Funnel control for a moving water tank[⋆]

Thomas Berger[a], Marc Puche[b], Felix L. Schwenninger[b,c]

[a]*Institut für Mathematik, Universität Paderborn, Warburger Str. 100, 33098 Paderborn, Germany*
[b]*Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany*
[c]*Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands*

Abstract

We study tracking control for a nonlinear moving water tank system modelled by the linearized Saint-Venant equations, where the output is given by the position of the tank and the control input is the force acting on it. For a given reference signal, the objective is that the tracking error evolves within a pre-specified performance funnel. Exploiting recent results in funnel control, this can be achieved by showing that *inter alia* the system's internal dynamics are bounded-input, bounded-output stable.

Keywords: Saint-Venant equations; sloshing; well-posed systems; adaptive control; funnel control.

1. Introduction

When a liquid-filled containment is subject to movement, the motion of the fluid may have a significant effect on the dynamics of the overall system and is known as *sloshing*. The latter phenomenon can be understood as internal dynamics of the system and it is of great importance in a range of applications such as aeronautics and control of containers and vehicles, and has been studied in engineering for a long time, see e.g. [11, 14, 17, 18, 35, 36].

The standard model for the one-dimensional movement of a fluid is given by the Saint-Venant equations, which is a system of nonlinear hyperbolic partial differential equations (PDEs). Models of a moving water tank involving these equations without friction have been studied in various articles, where the control is the acceleration and the output is the position of the tank. The first approach appears in [13] where a flat output for the linearized model is constructed. Several additional control problems related to this model are studied in [25] and it is proved that the linearization is steady-state controllable. Moreover, the seminal work [12] shows that the nonlinear model is locally controllable around any steady state. Different stabilization approaches by state and output feedback using Lyapunov functions are studied in [27]. In [1] observers are designed to estimate the horizontal currents by exploiting the symmetries in the Saint-Venant equations. Convergence of the estimates to the actual states is studied for the linearized model. In [11] a port-Hamiltonian formulation of the system is provided as a mixed finite-infinite dimensional system. For a recent numerical treatment of a truck with a fluid basin see e.g. [15].

In this note we consider output trajectory tracking for moving water tank systems by funnel control. The concept of funnel control was developed in [22], see also the survey [21]. The funnel controller is an adaptive controller of high-gain type and proved its potential for tracking problems in various applications, such as temperature control of chemical reactor models [24], control of industrial servo-systems [19] and underactuated multibody systems [5], voltage and current control of electrical circuits [8], control of peak inspiratory pressure [26] and adaptive cruise control [7]. We like to emphasize that the funnel controller is a model-free feedback controller, i.e., it does not require specific system parameters for feasibility. This makes it a suitable choice for the application to the water tank system, for which we assume that it contains a non-vanishing friction term as modeled in the Saint-Venant equations e.g. in [2], but the exact shape/magnitude of this term is unknown and not available to the controller.

While funnel control is known to work for a large class of functional differential equations with higher relative degree as shown in [4] (cf. also Section 2), it is often not clear if a particular system involving internal dynamics governed by PDEs are encompassed by these results. Recently [6], we have outlined an abstract framework to answer this question affirmatively. In the present work we follow this approach to show that tracking with prescribed transient behaviour of the moving tank — subject to sloshing effects modelled via the linearized shallow water equations — can indeed be achieved by funnel control.

1.1. Nomenclature

In the following let $\mathbb{N}$ denote the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}_{\geq 0} = [0, \infty)$. We write $\mathbb{C}_\omega = \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > \omega\}$ for $\omega \in \mathbb{R}$ and $\mathbb{C}_+ = \mathbb{C}_0$. For a Hilbert space X, $L^p(I; X)$ denotes the usual Lebesgue–Bochner space of (strongly) measurable functions $f : I \to X$, $I \subseteq \mathbb{R}$ an interval, where $p \in [1, \infty]$. We write $\| \cdot \|_\infty$ for $\| \cdot \|_{L^\infty(\mathbb{R}_{\geq 0}; X)}$. By $L^\infty_{\mathrm{loc}}(I; X)$ we denote the set of measurable and locally essentially bounded functions $f : I \to X$, by $W^{k,p}(I; X)$, $k \in \mathbb{N}_0$, the Sobolev space of k-times weakly differentiable functions $f : I \to X$ such that $f, \dot{f}, \ldots, f^{(k)} \in L^p(I; X)$, and by $\mathcal{C}^k(I; X)$ the set of k-times continuously differentiable functions $f : I \to X$, $k \in \mathbb{N}_0 \cup \{\infty\}$, where $\mathcal{C}(I; X) := \mathcal{C}^0(I; \mathbb{K}^n)$. By $\mathcal{B}(\mathcal{X}; \mathcal{Y})$, where $\mathcal{X}, \mathcal{Y}$ are Hilbert spaces, we denote the set of all bounded linear operators $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$. The symbol "$\lesssim$" is a placeholder for "$\leq c \cdot$" where the multiplicative constant c is independent of the variables occurring in the inequality.

1.2. The Model

In the present paper we study the horizontal movement of a water tank as depicted in Fig. 1, where we neglect the wheels' inertia and friction between the wheels and the ground. We assume that there is an external force
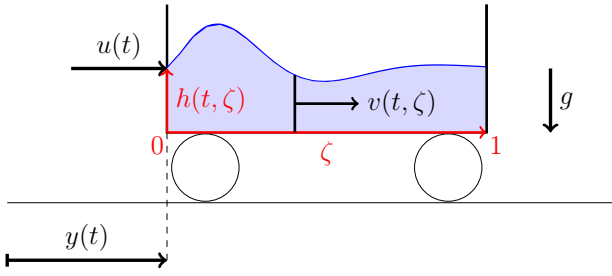

Figure 1: Horizontal movement of a water tank.

acting on the water tank, which we denote by $u(t)$ as this will be the control input of the resulting system, cf. also Section 1.3. The measurement output is the horizontal position $y(t)$ of the water tank, and the mass of the empty tank is denoted by m. The dynamics of the water under gravity g are described by the *Saint-Venant equations* (first derived in [30]; also called one-dimensional *shallow water equations*)

$$\partial_t h + \partial_\zeta(hv) = 0,$$
$$\partial_t v + \partial_\zeta\left(\frac{v^2}{2} + gh\right) + hS\left(\frac{v}{h}\right) = -\ddot{y} \quad (1)$$

with boundary conditions $v(t,0) = v(t,1) = 0$. Here $h : \mathbb{R}_{\geq 0} \times [0,1] \to \mathbb{R}$ denotes the height profile and $v : \mathbb{R}_{\geq 0} \times [0,1] \to \mathbb{R}$ the (relative) horizontal velocity profile, where the length of the container is normalized to 1. The friction term $S : \mathbb{R} \to \mathbb{R}$ is typically modeled as the sum of a high velocity coefficient of the form $C_S v^2/h^2$ and a viscous drag of the form $C_D v/h$ for some positive constants C_S, C_D. In the present paper, we do not specify the function S, but we do assume that $S(0) = 0$ and $S'(0) > 0$. The condition $S(0) = 0$ means that, whenever the velocity is zero, then there is no friction. The condition $S'(0) > 0$ means that the viscous drag does not vanish; this is the case in most real-world non-ideal situations, but sometimes neglected in the literature, see e.g. [2, Sec. 1.4].

For a derivation of the Saint-Venant equations (1) of a moving water tank we refer to [25, 11], see also the references therein. The friction term in the model is the general version of that used in [2, Sec. 1.4]. Let us emphasize that in our framework the input is the force acting on the water tank, which can be manipulated using an engine for instance. In contrast to this, in [12, 25] the acceleration of the tank is used as input, but this can usually not be influenced directly. Note that — in the presence of sloshing — the applied force does not equal the product of the tank's mass and acceleration. We also stress that, if the acceleration is used as input, then the input-output relation is given by the simple double integrator $\ddot{y} = u$, and the Saint-Venant equations (1) do not affect this relation.

As shown in [13, 25], the linearization of the Saint-Venant equations is relevant in the context of control since it provides a model which is much simpler to solve (both analytically and numerically) and still is an insightful approximation for motion planning purposes. Therefore, we restrict ourselves to the linearization of (1) around the steady state $(h_0, 0) = (\int_0^1 h(0,x)\mathrm{d}x, 0)$, given by

$$\partial_t z = Az + b\ddot{y} = -\begin{bmatrix} 0 & h_0\partial_\zeta \\ g\partial_\zeta & 2\mu \end{bmatrix} z + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\ddot{y} \quad (2)$$

with boundary conditions $z_2(t,0) = z_2(t,1) = 0$, $\mu = \frac{1}{2}S'(0) > 0$ and $b = (0,-1)^\top$. The state space in which $z(t)$ evolves is $X = L^2([0,1]; \mathbb{R}^2)$ and $A : \mathcal{D}(A) \subseteq X \to X$,

$$\mathcal{D}(A) = \left\{ (z_1, z_2) \in X \,\middle|\, \begin{matrix} z_1, z_2 \in W^{1,2}([0,1]; \mathbb{R}), \\ z_2(0) = z_2(1) = 0 \end{matrix} \right\}. \quad (3)$$

By conservation of mass in (2), $\int_0^1 z_1(t,x)\mathrm{d}x = h_0$ for all $t \geq 0$. The model is completed by the momentum

$$p(t) := m\dot{y}(t) + \int_0^1 z_1(t,\zeta)\big(z_2(t,\zeta) + \dot{y}(t)\big)\,\mathrm{d}\zeta, \ t \geq 0.$$

Substituting the absolute velocity $x_2 = z_2 + \dot{y}$ for z_2, $x_1 = z_1$ and using the balance law $\dot{p}(t) = u(t)$ and (2) we obtain

$$m\ddot{y}(t) = \frac{g}{2}x_1(t,\cdot)^2|_0^1 + 2\mu\langle x_1(t), x_2(t)\rangle - 2\mu h_0\dot{y}(t) + u(t),$$

where $\langle f, g \rangle = \int_0^1 f(s)g(s)\mathrm{d}s$. Altogether, the nonlinear model on the state space X reads

$$\partial_t x = A(x + b\dot{y}) \quad (4a)$$
$$m\ddot{y}(t) = \frac{g}{2}x_1(t,\cdot)^2|_0^1 + 2\mu\langle x_1(t), x_2(t)\rangle - 2\mu h_0\dot{y}(t) + u(t) \quad (4b)$$

with input u, state x and output y.

1.3. Control objective — funnel control

Our goal is to design an output error feedback of the form

$$u(t) = F\big(t, e(t), \dot{e}(t)\big), \quad (5)$$

where $e(t) = y(t) - y_{\mathrm{ref}}(t)$ is the tracking error and $y_{\mathrm{ref}} \in W^{2,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$ is a given reference position, which applied to (4) results in a closed-loop system that satisfies:

- the pair $(t, e(t))$ evolves within the prescribed set

$$\mathcal{F}_\varphi := \{ (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \,|\, \varphi(t)|e| < 1 \},$$

which is determined by a function φ belonging to

$$\Phi := \left\{ \varphi \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}) \,\middle|\, \begin{matrix} \varphi, \dot{\varphi} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \liminf_{\tau \to \infty} \varphi(\tau) > 0 \end{matrix} \right\}$$

and

- the signals $u, e, \dot{e}$ are uniformly bounded on $\mathbb{R}_{\geq 0}$.

The set $\mathcal{F}_\varphi$ is called the *performance funnel*. Its boundary, the *funnel boundary*, is given by the reciprocal of φ, see Fig. 2. The case $\varphi(0) = 0$ is explicitly allowed and puts no restriction on the initial value since $\varphi(0)|e(0)| < 1$; in this case the funnel boundary $1/\varphi$ has a pole at $t = 0$.

On the other hand, note that boundedness of φ implies that there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. This implies that signals evolving in $\mathcal{F}_\varphi$ are not forced to converge to 0 asymptotically. Furthermore, the funnel
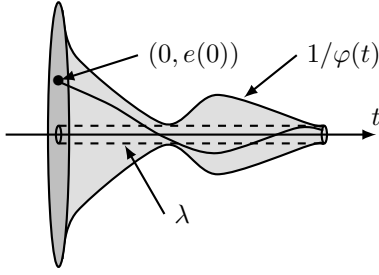
Figure 2: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $1/\varphi(t)$.

boundary is not necessarily monotonically decreasing and there are situations, like in the presence of periodic disturbances, where widening the funnel over some later time interval might be beneficial. It was shown in [4] that for $\varphi_0, \varphi_1 \in \Phi$, the following choice for F in (5)

$$F(t, e(t), \dot{e}(t)) = -k_1(t)\big(\dot{e}(t) + k_0(t)e(t)\big),$$
$$k_0(t) = \frac{1}{1 - \varphi_0(t)^2 \|e(t)\|^2},$$
$$k_1(t) = \frac{1}{1 - \varphi_1(t)^2 \|\dot{e}(t) + k_0(t)e(t)\|^2}, \qquad (6)$$

achieves the above control objective for a large class of nonlinear systems with relative degree two. In the present paper we extend this result and show feasibility of (6) for the model described by (4). We highlight that the functions φ_0, φ_1 are design parameters in the control law (6). Typically, the specific application dictates the constraints on the tracking error and thus indicates suitable choices.

In [6] — extending the findings from [4] — it was shown that the controller (6) is feasible for nonlinear systems of the form

$$\ddot{y}(t) = \mathcal{S}(y, \dot{y})(t) + \gamma\, u(t)$$
$$(y(0), \dot{y}(0)) = (y^0, y^1) \in \mathbb{R}^2, \qquad \text{(Sys)}$$

where, under structural assumptions, the operator $\mathcal{S}$ may in particular incorporate input-output dynamics from an infinite-dimensional well-posed linear system. We note that corresponding results hold for systems with relative degree other than two, but this special case is sufficient for the present article. General sufficient conditions on the operator $\mathcal{S}$ guaranteeing feasibility of the controller (6) were given in [4, 20, 23] and [22] before, while suitable adaptions allowing for truly infinite-dimensional internal dynamics were finally explored in [6]. For details on the structural assumptions on the systems class and the operator $\mathcal{S}$ and the relation to prior results we refer to [6].

1.4. Organization of the present paper

In Section 2 we formulate the main result of this article, stating that the funnel control objective for the model of the moving water tank is achieved in the sense of Section 1.3. For this, it suffices to verify the conditions identified in [6], which is done in Section 3 by considering the model in the framework of well-posed linear systems. The application of the controller to the moving water tank system is illustrated by a simulation in Section 4.

2. Main result

In this section we formulate how the funnel controller (6) described in Subsection 1.3 achieves the control objective for system (4) — this is the main result of the article. The initial conditions for (4) are

$$x(0) = x_0 \in X, \quad (y(0), \dot{y}(0)) = (y^0, y^1) \in \mathbb{R}^2. \qquad (7)$$

We call $(x, y) : [0, \omega) \to X \times \mathbb{R}$ a strong solution of (4)–(7) on an interval $[0, \omega)$, if [1]

- $y \in W^{2,1}_{\text{loc}}([0, \omega); \mathbb{R})$ and $x \in \mathcal{C}([0, \omega); X)$,

- the initial conditions (7) hold,

- $x \in W^{1,1}_{\text{loc}}([0, \omega); X_{-1})$ and (4a) holds for a.e. $t \in [0, \omega)$ as equation in X_{-1},

- y satisfies (4b) for a.a. $t \in [0, \omega)$.

In other words, x is a strong solution of (4a) and y is a Carathéodory solution of (4b). A solution (x, y) is called *classical*, if $x \in \mathcal{C}^1([0, \omega); X)$ and $y \in \mathcal{C}^2([0, \omega); \mathbb{R}^2)$; it is called *global*, if it can be extended to $\mathbb{R}_{\geq 0}$.

Theorem 2.1. *Let $y_{\text{ref}} \in W^{2,\infty}(\mathbb{R}_{\geq 0}; \mathbb{R})$, $\varphi_0, \varphi_1 \in \Phi$ and $(y^0, y^1) \in \mathbb{R}^2$, $x_0 \in X$ and $v_0 \in \mathbb{R}$ be such that*

$$\begin{aligned} &x_0 + bv_0 \in \mathcal{D}(A), \\ &\varphi_0(0)|y^0 - y_{\text{ref}}(0)| < 1, \quad \text{and} \\ &\varphi_1(0)|y^1 - \dot{y}_{\text{ref}}(0) + k_0(0)\big(y^0 - y_{\text{ref}}(0)\big)| < 1. \end{aligned} \qquad (8)$$

Then the closed-loop system (4)–(7) has a unique global strong solution $(x, y) : \mathbb{R}_{\geq 0} \to X \times \mathbb{R}$. Moreover, the following properties hold.

(i) *The functions k_0, k_1, u and $x, y, \dot{y}$ are bounded.*

(ii) *The error $e = y - y_{\text{ref}}$ is uniformly bounded away from the funnel boundary in the following sense:*

$$\exists\, \varepsilon > 0 \,\forall\, t > 0 : \; |e(t)| \leq \varphi_0(t)^{-1} - \varepsilon. \qquad (9)$$

(iii) *If $y^1 = v_0$, then the solution is a classical solution.*

Proof. Step 1: We rewrite (4), (7) in the form of equation (Sys), obtaining

$$\ddot{y}(t) = \mathcal{T}(\dot{y})(t) + \frac{u(t)}{m}, \qquad (10)$$

where the mapping $\mathcal{T}$ is formally given by

$$\begin{aligned} \mathcal{T}(\eta)(t) &= \frac{g}{2m} x_1(t, \cdot)^2 \big|_0^1 + \frac{2\mu}{m}\big(\langle x_1(t), x_2(t)\rangle - h_0\eta(t)\big) \\ &= F\big(\widetilde{\mathcal{T}}(\eta)(t), Cx(t)\big) \end{aligned}$$

with x being the strong solution of

$$\dot{x}(t) = A\big(x(t) + b\eta(t)\big), \quad x(0) = x_0, \qquad (11)$$

where A, b are defined in (2)–(3) and we use the notation

$$\begin{aligned} &F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}, (\alpha, \beta) \mapsto \frac{g}{2m}\beta_1\beta_2 + \frac{2\mu}{m}\alpha, \\ &\widetilde{\mathcal{T}} : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}) \to L^\infty_{\text{loc}}, \eta \mapsto \langle x_1, x_2\rangle - h_0\eta, \\ &C : \mathcal{D}(C) \subset X \to \mathbb{R}^2, x \mapsto (C_1 x, C_2 x)^\top, \\ &C_i x := (x_1(1) + (-1)^i x_1(0)), \quad i = \{1, 2\}. \end{aligned} \qquad (12)$$

Since the operator C acts as point evaluation of functions in space, the domain $\mathcal{D}(C) \subset X$ has to be chosen suitably, see (18). Also note that $\mathcal{T}$ depends on $x = x(t, \zeta)$ which in turn is given through η and x_0 as the solution of (11), the existence of which is an outcome of Step 3 below.

Step 2: We show that $\mathcal{T}$ is well-defined from $\mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$ to $L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R})$ and, in particular, that the mapping

$$\mathcal{F} : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; X \times \mathbb{R}^2), \ \eta \mapsto \begin{bmatrix} I \\ C \end{bmatrix} x,$$

associated with the PDE (11) is well-defined. Moreover we show that

$$\max\{\|x\|_\infty, \|Cx\|_\infty\} \lesssim c_{x_0} + \|\eta\|_\infty \qquad (13)$$

for all $\eta \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}) \cap L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})$, where $c_{x_0} = 0$ if $x_0 = 0$. This step is performed in Proposition 3.3 by showing that the triple $(A, Ab, \begin{bmatrix} I \\ C \end{bmatrix})$ defines a well-posed bounded-input, bounded-output stable linear system.

Step 3: Note that $\widetilde{\mathcal{T}}$ is uniformly Lipschitz on bounded sets, i.e., for any $R > 0$ there exists $L = L_{x_0} > 0$ such that

$$\|\widetilde{\mathcal{T}}(\eta) - \widetilde{\mathcal{T}}(\hat{\eta})\|_{L^\infty([0,t];\mathbb{R})} \leq L \|\eta - \hat{\eta}\|_{L^\infty([0,t];\mathbb{R})} \qquad (14)$$

for all $t > 0$ and $\eta, \hat{\eta} \in \{ \eta \in \mathcal{C}([0,t]; \mathbb{R}) \mid \|\eta\|_\infty \leq R \}$. This follows easily from (13) and the fact that an inner product restricted to bounded subsets is uniformly Lipschitz. Furthermore, it is clear that $\widetilde{\mathcal{T}}$ is causal, and, by (13), that for all $\eta \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}) \cap L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})$ we have

$$\|\widetilde{\mathcal{T}}(\eta)\|_\infty \lesssim \tilde{c}_{x_0} + \|\eta\|_\infty^2 + \|\eta\|_\infty.$$

Since $F \in \mathcal{C}^1(\mathbb{R} \times \mathbb{R}^2; \mathbb{R})$, we conclude from the above that $\mathcal{T}$ satisfies all the Properties (P1)–(P4) from [6, Def. 3.1] and thus [6, Thms. 2.1 & 3.3] imply the existence of a global strong solution (x, y) and, together with (13), that Assertions (i)–(ii) hold. Note that mild solutions of (4a) as considered in [6] are strong solutions, [31, Thm. 3.8.2].

Step 4: We show uniqueness of the solution. First recall that, as a consequence of Proposition 3.1, the unique strong solution of (11) is given by

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)B\eta(s)\, \mathrm{d}s, \quad t \geq 0, \quad (15)$$

where $B = A_{-1}b$. Thus, invoking $\eta = \dot{y}$, it suffices to show uniqueness of the solution y of (10). Assume that $\bar{y} \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R})$ is another solution of (10) with $\bar{y}(0) = y^0$ and $\dot{\bar{y}}(0) = y^1$, and let $\bar{x}$ be the strong solution of (4a) with $y = \bar{y}$ and $\bar{x}(0) = x_0$. Define $t_0 := \inf \{ t \geq 0 \mid \dot{y}(t) \neq \dot{\bar{y}}(t) \}$ and assume, seeking a contradiction, $t_0 < \infty$. Clearly, $x(t) = \bar{x}(t)$ for all $t \in [0, t_0]$, hence, by Proposition 3.3, there exists $c_0 > 0$ such that for all $t \in [t_0, t_0 + 1]$ we have $\|Cx(t) - C\bar{x}(t)\|_{\mathbb{R}^2} \leq c_0 \sup_{s \in [t_0, t_0 + t]} |\dot{y}(s) - \dot{\bar{y}}(s)|$. Furthermore, for $R := \max\{\|\dot{y}\|_\infty, \|\dot{\bar{y}}\|_\infty\}$ we obtain a constant $L > 0$ such that (14) holds. The preparations are completed by observing that, invoking Step 3 and (13), there exist compact subsets $K_1 \subseteq \mathbb{R}$ and $K_2 \subseteq \mathbb{R}^2$ such that $\widetilde{\mathcal{T}}(\dot{y})(s), \widetilde{\mathcal{T}}(\dot{\bar{y}})(s) \in K_1$ and $(Cx)(s), (C\bar{x})(s) \in K_2$ for all $s \in [t_0, t_0 + 1]$. Since F from (12) is in $\mathcal{C}^1(\mathbb{R} \times \mathbb{R}^2; \mathbb{R})$, there exists $c_1 > 0$ such that

$$|F(\alpha_1, \beta_1) - F(\alpha_2, \beta_2)| \leq c_1 (|\alpha_1 - \alpha_2| + \|\beta_1 - \beta_2\|_{\mathbb{R}^2})$$

for all $\alpha_1, \alpha_2 \in K_1$ and $\beta_1, \beta_2 \in K_2$. Now choose $\sigma \in (0, 1)$ such that $\sigma c_1 (L + c_0) < 1$. By definition of t_0, $\dot{y}(t_0) = \dot{\bar{y}}(t_0)$ and there exists $t \in [t_0, t_0 + \sigma]$ such that

$$|\dot{y}(t) - \dot{\bar{y}}(t)| = \sup_{s \in [t_0, t_0 + \sigma]} |\dot{y}(s) - \dot{\bar{y}}(s)| =: \varepsilon > 0.$$

Hence, integrating (10) yields the contradiction

$$\begin{aligned}
\varepsilon &= |\dot{y}(t) - \dot{\bar{y}}(t)| \\
&\leq \int_{t_0}^t \big| F(\widetilde{\mathcal{T}}(\dot{y})(s), Cx(s)) - F(\widetilde{\mathcal{T}}(\dot{\bar{y}})(s), C\bar{x}(s)) \big| \mathrm{d}s \\
&\leq \sigma c_1 (L + c_0) \sup_{s \in [t_0, t_0 + t]} |\dot{y}(s) - \dot{\bar{y}}(s)| < \varepsilon.
\end{aligned}$$

Thus, $\dot{y} = \dot{\bar{y}}$ and by $y(0) = y^0 = \bar{y}(0)$, it follows that $y = \bar{y}$.

Step 5: We show (iii). Note that $\dot{y} \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathbb{R})$ by Step 3 and $\dot{y}(0) = v_0$ by assumption. Invoking well-posedness together with $x(0) + b\dot{y}(0) \in \mathcal{D}(A)$ and [34, Prop. 4.6], the solution x of (11) is indeed a classical solution and $\mathcal{F}(\eta) \in \mathcal{C}(\mathbb{R}_{\geq 0}; X \times \mathbb{R}^2)$. This implies that $\mathcal{T}(\dot{y})(t)$ is continuous, whence $\ddot{y}$ is continuous by (10), which proves the claim. $\qquad\square$

3. Linearized model – abstract framework

In this section we collect and derive the results required for Step 2 in the proof of Theorem 2.1 by using the framework of well-posed linear systems and showing bounded-input, bounded-output stability of the considered systems.

In the following, $(T(t))_{t \geq 0}$ denotes a strongly continuous semigroup on a Banach space X with infinitesimal generator A and growth bound ω_A. For β in the resolvent set $\rho(A)$ of the generator A, we denote by X_{-1} the completion of X with respect to the norm $\| \cdot \|_{X_{-1}} = \|(\beta I - A)^{-1} \cdot \|_X$. Recall that X_{-1} is independent of the choice of β and that $(\beta I - A)$ uniquely extends to an surjective isometry $(\beta I - A_{-1}) \in \mathcal{B}(X; X_{-1})$. The semigroup $(T(t))_{t \geq 0}$ has a unique extension to a semigroup $(T_{-1}(t))_{t \geq 0}$ in X_{-1}, which is generated by A_{-1}. Furthermore, let X_1 be the space $\mathcal{D}(A)$ equipped with the graph norm of A. For Hilbert spaces U, Y, a triple $(A, B, C) \in \mathcal{B}(X_1; X) \times \mathcal{B}(U; X_{-1}) \times \mathcal{B}(X_1; Y)$ is called a *regular well-posed* system, if for some (hence all) $t > 0$,

(a) A is the generator of a semigroup $(T(t))_{t \geq 0}$ on X;

(b) $\int_0^t T_{-1}(t-s)Bu(s)\, \mathrm{d}s \in X$ for all $u \in L^2([0,t]; U)$;

(c) $(\mathcal{D}(A), \| \cdot \|_X) \to L^2([0,t]; Y), \ x \mapsto CT(\cdot)x$ is bounded;

(d) there exists a bounded function $\mathbf{G} : \mathbb{C}_\omega \to \mathcal{B}(U; Y)$, $\omega > \omega_A$, such that for all $r, s \in \mathbb{C}_\omega$,

$$\mathbf{G}(r) - \mathbf{G}(s) = C((rI - A)^{-1} - (sI - A)^{-1})B, \quad (16)$$

and $\lim_{s \to \infty, s \in \mathbb{R}} \mathbf{G}(s)v$ exists for every $v \in U$.

Operators satisfying (b) and (c) are called *admissible* in the literature and naturally appear in the theory of boundary control systems, cf. [31, 33]. Note that $\mathbf{G}$ is uniquely determined up to a constant.

From now on we will, without loss of generality, consider the complexification $L^2([0,1]; \mathbb{C}^2)$ of the state space X and the linear operator A defined in (2)–(3). The following is a simple exercise in the context of well-posed systems. We include a short proof for completeness.

Proposition 3.1. *Let A be defined as in (2)–(3), $B = A_{-1}b \in \mathcal{B}(\mathbb{R}; X_{-1})$ with $b = (0, -1)^\top$ and $C \in \mathcal{B}(X_1; \mathbb{C}^2)$ as defined in (12). Then A generates a contraction semigroup and the triple $(A, B, \left[\begin{smallmatrix} I \\ C \end{smallmatrix}\right])$ is a regular well-posed system. The transfer function $\mathbf{G} : \mathbb{C}_+ \to X \times \mathbb{C} \times \mathbb{C}$ is given by $\mathbf{G} = (\mathbf{L}, \mathbf{H}, 0)$ with*

$$\mathbf{L}(\lambda) = \frac{\lambda}{g\theta}\left(\frac{\cosh(\theta)-1}{\sinh(\theta)} \begin{bmatrix} \cosh(\theta\,\cdot) \\ -\frac{\lambda}{h_0\theta}\sinh(\theta\,\cdot) \end{bmatrix} + \begin{bmatrix} -\sinh(\theta\,\cdot) \\ \frac{\lambda}{h_0\theta}(\cosh(\theta\,\cdot)-1) \end{bmatrix} \right) - b$$

$$\mathbf{H}(\lambda) = -\sqrt{\frac{4h_0\lambda}{g(\lambda+2\mu)}}\,\tanh\left(\frac{\sqrt{\lambda(\lambda+2\mu)}}{2\sqrt{h_0 g}}\right), \quad \lambda \in \mathbb{C}_+, \qquad (17)$$

where $\theta = \theta(\lambda) = \frac{1}{\sqrt{h_0 g}}\sqrt{\lambda(\lambda+2\mu)}$.

Proof. By standard arguments (e.g. a Fourier ansatz or Lumer–Phillips theorem), A generates a contraction semigroup which even extends to a group, whence Property (a). Consider the holomorphic function $\mathbf{L}(\lambda) = (\lambda I - A)^{-1}B$ from $\mathbb{C}_+$ to X. By the resolvent identity, we have that

$$\mathbf{L}(\lambda) = \lambda(\lambda I - A)^{-1}b - b = \lambda z_\lambda - b,$$

where z_λ is computed by solving the ODE $\lambda z_\lambda - A z_\lambda = b$. Since $\mathbf{L}$ is bounded on the half-plane $\mathbb{C}_{\mu+1}$, it follows that B is admissible, Property (b), by [33, Thm. 5.2.2]. Thus, (A, B, I) is well-posed. Similarly, one can show that Property (c) holds for C using [33, Cor. 5.2.4]. Using the formula for $\mathbf{L}(\lambda)$ shows that $\mathbf{H}$ is indeed a transfer function for (A, B, C). As $\mathbf{H}$ is bounded and $\lim_{s\to\infty, s\in\mathbb{R}} \mathbf{G}(s)$ exists, Property (d) and the assertion follow. $\square$

Next we show that the inverse Laplace transform of $\mathbf{H}$ is a measure of bounded total variation on $\mathbb{R}_{\geq 0}$, i.e., $\mathbf{H} \in \mathrm{M}(\mathbb{R}_{\geq 0})$, where the total variation of $f \in \mathrm{M}(\mathbb{R}_{\geq 0})$ is denoted by $\|f\|_{\mathrm{M}(\mathbb{R}_{\geq 0})}$.

Lemma 3.2. *Let $\sigma_n = n\pi\sqrt{h_0 g}$, $n \in \mathbb{N}$. The function $\mathbf{H} : \mathbb{C}_+ \to \mathbb{C}$ defined in (17) can be represented as*

$$\mathbf{H}(\lambda) = -8h_0 \sum_{n\in\mathbb{N}} \mathbf{H}_n(\lambda) = -8h_0 \sum_{n\in 2\mathbb{N}_0+1} \frac{\lambda}{\lambda^2 + 2\mu\lambda + \sigma_n^2},$$

with inverse Laplace transform $\mathfrak{h} = \mathcal{L}^{-1}(\mathbf{H}) \in \mathrm{M}(\mathbb{R}_{\geq 0})$. Moreover,

$$\mathfrak{h} = \mathfrak{h}_{L^1} + \frac{1}{4c}\mathfrak{h}_\delta = \mathfrak{h}_{L^1} + \frac{1}{4c}\delta_0 + \frac{1}{2c}\sum_{k\in\mathbb{N}} (-1)^k e^{-k\frac{\mu}{c}}\delta_{\frac{k}{c}},$$

where $c = \sqrt{h_0 g}$, and $\mathfrak{h}_{L^1}(t) = e^{-\mu t}(t^2\mathfrak{f}_2(t) + t\mathfrak{f}_1(t) + \mathfrak{f}_0(t))$, $t \geq 0$, for some $\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{f}_2 \in L^\infty(\mathbb{R}_{\geq 0}; \mathbb{R})$.

Proof. The series representation of $\mathbf{H}$ follows from (17) and

$$\tanh(z) = 8z\sum_{k=1}^\infty \frac{1}{\pi^2(2k-1)^2 + 4z^2}, \quad z \notin i\pi(1+2\mathbb{Z}).$$

Next we study the inverse Laplace transform of $\mathbf{H}$; in particular, $\mathbf{H}_n(\lambda) = 0$ for $n \in 2\mathbb{N}_0$. It is clear that $\mathbf{H}$ is also continuous on $\overline{\mathbb{C}_+}$ and that the series converges locally uniformly along the imaginary axis. Thus, the partial sums converge to $\alpha \mapsto \mathbf{H}(i\alpha)$ in the distributional sense when considered as tempered distributions on $i\mathbb{R}$. By continuity of the Fourier transform $\mathcal{F}$, this gives that the series

$$-8h_0 \sum_{n\in\mathbb{N}} \mathcal{F}^{-1}(\mathbf{H}_n(i\cdot)) = -8h_0 \sum_{n\in\mathbb{N}} \mathcal{L}^{-1}(\mathbf{H}_n)$$

converges to $\mathfrak{h} = \mathcal{F}^{-1}(\mathbf{H}(i\cdot)) = \mathcal{L}^{-1}(\mathbf{H})$ in the distributional sense[2]. It remains to study $\mathcal{L}^{-1}(\mathbf{H}_n)$ and the limit of the corresponding sum. By known rules for the Laplace transform, $\mathcal{L}^{-1}(\mathbf{H}_n)(t) = e^{-\mu t}\mathfrak{g}_n(t)$ for $t \geq 0$, where

$$\mathfrak{g}_n(t) = \cos(\phi_n t) - \mu\phi_n^{-1}\sin(\phi_n t), \quad n \in 2\mathbb{N}_0+1,$$

and $\phi_n = \sqrt{\sigma_n^2 - \mu^2}$. The idea is to use Fourier series that are related to the frequencies σ_n in contrast to the 'perturbed' harmonics $\sin\phi_n$ and $\cos\phi_n$. We write

$$\mathfrak{g}_n(t) = [\cos(\phi_n t) - \cos(\sigma_n t)] + \frac{\mu}{\phi_n}[\sin(\sigma_n t) - \sin(\phi_n t)]$$
$$+ \cos(\sigma_n t) + \frac{\mu}{\phi_n}\sin(\sigma_n t).$$

By the mean value theorem there exist $\alpha_n, \beta_n \in [\phi_n, \sigma_n]$ and $\omega_n \in [\alpha_n, \sigma_n]$ such that

$$\cos(\phi_n t) - \cos(\sigma_n t) = t(\sigma_n - \phi_n)\sin(\alpha_n t) = \frac{\mu^2 t\sin(\alpha_n t)}{\sigma_n + \phi_n},$$
$$\sin(\alpha_n t) = t(\alpha_n - \sigma_n)\cos(\omega_n t) + \sin(\sigma_n t),$$
$$\sin(\sigma_n t) - \sin(\phi_n t) = t(\sigma_n - \phi_n)\cos(\beta_n t) = \frac{\mu^2 t\cos(\beta_n t)}{\sigma_n + \phi_n},$$

hence,

$$\mathfrak{g}_n(t) = t^2\frac{\mu^2(\alpha_n - \sigma_n)}{\sigma_n + \phi_n}\cos(\omega_n t) + \frac{\mu^3 t}{\phi_n(\sigma_n + \phi_n)}\cos(\beta_n t)$$
$$+ \cos(\sigma_n t) + \left(t(\sigma_n - \phi_n) + \frac{\mu}{\phi_n}\right)\sin(\sigma_n t).$$

The coefficient sequences of the first two terms in the sum,

$$a_n := \mu^2\frac{\alpha_n - \sigma_n}{\sigma_n + \phi_n}, \quad b_n := \frac{\mu^3}{\phi_n(\sigma_n + \phi_n)},$$

are absolutely summable sequences since

$$0 > a_n > \mu^2\frac{\phi_n - \sigma_n}{\sigma_n + \phi_n} = \frac{-\mu^4}{(\sigma_n + \phi_n)^2}.$$

Let us rewrite the coefficient of the last term, recalling that $\sigma_n^2 - \phi_n^2 = \mu^2$ implies that $\frac{1}{\sigma_n+\phi_n} - \frac{1}{2\sigma_n} = \frac{\mu^2}{2\sigma_n(\sigma_n+\phi_n)^2}$,

$$t(\sigma_n - \phi_n) = \frac{\mu^2 t}{\sigma_n + \phi_n} = \frac{\mu^4 t}{2\sigma_n(\sigma_n + \phi_n)^2} + \frac{\mu^2 t}{2\sigma_n},$$
$$\frac{\mu}{\phi_n} = \frac{\mu}{\phi_n} + \frac{\mu}{\sigma_n} - \frac{\mu}{\sigma_n} = \frac{\mu}{\sigma_n} + \frac{\mu^3}{\sigma_n\phi_n(\sigma_n + \phi_n)}.$$

Thus, with $c_n = \frac{\mu^4}{2\sigma_n(\sigma_n+\phi_n)^2}$ and $d_n = \frac{\mu^3}{\sigma_n\phi_n(\sigma_n+\phi_n)}$, which define absolutely summable sequences, we have

$$\mathfrak{g}_n(t) = t^2 a_n\cos(\omega_n t) + tb_n\cos(\beta_n t) + (tc_n + d_n)\sin(\sigma_n t)$$
$$+ \cos(\sigma_n t) + (\mu t + 2)\frac{\mu}{2\sigma_n}\sin(\sigma_n t).$$

As $\sigma_n = n\pi c$, the sum $\sum_{n\in 2\mathbb{N}_0+1}\frac{4c}{\sigma_n}\sin(\sigma_n t)$ converges to

$$H_0(t) = (-1)^k \quad \text{for } t \in [k/c, (k+1)/c), \; k \in \mathbb{N}_0,$$

---

[2] Here we identify functions on $\mathbb{R}_{\geq 0}$ with their trivial extension to $\mathbb{R}$ and use the relation between Fourier and Laplace transform.

for almost all $t \geq 0$. Therefore, for almost all $t \geq 0$,

$$\sum_{n \in 2\mathbb{N}_0 + 1} \frac{\mu}{2\sigma_n} \sin(\sigma_n t) = \frac{\mu}{8c} H_0(t).$$

Since the coefficients $\frac{\mu}{\sigma_n}$ are square summable, the series even converges in L^2 on any bounded interval and thus particularly in the distributional sense on $\mathbb{R}_{\geq 0}$.

Finally, by known facts on the Fourier series of delta distributions, $4c \sum_{n \in 2\mathbb{N}_0 + 1} \cos(\sigma_n \cdot)$ converges to the $\frac{2}{c}$-periodic extension of $(\delta_0 - 2\delta_{\frac{1}{c}} + \delta_{\frac{2}{c}})$ in the distributional sense as

$$\lim_{N \to \infty} \left\langle 4c \sum_{n=1,\, n\,\text{odd}}^{N} \cos(\sigma_n \cdot), \psi \right\rangle = \left\langle \delta_0 - 2\delta_{\frac{1}{c}} + \delta_{\frac{2}{c}}, \psi \right\rangle$$

for any function $\psi \in C^\infty([0, \frac{2}{c}]; \mathbb{R})$. Altogether, and as multiplying with $e^{-\mu t}$ preserves the distributional convergence, this yields that

$$\sum_{n \in 2\mathbb{N}_0 + 1} \mathcal{L}^{-1}(\mathbf{H}_n)(\cdot) = \sum_{n \in 2\mathbb{N}_0 + 1} e^{-\mu \cdot} \mathfrak{g}_n(\cdot) = \mathfrak{h}_{L^1}(\cdot) + \frac{1}{4c} \mathfrak{h}_\delta$$

with $\mathfrak{h}_{L^1}, \mathfrak{h}_\delta$ as in the assertion and where the functions

$$\mathfrak{f}_2(t) := \sum_{n \in 2\mathbb{N}_0 + 1} a_n \cos(\omega_n t)$$

$$\mathfrak{f}_1(t) := \frac{\mu^2}{8c} H_0(t) + \sum_{n \in 2\mathbb{N}_0 + 1} b_n \cos(\beta_n t) + c_n \sin(\sigma_n t),$$

$$\mathfrak{f}_0(t) := \frac{\mu}{4c} H_0(t) + \sum_{n \in \mathbb{N}} d_n \sin(\sigma_n t), \qquad t \geq 0,$$

are bounded since a_n, b_n, c_n, d_n are absolutely summable sequences. By this representation, $\mathfrak{h}_{L^1} \in L^1(\mathbb{R}_{\geq 0}; \mathbb{R})$ and can thus be identified with an element in $M(\mathbb{R}_{\geq 0})$, while $\mathfrak{h}_\delta \in M(\mathbb{R}_{\geq 0})$ as $\|\mathfrak{h}_\delta\|_{M(\mathbb{R}_{\geq 0})} = 1 + 2\sum_{k \in \mathbb{N}} e^{-\mu \frac{k}{c}} < \infty$. $\square$

For regular well-posed systems, it is convenient to consider an extension of the observation operator $C \in \mathcal{B}(X_1; Y)$, the so-called Λ-extension $C_\Lambda : \mathcal{D}(C_\Lambda) \to Y$, defined by

$$C_\Lambda x = \lim_{\lambda \to \infty} \lambda C(\lambda I - A)^{-1} x$$

with $\mathcal{D}(C_\Lambda) = \{ x \in X \mid \lim_{\lambda \to \infty} \lambda C(\lambda I - A)^{-1} x \text{ exists} \}$. It is easy to see that this indeed defines an extension of C, cf. [34]. In the following we will replace the operator C from (12) by its Λ-extension; thus, in particular, in (12),

$$\mathcal{D}(C) = \mathcal{D}(C_\Lambda). \tag{18}$$

Recall that the unique strong solution of (11) is given by (15).

Proposition 3.3. *Let* $x_0 \in X$ *and* $v_0 \in \mathbb{C}$ *such that* $x_0 + bv_0 \in \mathcal{D}(A)$. *Then*

$$\mathcal{F} : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{C}) \to L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}; X \times \mathbb{C}^2), \eta \mapsto \begin{bmatrix} I \\ C_\Lambda \end{bmatrix} x,$$

with x *as in* (15)*, is well-defined and*

$$\|\mathcal{F}(\eta)\|_\infty \lesssim \|x_0\| + \|Ax_0\| + \|\eta\|_\infty$$

for all $\eta \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{C}) \cap L^\infty(\mathbb{R}_{\geq 0}; \mathbb{C})$.

Proof. Since $\mu > 0$, the transfer function $\mathbf{G}$ of the regular well-posed system $(A, B, \begin{bmatrix} I \\ C \end{bmatrix})$, given in Proposition 3.1, is bounded on $\mathbb{C}_+$. Therefore, see e.g. [34, Thm. 5.3], $\mathcal{F}$ is well-defined on $L^2_{\text{loc}}(\mathbb{R}_{\geq 0}; \mathbb{C})$ and maps $L^2(\mathbb{R}_{\geq 0}; \mathbb{C})$ to $L^2(\mathbb{R}_{\geq 0}; X \times \mathbb{C}^2)$ with

$$\mathcal{L}(\mathcal{F}(\eta))(s) = \begin{bmatrix} I \\ C_\Lambda \end{bmatrix} (sI - A)^{-1} x_0 + \mathbf{G}(s) \cdot \mathcal{L}(\eta)(s)$$

for all $s \in \mathbb{C}_+, \eta \in L^2(\mathbb{R}_{\geq 0}; \mathbb{C})$. Identifying $\mathcal{L}^{-1}(\mathbf{G})$ and η with their extension by 0 on $\mathbb{R}$, we hence get for a.a. $t > 0$,

$$\mathcal{F}(\eta)(t) = \begin{bmatrix} I \\ C_\Lambda \end{bmatrix} T(t) x_0 + (\mathcal{L}^{-1}(\mathbf{G}) * \eta)(t)$$
$$= \begin{bmatrix} I \\ C_\Lambda \end{bmatrix} T(t)(x_0 + bv_0) + (\mathcal{L}^{-1}(\mathbf{G}) * \tilde{\eta})(t), \quad (19)$$

with $\tilde{\eta}(t) = e^t v_0 \chi_{\mathbb{R}_{\leq 0}}(t) + \eta(t)$, where $\chi_{\mathbb{R}_{\leq 0}}$ denotes the indicator function on $\mathbb{R}_{\leq 0}$. Here, we used that

$$- \begin{bmatrix} I \\ C_\Lambda \end{bmatrix} T(t) b = \begin{bmatrix} I \\ C_\Lambda \end{bmatrix} \int_{-\infty}^0 T_{-1}(t-s) B e^s \, ds$$
$$= (\mathcal{L}^{-1}(\mathbf{G}) * e_1 \chi_{\mathbb{R}_{\leq 0}})(t)$$

for a.a. $t > 0$, with $e_1(s) = e^s$, $s \in \mathbb{R}$. The first term on the right hand side of (19) is uniformly bounded on $\mathbb{R}_{\geq 0}$, because $x_0 + bv_0 \in \mathcal{D}(A)$ and $\|T(t)\| \leq 1$ for all $t \geq 0$ by Proposition 3.1. To show uniform boundedness of the second term, it suffices to show that $\mathcal{L}^{-1}(\mathbf{G})$ is a measure of bounded total variation, cf. [16]. By $\mathbf{G} = (\mathbf{L}, \mathbf{H}, 0)$ and Lemma 3.2, this reduces to proving that $\mathcal{L}^{-1}(\mathbf{L}) \in M(\mathbb{R}_{\geq 0})$. More precisely, we claim that $\mathcal{L}^{-1}(\mathbf{L})(t) = e^{-\epsilon t} f(t)$, $t > 0$, for some $f \in L^2(\mathbb{R}_{\geq 0}; X)$ and $\epsilon > 0$. By the vector-valued Paley-Wiener theorem, this holds if, and only if, $\mathbf{L}(\cdot - \epsilon)$ lies in the Hardy space $\mathcal{H}^2(\mathbb{C}_+; X)$ of holomorphic functions $G : \mathbb{C}_+ \to X$ such that $\sup_{x > 0} \|G(x + i \cdot)\|_{L^2(\mathbb{R}_{\geq 0}; X)} < \infty$. The latter finally follows for sufficiently small $\epsilon \in (0, \mu)$ by the formula of $\mathbf{L}$ from Proposition 3.1 and elementary estimates. $\square$

4. Simulations

In this section we illustrate the application of the funnel controller (6) to the system (4). Using the change of variables $z(t, \zeta) = Q \begin{pmatrix} \eta_1(t,\zeta) \\ \eta_2(t,\zeta) \end{pmatrix}$ with $Q := \begin{bmatrix} \frac{1}{g} & \frac{1}{g} \\ \frac{g}{c} & -\frac{g}{c} \end{bmatrix}$ in (2) enables us to solve the PDE corresponding to η_1 with an implicit finite difference method and the PDE corresponding to η_2 with an explicit finite difference method, respectively. For the simulation we have used the parameters $m = 1\text{kg}$, $h_0 = 0.5\text{m}$, $g = 9.81\text{ms}^{-2}$, $\mu = 0.1\text{Hz}$ and the reference signal $y_{\text{ref}}(t) = \tanh^2(\omega t)$ with $\omega = 0.06\pi f$, $f = \sqrt{g/h_0}$. The initial values (7) are chosen as $x_0(\zeta) = (h_0, 0.1 \sin^2(4\pi\zeta)\text{ms}^{-1})$ and $(y^0, y^1) = (0\text{m}, 0\text{ms}^{-1})$. For the controller (6) we chose the funnel functions $\varphi_0(t) = \varphi_1(t) = 100 \tanh(\omega t)$. Clearly, Condition (8) is satisfied. For the finite differences we used a grid in t with $M = 2000$ points for the interval $[0, 2\tau]$ with $\tau = f^{-1}$, and a grid in ζ with $N = \lfloor ML/(4c\tau) \rfloor$ points. The method has been implemented in Python and the simulation results are shown in Figs. 3 and 4.

It can be seen that even in the presence of sloshing effects a prescribed performance of the tracking error can be achieved with the funnel controller (6), while at the same time the generated input is bounded and exhibits an acceptable performance. Finally, we demonstrate that
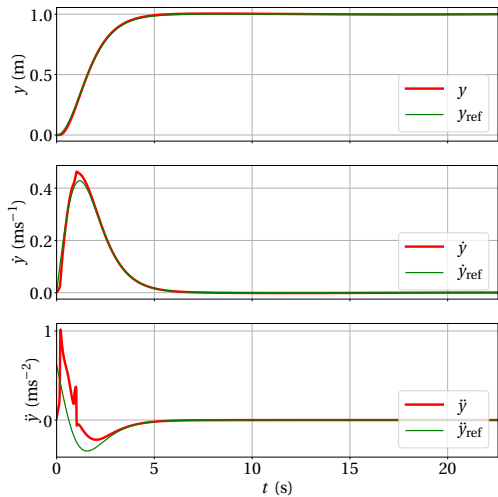
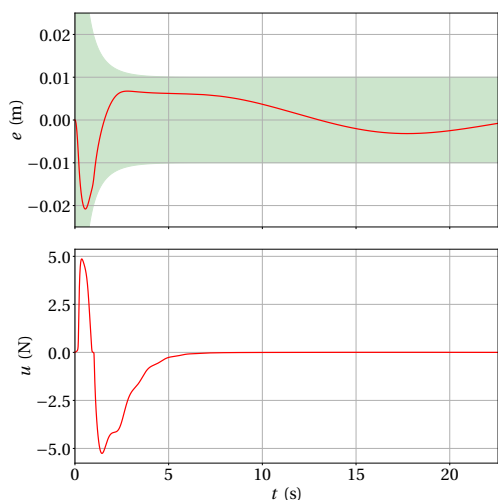Figure 3: Output y, reference signal y_{ref} and corresponding first and second derivatives.

Figure 4: Performance funnel with tracking error e and input u.

Figure 5: Reference signal y_{ref}, outputs y_n for (1), (6) and y_l for (2), (6) and corresponding first and second derivatives.

Figure 6: Performance funnel with tracking errors e_n and e_l and inputs u_n and u_l for (1), (6) and (2), (6), respectively.

the controller (6) is also feasible for the nonlinear Saint-Venant equations (1) in certain situations. For purposes of illustration we consider a friction term of the form $S(z) = C_D z + C_S z^2$ with $C_D = 2\mu$ and $C_S = 1$. Analogously as for linearized model, utilizing the momentum leads to the equation

$$m\ddot{y}(t) = u(t) + C_D \int_0^1 h(t, \zeta)v(t, \zeta)\,\mathrm{d}\zeta$$
$$+ C_S \int_0^1 v(t, \zeta)^2\,\mathrm{d}\zeta + \frac{g}{2}\big(h(t, 1)^2 - h(t, 0)^2\big),$$

which is used for the simulation instead of (4b) by applying the same method as described above. We choose the same parameters as in the first simulation, except for $\mu = 0.01\,\mathrm{Hz}$, $\omega = 0.025\,\mathrm{Hz}$ and $\varphi_0(t) = \varphi_1(t) = 10\tanh(\omega t)$. We compare simulations of the nonlinear Saint-Venant equations (1) with initial values $\big(h(0, \zeta), v(0, \zeta)\big) = (h_0, 0\mathrm{ms}^{-1})$ under control (6) with the linearized equations (4a) with initial values $x_0(\zeta) = (h_0, 0\mathrm{ms}^{-1})$ under control (6); the results are shown in Figs. 5 and 6. Note that compared to the first simulation, lower frequencies μ and ω and a zero initial velocity are chosen here.
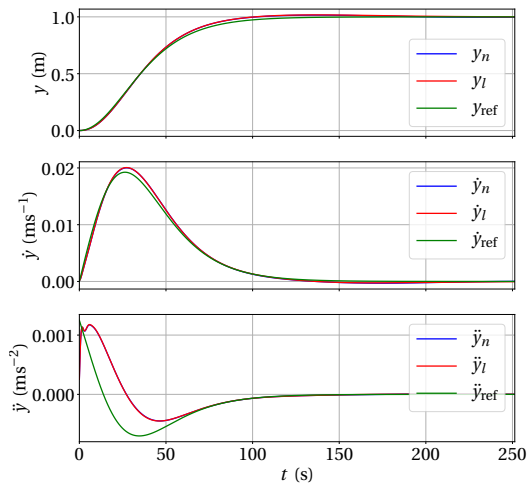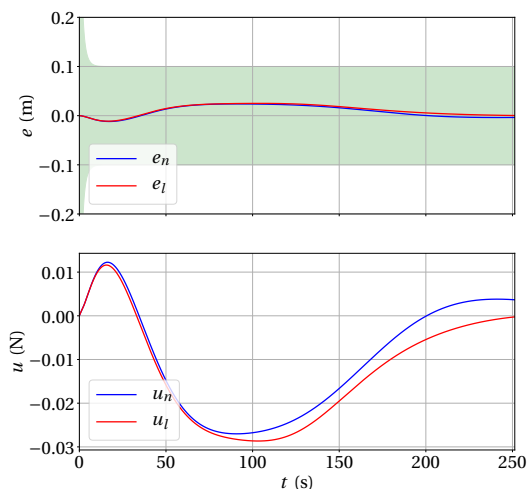
5. Conclusion

In the present paper we have shown that the controller (6) is feasible for the moving water tank system (4) which rests on the linearized Saint-Venant equations. We stress that the system is still nonlinear as the (linearized) sloshing effects influence the position of the cart through the momentum, which constitutes an internal feedback loop in (4). Nevertheless, the funnel controller is able to handle these effects as shown in Theorem 2.1 and in the simulations in Section 4.

We stress that the applicability of the results from [4, 6] on funnel control strongly rests on the fact that the original open-loop system can be viewed as an ODE-PDE coupling with an input-output relation allowing for a relative degree, i.e., the form (Sys) mentioned in Section 1.3. This, however, is in general not the case for systems governed by evolution equations and different approaches are required then, see [3, 28, 29]. Furthermore, we like to point out that the controller (6) requires that the derivative of the output is available for control. This may not be true in practice, and it may even be hard to obtain suitable estimates of the output derivative. This drawback may be resolved by combining the controller (6) with a funnel pre-

compensator as developed in [9, 10], which results in a pure output feedback.

Several extensions of the studied system (4) may be considered in future research, such as sloshing suppressing valves inside the tank, the interconnection of the tank with a truck as in [15] and, of course, the general nonlinear equations (1) as well as the higher-dimensional case.

Another issue is that we assume $\mu > 0$ for the friction term. This implies that the system's energy converges to the steady state exponentially. In the case $\mu = 0$ the statement of Theorem 2.1 is false in general. More precisely, if $\mu = 0$, then $\mathfrak{h} = \mathcal{L}^{-1}(\mathbf{H})$ derived in Lemma 3.2 does not have bounded total variation, by which $\widetilde{\mathcal{T}}$ from the proof of Theorem 2.1 is not bounded-input, bounded-output stable. It seems natural to assume some kind of damping in (1), but one may relax the assumption of exponential stability, even in the linearized case. This requires refined methods, whose development is a topic of future research. In this context, let us mention the recent work [32], where stabilization of a linearized 2D-shallow water system subject to polynomial damping was considered.

Acknowledgements

References

[1] Auroux, D., Bonnabel, S., 2011. Symmetry-based observers for some water-tank problems. IEEE Trans. Autom. Control 56, 1046–1058.

[2] Bastin, G., Coron, J.M., 2016. Stability and Boundary Stabilization of 1-D Hyperbolic Systems. volume 88 of *Progress in Nonlinear Differential Equations and Their Applications: Subseries in Control*. Birkhäuser.

[3] Berger, T., Breiten, T., Puche, M., Reis, T., 2019a. Funnel control for the monodomain equations with the FitzHugh-Nagumo model. Submitted for publication. Available at arXiv: https://arxiv.org/abs/1912.01847.

[4] Berger, T., Lê, H.H., Reis, T., 2018. Funnel control for nonlinear systems with known strict relative degree. Automatica 87, 345–357.

[5] Berger, T., Otto, S., Reis, T., Seifried, R., 2019b. Combined open-loop and funnel control for underactuated multibody systems. Nonlinear Dynamics 95, 1977–1998.

[6] Berger, T., Puche, M., Schwenninger, F.L., 2020. Funnel control in the presence of infinite-dimensional internal dynamics. Syst. Control Lett. 139, Article 104678.

[7] Berger, T., Rauert, A.L., 2020. Funnel cruise control. Automatica 119, Article 109061.

[8] Berger, T., Reis, T., 2014. Zero dynamics and funnel control for linear electrical circuits. J. Franklin Inst. 351, 5099–5132.

[9] Berger, T., Reis, T., 2018a. Funnel control via funnel precompensator for minimum phase systems with relative degree two. IEEE Trans. Autom. Control 63, 2264–2271.

[10] Berger, T., Reis, T., 2018b. The Funnel Pre-Compensator. Int. J. Robust & Nonlinear Control 28, 4747–4771.

[11] Cardoso-Ribeiro, F.L., Matignon, D., Pommier-Budinger, V., 2017. A port-Hamiltonian model of liquid sloshing in moving containers and application to a fluid-structure system. Journal of Fluids and Structures 69, 402–427.

[12] Coron, J.M., 2002. Local controllability of a 1-d tank containing a fluid modeled by the shallow water equations. ESAIM Control Optim. Calc. Var. 8, 513–554.

[13] Dubois, F., Petit, N., Rouchon, P., 1999. Motion planning and nonlinear simulations for a tank containing a fluid, in: Proc. 5th European Control Conf. (ECC) 1999, Karlsruhe, Germany, Karlsruhe, Germany. pp. 3232–3237.

[14] Feddema, J.T., Dohrmann, C.R., Parker, G.G., Robinett, R.D., Romero, V.J., Schmitt, D.J., 1997. Control for slosh-free motion of an open container. IEEE Control Systems Magazine 17, 29–36.

[15] Gerdts, M., Kimmerle, S.J., 2015. Numerical optimal control of a coupled ODE-PDE model of a truck with a fluid basin, in: Proc. 10th AIMS Int. Conf. Dynam. Syst. Diff. Equ. Appl., Madrid, Spain. pp. 515–524.

[16] Grafakos, L., 2014. Classical Fourier analysis. volume 249 of *Graduate Texts in Mathematics*. 3rd ed., Springer-Verlag, New York.

[17] Graham, E.W., Rodriguez, A.M., 1951. The characteristics of fuel motion which affect airplane dynamics. DTIC Document.

[18] Grundelius, M., Bernhardsson, B., 1999. Control of liquid slosh in an industrial packaging machine, in: Proc. IEEE Int. Conf. Control Appl., Hawai, USA. pp. 1654–1659.

[19] Hackl, C.M., 2017. Non-identifier Based Adaptive Control in Mechatronics–Theory and Application. volume 466 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Cham, Switzerland.

[20] Hackl, C.M., Hopfe, N., Ilchmann, A., Mueller, M., Trenn, S., 2013. Funnel control for systems with relative degree two. SIAM J. Control Optim. 51, 965–995.

[21] Ilchmann, A., Ryan, E.P., 2008. High-gain control without identification: a survey. GAMM Mitt. 31, 115–125.

[22] Ilchmann, A., Ryan, E.P., Sangwin, C.J., 2002. Tracking with prescribed transient behaviour. ESAIM: Control, Optimisation and Calculus of Variations 7, 471–493.

[23] Ilchmann, A., Selig, T., Trunk, C., 2016. The Byrnes-Isidori form for infinite-dimensional systems. SIAM J. Control Optim. 54, 1504–1534.

[24] Ilchmann, A., Trenn, S., 2004. Input constrained funnel control with applications to chemical reactor models. Syst. Control Lett. 53, 361–375.

[25] Petit, N., Rouchon, P., 2002. Dynamics and solutions to some control problems for water-tank systems. IEEE Trans. Autom. Control 47, 594–609.

[26] Pomprapa, A., Weyer, S., Leonhardt, S., Walter, M., Misgeld, B., 2015. Periodic funnel-based control for peak inspiratory pressure, in: Proc. 54th IEEE Conf. Decis. Control, Osaka, Japan, pp. 5617–5622.

[27] Prieur, C., de Halleux, J., 2004. Stabilization of a 1-d tank containing a fluid modeled by the shallow water equations. Syst. Control Lett. 52, 167–178.

[28] Puche, M., Reis, T., Schwenninger, F.L., 2019. Funnel control for boundary control systems. Submitted for publication. Available at arXiv: https://arxiv.org/abs/1903.03599.

[29] Reis, T., Selig, T., 2015. Funnel control for the boundary controlled heat equation. SIAM J. Control Optim. 53, 547–574.

[30] de Saint-Venant, A., 1871. Théorie du mouvement non permanent des eaux avec applications aux crues des rivières et à l'introduction des marées dans leur lit. Comptes Rendus de l'Académie des Sciences de Paris 73, 147–154 and 237–240.

[31] Staffans, O., 2005. Well-Posed Linear Systems. volume 103 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge.

[32] Su, P., Tucsnak, M., Weiss, G., 2020. Stabilizability properties of a linearized water waves system. Syst. Control Lett. 139, Article 104672.

[33] Tucsnak, M., Weiss, G., 2009. Observation and Control for Operator Semigroups. Birkhäuser Advanced Texts Basler Lehrbücher, Birkhäuser, Basel.

[34] Tucsnak, M., Weiss, G., 2014. Well-posed systems – the LTI case and beyond. Automatica 50, 1757–1779.

[35] Venugopal, R., Bernstein, D., 1996. State space modeling and active control of slosh, in: Proc. IEEE Int. Conf. Control Appl., Dearborn, MI. pp. 1072–1077.

[36] Yano, K., Yoshida, T., Hamaguchi, M., Terashima, K., 1996. Liquid container transfer considering the suppression of sloshing for the change of liquid level, in: Proc. 13th IFAC World Congress, San Francisco, CA. pp. 701–706.