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Isomorphism of critical and off-critical operator spaces in two-dimensional quantum field theory

Gesualdo Delfino[a,b] and Giuliano Niccoli[c] *

[a] *International School for Advanced Studies (SISSA)*
via Beirut 2-4, 34014 Trieste, Italy
[b] *INFN sezione di Trieste*
[c] *LPTM, Université de Cergy-Pontoise*
2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise, France
E-mail: delfino@sissa.it, giuliano.niccoli@desy.de

Abstract

For the simplest quantum field theory originating from a non-trivial fixed point of the renormalization group, the Lee-Yang model, we show that the operator space determined by the particle dynamics in the massive phase and that prescribed by conformal symmetry at criticality coincide.

* Present address: DESY Theory, Notkestr. 85, 22607 Hamburg, Germany.

1 Introduction

The local operators of a quantum field theory form an infinite-dimensional linear space which the transformation properties under space-time and internal symmetries naturally decompose into subspaces, each containing infinitely many operators with assigned spin and charge. For the generic quantum field theory describing a renormalization group trajectory flowing out of a non-trivial fixed point, a further, quantitative characterization of the operator space is a very difficult task. Particularly relevant is the characterization of the operators by their scaling dimension, as determined by the short distance behavior of the two point functions. The spectrum of the scaling dimensions, as well as the number of operators sharing the same dimension, are distinguishing data of the theory which remain, however, out of reach in the generic case.

Two-dimensional quantum field theory enjoys in this respect a special status. Here, the infinite-dimensional nature of conformal symmetry allows the solution of fixed point theories [1], revealing in particular a decomposition of the operator space into families containing operators with integer-spaced scaling dimensions. The number of families, the scaling dimensions and the degeneracy within each integer level are all known. Two non-negative integers, l and $\bar{l}$, determine, through their difference and their sum, the spin and (up to a constant characteristic of the family) the scaling dimension of an operator.

Perturbative considerations [2, 3, 4] suggest that, up to symmetry breaking effects, such a structure should survive a perturbation of the fixed point leading to a massive theory. The existence in two dimensions of massive integrable theories provides the chance of testing non-perturbatively this conjecture, but also poses a problem which is intriguing and challenging at the same time. Indeed, integrable massive theories are solved *on-shell*, through the determination of the exact S-matrix, and any information about the operators needs to be extracted from the particle dynamics. The monodromy properties and the singularity structure provide a set of equations [5, 6] for the matrix elements of the operators on asymptotic states (form factors) which have the S-matrix as their only input, and whose space of solutions is expected to coincide with the operator space of the theory.

The latter issue was first addressed by Cardy and Mussardo in [7], where the simplest case, that of the thermal Ising model, was investigated. They showed that the number of solutions of the form factor equations with spin s and with the mildest asymptotic behavior at high energy coincides with the number of chiral operators (i.e. having $\bar{l} = 0$) fixed by conformal field theory for spin $s = l$ at the ultraviolet fixed point[1]. Mildest asymptotic behavior is a reasonable conjecture for the chiral operators, which are the most relevant among the operators with given spin.

While clearly supporting the idea of the isomorphism of critical and off-critical operator spaces, this original study deferred to subsequent investigations some important issues. First of all, the counting of generic, non-chiral, operators requires the introduction in the form factor approach of some information about the levels, which are no longer uniquely specified by the

[1] Of course a similar analysis can be performed for the operators with $l = 0$ (antichiral).

spin. Moreover, as already pointed out in [7], the case of the thermal Ising model is "deceptively simple" due to its equivalence with a free fermionic theory[2]. In particular, all the form factor solutions can be generated by repeated action of the conserved quantities, a circumstance which does not persist for the generic integrable theory.

The problem of considering interacting integrable theories was tackled by Koubek [8], who extended the analysis of Cardy and Mussardo for the *conjectured* chiral solutions to several massive deformations of minimal conformal models. She also performed a more general counting of the form factor solutions according to the asymptotic behavior for given spin, obtaining very suggestive formal relations with the characters which in conformal field theory specify the structure of the operator families. The inability to establish a connection with the levels l and $\bar{l}$, however, prevented her from showing the isomorphism between the conformal and massive operator spaces. It was shown in [9, 10, 11] how this type of counting can be extended to the sine-Gordon model and its restrictions.

In this paper we show explicitly for the massive Lee-Yang model that the space of solutions of the form factor equations decomposes into subspaces labeled by a pair of non-negative integers $(l, \bar{l})$ related to the spin and to the asymptotic behavior of the form factors at high energy. We then show that the dimension of each subspace exactly coincides with the dimension of the subspace of the conformal operator space with levels $(l, \bar{l})$, proving in this way the isomorphism between the critical and off-critical operator spaces. The choice of the Lee-Yang model for a first time proof is obvious: on one hand, this model is fully representative of generic integrable quantum field theories with respect to the features which are of interest here (it is a massive, interacting theory originating from a non-trivial fixed point of the renormalization group, with an operator space which cannot be entirely generated by repeated action of conserved quantities on lowest level operators); on the other, it minimizes the technicalities and best illustrates the essential points due to the presence of the minimal number of operator families (two) and of a single species of massive particles.

The paper is organized as follows. After recalling in the next section the structure of the operator space of the Lee-Yang model at criticality, we turn in section 3 to the analysis of the space of solutions of the form factor equations in the massive theory. The comparison between the critical and off-critical operator spaces is then performed in section 4. Section 5 contains few final remarks while some technical parts of the proof are detailed in two appendices.

2 The conformal operator space

At a critical point the operators undergo the general conformal field theory classification [1]. A scaling operator $\Phi(x)$ is first of all labeled by a pair $(\Delta_\Phi, \bar{\Delta}_\Phi)$ of conformal dimensions which

[2] The spin sector considered in [7], however, is non-trivial due to non-locality with respect to the fermions.

determine the scaling dimension X_Φ and the spin s_Φ as

$$X_\Phi = \Delta_\Phi + \bar{\Delta}_\Phi \tag{2.1}$$

$$s_\Phi = \Delta_\Phi - \bar{\Delta}_\Phi . \tag{2.2}$$

There exist operator families associated to the lowest weight representations of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \, . \tag{2.3}$$

The L_n's generate the conformal transformations associated to the complex variable $z = x_1 + ix_2$, with the central charge c labeling the conformal theory. The same algebra, with the same value of c, holds for the generators $\bar{L}_n$ of the conformal transformations in the variable $\bar{z} = x_1 - ix_2$. The L_n's commute with the $\bar{L}_m$'s. Each operator family consists of a primary operator Φ_0 (which is annihilated by all the generators L_n and $\bar{L}_n$ with $n > 0$) and infinitely many descendant operators obtained through the repeated action on the primary of the Virasoro generators. A basis in the space of descendants of Φ_0 is given by the operators

$$L_{-i_1} \dots L_{-i_I} \bar{L}_{-j_1} \dots \bar{L}_{-j_J} \, \Phi_0 \tag{2.4}$$

with

$$0 < i_1 \leq i_2 \leq \dots \leq i_I \tag{2.5}$$

$$0 < j_1 \leq j_2 \leq \dots \leq j_J \, . \tag{2.6}$$

The *levels*

$$(l, \bar{l}) = \left(\sum_{n=1}^{I} i_n \, , \sum_{n=1}^{J} j_n \right) \tag{2.7}$$

determine the conformal dimensions of the descendants (2.4) in the form

$$(\Delta, \bar{\Delta}) = (\Delta_{\Phi_0} + l, \bar{\Delta}_{\Phi_0} + \bar{l}) \, . \tag{2.8}$$

In general the number of independent operators at level $(l, \bar{l})$ is $p(l)p(\bar{l})$, $p(l)$ being the number of partitions of l into positive integers. This number, however, is reduced in a model dependent way in presence of degenerate representations associated to primary operators $\phi_{r,s}$ which possess a vanishing linear combination of descendant operators (null vector) when l or $\bar{l}$ equals rs. So the dimensionality of the subspace with levels $(l, \bar{l})$ in the family of Φ_0 is usually written as $d_{\Phi_0}(l) d_{\Phi_0}(\bar{l})$, with the integers $d_{\Phi_0}(l)$ which can be encoded in the rescaled *character*

$$\chi_{\Phi_0}(q) = \sum_{l=0}^{\infty} d_{\Phi_0}(l) q^l \, . \tag{2.9}$$

The conformal field theory with the smallest operator content is the minimal model $\mathcal{M}_{2,5}$, with central charge $c = -22/5$, possessing only two primary operators: the identity $I = \phi_{1,1} = \phi_{1,4}$ with conformal dimensions $(0,0)$, and the operator $\varphi = \phi_{1,2} = \phi_{1,3}$ with conformal dimensions $(-1/5, -1/5)$. The negative values of the central charge and of X_φ show that the model does not satisfy reflection-positivity. It goes under the name of Lee-Yang model because it describes (see [12]) the universal properties of the edge singularity of the zeros of the partition function of the Ising model in an imaginary magnetic field [13, 14, 15].

The characters for the two operator families of the Lee-Yang model are [16, 17, 18]

$$\chi_I(q) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{2+5n})\,(1-q^{3+5n})} \tag{2.10}$$

$$\chi_\varphi(q) = \prod_{n=0}^{\infty} \frac{1}{(1-q^{1+5n})\,(1-q^{4+5n})}\,. \tag{2.11}$$

They also enjoy the following ('fermionic' [19]) representation based on the Rogers-Ramanujan identities [20]:

$$\chi_I(q) = G_{-1}, \quad \chi_\varphi(q) = G_0, \tag{2.12}$$

where

$$G_p = \sum_{k=0}^{\infty} \frac{q^{k(k-p)}}{(q)_k}, \tag{2.13}$$

$$(q)_k = \prod_{i=1}^{k}(1-q^i)\,. \tag{2.14}$$

This representation, when compared with the definition (2.9), yields the following expansions for the dimensions of the spaces of chiral descendants of level l

$$d_I(l) = \sum_{N=0}^{\infty} P(N, l-N(N+1)), \quad d_\varphi(l) = \sum_{N=0}^{\infty} P(N, l-N^2), \tag{2.15}$$

where $P(N,M)$ is the number of the partitions of the non-negative integer M into the integers $1, 2, \ldots, N$; it is generated by

$$\frac{1}{(q)_N} = \sum_{M=0}^{\infty} P(N,M)q^M. \tag{2.16}$$

We set

$$P(0,0)=1, \quad P(N,M)=0 \quad \text{for } N \geq 0 \text{ and } M<0, \quad P(0,M)=0 \quad \text{for } M>0; \tag{2.17}$$

notice that $P(N,0)=1$. The generating functions G_p satisfy the recursion relation

$$G_p = G_{p-1} + q^{1-p}G_{p-2}, \tag{2.18}$$

which implies

$$\chi_I(q) + \chi_\varphi(q) = G_1, \tag{2.19}$$

and leads to

$$d_I(l) + d_\varphi(l) = \sum_{N=0}^{\infty} P(N, l-N(N-1)). \tag{2.20}$$

The occurrence of different representations for the characters of rational conformal field theories is a general phenomenon. Fermionic representations have been derived for classes of rational conformal theories defined as cosets of affine Lie algebras [19] and in particular for the series of non-unitary minimal models $M(2, 2p+3)$ [21]. These representations follow by a quasi-particle interpretation based on the Bethe Ansatz description and give alternative representations of the characters with respect to the Feigin-Fuchs-Felder construction [22]. Their derivation uses generalizations of the Rogers-Ramanujan identities (the Gordon-Andrews identities [23]).

3 Operators in the massive theory

A renormalization group trajectory originating from the Lee-Yang conformal point is obtained perturbing the model $\mathcal{M}_{2,5}$ with its only non-trivial primary operator φ. This gives the massive Lee-Yang model with action

$$\mathcal{A} = \mathcal{A}_{CFT} + g \int d^2x\, \varphi(x)\,. \tag{3.1}$$

It follows from the general results of [2] about perturbed conformal field theories that the theory (3.1) belongs to the class of integrable quantum field theories. These are characterized by the existence of an infinite number of conserved quantities which induces the complete elasticity and factorization of the scattering processes, and allows the exact determination of the S-matrix [24]. It was shown in [25] that the massive Lee-Yang model has a mass spectrum containing a single species of neutral particles $A(\theta)$ with two-body scattering determined by the amplitude[3]

$$S(\theta) = \frac{\tanh\frac{1}{2}\left(\theta + \frac{2i\pi}{3}\right)}{\tanh\frac{1}{2}\left(\theta - \frac{2i\pi}{3}\right)}\,, \tag{3.2}$$

which, due to factorization, specifies the full S-matrix. The bound state pole located at $\theta = 2i\pi/3$ corresponds to the fusion process $AA \to A$ and has the residue

$$\text{Res}_{\theta=2i\pi/3} S(\theta) = i\Gamma^2, \tag{3.3}$$

where $\Gamma = i2^{1/2}3^{1/4}$ is the three-particle coupling; the fact that Γ is purely imaginary is again related to the lack of reflection-positivity [25].

Within integrable quantum field theory the operators are constructed determining their matrix elements on the asymptotic particle states. All the matrix elements of a given local operator $\Phi(x)$ can be obtained from the n-particle form factors

$$F_n^\Phi(\theta_1, \ldots, \theta_n) = \langle 0|\Phi(0)|\theta_1 \ldots \theta_n\rangle, \tag{3.4}$$

where $|0\rangle$ denotes the vacuum (i.e. zero-particle) state. The form factors satisfy a set of functional equations taking into account the spin s_Φ of the operator, the monodromy properties

[3] The on-shell two-momentum of a particle of mass m is parameterized by a rapidity variable θ as $(p^0, p^1) = (m\cosh\theta, m\sinh\theta)$. In (3.2) θ denotes the rapidity difference of the colliding particles.

under analytic continuation in rapidity space and the pole singularities associated to bound states and annihilation processes [5, 6]. For the Lee-Yang model these equations read

$$F_n^\Phi(\theta_1+\alpha,\dots,\theta_n+\alpha) = e^{s_\Phi \alpha} F_n^\Phi(\theta_1,\dots,\theta_n) \tag{3.5}$$

$$F_n^\Phi(\theta_1,\dots,\theta_i,\theta_{i+1},\dots,\theta_n) = S(\theta_i-\theta_{i+1})\, F_n^\Phi(\theta_1,\dots,\theta_{i+1},\theta_i,\dots,\theta_n) \tag{3.6}$$

$$F_n^\Phi(\theta_1+2i\pi,\theta_2,\dots,\theta_n) = F_n^\Phi(\theta_2,\dots,\theta_n,\theta_1) \tag{3.7}$$

$$\mathrm{Res}_{\theta'=\theta}\, F_{n+2}^\Phi(\theta'+\frac{i\pi}{3},\theta-\frac{i\pi}{3},\theta_1,\dots,\theta_n) = i\Gamma\, F_{n+1}^\Phi(\theta,\theta_1,\dots,\theta_n) \tag{3.8}$$

$$\mathrm{Res}_{\theta'=\theta+i\pi}\, F_{n+2}^\Phi(\theta',\theta,\theta_1,\dots,\theta_n) = i\left[1-\prod_{j=1}^{n} S(\theta-\theta_j)\right] F_n^\Phi(\theta_1,\dots,\theta_n)\,, \tag{3.9}$$

with $S(\theta)$ and Γ specified above. The space of solutions of these equations is linear in the operators and is expected to coincide with the infinite-dimensional operator space of the massive theory. It is our task to show that this space of solutions is isomorphic to the conformal operator space described in the previous section.

Let us start writing the general solution to the equations (3.6)–(3.9). It reads

$$F_n^\Phi(\theta_1,\dots,\theta_n) = U_n^\Phi(\theta_1,\dots,\theta_n) \prod_{1\le i<j\le n} \frac{F_{min}(\theta_i-\theta_j)}{\cosh\frac{\theta_i-\theta_j}{2}\left[\cosh(\theta_i-\theta_j)+\frac{1}{2}\right]}. \tag{3.10}$$

Here the factors in the denominator introduce the bound state and annihilation poles prescribed by (3.8) and (3.9), which are the only singularities of the form factors in rapidity space, while

$$F_{min}(\theta) = -i\sinh\frac{\theta}{2}\,\exp\left\{2\int_0^\infty \frac{dt}{t}\frac{\cosh\frac{t}{6}}{\cosh\frac{t}{2}\sinh t}\sin^2\frac{(i\pi-\theta)t}{2\pi}\right\} \tag{3.11}$$

is the solution of the equations

$$F(\theta) = S(\theta)F(-\theta) \tag{3.12}$$

$$F(\theta+2i\pi) = F(-\theta) \tag{3.13}$$

free of zeros and poles for $\mathrm{Im}\theta \in (0,2\pi)$; it behaves asymptotically as

$$\lim_{|\theta|\to\infty} e^{-|\theta|} F_{min}(\theta) = C_\infty, \tag{3.14}$$

with

$$C_\infty = \frac{-1}{4\gamma^2}\,, \qquad \gamma = \exp\left\{2\int_0^\infty \frac{dt}{t}\,\frac{\sinh\frac{t}{2}\sinh\frac{t}{3}\sinh\frac{t}{6}}{\sinh^2 t}\right\}. \tag{3.15}$$

All the information specifying the operator Φ is contained in the functions $U_n^\Phi(\theta_1,\dots,\theta_n)$. They are entire functions of the rapidities, symmetric and (up to a factor $(-1)^{n-1}$) $2\pi i$-periodic in all θ_j's, and homogeneous of degree s_Φ (i.e. they account for the property (3.5)). Of course the functions U_n^Φ with different n are related by the residue equations (3.8) and (3.9). These equations allow to build a solution starting from an initial condition for $n=1$, and then determining the matrix elements with a larger number of particles. In doing this, however, one should keep in mind that there can be more solutions corresponding to a given initial condition.

Indeed, N-particle matrix elements with vanishing residues on the bound state and kinematical poles are themselves initial conditions of *kernel* solutions which in a linear combination give no contribution for $n < N$. Enumerating the kernel solutions is then essential for counting the independent solutions of the form factor equations, as originally observed in [26].

We call *minimal* scalar N-kernel solution $K_n^N(\theta_1, ..., \theta_n)$ the solution of the form factor equations (3.5)–(3.9) with $s_\Phi = 0$ and initial condition

$$K_n^N(\theta_1, ..., \theta_n) \;=\; \begin{cases} 0 & \text{for } n < N \\ \\ \prod_{1 \leq i < j \leq N} F_{min}(\theta_i - \theta_j) & \text{for } n = N, \end{cases} \tag{3.16}$$

where $N \geq 2$. The initial condition of the general spin s N-kernel solution differs from this one by a multiplicative factor which is an entire function of the rapidities, symmetric and $2\pi i$-periodic in all θ_j's and homogeneous of degree s. After introducing the elementary symmetric polynomials $\sigma_i^{(n)}$ generated by

$$\prod_{i=1}^{n}(x + x_i) = \sum_{k=0}^{n} x^{n-k} \sigma_k^{(n)}(x_1, \ldots, x_n)\,, \tag{3.17}$$

with $x_i \equiv e^{\theta_i}$, a basis in the space of N-kernel solutions is provided by the solutions $K_n^{(a_1,..,a_{N-1}|A)}$ with initial condition

$$K_n^{(a_1,..,a_{N-1}|A)}(\theta_1, ..., \theta_n) = \begin{cases} 0 & \text{for } n < N \\ \\ (\sigma_N^{(N)})^A \prod_{1 \leq i \leq N-1} (\sigma_i^{(N)})^{a_i} K_N^N(\theta_1, ..., \theta_N) & \text{for } n = N, \end{cases} \tag{3.18}$$

where $a_1, \ldots a_{N-1}$ are non-negative integers and A is an integer. If we formally define the spin s '1-kernel' solution with initial condition

$$K_n^{(s)}(\theta) = \begin{cases} 0 & \text{for } n = 0 \\ \\ \left(\sigma_1^{(1)}\right)^s = e^{s\theta} & \text{for } n = 1, \end{cases} \tag{3.19}$$

and formally associate the identity solution $F_n^I = \delta_{n,0}$ to $N = 0$, we have that *all* the solutions of the form factor equations (3.5)–(3.9) can be written as linear combinations of the N-kernel solutions with $N \geq 0$. In the following we perform our analysis of the space of solutions within this basis.

It follows from (3.18) and (3.19) that the spin is

$$s = \sum_{i=1}^{N-1} i a_i + NA\,. \tag{3.20}$$

Since (3.14) implies

$$K_N^N\left(\theta_1 + \alpha, .., \theta_k + \alpha, \theta_{k+1}, ..., \theta_N\right) \sim e^{k(N-k)\alpha} \tag{3.21}$$

for $\alpha \to +\infty$, $N > 1$ and $1 \leq k \leq N-1$, we have[4], in the same limit and within the same restrictions on N and k,

$$K_N^{(a_1,..,a_{N-1}|A)}(\theta_1+\alpha,\ldots,\theta_k+\alpha,\theta_{k+1},\ldots,\theta_n) \sim e^{y_k\alpha}, \tag{3.22}$$

with

$$y_k = \sum_{i=1}^{k-1} i a_i + k\left(\sum_{i=k}^{N-1} a_i + A + N - k\right). \tag{3.23}$$

After defining

$$y = \mathrm{Max}\{y_k\}_{k=\{1,\ldots,N-1\}}, \tag{3.24}$$

we can attach to each solution $K_n^{(a_1,..,a_{N-1}|A)}$ two non-negative integers l and $\bar{l}$ in the following way

$$l = \mathrm{Max}\{s, y, 0\}, \tag{3.25}$$
$$\bar{l} = l - s\,. \tag{3.26}$$

By definition, the condition

$$y_k \leq l, \tag{3.27}$$

is satisfied and, if both l and $\bar{l}$ are non-vanishing, there certainly exists at least one value of k for which it holds as an equality. Since $y_{N-1} = -A + s + N - 1$, (3.27) with $k = N-1$ gives $A \geq N - \bar{l} - 1$, and then the parameterization

$$A = a_N + N - \bar{l} - 1, \tag{3.28}$$

with a_N a non-negative integer.

Then we see that to each $K_n^{(a_1,..,a_{N-1}|A)}$ with $N > 1$ in the basis of kernel solutions we can associate two non-negative intergers, l and $\bar{l}$, whose difference coincides with the spin. Moreover, taking into account (3.28), each solution is identified by N non-negative integers $a_1,\ldots,a_N$, and we will use the notation

$$K_n^{(a_1,..,a_N)}(\theta_1,\ldots,\theta_n) = K_n^{(a_1,..,a_{N-1}|A)}(\theta_1,\ldots,\theta_n)\,. \tag{3.29}$$

As for the cases $N = 0, 1$, since the set of y_k's is empty, we set $y \equiv 0$, so that l and $\bar{l}$ are still defined by (3.25) and (3.26). Notice that no N-kernel solution with $N > 1$ is compatible with $l = \bar{l} = 0$. Indeed, (3.20) with $s = 0$ gives $A \leq 0$, which contradicts (3.28) with $\bar{l} = 0$ and $N > 1$. Hence, there are only two independent solutions with $l = \bar{l} = 0$: the identity, which was associated to $N = 0$, and the solution with $N = 1$ and $s = 0$.

[4]In order to simplify the notation the dependence of y_k on $a_1,\ldots,a_{N-1},A$ is not indicated explicitly.

4 Isomorphism between critical and off-critical operator space

We now want to count how many independent solutions of the form factor equations (3.5)–(3.9) correspond to a given pair of non-negative integers l and $\bar{l}$ (we say these solutions are of type $(l,\bar{l})$). In the basis of the N-kernel solutions discussed in the previous section the problem reduces to counting how many sets of integers $a_1,\ldots,a_N$ can determine the given values of l and $\bar{l}$ through the relations (3.20), (3.25) and (3.26).

Let us consider first the case of solutions of type $(l,0)$, which is particularly simple. Since $l=s$, the only constraint comes from (3.20) which becomes

$$\sum_{i=0}^{N} i a_i = l - N(N-1), \tag{4.1}$$

and holds for all $N \geq 0$. For a given N, the number of ways of satisfying this condition with non-negative integers $a_1,\ldots,a_N$ coincides with the number of partitions of $l-N(N-1)$ into the positive integers $i=1,\ldots,N \leq N^{(l)}$, where $N^{(l)}$ is the largest integer ensuring the non-negativity of $l-N(N-1)$. We saw in section 2 that the number of such partitions is $P(N,l-N(N-1))$, so that the dimension of the space of solutions of type $(l,0)$ is

$$d(l,0) = \sum_{N=0}^{\infty} P(N,l-N(N-1))\,. \tag{4.2}$$

Comparison with (2.20) then gives

$$d(l,0) = d_I(l) + d_\varphi(l), \tag{4.3}$$

i.e. for any non-negative l the dimension of the space of solutions of type $(l,0)$ of the form factor equations for the massive theory identically coincides with the total dimension of the space of operators of level $(l,0)$ in the critical theory.

An analogous result holds for the space of solutions of type $(0,\bar{l})$. Indeed, we show in appendix A that, for each solution $K_n^{(a_1,..,a_N)}(\theta_1,\ldots,\theta_n)$ of type $(l,0)$, a corresponding solution $\bar{K}_n^{(a_1,..,a_N)}(\theta_1,\ldots,\theta_n)$ of type $(0,l)$ is obtained performing in (3.18) the substitution $\sigma_i^{(N)} \to \bar{\sigma}_i^{(N)}$, where $\bar{\sigma}_i^{(n)}$ stay for the symmetric polynomials computed in $\bar{x}_i \equiv e^{-\theta_i}$.

In principle the previous counting procedure can be extended to the solutions of type $(l,\bar{l})$ with both l and $\bar{l}$ non-vanishing. In this case, however, the analysis is substantially complicated by the fact that l now coincides with y, which is non-trivially determined through (3.23) and (3.24). There is, however, a simpler path. We show in appendix B that a solution $K_n^{(a_1,..,a_N)}$ of type $(l,\bar{l})$ satisfies the asymptotic factorization property

$$\lim_{\alpha\to+\infty} e^{-l\alpha} K_N^{(a_1,..,a_N)}(\theta_1+\alpha,...,\theta_R+\alpha,\theta_{R+1}...,\theta_N) =$$
$$(C_\infty)^{RL} K_R^{(a_1^{(l)},..,a_R^{(l)})}(\theta_1,...,\theta_R)\,\bar{K}_L^{(a_1^{(\bar{l})},..,a_L^{(\bar{l})})}(\theta_{R+1}...,\theta_N)\,, \tag{4.4}$$

where $K_R^{(a_1^{(l)},..,a_R^{(l)})}$ defines a solution of type $(l,0)$, $\bar{K}_L^{(a_1^{(\bar{l})},..,a_L^{(\bar{l})})}$ defines a solution of type $(0,\bar{l})$, $1 \leq R \leq N-1$, $L=N-R$, and the integers $a_1^{(l)},..,a_R^{(l)}$, $a_1^{(\bar{l})},..,a_L^{(\bar{l})}$ are determined by the

$a_1, .., a_N$ as

$$a_i^{(l)} = a_i \quad \text{for} \quad 1 \leq i \leq R-1, \tag{4.5}$$

$$a_{N-i}^{(\bar{l})} = a_i \quad \text{for} \quad R+1 \leq i \leq N-1, \tag{4.6}$$

$$a_R^{(l)} = \sum_{i=R}^{N} a_i - \bar{l} + 2L \,, \qquad a_L^{(\bar{l})} = - \sum_{i=R+1}^{N} a_i + \bar{l} - 2(L-1). \tag{4.7}$$

Inversely, the specification of the a_i's in terms of the $a_i^{(l)}$'s and $a_i^{(\bar{l})}$'s is completed by the relations

$$a_N = -\sum_{i=1}^{L} a_i^{(\bar{l})} + \bar{l} - 2(L-1), \tag{4.8}$$

$$a_R = a_R^{(l)} + a_L^{(\bar{l})} - 2. \tag{4.9}$$

The non-negativity of a_R implies

$$a_R^{(l)} + a_L^{(\bar{l})} \geq 2 \,, \tag{4.10}$$

while that of a_N follows from

$$a_N \geq -\sum_{i=1}^{L} i a_i^{(\bar{l})} + \bar{l} - 2(L-1) = (L-2)(L-1) \geq 0 \,. \tag{4.11}$$

Hence, equation (4.4) can be used to characterize all solutions of type $(l, \bar{l})$ in terms of those of type $(l, 0)$ and $(0, \bar{l})$. More precisely, given a solution of type $(l, 0)$ and one of type $(0, \bar{l})$, they specify a solution of type $(l, \bar{l})$ provided (4.10) is satisfied. The two conditions

$$a_R^{(l)} = 0, \qquad a_L^{(\bar{l})} \geq 2, \tag{4.12}$$

$$a_R^{(l)} \geq 1, \qquad a_L^{(\bar{l})} \geq 1, \tag{4.13}$$

exhaust all the independent possibilities and lead to the following expression for the dimension of the space of solutions of type $(l, \bar{l})$

$$d(l, \bar{l}) = d(0|(l,0)) d(2|(0,\bar{l})) + d(1|(l,0)) d(1|(0,\bar{l})), \tag{4.14}$$

where $d(0|(l,0))$ is the dimension of the subspace of solutions spanned by the R-kernels of type $(l, 0)$ with $a_R^{(l)} = 0$ and $R \leq N^{(l)}$, and $d(i|(l,0))$ with $i \geq 0$ is the dimension of the subspace of solutions spanned by the R-kernels of type $(l, 0)$ with $a_R^{(l)} \geq i$ and $R \leq N^{(l)}$ (analogous definitions for the dimensions of subspaces of type $(0, \bar{l})$ are understood). The formula for $d(i|(l,0))$ with $i \geq 0$ is simply derived redefining $a_R^{(l)} = a_R^{(l),i} + i$, with now $a_R^{(l),i} \geq 0$, so that

$$d(i|(l,0)) = \sum_{N=0}^{\infty} P(N, l - N(N+i-1)). \tag{4.15}$$

By definition $d(0|(l,0))$ gives the number of partitions of $l-R(R-1)$ into the integers $1,\ldots,R-1$ ($a_R^{(l)}=0$), so that

$$d(0|(l,0))=\sum_{N=1}^{\infty}P(N-1,l-N(N-1)), \tag{4.16}$$

which implies $d(0|(l,0))=d(2|(l,0))$. Finally, recalling (2.15) we obtain the identities

$$d(2|(l,0))=d_I(l), \quad d(1|(l,0))=d_\varphi(l), \tag{4.17}$$

so that (4.14) becomes

$$d(l,\bar{l})=d_I(l)d_I(\bar{l})+d_\varphi(l)d_\varphi(\bar{l})\,. \tag{4.18}$$

This formula completes our proof showing that the space of solutions of the form factor equations for the massive Lee-Yang model decomposes into subspaces labeled by pairs of non-negative integers l and $\bar{l}$ whose dimensionality coincides with that of the subspace of conformal operators of level $(l,\bar{l})$.

5 Conclusion

We have shown in this paper for the Lee-Yang model how the operator space reconstructed from the particle dynamics of the massive theory through the form factor equations (3.5)–(3.9) is a direct sum of subspaces with given levels which exactly coincides with the decomposition dictated by conformal symmetry at the fixed point.

It is worth stressing that we are able to achieve this result because, through (3.25) and (3.26), we are able to attach to each solution belonging to a basis for the whole space of solutions of the form factor equations two non-negative integers that, after the isomorphism has been shown, is natural to call levels. This notion of levels for the generic form factor solution is absent in previous investigations, and this is why an equation like (4.18) is not contained there.

At criticality conformal symmetry also naturally yields the notion of operator families corresponding the lowest weight representations of the Virasoro algebra. In the massive theory, in absence of internal symmetries which distinguish them, the two operator families of the Lee-Yang model cannot be disentangled using the form factor equations (3.5)–(3.9) only. Additional information is needed to pass from the classification of the solutions according to the levels to the identification of specific operators within them.

As far as the family of φ is concerned, this operator, being responsible for the breaking of conformal symmetry, is proportional to the trace Θ of the energy-momentum tensor, and on this ground the solution of the form factor equations corresponding to it was originally identified in [27, 3]. This solution coincides with that with $N=1$, $s=0$ in the N-kernel basis that in section 3 we identified as the only operator other than the identity with $l=\bar{l}=0$.

The first non-trivial representatives of the identity family appear at level 2. The solutions for the energy-momentum components T and $\bar{T}$ with levels $(2,0)$ and $(0,2)$, respectively, are immediately obtained from the solution for Θ through the energy-momentum conservation equations.

Hence, the first genuinely new solution in this family is that for the composite operator $T\bar{T}$ with levels $(2,2)$. Here it is worth recalling that, while at the conformal point the decoupling of holomorphic and anti-holomorphic components reduces non-chiral operators to trivial products of the chiral ones, in the massive theory the decoupling is lost[5] and non-chiral operators need to be suitably defined as regularized products. It is then particularly relevant that our proof of one-to-one correspondence between operators at and away from criticality includes the non-chiral ones. The form factor solution for $T\bar{T}$ in the Lee-Yang model was determined in [28] exploiting also some general properties of this operator obtained in [29].

In [30] all the operators with $l, \bar{l} \leq 7$ for the massive Lee-Yang model were obtained acting on the form factor solutions for Θ, T, $\bar{T}$ and $T\bar{T}$ with the first few conserved quantities of this integrable quantum field theory. For these values of l and $\bar{l}$, (4.18) was then reproduced with the two terms in the r.h.s. disentangled.

It is reasonable to expect that the approach illustrated here for the simplest non-trivial case can be generalized to more complicated integrable quantum field theories. In all massive integrable cases the analysis of the structure of the operator space reduces to the study of the space of solutions of a system of equations like (3.5)–(3.9), complicated in general by the presence of several species of particles. The form factor equations, instead, undergo substantial modifications when integrability is lost and particle production becomes possible. In the general two-dimensional case, however, both integrable and non-integrable renormalization group trajectories originate from a given fixed point. Then, up to symmetry breaking effects, the non-integrable form factor equations should yield the same operator space than the integrable ones.

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A Chiral and antichiral solutions

Notice that the generic N-kernel $K_n^{(a_1,..,a_N)}(\theta_1,...,\theta_n)$ of type $(0,\bar{l}>0)$ can be rewritten as

$$\bar{K}_n^{(\bar{a}_1,..,\bar{a}_N)}(\theta_1,...,\theta_n) = \begin{cases} 0 & \text{for } n<N, \\ (\bar{\sigma}_N^{(N)})^{N-1} \prod_{1\leq i\leq N} (\bar{\sigma}_i^{(N)})^{\bar{a}_i} K_N^N(\theta_1,...,\theta_N) & \text{for } n=N, \end{cases} \tag{A.1}$$

[5]Equation (4.4) expresses that the decoupling is recovered in the conformal, high energy limit.

where $\bar{\sigma}_i^{(N)}$ stay for the symmetric polynomials computed in $\bar{x}_i \equiv e^{-\theta_i}$, and that the integers $\bar{a}_1, .., \bar{a}_N$ given by

$$\bar{a}_i = a_{N-i}, \qquad n < N; \qquad \bar{a}_N = -\left(\sum_{i=1}^{N} a_i + 2(N-1) - \bar{l}\right) \tag{A.2}$$

are non-negative. This is obvious for $\bar{a}_i$ with $n < N$, while for $\bar{a}_N = -y_1(a_1, ..., a_N)$ it follows from (3.27) with $l = 0$. In addition, in terms of the $\bar{a}_i$ the condition $s = -\bar{l}$ is rewritten as

$$\sum_{i=1}^{N} i\bar{a}_i = \bar{l} - N(N-1). \tag{A.3}$$

Inversely, any solution $\bar{K}_n^{(\bar{a}_1,..,\bar{a}_N)}(\theta_1, ..., \theta_n)$ with $\bar{a}_1, .., \bar{a}_N$ non-negative integers satisfying (A.3) is a N-kernel of type $(0, \bar{l})$. Indeed, due to (A.3), $\bar{K}_n^{(\bar{a}_1,..,\bar{a}_N)}(\theta_1, ..., \theta_n)$ has spin $s = -\bar{l}$, while using (A.2) y_k can be rewritten in terms of $\bar{a}_1, .., \bar{a}_N$ as

$$y_k = -\left(\sum_{i=0}^{k-1}(k-i)\bar{a}_{N-i} + k(k-1)\right),$$

so that $y_k \leq 0$ for $1 \leq k \leq N-1$; then (3.25) implies $l = 0$.

Hence we see that the spaces of kernel solutions of type $(\bar{l}, 0)$ and $(0, \bar{l})$ are isomorphic. Indeed, N non-negative integers $\bar{a}_1, .., \bar{a}_N$ satisfying (A.3) define the solution $K_n^{(\bar{a}_1,..,\bar{a}_N)}(\theta_1, ..., \theta_n)$ of type $(\bar{l}, 0)$ (as discussed in section 4) as well as the solution $\bar{K}_n^{(\bar{a}_1,..,\bar{a}_N)}(\theta_1, ..., \theta_n)$ of type $(0, \bar{l})$.

B Asymptotic factorization

It follows from (3.14) that the minimal N-particle kernel (3.16) satisfies the asymptotic factorization property

$$\lim_{\alpha\to+\infty} e^{-RL\alpha} K_N^N(\theta_1+\alpha, ..., \theta_R+\alpha, \theta_{R+1}..., \theta_N) = (C_\infty)^{RL} K_R^R(\theta_1, ..., \theta_R) K_L^L(\theta_{R+1}..., \theta_N). \tag{B.1}$$

On the other hand the elementary symmetric polynomials enjoy the properties

$$\lim_{\alpha\to+\infty} e^{-k\alpha}\sigma_p^{(N)}(x_1 e^\alpha, .., x_k e^\alpha, x_{k+1}, .., x_N) = \sigma_k^{(k)}(x_1, \ldots, x_k)\sigma_{p-k}^{(N-k)}(x_{k+1}, \ldots, x_N), \quad k \leq p \leq N, \tag{B.2}$$

$$\lim_{\alpha\to+\infty} e^{-p\alpha}\sigma_p^{(N)}(x_1 e^\alpha, .., x_k e^\alpha, x_{k+1}, .., x_N) = \sigma_p^{(k)}(x_1, \ldots, x_k), \qquad p \leq k \leq N. \tag{B.3}$$

Using these equations it is simple to see that (4.4) holds for a N-kernel solution satisfying (3.27) as an equality for $k = R$. However, we still have to prove that the factors on the r.h.s. of (4.4) are indeed a R-kernel of type $(l, 0)$ and a L-kernel of type $(0, \bar{l})$, i.e. that the integers $a_1^{(l)}, .., a_R^{(l)}$ and $a_1^{(\bar{l})}, .., a_L^{(\bar{l})}$ are non-negative and satisfy the condition (4.1) and (A.3), respectively. Equation (4.1) for the $a_i^{(l)}$ follows from

$$\sum_{i=1}^{R} i a_i^{(l)} = \sum_{i=1}^{R-1} i a_i + R(\sum_{i=R}^{N} a_i - \bar{l} + 2L) = l - R(R-1), \tag{B.4}$$

where the last equality is due to the fact that (3.27) holds as an equality for $k = R$. Analogously,

$$\sum_{i=1}^{L} i a_i^{(\bar{l})} = -\sum_{j=R+1}^{N} (j-R)a_j + L(\bar{l} - 2(L-1)) = \bar{l} - L(L-1), \tag{B.5}$$

where we have used

$$\sum_{j=R+1}^{N} (j-R)a_j = -\bar{l} + L(\bar{l} + 1 - L), \tag{B.6}$$

a result which follows taking the difference of (3.20) and (3.27) for $k = R$.

The integers $a_i^{(l)}$ and $a_j^{(\bar{l})}$ with $1 \le i \le R-1$ and $1 \le j \le L-1$ are non-negative by (4.5)-(4.6) and so we have to prove only the non-negativity of $a_R^{(l)}$ and $a_L^{(\bar{l})}$. If we rewrite

$$a_R^{(l)} = (\sum_{i=1}^{R-1} i a_i + R \sum_{i=R}^{N} a_i) - (\sum_{i=1}^{R-1} i a_i + (R-1) \sum_{i=R}^{N} a_i) - \bar{l} + 2L,$$

we get $a_R^{(l)} \ge 0$ using (3.27) for $k = R$ and $k = R-1$. Similarly,

$$a_L^{(\bar{l})} = (\sum_{i=1}^{R} i a_i + R \sum_{i=R+1}^{N} a_i) - (\sum_{i=1}^{R} i a_i + (R+1) \sum_{i=R+1}^{N} a_i) + \bar{l} - 2(L-1),$$

so using (3.27) for $k = R$ and $k = R+1$ we get $a_L^{(\bar{l})} \ge 0$, in this way completing the proof.

Let us now prove the characterization of kernel solutions of type $(l, \bar{l})$ in terms of those of type $(l, 0)$ and $(0, \bar{l})$. We have just to prove that, given R non-negative integers $a_1^{(l)}, .., a_R^{(l)}$ and L non-negative integers $a_1^{(\bar{l})}, .., a_L^{(\bar{l})}$ satisfying (4.10) and, respectively, the conditions (4.1) and (A.3), then the integers $a_1, .., a_N$ determined by (4.5), (4.6), (4.8), (4.9) satisfy (3.20) and (3.27), with (3.27) which is an equality for $k = R$. For this purpose notice that the difference of (B.4) and (B.5) gives

$$\sum_{i=1}^{N} i a_i = \sum_{i=1}^{R} i a_i^{(l)} - \sum_{j=1}^{L} j a_j^{(\bar{l})} - N(2L - (\bar{l}+1)) + R - L = l - \bar{l} - N(N - (\bar{l}+1)), \tag{B.7}$$

where to derive the last equality we have used (4.1) for the $a_i^{(l)}$ and (A.3) for the $a_j^{(\bar{l})}$. This is the spin condition (3.20). The identity (3.27) for $k = R$ follows from

$$\sum_{i=1}^{R} i a_i + R \sum_{i=R+1}^{N} a_i = \sum_{i=1}^{R} i a_i^{(l)} - R(2L - \bar{l}) = l - R(N - R + (N - (\bar{l}+1))), \tag{B.8}$$

where to derive the last equality we have used the condition (4.1) for the $a_i^{(l)}$.

Consider now the inequality (3.27) for $k \ne R$. For $k < R$ we have

$$\sum_{i=1}^{k-1} i a_i + k \sum_{i=k}^{N} a_i = \sum_{i=1}^{k-1} i a_i^{(l)} + k \sum_{i=k}^{R} a_i^{(l)} - k(2L - \bar{l}) \le l - k((N-k) + (N - (\bar{l}+1))), \tag{B.9}$$

where to derive the last inequality we have used the condition (3.27) with $k < R$ for the integers $a_1^{(l)}, .., a_R^{(l)}$. For $k > R$ we have

$$\sum_{i=1}^{k-1} i a_i + k \sum_{i=k}^{N} a_i = \sum_{i=1}^{R} i a_i^{(l)} - \sum_{j=1+(N-k)}^{L} (j-N+k) a_j^{(\bar{l})} - 2R - k(2(L-1)-\bar{l}) \leq l - k((N-k)+(N-\bar{l}-1)), \tag{B.10}$$

where to obtain the last inequality we have used (4.1) for the $a_i^{(l)}$ and

$$\sum_{j=1+p}^{L} (j-p) a_j^{(\bar{l})} \geq -(L-p)(p+(L-1)); \tag{B.11}$$

this last inequality follows taking the difference of (A.3) and (3.27) applied to the $a_i^{(\bar{l})}$. (B.9) and (B.10) provide the conditions (3.27) for $k \neq R$, in this way completing the proof.

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