INPUT-TO-STATE STABILITY OF UNBOUNDED BILINEAR CONTROL SYSTEMS

RENÉ HOSFELD, BIRGIT JACOB, AND FELIX L. SCHWENNINGER

Abstract. We study input-to-state stability of bilinear control systems with possibly unbounded control operators. Natural sufficient conditions for integral input-to-state stability are given. The obtained results are applied to a bilinearly controlled Fokker-Planck equation.

1. Introduction

In this note we continue recent developments on *input-to-state stability (ISS)* for systems governed by evolution equations. This concept unifies both asymptotic stability with respect to the initial values and robustness with respect to the external inputs such as controls or disturbances. Loosely, if a system Σ is viewed as a mapping which sends initial values $x_0 \in X$ and inputs $u : [0, \infty) \to U$ to the time evolution $x : [0, T) \to X$ for some maximal $T > 0$, then Σ is ISS if $T = \infty$ and for all $t \in [0, \infty)$,

$$\|x(t)\|_X \leq \beta(\|x_0\|_X, t) + \gamma(\sup_{s\in[0,t]} \|u(s)\|_U), \qquad \forall x_0, u, \tag{1}$$

where the continuous functions $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are of Lyapunov class $\mathcal{KL}$ and $\mathcal{K}$ respectively[1]. Here X is called the state space and U the input space equipped with norms $\|\cdot\|_X$ and $\|\cdot\|_U$. For linear systems

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and $B : U \to X$ is a bounded linear operator, ISS is equivalent to uniform exponential stability of the semigroup [6, 12]. If B is not bounded as operator form U to X, which is typically the case for boundary controlled PDEs, the property of being ISS becomes non-trivial even for linear systems. In fact, this is closely related to suitable solution concepts see e.g. [12, 21, 29]. Along with the recent developments in ISS theory for infinite-dimensional systems [6, 7, 10, 16, 27], several partial results have been derived in the (semi)linear context, with a slight focus on parabolic equations, see e.g. [14, 17, 22, 23, 25, 36]. We refer to recent surveys on ISS for infinite-dimensional systems [26, 29] and the book [18]. The origin of ISS theory, introduced by Sontag in 1989 [30], are non-linear systems and we refer the reader to [31] for a survey on ISS for ODEs. Already seemingly harmless system classes such as bilinear systems

$$\dot{x}(t) = Ax(t) + B(u(t), x(t)), \tag{2}$$

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[1] I.e. $\beta(0, s) = \lim_{t\to\infty} \beta(r, t) = \gamma(0) = 0$ and $\beta(\cdot, t)$, $\beta\left(r, \frac{1}{\cdot}\right)$, γ are strictly increasing on $\mathbb{R}^+$ for all $r, s, t > 0$.

where $B(u,x) = \sum_{i=1}^m u_i(t)B_i x$ and matrices $A \in \mathbb{R}^{d\times d}$, $B_i \in \mathbb{R}^{d\times d}$, see [8], are typical counterexamples for ISS [32]. Nevertheless, the following variant of ISS [32] is satisfied by such systems; there exists functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that

$$\|x(t)\|_X \leq \beta(\|x_0\|_X, t) + \gamma_1 \left(\int_0^t \gamma_2(\|u(s)\|_U)\, \mathrm{d}s \right), \qquad \forall t > 0, \forall u, x_0, \tag{3}$$

which is called *integral input-to-state stable* (iISS), see also [32]. Note that the terms involving u in (1) and (3) cannot be compared for arbitrary $t > 0$, general functions u, and fixed functions $\gamma, \gamma_1, \gamma_2$. Still iISS and ISS are equivalent for infinite-dimensional linear systems with a bounded linear operator $B : U \to X$, [12, 24] as this reduces to uniform exponential stability of the uncontrolled system. The corresponding question for general infinite-dimensional systems seems to be much harder and notorious questions remain, see [12, 28, 14] and [35] for a negative result.
On the other hand in [24] the equivalence of iISS and uniform exponential stability is shown for a natural infinite-dimensional version of (2), with A generating a C_0-semigroup and $B : X \times U \to X$ satisfying a Lipschitz condition and being bounded in the sense that $\|B(x,u)\| \lesssim \|x\|\gamma(\|u\|)$ for some $\mathcal{K}$-function γ and all x and u. As indicated above, the property whether a system is ISS or iISS is more subtle when boundary controls are considered and consequently, the involved input operators become unbounded. This also applies for bilinear systems which — in the presence of boundary control — cannot be treated as in the references mentioned above.

In this article we establish the abstract theory to overcome such issues. More precisely, we study infinite-dimensional bilinear control systems of the abstract form

$$\dot{x}(t) = Ax(t) + B_1 F(x(t), u_1(t)) + B_2 u_2(t), \ t \geq 0, \tag{4}$$

where A generates a C_0-semigroup on a Banach space X and B_1 and B_2 are possibly unbounded linear operators defined on Banach spaces $\tilde{X}$ and U_2 respectively. The nonlinearity $F : X \times U_1 \to \tilde{X}$ is assumed to satisfy a Lipschitz condition and to be bounded in the sense that

$$\|F(x,u)\|_{\tilde{X}} \lesssim \|x\|_X \, \|u\|_{U_1} \qquad \forall x \in X, u \in U.$$

In Section 2 we present the abstract framework and prove the main theoretical results. Under weak conditions on the operators B_1 and B_2, we discuss existence of global (mild) solutions to (4) and provide several ISS estimates. Furthermore, we give conditions on B_1 assuring that uniform exponential stability of the semigroup and iISS are equivalent notions for System (4). We continue with an example of infinitely many scalar ODE's in Section 3 which justifies the use of Orlicz spaces in the abstract results from Section 2 The guiding example for this research work is the following bilinearly controlled Fokker–Planck equation with reflective boundary conditions, which has recently appeared in [5, 11],

$$\begin{aligned} \tfrac{\partial \rho}{\partial t}(x,t) &= \Delta\rho(x,t) + u_1(t)\nabla \cdot \Big(\rho(x,t)\nabla\alpha\Big), \\ \rho(x,0) &= \rho_0(x), \\ 0 &= (\nu\nabla\rho + \rho\nabla\alpha)\cdot\vec{n}, \end{aligned}$$

on a bounded domain Ω, where $\alpha : \Omega \to \mathbb{R}$ is sufficiently smooth and ρ_0 is an initial condition in $L^2(\Omega)$. In Section 4 we show that such systems satisfy suitable integral input-to-state estimates and provide a more general class of related examples tractable within the above abstract framework.

2. Input-to-state stability for bilinear systems

2.1. **System class and Notions.** In the following we study infinite-dimensional bilinear control systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 F(x(t), u_1(t)) + B_2 u_2(t), \ t \geq 0, \\ x(0) &= x_0, \end{aligned} \qquad (\Sigma(A, [B_1, B_2], F))$$

where

- X, $\tilde{X}$ and U_1, U_2 are Banach spaces and $x_0 \in X$,
- A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X,
- the input function u_1 is locally integrable function with values in U_1, that is, $u_1 \in L^1_{\text{loc}}(0, \infty; U_1)$ and $u_2 \in L^1_{\text{loc}}(0, \infty; U_2)$,
- the operators B_1 and B_2 are defined on $\tilde{X}$ and U_1 respectively. Both operators map into a space (see below) in which X is densely embedded,
- the nonlinear operator $F : X \times U_1 \to \tilde{X}$ is bounded in the sense that there exists a constant $m > 0$ such that

$$\|F(x, u)\|_{\tilde{X}} \leq m \|x\|_X \|u\|_{U_1} \qquad \forall x \in X, u \in U_1. \tag{5}$$

  and Lipschitz continuous in the first variable on bounded subsets of X, where the Lipschitz constant depends on the U_1-norm of the second argument, that is, for all bounded subsets $X_{\text{b}} \subset X$ there exists a constant $L_{X_{\text{b}}} > 0$, such that

$$\|F(x, u) - F(\tilde{x}, u)\|_{\tilde{X}} \leq L_{X_{\text{b}}} \|u\|_{U_1} \|x - \tilde{x}\|_X \qquad \forall x \in X_{\text{b}}, u \in U_1, \tag{6}$$

- $s \mapsto F(f(s), g(s))$ is measurable for any interval I and measurable functions $f : I \to X$, $g : I \to U_1$,
- we write $\Sigma(A, [0, B_2])$ if $B_1 = 0$ and thus System $\Sigma(A, [B_1, B_2], F)$ is linear.

Before explaining the details on the assumptions on B_1 and B_2 below, we list some examples for functions F and operators that fit our setting.

(a) $\tilde{X} = X$, $U = \mathbb{C}$ and $F(x, u) = xu$,
(b) $\tilde{X} = U = X$, $f \in X^*$, $F(x, u) = f(x)u$,
(c) $\tilde{X} = \mathbb{C}$, $U = X^*$, $F(x, u) = \langle x, u \rangle$.

Let X_{-1} be the completion of X with respect to the norm $\|x\|_{X_{-1}} = \|(\beta - A)^{-1}x\|_X$ for some β in the resolvent set $\rho(A)$ of A.

For a reflexive Banach space, X_{-1} can be identified with $(D(A^*))'$, the continuous dual of $D(A^*)$ with respect to the pivot space X. The operators B_1 and B_2 are assumed to map to X_{-1}, more precisely, $B_1 \in L(\tilde{X}, X_{-1})$ and $B_2 \in L(U_2, X_{-1})$, where $L(X, Y)$ refers to the bounded linear operators from X to Y. Only in the special case that B_1 or B_2 are in $L(\tilde{X}, X)$ or $L(U_2, X)$, we say that the respective operator is bounded. The C_0-semigroup $(T(t))_{t\geq 0}$ extends uniquely to a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} whose generator A_{-1} is the unique extension of A to an operator in $L(X, X_{-1})$, see e.g. [9]. Note that X_{-1} can be viewed as taking the role of a Sobolev space with negative index. With the above considerations we may consider System $\Sigma(A, [B_1, B_2], F)$ on the Banach space X_{-1}. We want to emphasize that our interest is primarily in the situation where B_1 and B_2 are not bounded — something that typically happens if the control enters through point boundary actuation. Note, however, that the assumptions imply that "the unboundedness of B_1 and B_2 is not worse than the one of A" — which particularly means that if $A \in L(X, X)$ then $B_1 \in L(\tilde{X}, X)$ and $B_2 \in L(U_2, X)$.

For zero-inputs u_1 and u_2, the solution theory for System $\Sigma(A, [B_1, B_2], F)$ is fully characterized by the property that A generates a C_0-semigroup as this reduces to solving a linear, homogeneous equation. For non-trivial inputs, the solution concept is a bit more delicate.

More precisely, for given $t_0, t_1 \in [0,\infty)$, $t_0 < t_1$, $x_0 \in X$, $u_1 \in L^1_{loc}(0,\infty;\tilde{X})$ and $u_2 \in L^1_{loc}(0,\infty;U_2)$, a continuous function $x : [t_0, t_1] \to X$ is called a *mild solution of* $\Sigma(A,[B_1,B_2],F)$ *on* $[t_0,t_1]$ if for all $t \in [t_0,t_1]$,

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T_{-1}(t-s)[B_1F(x(s),u_1(s)) + B_2u_2(s)]\,\mathrm{d}s. \tag{7}$$

We say that $x : [0,\infty) \to X$ is a *global mild solution* or a *mild solution on* $[0,\infty)$ of $\Sigma(A,[B_1,B_2],F)$ if $x|_{[0,t_1]}$ is a mild solution on $[0,t_1]$ for every $t_1 > 0$. We stress that existence of a mild solution is non-trivial, even when $u_1 = 0$. In this case, it is easy to see that $x \in C([0,\infty);X_{-1})$, but not necessarily $x(t) \in X$, $t > 0$, without further assumptions on B_2. The existence of a mild solutions to the linear System $\Sigma(A,[0,B_2])$ is closely related to the notion *admissibility* of the operator B_2 for the semigroup $(T(t))_{t\geq 0}$ and various sufficient and necessary conditions are available, see e.g. Proposition 2.4 and [12].

We need the following well-known function classes from Lyapunov theory.

$$\begin{aligned}
\mathcal{K} &= \{\mu \in C(\mathbb{R}_0^+,\mathbb{R}_0^+) \mid \mu(0) = 0, \mu \text{ strictly increasing}\},\\
\mathcal{K}_\infty &= \{\theta \in \mathcal{K} \mid \lim_{x\to\infty} \theta(x) = \infty\},\\
\mathcal{L} &= \{\gamma \in C(\mathbb{R}_0^+,\mathbb{R}_0^+) \mid \gamma \text{ strictly decreasing}, \lim_{t\to\infty} \gamma(t) = 0\},\\
\mathcal{KL} &= \{\beta : (\mathbb{R}_0^+)^2 \to \mathbb{R}_0^+ \mid \beta(\cdot,t) \in \mathcal{K}\ \forall t \geq 0, \beta(s,\cdot) \in \mathcal{L}\ \forall s > 0\}.
\end{aligned}$$

The following concept is central in this work. It originates from works by Sontag [30, 32]. We refer e.g. to [26, 27] for the infinite-dimensional setting.

Definition 2.1. *The system* $\Sigma(A,[B_1,B_2],F)$ *is called*

(i) input-to-state stable (ISS)*, if there exist functions* $\beta \in \mathcal{KL}$*,* $\mu_1,\mu_2 \in \mathcal{K}_\infty$ *such that for every* $x_0 \in X$*,* $u_1 \in L^\infty(0,\infty;U_1)$ *and* $u_2 \in L^\infty(0,\infty;U_2)$ *there exists a unique global mild solution* x *of* $\Sigma(A,[B_1,B_2],F)$ *and for every* $t \geq 0$

$$\|x(t)\| \leq \beta(\|x_0\|,t) + \mu_1(\|u_1\|_{L^\infty(0,t;U_1)}) + \mu_2(\|u_2\|_{L^\infty(0,t;U_2)});$$

(ii) integral input-to-state stable (iISS or integral ISS)*, if there exist functions* $\beta \in \mathcal{KL}$*,* $\theta_1,\theta_2 \in \mathcal{K}_\infty$ *and* $\mu_1,\mu_2 \in \mathcal{K}$ *such that for every* $x_0 \in X$*,* $u_1 \in L^\infty(0,\infty;U_2)$ *and* $u_2 \in L^\infty(0,\infty;U_2)$ *there exists a unique global mild solution* x *of* $\Sigma(A,[B_1,B_2],F)$ *and for every* $t \geq 0$

$$\|x(t)\| \leq \beta(\|x_0\|,t) + \theta_1\left(\int_0^t \mu_1(\|u_1(s)\|)\,\mathrm{d}s\right) + \theta_2\left(\int_0^t \mu_2(\|u_2(s)\|)\,\mathrm{d}s\right).$$

One may define some mixed type of these definitions like (ISS,iISS) (and (iISS,ISS)), in the sense that one has an ISS-estimate for u_1 *and some integral-ISS-estimate for* u_2 *(and vice versa).*

Although the terms involving u_1 and u_2 on the right-hand-side of the integral ISS estimate do not define norms in general, the following function spaces were shown to be naturally linked to integral ISS [12]. In this context we briefly introduce the Orlicz space $E_\Phi(I;Y)$ for an interval $I \subset \mathbb{R}$ and a Banach space Y. For more details on Orlicz spaces we refer to [1, 20, 19].

Let $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a *Young function*, i.e. Φ is continuous, increasing, convex with $\lim_{s\to 0} \frac{\Phi(s)}{s} = 0$ and $\lim_{s\to\infty} \frac{\Phi(s)}{s} = \infty$. Denote the set of Bochner-measurable functions $u : I \to Y$ for which there exists a constant $k > 0$ such that $\Phi(k\|u(\cdot)\|)$

integrable by $L_\Phi(I;Y)$. This vector space becomes a Banach space when equipped with the norm

$$\|u\|_{L_\Phi(I,Y)} = \inf\left\{k>0 \,\middle|\, \int_I \Phi\left(\frac{\|u(s)\|}{k}\right) \mathrm{d}s \le 1\right\}. \tag{8}$$

Despite the fact that $L_\Phi(I;Y)$ is typically referred to as "Orlicz space" in the literature, we prefer to call

$$E_\Phi(I;Y) = \overline{\{u \in L^\infty(I;Y) \mid \operatorname{ess\,supp} u \text{ is bounded}\}}^{\|\cdot\|_{L_\Phi(I;Y)}}$$

the *Orlicz space* associated with the Young function Φ. Note that $u \in E_\Phi(I;Y)$ implies that $\Phi \circ \|u(\cdot)\|$ is integrable. Typical examples of Orlicz spaces are L^p-spaces; for $\Phi(t) = t^p$ with $p \in (1,\infty)$ it holds that $E_\Phi(I;Y) = L^p(I;Y)$.
An important property of Φ in the characterization of Orlicz spaces is the Δ_2 condition. A Young function Φ is said to satisfy the Δ_2 *condition* if there exist $K>0$ and $s_0 \ge 0$ such that

$$\Phi(2s) \le K\Phi(s), \qquad s \ge s_0.$$

In particular note that $E_\Phi(I;Y) = L_\Phi(I;Y)$ if and only if Φ satisfies the Δ_2 condition. The Young functions $\Phi(s) = s^p$, $p \in (1,\infty)$, share this property, leading to $E_\Phi(I;Y) = L_\Phi(I;Y) = L^p(I;Y)$. For a Young function Φ let $\tilde{\Phi}$ denote the *complementary Young function*, a notion which can be seen as the Orlicz space analog of Hölder-conjugates. In fact, for $\Phi(s) = \frac{s^p}{p}$ it holds that $\tilde{\Phi}(s) = \frac{s^q}{q}$ for $p,q \in (1,\infty)$ with $\frac{1}{p}+\frac{1}{q} = 1$. As for L^p spaces, an equivalent norm to $\|\cdot\|_{L_\Phi(I;Y)}$ is given by

$$\|u\|_{\Phi,(I;Y)} = \sup\left\{\int_I \|u(s)\||v(s)|\,\mathrm{d}s \,\middle|\, v \text{ measurable}, \int_I \tilde{\Phi}(|v(s)|)\,\mathrm{d}s \le 1\right\}. \tag{9}$$

Furthermore, for a Young functions Φ and its complementary Young function $\tilde{\Phi}$ the following generalized Hölder inequality

$$\int_I \|u(s)\|\|v(s)\|\mathrm{d}s \le 2\|u\|_{L_\Phi}\|v\|_{L_{\tilde{\Phi}}}. \tag{10}$$

holds. This also implies the continuity of the embeddings

$$L^\infty(I;Y) \hookrightarrow L_\Phi(I;Y) \hookrightarrow L^1(I;Y)$$

if I is bounded. Although L^1 is not an Orlicz space, we will explicitly allow for $\Phi(t) = t$ in our notation referring to $E_\Phi(I;Y) = L^1(I;Y)$. Note that the definition of the norm (8) is indeed consistent with the L^1-norm and that Φ satisfies the Δ_2 condition. However, we will not define a "complementary Young function" for this particular Φ.
An essential property of Orlicz spaces is the absolute continuity of the E_Φ norm with respect to the length of the interval I (see e.g. [20, Thm. 3.15.6]), this is for $u \in E_\Phi(I;Y)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for each intervall I holds

$$\lambda(I) < \delta \quad \Rightarrow \quad \|u\|_{E_\Phi(I;Y)} < \varepsilon,$$

where λ referes to the Lebesgue-measure on $\mathbb{R}$.

Definition 2.2. *Let $(T(t))_{t\ge0}$ be a C_0-semigroup.*

(i) We say that $(T(t))_{t\ge0}$ is of type (M,ω) *if $M \ge 1$ and $\omega \in \mathbb{R}$ are such that*

$$\|T(t)\| \le M\mathrm{e}^{-\omega t}, \qquad t \ge 0. \tag{11}$$

(ii) We say that $(T(t))_{t\ge0}$ is (uniformly) exponentially stable *if $(T(t))_{t\ge0}$ is of type (M,ω) for some $\omega > 0$.*

(iii) *Let $Z = E_\Phi$ or $Z = L^\infty$. An operator $B \in L(U, X_{-1})$ is called Z-*admissible *for $(T(t))_{t\geq 0}$, if for every $t > 0$ and $u \in Z(0,t;U)$ it holds that*

$$\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \in X.$$

We will neglect the reference to $(T(t))_{t\geq 0}$ if this is clear from the context.

Recall that every C_0-semigroup is of type (M, ω) for some $M \geq 1$ and $\omega \in \mathbb{R}$. Note that any bounded operator B is Z-admissible for all Z considered above.

Remark 2.3. *Let $B \in L(U, X_{-1})$ be Z-admissible for $(T(t))_{t\geq 0}$ with $Z = E_\Phi$ or $Z = L^\infty$. Then for any $t > 0$ there exists a minimal constant $C_{t,B} > 0$ such that*

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \right\| \leq C_{t,B}\|u\|_{Z(0,t;U)}, \quad u \in Z(0,t;U). \tag{12}$$

This is a consequence of the closed graph theorem. Also note that B is Z-admissible for $(\mathrm{e}^{\delta t}T(t))_{t\geq 0}$ for any $\delta \in \mathbb{R}$. Furthermore, the function $t \mapsto C_{t,B}$ is non-decreasing and, if $(T(t))_{t\geq 0}$ is exponentially stable, even bounded, that is, $C_B := \sup_{t\geq 0} C_{t,B} < \infty$.

The following result clarifies on the relation between admissibility and (integral) ISS. Note in particular that the existence of mild solutions for E_Φ-admissible operators B_2 is based on the absolut continuity of the Orlicz norm with respect to the length of the interval and the strong continuity of the shift-semigroup on $E_\Phi(I;Y)$ for any interval I and any Banach space Y. The latter can be proven by similar methods one uses to prove the strong continuity of the Shift-semigroup on $L^p(I;Y)$.

Proposition 2.4 (Prop. 2.10 & Thm. 3.1 in [12])**.** *Let A generate the C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B_2 \in L(U_2, X_{-1})$.*

(i) *If B_2 is E_Φ-admissible, then for every $x_0 \in X$ and $u_2 \in E_{\Phi,\mathrm{loc}}(0,\infty;U_2)$ there exists a unique global mild solution x of System $\Sigma(A,[0,B_2])$, which is given by (7) with $B_1 = 0$.*

(ii) *System $\Sigma(A,[0,B_2])$ is ISS if and only if $(T(t))_{t\geq 0}$ is exponentially stable and B_2 is L^∞-admissible.*

(iii) *$\Sigma(A,[0,B_2])$ is iISS if and only if $(T(t))_{t\geq 0}$ is exponentially stable and B_2 is E_Φ-admissible for some Young function Φ.*

2.2. **Main results.** Whether ISS implies iISS for System $\Sigma(A,[0,B])$ is still an open question. This is true for B bounded, see e.g. [12, Prop. 2.14] or [24]. However, various conditions on A and the input spaces U are available under which iISS and ISS are equivalent [14] in the case of boundary control. In contrast to linear systems, the existence of mild solutions is less clear for bilinear control systems of the form $\Sigma(A,[B_1,B_2],F)$.

Sontag [32] showed that finite-dimensional bilinear systems are hardly ever ISS, but iISS if and only if the semigroup is exponentially stable. In [24] it was shown that this result generalizes to infinite-dimensional bilinear systems provided that B_1 and B_2 are bounded operators and $\tilde{X} = X$. The following results give sufficient conditions for ISS, iISS and some combination of ISS and iISS of $\Sigma(A,[B_1,B_2],F)$. We start with a result on existence of local solutions to $\Sigma(A,[B_1,B_2],F)$. The proof involves typical arguments in the context of mild solutions for semilinear equations.

Proposition 2.5. *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Suppose that $B_1 \in L(\tilde{X}, X_{-1})$ is E_Φ-admissible and that $B_2 \in L(U_2, X_{-1})$ is E_Ψ-admissible. Then for every $t_0 \geq 0$, $x_0 \in X$, $u_1 \in E_\Phi(0,\infty;U_1)$ and $u_2 \in E_\Psi(0,\infty;U_2)$ there exists $t_1 > t_0$ such that System $\Sigma(A,[B_1,B_2],F)$ possesses a unique mild solution x on $[t_0,t_1]$.*

Moreover, if $t_{max} > t_0$ denotes the supremum of all $t_1 > t_0$ such that System $\Sigma(A, [B_1, B_2], F)$ has a unique mild solution x on $[t_0, t_1]$, then $t_{max} < \infty$ implies that

$$\lim_{t\to t_{max}} \|x(t)\| = \infty.$$

Proof. We first show that for every $t_0 \geq 0, x_0 \in X$, $u_1 \in E_\Phi(0, \infty; U_1)$ and $u_2 \in E_\Psi(0, \infty; U_2)$ there exists $t_1 > t_0$ such that System $\Sigma(A, [B_1, B_2], F)$ possesses a unique mild solution on $[t_0, t_1]$ with initial condition x_0 and input functions u_1 and u_2. Moreover, we show that $t_1 = t_0 + \delta$ can be chosen such that δ is independent for any bounded sets of initial data x_0 and t_0. Let $T > 0$, $r > 0$, $u_1 \in E_\Phi(0, \infty; U_1)$ and $u_2 \in E_\Psi(0, \infty; U_2)$ be arbitrarily. We first recall the following property of Orlicz spaces. For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\max\{\|u_1\|_{E_\Phi(t,t+\delta;U_1)}, \|u_2\|_{E_\Psi(t,t+\delta;U_2)}\} < \varepsilon, \qquad \forall t \geq 0, \tag{13}$$

see e.g. [20, Thm. 3.15.6]. Let $t_0 \in [0, T]$, $t_1 > t_0$ and $x_0 \in K_r(0) = \{x \in X : \|x\| \leq r\}$ and define the mapping

$$\Phi_{t_0,t_1} : C([t_0, t_1]; X) \to C([t_0, t_1]; X)$$
$$(\Phi_{t_0,t_1}(x))(t) := T(t - t_0)x_0 + \int_{t_0}^{t} T_{-1}(t - s)[B_1F(x(s), u_1(s)) + B_2u_2(s)] \,\mathrm{d}s.$$

The strong continuity of $(T(t))_{t\geq 0}$ and Proposition 2.4 imply that Φ_{t_0,t_1} is well-defined, that is, $\Phi_{t_0,t_1}(x) \in C([t_0, t_1]; X)$ for every $x \in C([t_0, t_1]; X)$. Note that we applied Proposition 2.4 twice: To System $\Sigma(A, [0, B_2])$ with input u_2 and to System $\Sigma(A, [0, B_1])$ with input $F(x(\cdot), u_1(\cdot))$, where we set u_1, u_2, x zero on $(0, t_0)$.

Let $M, \omega > 0$ be such that $\|T(t)\| \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$ and choose $k = 4Mr+2M$. Set

$$M_k(t_0, t_1) := \{x \in C([t_0, t_1]; X) \mid \|x\|_{C([t_0,t_1];X)} \leq k\}.$$

We will show next that t_1 can be chosen such that Φ_{t_0,t_1} maps $M_k(t_0, t_1)$ to $M_k(t_0, t_1)$ and is contractive on this set. Let C_{t,B_1} and C_{t,B_2} refer to the admissibility constants such that (12) holds for B_1 and B_2 which can be chosen non-decreasing in t. Let m be the boundedness constant of F from (5) and let $L_{K_k(0)}$ be the Lipschitz constant of F such that (6) holds for the bounded set $X_b = \{x(t) \mid x \in M_k(t_0, t_1), t \in [t_0, t_1]\} \subset X$ which is equal to $K_k(0) = \{x \in X : \|x\| \leq k\}$. Now, let $t_1 = t_0 + \delta$ with $\delta \in (0, 1)$ be chosen such that for all $t_0 \in [0, T]$,

(i) $\mathrm{e}^{\omega(t_1-t_0)} = \mathrm{e}^{\omega\delta} \leq 2$,
(ii) $mC_{T+1,B_1}\|u_1\|_{E_\Phi(t_0,t_0+\delta,U_1)} \leq \frac{1}{2}$,
(iii) $C_{T,B_2}\|u_2\|_{E_\Psi(t_0,t_0+\delta;U_2)} \leq M$ and
(iv) $C_{T+\delta,B_1}L_{K_k(0)}\|u_1\|_{E_\Phi(t_0,t_0+\delta;U_1)} < 1$

holds, where we used (13) in (ii)-(vi). Note that apart from the parameters of the operators B_1, B_2, A, F, the choice of δ only depends on r and T, where the r-dependence of δ arises from the r-dependence of k. It follows that for all $t_0 \in [0, T]$, $x \in M_k(t_0, t_1)$ and $x_0 \in K_r(0)$

$$\begin{aligned}
&\|\Phi_{t_0,t_1}(x)\|_{C([t_0,t_1];X)} \\
&\leq M\mathrm{e}^{\omega(t_1-t_0)}\|x_0\| + C_{t_1,B_1}\|F(x, u_1)\|_{E_\Phi(t_0,t_1;\tilde{X})} + C_{t_1,B_2}\|u_2\|_{E_\Psi(t_0,t_1;U_2)} \\
&\leq 2M\|x_0\| + mC_{t_1,B_1}\|u_1\|_{E_\Phi(t_0,t_1,U_1)}\|x\|_{C([t_0,t_1];X)} + M \\
&\leq k,
\end{aligned}$$

where we used admissibility in the first inequality and (5) in the second inequality as well as the monotonicity of the Orlicz norm in both estimates. Hence, Φ_{t_0,t_1} maps $M_k(t_0, t_1)$ to $M_k(t_0, t_1)$. The contractivity follows since

$$\|\Phi_{t_0,t_1}(x) - \Phi_{t_0,t_1}(\tilde{x})\|_{C([t_0,t_1];X)}$$

$$
\begin{aligned}
&\leq \sup_{t\in[t_0,t_1]} \left\| \int_{t_0}^t T(t-s)B_1[F(x(s),u_1(s)) - F(\tilde{x}(s),u_1(s))]\,\mathrm{d}s \right\| \\
&\leq C_{t_1,B_1} L_{K_k(0)} \|u_1\|_{E_\Phi(t_0,t_1;U_1)} \|x-\tilde{x}\|_{C([t_0,t_1];X)},
\end{aligned}
$$

where we used again admissibility, the Lipschitz property of F and the monotonicity of the Orlicz norm. By Banach's fixed-point theorem, we conclude that System $\Sigma(A,[B_1,B_2],F)$ possesses a unique mild solution on $[t_0,t_1]$ with initial condition x_0 and input functions u_1 and u_2.
Now let $t_{\max}$ be the supremum of all t_1 such that there exists a mild solution x of $\Sigma(A,[B_1,B_2],F)$ on $[t_0,t_1]$ for every $t_1 < t_{\max}$, where $x_0 \in X$, $u_1 \in E_\Phi(0,\infty;U_1)$ and $u_2 \in E_\Psi(0,\infty;U_2)$ are given. Suppose that $t_{\max}$ is finite. We will show, that then $\lim_{t\to t_{\max}} \|x(t)\| = \infty$. If this is not the case, we have

$$
r = \sup_{t\in[t_0,t_{\max}]} \|x(t)\| < \infty.
$$

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers converging to $t_{\max}$ from below. Since $t_n \in [0,t_{\max}]$ and $\|x(t_n)\| \leq r$ for all $n\in\mathbb{N}$, there exists $\delta > 0$ independent of $n \in \mathbb{N}$ such that the equation

$$
\begin{aligned}
&\dot{y}(t) = Ay(t) + B_1F(y(t),u_1(t)) + B_2u_2(t),\\
&y(t_n) = x(t_n).
\end{aligned}
$$

has a mild solution y on $[t_n,t_n+\delta]$. Therefore, we can extend x by $x(t) = y(t)$, $t \in (t_n,t_n+\delta]$, to a solution of $\Sigma(A,[B_1,B_2],F)$ on $[t_0,t_n+\delta]$. This contradicts the maximality of $t_{\max}$ and hence, x has to be unbounded in $t_{\max}$. □

Theorem 2.6. *Let A generate a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Suppose that there exist Young functions Φ and Ψ such that*

- *$B_1 \in L(\tilde{X},X_{-1})$ is E_Φ-admissible, and*
- *$B_2 \in L(U_2,X_{-1})$ is E_Ψ-admissible.*

Then for all $x_0\in X$, $u_1 \in E_\Phi(0,\infty;U_1)$, $u_2 \in E_\Psi(0,\infty;U_2)$ there exists a unique global mild solution x of $\Sigma(A,[B_1,B_2],F)$ and it holds that

$$
\begin{aligned}
\|x(t)\| &\leq \beta(\|x_0\|,t) + \gamma_1(C_{t,B_1}\|u_1\|_{E_\Phi(0,t)}) + \gamma_2(C_{t,B_2}\mathrm{e}^{-\frac{\omega}{2}t}\|\mathrm{e}^{\frac{\omega}{2}\cdot}u_2\|_{E_\Psi(0,t)}), \qquad (14)\\
&\leq \beta(\|x_0\|,t) + \gamma_1(C_{t,B_1}\|u_1\|_{E_\Phi(0,t)}) + \gamma_2(C_{t,B_2}\sup_{r\in[0,t]}\mathrm{e}^{-\frac{\omega}{2}r}\|u_2\|_{E_\Psi(0,t)}),
\end{aligned}
$$

for all $t\geq 0$, where $\|\cdot\|_{E_\Phi(I)} = \|\cdot\|_{E_\Phi(I;U_1)}$ and $\|\cdot\|_{E_\Psi(I)} = \|\cdot\|_{E_\Psi(I;U_1)}$ and

$$
\begin{aligned}
\beta(s,t) &= M\mathrm{e}^{-\omega t}s + \tfrac{1}{2}M^2\mathrm{e}^{-\omega t}s^2 \sup_{r\in[0,t]} \mathrm{e}^{-\omega r},\\
\gamma_1(s) &= 4m^2s^2\mathrm{e}^{4ms},\\
\gamma_2(s) &= s + \tfrac{1}{2}s^2.
\end{aligned}
$$

Here, (M,ω) denotes the type of $(T(t))_{t\geq0}$, C_{t,B_i}, $i=1,2$, is the admissibility constant of B_i with respect to the semigroup $\left(\mathrm{e}^{\frac{\omega}{2}t}T(t)\right)_{t\geq 0}$ (see (12)), and m is the bound of F (see (5)).

Proof. By Remark 2.3 there exist $C_{t,B_1}, C_{t,B_2} > 0$ such that for every $t\geq 0$, $y \in E_\Phi(0,\infty;\tilde{X})$ and $\tilde{y} \in E_\Psi(0,\infty;U_2)$ we have

$$
\left\| \int_0^t \mathrm{e}^{\frac{\omega}{2}(t-s)} T_{-1}(t-s)B_1y(s)\,\mathrm{d}s \right\| \leq C_{t,B_1}\|y\|_{E_\Phi(0,t;\tilde{X})}
$$

and

$$
\left\| \int_0^t \mathrm{e}^{\frac{\omega}{2}(t-s)} T_{-1}(t-s)B_2\tilde{y}(s)\,\mathrm{d}s \right\| \leq C_{t,B_2}\|\tilde{y}\|_{E_\Psi(0,t;U_2)}.
$$

Let $x_0 \in X$, $u_1 \in E_\Phi(0,\infty;U_1)$ and $u_2 \in E_\Psi(0,\infty;U_2)$ and let $t_{\max}$ be the supremum over all t_1 such that $\Sigma(A,[B_1,B_2],F)$ possesses a unique x mild solution on $[0,t_1]$. Proposition 2.5 yields $t_{\max} > 0$. For $t \in [0,t_{\max})$ it follows that

$$\begin{aligned}
&\|x(t)\| \\
&= \left\| T(t)x_0 + \int_0^t T_{-1}(t-s)B_1F(x(s),u_1(s))\,\mathrm{d}s + \int_0^t T_{-1}(t-s)B_2u_2(s)\,\mathrm{d}s \right\| \\
&\le \|T(t)x_0\| + \mathrm{e}^{-\frac{\omega}{2}t}\left\| \int_0^t \mathrm{e}^{\frac{\omega}{2}(t-s)}T_{-1}(t-s)B_1(\mathrm{e}^{\frac{\omega}{2}s}F(x(s),u_1(s)))\,\mathrm{d}s \right\| \\
&\qquad\qquad + \mathrm{e}^{-\frac{\omega}{2}t}\left\| \int_0^t \mathrm{e}^{\frac{\omega}{2}(t-s)}T_{-1}(t-s)B_2\mathrm{e}^{\frac{\omega}{2}s}u_2(s)\,\mathrm{d}s \right\| \\
&\le M\mathrm{e}^{-\omega t}\|x_0\| + C_{t,B_1}\mathrm{e}^{-\frac{\omega}{2}t}\|\mathrm{e}^{\frac{\omega}{2}\cdot}F(x(\cdot),u_1(\cdot))\|_{E_\Phi(0,t;\tilde{X})} + C_{\omega,u_2,t},
\end{aligned} \tag{15}$$

where $C_{\omega,u_2,t} = C_{t,B_2}\mathrm{e}^{-\frac{\omega}{2}t}\left\|\mathrm{e}^{\frac{\omega}{2}\cdot}u_2\right\|_{E_\Psi(0,t;U_2)}$. The $\|\cdot\|_{E_\Phi}$-norm in the second term can be estimated by the boundedness of F,

$$\|\mathrm{e}^{\frac{\omega}{2}\cdot}F(x(\cdot),u_1(\cdot))\|_{E_\Phi(0,t;\tilde{X})} \le m\left\|\,\|u_1(\cdot)\|\,\mathrm{e}^{\frac{\omega}{2}\cdot}\,\|x(\cdot)\|\,\right\|_{E_\Phi(0,t)}.$$

Provided that $E_\Phi \neq L^1$, we can further pass over to the equivalent norm on E_Φ given in (9). Therefore, for $\varepsilon > 0$ there exists a function $g \in L_{\tilde{\Phi}}(0,t)$ with $\|g\|_{L_{\tilde{\Phi}}(0,t)} \le 1$ such that

$$\left\|\|u_1(\cdot)\|\mathrm{e}^{\frac{\omega}{2}\cdot}\|x(\cdot)\|\right\|_{E_\Phi(0,t)} \le \int_0^t \|u_1(s)\|\,|g(s)|\left(\mathrm{e}^{\frac{\omega}{2}s}\|x(s)\|\right)\,\mathrm{d}s + \varepsilon.$$

In the case that $E_\Phi = L^1$, the above estimate holds trivially with the constant function $g = 1$. Hence, by combining this with (15) gives

$$\begin{aligned}
\mathrm{e}^{\frac{\omega}{2}t}\|x(t)\| \quad &\le \quad M\mathrm{e}^{-\frac{\omega}{2}t}\|x_0\| + mC_{t,B_1}\varepsilon + \mathrm{e}^{\frac{\omega}{2}t}C_{\omega,u_2,t} \\
&\quad + mC_{t,B_1}\int_0^t \|u_1(s)\|\,|g(s)|\left(\mathrm{e}^{\frac{\omega}{2}s}\|x(s)\|\right)\,\mathrm{d}s.
\end{aligned}$$

Setting $\alpha(t) := M\mathrm{e}^{-\frac{\omega}{2}t}\|x_0\| + mC_{t,B_1}\varepsilon + \mathrm{e}^{\frac{\omega}{2}t}C_{\omega,u_2,t}$, Gronwall's inequality implies that

$$\begin{aligned}
\mathrm{e}^{\frac{\omega}{2}t}\|x(t)\| &\le \alpha(t) + mC_{t,B_1}\int_0^t \alpha(s)\|u_1(s)\|\,|g(s)|\mathrm{e}^{\left(mC_{t,B_1}\int_s^t \|u_1(r)\|\,|g(r)|\,\mathrm{d}r\right)}\mathrm{d}s \\
&\le \alpha(t) + \left(M\|x_0\| \sup_{r\in[0,t]} \mathrm{e}^{-\frac{\omega}{2}r} + mC_{t,B_1}\varepsilon + \mathrm{e}^{\frac{\omega}{2}t}C_{\omega,u_2,t} \right) \\
&\qquad \cdot 2mC_{t,B_1}\|u_1\|_{E_\Phi(0,t;U_1)}\mathrm{e}^{2mC_{t,B_1}\|u_1\|_{E_\Phi(0,t;U_1)}},
\end{aligned}$$

where we used the generalized Hölder inequality (10) in the case that $E_\Phi \neq L^1$. Thus, by letting ε tend to 0, multiplying with $\mathrm{e}^{-\frac{\omega}{2}t}$ and using $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ for $a,b \in \mathbb{R}$, we obtain

$$\begin{aligned}
\|x(t)\| \quad &\le \quad M\mathrm{e}^{-\omega t}\|x_0\| + \tfrac{1}{2}M^2\mathrm{e}^{-\omega t}\sup_{r\in[0,t]}\mathrm{e}^{-\omega r}\|x_0\|^2 \\
&\quad + 4m^2C_{t,B_1}^2\|u_1\|_{E_\Phi(0,t;U_1)}^2\mathrm{e}^{4mC_{t,B_1}\|u_1\|_{E_\Phi(0,t;U_1)}} \\
&\quad + C_{\omega,u_2,t} + \tfrac{1}{2}C_{\omega,u_2,t}^2.
\end{aligned}$$

This shows the first estimate in (14). The second inequality readily follows by monotonicity of the Orlicz norm,

$$\|\mathrm{e}^{\frac{\omega}{2}\cdot}u_2\|_{E_\Psi(0,t;U_2)} \le \sup_{r\in[0,t]}\mathrm{e}^{\frac{\omega}{2}r}\|u_2\|_{E_\Psi(0,t;U_2)}.$$

Moreover, the mild solution exists on $[0,\infty)$. Indeed, if this is not the case, we have $t_{\max}<\infty$ and Proposition 2.5 implies that x is unbounded in $t_{\max}$. This contradicts (14) since the right-hand-side is uniformly bounded in t on finite intervals $[0,t_{\max})$. □

Remark 2.7. *A similar result for the existence of the unique global mild solution in Proposition 2.5 were proved under slightly stronger conditions in* [4] *for L^p-admissible B_1, scalar-valued inputs u_1, $F(x,u_1)=u_1x$ and $B_2=0$. Our condition is more natural as the same condition guarantees the existence of continuous (and unique) global mild solutions of the linear systems $\Sigma(A,[0,B_1])$ and $\Sigma(A,[0,B_2])$, see Proposition 2.4.*

If, additionally, the semigroup is exponentially stable, then uniform estimates can be given for the terms depending on u_1 and u_2 in (14).

Corollary 2.8. *If in addition to the assumptions of Theorem 2.6, the semigroup $(T(t))_{t\geq0}$ is exponentially stable, i.e. $(T(t))_{t\geq0}$ is of type (M,ω) with $M\geq1$ and $\omega>0$, then for all $x_0\in X$, $u_1\in E_\Phi(0,\infty;U_1)$ and $u_2\in E_\Psi(0,\infty;U_2)$ the unique global mild solution x of $\Sigma(A,[B_1,B_2],F)$ satisfies*

$$\|x(t)\|\leq\beta(\|x_0\|,t)+\gamma_1\left(C_{B_1}\|u_1\|_{E_\Phi(0,t;U_1)}\right)+\gamma_2\left(C_{B_2}\|u_2\|_{E_\Psi(0,t;U_2)}\right),\tag{16}$$

for all $t\geq0$, with $C_{B_i}:=\sup_{t>0}C_{t,B_i}<\infty$, and $\beta,\gamma_1,\gamma_2,C_{t,B_i}$, $i=1,2$, as in Theorem 2.6. Moreover, $\beta\in\mathcal{KL}$ and $\gamma_1,\gamma_2\in\mathcal{K}_\infty$.

Proof. This is a direct consequence of Theorem 2.6. Indeed, Remark 2.3 implies that C_{t,B_1} and C_{t,B_2} are uniformly bounded in t since the semigroup $\left(\mathrm{e}^{\frac{\omega}{2}t}T(t)\right)_{t\geq0}$ is exponentially stable. Hence, (16) and that $\beta\in\mathcal{KL},\gamma_1,\gamma_2\in\mathcal{K}_\infty$ follow by (14) and since $\sup_{r\in[0,t]}\mathrm{e}^{-\frac{\omega}{2}r}=1$ if $\omega>0$. □

Remark 2.9. *In the situation of Theorem 2.6 note that Estimate* (14) *for $B_1=0$ is not optimal regarding the dependence on the norm of u_2. In fact, for the linear system $\Sigma(A,[0,B_2])$, we have by Remark 2.3 that*

$$\|x(t)\|\leq M\mathrm{e}^{-\omega t}\|x_0\|+C_{B_2}\|u_2\|_{E_\Psi(0,t;U_2)}.$$

Corollary 2.8 has the following variant.

Corollary 2.10. *Under the assumptions of Corollary 2.8 there exists some constant $C>0$, $\theta_2\in\mathcal{K}_\infty$ and $\mu_2\in\mathcal{K}$ such that the global mild solution x of $\Sigma(A,[B_1,B_2],F)$ satisfies*

$$\|x(t)\|\leq\beta(\|x_0\|,t)+\gamma_1(C_{B_1}\|u_1\|_{E_\Phi(0,t;U_1)})+\gamma_2(C\|u_2\|_{L^\infty(0,t;U_2)})$$

and

$$\|x(t)\|\leq\beta(\|x_0\|,t)+\gamma_1(C_{B_1}\|u_1\|_{E_\Phi(0,t;U_1)})+\theta_2\left(\int_0^t\mu_2(\|u_2(s)\|)\,\mathrm{d}s\right)\tag{17}$$

for all $t>0$, $x_0\in X$, $u_1\in E_\Phi(0,\infty;U_1)$, $u_2\in L^\infty(0,\infty;U_2)$, where β,γ_1,γ_2 defined in Theorem 2.6.

Proof. The first inequality follows from Theorem 2.6, (14), by realizing that there exists a constant $C>0$ such that

$$\mathrm{e}^{-\frac{\omega}{2}t}\|\mathrm{e}^{\frac{\omega}{2}\cdot}u_2\|_{E_\Psi(0,t;U_2)}\leq C\|u_2\|_{L^\infty(0,t;U_2)},\tag{18}$$

for all $u_2\in L^\infty(0,\infty;U_2)$ and $t>0$. To see this, let $\epsilon>0$ such that $\Psi(x)\leq x$ for all $x\in(0,\delta)$, which exists by the property that $\lim_{s\to0}\frac{\Psi(s)}{s}=0$. Therefore, choosing $C=\max\{\frac{1}{\epsilon},\frac{2}{\omega}\}$,

$$\int_0^t\Psi\left(C^{-1}\mathrm{e}^{-\frac{\omega}{2}s}\right)\mathrm{d}s\leq\int_0^tC^{-1}\mathrm{e}^{-\frac{\omega}{2}s}\mathrm{d}s\leq\frac{2}{C\omega}\leq1.$$

This implies that

$$\int_0^t \Psi\left(\frac{\mathrm{e}^{\frac{\omega}{2}s}\|u(s)\|}{C\mathrm{e}^{\frac{\omega}{2}t}\|u\|_{L^\infty(0,t;U_2)}}\right)\mathrm{d}s \leq \int_0^t \Psi\left(C^{-1}\mathrm{e}^{\frac{\omega}{2}(s-t)}\right)\mathrm{d}s \leq 1,$$

which yields (18) by the definition of the E_Ψ-norm. To show that (17) holds, note that by [12, Theorem 3.1] it follows that $\Sigma(A, [0\ B_2])$ is integral ISS. Using the respective estimates for the input function u_2 in (15) in the proof of Theorem 2.6 instead of the previously used E_Ψ-admissibility yields the adapted estimates for $u \in L^\infty(0,\infty;U_2)$. □

Remark 2.11. *This can be seen as some type of ISS and iISS in the linear part of $\Sigma(A,[B_1,B_2],F)$. One cannot expect such an ISS result for u_1 as the trivial finite-dimensional example $\dot{x} = -x + u_1 x$ shows.*

We now ask for conditions which guarantee iISS for $\Sigma(A,[B_1,B_2],F)$.

Corollary 2.12. *Under the assumptions of Corollary 2.8 the following statements hold.*

1. *If Φ satisfies the Δ_2 condition, then there exist $\beta \in \mathcal{KL}$, $\gamma_2, \theta_1 \in \mathcal{K}_\infty$ and $p \in (1,\infty)$ such that the unique global mild solution x of $\Sigma(A,[B_1,B_2],F)$ satisfies*

$$\|x(t)\| \leq \beta(\|x_0\|,t) + \theta_1\left(\int_0^t \|u_1(s)\|^p\,\mathrm{d}s\right) + \gamma_2\left(\|u_2\|_{E_\Psi(0,t;U_2)}\right)$$

for every $t \geq 0$, $u_1 \in L^p(0,\infty;U_1)$, and $u_2 \in E_\Psi(0,\infty;U_2)$. Moreover,

$$\|x(t)\| \leq \beta(\|x_0\|,t) + \theta_1\left(\int_0^t \|u_1(s)\|^p\,\mathrm{d}s\right) + \gamma_2(\|u_2\|_{L^\infty(0,t;U_2)})$$

and

$$\|x(t)\| \leq \beta(\|x_0\|,t) + \theta_1\left(\int_0^t \|u_1(s)\|^p\,\mathrm{d}s\right) + \theta_2\left(\int_0^t \mu_2(\|u_2(s)\|)\,\mathrm{d}s\right) \tag{19}$$

for all $t > 0$, $x_0 \in X$, $u_1 \in L^p(0,\infty;U_1)$ and $u_2 \in L^\infty(0,\infty;U_2)$, and some $\theta_2 \in \mathcal{K}_\infty$ and $\mu_2 \in \mathcal{K}$.
   In particular, $\Sigma(A,[B_1,B_2],F)$ is iISS.
2. *If Ψ satisfies the Δ_2 condition, then there exist $\beta \in \mathcal{KL}$, $\gamma_1, \theta_2 \in \mathcal{K}_\infty$ and $\mu_2 \in \mathcal{K}$ such that for every $t \geq 0$, $u_1 \in E_\Phi(0,\infty;U_1)$ and $u_2 \in E_\Psi(0,\infty;U_2)$ the unique global mild solution of $\Sigma(A,[B_1,B_2],F)$ satisfies Estimate* (17).
3. *If Φ and Ψ satisfy the Δ_2 condition, then there exist $\beta \in \mathcal{KL}$, $\theta_1, \theta_2 \in \mathcal{K}_\infty$, $p \in (1,\infty)$ and $\mu_2 \in \mathcal{K}$ such that for every $t \geq 0$, $u_1 \in L^p(1,\infty;U_1)$ and $u_2 \in E_\Psi(0,\infty;U_2)$ the unique global mild solution of $\Sigma(A,[B_1,B_2],F)$ satisfies Estimate* (19). *In particular, $\Sigma(A,[B_1,B_2],F)$ is iISS.*

Proof.
1. If Φ satisfies the Δ_2 condition, then there exists $p \in (1,\infty)$ such that the embedding $L^p(0,t;U_1) \hookrightarrow E_\Phi(0,t;U_1)$ is well-defined and continuous, see [19, p. 24-25] together with [20, Section 3.17]. Hence, B_1 is L^p-admissible and Corollary 2.8 applied with L^p-admissible B_1 yields the first assertion. The second assertion follows directly from Corollary 2.10.
2. If Ψ satisfies the Δ_2 condition, then [12, Theorem 3.2] shows that the integral ISS estimate in (17) even holds for all $u_2 \in E_\Psi(0,\infty;U_2)$.
3. This is clear by combining 1. and 2.

□

3. Example: Parabolic Diagonal System

Consider the following system of infinitely many bilinear ordinary differential equations

$$\begin{cases} \dot{x}_n(t) &= \lambda_n x_n(t) + u(t)\mu_n x_n(t), \\ & \\ x_n(0) &= x_{n,0}, \end{cases} \qquad n \in \mathbb{N}, t > 0, \tag{20}$$

for sequences $(\lambda_n)_{n\in\mathbb{N}}$, $(\mu_n)_{n\in\mathbb{N}}$ of complex numbers and scalar-valued functions x_n and u. We will discuss this system in the framework of sequences $(x_n(t))_{n\in\mathbb{N}}$ in $X = \ell^2 = \ell^2(\mathbb{N})$, but remark that this could be done more general[2]. Formally, one can rewrite (20) in the abstract form (4) with $F : X \times \mathbb{C} \to X$ given by $F(x, u) := ux$ and linear operators A and B_1 acting "diagonally" on the canonical basis $(e_n)_{n\in\mathbb{N}}$ of X, i.e. $Ae_n = \lambda_n e_n$ and $B_1 e_n = \mu_n e_n$ for all $n \in \mathbb{N}$. Let us from now on assume that $\sup_n \operatorname{Re} \lambda_n < \infty$, which guarantees the existence of solutions $x(t)$ in X for all initial values $x_0 \in X$ since A generates a C_0-semigroup in this case. If $(\mu_n)_{n\in\mathbb{N}}$ is a bounded sequence, then B is bounded and thus System (20) is integral ISS if and only if $\sup_n \operatorname{Re} \lambda_n < 0$. This is an easy exercise when using the explicit solution of (20), but can also be inferred from Corollary 2.12, as B is clearly L^1-admissible. If, however, $(\mu_n)_{n\in\mathbb{N}}$ is not bounded, the characterization does not hold any more.

With the following result we present an abstract example of a bilinear System $\Sigma(A, [B_1, B_2], F)$, where the control operator B_1 is Orlicz-admissible for some Young function Φ but not L^p-admissible for any $p \in [1, \infty)$. In the context of linear systems, such an example was already given in [12, Ex. 5.2] for an operator B defined on $\mathbb{C}$ using the connection between a Carleson-measure criterion and admissibility stated in [12], see also [13]. We will reformulate this example for B defined on X and show E_Φ-admissibility by using ideas from [35, Ex. 4.2.12]. Corollary 2.12 implies that such Φ can not satisfy the Δ_2 condition.
This result also shows that it is insufficient to consider Corollary 2.8 only in the case of L^p spaces and that it is necessary to consider Orlicz spaces for the assertion of the corollary.

Proposition 3.1. *There exists a system $\Sigma(A, [B_1, B_2], F)$ with*

$$F : X \times \mathbb{C} \to X, (x, u) \mapsto ux$$

such that the following properties hold.

- *B_1 and B_2 are E_Φ-admissible for some Young function Φ;*
- *B_1 and B_2 are not L^p-admissible for any $p \in [1, \infty)$;*
- *Estimate (16) holds with $\Phi = \Psi$.*

More precisely, on any separable Hilbert space X such a system can be defined by $\lambda_n = -2^n$ and $\mu_n = -\frac{\lambda_n}{n}$, $n \in \mathbb{N}$, and linear operators A, $B = B_1 = B_2$ given by

$$Ae_n = \lambda_n e_n, \quad Be_n = \mu_n e_n, \qquad n \in \mathbb{N},$$

with maximal domains, that is, $D(A) = \left\{\sum_n x_n e_n \in X \mid \sum_n |\lambda_n x_n|^2 < \infty\right\}$, and where $(e_n)_{n\in\mathbb{N}}$ refers to an orthonormal basis of X.

Proof. Without loss of generality let $X = \ell^2(\mathbb{N})$ and $(e_n)_{n\in\mathbb{N}}$ refer to the canonical basis. It is well-known that A generates an analytic exponentially stable C_0-semigroup $(T(t))_{t\geq 0}$ on X, given by $T(t)e_n = \mathrm{e}^{t\lambda_n} e_n$, $t > 0$, and that

$$(\ell^2)_{-1} = \left\{ x = (x_n)_{n\in\mathbb{N}} \mid \sum_n \frac{|x_n|^2}{|\lambda_n|^2} < \infty \right\}, \qquad \|x\|_{X_{-1}} = \|A^{-1}x\|_{\ell^2}.$$

[2] More generally, any separable Hilbert space can be chosen here and even more general, any space with a q-Riesz basis.

Hence, every bounded linear operator $b : \mathbb{C} \to (\ell^2)_{-1}$ can be identified with a sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ satisfying $\sum_n \frac{|b_n|^2}{|\lambda_n|^2} < \infty$.
To show the admissibility of B with respect to some Orlicz space consider

$$\tilde{\Phi}(x) = x \ln(\ln(x + \mathrm{e})).$$

It is easy to check $\tilde{\Phi}$ is a Young function. Let Φ be the complementary Young function and define the sequnce $k = (k_n)_{n\in\mathbb{N}}$ by $k_n = \frac{\ln(Cn)}{n}$, $n \in \mathbb{N}$, where $C = \ln(2) + \ln(2\mathrm{e}) > 1$. Choose n large enough, such that $k_n n = \ln(Cn) \geq 1$ holds. Then we have

$$\begin{aligned}
\tilde{\Phi}\left(\frac{2^n}{k_n n}\mathrm{e}^{-2^n t}\right) &= \frac{2^n}{k_n n}\mathrm{e}^{-2^n t}\ln\left(\ln\left(\frac{2^n}{k_n n}\mathrm{e}^{-2^n t} + \mathrm{e}\right)\right)\\
&\leq \frac{2^n}{k_n n}\mathrm{e}^{-2^n t}\ln\left(\ln\left(\frac{2^n}{k_n n}\left(\mathrm{e}^{-2^n t} + \mathrm{e}\right)\right)\right)\\
&\leq \frac{2^n}{k_n n}\mathrm{e}^{-2^n t}\ln\left(n\ln(2) - \ln(k_n n) + \ln(2\mathrm{e})\right)\\
&\leq \frac{2^n}{k_n n}\mathrm{e}^{-2^n t}\ln(Cn)\\
&= 2^n \mathrm{e}^{-2^n t}.
\end{aligned}$$

We deduce

$$\int_0^t \tilde{\Phi}\left(\frac{\mathrm{e}^{-2^n(t-s)}\frac{2^n}{n}}{k_n}\right)\mathrm{d}s \leq 1 - \mathrm{e}^{-2^n t} < 1$$

and hence $\|\mathrm{e}^{-2^n(t-\cdot)}\frac{2^n}{n}\|_{L_{\tilde{\Phi}}(0,t;\mathbb{C})} \leq k_n$ for sufficiently large n. Using the generalized Hölder inequality (10), we get for $u \in E_\Phi(0,t;\ell^2)$ and sufficiently large n

$$\begin{aligned}
\left|\left(\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right)(n)\right| &= \left|\int_0^t \mathrm{e}^{-2^n(t-s)}\frac{2^n}{n}(u(s))(n)\,\mathrm{d}s\right|\\
&\leq 2\left\|\mathrm{e}^{-2^n(t-\cdot)}\frac{2^n}{n}\right\|_{L_{\tilde{\Phi}}(0,t;\mathbb{C})}\|(u(\cdot))(n)\|_{E_\Phi(0,t;\mathbb{C})}\\
&\leq 2k_n\|u\|_{E_\Phi(0,t;\ell^2)},
\end{aligned}$$

where we used in the last inequality that

$$\int_0^t \Phi\left(\frac{|(u(s))(n)|}{k}\right)\mathrm{d}s \leq \int_0^t \Phi\left(\frac{\|u(s)\|_{\ell^2}}{k}\right)\mathrm{d}s.$$

Therefore, for some $M > 0$,

$$\left\|\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right\|_{\ell^2} \leq M\|k\|_{\ell^2}\|u\|_{E_\Phi(0,t;\ell^2)},$$

which shows that B is E_Φ-admissible. By Corollary 2.8, we conclude that Estimate (16) holds for $\Phi = \Psi$.
It remains to show that B is not L^p-admissible for any $p \in [1,\infty)$, see e.g. [12, Ex. 5.2]. Suppose on the contrary that B is L^p admissible for some $p \in [1,\infty)$. Since $L^{p'}$-admissibility implies L^p-admissibility for $1 \leq p' < p \leq \infty$ by the nesting properties of L^p spaces, we can without loss of generality assume that $p > 2$. By the definition of admissibility it follows that $b_x := Bx : \mathbb{C} \to (\ell^2)_{-1}$ is L^p-admissible for every $x \in \ell^2$. Taking $x = (\frac{1}{n})_{n\in\mathbb{N}}$, we can identifiy b_x with the sequence $(b_n)_{n\in\mathbb{N}} \in X_{-1}$ given by $b_n = \frac{2^n}{n^2}$. By the characterization of L^p-admissibility operators $b : \mathbb{C} \to X_{-1}$ for $2 < p < \infty$ from [13, Thm. 3.5], this implies that

$$\left(2^{-\frac{2n(p-1)}{p}}\mu(Q_n)\right)_{n\in\mathbb{Z}} \in \ell^{\frac{p}{p-2}}(\mathbb{N}),$$

where μ is the Dirac measure given by $\mu = \sum_n |b_n|^2 \delta_{-\lambda_n}$, δ_λ is centred at λ and $Q_n = \{z \in \mathbb{C} \mid 2^{n-1} < \mathrm{Re}(z) \leq 2^n\}$. This however leads to a contradiction since

$$\left(\left(2^{-\frac{2n(p-1)}{p}}\mu(Q_n)\right)^{\frac{p}{p-2}}\right)_{n\in\mathbb{N}} = \left(\frac{2^{\frac{2n}{p-2}}}{n^{\frac{4p}{p-2}}}\right)_{n\in\mathbb{N}} \notin \ell^1.$$

This completes the proof. □

4. Controlled Fokker–Planck equation

Following [5, 11] we consider a variant of the Fokker–Planck equation on a bounded domain $\Omega \subset \mathbb{R}^n$, with smooth boundary $\partial\Omega$, of the form

$$\begin{aligned} \tfrac{\partial \rho}{\partial t}(x,t) &= \nu\Delta\rho(x,t) + \nabla \cdot \Big(\rho(x,t)\nabla V(x,t)\Big), \\ \rho(x,0) &= \rho_0(x), \end{aligned} \tag{21}$$

where $x \in \Omega, t > 0$, with reflective boundary conditions

$$0 = (\nu\nabla\rho + \rho\nabla V) \cdot \vec{n}, \tag{22}$$

on $\partial\Omega \times (0,\infty)$ and where $\vec{n}$ refers to the outward-pointing unit normal vector on the boundary. Here ρ_0 denotes the initial probability distribution with $\int_\Omega \rho_0(x)\,\mathrm{d}x = 1$ and $\nu > 0$. Furthermore, the potential V is of the form

$$V(x,t) = W(x) + (\mathcal{M}(u(t)))\,(x), \tag{23}$$

where $W \in W^{2,\infty}(\Omega)$ and $\mathcal{M}(u)$ is assumed to satisfy the structural assumption

$$\nabla\mathcal{M}(u) \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega \times (0,\infty)$$

for all u from the input space U. To apply the results from Section 2, we introduce the following operators.

$$\begin{aligned} Af &= \nu\Delta f + \nabla \cdot (f\nabla W), \\ D(A) &= \{f \in H^1(\Omega) \mid \Delta f \in L^2(\Omega), (\nu\nabla f + f\nabla W)\cdot\vec{n} = 0 \text{ on } \partial\Omega\} \\ B &= \nabla\cdot = \mathrm{div} \\ D(B) &= H^1(\Omega)^n \end{aligned}$$

where the state space is $X = L^2(\Omega)$ and $H^1(\Omega)$, $H^2(\Omega)$ refer to the standard Sobolev spaces. Let

$$F : L^2(\Omega) \times U \to L^2(\Omega)^n, (\rho, u) \mapsto \rho\nabla\mathcal{M}(u), \tag{24}$$

with

$$\mathcal{M} : U \to \{g \in W^{1,\infty}(\Omega) \cap H^2(\Omega) \mid \nabla g \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}$$

and $\mathcal{M} \in \mathcal{L}(U, W^{1,\infty}(\Omega))$. For instance, $\mathcal{M}$ could be given by

$$\mathcal{M} : \mathbb{C} \to W^{1,\infty}(\Omega), u \mapsto \alpha(\cdot)u$$

with $\alpha \in W^{1,\infty}(\Omega) \cap H^2(\Omega)$ and $\nabla\alpha \cdot \vec{n} = 0$ on $\partial\Omega$. This example is studied in [5]. More generally, we can consider

$$\mathcal{M} : L^2(\Omega) \to W^{1,\infty}(\Omega), u \mapsto \left(x \mapsto \int_\Omega k(x,y)u(y)\,\mathrm{d}y\right),$$

where $k \in C^2(\overline{\Omega \times \Omega})$ satisfies $\nabla_x k(\cdot, y) \cdot \mathbf{n} = 0$ on $\partial\Omega$ for all $y \in \Omega$. Note that for $k(\cdot, y) = \alpha(\cdot)$ we arrive at the first example

$$\mathcal{M}(u) = \alpha(\cdot)\int_\Omega u(y)\,\mathrm{d}y,$$

upon identifying a function $u \in L^2(\Omega)$ with its integral.

Proposition 4.1. *The operator A generates a bounded semigroup on X, with discrete spectrum $\sigma(A) = \sigma_p(A) \subseteq (-\infty, 0]$ and $\rho_\infty = \mathrm{e}^{-\Phi}$ is an eigenfunction to the simple eigenvalue* 0.

To prove this, we define $\Phi = \ln \nu + \frac{W}{\nu}$ and let M be the multiplication operator with $\mathrm{e}^{\frac{\Phi}{2}}$ on $L^2(\Omega)$. Clearly, M is bounded on $L^2(\Omega)$ and leaves $H^1(\Omega)$ invariant. Moreover, M is invertible and the inverse is the multiplication operator with $\mathrm{e}^{-\frac{\Phi}{2}}$. Hence, $\tilde{A}$ given by

$$\begin{aligned} \tilde{A} =& MAM^{-1}, \\ D(\tilde{A}) =& MD(A) \end{aligned}$$

is well-defined. The proof of the following result is standard, wee e.g. [5]. For convenience, we sketch it.

Lemma 4.2. *The operator $\tilde{A}$ is self-adjoint and dissipative with compact resolvent and $\sigma(A) = \sigma_p(A) = \sigma_p(\tilde{A}) = \sigma(\tilde{A}) \subseteq (-\infty, 0]$. The eigenvalue 0 is simple and one corresponding eigenfunction is given by $e_0 = \mathrm{e}^{-\frac{\Phi}{2}}$. In particular, the eigenfunctions of $\tilde{A}$ form an orthonormal basis and $\tilde{A}$ generates a contraction semigroup on $L^2(\Omega)$.*

Proof. First we verify by form-methods that A generates a C_0-semigroup. It is shown in [3, Thm. 7.15] that Δ with $D(\Delta) = D(A)$ generates a C_0-semigroup on $L^2(\Omega)$. Since the mapping $\rho \mapsto \nabla \cdot (\rho \nabla W)$ is bounded considered as operator on $H^1(\Omega)$ mapping into $L^2(\Omega)$ [2, Prop. 7.2.1] implies that A and therefore $\tilde{A}$ generates a C_0-semigroup. Integration by parts will show the dissipativity and symmetry of $\tilde{A}$ and that $\ker(\tilde{A})$ (and therefore $\ker(A)$) is one-dimensional. In particular, $\sigma(\tilde{A}) \subset \mathbb{C}_-$ and $\tilde{A}$ generates a contraction semigroup. Hence, there exist $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $\lambda, \bar{\lambda} \in \rho(\tilde{A})$. Together with the symmetry, this implies the self-adjointness of $\tilde{A}$. Standard estimates such as the compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ show that $\tilde{A}$ and A have compact resolvent. □

To study admissibility of B we introduce the following well-known abstract interpolation and extrapolation spaces, see e.g. [34]. Let $\tilde{X}_1$ and $\tilde{X}_{-1}$ be defined in the same way as X_1 and X_{-1}, but using $\tilde{A}$ instead of A. We define $\tilde{X}_{-\frac{1}{2}}$ as the completion of $D(\tilde{A})$ with respect to the norm given by

$$\|z\|^2_{\tilde{X}_{\frac{1}{2}}} := \langle (I - \tilde{A})z, z \rangle$$

for $x \in D(\tilde{A})$. Furthermore denote by $\tilde{X}_{-\frac{1}{2}}$ the dual space of $\tilde{X}_{\frac{1}{2}}$ with respect to the pivot space X, i.e. it is the completion of X with respect to the norm

$$\|z\|^2_{\tilde{X}_{-\frac{1}{2}}} := \sup_{\|v\|_{\tilde{X}_{\frac{1}{2}}} \leq 1} |\langle z, v \rangle_X|.$$

The following embeddings are dense and continuous

$$\tilde{X}_1 \hookrightarrow \tilde{X}_{\frac{1}{2}} \hookrightarrow X \hookrightarrow \tilde{X}_{-\frac{1}{2}} \hookrightarrow \tilde{X}_{-1}.$$

Lemma 4.3. *The operator B extends uniquely to an L^2-admissible operator (for A) in $L(X^n, X_{-1})$.*

Proof. Let $\vec{M} : X^n \to X^n, (x_1, \ldots, x_n) \mapsto (Mx_1, \ldots, Mx_1)$, which is obviously invertible. We first prove that the operator $\tilde{B} := MB\vec{M}^{-1}$ defined on $D(\tilde{A})^n$ has a unique extension $\tilde{B} \in L(X^n, \tilde{X}_{-\frac{1}{2}})$ which is L^2-admissible. Integration by parts gives

$$\|v\|^2_{\tilde{X}_{\frac{1}{2}}} = \|v\|^2_{L^2} + \|\nabla \left(\mathrm{e}^{\frac{\Phi}{2}} v \right) e^{-\frac{\Phi}{2}} \|^2_{L^2}, \qquad v \in D(\tilde{A}).$$

For $\vec{f} \in D(\tilde{A})^n$ and $v \in D(\tilde{A})$, $\|v\|_{\tilde{X}_{\frac{1}{2}}} \leq 1$, we have that

$$\begin{aligned}
|\langle \tilde{B}\vec{f}, v\rangle_{L^2}| &= \left| \int_\Omega v\mathrm{e}^{\frac{\Phi}{2}} \nabla \cdot \left(\mathrm{e}^{-\frac{\Phi}{2}} \vec{f}\right) \mathrm{d}x \right| \\
&= \left| \int_{\partial\Omega} v\mathrm{e}^{\frac{\Phi}{2}} \mathrm{e}^{-\frac{\Phi}{2}} \vec{f} \cdot \vec{n} \, \mathrm{d}\sigma - \int_\Omega \nabla \left(v\mathrm{e}^{\frac{\Phi}{2}}\right) \cdot \left(\mathrm{e}^{-\frac{\Phi}{2}} \vec{f}\right) \mathrm{d}x \right| \\
&\leq \|\nabla \left(v\mathrm{e}^{\frac{\Phi}{2}}\right) \mathrm{e}^{-\frac{\Phi}{2}}\|^2_{L^2(\Omega)^n} \|\vec{f}\|^2_{L^2(\Omega)^n}
\end{aligned}$$

where σ is the surface measure on $\partial\Omega$. Thus $\tilde{B} \in L(X, \tilde{X}_{-\frac{1}{2}})$ and $\tilde{B}$ is L^2-admissible for $\tilde{A}$ which follows from [34, Prop. 5.1.3]. We have for $\beta \in \rho(A) = \rho(\tilde{A})$ and $f \in X$

$$\begin{aligned}
\|M^{-1}f\|_{X_{-1}} &= \|(\beta - A)^{-1} M^{-1} f\|_X \\
&= \|M^{-1}(\beta - \tilde{A})^{-1} f\|_X \leq \|M^{-1}\|_{L(X)} \|f\|_{\tilde{X}_{-1}}.
\end{aligned}$$

Thus, M^{-1} extends uniquely to an operator in $L(\tilde{X}_{-1}, X_{-1})$. The same argument yields a unique extension $M \in L(X_{-1}, \tilde{X}_{-1})$. Note that these extensions are inverse to each other, so it is natural to denote the extensions again by M and M^{-1}. Using these extensions we infer that $B = M^{-1}\tilde{B}\vec{M} \in L(X^n, X_{-1})$ is L^2-admissible for A. Indeed, if $(T(t))_{t\geq 0}$ is the semigroup generated by A, then $(S(t))_{t\geq 0}$ with $S(t) = MT(t)M^{-1}$ is the semigroup generated by $\tilde{A}$ and for $\vec{u} \in L^2(0, t; X^n)$ we have $\vec{M}\vec{u} \in L^2(0, t; X^n)$ and

$$\int_0^t T_{-1}(t-s)B\vec{u}(s)\,\mathrm{d}s = M^{-1} \int_0^t S(t-s)\tilde{B}(\vec{M}\vec{u})(s)\,\mathrm{d}s.$$

which proves the assertions. □

From what has been shown in this section so far, it can be shown that the bilinearly controlled Fokker–Planck system given by (21)-(23) can be written as a system $\Sigma(A, [B_1, B_2], F)$ with $B_2 = 0$, $B_1 = B$ and nonlinearity F satisfying the assumptions from Section 2. Therefore, the following result follows directly from Theorem 2.6.

Proposition 4.4. *The Fokker–Planck system* (21)-(23) *has a unique global mild solution ρ for any initial value $\rho_0 \in L^2(\Omega)$ and input function $u \in L^2(0, \infty; U)$. If $\int_\Omega \rho_0(x)\mathrm{d}x = 1$, then $\int_\Omega \rho(t, x)\mathrm{d}x = 1$ for all $t > 0$.*

Proof. The first assertion follows directly from Theorem 2.5. To see the second let ρ be the global mild solution for given $u \in L^2(0, \infty; U)$ and $\rho_0 \in L^2(\Omega)$ with $\int_\Omega \rho_0(x)\mathrm{d}x = 1$. Define $f = (t \mapsto f(t, \cdot))$ by

$$f(t, x) = BF(\rho(t, x), u(t)) = \nabla \cdot \rho(t, x) \left(\nabla \mathcal{M}(u(t))\right)(x),$$

which is an element of $L^1_{loc}(0, \infty; X_{-1})$. Since ρ is a global mild solution of the bilinear system, ρ is also a global mild solution of the linear equation

$$\dot{\rho}(t) = A\rho(t) + f(t), \quad t > 0, \quad \rho(0) = \rho_0. \tag{25}$$

By [33, Thm. 3.8.2(iii)] if follows that $\rho \in C([0, \infty); X) \cap W^{1,1}_{loc}(0, \infty; X_{-1})$ and that ρ satisfies the equation (25) pointwise almost everywhere in X_{-1}. Thus, for any $v \in D(A^*)$ we have that

$$\begin{aligned}
\langle \dot{\rho}(t), v\rangle_{X_{-1}\times D(A^*)} &= \langle A\rho(t), v\rangle_{X_{-1}\times D(A^*)} + \langle f(t), v\rangle_{X_{-1}\times D(A^*)} \\
&= \langle \rho(t), A^*v\rangle_{X_{-1}\times D(A^*)} + \int_\Omega (\nabla \cdot \rho(t)\nabla\mathcal{M}(u(t))) \cdot v\,\mathrm{d}x.
\end{aligned}$$

Letting $v \equiv 1 \in D(A^*)$, we conclude by $A^*v = 0$ and the structural assumption $\nabla\mathcal{M}(u) \cdot \mathbf{n} = 0$ on $\partial\Omega$ that $\int_\Omega \rho(t, x)\mathrm{d}x = \int_\Omega \rho_0(x)\mathrm{d}x = 1$ for all $t > 0$. □

However, recall from Lemma 4.2 that the kernel of A is one-dimensional with corresponding normalized eigenfunction $\rho_\infty = c\mathrm{e}^{-\Phi}$ where $c > 0$ is a suitable constant. Therefore A does not generate an exponentially stable semigroup and thus Theorem 2.6 is not applicable. This is why we consider the system around the stationary distribution ρ_∞ instead of the origin, see also [5]. This means that we consider the change of variables $y := \rho - \rho_\infty$, for which the Fokker–Planck equation then reads

$$\begin{aligned} \dot{y}(t) &= Ay(t) + B\left(y(t)\nabla\mathcal{M}(u(t))\right) + B\left(\rho_\infty\nabla\mathcal{M}(u)\right)\right), \ t \geq 0 \\ y(0) &= \rho_0 - \rho_\infty. \end{aligned}$$

In order to apply Corollary 2.12 we decompose X according to the projections

$$P : L^2(\Omega) \to L^2(\Omega), y \mapsto y - \int_\Omega y(x)\,\mathrm{d}x\rho_\infty \qquad \text{and} \qquad Q := I - P.$$

Note that $\operatorname{ran}(Q) = \ker(P) = \operatorname{span}\{\rho_\infty\}$ and $\ker(Q) = \operatorname{ran}(P)$. Define $\mathcal{X} = \operatorname{ran}(P)$. Using $y = y_P + y_Q$ with $y_P = Py \in \mathcal{X}$ and $y_Q = Qy \in \operatorname{span}\{\rho_\infty\}$ and following [5, Sec. 3.2], the Fokker–Planck equation can be rewritten as a system in $\mathcal{X}$,

$$\begin{aligned} \dot{y}_P(t) &= \mathcal{A}y_P(t) + \mathcal{B}_1\left(y_P(t)\nabla\mathcal{M}(u(t))\right) + \mathcal{B}_2\left(\rho_\infty\nabla\mathcal{M}(u(t))\right), \ t \geq 0, \\ y_P(0) &= P\rho_0, \\ y_Q(t) &= Q\rho_0 - \rho_\infty = 0, \ t \geq 0, \end{aligned} \tag{26}$$

where $\mathcal{A}$ is the restriction of A to $\mathcal{X}$, $\mathcal{B}_1$ is the restriction of B_1 to $\mathcal{X}^n$ respectively, and $\mathcal{B}_2 = B\rho_\infty$. See also [5, Eq. (3.12)]. We emphasize that $Q\rho_0 - \rho_\infty = 0$ follows by the assumption that $\int_\Omega \rho_0(x)\mathrm{d}x = 1$. Note that by [5] we have $PB_1 = B_1$ on $H^1(\Omega)^n$ and hence on $\mathcal{X}^n$ as well as $PB_2 = B_2$ on $\mathbb{R}$.

Lemma 4.5. *The operator $\mathcal{B}_1 = B|_{\mathcal{X}^n} \in \mathcal{L}(\mathcal{X}^n, \mathcal{X}_{-1})$ is L^2-admissible for $\mathcal{A}$ and $\mathcal{B}_2$ is L^1-admissible for $\mathcal{A}$.*

Proof. Since $\mathcal{B}_2 \in L(\mathbb{C}, \mathcal{X})$, admissibility of $\mathcal{B}_2$ is clear.
To infer the L^2-admissibility of $\mathcal{B}_1$ for $\mathcal{A}$, we refer to [15, Lem. 4.4] in combination with Lemma 4.3. We remark, that [15, Lem. 4.4] is true for $B_1 \in L(X, X_{-1})$, even though only $B_1 \in L(\mathbb{C}^m, X_{-1})$ is considered. More presicely, B_1 is L^2-admissible for A and P commutes with $T(t)$. To verify the latter it sufficies to prove that $\tilde{P} = MPM^{-1}$ commutes with the semigroup generated by $\tilde{A}$. This is clear because $\tilde{A}$ provides an orthonormal basis consisting of the eigenfunctions of $\tilde{A}$ and $\ker(\tilde{A}) = \operatorname{span}\{\mathrm{e}_0\}$ where $e_0 = \mathrm{e}^{-\frac{\Phi}{2}}$ is the eigenfunction to the eigenvalue 0. Then, by [15, Lem. 4.4] P has a unique extension to a projection $P \in L(X_{-1})$ commuting with $T(t)$ and with $\operatorname{ran}(P) = \mathcal{X}_{-1}$, where $\mathcal{X}_{-1}$ is considered with respect to the operator $\mathcal{A}$. Besides, $\mathcal{B}_1 = P\mathcal{B}_1$ as operator mapping from X to X_{-1}. By [15, Lem 4.4], $\mathcal{B}_1$ is L^2-admissible for $\mathcal{A}$. □

Corollary 2.8 now yields the following result on the considered class of Fokker–Plank systems.

Theorem 4.6. *There exists a constants $C, \omega > 0$ such that for any $\rho_0 \in L^2(\Omega)$ with $\int_\Omega \rho_0(x)\mathrm{d}x = 1$ and $u \in L^2(0,\infty;U)$, the global mild solution of the Fokker–Planck system* (21) *satisfies*

$$\|\rho(t) - \rho_\infty\|_{L^2} \leq C\mathrm{e}^{-\omega t}\left(\|\rho_0 - \rho_\infty\|_{L^2} + \|\rho_0 - \rho_\infty\|_{L^2}^2\right) + \gamma\left(\int_0^t \|u(s)\|_U^2 \mathrm{d}s\right),$$

where $\gamma(r) = Cr\mathrm{e}^{Cr^{\frac{1}{2}}} + Cr^{\frac{1}{2}} + Cr$.

Proof. This readily follows from Corollary 2.12 applied to System (26) using Lemma 4.5. Finally note that

$$y_P(t) = P\rho(t) = \rho(t) - \int_\Omega \rho(t,x)\mathrm{d}x\,\rho_\infty = \rho(t) - \rho_\infty,$$

where we applied Proposition 4.4 in the last identity. □

5. Conclusion

Bilinear systems appear naturally in control theory e.g. when considering multiplicative disturbances in feedback loops of linear systems. The results in this article draw a link between bilinear systems, which are a classical example class in (integral) ISS in finite-dimensions, and recent progress in ISS for infinite-dimensional systems. We emphasize that the most natural example in this context,

$$\dot{x}(t) = Ax(t) + u(t)x(t), \quad t > 0, \quad x(0) = x_0,$$

with A generating a C_0-semigroup T on X, is covered by the system class considered here. More precisely, by the results in Section 2, it follows that this system is integral ISS if and only if T is exponentially stable. More precisely, the sufficiency follows since the identity is L^1-admissible and hence the system is integral ISS by Corollary 2.12. It seems that prior works on integral ISS [24, Sec. 4.2] did not cover this comparably simple class as the bilinearity $x \mapsto xu$ fails to satisfy a Lipschitz condition uniform in u required there[3].

Moreover, our results generalize to integral ISS assessment for bilinearities arising from boundary control (or lumped control). We recap our findings by discussing the following conjecture.

Conjecture 5.1. *A bilinear system $\Sigma(A, [B_1, B_2], F)$ satisfying the hypothesis of Section 2 is integral ISS if the linear systems $\Sigma(A, [0, B_1])$ and $\Sigma(A, [0, B_2])$ are integral ISS.*

First, observe that this is known to be true if B_1, B_2 are bounded. In this case the converse holds true as well and moreover the condition is equivalent to A generating an exponentially stable semigroup. In general, the conjecture is open. We are, however, able to show slightly weaker variants of the statement. On the one hand we may sharpen the assumption on the linear systems by requiring that $\Sigma(A, [0, B_1])$ is E_Φ-ISS where Φ satisfies the Δ_2-condition, see Corollary 2.12. Whereas we show in Section 3 that this refined condition does not follow from integral ISS of $\Sigma(A, [B_1, B_2], F)$ in general, we stress that this seems to be no restriction for any more practical example, Section 4. On the other hand we can replace the hypothesis in the conjecture by inferring that $\Sigma(A, [B_1, B_2], F)$ is ISS with respect to some Orlicz space, Theorem 2.6. Note that it would already be interesting to know whether the conjecture holds true for the systems from Section 3.

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[3]However, it seems this can be overcome with a carefully refined argument in the proof of [24, Thm. 4.2].

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(RH) University of Wuppertal, School of Mathematics and Natural Siences, Gaussstr. 20, 42119 Wuppertal, Germany

E-mail address: `hosfeld@uni-wuppertal.de`

(BJ) University of Wuppertal, School of Mathematics and Natural Sciences, Gaussstr. 20, 42119 Wuppertal, Germany

E-mail address: `bjacob@uni-wuppertal.de`

(FLS) Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands and Department of Mathematics, University of Hamburg, Bundesstrasse 55, 20146 Hamburg, Germany

E-mail address: `f.l.schwenninger@utwente.nl`