DESY 15-038

Conformal constraints for anomalous dimensions of leading twist operators

A. N. Manashov[a,1,2,3], **M. Strohmaier**[b,2]

[1]Institut für Theoretische Physik Universität Hamburg, Luruper Chaussee 149, D-22761 Hamburg, Germany
[2]Institut für Theoretische Physik, University of Regensburg, D-93040 Regensburg, Germany
[3]Department of Theoretical Physics, St.-Petersburg State University, 199034 St.-Petersburg, Russia

Received: date / Accepted: date

Abstract Leading-twist operators have a remarkable property that their divergence vanishes in a free theory. Recently it was suggested that this property can be used for an alternative technique to calculate anomalous dimensions of leading-twist operators and allows one to gain one order in perturbation theory so that, i.e., two-loop anomalous dimensions can be calculated from one-loop Feynman diagrams, etc. In this work we study feasibility of this program on a toy-model example of the φ^3 theory in six dimensions. Our conclusion is that this approach is valid, although it does not seem to present considerable technical simplifications as compared to the standard technique. It does provide one, however, with a very nontrivial check of the calculation as the structure of the contributions is very different.

Keywords conformal invariance · anomalous dimensions

PACS 11.10.Hi · 11.25.Db · 12.38.Bx

1 Introduction

Calculation of anomalous dimensions of composite operators belongs to the standard tasks of any quantum field theory calculation. For example, in quantum chromodynamics, anomalous dimensions of leading twist two operators govern the scaling behavior of quark and gluon distributions in hadrons and have to be known with high precision. Nowadays the anomalous dimensions are known at three-loops, see Refs. [1–3] and references therein. Beyond the two-loop approximation such calculations are feasible only with the help of the advanced methods of computer algebra. Since the calculations are fully automated finding errors becomes highly nontrivial task and any approach which can provide a check of the final results is very helpful.

Given that the theory depends not only on the coupling constant but also some other parameters such as the dimension of an internal symmetry group, one can organize expansion over these parameters. The best known example of this kind is the $1/N$ expansion, see Ref. [4] for a review. Agreement between the results obtained in the perturbative and $1/N$ expansions serves as a powerful test for the validity of calculations. However, the calculations in the $1/N$ expansion are much harder than perturbative calculations: only two RG functions – indices of the basic fields in the nonlinear $\sigma-$model and Gross – Neveu model – are available at $1/N^3$ order [5–7]. In QCD the calculations rarely go beyond the leading order in $1/N_f$ (where N_f is the number of flavors). At leading order in $1/N_f$ the anomalous dimensions of twist two operators were calculated in Refs. [8, 9] but extension of these results to the next order is hardly possible.

A new approach for calculating the anomalous dimensions of leading twist operators was proposed in Refs. [10, 11]. It is still a perturbative approach, however, the contributing diagrams are completely different from those in the standard technique. The approach is based on a remarkable property of leading twist operators: namely, a divergence of such an operator, $\mathcal{O}_{\mu_1,\ldots\mu_j}$, vanishes in a free theory [12]

$$\partial^{\mu_1}\mathcal{O}_{\mu_1,\ldots\mu_j}(x)=0\,. \tag{1}$$

In the interacting theory the r.h.s. of Eq. (1) is nonzero but is proportional to the coupling constant. This identity allows to extract the $\ell-$loop contribution to the anomalous dimension of the operator $\mathcal{O}_{\mu_1,\ldots\mu_j}$ from $\ell-1$ loop diagrams only. In particular the one loop

[a]e-mail: alexander.manashov@desy.de
[b]e-mail: matthias.strohmaier@ur.de

anomalous dimensions of leading twist operators do not require calculation of loop integrals at all [11, 20].

The method developed in Ref. [11] is adjusted to local operators and relies heavily upon the so-called "conformal scheme" renormalization [13–15]. In our opinion it is more convenient to stay within the standard $\overline{\text{MS}}$ scheme and the formalism non-local (light-ray) operators technique. This technique proves to be more effective and flexible as we will demonstrate on the example of calculation of two – loop anomalous dimensions in the $su(n)$ symmetric φ^3 model [16].

The paper is organized as follows. In section 2 we introduce the model and fix notations. In section 3 we recall the light-ray operator technique. Section 4 is devoted to calculation of the divergence of conformal operator. The details of calculation of two – loop correlators are presented in section 5. Our conclusions are in section 7. In Appendices we explain some technical issues and details of the derivation.

2 Generalities

The $su(n)$ symmetric φ^3 model is a scalar field theory in $d = 6 - 2\epsilon \equiv 2\mu$ dimension with an action

$$S(\varphi) = \int d^dx \left[\frac{1}{2}(\partial\varphi^a)^2 + \frac{1}{6} g M^\epsilon d^{abc}\varphi^a\varphi^b\varphi^c\right], \qquad (2)$$

where $a = 1, \ldots, n^2 - 1$,

$$d^{abc} = 2\,\mathrm{tr}\, t^a\{t^b t^c\}, \qquad (3)$$

t^a are the generators of the $su(n)$ algebra normalized in the conventional way, $\mathrm{tr}\, t^a t^b = 1/2$. The theory is multiplicatively renormalizable

$$S_R(\varphi, g) = S(\varphi_0, g_0), \qquad (4)$$

where

$$\varphi_0 = Z_\varphi\, \varphi \qquad \text{and} \qquad g_0 = M^\epsilon Z_g g\,.$$

Two–loop expressions for the renormalization constants $Z_1 = Z_\varphi^2$ and $Z_3 = Z_g Z_\varphi^3$ can be found in Ref. [16]. The β–function of the charge $u = g^2/(4\pi)^3$ and the field anomalous dimension γ_φ are

$$\beta(u) = -2\epsilon u - u^2\frac{n^2 - 20}{2n} + \mathcal{O}(u^3),$$
$$\gamma_\varphi(u) = u\frac{n^2 - 4}{12n}\left(1 + u\frac{n^2 - 100}{36n}\right) + \mathcal{O}(u^3). \qquad (5)$$

At the critical point u_*, $\beta(u_*) = 0$,

$$u_* = 4n\epsilon/(20 - n^2) + O(\epsilon^2) \qquad (6)$$

theory enjoys the scale and conformal invariance. [1]

It is well known that in a conformal theory the form of two–point correlation function of conformal operators is fixed up to normalization. In particular, the correlator of the conformal traceless symmetric operators has the form

$$\left\langle \mathcal{O}_j^{(n)}(x)\mathcal{O}_{j'}^{(\bar n)}(y)\right\rangle = \delta_{jj'}\delta_{\Delta_j\Delta_{j'}}\frac{C_j I^j_{n,\bar n}(x-y)}{((x-y)^2)^{\Delta_j}}\,. \qquad (7)$$

Here j (j') is the spin of the operator. The vectors n and $\bar n$ are two light–like vectors, $n^2 = \bar n^2 = 0$, and $\mathcal{O}_j^{(n)}(x)$ ($\mathcal{O}_{j'}^{(\bar n)}(y)$) is a contraction of an operator with the vector n ($\bar n$), for instance

$$\mathcal{O}_j^{(n)}(x) = n^{\mu_1}\ldots n^{\mu_j}\mathcal{O}_{\mu_1\ldots\mu_j}(x)\,. \qquad (8)$$

Δ_j and $\Delta_{j'}$ are the scaling dimensions of the operators, C_j is a normalization constant and

$$I_{n,\bar n}(x) = (n\bar n) - \frac{2(nx)(\bar n x)}{x^2}\,. \qquad (9)$$

The scaling dimension of the operator is given by the sum of canonical and anomalous dimensions at a critical point, $\Delta_j = \Delta_j^{(0)} + \gamma_j$, where $\gamma_j \equiv \gamma_j(u_*)$. For the leading twist operators of spin j, $\Delta_j^{(0)} = 2\mu - 2 + j$.

The anomalous dimension of an operator in MS scheme, $\gamma_j(u)$, is a function of a coupling constant only. It can be restored from its critical value, $\gamma_j = \gamma_j(u_*)$, provided that the latter is known as a function of ϵ.

The correlation function for the divergence of the conformal operator

$$\partial\mathcal{O}_j^{(n)}(x) \equiv j\, n^{\mu_2}\ldots n^{\mu_j}\,\partial^{\mu_1}\mathcal{O}_{\mu_1\mu_2\ldots\mu_j}(x) \qquad (10)$$

can be obtained from Eq. (7). If x is chosen in the transverse plane, $(xn) = (x\bar n) = 0$, the ratio of the two correlation functions

$$\mathcal{T}_j(u_*) = x^2(n\bar n)\frac{\left\langle \partial\mathcal{O}_j^{(n)}(x)\partial\mathcal{O}_j^{(\bar n)}(0)\right\rangle}{\left\langle \mathcal{O}_j^{(n)}(x)\mathcal{O}_j^{(\bar n)}(0)\right\rangle} \qquad (11)$$

is a function of the anomalous dimension γ_j only

$$\mathcal{T}_j = 2j\gamma_j\left[\frac{(2\mu - 3 + j)(\mu - 1 + j)}{\mu - 2 + j} + 2\gamma_j\right]. \qquad (12)$$

This ratio, in full agreement with the result of Ref. [12], vanishes provided that $\gamma_j = 0$.

[1] Formally, a nontrivial critical point for $d < 6$ only exists for $n = 3, 4$. However, staying within perturbation theory one can consider n as a continuous parameter. In this sense all further results hold for arbitrary n.

The perturbation series for γ_j and $\mathcal{T}_j$

$$\gamma_j = u_*\gamma_j^{(1)} + u_*^2\gamma_j^{(2)} + \dots ,$$
$$\mathcal{T}_j = \varkappa_j\left(u_* T_j^{(1)} + u_*^2 T_j^{(2)} + \dots\right) , \tag{13}$$

are related to each other. For later convenience we chose the normalization factor $\varkappa_j$ as follows

$$\varkappa_j = \frac{2j(j+2)(j+3)}{j+1} . \tag{14}$$

Substituting the series (13) into Eq. (12) one finds the following relations between the expansion coefficients:

$$\gamma_j^{(1)} = T_j^{(1)} ,$$
$$\gamma_j^{(2)} = T_j^{(2)} - \frac{2(j+1)}{(j+2)(j+3)}(T_j^{(1)})^2 + \frac{2j^2+5j+1}{(j+1)(j+2)(j+3)}\epsilon^{(1)}\, T_j^{(1)} , \tag{15}$$

and so on. Here $\epsilon^{(1)} = (n^2-20)/4n$.

Since the divergence of the conformal operator is proportional to the coupling constant, $\partial\mathcal{O}_j^{(n)} \sim O(g)$, the ratio $\mathcal{T}_j$ contains a "kinematical" factor $u \sim g^2$. Thus, in order to determine $\mathcal{T}_j$ and, hence, the anomalous dimension γ_j, with $O(u^\ell)$ accuracy the corresponding correlation functions have to be calculated at one order in u less. In Ref. [11] one loop anomalous dimensions were reproduced by this method in φ^3 and $N=4$ SUSY models. Going beyond the leading order requires an effective technique for calculation of the two–point correlation functions otherwise one gains nothing in comparison with the standard approach.

3 Light-ray vs local operators

The first task is to find a convenient description for local operators. As we will argue the light-ray operator technique [17] is a most suitable one. The light-ray operator [2]

$$[\mathcal{O}(x;z_1,z_2)] = [\varphi^a(x+z_1 n)\varphi^a(x+z_2 n)] \tag{16}$$

is defined as the generating function for the renormalized local operators

$$[\mathcal{O}(x;z_1,z_2)] \equiv \sum_{k,m} z_1^k z_2^m [\mathcal{O}_{km}](x) = \sum_{k,m,k',m'} z_1^k z_2^m\, Z_{km}^{k'm'}\, \mathcal{O}_{k'm'}(x) . \tag{17}$$

[2]In order to make presentation more transparent we will consider the $su(n)$ scalar operator. The operators of other symmetry properties can be easily included into consideration, see Sect.6.

Here $[\mathcal{O}_{km}]$ is the renormalized (in MS scheme) local monomial $\mathcal{O}_{km} = \partial_+^k\varphi^a(x)\partial_+^m\varphi^a(x)/k!m!$ and $\partial_+ = (n\partial_x)$. The sum in Eq. (17) can be replaced by action of some integral operator on the bare operator

$$[\mathcal{O}(x;z)] = Z\mathcal{O}(x;z) . \tag{18}$$

Here we introduced a shorthand notation $z=\{z_1,z_2\}$. The integral operator Z can be written in the form [16]

$$Zf(z) = \int d\alpha d\beta\, Z(\alpha,\beta)\, f(z_{12}^\alpha, z_{21}^\beta) . \tag{19}$$

The renormalization kernel $Z(\alpha,\beta)$ is given by a series in $1/\epsilon$ and the coupling u. The light-ray operator (17) satisfies the RG equation

$$\Big(M\partial_M + \beta(u)\partial_u + \mathbb{H}(u)\Big)[\mathcal{O}(x;z)] = 0 , \tag{20}$$

where the evolution kernel $\mathbb{H}$ is an integral operator

$$\mathbb{H}(u) = -\left(M\frac{d}{dM}Z\right)Z^{-1} + 2\gamma_\varphi \tag{21}$$

which encodes all information on the anomalous dimension matrices for local operators.

At the critical point $u=u_*$ local operators can be classified according to the representations of the conformal group. An operator with the lowest scaling dimension in the representation is called a conformal operator. The leading twist operator [3], $\mathcal{O}_j$, is uniquely determined by its scaling dimension $\varDelta_j$. The expansion of the light-ray operator (17) over conformal operators and their descendants reads [16]

$$[\mathcal{O}(x;z)] = \sum_{jk} \varPsi_{jk}(z_1,z_2)\,\partial_+^k\mathcal{O}_j(x) , \tag{22}$$

where the coefficients $\varPsi_{jk}(z_1,z_2)$ are homogeneous polynomials of degree $j+k$ in z_1,z_2. These polynomials are eigenfunctions of the evolution kernel $\mathbb{H}(u_*)$

$$\mathbb{H}(u_*)\varPsi_{jk}(z) = \gamma_j\,\varPsi_{jk}(z) , \tag{23}$$

with the corresponding eigenvalues being the anomalous dimensions, $\gamma_j = \gamma_j(u_*)$. Since the theory enjoys conformal invariance at the critical point $u=u_*$ the evolution kernel commutes with three generators of the collinear subgroup of conformal group, [4]

$$[S_{\pm,0}, \mathbb{H}(u_*)] = 0. \tag{24}$$

[3]Note, that the operator $\mathcal{O}_j$ vanishes identically for odd j.

[4]Other generators acts on the operators in question trivially.

The generators, however, deviate from their canonical form (see Appendix Appendix A)

$$S_-^{(0)} = -\partial_{z_1} - \partial_{z_2}\,,$$
$$S_0^{(0)} = z_1\partial_{z_1} + z_2\partial_{z_2} + 2\,,$$
$$S_+^{(0)} = z_1^2\partial_{z_1} + z_2^2\partial_{z_2} + 2(z_1+z_2) \tag{25}$$

due to quantum corrections,

$$S_{\pm,0} = S_{\pm,0}^{(0)} + \mathit{\Delta} S_{\pm,0}. \tag{26}$$

Two of the generators are known to all orders,

$$\mathit{\Delta} S_- = 0\,, \qquad \mathit{\Delta} S_0 = -\epsilon + \frac{1}{2}\mathbb{H}(u_*)\,, \tag{27}$$

while corrections to the generator of special conformal transformations can be calculated order by order in perturbation theory [16]. The leading correction is

$$\mathit{\Delta} S_+ = (z_1+z_2)\left(-\epsilon + \frac{1}{2}\mathbb{H}(u_*)\right) + O(\epsilon^2)\,. \tag{28}$$

It follows from Eqs. (23), (24) that the operators $S_\pm$ act as raising (lowering) operators on the set of eigenfunctions Ψ_{jk},

$$S_\pm \Psi_{jk} \sim \Psi_{jk\pm 1}\,. \tag{29}$$

In turn S_0 counts conformal spin of the operator $\mathcal{O}_{jk}$,

$$S_0 \Psi_{jk} = j_{jk}\Psi_{jk} = \frac{1}{2}\left(\mathit{\Delta}_{jk} + S_{jk}\right)\Psi_{jk}\,, \tag{30}$$

where $\mathit{\Delta}_{jk} = \mathit{\Delta}_j + k$ and S_{jk} are the scaling dimension and spin of the operator, respectively. One derives immediately from (29) that the polynomial $\Psi_{jk=0}$ accompanying the conformal operator $\mathcal{O}_j$ in the expansion (22) is a simple power, $\Psi_{jk=0}(z_1,z_2) \sim (z_1 - z_2)^j$ and all other eigenfunctions have the form $\Psi_{jk}(z_1,z_2) \sim S_+^k (z_1 - z_2)^j$.

3.1 Scalar product

Eq. (23) can be considered as a standard quantum mechanical problem for the Hamiltonian $\mathbb{H}$. In order to make this analogy complete one needs to introduce a scalar product on the space of the eigenfunctions. Clearly, such a scalar product has to be adjusted to the symmetries of the problem. At leading order the answer is given by the standard $sl(2)$ invariant scalar product [18, 19]

$$(\psi,\phi)_0 = \frac{1}{\pi^2}\iint\limits_{|z_k|<1} d^2z_1 d^2z_2\,(\psi(z_1,z_2))^\dagger\,\phi(z_1,z_2)\,. \tag{31}$$

The integration goes over the unit disks $|z_k| < 1$, $k = 1,2$. The generator $S_0^{(0)}$ is a self-adjoint operator with respect to this scalar product and $(S_+^{(0)})^\dagger = -S_-^{(0)}$. We want to find a deformation of the scalar product (31) that keeps these relations for the complete generators, $S_0 = S_0^\dagger$ and $S_+^\dagger = -S_-$. Let us look for the solution in the form

$$(\psi,\phi)_\varpi = (\psi,\varpi\phi)_0\,, \qquad \varpi = \mathbb{1} + u_*\varpi^{(1)} + \ldots\,, \tag{32}$$

where $\varpi^{(1)}$ is a self-adjoint operator with respect to the scalar product (31). The conjugation conditions for the generators imply

$$\mathit{\Delta} S_0^{(1)} - (\mathit{\Delta} S_0^{(1)})^\dagger = [S_0^{(0)}, \varpi^{(1)}]\,,$$
$$\mathit{\Delta} S_+^{(1)} = [S_+^{(0)}, \varpi^{(1)}]\,. \tag{33}$$

The one loop corrections to the generators involve the kernel $\mathcal{H}^{(1)}$ $\left(\mathbb{H}(u) = \sum_k u^k \mathcal{H}^{(k)}\right)$ which is given by the following expression

$$\mathcal{H}^{(1)} = 2(\gamma_\varphi^{(1)} - \lambda_s \mathcal{H}^+)\,. \tag{34}$$

Here λ_s is a color factor, $\lambda_s = (n^2-4)/n$ and

$$\mathcal{H}^+\psi(z) = \int_0^1 d\alpha \int_0^{\bar\alpha} d\beta\,\psi(z_{12}^\alpha, z_{21}^\beta)\,. \tag{35}$$

The operator $\varpi^{(1)}$ is completely determined by Eqs. (33). The explicit expression for $\varpi^{(1)}$ and details of the derivation can be found in Appendix B [5].

Since the eigenfunctions Ψ_{jk} are mutually orthogonal w.r.t. the scalar product (32), one can represent the conformal operator as the scalar product of the coefficient function with the light-ray operator

$$\mathcal{O}_j(x) = (z_{12}^j, [\mathcal{O}(x,z)])_\varpi\,. \tag{36}$$

This representation for the conformal operator is the most convenient one for further analysis. We demonstrate it on the following example. The conformal operator is usually defined as an operator which vanishes under special conformal transformations, $K_{\bar n} = K\cdot\bar n$, $\delta_{K_{\bar n}}\mathcal{O}_j(0) = 0$. This property becomes transparent in the representation (36) if one takes into account that

$$\delta_{K_{\bar n}}[\mathcal{O}(z)] = 2(n\bar n)S_+[\mathcal{O}(z)],$$

(we put here $[\mathcal{O}(z)] = [\mathcal{O}(x=0;z)]$) and use that the generators S_+ and S_- are conjugate to each other w.r.t. the scalar product (32), $S_+^\dagger = -S_-$.

[5] It turns out that the corrections due to $\varpi^{(1)}$ cancel at $O(\epsilon^2)$ order in the ratio of the correlators (11). So that we do not need this explicit expression for the present purposes.

4 Divergence of conformal operator

In order to construct the divergence of the conformal operator $\partial\mathcal{O}_j^{(n)}$, Eq. (10), we calculate first the divergence of the light-ray operator (18)

$$[\partial\mathcal{O}(x;z)] \equiv \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial n_\mu}[\mathcal{O}(x;z)]\,. \tag{37}$$

Taking the $n-$derivative one cannot, however, keep $n^2 = 0$ any longer and has to take into account terms linear in n^2 which account for the trace subtraction in $[\mathcal{O}(x;z)]$. Taking the corresponding modification into account, see e.g. Refs. [11, 17, 21], one gets for (37)

$$[\partial\mathcal{O}(x;z)] = \frac{\partial}{\partial x^\mu}\nabla_\mu Z\varphi^a(x+z_1n)\varphi^a(x+z_2n) \tag{38}$$

where ∇^μ is a differential operator

$$\nabla_\mu = \frac{\partial}{\partial n^\mu} - \frac{1}{2}(\mu - 1 + n\cdot\partial_n)^{-1}n^\mu\frac{\partial^2}{\partial n^2}\,. \tag{39}$$

which commutes with the renormalization factor Z and acts on the fields directly. After a simple algebra one gets

$$\begin{aligned}[\partial\mathcal{O}(x;z)] = \frac{1}{2}(S_0^{(\epsilon)}-1)^{-1}Z\Big\{&S_+^{(\epsilon)}\,\partial_x^2\,\mathcal{O}(x;z)\\ &- L_{21}^{(\epsilon)}\,\partial^2\varphi^a(x+z_1n)\varphi^a(x+z_2n)\\ &- L_{12}^{(\epsilon)}\,\varphi^a(x+z_1n)\partial^2\varphi^a(x+z_2n)\Big\}\,,\end{aligned} \tag{40}$$

where $S_0^{(\epsilon)} = S_0^{(0)} - \epsilon$, $S_+^{(\epsilon)} = S_+^{(0)} - \epsilon(z_1+z_2)$ and

$$L_{21}^{(\epsilon)} = \partial_{z_2}z_{21}^2 - \epsilon z_{21}\,, \qquad L_{12}^{(\epsilon)} = \partial_{z_1}z_{12}^2 - \epsilon z_{12}\,. \tag{41}$$

Using equations of motion (EOM) one can replace in this expression

$$\partial^2\varphi^a(x) \mapsto \frac{1}{2}gM^\epsilon Z_3Z_1^{-1}d^{abc}\varphi^b(x)\varphi^c(x)\,. \tag{42}$$

We want to stress here that Eq. (40) holds for arbitrary coupling u but not only at the critical value. Since the l.h.s. of Eq. (40) is a finite (renormalized) operator the r.h.s. can be expressed in terms of renormalized operators with finite coefficients. These operators can be chosen as: the two-particle operator $\mathcal{O}_1 = \partial^2\mathcal{O}(x;z)$ and three particle operator

$$\begin{aligned}\mathcal{O}_2 &= \mathcal{O}^{(d)}(x;w)\\ &= gd^{abc}\varphi^a(x+w_1n)\varphi^b(x+w_2n)\varphi^c(x+w_3n)\,,\end{aligned} \tag{43}$$

where $w = \{w_1, w_2, w_3\}$. The operators $\mathcal{O}_1$ and $\mathcal{O}_2$ mix under renormalization. The mixing matrix (integral operator acting on fields variables) has an lower triangular form

$$[\mathcal{O}_k] = Z_{km}\mathcal{O}_m\,. \tag{44}$$

Here $Z_{11} = Z$ is the renormalization constant of the light-ray operator, Eq. (18), $Z_{12} = 0$, $Z_{21} = O(u^2)$ and the element Z_{22} is given, at the one loop order, by the sum of two - particles kernels

$$\begin{aligned}Z_{11} &= 1 - \frac{u}{\epsilon}\lambda_s\,\mathcal{H}_{12}^+ + O(u^2)\,,\\ Z_{22} &= \mathbb{1} + \frac{u}{2\epsilon}\sum\nolimits_{i<k}\left(\lambda_s\mathcal{H}_{ik}^d - \lambda_d\,\mathcal{H}_{ik}^+\right) + O(u^2)\,,\end{aligned} \tag{45}$$

where λ_s, λ_d are color factors

$$\lambda_s = (n^2-4)/n\,, \qquad \lambda_d = (n^2-12)/n\,. \tag{46}$$

The kernel $\mathcal{H}_{ik}^+$ is defined by Eq. (35) and $\mathcal{H}_{ik}^d$ has the form

$$\mathcal{H}_{12}^d f(z_1,z_2) = \int_0^1 d\alpha\,\alpha\bar\alpha\,f(z_{12}^\alpha, z_{12}^\alpha)\,. \tag{47}$$

The subscripts ik show the arguments the kernel acts on.

Using these results we can rewrite (40) as follows

$$[\partial\mathcal{O}(x;z)] = \frac{1}{2}(S_0^{(\epsilon)}-1)^{-1}\sum\nolimits_{k=1,2}A_k[\mathcal{O}_k(x;z)]\,. \tag{48}$$

The operators A_k have the following form:

$$\begin{aligned}A_1 &= Z_{11}S_+^{(\epsilon)}Z_{11}^{-1} - M^{-\epsilon}A_2Z_{21}Z_{11}^{-1}\,,\\ A_2 &= -\frac{1}{2}M^\epsilon Z_3Z_1^{-1}Z_{11}\Big(L_{12}^{(\epsilon)}S_2 + L_{21}^{(\epsilon)}S_1\Big)Z_{22}^{-1}\,,\end{aligned} \tag{49}$$

where the operators S_1, S_2 map functions of three variables to functions of two variables

$$\begin{aligned}[S_1f](z_1,z_2) &= f(z_1,z_1,z_2)\,,\\ [S_2f](z_1,z_2) &= f(z_1,z_2,z_2)\,.\end{aligned} \tag{50}$$

At one loop the operators A_k take the form

$$\begin{aligned}A_1 &= S_+(u) + u\Big(\lambda_s\mathcal{H}^+ - \gamma_\varphi^{(1)}\Big)(z_1+z_2) + O(u^2)\,,\\ A_2 &= -\frac{1}{2}M^\epsilon\Big(L_{12}S_2 + L_{21}S_1 + \Big(\epsilon - u\lambda_s\mathcal{H}^+\Big)z_{12}S_{12}\\ &\qquad - uz_{12}S_{12}\Big(\lambda_s\mathcal{H}_{13}^d - \lambda_d\mathcal{H}_{13}^+\Big) + O(u^2)\Big)\,.\end{aligned} \tag{51}$$

Here $S_+(u)$ is given by the expressions (26), (27) for arbitrary u, $u_* \to u$,

$$S_{12} = S_1 - S_2\,, \qquad L_{km} = L_{km}^{(\epsilon\mapsto 0)}$$

and deriving (51) we made use of the symmetry of the three particle operator $\mathcal{Q}^{(d)}(x;w_1,w_2,w_3)$ under permutation of w variables.

Let us stress again that the operators A_k do not contain singular terms for *arbitrary* u. Using one loop expressions for Z factors, Eq. (45), it can be checked that all pole terms cancel at order $O(u)$. In particular

$$Z_3 Z_1^{-1} Z_{11}\Big(L_{12}S_2 + L_{21}S_1\Big) Z_{22}^{-1} = \\ = L_{12}S_2 + L_{21}S_1 + O(u^2)\,. \quad (52)$$

Starting from the representation (36) for the conformal operator, we get for its divergence

$$\partial\mathcal{O}_j(x) = (z_{12}^j, [\partial\mathcal{O}(x;z)])_\varpi\,. \quad (53)$$

Making use of Eqs. (48) – (51) one finds that the divergence $\partial\mathcal{O}_j(x)$ is given by the sum of two–particle and three–particle (renormalized) operators

$$\partial\mathcal{O}_j(x) = \frac{1}{2(j+1-\epsilon)}\Big(\mathcal{R}_j^{(2)}(x) + \mathcal{R}_j^{(3)}(x)\Big)\,. \quad (54)$$

The prefactor on the r.h.s of Eq. (54) is the eigenvalue of the operator $(S_0^{(\epsilon)}-1)^{-1}$ on the function z_{12}^j. The two–particle term $\mathcal{R}_j^{(2)}$ has the form

$$\mathcal{R}_j^{(2)}(x) = \gamma_j\left(z_{12}^j,\, (z_1+z_2)[\mathcal{O}_1(x;z)]\right) + O(u_*^2). \quad (55)$$

We recall that $\gamma_j = \gamma_j(u_*)$ and $\mathcal{O}_1(x;z) = \partial^2\mathcal{O}(x;z)$. Let us note that the term $\sim S_+$ in the expression for A_1, Eq. (51), vanishes inside the scalar product since $(z_{12}^j, S_+\ldots) = -(S_- z_{12}^j,\ldots) = 0$. In turn, the expression for the three–particle contribution can be written as follows

$$\mathcal{R}_j^{(3)}(x) = \mathcal{R}_j^{(3,0)}(x) + \mathcal{R}_j^{(3,1)}(x) + O(\epsilon^2)\,, \quad (56)$$

where

$$\begin{aligned}\mathcal{R}_j^{(3,0)}(x) &= -M^\epsilon\big(z_{12}^j,\, L_{21}S_1[\mathcal{O}_2(x;z)]\big)_\varpi\,,\\ \mathcal{R}_j^{(3,1)}(x) &= -M^\epsilon\big(z_{12}^j, z_{12}S_1X_j[\mathcal{O}_2(x;z)]\big)_\varpi\end{aligned} \quad (57)$$

and the operator X_j has the form

$$X_j = \epsilon - \gamma_\varphi + \gamma_j/2 - u_*\big(\lambda_s\mathcal{H}_{13}^d - \lambda_d\mathcal{H}_{13}^+\big)\,. \quad (58)$$

This expression follows immediately from (51) if one takes into account that spin j is even and A_2 is symmetric under interchange $z_1 \leftrightarrow z_2$.

It is clear from (55) that the expansion of two particle term $\mathcal{R}_j^{(2)}$ over conformal operators does not contain the operator of spin j,

$$\mathcal{R}_j^{(2)} \sim \partial^2\left(\sum_{m=0}^{j-2} c_m\partial_+^{j-m-2}\mathcal{O}_m(x)\right)\,. \quad (59)$$

Taking into account that $\langle\mathcal{O}_j(x)\mathcal{O}_k(0)\rangle = 0$ for $k<j$ one derives that $\langle\mathcal{R}_j^{(2)}(x)\,\partial\mathcal{O}_j(0)\rangle = 0$. This, in virtue

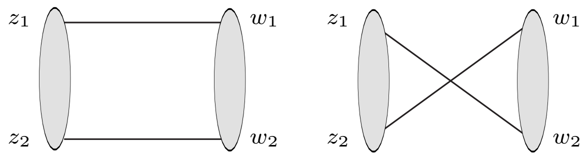

Fig. 1 The leading order diagrams for the correlator of two conformal operators, $\langle\mathcal{O}_j^{(n)}(x)\mathcal{O}_j^{(\bar n)}(0)\rangle$.

of Eq. (54), results in the following relation for the correlators

$$\langle\mathcal{R}_j^{(2)}(x)\mathcal{R}_j^{(2)}(0)\rangle = -\langle\mathcal{R}_j^{(2)}(x)\mathcal{R}_j^{(3)}(0)\rangle\,. \quad (60)$$

It can be shown that in the correlator $\langle\partial\mathcal{O}_j(x)\partial\mathcal{O}_j(0)\rangle$ one can replace (54) by a simpler expression

$$\partial\mathcal{O}_j(x) = \frac{1}{2(j+1)}\Big(\mathcal{R}_j^{(2)}(x) + \mathcal{R}_j^{(3,0)}(x)\Big)\,. \quad (61)$$

The omitted terms

$$\Delta\mathcal{X} = \frac{\epsilon}{j+1}\left(\mathcal{R}_j^{(3,1)}(x) + \frac{1}{j+1}\mathcal{R}_j^{(3,0)}(x)\right) + O(\epsilon^2)$$

give rise to the correction of order $O(\epsilon^3)$. In order to verify this it is sufficient to notice that $\Delta\mathcal{X}$ can be rewritten in the form $\big(z_{12}^j, F\,S_- z_{12}[\mathcal{O}_2(x;z)]\big)_\varpi$, where F is some operator whose explicit expression is not relevant. Inside the correlator the generator S_- acts, finally, on the function z_{12}^j nullifying it.

Thus in order to find the anomalous dimension γ_j at order $O(\epsilon^2)$ one has to calculate the three correlators

$$\langle\mathcal{O}_j\mathcal{O}_j\rangle, \qquad \langle\mathcal{R}_j^{(2)}\mathcal{R}_j^{(3,0)}\rangle, \qquad \langle\mathcal{R}_j^{(3,0)}\mathcal{R}_j^{(3,0)}\rangle$$

at one loop order. We will do it in the next section.

Finally, we note that for $j=2$ the r.h.s. of Eq. (54) has to vanish identically since the operator $\mathcal{O}_{\mu\nu}$ is, up to EOM terms, the energy-momentum tensor.The two particle term $\mathcal{R}_j^{(2)}$ is proportional to the anomalous dimension γ_j and therefore vanishes for $j=2$. In order to check it for $\mathcal{R}_j^{(3)}$, it is sufficient to take into account that only the linear term in the expansion of three particle operator

$$\begin{aligned}\mathcal{O}_2(x;w) &\sim (w_1+w_2+w_3)\cdot d^{abc}\partial_+\varphi^a(x)\varphi^b(x)\varphi^c(x)\\ &= \Big(S_+^{(1,1,1)}\cdot 1\Big)\; d^{abc}\partial_+\varphi^a(x)\varphi^b(x)\varphi^c(x)\end{aligned}$$

contributes to (56) for $j=2$. After simple algebra one finds that $\mathcal{R}_{j=2}^{(3)} = O(\epsilon^2)$.

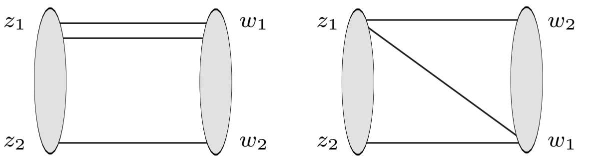

Fig. 2 The LO diagrams for the correlator of divergence of conformal operators, $\langle \partial\mathcal{O}_j^{(n)}(x)\partial\mathcal{O}_j^{(\bar n)}(0)\rangle$.

5 Correlators

5.1 LO correlators

In order to give a glimpse of the technique we start with calculation of the necessary correlators at leading order. The correlator $\langle \mathcal{O}_j^{(n)}(x)\mathcal{O}_j^{(\bar n)}(0)\rangle$ is given by the sum of two diagrams, shown schematically in Fig. 1. They are given by the product of the propagators and give rise to identical contributions to the correlation function. Assuming that x is chosen in a transverse plane, $(x,n)=(x,\bar n)=0$, one represents the propagator as

$$D(x+z_1n-\bar w_1\bar n)=D(x)(1-z_1\bar w_1 r)^{-(\mu-1)}\,, \tag{62}$$

where $r=2(n\bar n)/x^2$ and

$$D(x)=\Gamma(\mu-1)\,/(4\pi^\mu(x^2)^{\mu-1})\,. \tag{63}$$

At leading order one replaces $\mu-1\mapsto 2$ so that the second factor in (62) is nothing else as the reproducing kernel, $\mathcal{K}_{s=1}(z_1,w_1r)$, corresponding to the spin $s=1$, see Eq. (A.5). Therefore starting from Eq. (36) one gets for the correlator

$$\begin{aligned}\langle \mathcal{O}_j^{(n)}(x)\mathcal{O}_j^{(\bar n)}(0)\rangle &= 2\xi D^2(x)(z_{12}^j|\prod_{k=1}^2\mathcal{K}_1(z_k,w_kr)|w_{12}^j)\\ &=2\xi D^2(x)r^j\,||w_{12}^j||^2_{11}\,,\end{aligned} \tag{64}$$

where $\xi=n^2-1$ is the isotopic factor, the scalar products correspond to the conformal spin $s=1$ and we take into account the property of the reproducing kernel (A.6). The norm of w_{12}^j is given by the following expression

$$||w_{12}^j||^2_{s_1s_2}=j!\prod_{k=1}^2\frac{\Gamma(2s_k)}{\Gamma(j+2s_k)}\frac{\Gamma(2j+2(s_1+s_2)-1)}{\Gamma(j+2(s_1+s_2)-1)}\,.$$

The diagrams for the correlator $\langle \partial\mathcal{O}_j^{(n)}(x)\partial\mathcal{O}_j^{(\bar n)}(0)\rangle$ are shown in Fig. 2. On the leftmost diagram the points z_1n and $\bar w\bar n$ are connected by two propagators

$$\begin{aligned}D^2(x+z_1n-\bar w_1\bar n)&=D^2(x)(1-z_1\bar w_1r)^{-2(\mu-1)}\\ &=D^2(x)\mathcal{K}_{s=2}(z_1,w_1r)\,.\end{aligned} \tag{65}$$

Since $\partial\mathcal{O}_j^{(n)}\sim(z_{12}^j,L_{21}[\mathcal{O}(x,z_1,z_1,z_2)])_{11}$, see Eq. (57), the $z-$ scalar product has the form

$$(z_{12}^j,L_{21}\mathcal{K}_2(z_1,w_1r)\mathcal{K}_1(z_2,w_2r))_{11}\,. \tag{66}$$

The spins of the reproducing kernels and spins of the scalar product are in discord with each other. However, the operator $L_{21}=\partial_2z_{21}^2$ removes this mismatch. It intertwines the representations,

$$L_{21}D_2^+\otimes D_1^+=D_1^+\otimes D_1^+L_{21}$$

and it can be easily shown, see e.g. Ref. [20], that

$$(z_{12}^j,L_{21}\Phi(z_1,z_2))_{11}=-a_j(z_{12}^{j-1},\Phi(z_1,z_2))_{21}\,, \tag{67}$$

where

$$a_j=(j+1)||z_{12}^j||^2_{11}/||z_{12}^{j-1}||^2_{21}=\frac{j(j+2)(j+3)}{6}. \tag{68}$$

Thus the scalar product (66) takes the form

$$-a_j(z_{12}^{j-1},\mathcal{K}_2(z_1,w_1r)\mathcal{K}_1(z_2,w_2r))_{21}=-a_j(r\bar w_{12})^{j-1}. \tag{69}$$

Restoring all color and symmetry factors one gets for the first diagram

$$\begin{aligned}-\frac12\xi\lambda_sg^2D^3(x)a_j/(j+1)^2r^{j-1}(L_{21}\bar w_{12}^{j-1},w_{12}^j)_{11}=\\ =\frac12\xi\lambda_sg^2D^3(x)a_j/(j+1)r^{j-1}||w_{12}^j||^2_{11}\,.\end{aligned} \tag{70}$$

The calculation of the second diagram goes along the same line. One can combine propagators attached to the point z_1 using the Feynman's trick to get

$$\begin{aligned}(z_{12}^j,L_{21}\mathcal{K}_2(z_1,w_1r)\mathcal{K}_2(z_1,w_2r)\mathcal{K}_1(z_2,w_1r))_{11}=\\ =\frac{6(-1)^ja_j(r\bar w_{12})^{j-1}}{(j+1)(j+2)}\,.\end{aligned} \tag{71}$$

Finally, taking into account that second diagram enters with symmetry factor 2 one gets, in full agreement with (11),

$$\mathcal{T}_j(u_*)=u_*\varkappa_j\gamma_j^{(1)}+O(u_*^2)\,, \tag{72}$$

where the one–loop anomalous dimension is, see Eq. (34),

$$\gamma_j^{(1)}=\lambda_s\frac16\frac{(j-2)(j+5)}{(j+1)(j+2)}\,. \tag{73}$$

Thus the diagrams are easily calculated provided that the spins of the reproducing kernels match that of the scalar product. We will show that this scheme can be extended to loop diagrams as well.

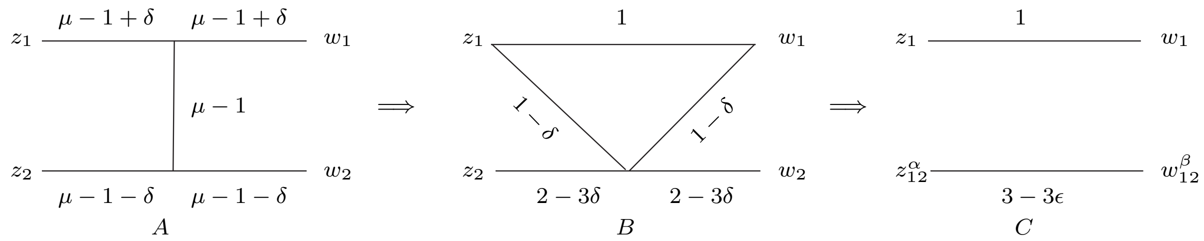

Fig. 3 NLO correction to the correlator of conformal operators $\langle \mathcal{O}_j^{(n)}(x)\mathcal{O}_j^{(\bar n)}(0)\rangle$. The parameter $\delta=\epsilon/2$.

5.2 NLO correlators

First, it is easy to see that the corrections due to modification of the scalar product cancel out in the ratio of correlators. Indeed, these corrections only influence the norm

$$||w_{12}^j||^2_{11} \mapsto ||w_{12}^j||^2_{\varpi} = (w_{12}^j, (1+\varpi^{(1)})w_{12}^j)_{11}.$$

entering the tree level expressions, Eqs. (64), (70), etc., which cancels out in the ratio of correlators, $\mathcal{T}_j$, irrespectively of the explicit form of ϖ. This cancellation is expected. Indeed, the problem can be reformulated as a standard quantum mechanical problem for a certain Hamiltonian. A modification of the scalar product produces corrections to the eigenstates (conformal operator). However, the energy shift at the leading order, $\delta E_\psi^{(1)} = \langle\psi^{(0)}|V|\psi^{(0)}\rangle$, is not sensitive to such corrections.

The calculation of $\langle \mathcal{R}_j^{(2)}\mathcal{R}_j^{(3,0)}\rangle$ is a bit more involved but straightforward. The corresponding contribution to $T_j^{(2)}$ reads

$$T_j^{(2\to3)} = -\frac{1}{2}(\gamma_j^{(1)})^2\,\frac{(j-1)(j^2+4j+9)}{(j+1)(j+2)(j+3)(j+5)}. \quad (74)$$

It is convenient to split one loop corrections to the correlators $\langle\mathcal{O}_j\mathcal{O}_j\rangle$ and $\langle\mathcal{R}_j^{(3,0)}\mathcal{R}_j^{(3,0)}\rangle$ into two groups

- Self-energy insertions to the propagators.
- All other loop diagrams

Taking into account the self-energy correction to the propagators is equivalent to the calculation of the tree–level diagrams with the exact (critical) propagator

$$D_c(x) = A(u_*)/(x^2)^{\Delta_\varphi}\,, \quad (75)$$

where $\Delta_\varphi = \mu-1+\gamma_\varphi$ is the critical dimension of the basic field and the residue $A(u_*)$ is

$$A(u_*) = \frac{\Gamma(\mu-1)}{4\pi^\mu\overline{M}^{2\gamma_\varphi}}\left(1-u_*\lambda_s\frac{5}{36}+O(u_*^2)\right), \quad (76)$$

where $\overline{M}^2 = \pi M^2 e^{\gamma_E}$. For the first diagram in Fig. 1 one gets

$$D_c^2(x)\; r^j\; (z_{12}^j|\mathcal{K}_s(z_1,w_1)\mathcal{K}_s(z_2,w_2)|w_{12}^j)_{(11),(11)}\,, \quad (77)$$

where $s=\Delta_\varphi/2 = 1-(\epsilon-\gamma_\varphi)/2$, the subscripts indicate the conformal spins of the z and w scalar products, respectively. The reproducing kernels in (77) correspond to spin s which does not match the spins of the scalar product. However, it is easy to see that, due to symmetry, the scalar product with modified spins

$$(z_{12}^j|\mathcal{K}_s(z_1,w_1)\mathcal{K}_s(z_2,w_2)|w_{12}^j)_{(s_+^\delta,s_+^\delta),(s_-^\delta,s_-^\delta)}\,, \quad (78)$$

where $s_\pm^\delta = 1\pm\delta$ is equal to that in (77) up to terms of order δ^2. Therefore, choosing $s_-^\delta = s$, ($\delta=(\epsilon-\gamma_\varphi)/2$), and evaluating the $w-$product in (78) one gets for (77)

$$D_c^2(x)\; r^j\; ||z_{12}^j||^2_{s_+^\delta,s_+^\delta} + O(u_*^2)\,. \quad (79)$$

The calculation of the diagrams in Fig. 2 goes along the same lines. Finally, the contribution to the ratio of correlators $\mathcal{T}_j$ from the leading order diagrams and self-energy diagrams can be written in the form (up to $O(\epsilon^2)$ terms)

$$\mathcal{T}_j^{SE} = \frac{2u_*\lambda_s}{\varkappa_j}\left(1+\frac{u_*}{2}\left(\lambda_d-\frac{7}{9}\lambda_s\right)\right)\frac{||z_{12}^{j-1}||^2_{2s_+^\delta,s_+^\delta}}{||z_{12}^j||^2_{s_+^\delta,s_+^\delta}} \times\left(1-\frac{2\Gamma(4s_-^\delta)(\Gamma(j-1+2s_-^\delta)}{\Gamma(2s_-^\delta)\Gamma(j-1+4s_-^\delta)}\right). \quad (80)$$

Expanding (80) we find for the corresponding contribution to the coefficient $T_j^{(2)}$

$$T_j^{SE} = \gamma_j^{(1)}\left\{\frac{1}{2}\left(\lambda_d-\frac{7}{9}\lambda_s\right) + 2\delta\left[S_{2j+2}-S_{j+3}-S_j+\frac{2}{3}\frac{4j^2+14j+9}{(j+2)(j+3)}\right]\right\} - \frac{4\lambda_s\,\delta}{(j+1)(j+2)}\left[S_{j+2}-\frac{(2j+1)(4j+7)}{3(j+1)(j+2)}\right], \quad (81)$$

where $S_j=\sum_{k=1}^j 1/k$ and

$$\delta = \frac{1}{4}\left(-\lambda_d+\frac{1}{3}\lambda_s\right)\,. \quad (82)$$

We recall that $\gamma_j^{(1)}$ is the one loop anomalous dimension, Eq. (73).

5.3 Loop diagrams

All loop diagrams can be calculated quite easily with the help of several simple tricks. Several examples are given below. Let us start with a correction to the correlator of two conformal operators. The corresponding contribution has the form

$$C(\epsilon) = 2\,(z_{12}^j|H(x;z,w)|w_{12}^j)_{(11),(11)}\,, \tag{83}$$

where the kernel $H(x;z;w)$ is given by the left diagram shown in Fig. 3 with the parameter $\delta \to 0$. We need to find $C(\epsilon)$ up to terms $O(\epsilon^0)$, $C(\epsilon) = \frac{1}{\epsilon}(c_0+\epsilon c_1+\ldots)$. To this end we proceed as follows. We modify the indices as shown in Fig. 3. This modification does not change the pole structure (see discussion in Ref. [26]) and, due to the symmetry $C(\epsilon,\delta) = C(\epsilon,-\delta)$, one concludes that

$$C(\epsilon,\delta) = \frac{1}{\epsilon}(c_0 + \epsilon c_1 + c_2\delta^2 + \ldots)\,. \tag{84}$$

The choice $\delta = \epsilon/2$ results in the uniqueness of the upper integration vertex and at the same time does not affect first two terms in (84). Using the star–triangle relation for the upper vertex one gets the diagram B in Fig. 3. Using Feynman formula for the left (right) propagators attached to the integration vertex one can perform the last integral that results in the diagram C in Fig. 3.

This diagram, up to a $x-$dependent factor, has the form

$$\mathcal{K}_{\frac{1}{2}}(z_1,w_1)\int_0^1 d\alpha d\beta(\bar\alpha\bar\beta)^{-\delta}(\alpha\beta)^{1-3\delta}\mathcal{K}_{\frac{3}{2}-3\delta}(z_{12}^\alpha,w_{12}^\beta)\,. \tag{85}$$

On the next step we want to get rid of the parametric integrals. To this end we use the properties of the reproducing kernel (A.6) and represent

$$\begin{aligned}\mathcal{K}_{s-\delta}(z_{12}^\alpha,w_{12}^\beta) &= \\ &\int d^2\xi'\,\mu_s(\xi')\mathcal{K}_s(z_{12}^\alpha,\xi')\int d^2\xi\,\mu_s(\xi)\mathcal{K}_{s-\delta}(\xi',\xi)\,\mathcal{K}_s(\xi,w_{12}^\beta) \\ &= \int d^2\xi\,\mu_{s+\delta}(\xi)\mathcal{K}_s(z_{12}^\alpha,\xi)\mathcal{K}_s(\xi,w_{12}^\beta) + O(\delta^2)\,,\end{aligned} \tag{86}$$

where $s = 3/2-2\delta$ and we used the relation (A.7). Using this expression one can carry out the integrals over α,β in (85). The resulting expression for $H(x;z,w)$ (up to a prefactor) takes the form of "$sl(2)$" diagram shown in Fig. 4.

Now we need to calculate the scalar product (83) with the kernel $H(x;z,w)$ given by the diagram on Fig. 4. The scalar product is a function of indices of left (right) propagators and conformal spins of z (w) scalar products, $S(a,b,s_1,s_2|a',b',s_1',s_2')$. We need this function up to $O(\epsilon)$ terms for $a = a' = \alpha$, $b = b' = \beta$ and $s_i = s_i' = 1$. As was explained in the previous section the integral with measure μ_s can be evaluated easily provided that the sum of the indices of propagators coming from this vertex is equal to $2s$. Taking this into account one finds that the scalar product with shifted indices

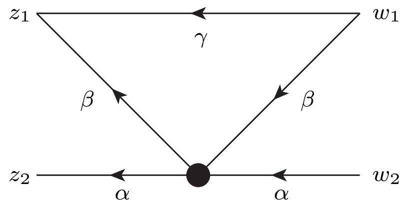

Fig. 4 The "sl(2)" diagram: an arrow line from w to z with index α stands for the propagator $(1-z\bar w)^{-\alpha}$. The indices have the following values: $\alpha = 2-3\epsilon/2$, $\beta = 1-\epsilon/2$ and $\gamma = 1$. The black circle denote an integration vertex with the $sl(2)$ invariant measure $\mu_{s+\delta}$, $s+\delta = 3/2-\epsilon/4$.

$$S(\alpha+\delta,\beta+\delta,1+2\delta,1+4\delta|\alpha-\delta,\beta-\delta,1-2\delta,1-4\delta)$$

can be straightforwardly calculated to

$$S(\epsilon) = \frac{j!\Gamma(3-\epsilon)}{\Gamma(j+3-\epsilon)}\,||z_{12}^j||^2_{1+\frac{1}{2}\epsilon,1+\epsilon} \tag{87}$$

and in the same time it differs from the scalar product in question by terms of order $O(\epsilon^2)$ only.

Restoring all factors one gets for $C(\epsilon)$

$$C(\epsilon) = u_*\,\lambda_s(n^2-1)D^2(x)(x^2\overline{M}^2)^\epsilon\, r^j\,\frac{2+\epsilon}{\epsilon}\,S(\epsilon)\,. \tag{88}$$

Since the ratio (11) does not depend on x, it is convenient to put $x^2\overline{M}^2 = 1$. Finally, subtracting the counterterms

$$\Delta C(\epsilon) = -\frac{1}{\epsilon}4u_*\,\lambda_s(n^2-1)D^2(x)\,r^j\,\frac{||z_{12}^j||^2_{1+\frac{1}{2}\epsilon,1+\frac{1}{2}\epsilon}}{(j+1)(j+2)}\,,$$

one obtains

$$\frac{C(\epsilon)+\Delta C(\epsilon)}{\langle\mathcal{O}_j(x)\mathcal{O}_j(0)\rangle_0} = u_*\lambda_s\frac{2(S_{2j+2}-S_{j+1})}{(j+1)(j+2)} + \cdots \tag{89}$$

where ellipses stand for higher order terms. The corresponding contribution from (89) to the coefficient $T_j^{(2)}$ in the ratio of the correlators, see Eq. (11), reads

$$T_j^{(o)} = -\gamma_j^{(1)}\lambda_s\frac{2(S_{2j+2}-S_{j+1})}{(j+1)(j+2)}\,. \tag{90}$$

The NLO diagrams contributing to the correlator $\langle\partial\mathcal{O}_j(x)\partial\mathcal{O}_j(0)\rangle$ are shown in Fig. 5 (type A) and Fig. 6

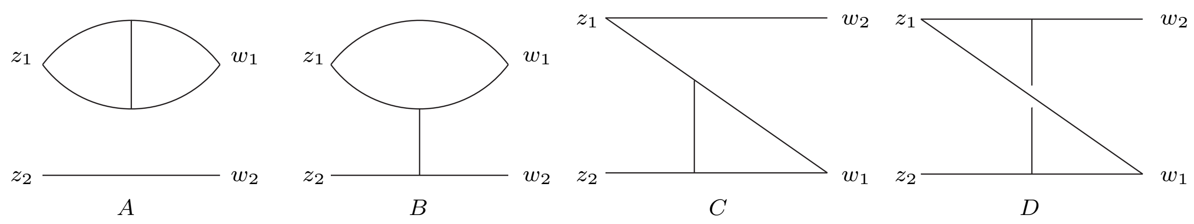

Fig. 5 One loop correction to the correlator of the divergence of conformal operators $\langle\partial\mathcal{O}_j^{(n)}(x)\partial\mathcal{O}_j^{(\bar{n})}(0)\rangle$.

(type B). These diagrams have different color factors: $\lambda_s\lambda_d$ for the A - diagrams, and λ_s^2 for the B - diagrams.

All diagrams of the A - type have the diagram we have just now discussed as a subgraph. The only difference is that in order to kill terms linear in δ one has to consider an average of the diagrams, $(D(\delta)+D(-\delta))/2$. For $\delta=\epsilon/2$ each of the diagrams, $D(\pm\delta)$, can be simplified with the help of the star–triangle relation and rewritten in the form of "$sl(2)$" diagrams. These diagrams in turn can be calculated up to $O(\epsilon^2)$ terms in a manner described above. So we skip all details and present the result for each diagram in Appendix C.

All diagrams of B-type shown schematically in Fig. 6 contain two $2\to 1$ subgraphs. The diagrams depicted on the right panel are finite while those on the left panel are divergent. The calculation of these diagrams does not present any problem so that we give only final results in Appendix C.

Finally, it follows from Eq. (52) that the sum of counterterm diagrams to the diagrams in Figs. 5 and 6 can be written in the form

$$2(Z_3Z_1^{-1}Z_j-1)\,\langle\partial\mathcal{O}_j^{(n)}(x)\partial\mathcal{O}_j^{(\bar{n})}(0)\rangle_0^{(\epsilon)}\,. \tag{91}$$

Here Z_j is the one–loop renormalization constant for the operator $\mathcal{O}_j$,

$$Z_j=1-\frac{u}{\epsilon}\frac{\lambda_s}{(j+1)(j+2)} \tag{92}$$

and

$$Z_1=1-\frac{u\lambda_s}{12\epsilon}+O(u^2),\quad Z_3=1-\frac{u\lambda_d}{4\epsilon}+O(u^2)\,. \tag{93}$$

We have put the superscript (ϵ) to the correlator in order to stress that even the tree–level correlator depends on ϵ through a space-time dimension $d=6-2\epsilon$.

6 Results

The coefficient $T_j^{(2)}$ in the ratio of the correlators is given by the sum of terms in Eqs. (74), (81), (89), (C.13) and (C.14). In the case of operators with other isotopic symmetry these expressions have to be modified. One can separate seven different isotopic projections

$$\mathcal{O}_f^{ab}(x;z_1,z_2)=(P_f)^{ab}_{a'b'}\varphi^{a'}(x+z_1n)\varphi^{b'}(x+z_2n), \tag{94}$$

where $f=1,\ldots,7$. The projectors P_f can be found in Ref. [16]. The one–loop anomalous dimension for the operator $\mathcal{O}_j^{(f)}$ is given by

$$\gamma_j^{f(1)}=\frac{1}{6}\left(\lambda_1-\frac{12\lambda_f}{(j+1)(j+2)}\right)\,, \tag{95}$$

where λ_f are the eigenvalues of the operator $\mathbb{R}^{ab}_{a'b'}=d^{aa'c}d^{bb'c}$ on the invariant subspaces, $\mathbb{R}P_f=\lambda_fP_f$. The explicit expressions for λ_f can be found in [16]. We note also that $\lambda_s=\lambda_1$ and $\lambda_d=2\lambda_3$.

The modifications of the expressions (74), (81), (89), (C.13) and (C.14) for the case of arbitrary projections, P_f, are the following:

- Replace $\gamma_j^{(1)}\to\gamma_j^{(1,f)}$ in all expressions.
- Replace $\lambda_s\to\lambda_f$ in the expressions for $T_j^{(o)}$, $T_j^{(5,B)}$, $T_j^{(5,C)}$, $T_j^{(6,A)}$ and in the last line of T_j^{SE}, Eq. (81).
- Replace $\lambda_s\lambda_d\to 2\nu_f$ in the expression for $T_j^{(5,D)}$,

where ν_f are the eigenvalues of the invariant operator $\mathbb{T}^{ab}_{a'b'}=(\mathbb{R}^2)^{a,b'}_{a'b}$, see Ref. [16].

Representing the ratio of the correlators in the form

$$\mathcal{T}_j^f(u_*)=\varkappa_j\Big(u_*\,T_j^{f(1)}\;+\;u_*^2\,T_j^{f(2)}+\ldots\Big) \tag{96}$$

one obtains for the coefficient $T_j^{f(2)}$:

$$T_j^{f(2)}=\sum_{ab}\lambda_a\,\lambda_b\,T_{j,ab}^{(2)}+\nu_f\,T_{j,f}^{(2)} \tag{97}$$

where

$$T^{(2)}_{j,ss} = -\frac{1}{72}\left(\frac{11}{3} + \frac{2j^2+7j-1}{(j+1)(j+2)(j+3)}\right),$$

$$T^{(2)}_{j,sd} = \frac{1}{12}\left(\frac{2}{3} + \frac{2j^2+5j+1}{(j+1)(j+2)(j+3)}\right),$$

$$T^{(2)}_{j,ff} = 2\frac{j^3+8j^2+10j+1}{(j+1)^3(j+2)^3(j+3)},$$

$$T^{(2)}_{j,fd} = \frac{1}{(j+1)(j+2)}\left(S_{j+1} - 3 + \frac{4}{(j+1)(j+3)}\right),$$

$$T^{(2)}_{j,fs} = -\frac{1}{3(j+1)(j+2)}\left(S_{j+3} - 4 + \frac{j^2+2j+5}{(j+1)(j+2)(j+3)}\right) \tag{98}$$

and

$$T^{(2)}_{j,f} = -\frac{2}{(j+1)^2(j+2)^2}. \tag{99}$$

Finally, comparing (96) with the r.h.s. of Eq. (11) we get for the anomalous dimension

$$\gamma^f_j(u_*) = u_*\gamma^{f,(1)}_j + u_*^2\gamma^{f,(2)}_j + \ldots, \tag{100}$$

where $\gamma^{f(1)}_j$ is given by Eq. (95) and $\gamma^{f(2)}_j$ has the form

$$\gamma^{f(2)}_j = 2\gamma^{(2)}_\varphi + \frac{2\nu_f}{(j+1)^2(j+2)^2} + 2\lambda_f^2\frac{j^2+j-1}{(j+1)^3(j+2)^3} - \frac{\lambda_f\lambda_s}{3(j+1)(j+2)}\left[S_{j+2} - 4 + \frac{2j+3}{2(j+1)(j+2)}\right] + \frac{\lambda_f\lambda_d}{(j+1)(j+2)}\left[S_{j+2} - 3 + \frac{1}{j+1}\right]. \tag{101}$$

This expression completely agrees with the anomalous dimensions reconstructed from the two–loop evolution kernels [16]. We have also checked that the large j expansion of the anomalous dimensions $\gamma^f_j(u_*)$ in terms of the quadratic Casimir

$$J^2 = \left(j + 2 - \epsilon + \frac{1}{2}\gamma^f_j(u_*)\right)\left(j + 1 - \epsilon + \frac{1}{2}\gamma^f_j(u_*)\right)$$

contains only even powers of $1/J$ [27, 28].

7 Summary

We have calculated two–loop anomalous dimensions of the leading–twist operators in the φ^3 model using the approach proposed in Refs. [10, 11]. Formally this method allows one to gain one order in the perturbation theory. However, this advantage is illusory since one has to calculate diagrams including finite parts instead of the pole terms in the standard approach. There is also no essential gain in a complexity of calculations.

Nevertheless, both the contributing diagrams and the methods of calculation are quite different in the two approaches. Therefore the calculation of anomalous dimensions performed in this approach could provide an additional check of the results obtained within the standard approach. Of course, going to the next order is only possible with the advanced methods of computer algebra.

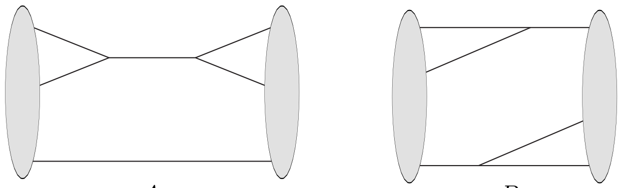

Fig. 6 NLO diagrams for the correlator of the divergence of conformal operators $\langle\partial\mathcal{O}^{(n)}_j(x)\partial\mathcal{O}^{(\bar n)}_j(0)\rangle$.

Acknowledgements We are grateful to Vladimir Braun for useful discussions. This work has been supported by Deutsche Forschungsgemeinschaft through grant MO 1801/1-1.

Appendix A: Discrete series representations of the $su(1,1)$ group

At the leading order the light-ray operator $\mathcal{O}(x; z_1, z_2)$ transforms according to the tensor product of discrete series representations of the group $su(1,1)$. In this Appendix we recall their basis properties. The discrete series representation of the $su(1,1)$ group, D^+_s in the standard notations [18], is defined on the space of functions analytic inside the unit circle, $|z| < 1$,

$$D^+_s(g^{-1})f(z) = (\bar b z + \bar a)^{-2s} f(z'), \tag{A.1}$$

where $z' = (az+b)/(\bar b z + \bar a)$, and $g = \begin{pmatrix} a & b \\ \bar b & \bar a\end{pmatrix} \in su(1,1)$ and s is an integer or half-integer. The generators of the group have the form

$$S_- = -\partial_z, \quad S_0 = z\partial_z + s, \quad S_+ = z^2\partial_z + 2sz. \tag{A.2}$$

The invariant scalar product [18, 19]

$$(D^+_s(g)f, D^+_s(g)\psi)_s = (f, \psi)_s$$

is uniquely defined (up to unitary equivalence)

$$(f,\psi)_s = \int d^2z\, \mu_s(z)\, \overline{f(z)}\, \psi(z), \tag{A.3}$$

where

$$\mu_s(z) = \frac{2s-1}{\pi}(1-|z|^2)^{2s-2}\theta(1-|z|).$$

The space of analytic functions on the unit disk with the scalar product (A.3) is called a holomorphic Hilbert space, D_s^+, see for a review Ref. [22]. The powers of z form an orthogonal basis, $\{e_k(z) = z^k,\ k = 0, 1, \dots\}$, in this space,

$$(e_m, e_k)_s = \delta_{km}||e_k||_s^2 = \delta_{mk}\frac{\Gamma(2s)k!}{\Gamma(k+2s)}\,. \tag{A.4}$$

The unit operator (the reproducing kernel) has the form

$$\mathcal{K}_s(z,w) = \sum_{k=0}^{\infty} e_k(z)\overline{e_k(w)}/||e^k||_s^2 = (1-z\bar{w})^{-2s} \tag{A.5}$$

and for an arbitrary function $f \in D_s^+$ the following identity holds

$$f(z) = \int d^2w\, \mu_s(w)\, \mathcal{K}_s(z,w)\, f(w)\,. \tag{A.6}$$

We note here that all formulae (A.3) – (A.5) have a perfect sense for any $s \geq 1/2$.

Finally, we give a relation that turned out to be very useful in the calculations

$$\int d^2w\, \mu_{s+\epsilon}(w)\, \mathcal{K}_s(z,w)\, f(w) = \\ = \int d^2w\, \mu_s(w)\, \mathcal{K}_{s-\epsilon}(z,w)\, f(w) + O(\epsilon^2)\,. \tag{A.7}$$

It follows immediately from (A.6) if one replaces $s \to s+\epsilon$ and expands it in ϵ.

Appendix B: One loop scalar product

The one loop correction $\varpi^{(1)}$ to the scalar product (32) is determined by Eqs. (33). To find the solution we note that the one loop kernel $\mathcal{H}^{(1)}$ can be represented in the factorized form [6]

$$\mathcal{H}^{(1)} = \overline{\mathcal{F}}_{12}\mathcal{F}_{12} = \overline{\mathcal{F}}_{21}\mathcal{F}_{21}\,, \tag{B.1}$$

where

$$\mathcal{F}_{12} = (\partial_2 z_{21})^{-1}\,, \qquad \overline{\mathcal{F}}_{12} = \left(z_{12}^{-1}\partial_1 z_{12}^2\right)^{-1}\,, \\ \mathcal{F}_{21} = (\partial_1 z_{12})^{-1}\,, \qquad \overline{\mathcal{F}}_{21} = \left(z_{21}^{-1}\partial_2 z_{21}^2\right)^{-1}\,. \tag{B.2}$$

[6] Let us remark that the evoluion kernel $\mathcal{H}^{(1)}$ can be identified with the $sl(2)-$invariant $\mathcal{R}-$operator for a special value of spectral parameter, $\mathcal{H}^{(1)} = \mathcal{R}_{s_1=1,s_2=1}(u=-i)$. The factorization of $\mathcal{H}^{(1)}$ is a consequence of a factorization property of the $\mathcal{R}-$operator [23], see also Ref. [24].

In order to obtain (B.1) it is sufficient to notice that $\mathcal{H}^{(1)}$ is nothing else as the inverse Casimir operator,

$$(\mathcal{H}^{(1)})^{-1} = -\partial_1\partial_2 z_{12}^2 = \mathcal{F}_{12}^{-1}\,\overline{\mathcal{F}}_{12}^{-1} = \mathcal{F}_{21}^{-1}\,\overline{\mathcal{F}}_{21}^{-1}\,.$$

While the operator $\mathcal{H}^{(1)} : D_+^{(1)} \otimes D_+^{(1)} \mapsto D_+^{(1)} \otimes D_+^{(1)}$, the $\mathcal{F}$ operators intertwine the representations with different spins. Namely,

$$\mathcal{F}_{12}\, D_+^{(1)} \otimes D_+^{(1)} = D_+^{(\frac{3}{2})} \otimes D_+^{(\frac{1}{2})}\, \mathcal{F}_{12}\,, \\ \overline{\mathcal{F}}_{12}\, D_+^{(\frac{3}{2})} \otimes D_+^{(\frac{1}{2})} = D_+^{(1)} \otimes D_+^{(1)}\, \overline{\mathcal{F}}_{12}\,, \tag{B.3}$$

and similar for $\mathcal{F}_{21}$. It results in the intertwining relations for two-particle generators, $S_\alpha^{(s_1,s_2)} = S_\alpha^{(s_1)} + S_\alpha^{(s_2)}$,

$$\mathcal{F}_{12}S_\alpha^{(1,1)} = S_\alpha^{(\frac{3}{2},\frac{1}{2})}\mathcal{F}_{12}\,, \quad \mathcal{F}_{21}S_\alpha^{(1,1)} = S_\alpha^{(\frac{1}{2},\frac{3}{2})}\mathcal{F}_{21} \tag{B.4}$$

and so on.

Next, we introduce one particle operator

$$W^{(s)}f(z) = \int_0^1 d\alpha\,\frac{\bar{\alpha}^{2s-1}}{\alpha}\Big(f(z) - f(\bar{\alpha}z)\Big)\,, \tag{B.5}$$

such that $W^{(s)}z^n = (\psi(n+2s) - \psi(2s))z^n$. This operator commutes with the generator $S_0^{(s)}$ while

$$[S_+^{(s)}, W^{(s)}] = -z\,. \tag{B.6}$$

Now let us check that

$$\varpi^{(1)} = (\epsilon - u_*\gamma_\varphi^{(1)})\Big(W_1^{(1)} + W_2^{(1)}\Big) \\ + \lambda_s u_*\Big(\overline{\mathcal{F}}_{12}W_2^{(\frac{1}{2})}\mathcal{F}_{12} + \overline{\mathcal{F}}_{21}W_1^{(\frac{1}{2})}\mathcal{F}_{21}\Big) \tag{B.7}$$

gives solution to Eqs. (33). The first equation is obviously satisfied, $[S_0^{(0)}, \varpi^{(1)}] = 0$. Next, making use of Eqs. (B.4), (B.6) one gets for the commutator

$$[S_+^{(0)}, \varpi^{(1)}] = -(\epsilon - u_*\gamma_\varphi^{(1)})(z_1 + z_2) \\ - \lambda_s u_*\Big(\overline{\mathcal{F}}_{12}z_2\mathcal{F}_{12} + \overline{\mathcal{F}}_{21}z_1\mathcal{F}_{21}\Big)\,. \tag{B.8}$$

Finally, one casts the r.h.s. into the necessary form taking into account that $[\overline{\mathcal{F}}_{12}, z_2] = [\overline{\mathcal{F}}_{21}, z_1] = 0$.

The scalar product can also be written in the form

$$(f,\psi)_\varpi = (f,\psi)_{s_*,s_*} + \lambda_s u_*\Big[(\mathcal{F}_{12}f, W_2^{(\frac{1}{2})}\mathcal{F}_{12}\psi)_{\frac{3}{2},\frac{1}{2}} \\ + (\mathcal{F}_{21}f, W_1^{(\frac{1}{2})}\mathcal{F}_{21}\psi)_{\frac{1}{2},\frac{3}{2}}\Big] + O(\epsilon^2)\,, \tag{B.9}$$

where $s_* = 2 - \epsilon + \gamma_\varphi^*$ is the conformal spin of the basic field at the critical point and $(f,\psi)_{s_1,s_2}$ stays for the two particle scalar product. We have to mention here that the solution of Eqs. (33) is not unique. For instance, at one loop order $\varpi' = \varpi^{(1)} + Z$, where Z is an invariant operator, $[Z, S_\alpha^{(0)}] = 0$, also satisfies Eqs. (33).

Closing this section we give the standard representation for the conformal operator,

$$\mathcal{O}_j(x) = P_j(\partial_{z_1}, \partial_{z_2})[\mathcal{O}(x; z_1, z_2)]\Big|_{z_1=z_2=0}. \qquad (B.10)$$

The operator $\mathcal{O}_j$ is completely determined by a polynomial $P_j(z_1, z_2)$. It was known a long ago [25] that at the leading order

$$P_j(z_1, z_2) \sim (z_1 + z_2)^j C_j^{(3/2)}\left(\frac{z_1 - z_2}{z_1 + z_2}\right),$$

where $C_j^{(3/2)}$ is the Gegenbauer polynomial. Beyond the leading order one derives from Eqs. (36) and (B.9)

$$P_j(z_1, z_2) = (z_1 + z_2)^j \Bigg\{ p_j^{(\lambda)}(z_1, z_2) - 4\, u_* \lambda_s \sum_{k=0,2\ldots}^{j} b_k^j\, p_k^{(\lambda)}(z_1, z_2) \Bigg\}, \qquad (B.11)$$

where $\lambda = 2s_* - 1/2$,

$$p_j^{(\lambda)}(z_1, z_2) = \frac{j!\,\Gamma(2\lambda)\Gamma(\lambda + \frac{1}{2})}{\Gamma(j + 2\lambda)\Gamma(j + \lambda + \frac{1}{2})} C_j^{(\lambda)}\left(\frac{z_1 - z_2}{z_1 + z_2}\right),$$

and the expansion coefficients have the form

$$b_j^j = \frac{1}{(j+1)(j+2)}\left[S_{2j+2} - 2S_{j+1} - \frac{1}{(j+1)(j+2)}\right],$$
$$b_{k<j}^j = \frac{(2k+3)}{(j-k)(j+k+3)} \frac{(k+1)!}{(j+2)!(j+1)}. \qquad (B.12)$$

The expression (B.11) agrees with the expression for the conformal operator obtained in [11, 13].

Appendix C: Loop diagrams

We will present an answer for a diagram $D^{(a)}$ minus the counterterm $\Delta D^{(a)}$ in the form

$$x^2(n\bar{n})\,(D^{(a)} - \Delta D^{(a)}) = u_*^2 \cdot \varkappa_j T_j^{(a)} \langle \mathcal{O}_j^{(n)}(x) \mathcal{O}_j^{(\bar{n})} \rangle_0 .$$

Here $\langle \mathcal{O}_j^{(n)}(x) \mathcal{O}_j^{(\bar{n})} \rangle_0$ is the tree level correlator. For the diagrams in Fig. 5 and 6 we obtain

$$T_j^{(5,A)} = \frac{\lambda_s \lambda_d}{12}\left[S_{2j+2} - S_{j+3} - S_{j+2} + \frac{7}{3}\right],$$
$$T_j^{(5,B)} = \frac{\lambda_s \lambda_d}{(j+1)(j+2)}\left[S_{2j+2} - S_{j+3} + \frac{1}{j+1}\right],$$
$$T_j^{(5,C)} = -\frac{2\lambda_s \lambda_d}{(j+1)(j+2)}\left[S_{2j+2} - S_{j+3} - \frac{1}{2}S_j + 1\right],$$
$$T_j^{(5,D)} = -\frac{\lambda_s \lambda_d}{(j+1)^2(j+2)^2}, \qquad (C.13)$$

and

$$T_j^{(6,A)} = -\gamma_j^{(1)} \Bigg\{ \gamma_j^{(1)} \left[S_{2j+2} - S_{j+3} - S_{j+2} + \frac{5}{3}\right] - \frac{2\lambda_s}{(j+1)(j+2)}\left[S_{j+2} - \frac{(2j+1)(4j+7)}{3(j+1)(j+2)}\right] \Bigg\},$$
$$T_j^{(6,B)} = -\frac{1}{2}(\gamma_j^{(1)})^2 \frac{j^2 + 3j + 4}{(j+1)(j+5)}. \qquad (C.14)$$

References

1. S. Moch, J. A. M. Vermaseren and A. Vogt, *The Three loop splitting functions in QCD: The Nonsinglet case,* Nucl. Phys. B **688** (2004) 101.
2. A. Vogt, S. Moch and J. A. M. Vermaseren, *The Three-loop splitting functions in QCD: The Singlet case,* Nucl. Phys. B **691** (2004) 129.
3. S. Moch, J. A. M. Vermaseren and A. Vogt, *The Three-Loop Splitting Functions in QCD: The Helicity-Dependent Case,* Nucl. Phys. B **889** (2014) 351.
4. M. Moshe and J. Zinn-Justin, *Quantum field theory in the large N limit: A Review,* Phys. Rept. **385**, 69 (2003) [hep-th/0306133].
5. A. N. Vasiliev, Y. M. Pismak and Y. R. Khonkonen, *1/n Expansion: Calculation Of The Exponent Eta In The Order* $1/n^3$ *By The Conformal Bootstrap Method,* Theor. Math. Phys. **50** (1982) 127 [Teor. Mat. Fiz. **50** (1982) 195].
6. A. N. Vasiliev, S. E. Derkachov, N. A. Kivel and A. S. Stepanenko, *The 1/n expansion in the Gross-Neveu model: Conformal bootstrap calculation of the index eta in order* $1/n^3$, Theor. Math. Phys. **94** (1993) 127 [Teor. Mat. Fiz. **94** (1993) 179].
7. J. A. Gracey, *Computation of critical exponent eta at* $O(1/N^3)$ *in the four Fermi model in arbitrary dimensions,* Int. J. Mod. Phys. A **9** (1994) 727.
8. J. A. Gracey, *Anomalous dimension of nonsinglet Wilson operators at* $O(1/N(f))$ *in deep inelastic scattering,* Phys. Lett. B **322** (1994) 141.
9. J. F. Bennett and J. A. Gracey, *Determination of the anomalous dimension of gluonic operators in deep inelastic scattering at* $O(1/N(f))$, Nucl. Phys. B **517** (1998) 241.
10. D. Anselmi, *The N=4 quantum conformal algebra,* Nucl. Phys. B **541** (1999) 369.
11. A. V. Belitsky, J. Henn, C. Jarczak, D. Mueller and E. Sokatchev, *Anomalous dimensions of leading twist conformal operators,* Phys. Rev. D **77** (2008) 045029.
12. S. Ferrara, A. F. Grillo, G. Parisi and R. Gatto, *Canonical scaling and conformal invariance*, Phys. Lett. **B38**, 333 (1972).

13. D. Müller, *Constraints for anomalous dimensions of local light cone operators in ϕ^3 in six-dimensions theory*, Z. Phys. C **49** (1991) 293.
14. D. Müller, *Conformal constraints and the evolution of the nonsinglet meson distribution amplitude*, Phys. Rev. D **49** (1994) 2525.
15. D. Müller, *Restricted conformal invariance in QCD and its predictive power for virtual two photon processes*, Phys. Rev. D **58** (1998) 054005.
16. V. M. Braun and A. N. Manashov, *Evolution equations beyond one loop from conformal symmetry,* Eur. Phys. J. C **73** (2013) 2544.
17. I. I. Balitsky and V. M. Braun, *Evolution Equations for QCD String Operators,* Nucl. Phys. B **311** (1989) 541.
18. I.M. Gelfand, M.I. Graev, N.Ya. Vilenkin, *Generalized functions. Vol. 5: Integral geometry and representation theory,* Academic Press (New York, 1966).
19. Vilenkin, N. Ya., *Special Functions and the Theory of Group Representations.* Translations of Mathematical Monographs 22, Amer. Math. Soc., Providence, Rhode Island, 1968.
20. V. M. Braun and A. N. Manashov, *Operator product expansion in QCD in off-forward kinematics: Separation of kinematic and dynamical contributions,* JHEP **1201** (2012) 085.
21. V. Bargmann and I. T. Todorov, *Spaces of Analytic Functions on a Complex Cone as Carries for the Symmetric Tensor Representations of $SO(N)$,* J. Math. Phys. **18** (1977) 1141.
22. B.C. Hall, *Holomorphic Methods in Mathematical Physics," Contemporary Mathematics,* Volume 260, pp. 1-59 [quant-ph/9912054].
23. S. E. Derkachov, *Factorization of the R-matrix. I.,* math/0503396 [math-qa].
24. S. E. Derkachov and A. N. Manashov, *Factorization of the transfer matrices for the quantum sl(2) spin chains and Baxter equation,* J. Phys. A **39** (2006) 4147.
25. Y. M. Makeenko, *Conformal Operators In Quantum Chromodynamics,* Sov. J. Nucl. Phys. **33** (1981) 440 [Yad. Fiz. **33** (1981) 842].
26. M. Ciuchini, S. E. Derkachov, J. A. Gracey and A. N. Manashov, *Computation of quark mass anomalous dimension at $O(1/N_f^2)$ in quantum chromodynamics,* Nucl. Phys. B **579** (2000) 56.
27. B. Basso and G. P. Korchemsky, *Anomalous dimensions of high-spin operators beyond the leading order,* Nucl. Phys. B **775** (2007) 1.
28. L. F. Alday, A. Bissi and T. Lukowski, *Large spin systematics in CFT,* arXiv:1502.07707 [hep-th].