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# The Robustness of $n_s \lesssim 0.95$ in Racetrack Inflation

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**Abstract**

A spectral index $n_s \lesssim 0.95$ appears to be a generic prediction of racetrack inflation models. Reducing a general racetrack model to a single-field inflation model with a simple potential, we obtain an analytic expression for the spectral index, which explains this result. By considering the limits of validity of the derivation, possible ways to achieve higher values of the spectral index are described, although these require further fine-tuning of the potential.

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## 1 Introduction

Racetrack inflation [1] is an explicit realization of modular inflation within the context of KKLT volume stabilisation [2]. It employs a superpotential of the double-exponential

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racetrack form. The inflaton is then the imaginary part of a geometric modulus. Inflation begins near a saddle in the moduli potential, and ends when the inflaton fast rolls towards a local minimum of the potential. A big success of this class of models is the seemingly very robust prediction $n_s \lesssim 0.95$ for the spectral index, which is very close to the latest WMAP results [3]. In this letter we explain the origin of this upper bound on the spectral index, which is a generic feature of double- as well as many-exponential superpotentials. In addition, we will show that to a very good approximation the spectral index only depends on one parameter, namely the value of the slow roll parameter $\eta$ at the saddle point.

## 2 Racetrack inflation

Modular inflation occurs in the KKLT scenario at sufficiently flat saddle points of the potential. There the real part of the volume modulus $T$ is stabilised, and the tachyonic, imaginary part plays the role of the inflaton. The KKLT [2] racetrack potential comes from a supergravity model with

$$K = -3\ln(T+\bar{T})\,, \qquad W = W_0 + A\mathrm{e}^{-aT} + B\mathrm{e}^{-bT}\,, \tag{1}$$

together with an non-supersymmetric lifting term $V_{\rm lift} = 2^\alpha E/(T+\bar{T})^\alpha$. The constants $a, b$ depend on the specifics of the non-perturbative physics, which can come from gaugino condensation or Euclidean instantons. The lifting term is adjusted to obtain a Minkowski vacuum with a vanishing cosmological constant. Notice that the lifting term breaks supersymmetry explicitly. It can be realised in a string context by putting an anti-brane in the bulk ($\alpha = 2$) or at the tip of a warped throat ($\alpha = 3$). Defining $T = X + \mathrm{i}Y$, the resulting potential is

$$\begin{aligned} V = \frac{E}{X^\alpha} + \frac{1}{6X^2}\Big\{ & A^2 a[aX+3]\mathrm{e}^{-2aX} + B^2 b[bX+3]\mathrm{e}^{-2bX} + 3W_0 Aa\mathrm{e}^{-aX}\cos(aY) \\ & + 3W_0 Bb\mathrm{e}^{-bX}\cos(bY) + AB[2abX + 3(a+b)]\mathrm{e}^{-(a+b)X}\cos([a-b]Y)\Big\}. \end{aligned} \tag{2}$$

It should be noted that the kinetic terms $\mathcal{L}_{\rm kin} = (3/4X^2)[(\partial X)^2 + (\partial Y)^2]$ are non-canonical. The axion field $Y$ is the inflaton [1]. The potential is periodic in $Y$, giving rise to saddle points between degenerate local minima. If its initial value is close enough to the saddle point, the rolling of the $T$ modulus along the unstable $Y$ direction produces inflation (see figure 1 in [1] for the shape of the potential).

In [4] an improved racetrack inflation model was proposed with two geometric moduli:

$$K = -2\ln\left[\frac{(T_2-\bar{T}_2)^{3/2} - (T_1-\bar{T}_1)^{3/2}}{36}\right], \qquad W = W_0 + A\mathrm{e}^{-aT_1} + B\mathrm{e}^{-bT_2}\,. \tag{3}$$

In this case the inflaton is a linear combination of the two axions (the orthogonal combination is a flat direction), and the inflaton potential has a structure similar to (2).

Alternatively, racetrack inflation can be obtained without the need for non-supersymmetric terms by using $D$-term uplifting [5]. Gauge invariance requires additional meson fields $\chi$. The model is

$$K = -3\ln(T+\bar{T}) + |\chi|^2\,, \qquad W = W_0 + A\chi^{-ra}\mathrm{e}^{-aT} + B\chi^{-rb}\mathrm{e}^{-bT}\,. \tag{4}$$

The $D$-term potential is similar in form to $V_{\rm lift}$, and serves to uplift the minimum to a Minkowski vacuum. Although the form of the potential is more complicated [6], its structure of minima and maxima is similar to (2). In particular, there are still saddle points whose unstable direction is almost coincident with the $Y$ direction. Like the field $X$, the meson field $\chi$ remains roughly constant during inflation. This was found to be the case for all models and parameters studied in [6].

All these different realizations of racetrack inflation are effectively single field inflation models with the axion field as the inflaton. Although other fields are present — the $X_i$ and the meson fields — they are fixed during the period of inflation when WMAP scales leave the horizon. One can therefore approximate the effective inflaton potential as

$$V(Y) = V_0 + \sum_i A_i \cos a_i Y\,, \tag{5}$$

where the overall phase is chosen such that the saddle/maximum is at $Y=0$. For the double-exponential superpotentials discussed above, $i=3$. More generally, the superpotential can have more than two exponentials, and $i>3^*$. Inflation can occur when the saddle point is such that $\eta_0 = V''(Y)/V(Y)|_{Y=0}$ is much smaller than one, guaranteeing that the slow roll conditions are satisfied close enough to the saddle point. In the following, we will use a further approximation of this effective potential to show that, quite generically in racetrack inflation, the spectral index is determined by $\eta_0$ only, and is bounded from above.

# 3 Inflationary dynamics

For all the racetrack models mentioned in the previous section, the physics which determines the cosmic microwave background parameters occurs close to the saddle point at $Y=0$, where the inflaton is the only field that evolves significantly. Taylor expanding the potential at the saddle gives

$$V = V_{\rm sad}\left(1 + \eta_0\frac{y^2}{2} + C\frac{y^4}{4} + \cdots\right)\,, \tag{6}$$

where $\eta_0$ is the value of $\eta$ at the saddle point, and $y$ is the canonically normalised inflaton field, which e.g. for the racetrack models (1, 4) is $y=\sqrt{3/2}\,Y/X$. Using this approximate

*In model (3) there is more than one axion if there are more than two moduli fields. In this case the potential only reduces to the form (5) in the limit of single field inflation.

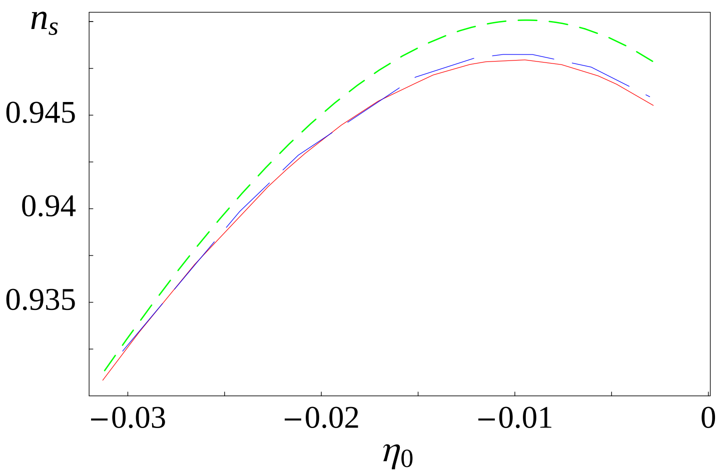


Figure 1: Plot of $n_s(\eta_0)$ for Taylor expanded, approximate cosine, and full racetrack potentials (curves from top to bottom).

potential we can calculate the number of e-folds before the end of inflation $N = -\ln a$, as a function of $y$ in the slow roll approximation:

$$N(y) = \int_{y_{\text{end}}}^{y} \frac{V\,dy}{V'} \approx \left[\frac{1}{2\eta_0} \log \frac{y^2}{y^2 + \eta_0/C}\right]_{y_{\text{end}}}^{y} . \tag{7}$$

Inflation ends when the slow roll parameter $\epsilon = (1/2)(V'/V)^2 \approx 1$, which for the above potential happens at $y_{\text{end}} \sim C^{1/3}$. Inverting the above equation to get $y(N)$, and substituting it into the expression for the slow roll parameter $\eta = V''/V$, we obtain

$$\eta \approx \eta_0 - 3\eta_0 \left(1 - \mathrm{e}^{-2\eta_0 N} - \frac{\eta_0}{C y_{\text{end}}^2} \mathrm{e}^{-2\eta_0 N}\right)^{-1} . \tag{8}$$

In the racetrack models $|\eta_0| \ll 1$ is tuned small, whereas the coefficient $C$ is not. Hence we can expand the above expression in $|\eta_0/(C y_{\text{end}}^2)| \ll 1$. To lowest order the spectral index $n_s \approx 1 + 2\eta$ is then

$$n_s = 1 + 2\eta_0 - \frac{6\eta_0}{1 - \mathrm{e}^{-2N\eta_0}} \tag{9}$$

evaluated $N = N_* \approx 55$ e-folds before the end of inflation. For the parameters used in the original racetrack model [1] $\eta_0 \approx -0.0061$, $C \approx 293$ and $y_{\text{end}} \approx 0.12$, while for the $D$-term uplifted racetrack [6] $\eta_0 \approx -0.0095$, $C \approx 999$ and $y_{\text{end}} \approx 0.10$. Hence our expansion in $\eta_0/(C y_{\text{end}}^2)$ is justified. In fact, it is a particularly good approximation since the error in the spectral index from neglecting higher order terms is only of the order $10^{-4}$. As a result $n_s$ is practically a function of $\eta_0$ (and $N_*$) only.

## 4 Discussion

Figure 1 shows $n_s(\eta_0)$ with $N = 55$ for $D$-term uplifted racetrack inflation (4), for the effective potential (5), and for our analytic result (9). The results for the cosine potential are nearly identical to the full racetrack model, confirming that the dynamics of the "spectator fields" ($X_i$ and $\chi$, which only evolve significantly towards the end of inflation) have a negligible effect on the inflationary predictions. Freezing the spectator fields all the different models proposed (1, 3, 4) reduce to the same effective model, so it is not surprising they give the same inflationary predictions. Hence the predictions of racetrack inflation are very robust.

The approximation (6) of the full racetrack potential is simple yet very effective. It gives $n_s(\eta_0)$ in very good agreement with the results of the full racetrack potential, the error in $n_s$ is of the order $10^{-3}$. Accordingly it correctly predicts an upper bound on the spectral index $n_s \lesssim 0.95$ for generic cases, i.e. with the coefficient $C$ not fine-tuned. It further explains the puzzling result of the racetrack models that the spectral index is a function of $\eta_0$ only; the dependence on $C$ and $y_{\rm end}$ drops out in the $Cy_{\rm end}^2 \gg |\eta_0|$ limit. The reason for the small disagreement between the analytic results and the full potential is that although (6) is a very good approximation when observable scales leave the horizon, it becomes worse towards the end of inflation when $y$ is larger. To improve upon our results we should take higher terms into account when calculating $N(y)$, which only play a role in the integration region near $y_{\rm end}$. We reemphasise that higher order corrections in $1/C$ only affect the spectral index at the level of $10^{-4}$, and are not the cause of the small mismatch between the analytic approximation and the racetrack model.

Our analytical model also suggests ways to get around the upper bound $n_s \lesssim 0.95$. These require extra tuning, in addition to that needed to get $|\eta_0| \ll 1$. One possibility is if inflation is multi-field, in which case our approximation (5) breaks down. This is probably most easily achieved in a set-up analogous to (3), with additional moduli fields, and so more than one axion field. However, the parameters need to be tuned to arrange that the saddle point has more than one unstable directions with similar curvature if multiple axion fields are to act as inflatons.

An alternative way to avoid the upper bound is by fine-tuning the coefficient of the quartic term in the expansion (6). Setting $C \approx 0$ would lead to a model where the $y^6$ term is dominant. More generally we can consider a model with $V = V_{\rm sad}(1+\eta_0 y^2/2+\sum_{n>1} c_n y^{2n})$, with the coefficients $|c_n| \lesssim |\eta_0| y_{\rm end}^{2(1-n)}$ for $1 < n < p$ , and so negligible during inflation. Since $y_* \equiv y(N_*) \ll y_{\rm end}$, the evolution of $y$ is then dominated by the $y^2$ and $y^{2p}$ terms. The approximation (9) then generalises to

$$n_s = 1 + 2\eta_0 - \frac{2(2p-1)\eta_0}{1 - {\rm e}^{-2(p-1)N\eta_0}} \,. \tag{10}$$

Figure 2 shows the above function for various values of $p$. We see that as $p$ is increased, the bound on the spectral index is relaxed: $n_s \lesssim 0.950, 0.968, 0.975, 0.980$ for $p = 2, 3, 4, 5$ respectively. Larger values of $p$ require more fine-tuning, and hence progressively more

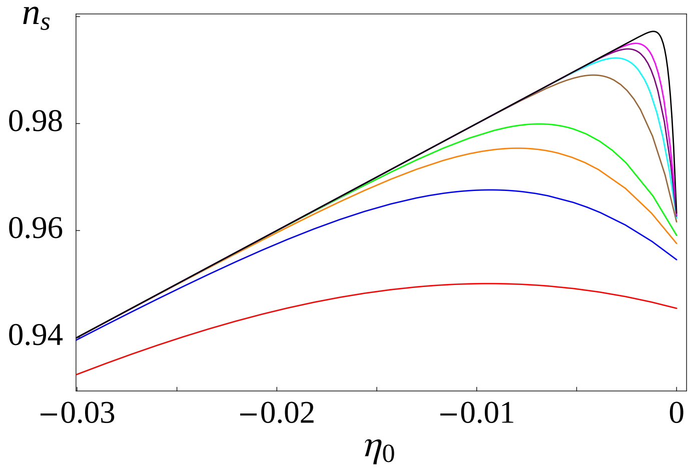


Figure 2: Plot of $n_s(\eta_0)$ given in (10) for $2p = 4, 6, 8, 10, 20, 30, 40, 50, 100$ (increasing $p$ corresponds to larger spectral index).

exponential terms are needed in the superpotential, in order to have enough parameters to tune.

In [4] it was argued that anthropic considerations may favour models with the largest possible spectral index. In principle, double exponential superpotentials (1) contain enough parameters to simultaneously fine-tune both $\eta_0$ and $C$ to be small, as well as match the COBE normalisation for the power spectrum ($P = V/[150\pi^2\epsilon] \approx 4 \times 10^{-10}$ at $N = N_*$). Hence $n_s \sim 0.97$ may be possible for the racetrack models (1, 3, 4). However, it will come at the cost of even more severe fine-tuning.

To summarise, in this paper we have reduced a general racetrack model to a single field inflation model. This allowed us to derive a simple analytic expression for the spectral index (9), and show that it is bounded from above. Barring exceptional fine-tuning, i.e. any tuning beyond that needed to get $\eta$ small, we predict $n_s \lesssim 0.95$ for all racetrack inflation models in agreement with the WMAP3 data.

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