Baxterization of $GL_q(2)$ and its application to the Liouville model and some other models on a lattice

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Abstract

We develop the Baxterization approach to (an extension of) the quantum group $GL_q(2)$. We introduce two matrices which play the role of spectral parameter dependent L–matrices and observe that they are naturally related to two different comultiplications. Using these comultiplication structures, we find the related fundamental R–operators in terms of powers of coproducts and also give their equivalent forms in terms of quantum dilogarithms. The corresponding quantum local Hamiltonians are given in terms of logarithms of positive operators. An analogous construction is developed for the q–oscillator and Weyl algebras using that their algebraic and coalgebraic structures can be obtained as reductions of those for the quantum group. As an application, the lattice Liouville model, the q–DST model, the Volterra model, a lattice regularization of the free field, and the relativistic Toda model are considered.

Contents

1 Introduction: motivation and outline of main results

A quantum model is a system $(\mathcal{H}, \mathcal{A}, \mathsf{H})$, with a Hilbert space $\mathcal{H}$, an algebra of observables $\mathcal{A}$, and a Hamiltonian H. The model is *integrable* if there exists a complete set of quantum integrals of motion, i.e., a set of self–adjoint elements of $\mathcal{A}$ which commute with each other and with the Hamiltonian. For *homogeneous* one–dimensional lattice models one has $\mathcal{H} = \mathcal{K}^{\otimes \mathrm{N}}$, $\mathcal{A} = \mathcal{B}^{\otimes \mathrm{N}}$, with one copy of Hilbert space $\mathcal{K}$ and algebra of local observables $\mathcal{B}$ being associated to each of the N sites of a one–dimensional lattice. $\mathcal{K}$ is usually characterized as a representation of an algebra $\mathfrak{U}$ of "symmetries", and $\mathcal{B}$ is generated from the operators which represent the elements of $\mathfrak{U}$ on $\mathcal{K}$.

A key step in constructing an integrable lattice model is to find an L–matrix $L(\lambda) \in \mathrm{Mat}(n) \otimes \mathcal{B}$ and an auxiliary R–matrix $R(\lambda) \in \mathrm{Mat}(n)^{\otimes 2}$ such that the following matrix commutation relation

$$R_{12}(\lambda)\, L_{13}(\lambda\mu)\, L_{23}(\mu) = L_{23}(\mu)\, L_{13}(\lambda\mu)\, R_{12}(\lambda)\,, \tag{1}$$

where $\lambda, \mu \in \mathbb{C}$, is equivalent to the defining relations of $\mathfrak{U}$. Here and below we use the standard notation: subscripts indicate nontrivial components in tensor product, e.g., $R_{12} \equiv R \otimes 1$, etc. For further details on the R–matrix approach to quantum integrability we refer the reader to the review [F1].

For a model on a *closed* one–dimensional lattice, i.e., with periodic boundary conditions, a set of quantum integrals of motion is generated by the *auxiliary* transfer–matrix $T(\lambda) = \mathrm{tr}_a\big(L_{a,\mathrm{N}}(\lambda) \dots L_{a,1}(\lambda)\big)$. However, these integrals are in general non–local, i.e., they are not representable as a sum of terms each containing nontrivial contributions only from several nearest sites. The recipe [FT2] for constructing *local* integrals of motion for a model with a given L–matrix is to find first the corresponding *fundamental* R–operator $\mathsf{R}(\lambda) \in \mathcal{B}^{\otimes 2}$, which satisfies the following intertwining relation (here and below we will use it in the braid form):

$$\mathsf{R}_{23}(\lambda)\, L_{12}(\lambda\mu)\, L_{13}(\mu) = L_{12}(\mu)\, L_{13}(\lambda\mu)\, \mathsf{R}_{23}(\lambda)\,. \tag{2}$$

The corresponding transfer–matrix is constructed as $\mathsf{T}(\lambda) = \mathrm{tr}_a\big(\mathsf{R}_{a\mathrm{N}}(\lambda)\mathsf{P}_{a\mathrm{N}} \dots \mathsf{R}_{a1}(\lambda)\mathsf{P}_{a1}\big)$, where the subscript a stands now for an auxiliary copy of $\mathcal{B}$, and P is the unitary operator permuting tensor factors in $\mathcal{B}^{\otimes 2}$. The fundamental R–operator is usually *regular*, that is, after appropriate normalization, it satisfies the relation

$$\mathsf{R}(1) = 1 \otimes 1\,. \tag{3}$$

If the regularity condition holds, then first and higher order logarithmic derivatives of $\mathsf{T}(\lambda)$ at $\lambda{=}1$ are local integrals of motion for the periodic homogeneous model in question. In particular, the Hamiltonian is often chosen as the most local integral which involves only nearest neighbour interaction:

$$\mathsf{H} = i\frac{\partial}{\partial\lambda}\big[\log \mathsf{T}(\lambda)\big]_{\lambda=1} = \sum_{n=1}^{\mathrm{N}} H_{n,n+1} = \sum_{n=1}^{\mathrm{N}} i\frac{\partial}{\partial\lambda}\big[\mathsf{R}_{n+1,n}(\lambda)\big]_{\lambda=1}\,, \tag{4}$$

where the summation assumes that $\mathrm{N}{+}1 \equiv 1$.

Thus, finding the fundamental R–operator for a given L–matrix is an important part of the R–matrix approach to quantum integrable models. Furthermore, this problem is closely related to the problem of constructing the corresponding evolution operators and Q–operators. However, there is no general method for solving equation (2). The particular difficulty here is that it is not clear apriori on which operator argument(s) the function $\mathsf{R}(\lambda)$ depends.

Among the few known examples of constructing a fundamental R–operator the most algebraically transparent are those related to the case where the symmetry $\mathfrak{U}$ admits the structure of a bialgebra. Such examples include the XXX spin chain [KRS] and closely related nonlinear Schrödinger model [FT2], where $\mathfrak{U} = \mathcal{U}(\mathfrak{sl}_2)$; and the XXZ spin chain [J1] and closely related sine–Gordon model [FT2, T1], where $\mathfrak{U} = \mathcal{U}_q(\mathfrak{sl}_2)$. A crucial observation for solving (2) in these cases is that the operator argument of $\mathsf{R}(\lambda)$ is $\Delta(C_q)$, where C_q is the Casimir element of $\mathfrak{U}$ and Δ is the comultiplication that defines the bialgebra structure of $\mathfrak{U}$. The corresponding solutions to (2) are expressed, respectively, in terms of the Gamma function or its q–analogue (see [J1, T1, F1, B2] for more details in the latter case).

The aim of the present article is to develop a similar algebraic construction of fundamental R–operators for models whose underlying symmetry corresponds, in the sense of Eq. (1), to the *quantum group* $GL_q(2)$. More precisely, we introduce the quantum group $\widetilde{GL}_q(2)$ with generators a, b, c, d, θ, where θ may be chosen to be the inverse to b or c. It will be important to consider special *positive* representations of $\widetilde{GL}_q(2)$ which ensures that the operators that we use are positive self–adjoint. These properties are crucial for constructing fundamental R–operators since we will need non–polynomial functions of generators and their coproducts.

The article is organized as follows. First, we discuss *Baxterization* of $GL_q(2)$ and $\widetilde{GL}_q(2)$, presenting their defining relations in the form (1). The two matrices, $g(\lambda)$ and $\hat{g}(\lambda)$, which play the role of an L–matrix for $\widetilde{GL}_q(2)$, will be our main objects of consideration. Next, we show that, besides the standard comultiplication Δ, there is another algebra homomorphism $\delta : \widetilde{GL}_q(2) \to \widetilde{GL}_q(2)^{\otimes 2}$. Further, we solve Eq. (2) for $g(\lambda)$ and $\hat{g}(\lambda)$. The corresponding fundamental R–operators are given (up to some twists) by powers of, respectively, $\Delta(bc)$ and $\delta(ad-qbc)$. Next, we show that the L–matrices of the lattice Liouville model and the q–DST model are nothing but $g(\lambda)$ and $\hat{g}(\lambda)$ with appropriately chosen representations of generators. Using this observation, we construct the corresponding local lattice Hamiltonians. Finally, we consider some reductions of $\widetilde{GL}_q(2)$, including the q–oscillator algebra $\mathcal{A}_q$ and the Weyl algebra $\mathcal{W}_q$. Following the same scheme, we introduce reductions of $g(\lambda)$ and $\hat{g}(\lambda)$, and of Δ and δ, and then construct the corresponding fundamental R–operators by solving Eq. (2). We discuss relation of these R–operators and of the corresponding local lattice Hamiltonians to the Volterra model, the relativistic Toda model, and a lattice regularization of the free field.

Let us remark that, although our construction based on the use of the comultiplication structure yields expressions for fundamental R–operators mainly as powers of coproducts of some elements, it is often useful to rewrite these expressions in terms of the *quantum dilogarithm* function or, more precisely, its self–dual form [F2, F3] which is suitable for dealing with the $|q| = 1$ case. A brief account on this function along with several related statements which we use in the main text are given in the Appendix.

2 $GL_q(2)$ and its Baxterization

Let $q = e^{i\gamma}$, where $\gamma \in (0, \pi)$. We will use the abbreviated notation $GL_q(2)$ for the algebra of regular functions on the quantum group, $Fun\big(GL_q(2)\big)$ (see [V1, CP, KS]).

Definition 1. *$GL_q(2)$ is a unital associative algebra with generators a, b, c, d, and defining relations*

$$\begin{gathered} [a,d] = (q-q^{-1})\, b\, c\,, \qquad [b,c] = 0\,, \\ a\, b = q\, b\, a\,, \quad a\, c = q\, c\, a\,, \quad b\, d = q\, d\, b\,, \quad c\, d = q\, d\, c\,. \end{gathered} \tag{5}$$

$SL_q(2)$ is the factor algebra of $GL_q(2)$ over the ideal generated by the relation $ad - qbc = 1$.

Following the R–matrix approach to quantum groups [FRT], the generators of $GL_q(2)$ can be assembled into a matrix, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then, by direct inspection of 16 quadratic exchange relations, one can verify the following assertion (see, e.g. [CP, KS]).

Lemma 1. *The defining relations (5) are equivalent to the following relation*

$$R_{12}\, g_{13}\, g_{23} = g_{23}\, g_{13}\, R_{12}\,, \tag{6}$$

where the auxiliary R–matrix is given by either of the following matrices

$$R^+ = \begin{pmatrix} q & & & \\ & 1 & & \\ & q-q^{-1} & 1 & \\ & & & q \end{pmatrix}, \qquad R^- = \big(R^+_{21}\big)^{-1} = \begin{pmatrix} q^{-1} & & & \\ & 1 & q^{-1}-q & \\ & & 1 & \\ & & & q^{-1} \end{pmatrix}. \tag{7}$$

In what follows we will also need the following spectral parameter dependent R–matrices

$$\hat{R}(\lambda) = \lambda R^+ - \lambda^{-1} R^- = \begin{pmatrix} \varpi(q\lambda) & & & \\ & \varpi(\lambda) & \lambda^{-1}\varpi(q) & \\ & \lambda\,\varpi(q) & \varpi(\lambda) & \\ & & & \varpi(q\lambda) \end{pmatrix}, \tag{8}$$

$$R(\lambda) = \lambda^{\frac{1}{2}\sigma_3 \otimes 1}\, \hat{R}(\lambda)\, \lambda^{-\frac{1}{2}\sigma_3 \otimes 1} = \begin{pmatrix} \varpi(q\lambda) & & & \\ & \varpi(\lambda) & \varpi(q) & \\ & \varpi(q) & \varpi(\lambda) & \\ & & & \varpi(q\lambda) \end{pmatrix}, \tag{9}$$

where $\varpi(\lambda) \equiv \lambda - \lambda^{-1}$ and $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

In the theory of quantum groups, the notion of *Baxterization* was originally introduced by V. Jones [J2] in the context of knot theory. It refers to the procedure of constructing spectral parameter dependent solutions to the Yang–Baxter equation out of solutions to the constant (spectral parameter independent) Yang–Baxter equation. An example is provided by the expression for $\hat{R}(\lambda)$ in terms of $R^{\pm}$ in formula (8). Analogously, an L–matrix satisfying the RLL relation (1) can be regarded as Baxterized if it is constructed from L–matrices that satisfy the constant RLL relation. For instance, the L–matrix of the XXZ model (see, e.g., [F1]) has the form

$$L_{\mathrm{XXZ}}(\lambda) = \lambda\, L_+ + \lambda^{-1} L_-\,, \tag{10}$$

where $L_{\pm}$ satisfy the constant RLL relation with constant R–matrices given by (7).

In the theory of quantum integrable models it is crucial that an R–matrix is spectral dependent (see Introduction), and so the Baxterization procedure serves as quite a common technique for constructing new solutions to the Yang–Baxter equation and hence new integrable models. However, what concerns the Baxterization of L–matrices, the vast majority of examples occur in the cases where the symmetry $\mathfrak{U}$ is a quantum algebra, typically the universal enveloping of a quantum Lie algebra, like $\mathfrak{U} = \mathcal{U}_q(\mathfrak{sl}_2)$ for the XXZ model.

Quantum groups, in particular $GL_q(2)$, are usually not considered from the point of view of Baxterization of L–matrices. In the present paper, we will try to fill this gap a bit. Let us commence with observation that equation (6) can be Baxterized, albeit in a somewhat weaker sense than it is usually meant. For this purpose, we assemble the generators of $GL_q(2)$ into two matrices:

$$\mathrm{g}(\lambda) = \begin{pmatrix} a & \lambda\, b \\ \lambda^{-1} c & d \end{pmatrix}, \qquad \hat{\mathrm{g}}(\lambda) = \begin{pmatrix} \lambda^{-1} c & \lambda^{-1} d \\ \lambda\, a & \lambda\, b \end{pmatrix}. \tag{11}$$

Proposition 1. *Each of the following matrix relations*

$$R_{12}(\lambda)\, \mathrm{g}_{13}(\lambda\mu)\, \mathrm{g}_{23}(\mu) = \mathrm{g}_{23}(\mu)\, \mathrm{g}_{13}(\lambda\mu)\, R_{12}(\lambda)\,, \tag{12}$$

$$\hat{R}_{12}(\lambda)\, \hat{\mathrm{g}}_{13}(\lambda\mu)\, \hat{\mathrm{g}}_{23}(\mu) = \hat{\mathrm{g}}_{23}(\mu)\, \hat{\mathrm{g}}_{13}(\lambda\mu)\, \hat{R}_{12}(\lambda)\,, \tag{13}$$

holds if and only if the elements a, b, c, d satisfy the defining relations (5).

Proof. Matrices (11) are related to each other and to the matrix g as follows

$$\mathrm{g}(\lambda) = \lambda^{\frac{1}{2}\sigma_3}\, g\, \lambda^{-\frac{1}{2}\sigma_3}\,, \qquad \hat{\mathrm{g}}(\lambda) = \lambda^{-\frac{1}{2}\sigma_3}\, \sigma_1\, \mathrm{g}(\lambda)\, \lambda^{\frac{1}{2}\sigma_3}\,, \tag{14}$$

where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Notice also that

$$[R(\lambda)\,,\, \sigma_3 \otimes 1 + 1 \otimes \sigma_3] = 0\,, \qquad [R(\lambda)\,,\, \sigma_1 \otimes \sigma_1] = 0\,. \tag{15}$$

Substituting the first of relations (14) into (13), using the first of relations (15), and taking into account relation (9) between $\hat{R}(\lambda)$ and $R(\lambda)$, it is easy to see that (12) is equivalent to the relation $\hat{R}_{12}(\lambda) g_{13} g_{23} = g_{23} g_{13} \hat{R}_{12}(\lambda)$, which is nothing but a linear combination of the $R{=}R_+$ and $R{=}R_-$ versions of Eq. (6). Since λ is arbitrary here, we conclude that (12) is equivalent to (6) and hence, by Lemma 1, to (5). Similarly, substituting the second relation in (14) into (13), using (9) to replace $\hat{R}(\lambda)$ with $R(\lambda)$, and then taking into account both relations (15), it is easy to see that (13) is equivalent to (12), and hence to (5). □

The proof shows that the Baxterization in (11) is not a true one in the sense that it can be removed by the twist transformations (14). Furthermore, for $\mathrm{g}(\lambda)$, the transfer–matrix $T_{\mathrm{g}}(\lambda) = \mathrm{tr}_a\big(\mathrm{g}_{a,\mathrm{N}}(\lambda) \ldots \mathrm{g}_{a,1}(\lambda)\big)$ does not actually depend on λ and thus it is not a generating function for integrals of motion. However, the corresponding transfer–matrix $T_{\hat{\mathrm{g}}}(\lambda)$ for $\hat{\mathrm{g}}(\lambda)$ depends on λ nontrivially, and the operator coefficients T_n in its expansion, $T_{\hat{\mathrm{g}}}(\lambda) = \sum_n \lambda^n T_n$, form a set of mutually commuting elements of $GL_q(2)^{\otimes \mathrm{N}}$.

3 $\widetilde{GL}_q(2)$ and related lattice models

3.1 Definition of $\widetilde{GL}_q(2)$ and its Baxterization

Let us introduce the following extension of the quantum group $GL_q(2)$.

Definition 2. *$\widetilde{GL}_q(2)$ is a unital associative algebra with generators a, b, c, d, θ, and defining relations (5) and*

$$a\,\theta = q^{-1}\,\theta\,a\,, \quad \theta\,d = q^{-1}\,d\,\theta\,, \quad [b,\theta] = 0\,, \quad [\theta,c] = 0\,. \tag{16}$$

Lemma 2. *For a generic q, the center of $\widetilde{GL}_q(2)$ is generated by the following elements*

$$D_q \equiv a\,d - q\,b\,c\,, \qquad \eta_q' \equiv \theta\,b\,, \qquad \eta_q'' \equiv \theta\,c\,. \tag{17}$$

Proof. First, it is straightforward to check that D_q, η_q', and η_q'' commute with the generators of $\widetilde{GL}_q(2)$. Next, any central element C of $\widetilde{GL}_q(2)$ can be represented as a linear combination of monomials $a^{n_a}d^{n_d}b^{n_b}c^{n_c}\theta^{n_\theta}$, where all n's are non–negative integers. Commutativity of C with b, c, and θ implies that $n_a = n_d$. Therefore, C is equivalently represented as a linear combination of monomials $D_q^n b^m c^k \theta^l$. Commutativity of C with a and d implies that $m + k = l$. Hence, using (17), we conclude that C is represented as a linear combination of monomials $D_q^n(\eta_q')^m(\eta_q'')^k$. □

Lemma 3. *The defining relations (5) and (16) are equivalent to the following set of equations:*

$$R_{12}\,g^{\pm}_{13}\,g^{\pm}_{23} = g^{\pm}_{23}\,g^{\pm}_{13}\,R_{12}\,, \qquad R^{+}_{12}\,g^{+}_{13}\,g^{-}_{23} = g^{-}_{23}\,g^{+}_{13}\,R^{+}_{12}\,, \tag{18}$$

where $g^+ = \left(\begin{smallmatrix}\theta & 0\\ a & b\end{smallmatrix}\right)$ and $g^- = \left(\begin{smallmatrix}c & d\\ 0 & 0\end{smallmatrix}\right)$, the auxiliary matrices $R^{\pm}$ are given by (7), and R in the first relation is either of them.

Proof. Direct inspection. □

Let us assemble the generators of $\widetilde{GL}_q(2)$ into two matrices

$$g(\lambda) = \begin{pmatrix} a & \lambda\,b \\ \lambda\,\theta + \lambda^{-1}c & d \end{pmatrix}, \qquad \hat{g}(\lambda) = \begin{pmatrix} \lambda\,\theta + \lambda^{-1}c & \lambda^{-1}d \\ \lambda\,a & \lambda\,b \end{pmatrix}. \tag{19}$$

Proposition 2. *Each of the following matrix relations*

$$R_{12}(\lambda)\,g_{13}(\lambda\mu)\,g_{23}(\mu) = g_{23}(\mu)\,g_{13}(\lambda\mu)\,R_{12}(\lambda)\,, \tag{20}$$

$$\hat{R}_{12}(\lambda)\,\hat{g}_{13}(\lambda\mu)\,\hat{g}_{23}(\mu) = \hat{g}_{23}(\mu)\,\hat{g}_{13}(\lambda\mu)\,\hat{R}_{12}(\lambda)\,, \tag{21}$$

holds if and only if the elements a, b, c, d, θ satisfy the defining relations (5) and (16).

Proof. Notice that the second relation in (14) remains true for $g(\lambda)$ and $\hat{g}(\lambda)$ given by (19). Therefore, the same line of arguments as in the proof of Proposition 1 establishes equivalence of relations (20) and (21). Thus, it suffices to prove only (21). For this aim we observe that

$$\hat{g}(\lambda) = \lambda\,g^+ + \lambda^{-1}\,g^-\,, \tag{22}$$

where $g^{\pm}$ were defined in Lemma 3. Substitute now (8) and (22) into (21) and match coefficients at different powers of λ and μ. It is not difficult to check that resulting matrix relations are exactly those contained in (18). (For the coefficient at $\lambda^0\mu^0$, we have to take into account the relation $R^+ = P(R^-)^{-1}P$ along with the Hecke identity $R^+ - R^- = (q-q^{-1})P$, where P is the permutation in $\mathrm{Mat}(2)^{\otimes 2}$, i.e., $Pg^{\pm}_{13}P = g^{\pm}_{23}$.) Thus, relations (20) and (21) are equivalent to (18), and hence, by Lemma 3, to the defining relations of $\widetilde{GL}_q(2)$. □

Unlike their $GL_q(2)$ prototypes (11), matrices (19) are true Baxterizations of g^+ and g^-. Indeed, their q–determinants (see, e.g., [BT2], Appendix C) are

$$\mathrm{qdet}\, g(\lambda) = -\,\mathrm{qdet}\, \hat{g}(\lambda) = D_q - q^{-1}\,\lambda^2\,\eta'_q\,, \tag{23}$$

which implies that the dependence of $g(\lambda)$ and $\hat{g}(\lambda)$ on λ cannot be removed by transformations of the type (14).

Let us emphasize a close similarity between our L–matrices for $\widetilde{GL}_q(2)$ and those for $\mathcal{U}_q(\mathfrak{sl}_2)$. Indeed, $\hat{g}(\lambda)$ in (22) and L_{XXZ} in (10) are constructed in the same way from their constant counterparts and they satisfy the RLL relations with the same auxiliary R–matrices. Such a similarity seems quite natural in view of a duality between $SL_q(2)$ and $\mathcal{U}_q(\mathfrak{sl}_2)$ (see [CP, KS]). However, this similarity is not absolute because the constant matrices $L_{\pm}$ in (10) are nondegenerate and generate the Borel subalgebras of $\mathcal{U}_q(\mathfrak{sl}_2)$, whereas g_- is degenerate and division of $\widetilde{GL}_q(2)$ into the subgroups generated by $g_{\pm}$ looks somewhat asymmetric.

3.2 Standard and non–standard comultiplications for $\widetilde{GL}_q(2)$

Recall that the linear homomorphism $\Delta\colon GL_q(2) \to GL_q(2)^{\otimes 2}$ defined on generators as follows

$$\begin{aligned} \Delta(a) &= a\otimes a + b\otimes c\,, \qquad & \Delta(b) &= a\otimes b + b\otimes d\,,\\ \Delta(c) &= c\otimes a + d\otimes c\,, \qquad & \Delta(d) &= c\otimes b + d\otimes d\,. \end{aligned} \tag{24}$$

is a coassociative algebra homomorphism, i.e., its homomorphism property $\Delta(x\,y) = \Delta(x)\Delta(y)$ is compatible with the defining relations (5), and it satisfies the coassociativity property

$$(id\otimes\Delta)\Delta(x) = (\Delta\otimes id)\Delta(x)\,. \tag{25}$$

The proof of these assertions is very simple in the R–matrix approach due to an observation that (24) can be rewritten in the matrix form as follows:

$$(id\otimes\Delta)\,g = g_{12}\,g_{13}\,. \tag{26}$$

The fact that the Casimir element of $GL_q(2)$ is a group–like element w.r.t. the map Δ, that is

$$\Delta(D_q) = D_q\otimes D_q\,, \tag{27}$$

implies that the same map (24) defines also a coassociative algebra homomorphism for $SL_q(2)$.

$GL_q(2)$ can be equipped with a bialgebra structure if, in addition to the map Δ, the linear homomorphism $\epsilon\colon GL_q(2)\to\mathbb{C}$ is defined on generators as follows: $\epsilon(g) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Then Δ and ϵ become *comultiplication* and *counit* maps, respectively.

A natural question about the algebra $\widetilde{GL}_q(2)$ is whether we can introduce for it a comultiplication map, and, in particular, whether we can extend the definition (24) to $\widetilde{GL}_q(2)$. It appears that to define $\Delta(\theta)$ compatible with (5) and (24) in a purely algebraic manner is not straightforward. However, for our purposes it will be sufficient to define $\Delta(\theta)$ for (special positive representations of) real forms of certain factor algebras of $\widetilde{GL}_q(2)$.

Definition 3. *$\widetilde{GL}_q(2,\mathbb{R})$ is a real form of $\widetilde{GL}_q(2)$ equipped with an anti–involution * defined on generators by*

$$a^* = a\,, \quad b^* = b\,, \quad c^* = c\,, \quad d^* = d\,, \quad \theta^* = \theta\,. \tag{28}$$

$\widetilde{GL}'_q(2,\mathbb{R})$ and $\widetilde{GL}''_q(2,\mathbb{R})$ are the factor algebras of $\widetilde{GL}_q(2,\mathbb{R})$ over the ideals generated, respectively, by the relations $\eta'_q = 1$ and $\eta''_q = 1$.

Apparently, the algebras $\widetilde{GL}'_q(2,\mathbb{R})$ and $\widetilde{GL}''_q(2,\mathbb{R})$ are isomorphic; the corresponding isomorphism map ι is defined on generators as follows: $\iota(a) = a$, $\iota(d) = d$, $\iota(b) = c$, $\iota(c) = b$, $\iota(\theta) = \theta$.

Definition 4. *Let $\mathcal{B}$ be an algebra of linear operators acting on a Hilbert space $\mathcal{K}$. Let $\mathfrak{U}$ stand for $\widetilde{GL}'_q(2,\mathbb{R})$ or $\widetilde{GL}''_q(2,\mathbb{R})$. An irreducible representation $\pi\colon \mathfrak{U} \to \mathcal{B}$ is called* positive *if the following operators are* self–adjoint *and* strictly positive *on $\mathcal{K}$:*
i) $\pi(x)$ for $x = a, b, c, d, \theta, D_q$;
ii) $q^{\frac{1}{2}}\pi(a)\big(\pi(x)\big)^{-1}$ and $q^{\frac{1}{2}}\big(\pi(x)\big)^{-1}\pi(d)$ for $x = b, c$.

Remark 1. In Definition 4, elements of $\mathfrak{U}$ are realized by *unbounded* operators. Following [W1, W2], we will understand the Weyl–type relations $xy = e^{i\gamma}yx$ in the defining relations (5) and (16) in the sense that, for a given pair of positive self–adjoint operators $\pi(x)$ and $\pi(y)$, the following unitary equivalence relations $\pi(x)^{it}\pi(y)\pi(x)^{-it} = e^{-\gamma t}\pi(y)$ and $\pi(y)^{it}\pi(x)\pi(y)^{-it} = e^{\gamma t}\pi(x)$ hold for all $t \in \mathbb{R}$ and admit analytic continuation to complex values of t.

Remark 2. Condition $ii)$ in Definition 4 ensures that, for a pair of generators x and y which satisfy the Weyl–type relation, the sum $\pi(x)+\pi(y)$ is a positive self–adjoint operator. Indeed, let u and v be positive self–adjoint operators satisfying relation $uv = q^2vu$. Then, in general, the sum $u+v$ is a symmetric but not necessarily self–adjoint operator [S1]. If, following [W1, W2], we require that the operator $qu^{-1}v$ is positive self–adjoint, then property (162) of the quantum dilogarithm function $S_\omega(t)$ (see Appendix A.1) implies that $S_\omega(qu^{-1}v)$ is a unitary operator. In this case Eq. (164) shows that $u+v$ is unitarily equivalent to both u and v and hence is a positive self–adjoint operator. Let us remark also that understanding relation $uv = q^2vu$, $q = i\gamma$ in the sense of Remark 1 is equivalent to say that $[\log u, \log v] = 2i\gamma$. Then, restricting our consideration to the case $\gamma \in (0, \pi)$, again ensures self–adjointness of $u+v$, by Proposition A.2 in [S1].

An example of a positive representation of $\mathfrak{U}$ will be given below in Section 3.5. Notice that $\pi(\eta'_q)$ and $\pi(\eta''_q)$ are also represented by positive self–adjoint operators. Moreover, we have $\pi(\theta) = \big(\pi(b)\big)^{-1}$ for $\mathfrak{U} = \widetilde{GL}'_q(2,\mathbb{R})$ and $\pi(\theta) = \big(\pi(c)\big)^{-1}$ for $\mathfrak{U} = \widetilde{GL}''_q(2,\mathbb{R})$.

Proposition 3. *Let $\mathcal{B}$, $\mathcal{K}$, and $\mathfrak{U}$ be as in Definition 4 and let π be a positive representation of $\mathfrak{U}$. Define the map $\Delta_\pi : \mathfrak{U} \to \mathcal{B}^{\otimes 2}$ as a linear homomorphism such that:*
i) $\Delta_\pi(x) = (\pi\otimes\pi)\big(\Delta(x)\big)$ for $x = a, b, c, d$ with $\Delta(x)$ given by (24);
ii) $\Delta_\pi(\theta) = \big(\Delta_\pi(b)\big)^{-1}$ for $\mathfrak{U} = \widetilde{GL}'_q(2,\mathbb{R})$ and $\Delta_\pi(\theta) = \big(\Delta_\pi(c)\big)^{-1}$ for $\mathfrak{U} = \widetilde{GL}''_q(2,\mathbb{R})$.
Then Δ_π is an algebra homomorphism *and a* *–homomorphism *w.r.t. the anti–involution (28).*

Proof. The crucial property of $\Delta_\pi(x)$ for $x = a, b, c, d$ is that each of these operators is of the form $u_x + v_x$, where u_x and v_x are positive self–adjoint operators satisfying the relation $u_x v_x = q^2 v_x u_x$, e.g. $u_b = \pi(a)\otimes\pi(b)$ and $v_b = \pi(b)\otimes\pi(d)$ for $x = b$. Furthermore, it is easy to check that $q u_x^{-1} v_x$ for $x = a, b, c, d$ are positive self–adjoint operators thanks to condition $ii)$ in Definition 4. According to Remark 2, these facts together imply that $\Delta_\pi(x)$ for $x = a, b, c, d$ are also positive self–adjoint and hence *invertible* operators. This, in particular, means that the inverse operators in the part $ii)$ in the definition of Δ_π are well defined.

Since Δ_π is a homomorphism, it suffices to verify its properties for the generators. In particular, the *–homomorphism property, which is $\big(\Delta_\pi(x)\big)^* \equiv (*\otimes *)\Delta_\pi(x) = \Delta(x^*)$ is obvious. The algebra homomorphism property of Δ_π for $x = a, b, c, d$ is inherited from that of Δ for $GL_q(2)$. Finally, applying $\Delta_\pi\otimes\Delta_\pi$ to (16) and multiplying the resulting relations with $\Delta_\pi(b)$ (or $\Delta_\pi(c)$), we see that they are equivalent to correct relations between $\Delta_\pi(b)$ (resp. $\Delta_\pi(c)$) and $\Delta_\pi(x)$ for $x = a, b, c, d$. □

Remark 3. Using the $u_x + v_x$ form of $\Delta_\pi(x)$ along with Eq. (164), we can write an explicit expression for $\Delta_\pi(\theta)$. For instance, in the case of $\mathfrak{U} = \widetilde{GL}'_q(2,\mathbb{R})$ we have

$$\Delta_\pi(\theta) = S_\omega(\mathsf{w})\big(\pi(a)\otimes\pi(b)\big)^{-1}\big(S_\omega(\mathsf{w})\big)^{-1}\,, \qquad \mathsf{w} = \pi(b)\big(\pi(a)\big)^{-1}\otimes\big(\pi(b)\big)^{-1}\pi(d)\,. \quad (29)$$

We introduced the map Δ_π by extending the standard comultiplication (24) to $\widetilde{GL}_q(2)$. Now we will show that $\widetilde{GL}_q(2)$ admits another "comultiplication" δ which is not related to Δ.

Proposition 4. *The linear homomorphism $\delta\colon \widetilde{GL}_q(2) \to \widetilde{GL}_q(2)^{\otimes 2}$ defined on generators as follows*

$$\begin{aligned} &\delta(a) = a\otimes\theta + b\otimes a\,, \qquad \delta(\theta) = \theta\otimes\theta\,,\\ &\delta(c) = c\otimes c\,, \qquad \delta(b) = b\otimes b\,, \qquad \delta(d) = c\otimes d\,, \end{aligned} \quad (30)$$

is a coassociative algebra homomorphism *and a* *–homomorphism *w.r.t. the anti–involution (28).*

Proof. First, for the *–homomorphism property, it suffices to notice that it obviously holds on generators. Next, we notice that

$$(id\otimes\delta)\,g^\pm = g^\pm_{12}\,g^\pm_{13}\,, \quad (31)$$

where $g^\pm$ were defined in Lemma 3. This allows us to use the same approach as in the case of $GL_q(2)$. Namely, the coassociativity property (25) follows immediately if we apply $\delta_2 \equiv (id\otimes\delta\otimes id)$ and $\delta_3 \equiv (id\otimes id\otimes\delta)$ to (31). In order to prove compatibility of the homomorphism property of δ with the defining relations (5) and (16), we recall that, by Lemma 3, these relations are equivalent to relations (18). Therefore, it suffices to apply δ_3 to (18), use (31), and then to verify the resulting R–matrix relations. The latter task simply amounts to using (18) twice, for instance: $\delta_3(R^+_{12}g^+_{13}g^-_{23}) = R^+_{12}g^+_{13}g^+_{14}g^-_{23}g^-_{24} = R^+_{12}g^+_{13}g^-_{23}g^+_{14}g^-_{24} = g^-_{23}g^+_{13}g^-_{24}g^+_{14}R^+_{12} = g^-_{23}g^-_{24}g^+_{13}g^+_{14}R^+_{12} = \delta_3(g^-_{23}g^+_{13}R^+_{12})$. □

Notice that for δ there exists no counit ϵ because the bialgebra axiom $(id\otimes\epsilon)\circ\delta = id$ cannot be fulfilled as seen from the action of δ on d. Nevertheless, Proposition 4 justifies referring to δ as a (non–standard) "comultiplication" for the sake of brevity.

An important difference of the non–standard "comultiplication" from Δ is that the generators b, c, and θ are group–like w.r.t. δ. Therefore, so are the central elements (17):

$$\delta(\eta_q') = \eta_q' \otimes \eta_q' \,, \qquad \delta(\eta_q'') = \eta_q'' \otimes \eta_q'' \,. \tag{32}$$

On the other hand, the Casimir element D_q is now not group–like. Instead, we have

$$\delta(D_q) = ac \otimes \theta d + bc \otimes D_q \,. \tag{33}$$

Therefore, the relation $D_q = 1$ cannot be imposed as a representation independent condition on generators.

Although both matrices $g(\lambda)$ and $\hat{g}(\lambda)$ define, according to Proposition 2, the same algebra $\widetilde{GL}_q(2)$, the map δ is in a sense more related to $\hat{g}(\lambda)$. Indeed, formulae (22) and (31) have strong similarity with (10) and the formula $(id \otimes \Delta)L_\pm = (L_\pm)_{12}(L_\pm)_{13}$, which holds for the standard comultiplication of $\mathcal{U}_q(\mathfrak{sl}_2)$. We will see below that the construction of the fundamental R–operator for $\hat{g}(\lambda)$ indeed requires invoking the map δ, whereas the corresponding construction for $g(\lambda)$ uses the map Δ_π.

3.3 Fundamental R–operator for $g(\lambda)$

According to Proposition 2, both matrices $g(\lambda)$ and $\hat{g}(\lambda)$ can serve as an L–matrix for the algebra $\mathfrak{U} = \widetilde{GL}_q(2)$. Following the general scheme outlined in Introduction, we have now to find their corresponding fundamental R–operators, i.e., to solve Eq. (2). In this context, the following preliminary remark is in order. In the case of $\mathfrak{U} = \mathcal{U}_q(\mathfrak{sl}_2)$, the L–matrices for the XXZ model and for the sinh–Gordon model are related in essentially the same way as $g(\lambda)$ and $\hat{g}(\lambda)$ (cf. the second relation in Eq. (14)) and, as a consequence, their fundamental R–operators are also closely related [FT2, T1, BT2]. But in our case there will be no such a relationship between the fundamental R–operators for $g(\lambda)$ and $\hat{g}(\lambda)$. To explain this difference between our case and the $\mathcal{U}_q(\mathfrak{sl}_2)$ case, let us formulate the following statement.

Lemma 4. *Let s be a constant invertible matrix. Suppose that matrices $L(\lambda)$ and $\hat{L}(\lambda) = s \cdot L(\lambda)$ satisfy Eq. (1) and define the same algebra $\mathfrak{U}$. If there exists an automorphism ι of $\mathfrak{U}$ such that*

$$s \cdot L(\lambda) \cdot s = (id \otimes \iota)\, L(\lambda) \,, \tag{34}$$

then the fundamental R–operators corresponding to $L(\lambda)$ and $\hat{L}(\lambda)$ are related as follows

$$\mathsf{R}(\lambda) = (\iota^{-1} \otimes id)\, \hat{\mathsf{R}}(\lambda) \,. \tag{35}$$

Proof. Consider Eq. (2) for $\hat{L}(\lambda)$, substitute all $\hat{L}(\lambda)$ with $s \cdot L(\lambda)$, and use (34). □

The structure of the L–matrices for the XXZ model and the sinh–Gordon model is such that the automorphism ι does exist (for the generators of $\mathcal{U}_q(\mathfrak{sl}_2)$ it reads: $\iota(E) = F$, $\iota(F) = E$, $\iota(K) = K^{-1}$). But for $g(\lambda)$ given by (19) and $s = \sigma_1$, matrix entries of the l.h.s. and the r.h.s. in (34) have different functional dependence on λ. This means that there is no automorphism ι that would resolve (34) in our case and so we have to solve Eq. (2) separately for $g(\lambda)$ and $\hat{g}(\lambda)$.

Now we will solve Eq. (2) for $g(\lambda)$. For brevity of notations, we will write $x \otimes y$ instead of $\pi(x) \otimes \pi(y)$.

Theorem 1. *Let $\mathcal{B}$, $\mathcal{K}$, and $\mathfrak{U}$ be as in Definition 4 and let π be a positive representation of $\mathfrak{U}$. Let $g(\lambda) \in \mathrm{Mat}(2) \otimes \mathcal{B}$ be as in (19). Then the operator $\mathsf{R}(\lambda) \in \mathcal{B}^{\otimes 2}$ acting on $\mathcal{K} \otimes \mathcal{K}$ and defined by the formula*

$$\mathsf{R}(\lambda) = (c \otimes b)^{-\frac{\alpha}{2} \log \lambda} \left((a \otimes b + b \otimes d)(c \otimes a + d \otimes c) \right)^{\alpha \log \lambda} (c \otimes b)^{-\frac{\alpha}{2} \log \lambda}, \tag{36}$$

where $\alpha \equiv \frac{1}{\log q} = \frac{1}{i\gamma}$, satisfies the equation

$$\mathsf{R}_{23}(\lambda)\, g_{12}(\lambda\mu)\, g_{13}(\mu) = g_{12}(\mu)\, g_{13}(\lambda\mu)\, \mathsf{R}_{23}(\lambda)\,. \tag{37}$$

If the tensor product $\pi \otimes \pi$ is multiplicity free, then (36) is the unique solution of (37) up to multiplication by a scalar factor.

Proof. Matching coefficients at different powers of μ, it is easy to see that (37) is equivalent to the following set of equations:

$$[\mathsf{R}(\lambda), (\theta \otimes b)] = 0\,, \qquad [\mathsf{R}(\lambda), (b \otimes \theta)] = 0\,, \tag{38}$$

$$\mathsf{R}(\lambda)\,(c \otimes a + \lambda\, d \otimes c) = (\lambda\, c \otimes a + d \otimes c)\,\mathsf{R}(\lambda)\,, \tag{39}$$

$$\mathsf{R}(\lambda)\,(a \otimes b + \lambda\, b \otimes d) = (\lambda\, a \otimes b + b \otimes d)\,\mathsf{R}(\lambda)\,, \tag{40}$$

$$\mathsf{R}(\lambda)\,(a \otimes a + \lambda\, b \otimes c) = (a \otimes a + \lambda^{-1} b \otimes c)\,\mathsf{R}(\lambda)\,, \tag{41}$$

$$\mathsf{R}(\lambda)\,(d \otimes d + \lambda^{-1} c \otimes b) = (d \otimes d + \lambda\, c \otimes b)\,\mathsf{R}(\lambda)\,, \tag{42}$$

$$\mathsf{R}(\lambda)\,(\lambda\, \theta \otimes a + d \otimes \theta) = (\theta \otimes a + \lambda\, d \otimes \theta)\,\mathsf{R}(\lambda)\,. \tag{43}$$

It is now easy to recognize in (39)–(42) a structure related to the comultiplication Δ (cf. (24)). To make this structure more transparent, we introduce $\tilde{\mathsf{R}}(\lambda) = (c \otimes b)^{\frac{\alpha}{2} \log \lambda}\, \mathsf{R}(\lambda)\, (c \otimes b)^{\frac{\alpha}{2} \log \lambda}$. Then equations (38)–(43) acquire the following form:

$$[\tilde{\mathsf{R}}(\lambda), b \otimes \theta] = [\tilde{\mathsf{R}}(\lambda), \theta \otimes b] = 0\,, \tag{44}$$

$$\tilde{\mathsf{R}}(\lambda)\,\Delta_\pi(b) = \Delta_\pi(b)\,\tilde{\mathsf{R}}(\lambda)\,, \qquad \tilde{\mathsf{R}}(\lambda)\,\Delta_\pi(c) = \Delta_\pi(c)\,\tilde{\mathsf{R}}(\lambda)\,, \tag{45}$$

$$\tilde{\mathsf{R}}(\lambda)\,\Delta_\pi(a) = \lambda^{-2}\Delta_\pi(a)\,\tilde{\mathsf{R}}(\lambda)\,, \qquad \tilde{\mathsf{R}}(\lambda)\,\Delta_\pi(d) = \lambda^{2}\,\Delta_\pi(d)\,\tilde{\mathsf{R}}(\lambda)\,, \tag{46}$$

$$\tilde{\mathsf{R}}(\lambda)\,(\lambda\, \theta \otimes a + \lambda^{-1} d \otimes \theta) = (\lambda^{-1}\theta \otimes a + \lambda\, d \otimes \theta)\,\tilde{\mathsf{R}}(\lambda)\,. \tag{47}$$

where Δ_π is the algebra homomorphism introduced in Proposition 3. Next, observing that

$$[\Delta_\pi(a), b \otimes \theta] = 0\,, \qquad \Delta_\pi(b)\,(b \otimes \theta) = q\,(b \otimes \theta)\,\Delta_\pi(b)\,, \tag{48}$$

$$[\Delta_\pi(d), b \otimes \theta] = 0\,, \qquad \Delta_\pi(c)\,(b \otimes \theta) = q^{-1}(b \otimes \theta)\,\Delta_\pi(c)\,, \tag{49}$$

are consequences of (5), (16), and (24), we infer that Eqs. (44)–(45) are satisfied if $\tilde{\mathsf{R}}(\lambda)$ is taken to be a function of $\Delta_\pi(ad)$ and $\Delta_\pi(bc)$. Furthermore, due to Eq. (27) we have $\Delta_\pi(ad) = q\Delta_\pi(bc) + D_q \otimes D_q$, where the last term is a multiple of the unit operator. This implies that we can take $\tilde{\mathsf{R}}(\lambda)$ to be a function of $\Delta_\pi(bc)$ only. Then Eqs. (44)–(46) are solved easily:

$$\tilde{\mathsf{R}}(\lambda) = \left(\Delta_\pi(bc)\right)^{\alpha \log \lambda}, \qquad \alpha = \tfrac{1}{\log q}\,. \tag{50}$$

It remains to verify (47). For this aim we notice that, since $\Delta_\pi(b)$ is invertible, Eq. (47) is equivalent to the relation

$$\tilde{\mathsf{R}}(\lambda)\,\mathsf{X}(\lambda) = \mathsf{X}(\lambda^{-1})\,\tilde{\mathsf{R}}(\lambda)\,, \tag{51}$$

where we denoted $\mathsf{X}(\lambda) \equiv \Delta_\pi(b)(\lambda\,\theta \otimes a + \lambda^{-1} d \otimes \theta)$. Now, using (24), we find

$$\mathsf{X}(\lambda) = q^{-1}\lambda(\theta \otimes b)\,\Delta_\pi(a) + q\,\lambda^{-1}(b \otimes \theta)\,\Delta_\pi(d) + \lambda\,\eta_q' \otimes D_q + \lambda^{-1}\,D_q \otimes \eta_q'\,. \tag{52}$$

The sum of the last two terms here obviously satisfies (51). The first two terms satisfy (51) as a consequence of relations (44) and (46). Thus, Eq. (47) is proven and we have shown that (36) indeed solves Eq. (37).

Let us prove the uniqueness of $\mathsf{R}(\lambda)$. Notice that $\tilde{\mathsf{R}}(\lambda)$ is an invertible operator due to the properties of π. Suppose that there exists another solution, $\tilde{\mathsf{R}}'(\lambda)$, to Eqs. (44)–(46). Then it follows from (45)–(46) that $\mathsf{F}(\lambda) \equiv (\tilde{\mathsf{R}}(\lambda))^{-1}\tilde{\mathsf{R}}'(\lambda)$ commutes with $\Delta_\pi(x)$ for all $x \in \mathfrak{U}$. Under the assumption that $\pi \otimes \pi$ is multiplicity free, we invoke Lemma 2 and infer that $\mathsf{F}(\lambda)$ can be a function only of $\Delta_\pi(\eta_q'')$. But it follows from (48)–(49) that $\Delta_\pi(\eta_q'')$ does not commute with $b \otimes \theta$. Thus, $\mathsf{F}(\lambda)$ satisfying (44) cannot depend non–trivially on $\Delta_\pi(\eta_q'')$ and therefore it must be just a scalar function. □

Remark 4. The positivity property of the representation π is crucial for the assertion that (50) solves Eqs. (45)–(46). Indeed, it ensures that $x = \Delta_\pi(bc)$ and $y = \Delta_\pi(z)$ for $z = a, b, c, d$ are positive self–adjoint operators (cf. Remark 2) and therefore (50) solves Eqs. (45)–(46) in the sense clarified in Remark 1. Notice also that on the same ground we have $\big(\Delta_\pi(b)\Delta_\pi(c)\big)^t = \big(\Delta_\pi(bc)\big)^t$.

Remark 5. For lattice integrable models, the function that most commonly appears in solutions for fundamental R–operators is the *quantum dilogarithm* (see Appendix A.1). Lemma 12 (see the same Appendix) allows us to rewrite our solution (36) in a form involving quantum dilogarithms:

$$\begin{aligned}\mathsf{R}(\lambda) &= \frac{S_\omega(\lambda^{-1}\mathsf{w})}{S_\omega(\lambda\,\mathsf{w})}\,(a \otimes a)^{\alpha\log\lambda}\,\frac{S_\omega(\lambda^{-1}\tilde{\mathsf{w}})}{S_\omega(\lambda\,\tilde{\mathsf{w}})} \\ &= \frac{S_\omega(\lambda^{-1}\mathsf{w}^{-1})}{S_\omega(\lambda\,\mathsf{w}^{-1})}\,(d \otimes d)^{\alpha\log\lambda}\,\frac{S_\omega(\lambda^{-1}\tilde{\mathsf{w}}^{-1})}{S_\omega(\lambda\,\tilde{\mathsf{w}}^{-1})}\,,\end{aligned} \tag{53}$$

where $\mathsf{w} = ba^{-1} \otimes b^{-1}d$ and $\tilde{\mathsf{w}} = dc^{-1} \otimes a^{-1}c$.

Fundamental R–operator (36) is regular in the sense of Eq. (3) and has the following properties:

$$\big(\mathsf{R}(\bar{\lambda})\big)^* = \mathsf{R}(\lambda^{-1}) = \mathsf{R}^{-1}(\lambda)\,. \tag{54}$$

Application of formula (4) to (36) yields the following lattice Hamiltonian density:

$$\gamma\,H_{n,n+1} = \log\big((a_{n+1}b_n + b_{n+1}d_n)(c_{n+1}a_n + d_{n+1}c_n)\big) - \log(b_n b_{n+1})\,. \tag{55}$$

Definition 4 along with Remark 2 ensure that the arguments of the logarithms here are products of commuting positive self–adjoint operators.

3.4 Fundamental R–operator for $\hat{g}(\lambda)$

Now we will solve Eq. (2) for $\hat{g}(\lambda)$. For brevity of notations, we will write $x \otimes y$ instead of $\pi(x) \otimes \pi(y)$ and δ instead of $(\pi \otimes \pi) \circ \delta$.

Theorem 2. *Let $\mathcal{B}$, $\mathcal{K}$, and $\mathfrak{U}$ be as in Definition 4 and let π be a positive representation of $\mathfrak{U}$. Let $\hat{g}(\lambda) \in \mathrm{Mat}(2) \otimes \mathcal{B}$ be as in (19). Then the operator $\hat{\mathsf{R}}(\lambda) \in \mathcal{B}^{\otimes 2}$ acting on $\mathcal{K} \otimes \mathcal{K}$ and defined by the formula*

$$\hat{\mathsf{R}}(\lambda) = \big(ac \otimes \theta d + bc \otimes ad - qbc \otimes bc\big)^{\alpha\log\lambda}\,, \tag{56}$$

where $\alpha \equiv \frac{1}{\log q} = \frac{1}{i\gamma}$*, satisfies the equation*

$$\hat{\mathsf{R}}_{23}(\lambda)\,\hat{g}_{12}(\lambda\mu)\,\hat{g}_{13}(\mu) = \hat{g}_{12}(\mu)\,\hat{g}_{13}(\lambda\mu)\,\hat{\mathsf{R}}_{23}(\lambda)\,. \tag{57}$$

If the tensor product $\pi \otimes \pi$ *is multiplicity free, then (56) is the unique solution of (57) up to multiplication by a scalar factor.*

Proof. Substituting the Baxterized form (22) of $\hat{g}(\lambda)$ into (57) and matching coefficients at different powers of μ, we see that (57) is equivalent to the following set of matrix equations:

$$\hat{\mathsf{R}}_{23}(\lambda)\,\hat{g}^{\pm}_{12}\,\hat{g}^{\pm}_{13} = \hat{g}^{\pm}_{12}\,\hat{g}^{\pm}_{13}\,\hat{\mathsf{R}}_{23}(\lambda)\,, \tag{58}$$

$$\hat{\mathsf{R}}_{23}(\lambda)\,(\lambda\,\hat{g}^{+}_{12}\,\hat{g}^{-}_{13} + \lambda^{-1}\hat{g}^{-}_{12}\,\hat{g}^{+}_{13}) = (\lambda^{-1}\hat{g}^{+}_{12}\,\hat{g}^{-}_{13} + \lambda\,\hat{g}^{-}_{12}\,\hat{g}^{+}_{13})\,\hat{\mathsf{R}}_{23}(\lambda)\,. \tag{59}$$

Comparing (58) with (31), we conclude that

$$[\,\hat{\mathsf{R}}(\lambda)\,,\,\delta(x)\,] = 0 \tag{60}$$

for all generators and hence for all $x \in \mathfrak{U}$. This suggests to seek $\hat{\mathsf{R}}(\lambda)$ as a function of $\delta(D_q)$.

Matrix equation (59) is equivalent to the following set of equations:

$$\hat{\mathsf{R}}(\lambda)\,(a\otimes d) = \lambda^{-2}(a\otimes d)\,\hat{\mathsf{R}}(\lambda)\,, \qquad \hat{\mathsf{R}}(\lambda)\,(a\otimes c) = \lambda^{-2}(a\otimes c)\,\hat{\mathsf{R}}(\lambda)\,, \tag{61}$$

$$\hat{\mathsf{R}}(\lambda)\,(\lambda\,\theta\otimes d + \lambda^{-1} d\otimes b) = (\lambda^{-1}\theta\otimes d + \lambda\, d\otimes b)\,\hat{\mathsf{R}}(\lambda)\,, \tag{62}$$

$$\hat{\mathsf{R}}(\lambda)\,(\lambda^{-1} d\otimes a + \lambda^{-1} c\otimes\theta + \lambda\,\theta\otimes c) = (\lambda\, d\otimes a + \lambda\, c\otimes\theta + \lambda^{-1}\theta\otimes c)\,\hat{\mathsf{R}}(\lambda). \tag{63}$$

Noticing that

$$\delta(D_q)\,(a\otimes d) = q^{-2}(a\otimes d)\,\delta(D_q)\,, \qquad \delta(D_q)\,(a\otimes c) = q^{-2}(a\otimes c)\,\delta(D_q)\,, \tag{64}$$

we infer that a solution to (61) is given by

$$\hat{\mathsf{R}}(\lambda) = \big(\delta(D_q)\big)^{\alpha\log\lambda}\,, \qquad \alpha \equiv \tfrac{1}{\log q}\,. \tag{65}$$

Lemma 5. $\hat{\mathsf{R}}(\lambda)$ *given by (65) satisfies relation (62).*

The proof is given in Appendix B. It remains to prove that $\hat{\mathsf{R}}(\lambda)$ satisfies Eq. (63). For this aim, we notice that since $\delta(b)$ is represented by an invertible element, Eq. (63) is equivalent to the relation

$$\hat{\mathsf{R}}(\lambda)\,\mathsf{X}(\lambda) = \mathsf{X}(\lambda^{-1})\,\hat{\mathsf{R}}(\lambda)\,, \tag{66}$$

where we denoted $\mathsf{X}(\lambda) \equiv q^{-1}\delta(b)\,(\lambda^{-1}d\otimes a + \lambda^{-1}c\otimes\theta + \lambda\,\theta\otimes c)$ Now we observe that

$$\mathsf{X}(\lambda) = (\lambda\,\theta\otimes d + \lambda^{-1} d\otimes b)\,\delta(a) - q\,\lambda\,\delta(\theta)\,(a\otimes d) - \lambda\, D_q\otimes\eta'_q - \lambda^{-1}\eta'_q\otimes D_q\,. \tag{67}$$

The sum of the last two terms here obviously satisfies (66). The first two terms satisfy (66) as a consequence of relations (61) and (62). Thus, (63) is proven and we have shown that (65) indeed solves Eq. (57).

Eq. (60) implies that $\hat{\mathsf{R}}(\lambda)$ is essentially unique. Indeed, under the assumption that $\pi\otimes\pi$ is multiplicity free, we invoke Lemma 2 and infer that $\hat{\mathsf{R}}(\lambda)$ can be a function only of $\delta(D_q)$ and $\Delta_\pi(\eta''_q)$. Furthermore, Eq. (32) implies that $(\pi\otimes\pi)\delta(\eta''_q)$ is just a multiple of unity, so $\hat{\mathsf{R}}(\lambda)$ must be a function of $\delta(D_q)$ only. Finally, it is clear that such a function satisfying (64) is given by (65) uniquely up to a scalar factor. □

Remark 6. Lemma 12 allows us to rewrite our solution (56) in terms of quantum dilogarithms:

$$\hat{\mathsf{R}}(\lambda) = \big(bc \otimes D_q\big)^{\alpha \log \lambda} \, \frac{S_\omega(\mathsf{r})}{S_\omega\big(\lambda^2 \mathsf{r}\big)} \,, \qquad \mathsf{r} = \big(D_q\big)^{-1} \, b^{-1} a \otimes \theta d \,. \tag{68}$$

Fundamental R–operator (56) is regular and has the properties (54). Application of formula (4) yields the following lattice Hamiltonian density:

$$\gamma \, \hat{H}_{n,n+1} = \log\big(a_{n+1} c_{n+1} \theta_n d_n + b_{n+1} c_{n+1} (D_q)_n\big) \,. \tag{69}$$

Definition 4 along with Remark 2 ensure that the argument of the logarithm here is a positive self–adjoint operator.

3.5 Lattice Liouville model

A one–parameter family π_κ of positive representations of $\mathfrak{U} = \widetilde{GL}'_q(2,\mathbb{R})$ or $\mathfrak{U} = \widetilde{GL}''_q(2,\mathbb{R})$ on the Hilbert space $\mathcal{K} = L^2(\mathbb{R})$ can be constructed as follows (it is closely related to one of the representations of $SL_q(2,\mathbb{R})$ listed in [S1]):

$$\begin{aligned} \pi_\kappa(a) = e^{\frac{\beta}{8}\,\Pi} \left(\kappa^{-1} + \kappa \, e^{-\beta\Phi}\right) e^{\frac{\beta}{8}\,\Pi} \,, \qquad \pi_\kappa(d) = \kappa^{-1} e^{-\frac{\beta}{4}\,\Pi} \,, \\ \pi_\kappa(b) = \pi_\kappa(c) = e^{-\frac{\beta}{2}\,\Phi} \,, \qquad \pi_\kappa(\theta) = e^{\frac{\beta}{2}\,\Phi} \,. \end{aligned} \tag{70}$$

Here $\kappa > 0$, $\beta \equiv \sqrt{8\gamma} = 2\omega\sqrt{2\pi} > 0$, and Π and Φ are self–adjoint operators on $L^2(\mathbb{R})$ which satisfy $[\Pi, \Phi] = -i$. Since elements of $\mathfrak{U}$ are realized by unbounded operators on $L^2(\mathbb{R})$, it is necessary to consider suitable subspaces $\mathcal{T}_\kappa \subset L^2(\mathbb{R})$ of test–functions on which all operators $\pi_\kappa(x)$, $x \in \mathfrak{U}$ are well defined. Similar consideration was done for $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ in [PT, BT1]. We will provide analogous analytic details for π_κ elsewhere.

Let us now introduce the following L–matrix: $L^{\mathrm{L}}(\lambda) = \kappa \, \pi_\kappa\big(g(\lambda)\big)$. In order to construct the corresponding lattice model we assign a copy of this matrix to each site of the lattice, i.e. for $n = 1, \ldots, \mathrm{N}$ we have

$$L^{\mathrm{L}}_n(\lambda) = \begin{pmatrix} e^{\frac{\beta}{8}\,\Pi_n} \left(1 + \kappa^2 \, e^{-\beta\Phi_n}\right) e^{\frac{\beta}{8}\,\Pi_n} & \kappa \, \lambda \, e^{-\frac{\beta}{2}\Phi_n} \\ \kappa \left(\lambda \, e^{\frac{\beta}{2}\Phi_n} + \lambda^{-1} e^{-\frac{\beta}{2}\Phi_n}\right) & e^{-\frac{\beta}{4}\,\Pi_n} \end{pmatrix} , \tag{71}$$

where Φ_n and Π_n act non–trivially only on the n–th tensor factor in the Hilbert space $\mathcal{H} = \big(L^2(\mathbb{R})\big)^{\otimes \mathrm{N}}$ and therefore satisfy the relation $[\Pi_n, \Phi_m] = -i\delta_{nm}$.

In the pioneering work [FT3], a close analogue of (71) was constructed as a special limit of the L–matrix for the sine–Gordon model and put forward as an L–matrix describing a lattice version of the Liouville model with Φ_n and Π_n being discrete counterparts of the field and its conjugate momentum variables. In its present form, L–matrix (71) was obtained in [BT2] by analogous limit applied to the sinh–Gordon model.

The *continuum limit* of a classical lattice integrable model is usually constructed as the limit of vanishing lattice spacing ($\mathrm{N} \to \infty$, $\kappa \to 0$ with $\kappa \, \mathrm{N}$ kept fixed) combined with the standard recipe [FST] of replacement of lattice canonical variables by their continuum counterparts:

$$\Pi_n \to \kappa \, \Pi(x) \,, \quad \Phi_n \to \Phi(x) \,, \quad x = n \, \kappa \,, \tag{72}$$

which leads to the canonical Poisson brackets, $\{\Pi(x), \Phi(y)\} = \delta(x-y)$. In this classical continuum limit we have $L_n^{\rm L}(\lambda) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \kappa \left(U_+(\lambda) + U_-(\lambda)\right) + O(\kappa^2)$, where ($\partial_\pm \equiv \partial_t \pm \partial_x$)

$$U_+(\lambda) = \begin{pmatrix} \frac{\beta}{8}\,\partial_+\Phi & \lambda\, e^{-\frac{\beta}{2}\Phi} \\ \lambda\, e^{\frac{\beta}{2}\Phi} & -\frac{\beta}{8}\,\partial_+\Phi \end{pmatrix}, \quad U_-(\lambda) = \begin{pmatrix} \frac{\beta}{8}\,\partial_-\Phi & 0 \\ \lambda^{-1} e^{-\frac{\beta}{2}\Phi} & -\frac{\beta}{8}\,\partial_-\Phi \end{pmatrix}. \tag{73}$$

These matrices satisfy the zero curvature equation, $\partial_- U_+(\lambda) + \partial_+ U_-(\lambda) = 2[U_+(\lambda), U_-(\lambda)]$, provided that Φ satisfies the equation of motion of the Liouville field: $\Box\,\Phi = \frac{8}{\beta} e^{-\beta\Phi}$. On this ground it was suggested in [FT3] that (71) corresponds to a lattice version of the Liouville model. However, a direct verification of this claim, i.e., construction of a lattice Hamiltonian that $i)$ commutes with the transfer matrix for (71), and $ii)$ turns in the continuum limit into the Hamiltonian of the continuum Liouville model, has been missing until now although some partial results have been obtained. In particular, it was shown in [BT2] that applying to the Hamiltonian of the lattice sinh–Gordon model first the special limit procedure described in [FT3] and then taking the continuum limit, we indeed obtain the Hamiltonian of the continuum Liouville model. Another computation [S3] demonstrated that, unlike for the sinh–Gordon model, the factorization method [IK] of constructing integrals of motion applied to (71) yields a lattice analogue only of the chiral combination $(H + P)$ of the Liouville Hamiltonian and momentum operator.

Results of Section 3.3 imply that

$$H^{\rm L}_{n,n+1} = (\pi_\kappa \otimes \pi_\kappa) H_{n,n+1}\,, \tag{74}$$

where $H_{n,n+1}$ is given by (55), is a quantum nearest–neighbour lattice Hamiltonian corresponding to L–matrix (71). In order to show that (74) is a lattice analogue of the Hamiltonian for the continuous Liouville model we consider first its classical limit where Φ_n and Π_n become canonical variables on the phase space equipped with the Poisson bracket $\{\Pi_n, \Phi_m\} = \delta_{nm}$. A direct computation using (70) yields (up to an additive constant)

$$\begin{aligned} H^{\rm L,cl}_{n,n+1} = \tfrac{1}{\gamma} \log\Big(& \tfrac{1}{2} \cosh \tfrac{\beta}{4}\,(\Pi_n + \Pi_{n+1}) + \tfrac{1}{2}\, \cosh \tfrac{\beta}{2}\,(\Phi_n - \Phi_{n+1}) \\ & + \tfrac{\kappa^2}{2}\, e^{-\frac{\beta}{2}(\Phi_n + \Phi_{n+1})} \left(1 + e^{\frac{\beta}{4}(\Pi_n + \Pi_{n+1})}\, \cosh \tfrac{\beta}{2}(\Phi_n - \Phi_{n+1})\right) \\ & + \tfrac{\kappa^4}{4}\, e^{\frac{\beta}{4}(\Pi_n + \Pi_{n+1})} e^{-\beta(\Phi_n + \Phi_{n+1})} \Big)\,. \end{aligned} \tag{75}$$

Let us remark that the difference between (75) and the analogous expression obtained by a "naive" limit in [BT2] is only in the last term. Taking now the continuum limit of (75) according to (72), we obtain (again up to an additive constant)

$$\lim_{\kappa\to 0} \sum_n \tfrac{1}{\kappa} H^{\rm L,cl}_{n,n+1} = \int dx\, \left(\tfrac{1}{2}\,\Pi^2 + \tfrac{1}{2}\,(\partial_x \Phi)^2 + \tfrac{1}{\gamma}\, e^{-\beta\,\Phi}\right), \tag{76}$$

which is the Hamiltonian of the classical continuum Liouville model.

4 Reductions of $\widetilde{GL}_q(2)$ and related lattice models

Defining relations of $\widetilde{GL}_q(2)$ admit the following reductions: $i)$ $\theta = 0$; $ii)$ $b = c$; $iii)$ $b = 0$, and $iv)$ $c = 0$. Below we will consider the problem of constructing the fundamental R–matrices for the corresponding reductions of matrices $g(\lambda)$ and $\hat{g}(\lambda)$ in each of these cases.

4.1 $\theta = 0$

For $\theta = 0$, matrix $g(\lambda)$ reduces back to $\mathrm{g}(\lambda)$ given by (11). In this case we take π to be a positive representation of $GL_q(2)$ (modification of Definition 4 is obvious). As we discussed at the end of Section 2, dependence on the spectral parameter of the auxiliary transfer–matrix for $\mathrm{g}(\lambda)$ can be removed with the help of the twist transformation (14). However, Eq. (37) for the corresponding fundamental R–matrix cannot be transformed by a similar means to a spectral parameter independent form.

Let $\mathsf{R}_{\mathfrak{o}}(\lambda)$ denote a solution to (37) where $g(\lambda)$ is replaced with $\mathrm{g}(\lambda)$. Introduce $\tilde{\mathsf{R}}_{\mathfrak{o}}(\lambda) = (c \otimes c)^{\frac{\alpha}{2}\log\lambda}\, \mathsf{R}_{\mathfrak{o}}(\lambda)\, (c \otimes c)^{\frac{\alpha}{2}\log\lambda}$. Evidently, $\mathsf{R}_{\mathfrak{o}}(\lambda)$ must satisfy only Eqs. (39)–(42), and $\tilde{\mathsf{R}}_{\mathfrak{o}}(\lambda)$ must satisfy only Eqs. (45)–(46). For the latter we have a one–parameter family of solutions:

$$\tilde{\mathsf{R}}_{\mathfrak{o}}(\lambda;\beta) = \big(\Delta_\pi(b)\big)^{(\alpha-\beta)\log\lambda} \big(\Delta_\pi(c)\big)^{(\alpha+\beta)\log\lambda}, \qquad \alpha = \tfrac{1}{\log q}\,. \tag{77}$$

Remark 7. The reason why the proof of essential uniqueness given for $\tilde{\mathsf{R}}(\lambda)$ in Section 3.3 does not apply to the case of $\tilde{\mathsf{R}}_{\mathfrak{o}}(\lambda)$ despite that, by Lemma 2, the center of $GL_q(2)$ is generated only by the quadratic Casimir element D_q, is that the ratio of two solutions, $\mathsf{F}(\lambda) = \big(\tilde{\mathsf{R}}_{\mathfrak{o}}(\lambda;\beta_1)\big)^{-1}\tilde{\mathsf{R}}_{\mathfrak{o}}(\lambda;\beta_2) = \big(\Delta_\pi(bc^{-1})\big)^{(\beta_1-\beta_2)\log\lambda}$ depends non–trivially on the *non–polynomial* Casimir element, bc^{-1}, which can formally be written as $\lim_{\theta\to 0}\eta'/\eta''$.

Next we consider the $\theta = 0$ counterpart of matrix $\hat{g}(\lambda)$ which is $\hat{\mathrm{g}}(\lambda)$ given by (11). We again take π to be a positive representation of $GL_q(2)$.

Let $\hat{\mathsf{R}}_{\mathfrak{o}}(\lambda)$ denote a solution to (57) where $\hat{g}(\lambda)$ is replaced with $\hat{\mathrm{g}}(\lambda)$. Apparently, $\hat{\mathsf{R}}_{\mathfrak{o}}(\lambda)$ must satisfy (58) and as a consequence it is a function of $\delta(D_q)$ only (the Casimir element bc^{-1} is group–like w.r.t. δ and hence is represented by a multiple of the unity). Further, $\hat{\mathsf{R}}_{\mathfrak{o}}(\lambda)$ must satisfy Eq. (61) and the relations that replace Eqs. (62)–(63), namely $\hat{\mathsf{R}}_{\mathfrak{o}}(\lambda)\,(d\otimes x) = \lambda^2 (d\otimes x)\,\hat{\mathsf{R}}_{\mathfrak{o}}(\lambda)$ for $x = a, b$. It is easy to see that the unique (up to a scalar factor) solution to these equations is given by the same formula (65). But for $\theta = 0$ we have $\delta(D_q) = bc \otimes D_q$, which has nontrivial operator dependence only in its first tensor component. This makes $\hat{\mathsf{R}}_{\mathfrak{o}}(\lambda)$ rather useless for constructing integrals of motion since it produces only those that have no interaction between different sites of the lattice (cf. (69) for $\theta = 0$).

Thus, we see a kind of dual pictures for matrices $\mathrm{g}(\lambda)$ and $\hat{\mathrm{g}}(\lambda)$: it is the fundamental transfer–matrix for the former and the auxiliary transfer–matrix for the latter that generate a set of mutually commuting elements of $\mathcal{B}^{\mathrm{N}}$.

4.2 q–oscillator algebra $\mathcal{A}_q$

Interrelations between deformed oscillator algebras and quantum Lie algebras are well known (see, e.g. [CP, KS]). Relation of the former to quantum groups is also known, see e.g. [S1, DK], but has been employed in the context of integrable models less extensively. Here we will show that a reduction of $\widetilde{GL}_q(2)$ yields a q–oscillator algebra. This will allow us to adapt the results of the previous sections, in particular, the constructions of fundamental R–operators, to the case of the q–oscillator algebra. Recall that, as above, we deal with the case $q = e^{i\gamma}$, $\gamma \in (0, \pi)$.

Definition 5. *The q–oscillator algebra $\mathcal{A}_q$ is a unital associative algebra with generators e, f, k, k^{-1} and defining relations $k\,k^{-1} = k^{-1}k = 1$, and*

$$e\,k = q\,k\,e\,, \qquad f\,k = q^{-1}\,k\,f\,, \qquad [e,f] = (q-q^{-1})\,k^2\,, \tag{78}$$

*and equipped with an anti–involution * defined on generators by*

$$e^* = e\,, \quad f^* = f\,, \quad k^* = k\,, \quad (k^{-1})^* = k^{-1}\,. \tag{79}$$

Lemma 6. *For a generic q, the center of $\mathcal{A}_q$ is generated by the Casimir element*

$$C_q \equiv e\,f - q\,k^2\,. \tag{80}$$

The lemma is standard [CP, KS]. Now we need the following simple but useful statements which are straightforward to verify:

Lemma 7. *Let $\mathfrak{U}$ be $\widetilde{GL}'_q(2,\mathbb{R})$ or $\widetilde{GL}''_q(2,\mathbb{R})$. The linear homomorphism $\mathcal{Q}: \mathfrak{U} \to \mathcal{A}_q$ defined on generators as follows*

$$\mathcal{Q}(a) = e\,, \quad \mathcal{Q}(c) = k\,, \quad \mathcal{Q}(b) = k\,, \quad \mathcal{Q}(\theta) = k^{-1}\,, \quad \mathcal{Q}(d) = f \tag{81}$$

is an algebra homomorphism.

Lemma 8. *The defining relations (78) are equivalent to the following relation*

$$R_{12}\;\mathcal{Q}(g)_{13}\;\mathcal{Q}(g)_{23} = \mathcal{Q}(g)_{23}\;\mathcal{Q}(g)_{13}\;R_{12}\,, \tag{82}$$

where $\mathcal{Q}(g) = \left(\begin{smallmatrix} e & k \\ k & f \end{smallmatrix}\right)$, and the auxiliary R–matrix is given by either of the matrices in (7).

For $\mathcal{Q}$–images of the Casimir elements we have $\mathcal{Q}(D_q) = C_q$ and $\mathcal{Q}(\eta'_q) = \mathcal{Q}(\eta''_q) = 1$. The latter equalities mean that we identified θ as the inverse to *both* b and c.

Let us introduce the following $\mathcal{Q}$–images of $g(\lambda)$ and $\hat{g}(\lambda)$:

$$L^{\mathcal{A}}(\lambda) = \begin{pmatrix} e & \lambda\,k \\ \lambda\,k^{-1} + \lambda^{-1}k & f \end{pmatrix}, \qquad \hat{L}^{\mathcal{A}}(\lambda) = \begin{pmatrix} \lambda\,k^{-1} + \lambda^{-1}k & \lambda^{-1}f \\ \lambda\,e & \lambda\,k \end{pmatrix}. \tag{83}$$

Proposition 5. *Each of the following matrix relations*

$$R_{12}(\lambda)\;L^{\mathcal{A}}_{13}(\lambda\mu)\;L^{\mathcal{A}}_{23}(\mu) = L^{\mathcal{A}}_{23}(\mu)\;L^{\mathcal{A}}_{13}(\lambda\mu)\;R_{12}(\lambda)\,, \tag{84}$$

$$\hat{R}_{12}(\lambda)\;\hat{L}^{\mathcal{A}}_{13}(\lambda\mu)\;\hat{L}^{\mathcal{A}}_{23}(\mu) = \hat{L}^{\mathcal{A}}_{23}(\mu)\;\hat{L}^{\mathcal{A}}_{13}(\lambda\mu)\;\hat{R}_{12}(\lambda)\,, \tag{85}$$

where the auxiliary R–matrices are given by (9) and (8), respectively, holds if and only if the elements e, f, k satisfy relations (78) and k^{-1} satisfies the following relations:

$$e\,k^{-1} = q^{-1}k^{-1}e\,, \qquad f\,k^{-1} = q\,k^{-1}\,f\,, \qquad [k,k^{-1}] = 0\,. \tag{86}$$

Proof. First, applying Lemma 7 to Eqs. (20)–(21), we conclude that relations (84)–(85) do hold. Next, it is easy to see that all the steps in the proof of Proposition 2 remain valid. Therefore, each of relations (84)–(85) is equivalent to (86) together with (82). The latter matrix relation is equivalent to (78) by Lemma 8. □

Notice that the comultiplication Δ has no consistent reduction to $\mathcal{A}_q$ since $\big(\mathcal{Q}\otimes\mathcal{Q}\big)\Delta(b) \neq \big(\mathcal{Q}\otimes\mathcal{Q}\big)\Delta(c)$. Nevertheless, it is useful to observe the following.

Proposition 6. *The linear homomorphism* $\Delta_{\mathcal{A}}\colon GL_q(2,\mathbb{R}) \to \mathcal{A}_q^{\otimes 2}$ *defined on generators as follows:* $\Delta_{\mathcal{A}}(x) = \big(\mathcal{Q}\otimes\mathcal{Q}\big)\Delta(x)$ *for* $x = a, b, c, d$, *is an* algebra homomorphism *and a* *–homomorphism *w.r.t. the anti–involution (79).*

Proof. The assertion follows by combining Lemma 7 with the properties of the standard comultiplication Δ for $GL_q(2)$. □

For the non–standard "comultiplication" we have the following reduction of δ to $\mathcal{A}_q$.

Proposition 7. *The linear homomorphism* $\delta_{\mathcal{A}}\colon \mathcal{A}_q \to \mathcal{A}_q^{\otimes 2}$ *defined on generators as follows*

$$\delta_{\mathcal{A}}(e) = e\otimes k^{-1} + k\otimes e\,, \qquad \delta_{\mathcal{A}}(f) = k\otimes f\,, \qquad \delta_{\mathcal{A}}(k^{\pm 1}) = k^{\pm 1}\otimes k^{\pm 1} \tag{87}$$

is a coassociative algebra homomorphism *and a* *–homomorphism *w.r.t. the anti–involution (79).*

Proof. Notice that $\delta_{\mathcal{A}}\circ\mathcal{Q} = (\mathcal{Q}\otimes\mathcal{Q})\circ\delta$ is a linear homomorphism from $\mathfrak{U}$ to $\mathcal{A}_q^{\otimes 2}$, where $\mathfrak{U}$ is $\widetilde{GL}_q'(2,\mathbb{R})$ or $\widetilde{GL}_q''(2,\mathbb{R})$. Therefore, applying $\mathcal{Q}\otimes\mathcal{Q}$ to (31), we infer that

$$(id\otimes\delta_{\mathcal{A}})\,\mathcal{Q}(g^{\pm}) = \mathcal{Q}(g^{\pm})_{12}\,\mathcal{Q}(g^{\pm})_{13}\,, \tag{88}$$

where $\mathcal{Q}(g^+) = \left(\begin{smallmatrix} k^{-1} & 0 \\ e & k \end{smallmatrix}\right)$ and $\mathcal{Q}(g^-) = \left(\begin{smallmatrix} k & f \\ 0 & 0 \end{smallmatrix}\right)$. Further we can proceed exactly as in the proof of Proposition 4. □

Definition 6. *Let* $\mathcal{B}$ *be an algebra of linear operators acting on a Hilbert space* $\mathcal{K}$. *An irreducible representation* $\pi_{\mathcal{A}}\colon \mathcal{A}_q \to \mathcal{B}$ *is called* positive *if the following operators are* self–adjoint *and* strictly positive *on* $\mathcal{K}$*:*
i) $\pi_{\mathcal{A}}(x)$ *for* $x = e, f, k, C_q$*;*
ii) $q^{\frac{1}{2}}\pi_{\mathcal{A}}(e)\big(\pi_{\mathcal{A}}(k)\big)^{-1}$ *and* $q^{\frac{1}{2}}\big(\pi_{\mathcal{A}}(k)\big)^{-1}\pi_{\mathcal{A}}(f)$.

Proposition 8. *Let* $\mathcal{B}$ *and* $\mathcal{K}$ *be as in Definition 6 and let* $\pi_{\mathcal{A}}$ *be a positive representation of* $\mathcal{A}_q$. *Let* $L^{\mathcal{A}}(\lambda), \hat{L}^{\mathcal{A}}(\lambda) \in \mathrm{Mat}(2)\otimes\mathcal{B}$ *be as in (83). Then the operators* $\mathsf{R}^{\mathcal{A}}(\lambda), \hat{\mathsf{R}}^{\mathcal{A}}(\lambda) \in \mathcal{B}^{\otimes 2}$ *acting on* $\mathcal{K}\otimes\mathcal{K}$ *and defined by the formulae*

$$\mathsf{R}^{\mathcal{A}}(\lambda) = (k\otimes k)^{-\frac{\alpha}{2}\log\lambda}\left((e\otimes k + k\otimes f)(k\otimes e + f\otimes k)\right)^{\alpha\log\lambda}(k\otimes k)^{-\frac{\alpha}{2}\log\lambda}\,, \tag{89}$$

$$\hat{\mathsf{R}}^{\mathcal{A}}(\lambda) = \left(ek\otimes k^{-1}f + k^2\otimes ef - qk^2\otimes k^2\right)^{\alpha\log\lambda}\,, \tag{90}$$

where $\alpha \equiv \frac{1}{\log q}$, *satisfy the equations*

$$\mathsf{R}^{\mathcal{A}}_{23}(\lambda)\,L^{\mathcal{A}}_{12}(\lambda\mu)\,L^{\mathcal{A}}_{13}(\mu) = L^{\mathcal{A}}_{12}(\mu)\,L^{\mathcal{A}}_{13}(\lambda\mu)\,\mathsf{R}^{\mathcal{A}}_{23}(\lambda)\,, \tag{91}$$

$$\hat{\mathsf{R}}^{\mathcal{A}}_{23}(\lambda)\,\hat{L}^{\mathcal{A}}_{12}(\lambda\mu)\,\hat{L}^{\mathcal{A}}_{13}(\mu) = \hat{L}^{\mathcal{A}}_{12}(\mu)\,\hat{L}^{\mathcal{A}}_{13}(\lambda\mu)\,\hat{\mathsf{R}}^{\mathcal{A}}_{23}(\lambda)\,. \tag{92}$$

If the tensor product $\pi_{\mathcal{A}}\otimes\pi_{\mathcal{A}}$ *is multiplicity free, then (90) is the unique solution of (92) up to multiplication by a scalar factor.*

Proof. First, $\pi_{\mathfrak{o}} \equiv \pi_{\mathcal{A}}\otimes\mathcal{Q}\colon \mathfrak{U}\to\mathcal{B}$ is clearly a positive representation for $\mathfrak{U} = \widetilde{GL}_q'(2,\mathbb{R})$ (as well as for $\mathfrak{U} = \widetilde{GL}_q''(2,\mathbb{R})$). Next, it is obvious that $\mathsf{R}^{\mathcal{A}}(\lambda)$ solving (91) is a solution of Eqs. (38)–(43), or, equivalently, $\tilde{\mathsf{R}}^{\mathcal{A}}(\lambda) = (k\otimes k)^{\frac{\alpha}{2}\log\lambda}\,\mathsf{R}^{\mathcal{A}}(\lambda)\,(k\otimes k)^{\frac{\alpha}{2}\log\lambda}$ is a solution of Eqs. (44)–(47), where each term $x\otimes y$ is understood as $\pi_{\mathfrak{o}}(x)\otimes\pi_{\mathfrak{o}}(y)$ and Δ_π is replaced with $\Delta_{\pi_{\mathfrak{o}}} \equiv (\pi_{\mathcal{A}}\otimes\pi_{\mathcal{A}})\circ\Delta_{\mathcal{A}}$.

Notice that the definition of $\Delta_{\mathcal{A}}$ given in Proposition 6 is sufficient because $\Delta_\pi(\theta)$ does not enter Eqs. (44)–(47). Now, it is easy to see that the $\pi_\circ$ counterparts of Eqs. (48)-(49) hold. This, along with Proposition 6, implies that the $\pi_\circ$ counterpart of formula (50) holds as well. Whence we obtain formula (89) as the $\pi_\circ$ counterpart of formula (36). Finally, it is easy to see that the remaining verification of Eq. (51) in Theorem 1 is valid for the $\pi_\circ$ counterpart of $\mathsf{X}(\lambda)$.

Analogous consideration for the $\pi_\circ$ counterparts of Eqs. (60)–(67), where δ (which actually stands for δ_π) is replaced with $\delta_{\pi_\circ} \equiv (\pi_\circ \otimes \pi_\circ) \circ \delta$, is straightforward because, by Proposition 7, $\delta_{\pi_\circ}$ has the same algebra homomorphism properties as δ_π. For the same reason, the $\pi_\circ$ analogue of the uniqueness part of Theorem 2 is valid if we invoke Lemma 6 instead of Lemma 2. □

Fundamental R–operators (89) and (90) are regular and have the properties given in (54). The corresponding local Hamiltonian densities constructed via (4) are $\mathcal{Q}$–images of those in (55) and (69), namely

$$H^{\mathcal{A}}_{n,n+1} = \log\big((e_{n+1}k_n + k_{n+1}f_n)(k_{n+1}e_n + f_{n+1}k_n)\big) - \log(k_n k_{n+1})\,, \tag{93}$$

$$\hat{H}^{\mathcal{A}}_{n,n+1} = \log\big(e_{n+1}\,k_{n+1}k_n^{-1}f_n\; + k_{n+1}^2\,(C_q)_n\big)\,. \tag{94}$$

As before, the arguments of the logarithms here are positive self–adjoint operators.

4.3 q–DST model

The discrete self–trapping model, which describes a chain of N coupled anharmonic oscillators, is know to be integrable [E1, KSS]. The corresponding L–matrix satisfies a counterpart of Eq. (1) with additive spectral parameter and the rational auxiliary R–matrix, which is obtained from (9) in the limit $q \to 1$. It was suggested in [PS, KP] that the following L–matrix

$$L_n^{\mathrm{qDST}}(\lambda) = \begin{pmatrix} \lambda\,k_n^{-1} + \lambda^{-1}k_n & f_n \\ e_n & \lambda\,k_n \end{pmatrix}, \tag{95}$$

where each triple $(e_n,\ f_n,\ k_n)$ satisfies relations (78) and operators assigned to different sites commute, can be regarded as an L–matrix associated with a q–deformed discrete self–trapping (q–DST) model. The expansion of the corresponding auxiliary transfer–matrix about the point $\lambda = 0$ yields

$$T(\lambda) = \lambda^{-\mathrm{N}}\,Q + \lambda^{2-\mathrm{N}}\,Q\cdot H^{\mathrm{qDST}} + \dots\,, \tag{96}$$

$$Q = \prod_{n=1}^{\mathrm{N}} k_n\,, \qquad H^{\mathrm{qDST}} = \sum_{n=1}^{\mathrm{N}} k_n^{-2} + k_n^{-1}e_n k_{n+1}^{-1}f_{n+1}\,. \tag{97}$$

Here $Q = e^{\gamma\,h}$ with h being the number of particles operator and H^{qDST} is a nearest–neighbour Hamiltonian for the q–DST model.

Let us remark that (95) is related to $\hat{L}^{\mathcal{A}}(\lambda)$ in (83) via a twist in either the auxiliary or in the quantum space:

$$L^{qDST}(\lambda) = \lambda^{\frac{1}{2}\sigma_3}\,\hat{L}^{\mathcal{A}}(\lambda)\,\lambda^{-\frac{1}{2}\sigma_3} = k^{\alpha\log\lambda}\,\hat{L}^{\mathcal{A}}(\lambda)\,k^{-\alpha\log\lambda}\,, \qquad \alpha = \tfrac{1}{\log q}\,. \tag{98}$$

The first equality here implies that $L^{qDST}(\lambda)$ satisfies Eq. (1) with the same auxiliary R–matrix (8). The second equality implies that the fundamental R–operator for $L^{qDST}(\lambda)$ is related to that for

$\hat{L}^{\mathcal{A}}(\lambda)$ as follows (notice that (90) commutes with $k \otimes k$)

$$\mathsf{R}^{\mathrm{qDST}}(\lambda) = (1 \otimes k)^{\alpha \log \lambda}\, \hat{\mathsf{R}}^{\mathcal{A}}(\lambda)\, (k \otimes 1)^{-\alpha \log \lambda}\,. \tag{99}$$

Using (90) and applying formula (4) to (99), we find a nearest–neighbour Hamiltonian different from (97) which corresponds to L–matrix (95):

$$\tilde{H}^{\mathrm{qDST}} = \tfrac{1}{\gamma} \sum_n \Big(\log\big(C_q\, k_{n+1}^2 + e_{n+1} k_{n+1} k_n^{-1} f_n\big) + \log(k_n k_{n+1}^{-1}) \Big)\,. \tag{100}$$

Notice that the term $\log(k_n k_{n+1}^{-1})$ does not contribute to the total Hamiltonian in the case of a periodic chain.

Remark 8. Substituting (68) into (99), we obtain (omitting a scalar factor):

$$\mathsf{R}^{\mathrm{qDST}}(\lambda) = \big(k \otimes k\big)^{\alpha \log \lambda}\, \frac{S_\omega\big(\lambda^{-1}\mathsf{r}\big)}{S_\omega(\lambda\,\mathsf{r})}\,, \qquad \mathsf{r} = \big(C_q\big)^{-1}\, k^{-1} e \otimes k^{-1} f\,. \tag{101}$$

An analogous formula was proposed in [KP] in the case of $|q| < 1$ in terms of the compact quantum dilogarithm $S(x)$.

4.4 Weyl algebra

For the factor algebras of $\widetilde{GL}_q'(2,\mathbb{R})$ and $\widetilde{GL}_q''(2,\mathbb{R})$ over the ideals generated by the relations $c = 0$ and $b = 0$, respectively, the only nontrivial defining relations are of the Weyl type. These factor algebras are isomorphic to the following algebra.

Definition 7. *The Weyl algebra $\mathcal{W}_q$ is a unital associative algebra with generators u, $\tilde{u}$, v, v^{-1} and defining relations $v\, v^{-1} = v^{-1} v = 1$ and*

$$u\,\tilde{u} = \tilde{u}\,u\,, \qquad u\,v = q\,v\,u\,, \qquad \tilde{u}\,v = q^{-1}\,v\,\tilde{u} \tag{102}$$

*and equipped with an anti–involution * defined on generators by*

$$u^* = u\,, \qquad \tilde{u}^* = \tilde{u}\,, \qquad v^* = v\,, \qquad (v^{-1})^* = v^*\,. \tag{103}$$

The following statements are straightforward to verify.

Lemma 9. *For a generic q, the center of $\mathcal{W}_q$ is generated by the element $Z_q = u\,\tilde{u}$.*

Lemma 10. *The linear homomorphisms $\mathcal{Q}' : \widetilde{GL}_q'(2,\mathbb{R}) \to \mathcal{W}_q$ and $\mathcal{Q}'' : \widetilde{GL}_q''(2,\mathbb{R}) \to \mathcal{W}_q$ defined on generators as follows*

$$\mathcal{Q}'(a) = u\,, \quad \mathcal{Q}'(b) = v\,, \quad \mathcal{Q}'(c) = 0\,, \quad \mathcal{Q}'(\theta) = v^{-1}\,, \quad \mathcal{Q}'(d) = \tilde{u}\,, \tag{104}$$

$$\mathcal{Q}''(a) = u\,, \quad \mathcal{Q}''(b) = 0\,, \quad \mathcal{Q}''(c) = v\,, \quad \mathcal{Q}''(\theta) = v^{-1}\,, \quad \mathcal{Q}''(d) = \tilde{u} \tag{105}$$

are algebra homomorphisms.

Now we will introduce contractions of the maps Δ and δ suitable for $\mathcal{W}_q$.

Definition 8. *Let $\mathcal{B}$ be an algebra of linear operators acting on a Hilbert space $\mathcal{K}$. An irreducible representation $\pi_{\mathcal{W}}\colon \mathcal{W}_q \to \mathcal{B}$ is called* positive *if the following operators are* self–adjoint *and* strictly positive *on $\mathcal{K}$:*
i) $\pi_{\mathcal{W}}(x)$ *for* $x = u, \tilde{u}, v$*;*
ii) $q^{\frac{1}{2}}\pi_{\mathcal{W}}(u)\big(\pi_{\mathcal{W}}(v)\big)^{-1}$ *and* $q^{\frac{1}{2}}\big(\pi_{\mathcal{W}}(v)\big)^{-1}\pi_{\mathcal{W}}(\tilde{u})$.

Proposition 9. *Let $\mathcal{B}$ and $\mathcal{K}$ be as in Definition 8 and let $\pi_{\mathcal{W}}$ be a positive representation of $\mathcal{W}_q$. The linear homomorphism $\Delta_{\mathcal{W}} : \mathcal{W}_q \to \mathcal{B}^{\otimes 2}$ defined on generators as follows:*

$$\begin{aligned} &\Delta_{\mathcal{W}}(u) = \pi_{\mathcal{W}}(u)\otimes\pi_{\mathcal{W}}(u)\,, \qquad \Delta_{\mathcal{W}}(\tilde{u}) = \pi_{\mathcal{W}}(\tilde{u})\otimes\pi_{\mathcal{W}}(\tilde{u})\,, \\ &\Delta_{\mathcal{W}}(v) = \pi_{\mathcal{W}}(u)\otimes\pi_{\mathcal{W}}(v) + \pi_{\mathcal{W}}(v)\otimes\pi_{\mathcal{W}}(\tilde{u})\,, \qquad \Delta_{\mathcal{W}}(v^{-1}) = \big(\Delta_{\mathcal{W}}(v)\big)^{-1} \end{aligned} \tag{106}$$

is an algebra homomorphism *and a* *–homomorphism *w.r.t. the anti–involution (103).*

Proof. Notice that $\Delta_{\mathcal{W}} \circ \mathcal{Q}' = \big((\pi_{\mathcal{W}} \circ \mathcal{Q}') \otimes (\pi_{\mathcal{W}} \circ \mathcal{Q}')\big) \circ \Delta$ is, by Lemma 10, a linear homomorphism from $\widetilde{GL}'_q(2,\mathbb{R})$ to $\mathcal{B}$. Therefore, for $x = u, \tilde{u}, v$, the claimed properties of $\Delta_{\mathcal{W}}$ are inherited from those of Δ. For $\Delta_{\mathcal{W}}(v^{-1})$, a consideration analogous to that in the proof of Proposition 3 applies since, by Remark 2, $\Delta_{\mathcal{W}}(v)$ is a positive self–adjoint and hence invertible operator. □

Remark 9. We used in this proof that $\Delta_{\mathcal{W}}$ is related to Δ via $\mathcal{Q}'$. The *opposite* comultiplication $\Delta'_{\mathcal{W}}$ (obtained by exchanging the tensor factors in $\Delta_{\mathcal{W}}(x)$) is similarly related to Δ via $\mathcal{Q}''$, namely, $\Delta'_{\mathcal{W}} \circ \mathcal{Q}'' = \big((\pi_{\mathcal{W}} \circ \mathcal{Q}'') \otimes (\pi_{\mathcal{W}} \circ \mathcal{Q}'')\big) \circ \Delta$.

Remark 10. Using the relation between Δ and $\Delta_{\mathcal{W}}$, we can write an explicit expression for $\Delta_{\mathcal{W}}(v^{-1})$. Namely, applying $\mathcal{Q}'$ to (29), we obtain

$$\Delta_{\mathcal{W}}(v^{-1}) = \big(\pi_{\mathcal{W}}\otimes\pi_{\mathcal{W}}\big)\left(S_\omega(\mathsf{w})\,\big(u^{-1}\otimes v^{-1}\big)\,\big(S_\omega(\mathsf{w})\big)^{-1}\right), \tag{107}$$

where $\mathsf{w} = vu^{-1}\otimes v^{-1}\tilde{u}$ and $\pi_{\mathcal{W}}(u^{-1}) = \pi_{\mathcal{W}}(\tilde{u})\big(\pi_{\mathcal{W}}(Z_q)\big)^{-1}$.

Proposition 10. *The linear homomorphism $\delta_{\mathcal{W}}\colon \mathcal{W}_q \to \mathcal{W}_q^{\otimes 2}$ defined on generators as follows:*

$$\delta_{\mathcal{W}}(u) = u\otimes v^{-1}\,, \qquad \delta_{\mathcal{W}}(\tilde{u}) = v\otimes\tilde{u}\,, \qquad \delta_{\mathcal{W}}(v^{\pm 1}) = v^{\pm 1}\otimes v^{\pm 1} \tag{108}$$

is a coassociative algebra homomorphism *and a* *–homomorphism *w.r.t. the anti–involution (103).*

Proof. Straightforward. However, it is instructive to notice that $\delta_{\mathcal{W}} \circ \mathcal{Q}'' = (\mathcal{Q}''\otimes\mathcal{Q}'') \circ \delta$ is a linear homomorphism from $\widetilde{GL}''_q(2,\mathbb{R})$ to $\mathcal{W}_q^{\otimes 2}$. Therefore, applying $\mathcal{Q}''\otimes\mathcal{Q}''$ to (31), we infer that

$$(id\otimes\delta_{\mathcal{W}})\,\mathcal{Q}''(g^{\pm}) = \mathcal{Q}''(g^{\pm})_{1\,2}\,\mathcal{Q}''(g^{\pm})_{1\,3}\,, \tag{109}$$

where $\mathcal{Q}''(g^+) = \left(\begin{smallmatrix} v^{-1} & 0 \\ u & 0 \end{smallmatrix}\right)$ and $\mathcal{Q}''(g^-) = \left(\begin{smallmatrix} v & \tilde{u} \\ 0 & 0 \end{smallmatrix}\right)$. □

It is easy to check that any monomial in $\mathcal{W}_q^{\otimes 2}$ which commutes with $\delta_{\mathcal{W}}(x)$, $x = u, \tilde{u}, v$ is a power of $\delta_{\mathcal{W}}(Z_q)$. But the centralizer of $\Delta_{\mathcal{W}}\big(\mathcal{W}_q\big)$ contains not only functions of $\Delta_{\mathcal{W}}(Z_q)$.

Lemma 11. *Denote $\mathsf{z} = uv\otimes uv^{-1}$. Then for all $x \in \mathcal{W}_q$ we have*

$$[(\pi_{\mathcal{W}}\otimes\pi_{\mathcal{W}})(\mathsf{z}), \Delta_{\mathcal{W}}(x)] = 0\,. \tag{110}$$

Proof. It suffices to verify (110) for the generators $u, \tilde{u}, v$, which is straightforward. □

4.5 Fundamental R–operators for $g'(\lambda)$ and $\check{g}'(\lambda)$, Volterra and lattice free field models

Let us introduce the following $\mathcal{Q}'$–images of $g(\lambda)$ and $\check{g}(\lambda) = \sigma_1 g(\lambda)$:

$$g'(\lambda) = \begin{pmatrix} u & \lambda\, v \\ \lambda\, v^{-1} & \tilde{u} \end{pmatrix}, \qquad \check{g}'(\lambda) = \begin{pmatrix} \lambda\, v^{-1} & \tilde{u} \\ u & \lambda\, v \end{pmatrix}. \tag{111}$$

Matrix $g'(\lambda)$ is the L–matrix for the Volterra model [V2] and is also related to the lattice sine–Gordon model [V2, F1, F2]. We will see below that $\check{g}'(\lambda)$ is the L–matrix for the Volterra model for a dual dynamical variable (we use $\check{g}'(\lambda)$ rather than $\mathcal{Q}'(\hat{g}(\lambda))$ to make the duality most transparent; the corresponding fundamental R–operators differ only by a twist). In the compact case, a fundamental R–operator for $g'(\lambda)$ was found in [V2]. Here we will give an alternative derivation, which exhibits transparently the underlying comultiplication structure. For brevity of notations, we will write $x \otimes y$ instead of $\pi_{\mathcal{W}}(x) \otimes \pi_{\mathcal{W}}(y)$.

Theorem 3. *Let $\mathcal{B}$ and $\mathcal{K}$ be as in Definition 8 and let $\pi_{\mathcal{W}}$ be a positive representation of $\mathcal{W}_q$. Let $g'(\lambda), \check{g}'(\lambda) \in \mathrm{Mat}(2) \otimes \mathcal{B}$ be as in (111). Then the operators $\mathsf{R}'(\lambda), \check{\mathsf{R}}'(\lambda) \in \mathcal{B}^{\otimes 2}$ acting on $\mathcal{K} \otimes \mathcal{K}$ and defined by the formulae*

$$\mathsf{R}'(\lambda) = r(\mathsf{z}, \lambda)\, \tilde{\mathsf{z}}^{\frac{\alpha}{4}\log\lambda} \left(u \otimes v + v \otimes \tilde{u}\right)^{\alpha \log\lambda} \tilde{\mathsf{z}}^{\frac{\alpha}{4}\log\lambda}, \tag{112}$$

$$\check{\mathsf{R}}'(\lambda) = r(\tilde{\mathsf{z}}, \lambda)\, \mathsf{z}^{\frac{\alpha}{4}\log\lambda} \left(\tilde{u} \otimes v + v^{-1} \otimes \tilde{u}\right)^{\alpha \log\lambda} \mathsf{z}^{\frac{\alpha}{4}\log\lambda}, \tag{113}$$

where $\mathsf{z} = uv \otimes uv^{-1}$, $\tilde{\mathsf{z}} = \tilde{u}v^{-1} \otimes uv^{-1}$, $\alpha \equiv \frac{1}{\log q}$, satisfy the equations

$$\mathsf{R}'_{23}(\lambda)\, g'_{12}(\lambda\mu)\, g'_{13}(\mu) = g'_{12}(\mu)\, g'_{13}(\lambda\mu)\, \mathsf{R}'_{23}(\lambda), \tag{114}$$

$$\check{\mathsf{R}}'_{23}(\lambda)\, \check{g}'_{12}(\lambda\mu)\, \check{g}'_{13}(\mu) = \check{g}'_{12}(\mu)\, \check{g}'_{13}(\lambda\mu)\, \check{\mathsf{R}}'_{23}(\lambda), \tag{115}$$

for any choice of the function $r(t, \lambda)$.

Proof. Eq. (114) can be regarded as a $\mathcal{Q}' \otimes \mathcal{Q}'$–image of (37). It is easy to see, that Eqs. (38)–(43) turn into the following relations:

$$[\mathsf{R}'(\lambda), v \otimes v^{-1}] = [\mathsf{R}'(\lambda), v^{-1} \otimes v] = [\mathsf{R}'(\lambda), u \otimes u] = [\mathsf{R}'(\lambda), \tilde{u} \otimes \tilde{u}] = 0, \tag{116}$$

$$\mathsf{R}'(\lambda)\, (u \otimes v + \lambda\, v \otimes \tilde{u}) = (\lambda\, u \otimes v + v \otimes \tilde{u})\, \mathsf{R}'(\lambda), \tag{117}$$

$$\mathsf{R}'(\lambda)\, (\lambda\, v^{-1} \otimes u + \tilde{u} \otimes v^{-1}) = (v^{-1} \otimes u + \lambda\, \tilde{u} \otimes v^{-1})\, \mathsf{R}'(\lambda). \tag{118}$$

To exhibit maximally the structure of these equations related to the comultiplication $\Delta_{\mathcal{W}}$, we introduce $\tilde{\mathsf{R}}'(\lambda) = \tilde{\mathsf{z}}^{-\frac{\alpha}{4}\log\lambda}\, \mathsf{R}'(\lambda)\, \tilde{\mathsf{z}}^{-\frac{\alpha}{4}\log\lambda}$. Then equations (116)–(118) acquire the following form:

$$\tilde{\mathsf{R}}'(\lambda)\, (v \otimes v^{-1}) = \lambda\, (v \otimes v^{-1})\, \tilde{\mathsf{R}}'(\lambda), \qquad \tilde{\mathsf{R}}'(\lambda)\, \Delta_{\mathcal{W}}(v) = \Delta_{\mathcal{W}}(v)\, \tilde{\mathsf{R}}'(\lambda), \tag{119}$$

$$\tilde{\mathsf{R}}'(\lambda)\, \Delta_{\mathcal{W}}(u) = \lambda^{-1} \Delta_{\mathcal{W}}(u)\, \tilde{\mathsf{R}}'(\lambda), \qquad \tilde{\mathsf{R}}'(\lambda)\, \Delta_{\mathcal{W}}(\tilde{u}) = \lambda\, \Delta_{\mathcal{W}}(\tilde{u})\, \tilde{\mathsf{R}}'(\lambda). \tag{120}$$

It is now easy to see that (119)–(120) are solved by

$$\tilde{\mathsf{R}}'(\lambda) = r(\mathsf{z}, \lambda)\, \left(\Delta_{\mathcal{W}}(v)\right)^{\alpha \log\lambda}, \tag{121}$$

where $r(t,\lambda)$ can be an arbitrary function thanks to Lemma 11 and the fact that $[\mathsf{z}, v\otimes v^{-1}]=0$. Thus, we established that (112) satisfies (116)–(117). Taking into account that $[\mathsf{z}, v^{-1}\otimes u] = [\mathsf{z}, \tilde{u}\otimes v^{-1}]=0$, it remains to prove that $\mathsf{R}'_{\circ}(\mathsf{w},\lambda) = \tilde{\mathsf{z}}^{\frac{\alpha}{4}\log\lambda}\big(\Delta_{\mathcal{W}}(v)\big)^{\alpha\log\lambda}\tilde{\mathsf{z}}^{\frac{\alpha}{4}\log\lambda}$ satisfies (118). For this purpose we apply Lemma 12 and rewrite it as follows:

$$\mathsf{R}'_{\circ}(\mathsf{w},\lambda) = \big(Z_q\big)^{\alpha\log\lambda}\,\frac{S_\omega\big(\lambda^{-1}\mathsf{w}\big)}{S_\omega\big(\lambda\,\mathsf{w}\big)}\,\mathsf{w}^{-\frac{\alpha}{2}\log\lambda}\,,\qquad \mathsf{w}=vu^{-1}\otimes v^{-1}\tilde{u}\,, \tag{122}$$

where we used that $\mathsf{w}\,\tilde{\mathsf{z}} = q^4\tilde{\mathsf{z}}\,\mathsf{w}$. Multiplying (118) with $v\otimes\tilde{u}$ from the right, we obtain the following functional equation on function $\mathsf{R}'_{\circ}(w,\lambda)$:

$$\mathsf{R}'_{\circ}(w,\lambda)\,(\lambda+q^{-1}w) = (1+\lambda\,q^{-1}w)\,\mathsf{R}'_{\circ}(q^{-2}w,\lambda)\,, \tag{123}$$

which is easy to verify using Eq. (159).

To prove that (113) satisfies (115), we observe that $g'(\lambda)$ and $\check{g}'(\lambda)$ are related in a way which fits the hypotheses of Lemma 4 (namely, $s=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and the automorphism ι is defined by $\iota(u)=\tilde{u}$, $\iota(\tilde{u})=u$, $\iota(v^{\pm1})=v^{\mp1}$). Therefore, according to Eq. (35), the fundamental R–operator for $\check{g}'(\lambda)$ is $\check{\mathsf{R}}'(\lambda)=(\iota\otimes id)\mathsf{R}'(\lambda)$. Noticing that $\iota(\tilde{\mathsf{z}})=\mathsf{z}$ and $\iota(\mathsf{z})=\tilde{\mathsf{z}}$, we obtain formula (113). □

Remark 11. In [V2, F2, FV2], another solution to Eq. (123) was given, namely

$$\tilde{\mathsf{R}}'_{\circ}(\mathsf{w},\lambda) = \frac{S_\omega(\mathsf{w})\,S_\omega(\mathsf{w}^{-1})}{S_\omega(\lambda\,\mathsf{w})\,S_\omega(\lambda\,\mathsf{w}^{-1})}\,. \tag{124}$$

Eq. (166) in Appendix A.1 shows that (122) and (124) coincide up to a factor independent of w.

Fundamental R–operators (112) and (113) are regular the sense of Eq. (3) if $r(t,1)=1$. Furthermore, they have the properties given in (54) provided that $\bar{r}(t,\lambda)=r(t,\lambda^{-1})=1/r(t,\lambda)$ for $t,\lambda>0$ (notice that $\mathsf{z}^*=\mathsf{z}$, $\tilde{\mathsf{z}}^*=\tilde{\mathsf{z}}$, and $[\mathsf{z},\tilde{\mathsf{z}}]=0$). The corresponding local Hamiltonian densities constructed via (4) are given by ($r'(t)$ stands for the derivative of $r(t,\lambda)$ w.r.t. λ at $\lambda=0$)

$$\gamma\,H'_{n,n+1} = \log\big(v_nu_{n+1}+\tilde{u}_nv_{n+1}\big) - \tfrac{1}{2}\log(v_nv_{n+1}) + r'(\mathsf{z}_{n+1,n}) + \tfrac{1}{2}\log(u_n\tilde{u}_{n+1})\,, \tag{125}$$

$$\gamma\,\check{H}'_{n,n+1} = \log\big(v_n\tilde{u}_{n+1}+\tilde{u}_nv^{-1}_{n+1}\big) + \tfrac{1}{2}\log(u_nu_{n+1}) + r'(\tilde{\mathsf{z}}_{n+1,n}) + \tfrac{1}{2}\log(v_n^{-1}v_{n+1})\,. \tag{126}$$

The arguments of the logarithms here are positive self–adjoint operators. Notice that the last terms in (125) and (126) add only a constant to the total Hamiltonian in the case of a periodic chain.

Consider the following positive representations of $\mathcal{W}_q$ on the Hilbert space $\mathcal{K}=L^2(\mathbb{R})$

$$\pi_+(u)=e^{p}\,,\qquad \pi_+(\tilde{u})=e^{-p}\,,\qquad \pi_+(v)=e^{-2\phi}\,, \tag{127}$$

$$\pi_-(u)=e^{-2\phi}\,,\qquad \pi_-(\tilde{u})=e^{2\phi}\,,\qquad \pi_-(v)=e^{-p}\,, \tag{128}$$

were p and ϕ are self–adjoint operators which satisfy $[p,\phi]=\frac{\gamma}{2i}$, $\gamma\in(0,\pi)$. For these representations, the classical limit of (125)–(126) acquires the following form (up to additive constants):

$$\gamma\,\pi_+(H'_{n,n+1}) = \log\cosh s_+ + r'(e^{2s_-})\,, \tag{129}$$

$$\gamma\,\pi_-(\check{H}'_{n,n+1}) = \log\cosh s_- + r'(e^{2s_+})\,. \tag{130}$$

where $s_\pm\equiv\frac{1}{2}p_n+\frac{1}{2}p_{n+1}\pm\phi_{n+1}\mp\phi_n$. It was shown in [V2] that $s_\pm$ are related (in a nonultralocal way, via a discretized Miura transformation) to the dual dynamical variables of the Volterra

model, and that (129) for $r(t,\lambda)=0$ coincides with the Hamiltonian of the Volterra model for s_+. The obvious symmetry between (129) and (130) makes it clear that the matrix $\check{g}'(\lambda)$ can as well be taken as an L–matrix for the Volterra model and that the corresponding Hamiltonian (130) for $r(t,\lambda)=0$ is the Hamiltonian of the Volterra model for the dual dynamical variable s_-.

Let us demonstrate that $g'(\lambda)$ can be regarded also as an L–matrix for a lattice regularization of the free field. For this goal we have to choose such $r(t,\lambda)$ in (112) that $r'(e^{2t}) = \log\cosh t + \mathrm{const}$ in the classical limit. For instance, we can take (cf. (122) and notice that $[\mathsf{z},\mathsf{w}]=0$ and $\mathsf{z}_{n,n+1}\mathsf{w}_{n+1,n}=Z_q$)

$$\mathsf{R}'(\lambda) = \frac{S_\omega\big(\lambda^{-1}Z_q\,\mathsf{z}^{-1}\big)}{S_\omega\big(\lambda\,Z_q\,\mathsf{z}^{-1}\big)}\,\big(\mathsf{z}\,\mathsf{w}^{-1}\big)^{\frac{\alpha}{2}\log\lambda}\,\frac{S_\omega\big(\lambda^{-1}\mathsf{w}\big)}{S_\omega\big(\lambda\,\mathsf{w}\big)}\,. \tag{131}$$

Then (129) acquires the following form:

$$\gamma\,\pi_+(H'_{n,n+1}) = \log\cosh s_+ + \log\cosh s_-\,. \tag{132}$$

In the continuum limit (72), we have $s_\pm = \kappa\big(p(x)\pm\partial_x\phi(x)\big)+o(\kappa)$ (κ stands for the lattice spacing) and (132) turns into $H'_{n,n+1} = \mathrm{const} + \frac{\kappa^2}{\gamma}\big(p^2+(\partial_x\phi)^2\big)+o(\kappa^2)$, i.e., it becomes the Hamiltonian density of the free field. Furthermore, assining a copy of $L^{\mathrm{f}}(\lambda)=\pi_+\big(g'(\kappa\lambda)\big)$ to each site of the lattice, we get the following continuum limit of this L–matrix: $L^{\mathrm{f}}_n(\lambda) = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right) + \kappa\,\big(U_+(\lambda)+U_-(\lambda)\big)+O(\kappa^2)$, where

$$U_+(\lambda) = \begin{pmatrix} \frac{1}{2}\,\partial_+\phi & \lambda\,e^{-2\phi} \\ \lambda\,e^{2\phi} & -\frac{1}{2}\,\partial_+\phi \end{pmatrix}, \quad U_-(\lambda) = \begin{pmatrix} \frac{1}{2}\,\partial_-\phi & 0 \\ 0 & -\frac{1}{2}\,\partial_-\phi \end{pmatrix}. \tag{133}$$

These matrices satisfy the zero curvature equation, $\partial_- U_+(\lambda)+\partial_+U_-(\lambda) = 2[U_+(\lambda),U_-(\lambda)]$, provided that ϕ satisfies the equation of motion of the free field: $\Box\,\phi=0$.

4.6 Fundamental R–operator for $g''(\lambda)$, lattice free field model

Let us introduce the following $\mathcal{Q}''$–image of $g(\lambda)$:

$$g''(\lambda) = \begin{pmatrix} u & 0 \\ \lambda\,v^{-1}+\lambda^{-1}v & \tilde{u} \end{pmatrix}. \tag{134}$$

Theorem 4. *Let $\mathcal{B}$ and $\mathcal{K}$ be as in Definition 8 and let $\pi_{\mathcal{W}}$ be a positive representation of $\mathcal{W}_q$. Let $g''(\lambda)\in\mathrm{Mat}(2)\otimes\mathcal{B}$ be as in (134). Then the operator $\mathsf{R}''(\lambda)\in\mathcal{B}^{\otimes 2}$ acting on $\mathcal{K}\otimes\mathcal{K}$ and defined by the formula*

$$\mathsf{R}''(\lambda) = \hat{\mathsf{z}}^{\frac{\alpha}{2}\log\lambda}\,\Big(\big(v\otimes u+\tilde{u}\otimes v\big)\,\big(u\otimes v+v\otimes\tilde{u}\big)\Big)^{2\alpha\log\lambda}\,\hat{\mathsf{z}}^{\frac{\alpha}{2}\log\lambda}\,, \tag{135}$$

where $\hat{\mathsf{z}}=\tilde{u}v^{-2}\otimes uv^{-2}$ and $\alpha\equiv\frac{1}{\log q}$, satisfies the equation

$$\mathsf{R}''_{2_3}(\lambda)\,g''_{1_2}(\lambda\mu)\,g''_{1_3}(\mu) = g''_{1_2}(\mu)\,g''_{1_3}(\lambda\mu)\,\mathsf{R}''_{2_3}(\lambda)\,. \tag{136}$$

Proof. As always, for brevity of notations, we write $x \otimes y$ instead of $\pi_{\mathcal{W}}(x) \otimes \pi_{\mathcal{W}}(y)$. Eq. (136) can be regarded as a $\mathcal{Q}'' \otimes \mathcal{Q}''$–image of (37). It is easy to see, that Eqs. (38)–(43) turn into the following relations:

$$\mathsf{R}''(\lambda)\,(v \otimes u + \lambda\,\tilde{u} \otimes v) = (\lambda\,v \otimes u + \tilde{u} \otimes v)\,\mathsf{R}''(\lambda)\,, \tag{137}$$

$$[\mathsf{R}''(\lambda), u \otimes u] = 0\,, \qquad [\mathsf{R}''(\lambda), \tilde{u} \otimes \tilde{u}] = 0\,, \tag{138}$$

$$\mathsf{R}''(\lambda)\,(\lambda\,v^{-1} \otimes u + \tilde{u} \otimes v^{-1}) = (v^{-1} \otimes u + \lambda\,\tilde{u} \otimes v^{-1})\,\mathsf{R}''(\lambda)\,. \tag{139}$$

It is easy to recognize in (137)–(138) a structure related to the opposite comultiplication $\Delta'_{\mathcal{W}}$ in accordance with Remark 9. To make this structure more transparent, we introduce

$$\tilde{\mathsf{R}}''(\lambda) = (v \otimes v)^{\frac{\alpha}{2}\log\lambda}\,\mathsf{R}''(\lambda)\,(v \otimes v)^{\frac{\alpha}{2}\log\lambda}\,. \tag{140}$$

Then equations (137)–(139) acquire the following form:

$$\tilde{\mathsf{R}}''(\lambda)\,\Delta'_{\mathcal{W}}(v) = \Delta'_{\mathcal{W}}(v)\,\tilde{\mathsf{R}}''(\lambda)\,, \tag{141}$$

$$\tilde{\mathsf{R}}''(\lambda)\,\Delta'_{\mathcal{W}}(u) = \lambda^{-2}\Delta'_{\mathcal{W}}(u)\,\tilde{\mathsf{R}}''(\lambda)\,, \qquad \tilde{\mathsf{R}}''(\lambda)\,\Delta'_{\mathcal{W}}(\tilde{u}) = \lambda^{2}\,\Delta'_{\mathcal{W}}(\tilde{u})\,\tilde{\mathsf{R}}''(\lambda)\,, \tag{142}$$

$$\tilde{\mathsf{R}}''(\lambda)\,(\lambda\,v^{-1} \otimes u + \lambda^{-1}\tilde{u} \otimes v^{-1}) = (\lambda^{-1}v^{-1} \otimes u + \lambda\,\tilde{u} \otimes v^{-1})\,\tilde{\mathsf{R}}''(\lambda)\,. \tag{143}$$

According to Lemma 11, a solution to Eqs. (141)–(142) may contain as a factor an arbitrary function of $\check{\mathsf{z}} = uv^{-1} \otimes uv$. Actually, it is more convenient to introduce $\mathsf{w} \equiv Z_q\,\check{\mathsf{z}}^{-1} = vu^{-1} \otimes v^{-1}\tilde{u}$. Then (141)–(142) are solved by

$$\tilde{\mathsf{R}}''(\lambda) = \big(\Delta'_{\mathcal{W}}(v)\big)^{2\alpha\log\lambda}\,\check{\mathsf{R}}''(\mathsf{w},\lambda)\,, \tag{144}$$

where $\check{\mathsf{R}}''(t,\lambda)$ is yet undetermined function. Noticing that $[\Delta'_{\mathcal{W}}(v), v^{-1} \otimes u] = [\Delta'_{\mathcal{W}}(v), \tilde{u} \otimes v^{-1}] = 0$, we infer that $\check{\mathsf{R}}''(\mathsf{w},\lambda)$ must solve (143). Multiplying (143) with $u \otimes v$ from the right, we obtain the following functional equation on function $\check{\mathsf{R}}''(w,\lambda)$:

$$\check{\mathsf{R}}''(w,\lambda)\,(\lambda\,qw^{-1} + \lambda^{-1}) = (\lambda^{-1}qw^{-1} + \lambda)\,\check{\mathsf{R}}''(q^{-2}w,\lambda)\,. \tag{145}$$

Comparing this equation with (121)–(123), we conclude that

$$\check{\mathsf{R}}''(\mathsf{w},\lambda) = \big(Z_q\big)^{2\alpha\log\lambda}\,\frac{S_\omega\big(\lambda^{-2}\mathsf{w}\big)}{S_\omega\big(\lambda^{2}\mathsf{w}\big)}\,\mathsf{w}^{-\alpha\log\lambda} = \tilde{\mathsf{z}}^{\frac{\alpha}{2}\log\lambda}\big(\Delta_{\mathcal{W}}(v)\big)^{2\alpha\log\lambda}\tilde{\mathsf{z}}^{\frac{\alpha}{2}\log\lambda}\,, \tag{146}$$

where $\tilde{\mathsf{z}} = \tilde{u}v^{-1} \otimes uv^{-1}$. Notice that $\Delta'_{\mathcal{W}}(v)$ commutes with $\Delta_{\mathcal{W}}(v)$ and $\tilde{\mathsf{z}}$. Combining (140), (144), and (146), we obtain formula (135). □

Fundamental R–operator (135) is regular and has the properties given in (54). The corresponding local Hamiltonian density constructed via (4) is given by

$$\gamma\,H''_{n,n+1} = 2\log\Big(\big(v_{n+1}u_n + \tilde{u}_{n+1}v_n\big)\,\big(u_{n+1}v_n + v_{n+1}\tilde{u}_n\big)\Big) + \log(\tilde{u}_{n+1}v_{n+1}^{-2}u_nv_n^{-2})\,. \tag{147}$$

Definition 8 along with Remark 2 ensure that the argument of the first logarithm here is a product of commuting positive self–adjoint operators. In the classical limit (147) can be written as follows:

$$\gamma\,H''^{,\mathrm{cl}}_{n,n+1} = 2\log\Big(u_nu_{n+1} + \tilde{u}_n\tilde{u}_{n+1} + Z_q\big(v_n^{-1}v_{n+1} + v_nv_{n+1}^{-1}\big)\Big) + \log(\tilde{u}_{n+1}u_n)\,. \tag{148}$$

Consider the following one–parameter family π_κ of positive representations of $\mathcal{W}_q$:

$$\pi_\kappa(u) = \tfrac{1}{\kappa}\, e^{\frac{\beta}{4}\,\Pi} \qquad \pi_\kappa(\tilde{u}) = \tfrac{1}{\kappa}\, e^{-\frac{\beta}{4}\,\Pi}\,, \qquad \pi_\kappa(v) = e^{-\frac{\beta}{2}\,\Phi}\,, \tag{149}$$

where κ, β, Π, and Φ are as in (70). Let us introduce the following L–matrix: $L^{\rm F}(\lambda) = \kappa\, \pi_\kappa\big(g''(\lambda)\big)$ and assign a copy of this matrix to each site of the lattice,

$$L^{\rm F}_n(\lambda) = \begin{pmatrix} e^{\frac{\beta}{4}\,\Pi_n} & 0 \\ \kappa\,\big(\lambda\, e^{\frac{\beta}{2}\Phi_n} + \lambda^{-1} e^{-\frac{\beta}{2}\Phi_n}\big) & e^{-\frac{\beta}{4}\,\Pi_n} \end{pmatrix}, \tag{150}$$

where $[\Pi_n, \Phi_m] = -i\delta_{nm}$. L–matrix (150) can be obtained from the Liouville L–matrix if in Eq. (71) we shift the zero mode of the field: $\Phi_n \to \Phi_n + \xi$, rescale the spectral parameter: $\lambda \to \lambda\, e^{-\xi\frac{\beta}{2}}$, and take the limit $\xi \to +\infty$.

One may expect that in such a limit the Liouville model turns into the free field. Indeed, for the representation (149), Eq. (148) acquires the following form:

$$\gamma\, H''^{,{\rm F},{\rm cl}}_{n,n+1} = 2\log\big(2\cosh\tfrac{\beta}{4}(\Pi_n + \Pi_{n+1}) + 2\cosh\tfrac{\beta}{2}(\Phi_{n+1} - \Phi_n)\big) + {\rm const}\,, \tag{151}$$

where we omitted the last term in (148) since it does not contribute to the total Hamiltonian in the case of a periodic chain. In the continuum limit (72), we recover from (151) the Hamiltonian density of the free field: $H''^{,{\rm F},{\rm cl}}_{n,n+1} = {\rm const} + \kappa^2\big(\Pi^2 + (\partial_x\Phi)^2\big) + o(\kappa^2)$. Furthermore, in the continuum limit we have $L^{\rm F}_n(\lambda) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \kappa\,\big(U_+(\lambda) + U_-(\lambda)\big) + O(\kappa^2)$, where

$$U_+(\lambda) = \begin{pmatrix} \frac{\beta}{8}\,\partial_+\Phi & 0 \\ \lambda\, e^{\frac{\beta}{2}\Phi_n} & -\frac{\beta}{8}\,\partial_+\Phi \end{pmatrix}, \quad U_-(\lambda) = \begin{pmatrix} \frac{\beta}{8}\,\partial_-\Phi & 0 \\ \lambda^{-1} e^{-\frac{\beta}{2}\Phi_n} & -\frac{\beta}{8}\,\partial_-\Phi \end{pmatrix}. \tag{152}$$

These matrices satisfy the zero curvature equation, $\partial_- U_+(\lambda) + \partial_+ U_-(\lambda) = 2[U_+(\lambda), U_-(\lambda)]$, provided that Φ satisfies the equation of motion of the free field: $\Box\,\Phi = 0$.

Remark 12. Let us remark that the two fundamental R–operators that we have found for the lattice free field are quite similar. Namely, it is straightfoward to check that

$$\mathsf{R}''(\lambda) = (u^{-1}\otimes u)^{\frac{\alpha}{2}\log\lambda}\; \mathsf{R}'(\lambda^2)\; (u^{-1}\otimes u)^{\frac{\alpha}{2}\log\lambda}\,, \tag{153}$$

where $\mathsf{R}''(\lambda)$ is given by (135) and $\mathsf{R}'(\lambda)$ is given by (131). Notice that, for a periodic chain, the factors $(u^{-1}\otimes u)^{\frac{\alpha}{2}\log\lambda}$ do not contribute to the total Hamiltonian.

4.7 Fundamental R–operator for $\hat{g}''(\lambda)$, relativistic Toda model

Let us introduce the following $\mathcal{Q}''$–image of $\hat{g}(\lambda)$:

$$\hat{g}''(\lambda) = \begin{pmatrix} \lambda\, v^{-1} + \lambda^{-1} v & \lambda^{-1}\tilde{u} \\ \lambda\, u & 0 \end{pmatrix}. \tag{154}$$

This matrix is related via a twist (cf. (98)) to the L–matrix of the relativistic Toda model [KT, PS]:

$$L^{\rm rT}(\lambda) = (\pi_-\otimes\pi_-)\big(\tfrac{1}{i} v^{\alpha\log\lambda}\,\hat{g}''(i\lambda)\, v^{-\alpha\log\lambda}\big) = \begin{pmatrix} \lambda\, e^{p} - \lambda^{-1} e^{-p} & -e^{2\phi} \\ e^{-2\phi} & 0 \end{pmatrix}, \tag{155}$$

where π_- is the positive representation (128) of $\mathcal{W}_q$. A suitable limit of (155) for $q \to 1$ yields the L–matrix of the ordinary Toda chain model, which satisfies a counterpart of Eq. (1) with additive spectral parameter and a rational auxiliary R–matrix.

Integrals of motion both for the ordinary and relativistic Toda models are constructed by means of expanding the auxiliary transfer–matrix $T(\lambda)$ (cf. Section 4.3). Results of the present article explain why the corresponding fundamental R–operators cannot be employed for this purpose.

Theorem 5. *Let $\mathcal{B}$ and $\mathcal{K}$ be as in Definition 8 and let $\pi_{\mathcal{W}}$ be a positive representation of $\mathcal{W}_q$. Let $\hat{g}''(\lambda) \in \mathrm{Mat}(2) \otimes \mathcal{B}$ be as in (154). Then the operator $\hat{\mathsf{R}}''(\lambda) \in \mathcal{B}^{\otimes 2}$ acting on $\mathcal{K} \otimes \mathcal{K}$ and defined by the formula*

$$\hat{\mathsf{R}}''(\lambda) = \big(\delta_{\mathcal{W}}(Z_q)\big)^{\alpha \log \lambda} = (u\, v \otimes v^{-1} \tilde{u})^{\alpha \log \lambda}\,, \tag{156}$$

where $\alpha \equiv \frac{1}{\log q}$, satisfies the equations

$$\hat{\mathsf{R}}''_{23}(\lambda)\, \hat{g}''_{12}(\lambda\mu)\, \hat{g}''_{13}(\mu) = \hat{g}''_{12}(\mu)\, \hat{g}''_{13}(\lambda\mu)\, \hat{\mathsf{R}}''_{23}(\lambda)\,. \tag{157}$$

Proof. Reexamining the proof of Theorem 2 in the case of $b = 0$, we see that an analogue of Eq. (60) holds in the form $[\hat{\mathsf{R}}''(\lambda), \delta_{\mathcal{W}}(x)]$, $x = u, \tilde{u}, v$. Unlike $\Delta_{\mathcal{W}}$, the centralizer of $\delta_{\mathcal{W}}(x)$, $x \in \mathcal{W}_q$ is generated only by $\delta_{\mathcal{W}}(Z_q)$. Therefore, $\hat{\mathsf{R}}''(\lambda)$ must be a function of $\delta_{\mathcal{W}}(Z_q)$. It is easy to see that that the $b = 0$ counterparts of Eqs. (61)–(63) determine this function uniquely (up to a scalar factor) and lead to formula (156). □

Fundamental R–operator (156) is regular and has the properties given in (54). However, the corresponding local Hamiltonian density constructed via (4),

$$\gamma\, H''_{n,n+1} = \log\big(v_n^{-1} \tilde{u}_n u_{n+1} v_{n+1}\big)\,, \tag{158}$$

leads to a trivial total Hamiltonian in the case of a periodic chain.

5 Conclusion

We have developed the Baxterization approach to the quantum group $GL_q(2)$ and emphasized the role of the standard and non–standard comultiplications for constructing the corresponding fundamental R–operators. Our results imply that the quantum symmetry algebra for a number of integrable lattice models is the quantum group $GL_q(2)$ or its reductions for which the comultiplication structure is a reduction of those for $GL_q(2)$. This is especially remarkable in the case of the lattice Liouville model because the quantum group $GL_q(2)$ itself emerged for the first time exactly in the study of relations for the monodromy matrix of the lattice Liouville model [FT1]. For the Volterra model, we have shown that the two dual L–matrices lead to the same Hamiltonian but for the dual dynamical variables. We have also emphasized the role of the ambiguity in the solution for the corresponding fundamental R–operators: fixing it in a trivial way yields the Hamiltonian of the Volterra model, whereas fixing it in a self–dual way yields the Hamiltonian of a lattice regularization of the free field. For the latter model we have also found another L–matrix which can be regarded as a limit of that for the lattice Liouville model. It is interesting that, although the

free field in continuum is a very simple model, the fundamental R–operators related to its lattice regularization have quite a nontrivial structure.

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A Appendix

A.1 Quantum dilogarithm

Consider the functional equation

$$S(q^{-1}x) = (1+x)\,S(q\,x)\,. \tag{159}$$

Its solution is given by the product $S(x) = \prod_{n=1}^{\infty}(1 + x\,q^{2n-1})$, which is convergent for $|q| < 1$. This function appears in various related forms in lattice integrable models [T1, V2, FV1] and was coined "quantum dilogarithm" in [FK1]. It was observed in [F2, F3] that, for $q = e^{i\pi\omega^2}$, $\omega \in (0,1)$, a well–defined solution to (159) is given by

$$S_\omega(x) = \prod_{n=1}^{\infty} \frac{(1 + x\,q^{2n-1})}{(1 + x^{\omega^{-2}}\,\hat{q}^{2n-1})} = \exp\left\{ \int\limits_{\Omega} \frac{dt}{4\,t}\, \frac{e^{\frac{t}{i\pi\omega}\,\log x}}{\sinh\omega t\,\sinh\frac{t}{\omega}} \right\}, \tag{160}$$

where $\hat{q} \equiv e^{-i\pi\omega^{-2}}$ and $\Omega = \mathbb{R} + i0$. Among the important properties of $S_\omega(x)$ are

$$\text{self–duality:} \qquad S_\omega(x^\omega) = S_{\omega^{-1}}(x^{\omega^{-1}})\,, \tag{161}$$

$$\text{unitarity:} \qquad \overline{S_\omega(x)}\,S_\omega(x) = 1 \quad \text{for } x \in \mathbb{R}_+\,. \tag{162}$$

This function is closely related to the Barnes double gamma function [B1] and plays an important role in studies of non–compact quantum groups [F4, PT, W1, S2, BT1, W2, V3] and related integrable models [KLS, FK2, T2, BT2].

The following lemma proves to be useful for converting powers of coproducts in formulae for fundamental R–operators into expressions involving quantum dilogarithms.

Lemma 12. *Let u and v be a pair of positive self–adjoint operators satisfying, in the sense of Remark 1, the Weyl relation: $u\,v = q^2\,v\,u$, where $q = e^{i\pi\omega^2}$, $\omega \in (0,1)$. Suppose that $\mathsf{w} \equiv q\,u^{-1}v$ is positive self–adjoint. Then the following identity holds:*

$$(u+v)^t = u^{\frac{t}{2}}\,\frac{S_\omega(q^{-t}\mathsf{w})}{S_\omega(q^t\,\mathsf{w})}\,u^{\frac{t}{2}} = v^{\frac{t}{2}}\,\frac{S_\omega(q^{-t}\mathsf{w}^{-1})}{S_\omega(q^t\,\mathsf{w}^{-1})}\,v^{\frac{t}{2}}\,. \tag{163}$$

Proof. Using relations $u\,f(\mathsf{w}) = f(q^2\mathsf{w})\,u$ and $v\,f(\mathsf{w}^{-1}) = f(q^{-2}\mathsf{w}^{-1})\,v$, it is easy to verify the following identities

$$u + v = S_\omega(\mathsf{w})\,u\,\big(S_\omega(\mathsf{w})\big)^{-1} = \big(S_\omega(\mathsf{w}^{-1})\big)^{-1}\,v\,S_\omega(\mathsf{w}^{-1})\,. \tag{164}$$

These are relations of unitary equivalence, thanks to the property (162). Whence we infer that

$$(u+v)^t = S_\omega(\mathsf{w})\, u^{\frac{t}{2}}\, u^{\frac{t}{2}}\, \big(S_\omega(\mathsf{w})\big)^{-1} = u^{\frac{t}{2}}\, \frac{S_\omega(q^{-t}\mathsf{w})}{S_\omega(q^t\mathsf{w})}\, u^{\frac{t}{2}}\,, \tag{165}$$

holds in the sense of Remark 1. The second equality in (163) can be derived analogously. □

Remark 13. Equality of the two expressions involving quantum dilogarithms in (163) allows us to obtain the following functional identities:

$$w^t = \frac{S_\omega(q^{-t}w)\, S_\omega(q^t w^{-1})}{S_\omega(q^t w)\, S_\omega(q^{-t}w^{-1})} \;=\; q^{t^2}\, \frac{S_\omega(q^{-2t}w)\, S_\omega(q^{2t}w^{-1})}{S_\omega(w)\, S_\omega(w^{-1})}\,. \tag{166}$$

For the proof of Theorem 2 we will need the following lemma (for the sake of brevity we will write x instead of $\pi_{\mathcal{A}}(x)$).

Lemma 13. *Let e, f, and k generate a positive representation $\pi_{\mathcal{A}}$ of the q–oscillator algebra $\mathcal{A}_q$ (cf. Definition 6). Then the following relation holds*

$$\frac{S_\omega(\lambda^{-1}f)}{S_\omega\big(\lambda\, f\big)}\, \big(\lambda\, k^2 + e\big) = \big(\lambda^{-1}k^2 + e\big)\, \frac{S_\omega(\lambda^{-1}f)}{S_\omega\big(\lambda\, f\big)}\,. \tag{167}$$

For a positive representation, we can write $e = u_e + v_e$, where $u_e = f^{-1}C_q$ and $v_e = q^{-1}f^{-1}k^2$ are positive self–adjoint operators satisfying relation $u_e v_e = q^2 v_e u_e$ (hence, by Remark 2, e is positive self–adjoint; a rigorous operator–theoretic consideration of the formula $e = f^{-1}(C_q + q^{-1}k^2)$ is given in [S1]). Therefore, if $G(t)$ is a sufficiently nice function (i.e. $G(f)$ has a suitable domain, cf. the discussion in [S1]), then we have $k^2 G(f) = G(q^2 f)k^2$ and $eG(f) = G(f)C_q f^{-1} + q^{-1}G(q^2 f)f^{-1}k^2$. Taking these relations into account, we infer that the operator equation

$$G(f,\lambda)\, \big(\lambda\, k^2 + e\big) = \big(\lambda^{-1}k^2 + e\big)\, G(f,\lambda) \tag{168}$$

is equivalent to the following functional one:

$$G(f,\lambda)\, \big(\lambda + q^{-1}f^{-1}\big) = \big(\lambda^{-1} + q^{-1}f^{-1}\big)\, G(q^2 f,\lambda)\,. \tag{169}$$

Using (159), it is straightforward to check that $G(f,\lambda) = \dfrac{S_\omega(\lambda^{-1}f)}{S_\omega\big(\lambda\, f\big)}$ solves (169). □

A.2 Proof of Lemma 5

Formula (68) can be rewritten as follows:

$$\hat{\mathsf{R}}(\lambda) = \big(D_q\big)^{\alpha \log \lambda}\, \big(b \otimes 1\big)^{\alpha \log \lambda}\, \check{\mathsf{R}}(\mathsf{r};\lambda)\, \big(c \otimes 1\big)^{\alpha \log \lambda}\,, \tag{170}$$

where

$$\check{\mathsf{R}}(\mathsf{r};\lambda) = \frac{S_\omega(\lambda^{-1}\mathsf{r})}{S_\omega(\lambda\, \mathsf{r})}\,, \qquad \mathsf{r} = \big(D_q\big)^{-1}\, \big(q^{-\frac{1}{2}}b^{-1}a\big) \otimes \big(q^{\frac{1}{2}}\theta d\big)\,. \tag{171}$$

Substituting (170) in (62), it is easy to check that Lemma 5 is equivalent to the assertion that $\check{\mathsf{R}}(r;\lambda)$ satisfies the following relation:

$$\check{\mathsf{R}}(\mathsf{r};\lambda)\, (\lambda\, \theta \otimes d + d \otimes b) = (\lambda^{-1}\theta \otimes d + d \otimes b)\, \check{\mathsf{R}}(\mathsf{r};\lambda)\,. \tag{172}$$

Notice that $d \otimes b$, $\theta \otimes d$, and r are positive self–adjoint operators. Now, a simple computation (using, in particular, the identity $q\, da - q^{-1} ad = (q - q^{-1}) D_q$) yields

$$(d \otimes b)\, \mathsf{r} - \mathsf{r}\, (d \otimes b) = (q - q^{-1})\, \theta \otimes d \tag{173}$$

$$(d \otimes b)\, (\theta \otimes d) = q^2\, (\theta \otimes d)\, (d \otimes b)\,, \qquad \mathsf{r}\, (\theta \otimes d) = q^{-2} (\theta \otimes d)\, \mathsf{r}\,. \tag{174}$$

Comparing these relations with (78) we see that $\hat{e} = d \otimes b$, $\hat{f} = \mathsf{r}$, and $\hat{k}^2 = \theta \otimes d$ generate a positive representation of the algebra $\mathcal{A}_q$ ($\hat{k}$ can be defined as the unique positive self–adjoint square root of $\theta \otimes d$). Invoking Lemma 13, we establish validity of Eq. (172) and hence of Lemma 5.

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