2, z_1 \leftrightarrow z_2), && (198)\\
C_{b_2,\Phi_2} &= B_{b_1,\Phi_2}(b_1 \leftrightarrow b_2, z_1 \leftrightarrow z_2), && (199)\\
D_{b_2,\Phi_2} &= 0, && (200)
\end{aligned}
$$

$$
\begin{aligned}
A_{c,\Phi_2} &= \frac{1-c}{b_1+b_2-c+1}, && (201)\\
B_{c,\Phi_2} &= -\frac{1-c}{z_1\,(b_1+b_2-c+1)}, && (202)\\
C_{c,\Phi_2} &= -\frac{1-c}{z_2\,(b_1+b_2-c+1)}, && (203)\\
D_{c,\Phi_2} &= 0. && (204)
\end{aligned}
$$

A.2.3 Function $\Phi_3(b, c, z_1, z_2)$

$$
\Phi_3(\text{b,c},z_1,z_2) = \sum_{n_1,n_2} \frac{(b)_{n_1}}{n_1! n_2! (c)_{n_1+n_2}} z_1^{n_1} z_2^{n_2}. \tag{205}
$$

The additional PDE reads:

$$
\theta_1\theta_2\Phi_3(\vec{\gamma};\vec{\sigma};z_1,z_2) = (\frac{z_2}{z_1}\theta_1 - b\theta_2)\Phi_3(\vec{\gamma};\vec{\sigma};z_1,z_2). \tag{206}
$$

$$
\begin{aligned}
A_{b,\Phi_3} &= 1, && (207)\\
B_{b,\Phi_3} &= 0, && (208)\\
C_{b,\Phi_3} &= -\frac{z_1}{z_2}, && (209)\\
D_{b,\Phi_3} &= 0, && (210)
\end{aligned}
$$

$$
\begin{aligned}
A_{c,\Phi_3} &= 0, && (211)\\
B_{c,\Phi_3} &= 0, && (212)\\
C_{c,\Phi_3} &= \frac{c-1}{z_2}, && (213)\\
D_{c,\Phi_3} &= 0. && (214)
\end{aligned}
$$

A.2.4 Function $\Psi_1(a, b, c_1, c_2, z_1, z_2)$

$$\Psi_1(\mathrm{a},\mathrm{b},c_1, c_2, z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1+n_2}(b)_{n_1}}{n_1!n_2!\,(c_1)_{n_1}\,(c_2)_{n_2}} z_1^{n_1} z_2^{n_2}, \tag{215}$$

$$\begin{aligned}
A_{a,\Psi_1} &= \frac{bz_1}{a-c_1+1} + \frac{z_2}{a-c_2+1} + 1, & (216)\\
B_{a,\Psi_1} &= \frac{\frac{z_2(-2a+c_1+c_2-2)}{(a-c_2+1)(a-c_1-c_2+2)} + z_1 - 1}{a-c_1+1}, & (217)\\
C_{a,\Psi_1} &= \frac{\frac{bz_1(-2a+c_1+c_2-2)}{(a-c_1+1)(a-c_1-c_2+2)} - 1}{a-c_2+1}, & (218)\\
D_{a,\Psi_1} &= -\frac{(z_1-1)\,(2a-c_1-c_2+2)}{(a-c_1+1)\,(a-c_2+1)\,(a-c_1-c_2+2)}, & (219)
\end{aligned}$$

$$\begin{aligned}
A_{b,\Psi_1} &= \frac{az_1}{b-c_1+1} + 1, & (220)\\
B_{b,\Psi_1} &= \frac{z_1-1}{b-c_1+1}, & (221)\\
C_{b,\Psi_1} &= \frac{z_1}{b-c_1+1}, & (222)\\
D_{b,\Psi_1} &= 0, & (223)
\end{aligned}$$

$$\begin{aligned}
A_{c_1,\Psi_1} &= \frac{(c_1-1)\,(a+b-c_1+1)}{(a-c_1+1)\,(-b+c_1-1)}, & (224)\\
B_{c_1,\Psi_1} &= \frac{(c_1-1)\,((z_1-1)\,(a-c_1-c_2+2)-z_2)}{z_1\,(a-c_1+1)\,(a-c_1-c_2+2)\,(-b+c_1-1)}, & (225)\\
C_{c_1,\Psi_1} &= \frac{b\,(c_1-1)}{(-a+c_1-1)\,(-a+c_1+c_2-2)\,(b-c_1+1)}, & (226)\\
D_{c_1,\Psi_1} &= \frac{(c_1-1)\,(z_1-1)}{z_1\,(a-c_1+1)\,(a-c_1-c_2+2)\,(b-c_1+1)}, & (227)
\end{aligned}$$

$$\begin{aligned}
A_{c_2,\Psi_1} &= \frac{a}{-a+c_2-1} + 1, & (228)\\
B_{c_2,\Psi_1} &= \frac{c_2-1}{(-a+c_2-1)\,(-a+c_1+c_2-2)}, & (229)\\
C_{c_2,\Psi_1} &= \frac{(c_2-1)\,(a+bz_1-c_1-c_2+2)}{z_2\,(a-c_2+1)\,(a-c_1-c_2+2)}, & (230)\\
D_{c_2,\Psi_1} &= \frac{(c_2-1)\,(z_1-1)}{z_2\,(-a+c_2-1)\,(-a+c_1+c_2-2)}. & (231)
\end{aligned}$$

A.2.5 Function $\Psi_2(a, c_1, c_2, z_1, z_2)$

$$\Psi_2(\text{a},c_1, c_{2,}z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1+n_2}}{n_1!n_2!\,(c_1)_{n_1}\,(c_2)_{n_2}} z_1^{n_1} z_2^{n_2}, \tag{232}$$

$$A_{a,\Psi_2} = \frac{z_1}{a-c_1+1} + \frac{z_2}{a-c_2+1} + 1, \tag{233}$$

$$B_{a,\Psi_2} = \frac{\frac{z_2(-2a+c_1+c_2-2)}{(a-c_2+1)(a-c_1-c_2+2)} - 1}{a-c_1+1}, \tag{234}$$

$$C_{a,\Psi_2} = \frac{\frac{z_1(-2a+c_1+c_2-2)}{(a-c_1+1)(a-c_1-c_2+2)} - 1}{a-c_2+1}, \tag{235}$$

$$D_{a,\Psi_2} = -\frac{-2(a+1)+c_1+c_2}{(a-c_1+1)\,(a-c_2+1)\,(a-c_1-c_2+2)}, \tag{236}$$

$$A_{c_1,\Psi_2} = \frac{a}{-a+c_1-1} + 1, \tag{237}$$

$$B_{c_1,\Psi_2} = \frac{(c_1-1)\,(a-c_1-c_2+z_2+2)}{z_1\,(a-c_1+1)\,(a-c_1-c_2+2)}, \tag{238}$$

$$C_{c_1,\Psi_2} = \frac{c_1-1}{(-a+c_1-1)\,(-a+c_1+c_2-2)}, \tag{239}$$

$$D_{c_1,\Psi_2} = -\frac{c_1-1}{z_1\,(-a+c_1-1)\,(-a+c_1+c_2-2)}, \tag{240}$$

$$A_{c_2,\Psi_2} = A_{c_1,\Psi_2}(c_1 \leftrightarrow c_2, z_1 \leftrightarrow z_2), \tag{241}$$

$$B_{c_2,\Psi_2} = C_{c_1,\Psi_2}(c_1 \leftrightarrow c_2, z_1 \leftrightarrow z_2), \tag{242}$$

$$C_{c_2,\Psi_2} = B_{c_1,\Psi_2}(c_1 \leftrightarrow c_2, z_1 \leftrightarrow z_2), \tag{243}$$

$$D_{c_2,\Psi_2} = D_{c_1,\Psi_2}(c_1 \leftrightarrow c_2, z_1 \leftrightarrow z_2). \tag{244}$$

A.2.6 Function $\Theta_1(a_1, a_2, b, c, z_1, z_2)$

$$\Theta_1\,(a_1, a_2,\text{b,c},z_1, z_2) = \sum_{n_1,n_2} \frac{(a_1)_{n_1}\,(a_2)_{n_2}\,(b)_{n_1}}{n_1!n_2!(c)_{n_1+n_2}} z_1^{n_1} z_2^{n_2}, \tag{245}$$

$$A_{a_1,\Theta_1} = \frac{bz_1}{a_1+a_2-c+1} + 1, \tag{246}$$

$$B_{a_1,\Theta_1} = -\frac{1-z_1}{a_1+a_2-c+1}, \tag{247}$$

$$C_{a_1,\Theta_1} = -\frac{bz_1}{z_2\,(a_1+a_2-c+1)}, \tag{248}$$

$$D_{a_1,\Theta_1} = -\frac{z_1}{z_2\,(a_1+a_2-c+1)}, \tag{249}$$

$$A_{a_2,\Theta_1} = -\frac{(a_2+b-c+1)(-a_2+c-z_2-1)-a_1(a_2+b-c+z_2+1)}{(a_1+a_2-c+1)(a_2+b-c+1)}, \quad (250)$$

$$B_{a_2,\Theta_1} = \frac{(z_1-1)z_2}{z_1(a_1+a_2-c+1)(a_2+b-c+1)}, \quad (251)$$

$$C_{a_2,\Theta_1} = -\frac{a_1+a_2+b-c+1}{(a_1+a_2-c+1)(a_2+b-c+1)}, \quad (252)$$

$$D_{a_2,\Theta_1} = -\frac{1}{(a_1+a_2-c+1)(a_2+b-c+1)}, \quad (253)$$

$$A_{b,\Theta_1} = \frac{a_1 z_1}{a_2+b-c+1}+1, \quad (254)$$

$$B_{b,\Theta_1} = \frac{z_1-1}{a_2+b-c+1}, \quad (255)$$

$$C_{b,\Theta_1} = -\frac{a_1 z_1}{z_2(a_2+b-c+1)}, \quad (256)$$

$$D_{b,\Theta_1} = -\frac{z_1}{z_2(a_2+b-c+1)}, \quad (257)$$

$$A_{c,\Theta_1} = -\frac{(c-1)(a_1+a_2+b-c+1)}{(a_1+a_2-c+1)(a_2+b-c+1)}, \quad (258)$$

$$B_{c,\Theta_1} = -\frac{(c-1)(z_1-1)}{z_1(a_1+a_2-c+1)(a_2+b-c+1)}, \quad (259)$$

$$C_{c,\Theta_1} = \frac{(c-1)(a_1+a_2+b-c+1)}{z_2(a_1+a_2-c+1)(a_2+b-c+1)}, \quad (260)$$

$$D_{c,\Theta_1} = \frac{c-1}{z_2(a_1+a_2-c+1)(a_2+b-c+1)}. \quad (261)$$

A.2.7 Function $\Theta_2(a,b,c,z_1,z_2)$

$$\Theta_2(\mathrm{a,b,c},z_1,z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1}(b)_{n_1}}{n_1!n_2!(c)_{n_1+n_2}} z_1^{n_1} z_2^{n_2}, \quad (262)$$

$$A_{a,\Theta_2} = 1, \quad (263)$$

$$B_{a,\Theta_2} = 0, \quad (264)$$

$$C_{a,\Theta_2} = -\frac{bz_1}{z_2}, \quad (265)$$

$$D_{a,\Theta_2} = -\frac{z_1}{z_2}, \quad (266)$$

$$
\begin{aligned}
A_{b,\Theta_2} &= 1, &(267)\\
B_{b,\Theta_2} &= 0, &(268)\\
C_{b,\Theta_2} &= -\frac{az_1}{z_2}, &(269)\\
D_{b,\Theta_2} &= -\frac{z_1}{z_2}, &(270)
\end{aligned}
$$

$$
\begin{aligned}
A_{c,\Theta_2} &= 0, &(271)\\
B_{c,\Theta_2} &= 0, &(272)\\
C_{c,\Theta_2} &= \frac{c-1}{z_2}, &(273)\\
D_{c,\Theta_2} &= 0. &(274)
\end{aligned}
$$

A.2.8 Function $\Gamma_1(a, b_1, b_2, z_1, z_2)$

$$
\Gamma_1(\mathrm{a}, b_1, b_2, z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1}\,(b_1)_{n_2-n_1}\,(b_2)_{n_1-n_2}}{n_1!n_2!} z_1^{n_1} z_2^{n_2}. \tag{275}
$$

The additional PDE reads:

$$
\theta_1\theta_2\Gamma_1(\vec{\gamma}; z_1, z_2) = (az_1z_2 + z_1z_2\theta_1)\Gamma_1(\vec{\gamma}; z_1, z_2). \tag{276}
$$

$$
\begin{aligned}
A_{a,\Gamma_1} &= \frac{z_1\,(z_2-b_2)}{a+b_1} + 1, &(277)\\
B_{a,\Gamma_1} &= -\frac{z_1+1}{a+b_1}, &(278)\\
C_{a,\Gamma_1} &= \frac{z_1}{a+b_1}, &(279)\\
D_{a,\Gamma_1} &= 0, &(280)
\end{aligned}
$$

$$
\begin{aligned}
A_{b_1,\Gamma_1} &= \frac{b_1\,(a+b_1+b_2-z_2)}{(b_1+b_2)\,(a+b_1)}, &(281)\\
B_{b_1,\Gamma_1} &= \frac{b_1\,(z_1+1)}{(b_1+b_2)\,z_1\,(a+b_1)}, &(282)\\
C_{b_1,\Gamma_1} &= -\frac{b_1}{(b_1+b_2)\,(a+b_1)}, &(283)\\
D_{b_1,\Gamma_1} &= 0, &(284)
\end{aligned}
$$

$$
\begin{aligned}
A_{b_2,\Gamma_1} &= \frac{b_2}{b_1+b_2}, & (285)\\
B_{b_2,\Gamma_1} &= 0, & (286)\\
C_{b_2,\Gamma_1} &= \frac{b_2}{(b_1+b_2)\,z_2}, & (287)\\
D_{b_2,\Gamma_1} &= 0. & (288)
\end{aligned}
$$

A.2.9 Function $\boldsymbol{\Gamma_2(b_1, b_2, z_1, z_2)}$

$$
\Gamma_2\left(b_1, b_2, z_1, z_2\right) = \sum_{n_1,n_2} \frac{\left(b_1\right)_{n_2-n_1}\left(b_2\right)_{n_1-n_2}}{n_1!n_2!} z_1^{n_1} z_2^{n_2}. \tag{289}
$$

The additional PDE reads:

$$
\theta_1\theta_2\Gamma_2(\vec{\gamma}; z_1, z_2) = z_1 z_2 \Gamma_2(\vec{\gamma}; z_1, z_2). \tag{290}
$$

$$
\begin{aligned}
A_{b_1,\Gamma_2} &= \frac{b_1}{b_1+b_2}, & (291)\\
B_{b_1,\Gamma_2} &= \frac{b_1}{(b_1+b_2)\,z_1}, & (292)\\
C_{b_1,\Gamma_2} &= 0, & (293)\\
D_{b_1,\Gamma_2} &= 0, & (294)
\end{aligned}
$$

$$
\begin{aligned}
A_{b_2,\Gamma_2} &= A_{b_1,\Gamma_2}(b_1 \leftrightarrow b_2, z_1 \leftrightarrow z_2), & (295)\\
B_{b_2,\Gamma_2} &= 0, & (296)\\
C_{b_2,\Gamma_2} &= B_{b_1,\Gamma_2}(b_1 \leftrightarrow b_2, z_1 \leftrightarrow z_2), & (297)\\
D_{b_2,\Gamma_2} &= 0. & (298)
\end{aligned}
$$

A.2.10 Function $\boldsymbol{H_1(a, b, c, z_1, z_2)}$

$$
H_1(\mathrm{a,b,c},z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1-n_2}(b)_{n_1+n_2}}{n_1!n_2!(c)_{n_1}} z_1^{n_1} z_2^{n_2}, \tag{299}
$$

$$A_{a,H_1^c} = \frac{a\left(a+4bz_1+b-2c+2\right)}{(a+b)(a+b-2c+2)}, \tag{300}$$

$$B_{a,H_1^c} = \frac{2a\left(2z_1-1\right)}{(a+b)(a+b-2c+2)}, \tag{301}$$

$$C_{a,H_1^c} = \frac{a\left(\frac{4z_1}{a+b-2c+2}+\frac{1}{z_2}\right)}{a+b}, \tag{302}$$

$$D_{a,H_1^c} = -\frac{2a}{z_2(a+b)(a+b-2c+2)}, \tag{303}$$

$$A_{b,H_1^c} = -\frac{2(b+1)z_2z_1(a+3b-2c+2)}{(a+b)(b-c+1)(a+b-2c+2)}+\frac{az_1}{b-c+1}+\frac{a+b-z_2}{a+b}, \tag{304}$$

$$B_{b,H_1^c} = \frac{-\frac{(2z_1-1)z_2(a+3b-2c+2)}{(a+b)(a+b-2c+2)}+z_1-1}{b-c+1}, \tag{305}$$

$$C_{b,H_1^c} = -\frac{(a+b-2c+2)\left(z_1(a+b)+b-c+1\right)+2z_1z_2(a+3b-2c+2)}{(a+b)(b-c+1)(a+b-2c+2)}, \tag{306}$$

$$D_{b,H_1^c} = \frac{a+3b-2c+2}{(a+b)(b-c+1)(a+b-2c+2)}, \tag{307}$$

$$A_{c,H_1^c} = -\frac{(c-1)\left(a^2+a(3b-4c+5)+2(2b-2c+3)(b-c+1)-2bz_2\right)}{(b-c+1)(a+b-2c+2)(a+b-2c+3)}, \tag{308}$$

$$B_{c,H_1^c} = \frac{(c-1)\left(-z_1\left(a+3b-4c-2z_2+5\right)+a+2b-3c-z_2+4\right)}{z_1(b-c+1)(a+b-2c+2)(a+b-2c+3)}, \tag{309}$$

$$C_{c,H_1^c} = \frac{(c-1)\left(a-b+2z_2+1\right)}{(b-c+1)(a+b-2c+2)(a+b-2c+3)}, \tag{310}$$

$$D_{c,H_1^c} = \frac{(c-1)\left(b-c-z_2+1\right)}{z_1z_2(b-c+1)(a+b-2c+2)(a+b-2c+3)}. \tag{311}$$

A.2.11 Function $H_2(a,b,c,d,z_1,z_2)$

$$H_2(\text{a,b,c,d},z_1,z_2) = \sum_{n_1,n_2}\frac{(a)_{n_1-n_2}(b)_{n_1}(c)_{n_2}}{n_1!n_2!(d)_{n_1}}z_1^{n_1}z_2^{n_2}, \tag{312}$$

$$A_{a,H_2^c} = \frac{a\,(a+bz_1+c-d+1)}{(a+c)(a+c-d+1)}, \tag{313}$$

$$B_{a,H_2^c} = \frac{a\,(z_1-1)}{(a+c)(a+c-d+1)}, \tag{314}$$

$$C_{a,H_2^c} = \frac{a}{z_2(a+c)}, \tag{315}$$

$$D_{a,H_2^c} = -\frac{a}{z_2(a+c)(a+c-d+1)}, \tag{316}$$

$$A_{b,H_2^c} = \frac{az_1}{b-d+1}+1, \tag{317}$$

$$B_{b,H_2^c} = \frac{z_1-1}{b-d+1}, \tag{318}$$

$$C_{b,H_2^c} = -\frac{z_1}{b-d+1}, \tag{319}$$

$$D_{b,H_2^c} = 0, \tag{320}$$

$$A_{c,H_2^c} = 1-\frac{z_2\,(a+bz_1+c-d+1)}{(a+c)(a+c-d+1)}, \tag{321}$$

$$B_{c,H_2^c} = -\frac{(z_1-1)\,z_2}{(a+c)(a+c-d+1)}, \tag{322}$$

$$C_{c,H_2^c} = -\frac{1}{a+c}, \tag{323}$$

$$D_{c,H_2^c} = \frac{1}{(a+c)(a+c-d+1)}, \tag{324}$$

$$A_{d,H_2^c} = -\frac{(d-1)(a+b+c-d+1)}{(b-d+1)(a+c-d+1)}, \tag{325}$$

$$B_{d,H_2^c} = -\frac{(d-1)\,(z_1-1)}{z_1(b-d+1)(a+c-d+1)}, \tag{326}$$

$$C_{d,H_2^c} = 0, \tag{327}$$

$$D_{d,H_2^c} = \frac{d-1}{z_1z_2(b-d+1)(a+c-d+1)}. \tag{328}$$

A.2.12 Function $H_3(a,b,c,z_1,z_2)$

$$H_3(\text{a,b,c},z_1,z_2) = \sum_{n_1,n_2}\frac{(a)_{n_1-n_2}(b)_{n_1}}{n_1!n_2!(c)_{n_1}}z_1^{n_1}z_2^{n_2}, \tag{329}$$

$$
\begin{aligned}
A_{a,H_3^c} &= 0, & (330)\\
B_{a,H_3^c} &= 0, & (331)\\
C_{a,H_3^c} &= \frac{a}{z_2}, & (332)\\
D_{a,H_3^c} &= 0, & (333)
\end{aligned}
$$

$$
\begin{aligned}
A_{b,H_3^c} &= \frac{az_1}{b-c+1}+1, & (334)\\
B_{b,H_3^c} &= \frac{z_1-1}{b-c+1}, & (335)\\
C_{b,H_3^c} &= -\frac{z_1}{b-c+1}, & (336)\\
D_{b,H_3^c} &= 0, & (337)
\end{aligned}
$$

$$
\begin{aligned}
A_{c,H_3^c} &= \frac{b}{-b+c-1}+1, & (338)\\
B_{c,H_3^c} &= 0, & (339)\\
C_{c,H_3^c} &= 0, & (340)\\
D_{c,H_3^c} &= \frac{c-1}{z_1 z_2(b-c+1)}. & (341)
\end{aligned}
$$

A.2.13 Function $H_4(a, b, c, z_1, z_2)$

$$
H_4(\text{a,b,c},z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1-n_2}(b)_{n_2}}{n_1!n_2!(c)_{n_1}} z_1^{n_1} z_2^{n_2}, \tag{342}
$$

$$
\begin{aligned}
A_{a,H_4^c} &= \frac{a\,(a+b-c+z_1+1)}{(a+b)(a+b-c+1)}, & (343)\\
B_{a,H_4^c} &= -\frac{a}{(a+b)(a+b-c+1)}, & (344)\\
C_{a,H_4^c} &= \frac{a}{z_2(a+b)}, & (345)\\
D_{a,H_4^c} &= -\frac{a}{z_2(a+b)(a+b-c+1)}, & (346)
\end{aligned}
$$

$$A_{b,H_4^c} = 1 - \frac{z_2\,(a+b-c+z_1+1)}{(a+b)(a+b-c+1)}, \quad (347)$$

$$B_{b,H_4^c} = \frac{z_2}{(a+b)(a+b-c+1)}, \quad (348)$$

$$C_{b,H_4^c} = -\frac{1}{a+b}, \quad (349)$$

$$D_{b,H_4^c} = \frac{1}{(a+b)(a+b-c+1)}, \quad (350)$$

$$A_{c,H_4^c} = \frac{1-c}{a+b-c+1}, \quad (351)$$

$$B_{c,H_4^c} = \frac{c-1}{z_1(a+b-c+1)}, \quad (352)$$

$$C_{c,H_4^c} = 0, \quad (353)$$

$$D_{c,H_4^c} = \frac{c-1}{z_1 z_2(a+b-c+1)}. \quad (354)$$

A.2.14 Function $H_5(a, b, z_1, z_2)$

$$H_5(\text{a,b},z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1-n_2}}{n_1!n_2!(b)_{n_1}} z_1^{n_1} z_2^{n_2}, \quad (355)$$

$$A_{a,H_5^c} = 0, \quad (356)$$

$$B_{a,H_5^c} = 0, \quad (357)$$

$$C_{a,H_5^c} = \frac{a}{z_2}, \quad (358)$$

$$D_{a,H_5^c} = 0, \quad (359)$$

$$A_{b,H_5^c} = 0, \quad (360)$$

$$B_{b,H_5^c} = 0, \quad (361)$$

$$C_{b,H_5^c} = 0, \quad (362)$$

$$D_{b,H_5^c} = \frac{b-1}{z_1 z_2}. \quad (363)$$

A.2.15 Function $H_6(a, b, z_1, z_2)$

$$H_6\left(a,b,z_1,z_2\right) = \sum_{n_1,n_2} \frac{(a)_{2n_1+n_2}}{n_1!n_2!(b)_{n_1+n_2}} z_1^{n_1} z_2^{n_2}. \tag{364}$$

The additional PDE reads:

$$(-z_2\theta_1 + az_1\theta_2 + z_1{\theta_2}^2 + 2z_1\theta_1\theta_2)H_6(\vec{\gamma};\vec{\sigma};z_1,z_2) = 0. \tag{365}$$

$$A_{a,H_6^c} = \frac{2(a+1)z_1 + a - b + z_2 + 1}{a-b+1}, \tag{366}$$

$$B_{a,H_6^c} = \frac{4z_1 - 1}{a-b+1}, \tag{367}$$

$$C_{a,H_6^c} = -\frac{z_1\left(a - 2z_2\right) + z_2}{z_2(a-b+1)}, \tag{368}$$

$$D_{a,H_6^c} = 0. \tag{369}$$

$$A_{b,H_6^c} = \frac{(b-1)\left(2az_1 + z_2\right)}{z_2(-a+b-1)}, \tag{370}$$

$$B_{b,H_6^c} = -\frac{(b-1)\left(4z_1 - 1\right)}{z_2(a-b+1)}, \tag{371}$$

$$C_{b,H_6^c} = \frac{(b-1)\left(z_2 - z_1\left(a - 2b + 2z_2 + 3\right)\right)}{z_2^2(a-b+1)}, \tag{372}$$

$$D_{b,H_6^c} = 0. \tag{373}$$

A.2.16 Function $H_7(a, c, d, z_1, z_2)$

$$H_7(\text{a,c,d},z_1,z_2) = \sum_{n_1,n_2} \frac{(a)_{2n_1+n_2}}{n_1!n_2!(c)_{n_1}(d)_{n_2}} z_1^{n_1} z_2^{n_2}, \tag{374}$$

$$A_{a,H_7^c} = \frac{4(a+1)z_2z_1(-2a+2c+d-3)}{(a-2c+2)(a-d+1)(a-2c-d+3)} + \frac{4(a+1)z_1}{a-2c+2} + \frac{a-d+z_2+1}{a-d+1}, \quad (375)$$

$$B_{a,H_7^c} = \frac{2\left(-\frac{(4z_1+1)z_2(2a-2c-d+3)}{(a-d+1)(a-2c-d+3)} + 4z_1 - 1\right)}{a-2c+2}, \quad (376)$$

$$C_{a,H_7^c} = \frac{4z_1\left(z_2(-2a+2c+d-3) + a(-a+d-3) + 2c + d - 3\right)}{(a-2c+2)(a-d+1)(a-2c-d+3)} + \frac{1}{-a+d-1}, \quad (377)$$

$$D_{a,H_7^c} = -\frac{2\left(4z_1-1\right)\left(2a-2c-d+3\right)}{(a-2c+2)(a-d+1)(a-2c-d+3)}, \quad (378)$$

$$A_{c,H_7^c} = \frac{2a(c-1)z_2\left(a-2c+z_2+2\right)}{(a-2c+2)(a-2c+3)(a-2c-d+3)(a-2c-d+4)} - \frac{2(c-1)(2a-2c+3)}{(a-2c+2)(a-2c+3)}, \quad (379)$$

$$B_{c,H_7^c} = \frac{4(c-1)z_2}{(a-2c+3)(a-2c-d+3)(a-2c-d+4)} + \frac{(c-1)z_2\left(3a-2(3c+d-5)+\left(4z_1+1\right)z_2\right)}{z_1(a-2c+2)(a-2c+3)(a-2c-d+3)(a-2c-d+4)} - \frac{(c-1)\left(4z_1-1\right)}{z_1(a-2c+2)(a-2c+3)}, \quad (380)$$

$$C_{c,H_7^c} = \frac{4(a+1)(c-1)}{(a-2c+2)(a-2c-d+3)(a-2c-d+4)} - \frac{2(c-1)\left(d-z_2\right)\left(2a-2c+z_2+3\right)}{(a-2c+2)(a-2c+3)(a-2c-d+3)(a-2c-d+4)}, \quad (381)$$

$$D_{c,H_7^c} = \frac{(c-1)\left(4z_1-1\right)\left(2a-4c-d+z_2+6\right)}{z_1(a-2c+2)(a-2c+3)(a-2c-d+3)(a-2c-d+4)}, \quad (382)$$

$$A_{d,H_7^c} = \frac{(d-1)\left(4az_1-a+2c+d-3\right)}{(a-d+1)(a-2c-d+3)}, \quad (383)$$

$$B_{d,H_7^c} = \frac{2(d-1)\left(4z_1+1\right)}{(a-d+1)(a-2c-d+3)}, \quad (384)$$

$$C_{d,H_7^c} = \frac{(d-1)\left(4z_1\left(a-d+z_2+2\right)+a-2c-d+3\right)}{z_2(a-d+1)(a-2c-d+3)}, \quad (385)$$

$$D_{d,H_7^c} = \frac{2(d-1)\left(4z_1-1\right)}{z_2(a-d+1)(a-2c-d+3)}. \quad (386)$$

A.2.17 Function $H_8(a, b, z_1, z_2)$

$$H_8\left(a,b,z_1,z_2\right)=\sum_{n_1,n_2}\frac{(a)_{2n_1-n_2}(b)_{n_2-n_1}}{n_1!n_2!}z_1^{n_1}z_2^{n_2}. \tag{387}$$

The additional PDE reads:

$$(-az_1z_2-2z_1z_2\theta_1+z_1z_2\theta_2+\theta_1\theta_2)H_8(\vec{\gamma};z_1,z_2)=0. \tag{388}$$

$$A_{a,H_8^c} = \frac{a\left(z_1\left(z_2-2(a+1)\right)+a+2b\right)}{(a+b)(a+2b)}, \tag{389}$$

$$B_{a,H_8^c} = -\frac{a\left(4z_1+1\right)}{(a+b)(a+2b)}, \tag{390}$$

$$C_{a,H_8^c} = \frac{a\left(a+2b+2z_1z_2\right)}{z_2(a+b)(a+2b)}, \tag{391}$$

$$D_{a,H_8^c} = 0, \tag{392}$$

$$A_{b,H_8^c} = \frac{b\left(2(a+b)(2a+2b+1)-z_2\left(z_1\left(z_2-2a\right)+2a+3b+1\right)\right)}{(a+b)(a+2b)(a+2b+1)}, \tag{393}$$

$$B_{b,H_8^c} = \frac{b\left(4z_1+1\right)\left(a+b+z_1z_2\right)}{z_1(a+b)(a+2b)(a+2b+1)}, \tag{394}$$

$$C_{b,H_8^c} = -\frac{b\left(3a+4b+2z_1z_2+1\right)}{(a+b)(a+2b)(a+2b+1)}, \tag{395}$$

$$D_{b,H_8^c} = 0. \tag{396}$$

A.2.18 Function $H_9(a, b, c, z_1, z_2)$

$$H_9(\text{a,b,c},z_1,z_2)=\sum_{n_1,n_2}\frac{(a)_{2n_1-n_2}(b)_{n_2}}{n_1!n_2!(c)_{n_1}}z_1^{n_1}z_2^{n_2}, \tag{397}$$

$$A_{a,H_9^c} = \frac{a\left(4(a+1)z_1+a+b-2c+2\right)}{(a+b)(a+b-2c+2)}, \tag{398}$$

$$B_{a,H_9^c} = \frac{2a\left(4z_1-1\right)}{(a+b)(a+b-2c+2)}, \tag{399}$$

$$C_{a,H_9^c} = \frac{a\left(\frac{1}{z_2}-\frac{4z_1}{a+b-2c+2}\right)}{a+b}, \tag{400}$$

$$D_{a,H_9^c} = -\frac{2a}{z_2(a+b)(a+b-2c+2)}, \tag{401}$$

$$A_{b,H_9^c} = 1 - \frac{z_2\,(4az_1 + a + b - 2c + 2)}{(a+b)(a+b-2c+2)}, \tag{402}$$

$$B_{b,H_9^c} = \frac{2\,(1-4z_1)\,z_2}{(a+b)(a+b-2c+2)}, \tag{403}$$

$$C_{b,H_9^c} = -\frac{a+b-2c-4z_1z_2+2}{(a+b)(a+b-2c+2)}, \tag{404}$$

$$D_{b,H_9^c} = \frac{2}{(a+b)(a+b-2c+2)}, \tag{405}$$

$$A_{c,H_9^c} = \frac{2(c-1)\,(ab - z_2(2a+b-2c+3))}{z_2(a+b-2c+2)(a+b-2c+3)}, \tag{406}$$

$$B_{c,H_9^c} = \frac{(c-1)\,(4z_1-1)\,(b-z_2)}{z_1z_2(a+b-2c+2)(a+b-2c+3)}, \tag{407}$$

$$C_{c,H_9^c} = -\frac{2(c-1)\,(b-z_2)}{z_2(a+b-2c+2)(a+b-2c+3)}, \tag{408}$$

$$D_{c,H_9^c} = \frac{(c-1)\,(a-2c+z_2+3)}{z_1z_2^2(a+b-2c+2)(a+b-2c+3)}. \tag{409}$$

A.2.19 Function $H_{10}(a, c, z_1, z_2)$

$$H_{10}(\text{a,c},z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{2n_1-n_2}}{n_1!n_2!(c)_{n_1}} z_1^{n_1} z_2^{n_2}, \tag{410}$$

$$A_{a,H_{10}^c} = 0, \tag{411}$$

$$B_{a,H_{10}^c} = 0, \tag{412}$$

$$C_{a,H_{10}^c} = \frac{a}{z_2}, \tag{413}$$

$$D_{a,H_{10}^c} = 0, \tag{414}$$

$$A_{c,H_{10}^c} = \frac{2a(c-1)}{z_2}, \tag{415}$$

$$B_{c,H_{10}^c} = \frac{(c-1)\,(4z_1-1)}{z_1z_2}, \tag{416}$$

$$C_{c,H_{10}^c} = \frac{2-2c}{z_2}, \tag{417}$$

$$D_{c,H_{10}^c} = \frac{(c-1)(a-2c+3)}{z_1z_2^2}. \tag{418}$$

A.2.20 Function $H_{11}(a, b, c, d, z_1, z_2)$

$$H_{11}(\text{a,b,c,d},z_1, z_2) = \sum_{n_1,n_2} \frac{(a)_{n_1-n_2}(b)_{n_2}(c)_{n_2}}{n_1!n_2!(d)_{n_1}} z_1^{n_1} z_2^{n_2}, \tag{419}$$

$$\begin{aligned} A_{a,H_{11}^c} &= \frac{z_1\left(a^2 + a(b + 2c - d + 1) + c(-b + c - d + 1)\right)}{(a + c)(a + b - d + 1)(a + c - d + 1)} \\ &+ \frac{bcz_1(b - c + d - 1)}{(a + b)(a + c)(a + b - d + 1)(a + c - d + 1)} \\ &+ \frac{a(a + b + c)}{(a + b)(a + c)}, \end{aligned} \tag{420}$$

$$B_{a,H_{11}^c} = \frac{a\left(\frac{c}{(a+b)(a+b-d+1)} - \frac{b}{(a+c)(a+c-d+1)}\right)}{b - c}, \tag{421}$$

$$\begin{aligned} C_{a,H_{11}^c} &= \frac{az_1(2a + b + c - d + 1)}{(a + b)(a + c)(a + b - d + 1)(a + c - d + 1)} \\ &+ \frac{a\,(z_2 + 1)}{z_2(a + b)(a + c)}, \end{aligned} \tag{422}$$

$$D_{a,H_{11}^c} = -\frac{a\,(z_2 + 1)\,(2a + b + c - d + 1)}{z_2(a + b)(a + c)(a + b - d + 1)(a + c - d + 1)}, \tag{423}$$

$$A_{b,H_{11}^c} = 1 - \frac{cz_2\,(a + b - d + z_1 + 1)}{(a + b)(a + b - d + 1)}, \tag{424}$$

$$B_{b,H_{11}^c} = \frac{cz_2}{(a + b)(a + b - d + 1)}, \tag{425}$$

$$C_{b,H_{11}^c} = -\frac{z_2\,(a + b - d + z_1 + 1) + a + b - d + 1}{(a + b)(a + b - d + 1)}, \tag{426}$$

$$D_{b,H_{11}^c} = \frac{z_2 + 1}{(a + b)(a + b - d + 1)}, \tag{427}$$

$$A_{c,H_{11}^c} = A_{b,H_{11}^c}(b \leftrightarrow c), \tag{428}$$

$$B_{c,H_{11}^c} = B_{b,H_{11}^c}(b \leftrightarrow c), \tag{429}$$

$$C_{c,H_{11}^c} = C_{b,H_{11}^c}(b \leftrightarrow c), \tag{430}$$

$$D_{c,H_{11}^c} = D_{b,H_{11}^c}(b \leftrightarrow c), \tag{431}$$

$$A_{d,H^c_{11}} = -\frac{(d-1)(a+b+c-d+1)}{(a+b-d+1)(a+c-d+1)}, \quad (432)$$

$$B_{d,H^c_{11}} = \frac{(d-1)(a+b+c-d+1)}{z_1(a+b-d+1)(a+c-d+1)}, \quad (433)$$

$$C_{d,H^c_{11}} = \frac{1-d}{(a+b-d+1)(a+c-d+1)}, \quad (434)$$

$$D_{d,H^c_{11}} = \frac{(d-1)(z_2+1)}{z_1 z_2(a+b-d+1)(a+c-d+1)}. \quad (435)$$

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Table 1: List of Horn-type hypergeometric functions implemented in the program package **HornFunctions** and their respective operators.

Horn-type function	HYPERDIRE
$G_1(a,b_1,b_2,z_1,z_2)$	G1IndexChange[...]
$G_2(a_1,a_2,b_1,b_2,z_1,z_2)$	G2IndexChange[...]
$G_3(a_1,a_2,z_1,z_2)$	G3IndexChange[...]
$H_1(a,b,c,d,z_1,z_2)$	H1IndexChange[...]
$H_2(a,b,c,d,e,z_1,z_2)$	H2IndexChange[...]
$H_3(a,b,c,z_1,z_2)$	H3IndexChange[...]
$H_4(a,b,c,d,z_1,z_2)$	H4IndexChange[...]
$H_5(a,b,c,z_1,z_2)$	H5IndexChange[...]
$H_6(a,b,c,z_1,z_2)$	H6IndexChange[...]
$H_7(a,b,c,d,z_1,z_2)$	H7IndexChange[...]
confluent series:	
$\Phi_1(a,b,c,z_1,z_2)$	Phi1IndexChange[...]
$\Phi_2(b_1,b_2,z_1,z_2)$	Phi2IndexChange[...]
$\Phi_3(b,c,z_1,z_2)$	Phi3IndexChange[...]
$\Psi_1(a,b,c_1,c_2,z_1,z_2)$	Psi1IndexChange[...]
$\Psi_2(a,b,c,z_1,z_2)$	Psi2IndexChange[...]
$\Theta_1(a_1,a_2,b,c,z_1,z_2)$	Theta1IndexChange[...]
$\Theta_2(a,b,c,z_1,z_2)$	Theta2IndexChange[...]
$\Gamma_1(a,b_1,b_2,c,z_1,z_2)$	Gamma1IndexChange[...]
$\Gamma_2(b_1,b_2,c,z_1,z_2)$	Gamma2IndexChange[...]
$H_1(a,b,c,z_1,z_2)$	H1cIndexChange[...]
$H_2(a,b,c,d,z_1,z_2)$	H2cIndexChange[...]
$H_3(a,b,c,z_1,z_2)$	H3cIndexChange[...]
$H_4(a,c,d,z_1,z_2)$	H4cIndexChange[...]
$H_5(a,b,z_1,z_2)$	H5cIndexChange[...]
$H_6(a,c,z_1,z_2)$	H6cIndexChange[...]
$H_7(a,c,d,z_1,z_2)$	H7cIndexChange[...]
$H_8(a,b,z_1,z_2)$	H8cIndexChange[...]
$H_9(a,b,c,z_1,z_2)$	H9cIndexChange[...]
$H_{10}(a,b,z_1,z_2)$	H10cIndexChange[...]
$H_{11}(a,b,c,d,z_1,z_2)$	H11cIndexChange[...]

Table 2: Sets of exceptional parameters for the Horn-type hypergeometrc functions implemented in the program package **HornFunctions**.

Horn-type function	set of exceptional parameters
$G_1(a,b_1,b_2,z_1,z_2)$	$\{a, a+b_i, b_1+b_2\} \in \mathbb{Z}$
$G_2(a_1,a_2,b_1,b_2,z_1,z_2)$	$\{a_i, a_i+b_i, b_1+b_2\} \in \mathbb{Z}$
$G_3(a_1,a_2,z_1,z_2)$	$\{2a_1+a_2, 2a_2+a_1\} \in \mathbb{Z}$
$H_1(a,b,c,d,z_1,z_2)$	$\{b, c, a+b, a+c, a+b-2d, a+c-d, b-d\} \in \mathbb{Z}$
$H_2(a,b,c,d,e,z_1,z_2)$	$\{b, c, d, a+c, a+d, a+c-e, a+d-e, b-e\} \in \mathbb{Z}$
$H_3(a,b,c,z_1,z_2)$	$\{a, b, a-c, a+b-2c\} \in \mathbb{Z}$
$H_4(a,b,c,d,z_1,z_2)$	$\{a, b, a-2c, a-d, a-2c-d, b-d\} \in \mathbb{Z}$
$H_5(a,b,c,z_1,z_2)$	$\{a, a+2b, a-c, a+2b-3c\} \in \mathbb{Z}$
$H_6(a,b,c,z_1,z_2)$	$\{c, a+b, a+c, a+2b\} \in \mathbb{Z}$
$H_7(a,b,c,d,z_1,z_2)$	$\{b, c, a+b, a+c, a+b-2d, a+c-2d\} \in \mathbb{Z}$
confluent series:	
$\Phi_1(a,b,c,z_1,z_2)$	$\{a, b, a-c\} \in \mathbb{Z}$
$\Phi_2(b_1,b_2,z_1,z_2)$	$\{b_i, b_1+b_2-c\} \in \mathbb{Z}$
$\Phi_3(b,c,z_1,z_2)$	$\{b\} \in \mathbb{Z}$
$\Psi_1(a,b,c_1,c_2,z_1,z_2)$	$\{a, b, a-c_i, a-c_1-c_2, b-c_1\} \in \mathbb{Z}$
$\Psi_2(a,b,c,z_1,z_2)$	$\{a, a-c_i, a-c_1-c_2\} \in \mathbb{Z}$
$\Theta_1(a_1,a_2,b,c,z_1,z_2)$	$\{a_i, b, a_1+a_2-c, b+a_2-c\} \in \mathbb{Z}$
$\Theta_2(a,b,c,z_1,z_2)$	$\{a, b\} \in \mathbb{Z}$
$\Gamma_1(a,b_1,b_2,c,z_1,z_2)$	$\{a, a+b_1, b_1+b_2\} \in \mathbb{Z}$
$\Gamma_2(b_1,b_2,c,z_1,z_2)$	$\{b_1+b_2\} \in \mathbb{Z}$
$H_1(a,b,c,z_1,z_2)$	$\{b, a+b, a+b-2c, b-c\} \in \mathbb{Z}$
$H_2(a,b,c,d,z_1,z_2)$	$\{b, c, a+c, a+c-d, b-d\} \in \mathbb{Z}$
$H_3(a,b,c,z_1,z_2)$	$\{b, b-c\} \in \mathbb{Z}$
$H_4(a,c,d,z_1,z_2)$	$\{b, a+b, a+b-c\} \in \mathbb{Z}$
$H_5(a,b,z_1,z_2)$	
$H_6(a,c,z_1,z_2)$	$\{a, a-b\} \in \mathbb{Z}$
$H_7(a,c,d,z_1,z_2)$	$\{a, a-d, a-2c, a-2c-d\} \in \mathbb{Z}$
$H_8(a,b,z_1,z_2)$	$\{a+b, a+2b\} \in \mathbb{Z}$
$H_9(a,b,c,z_1,z_2)$	$\{b, a+b, a+b-2c\} \in \mathbb{Z}$
$H_{10}(a,b,z_1,z_2)$	
$H_{11}(a,b,c,d,z_1,z_2)$	$\{b, c, a+b, a+c, a+b-d, a+c-d\} \in \mathbb{Z}$

Table 3: Loci of singularities of the homogeneous linear systems of PDEs of second order with two variables for the Horn-type hypergeometric functions implemented in the program package **HornFunctions**.

Horn-type function	singularity surfaces
$G_1(a,b_1,b_2,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{1+z_1+z_2=0\}\cup\{4z_1z_2=1\}$
$G_2(a_1,a_2,b_1,b_2,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup_{i=1}^{2}\{1+z_i=0\}\cup\{z_1z_2=1\}$
$G_3(a_1,a_2,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1-4z_1-4z_2-18z_1z_2+27z_1^2z_2^2=0\}$
$H_1(a,b,c,d,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup_{i=1}^{2}\{1-z_i=0\}\cup\{1+z_2=0\}$
$H_2(a,b,c,d,e,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{1-z_1=0\}\cup\{1+z_2=0\}$
$H_3(a,b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{1-4z_1=0\}\cup\{z_1-z_2+z_2^2=0\}$
$H_4(a,b,c,d,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{1-4z_1=0\}\cup\{1-z_2=0\}$
$H_5(a,b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+z_2+z_1(-4+6z_2)=0\}$
$H_6(a,b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{1+4z_1=0\}\cup\{1+z_2-z_1z_2^2=0\}$
$H_7(a,b,c,d,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+4z_1=0\}\cup\{1+z_2=0\}$
confluent series:	
$\Phi_1(a,b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+z_1=0\}$
$\Phi_2(b_1,b_2,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{z_1-z_2=0\}$
$\Phi_3(b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}$
$\Psi_1(a,b,c_1,c_2,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+z_1=0\}$
$\Psi_2(a,b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}$
$\Theta_1(a_1,a_2,b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+z_1=0\}$
$\Theta_2(a,b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+z_1=0\}$
$\Gamma_1(a,b_1,b_2,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{1+z_1=0\}$
$\Gamma_2(b_1,b_2,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}$
$H_1(a,b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+z_1=0\}$
$H_2(a,b,c,d,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+z_1=0\}$
$H_3(a,b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+z_1=0\}$
$H_4(a,c,d,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}$
$H_5(a,b,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}$
$H_6(a,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+4z_1=0\}$
$H_7(a,c,d,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+4z_1=0\}$
$H_8(a,b,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{1+4z_1=0\}$
$H_9(a,b,c,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+4z_1=0\}$
$H_{10}(a,b,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{-1+4z_1=0\}$
$H_{11}(a,b,c,d,z_1,z_2)$	$\cup_{i=1}^{2}\{z_i=0\}\cup\{1+z_2=0\}$