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On string solutions of Bethe equations in $\mathcal{N}=4$ supersymmetric Yang–Mills theory

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Abstract

The Bethe equations, arising in description of the spectrum of the dilatation operator for the $su(2)$ sector of the $\mathcal{N}=4$ supersymmetric Yang–Mills theory, are considered in the anti–ferromagnetic regime. These equations are deformation of those for the Heisenberg XXX magnet. It is proven that in the thermodynamic limit roots of the deformed equations group into strings. It is proven that the corresponding Yang's action is convex, which implies uniqueness of solution for centers of the strings. The state formed of strings of length $(2n+1)$ is considered and the density of their distribution is found. It is shown that the energy of such a state decreases as n grows. It is observed that non–analyticity of the left hand side of the Bethe equations leads to an additional contribution to the density and energy of strings of even length. Whence it is concluded that the structure of the anti–ferromagnetic vacuum is determined by the behaviour of exponential corrections to string solutions in the thermodynamic limit and possibly involves strings of length 2.

1 Introduction

Integrable models, in particular, spin chains, appear in several problems of high energy physics as effective models of interaction. For example, the Hamiltonian of the XXX spin chain with the non–compact representation of spin $s=-1$ arises in description of scattering of hadrons at high energies [1, 2], and also in description of mixing of composite operators under renormalization in QCD [3].

Mixing of composite operators under renormalization in the super–symmetric Yang–Mills theory also gives rise to an XXX–chain but with a compact representation. In this theory, one considers locally invariant operators of the form

$$\mathcal{O} = tr(Z^{J_1} W^{J_2} + \text{permutations}), \tag{1}$$

where Z and W are two complex scalar fields from the supermultiplet. The conformal dimensions Δ of this operators comprise the spectrum of the dilatation operator D. It is convenient to describe mixing of operators (1) under renormalization with the help of an analogy with the quantum spin chain of length $L = J_1 + J_2$, where each occurrence of Z is represented by a spin up, and each occurrence of W is is represented by a spin down. For example, the state $ZZZWWZWZ$ corresponds to the following spin chain $\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\downarrow\uparrow|\rangle$. An important observation made in [4] was that, in the $su(2)$–sector of the theory in the one–loop approximation (i.e., in the first order in $\lambda = g_{\rm YM}^2 N$, where $g_{\rm YM}$ is the Yang–Mills coupling constant, and N is the number of colours) the dilatation operator D can be expressed via the XXX Hamiltonian of spin $s=\frac{1}{2}$ for the described above chain:

$$D = \text{const} - \lambda\, H_{\rm XXX} + O(\lambda^2)\,. \tag{2}$$

Therefore, determining the spectrum of D in this approximation is reduced to investigating the Bethe equations for the Heisenberg magnet (see [5, 6, 7]),

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}\,, \tag{3}$$

where u_i are the rapidities of elementary excitations.

The high–loop corrections to (2) were found in articles that followed [4], and there it was shown that the corresponding expressions are also integrable Hamiltonians for the spin chain (those that include interaction between several nearest sites). In these approximations, the spectrum of D is determined not by equations (3) but by their "deformations" which explicitely contain the parameter λ in the left–hand side. Assuming that integrability of D takes place in all orders, Beisert, Dippel, and Staudacher [8] argued that the exact Bethe equations determining the spectrum of D look like following

$$\left(\frac{x(u_j + i/2)}{x(u_j - i/2)}\right)^L = \prod_{k \neq j} \frac{u_j - u_k + i}{u_j - u_k - i}\,, \tag{4}$$

where

$$x(u) = \frac{1}{2}\left(u + \sqrt{u^2 - \chi^2}\right)\,, \qquad \chi^2 \equiv \frac{\lambda}{4\pi^2}\,. \tag{5}$$

For the spin chain of length L, the spectrum of the Hamiltonian is bounded by the energies of the ferromagnetic and the antiferromagnetic vacua. Therefore, the spectrum of the operator D, i.e. the dimensions of operators (1), belongs to the interval

$$L \leqslant \Delta \leqslant \Delta_{\max}\,. \tag{6}$$

The lower bound here follows obviously from the structure of the antiferromagnetic vacuum (all spins up), whereas determining the upper bound requires a non–trivial evaluation of the energy of the antiferromagnetic vacuum. The value of $\Delta_{\max}$ in the thermodynamic limit was found in [9] and [10] by means of the standard technique of passing to the limit $L \to \infty$ in the Bethe equations. In this procedure it was assumed that, like for equations (3), the antiferromagnetic vacuum is formed of strings of length 1, i.e., real roots of equations (4). However, the example of the XXX spin chain of higher spin [11,12] shows that the antiferromagnetic vacuum for Bethe equations whose left–hand side differs from (3) can have other structure, for instance, it can be filled by strings of greater length.

The aim of the present work is to prove existence of solutions corresponding to strings of length greater than 1 for the equations (4) in the thermodynamic limit and to check validity of the assumption about the structure of the antiferromagnetic vacuum.

2 Existence of string solutions

To make from $x(u)$ an analytic single–valued function, we fix in $\mathbb{C}$ the cut $[-\chi, \chi]$ and define $x(u)$ on $\mathbb{C}/[-\chi, \chi]$ as follows:

$$x(u) = \frac{1}{4}\left(\sqrt{r_1}\, e^{\frac{i}{2}\theta_1} + \sqrt{r_2}\, e^{\frac{i}{2}\theta_2}\right)^2\,, \tag{7}$$

where $r_1,\ r_2 \in \mathbb{R}_+$ and $\theta_1,\ \theta_2 \in]-\pi, \pi]$ are determined from the relations $u = \chi + r_1 e^{i\theta_1} = -\chi + r_2 e^{i\theta_2}$. Notice that the signs of the imaginary parts of u and $x(u)$ coincide, that is $x(u)$ maps a point from the upper/lower half–plane to the upper/lower half–plane, respectively.

Let us prove that the following relations

$$\left|\frac{x(u+i\delta)}{x(u-i\delta)}\right| \quad \begin{cases} >1, & \text{for } \operatorname{Im} u > 0, \\ =1, & \text{for } \operatorname{Im} u = 0, \\ <1, & \text{for } \operatorname{Im} u < 0. \end{cases} \tag{8}$$

hold for every $\delta > 0$.

Let us denote $s = \mathrm{Re}(u)$, $t = \mathrm{Im}(u)$, $a = \mathrm{Re}\big(x(u)\big)$, $b = \mathrm{Im}\big(x(u)\big)$. It follows from (7) that $|x(u)|$ is a function continuous in u (in particular, even when u crosses the cut), and $|x(\overline{u})| = |x(u)|$. Therefore, with s being fixed, the function $|x(s+it)|$ is continuous and symmetric in t. We will prove that this function is convex, of which (8) will then be an obvious consequence.

It follows from (5) that the function inverse to $x(u)$ is given by $u(x) = x + \frac{\chi^2}{4x}$. Whence it is easy to derive the following relations

$$s = \left(1 + \frac{\chi^2}{a^2+b^2}\right) a, \quad t = \left(1 - \frac{\chi^2}{a^2+b^2}\right) b. \tag{9}$$

Let ∂_t denote the partial derivative w.r.t t (i.e. $\partial_t s = 0$). Applying it to (9) and solving the system of equations for $\partial_t a$ and $\partial_t b$, we find

$$\partial_t a = \frac{2\chi^2 ab}{D}, \quad \partial_t b = \frac{1}{D}\left((a^2+b^2)^2 + \chi^2(b^2-a^2)\right), \tag{10}$$

where $D = ((\chi-a)^2+b^2)((\chi+a)^2+b^2)$. Whence we obtain

$$\partial_t |x(u)|^2 = 2\,\frac{a^2+b^2}{D}\,(a^2+b^2+\chi^2)\,b. \tag{11}$$

Due to the remark made after equation (7), t and b are of the same sign. Therefore, expression (11) is positive/negative in the upper/lower half–plane, respectively. Hence $|x(u)|$ is convex in $\mathrm{Im}(u)$, which completes the proof of (8).

Relations (8) allow us to adapt the analysis of complex roots of equations (3) (see [5,7]) to the case of equations (4). Namely, it follows from (8) that, in the $L \to \infty$ limit, the absolute value of the l.h.s. of (4) tends to ∞ when $\mathrm{Im}(u_j) > 0$ and to 0 when $\mathrm{Im}(u_j) < 0$. This implies that r.h.s. has, respectively, a pole or a zero, i.e. there must exist also the root $u_j - i$ in the first case and the root $u_j + i$ in the second case. Thus, like in case of the XXX magnet, roots of equations (4) group in the thermodynamic limit into "strings" which are complexes of the form $u_{j,m} = u_j + im$, where $u_j \in \mathbb{R}$ and $2m \in \mathbb{Z}$.

3 Strings of odd length

3.1 Equation for the centers of strings

Now we will investigate what state has the maximal energy in the case when the vacuum is filled with strings of length $2n+1$, where n is integer. The number of these strings ν_n is fixed by the condition $(2n+1)\nu_n = L/2$. Following [5–7], we multiply Bethe equations (4) along a string of length $2n+1$. Since the right–hand side of these equations is the same as in the "undeformed" Bethe equations, strings will have the same form, i.e. $u_j = u_j^n + im$, $m \in \mathbb{Z}$. Further, considering the thermodynamic limit ($L \to \infty$), we obtain that the

centers of strings are arranged along the real axis with some density which satisfies certain integral equation. Having found this density, one can compute the energy of the ground state (see [5,6]).

Thus, we obtain the following equation for the centers of strings:

$$\frac{i}{2}L\ln\frac{x\left(u_j^n+(2n+1)\frac{i}{2}\right)}{x\left(u_j^n-(2n+1)\frac{i}{2}\right)}=\pi Q_j^n+\sum_{k=1}^{\nu_n}\Phi_{n,n}(u_j^n-u_k^n), \tag{12}$$

where

$$\Phi_{n,n}(u)=\arctan\frac{u}{2n+1}+2\sum_{m=0}^{2n-1}\arctan\frac{u}{m+1}. \tag{13}$$

3.2 Yang's action

To prove the uniqueness of a solution to equations (12) for a given set of integer numbers Q_j^n, we will use the Yang's action. As in the case of the XXX magnet [7], there exists a functional S (called Yang's action) such that equations (12) are the conditions of its extremum, $\partial_{u_\alpha}S=0$. Let us consider the quadratic form for the matrix of second derivatives of S:

$$\sum_{\alpha,\beta}v_\alpha\frac{\partial^2 S}{\partial_{u_\alpha}\partial_{u_\beta}}v_\beta=\frac{i}{2}L\sum_\alpha\partial_{u_\alpha}\ln\frac{x(u_\alpha+\frac{i}{2}(2n+1))}{x(u_\alpha-\frac{i}{2}(2n+1))}v_\alpha^2+\sum_{\alpha>\beta}\frac{1}{(u_\alpha-u_\beta)^2+1}(v_\alpha-v_\beta)^2, \tag{14}$$

where $v_\alpha\in\mathbb{R}$. It is obvious that the second term is always positive. Let us prove the positivity of the first one:

$$\frac{i}{2}\partial_s\log\frac{x(s+it)}{x(s-it)}=\partial_s\arctan\frac{a}{b}=\frac{b\partial_s a-a\partial_s b}{(a^2+b^2)}=\frac{a\partial_t a+b\partial_t b}{(a^2+b^2)}=\frac{\partial_t|x(s+it)|^2}{2(a^2+b^2)}. \tag{15}$$

Here we used the same notation as in Section 2 and applied the Cauchy equations for derivatives of an analytic function. Relation (11) shows that (15) is positive for $b>0$. Therefore the quadratic form (14) is positive definite and, consequently, the action S has a unique minimum.

3.3 Thermodynamic limit

Let us go to the thermodynamic limit now. Taking $L\to\infty$ and differentiating (12) with respect to u, we obtain for the left–hand side:

$$l.h.s.=\frac{i}{2}\left(\frac{1}{\sqrt{\left(\frac{(2n+1)}{2}i+u\right)^2-\chi^2}}-\frac{1}{\sqrt{\left(\frac{(2n+1)}{2}i-u\right)^2-\chi^2}}\right). \tag{16}$$

Now we introduce the root density $\rho(u)$:

$$\rho(u)=\frac{1}{\left(\frac{du}{dq}\right)_{q=q(u)}}, \tag{17}$$

where $q(u) = Q_j/L$. $\rho(u)$ plays the role of density of numbers $q(u)$ on the interval du. Having introduced this density, we can rewrite the l.h.s. of (12) as follows

$$r.h.s. = \pi\rho(u) \int\limits_{-\infty}^{\infty} d\mu\, \rho(\mu) \Big[\frac{2n+1}{(2n+1)^2 + (u-\mu)^2} + 2 \sum_{m=0}^{2n-1} \frac{m+1}{(m+1)^2 + (u-\mu)^2} \Big]. \qquad (18)$$

This integral equation can be solved by means of the Fourier transform. To this end we compute first

$$\int_{-\infty}^{\infty} du\, \frac{\mathrm{e}^{iku}}{\sqrt{(u+il)^2 - \chi^2}} = \theta(-kl)\, \mathrm{e}^{-|kl|} \oint \frac{dq}{q} \exp\left(ik \left(q + \frac{\chi^2}{4q} \right) \right)$$
$$= \ \mathrm{sign}\,(-l)\theta(-kl)\, \mathrm{e}^{-|kl|} \int_0^{2\pi} id\varphi\, \mathrm{e}^{ik\chi\cos\varphi} = 2\pi i\, \mathrm{sign}\,(-l)\theta(-kl)\, \mathrm{e}^{-|kl|}\, J_0(\chi k), \quad (19)$$

where $J_0(k)$ is the Bessel function of the first kind. Thus, after the Fourier transform, the left–hand side of our integral equation acquires the following form

$$F[l.h.s] = \pi\, \mathrm{e}^{-\frac{2n+1}{2}|k|}\, J_0(\chi k)\,. \qquad (20)$$

On the r.h.s. the Fourier integral is divided into the following terms

$$\int\limits_{-\infty}^{\infty} d\mu\, \rho(\mu) \int\limits_{-\infty}^{\infty} \mathrm{e}^{iku}\, \frac{A}{A^2 + (u-\mu)^2} du = \pi \int\limits_{-\infty}^{\infty} d\mu\, \rho(\mu)\, \mathrm{e}^{i\mu k}\, \mathrm{e}^{-|k|A} = \pi\, \mathrm{e}^{-|k|A}\, \widetilde{\rho}(k), \qquad (21)$$

where $\widetilde{\rho}(k)$ stands for the Fourier transform of the density $\rho(u)$. These terms can be summed up:

$$\pi\widetilde{\rho}(k) \left(2 \sum_{m=0}^{2n-1} \mathrm{e}^{-|k|(m+1)} + \mathrm{e}^{-|k|(2n+1)} \right) = \pi\widetilde{\rho}(k) \left(\frac{2 - \mathrm{e}^{-2n|k|} - \mathrm{e}^{-(2n+1)|k|}}{\mathrm{e}^{|k|} - 1} \right). \qquad (22)$$

As a result, we obtain the following expression for the Fourier transformed density $\widetilde{\rho}(k)$:

$$\widetilde{\rho}(k) = J_0(\chi k) \frac{\mathrm{e}^{|k|} - 1}{\mathrm{e}^{-\frac{2n+1}{2}|k|}(1 + \mathrm{e}^{|k|})(\mathrm{e}^{(2n+1)|k|} - 1)} = \frac{J_0(\chi k) \tanh\frac{|k|}{2}}{2\sinh\left(n + \frac{1}{2}\right)|k|}. \qquad (23)$$

This yields the solution of the integral equation,

$$\rho(u) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \frac{J_0(\chi k) \tanh\frac{|k|}{2}}{2\sinh\left(n + \frac{1}{2}\right)|k|}\, \mathrm{e}^{iku}\, dk. \qquad (24)$$

For $n = 0$ (i.e. strings of length 1) this expression coincides with the expression for the density that was obtained in [9, 10].

3.4 Energy of strings of odd length

Now we will compute the energy (or, equivalently, the maximal dimension of the dilatation operator $\Delta_{\max}$) for the state filled with strings of length $2n+1$, where n is integer. The

dispersion in the considered theory differs from that of the XXX magnet, the corresponding energy density is given by [9]

$$\frac{\Delta_{\max}}{L} = 1 + \frac{i\lambda}{8\pi^2} \int\limits_{-\infty}^{\infty} du\, \rho(u) \left(\frac{1}{x\left(u + \frac{2n+1}{2} i\right)} - \frac{1}{x\left(u - \frac{2n+1}{2} i\right)} \right). \tag{25}$$

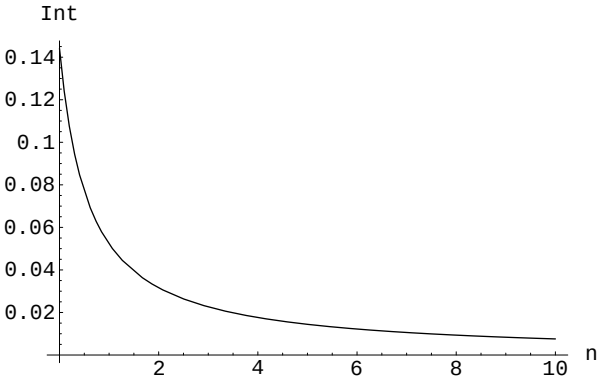
Substituting here the expression (24) for the density, we obtain:

$$\frac{\Delta_{\max}}{L} = 1 + \frac{\sqrt{\lambda}}{\pi} \int\limits_{0}^{\infty} \frac{dk}{k} \frac{J_0(\chi k) J_1(\chi k) \tanh \frac{k}{2}}{\mathrm{e}^{(2n+1)k} - 1}. \tag{26}$$

Here we used the following relation:

$$\frac{\chi^2}{2} \left(\frac{i}{\sqrt{(u-il)^2 - \chi^2}} - \frac{i}{\sqrt{(u+il)^2 - \chi^2}} \right) = \frac{1}{2} \chi \partial_\chi \left[\frac{\chi^2}{2i} \left(\frac{1}{x(u+il)} - \frac{1}{x(u-il)} \right) \right]. \tag{27}$$

Below we provide a plot which shows how the second term in (26) depends on n for a fixed value of λ.



It is apparent that the maximum of the integral (and thus the maximum of $\Delta_{\max}$) is attained at $n = 0$. In the Appendix we give a strict proof that $\Delta_{\max}$ decreases monotonously as n grows.

Thus, we conclude that the state with maximal energy in the sector of strings of odd length indeed corresponds to strings of length 1, as it was assumed in [9, 10]. However, in the next section we will show that the real anti–ferromagnetic vacuum might have a more complicated structure and possibly corresponds to strings of length 2.

4 Strings of even length

Unlike the absolute value $|x(u)|$, the phase of $x(u)$ is not a continuous function. Its value changes by a finite amount when u crosses the cut. Using (7) it is easy to derive that for $u \in [-\chi, \chi]$ we have the following relation:

$$\lim_{\epsilon \to 0} \ln \frac{x(u - i\epsilon)}{x(u + i\epsilon)} = 2i\, \nu_\epsilon \, \arctan \frac{\sqrt{\chi^2 - u^2}}{u}, \tag{28}$$

where $\nu_\epsilon = -1$ if ϵ tends to the zero from the right and $\nu_\epsilon = 1$ if ϵ tends to the zero from the left.

For a chain of a large but finite length L, the roots of Bethe equations group into strings only up to exponential deviations: $u_{j,m} = u_j + i(m + \epsilon_m)$, where m is integer or half–integer and $\epsilon_m = O(e^{-\omega_m L})$, $\omega_m > 0$. The total energy of a string is given by $\mathcal{E}_j = \frac{i}{2}\sum_m \Big(\frac{1}{x(u_{j,m}+\frac{i}{2})} - \frac{1}{x(u_{j,m}-\frac{i}{2})}\Big)$. Computing this expression for a string of even length and taking relation (28) into account, we observe that the roots

$$u_j + i\Big(\frac{1}{2}+\epsilon\Big), \quad u_j - i\Big(\frac{1}{2}+\epsilon\Big) \tag{29}$$

give additional contributions,

$$\mathcal{E}_j = \lim_{\epsilon\to 0}\frac{i}{2}\left(\frac{1}{x(u_j - i\epsilon)} - \frac{1}{x(u_j + i\epsilon)} + + \frac{1}{x(u_j + \frac{2n+1}{2}i)} - \frac{1}{x(u_j - \frac{2n+1}{2}i)}\right). \tag{30}$$

It is important to remark here that, for the regime corresponding to $\nu_\epsilon = -1$, the first two terms in (30) give a negative contribution to $\mathcal{E}_j$. Furthermore, it can be shown that in this regime $\mathcal{E}_j$ is positive not everywhere on the real axis. As a consequence, the anti–ferromagnet vacuum in this case has a more complicated structure — it has to be filled with strings only in the intervals where $\mathcal{E}_j > 0$. In the present work we will consider only the regime corresponding to $\nu_\epsilon = 1$. Let us remark that in this case the proof of convexity of the Yang's action given in Section 3.2 remains valid.

Taking product of the left hand sides of the Bethe equations (4) along a string of even length and taking into account the additional contributions due to the roots (29), we find that the l.h.s. of the equation for the centers of strings looks like following

$$\lim_{\epsilon\nearrow 0}\left(\frac{x(u_j - i\epsilon)}{x(u_j + i\epsilon)}\right)^L \left(\frac{x(u_j + \frac{i}{2}(2n+1))}{x(u_j - \frac{i}{2}(2n+1))}\right)^L. \tag{31}$$

Taking logarithm of this expression, we obtain equations (12) but with an additional term on the left hand side. Differentiating (28) w.r.t. u, then making the Fourier transformation,

$$F\Big[\partial_u \lim_{\epsilon\to 0}\ln\frac{x(u-i\epsilon)}{x(u+i\epsilon)}\Big] = -2i\,\nu_\epsilon\int_{-\chi}^{\chi}\frac{e^{iku}\,du}{\sqrt{\chi^2-u^2}} = -2\pi i\,\nu_\epsilon\,J_0(\chi k), \tag{32}$$

and comparing with formula (19), we see that, for strings of even length in the regime $\nu_\epsilon = 1$, equation (20) acquires the following form

$$F[l.h.s] = \pi\big(e^{-\frac{2n+1}{2}|k|} + 1\big)J_0(\chi k). \tag{33}$$

Repeating now the computation of the Fourier image of the density as was done in Section 3.3, we find

$$\rho_\epsilon(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{J_0(\chi k)\big(1 + e^{\frac{2n+1}{2}|k|}\big)\tanh\frac{|k|}{2}}{2\sinh\big(n+\frac{1}{2}\big)|k|}\,e^{iku}dk. \tag{34}$$

Formula (33) shows that the additional contributions from the roots (29) yield the same effect as if the vacuum were filled with mixture of strings of length $(2n+1)$ and strings of length 0 (which formally corresponds to $n = -\frac{1}{2}$). In this context we remark that expressions of the type (31) appear in the left hand sides of Bethe equations for chains with alternating spins (see, e.g. [13, 14]).

Formula (30) implies that, in order to compute the total energy of the chain, we have to replace (25) with

$$\frac{\Delta}{L} = 1 + \frac{i\chi^2}{2} \lim_{\epsilon \nearrow 0} \int\limits_{-\infty}^{\infty} du\, \rho_\epsilon(u) \left(\frac{1}{x(u-i\epsilon)} - \frac{1}{x(u+i\epsilon)} + \frac{1}{x\left(u+\frac{2n+1}{2}i\right)} - \frac{1}{x\left(u-\frac{2n+1}{2}i\right)} \right). \tag{35}$$

Substituting here (34) and making the same computation in the Section 3.4, we obtain

$$\frac{\Delta}{L} = 1 + 2\chi \int\limits_{0}^{\infty} \frac{dk}{k}\, J_0(\chi k) J_1(\chi k)\, \tanh\frac{k}{2}\, \coth\frac{(2n+1)k}{4}. \tag{36}$$

Using the method described in the Appendix, one can show that for $n \geq \frac{1}{2}$ expression (36) decreases monotonously as n grows. The maximal value of (36) at $n = \frac{1}{2}$ is given by

$$\frac{\Delta_{\max}}{L} = 1 + 2\chi \int\limits_{0}^{\infty} \frac{dk}{k}\, J_0(\chi k) J_1(\chi k). \tag{37}$$

In order to compare this expression with the value of $\frac{\Delta_{\max}}{L}$ corresponding to strings of length 1, one can subtract the value of (26) for $n = 0$ from (37). The resulting expression has the form of the integral in (38) with monotonously decreasing in k function $f(k)$. As shown in the Appendix, such an integral is positive.

Thus, filling the vacuum with strings of length 2 in the regime $\nu_\epsilon = 1$ yields larger value for $\frac{\Delta_{\max}}{L}$ than filling it with strings of length 1. However, what regime is indeed realized for string of even length in the thermodynamic limit, remains at present an open question. Its resolution requires quite subtle analysis of the exponential corrections ϵ in (29) for $L \to \infty$.

Appendix

Let us denote $\alpha = (2n+1)$, $\chi = \sqrt{\lambda}/(2\pi)$ and make in (26) a substitution $k' = \chi k$. Then

$$\partial_\alpha \Big(\frac{\Delta_{\max}}{L} \Big) = -2 \int\limits_{0}^{\infty} dk\, J_0(k)\, J_1(k)\, f(k), \tag{38}$$

where

$$f(k) = \left(\frac{1}{e^{\frac{k}{\chi}} + 1} \right) \left(\frac{e^{\frac{\alpha k}{\chi}}}{e^{\frac{\alpha k}{\chi}} - 1} \right) \left(\frac{e^{\frac{k}{\chi}} - 1}{e^{\frac{\alpha k}{\chi}} - 1} \right). \tag{39}$$

In this form, it is evident that $f(k)$ decreases monotonously as k grows for all $\alpha > 1$ and $\chi > 0$. Using this we will show that the integral in (38) is positive.

Let $0 < t_1 < t_3 < t_5 \ldots$ be the ordered set of roots of $J_0(t)$, and $0 = t_0 < t_2 < t_4 \ldots$ be the ordered set of roots of $J_1(t)$. It follows from the relation

$$\partial_t J_0(t) = -J_1(t) \tag{40}$$

that $t_{2n} < t_{2n+1} < t_{2n+2}$. Taking into account that $J_0(t)J_1(t)$ is positive on $]t_{2n}, t_{2n+1}[$ and negative on $]t_{2n+1}, t_{2n+2}[$, and the function $f(k)$ is positive and monotonously decreasing

for all $k > 0$, we obtain the following estimate:

$$\int_0^\infty dk\ J_0(k)\, J_1(k)\, f(k) = \sum_{n=0}^{\infty} \int_{t_{2n}}^{t_{2n+2}} dk\, J_0(k)\, J_1(k)\, f(k) > \tag{41}$$

$$> \sum_{n=0}^{\infty} \Big[f(t_{2n+1}) \int_{t_{2n}}^{t_{2n+2}} dk\, J_0(k)\, J_1(k) \Big] = \frac{1}{2} \sum_{n=0}^{\infty} \Big[f(t_{2n+1}) \left(J_0^2(t_{2n}) - J_0^2(t_{2n+2}) \right) \Big].$$

In the last equality we used relation (40). Now, since roots of $J_1(t)$ are the points of local extrema for $J_0(t)$, and the values of $|J_0(t)|$ at these points form a decreasing sequence, we conclude that the sum on the r.h.s. of (41) and hence the initial integral are positive. Thus, $\partial_\alpha \left(\frac{\Delta_{\max}}{L} \right) < 0$, which implies that $\frac{\Delta_{\max}}{L}$ decreases monotonously as n grows.

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