G-isotropy of T-relative equilibria within manifolds tangent to spaces with linearly independent weights

Mara Sommerfeld
mara.sommerfeld@uni-hamburg.de

Departement of Mathematics, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany

Abstract

We investigate the generic local structure of relative equilibria in Hamiltonian systems with symmetry G near a completely symmetric equilibrium, where G is compact and connected. Fix a maximal torus $T \subset G$ and identify the equilibrium with the origin within a symplectic representation of G. By a previous result, generically, for each $\xi \in \mathfrak{t}$ such that $V_0 := \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ has linearly independent weights, there is a manifold tangent to V_0 that consists of relative equilibria with generators in $\mathfrak{t}$. Here we determine their isotropy with respect to G. The main result asserts that for each of these manifolds of T-relative equilibria, there is a local diffeomorphism to its tangent space at 0 that preserves the isotropy groups. We will then deduce that the G-orbit of the union of these manifolds is stratified by isotropy type. The stratum given by the relative equilibria of type (H) has the dimension $\dim G - \dim H + \dim(\mathfrak{t}')^L$, where $\mathfrak{t}' \subset \mathfrak{t}$ is the orthogonal complement of $\mathfrak{h} \cap \mathfrak{t}$ and L is the minimal adjoint isotropy subgroup of an element of $\mathfrak{t}$ with $H \subset L$. In the end, we consider some examples of these manifolds of T-relative equilibria that contain points with the same isotropy type with respect to the T-action but different isotropy type with respect to G.

Introduction

Symmetries play a central role in the description of physical systems. In classical mechanics, we usually consider actions with compact isotropy groups of rank 1, but in non-classical physics we often have to deal with groups of higher rank. For instance in gauge theories, the symmetries of the vacuum state are typically products of special unitary groups.

In my thesis [19], I investigated classical finite-dimensional Hamiltonian systems with proper group actions, but with connected isotropy groups of rank greater than 1. Even though physicists usually consider infinite dimensional systems, this could be a step to a better understanding of the dynamics caused by high dimensional symmetries.

The main objects of interest of [19] are *relative equilibria*: states whose trajectories are contained in their group orbit. In classical mechanics, they often correspond to solutions of constant shape. If the physical system is modeled by a gauge system, relative equilibria describe stationary physical states.

One of the main results of [19] is of local nature: We consider an isolated equilibrium with connected isotropy group G and investigate the generic structure of relative equilibria in a small neighborhood of the equilibrium. If the equilibrium is a minimum of the Hamiltonian, this means that we only consider low energy states. Since we are only interested in the local structure, it suffices to consider a G-representation V with a G-invariant symplectic form and a G-invariant Hamiltonian function h with a critical point at the origin.

The symplectic form induces a G-invariant complex structure on the center space V^c of $\mathrm{d}X_h(0)$, which is considered as a complex representation.

The approach to find relative equilibria is to fix a maximal torus T of G and to search for invariant T-orbits. Due to the conjugation theorem, their G-orbits form the set of relative equilibria with respect to the G-action.

As discussed in [19, Section 5.2.2], the equivariant Weinstein-Moser theorem, introduced by Montaldi, Roberts, and Stewart [16] already implies that generically for each irreducible subrepresentation W of V^c and each weight that occurs with multiplicity 1 in W there is a local manifold of invariant T-orbits that is at the origin tangent to the corresponding weight space in W. Alternatively this follows from a theorem by Ortega and Ratiu [17, Theorem 4.1].

The main observation in [19] is that, in the case $\mathrm{rank}(G) > 1$, also for some particular sums of weight spaces there are manifolds of invariant T-orbits whose tangent spaces at the origin coincide with these sums: Generically, we obtain such a manifold for each kernel of the form $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ that has only linearly independent weights that occur with multiplicity 1. (Here $\mathbf{J}^\xi$ denotes the momentum map $\mathbf{J}$ evaluated at $\xi \in \mathfrak{t}$.) Moreover, the union of these manifolds forms a Whitney-stratified set. The proof of this statement is based on an application of equivariant transversality theory, which was developed independently by Bierstone [1, 2] and Field [8, 10] in the 70's of the last century. The method is very similar to Field and Richardson's approach to equivariant bifurcation theory (see [11, 7, 9]).

This article deals with the computation of the isotropy types of these relative equilibria. The isotropy type is not only an interesting property of the solution but also gives information about the overall structure of relative equilibria, since the G-orbits of these manifolds consist of relative equilibria.

Moreover, we will need to compute the isotropy types of the generators and momenta, which also play a key role in the description of the structure of relative equilibria in Hamiltonian systems: Patrick and Roberts [18] have shown that the relative equilibria of Hamiltonian systems with a compact connected symmetry group G that acts freely are generically stratified by the isotropy type $(K) := (G_\mu \cap G_\xi)$ of their momentum generator pair (μ, ξ) with respect to the sum of the coadjoint and adjoint action on $\mathfrak{g}^* \oplus \mathfrak{g}$. The corresponding stratum has the dimension $\dim G + 2 \dim Z(K) - \dim K$, where $Z(K)$ is the center of K. To prove this, they define a generic transversality condition. As pointed out in [19], this condition is a special case of a transversality condition that is natural in terms of equivariant transversality theory and is also valid for non-free actions. This implies that Patrick and Robert's result generically holds for the action of the group $\left(N(H)/H\right)^\circ$ on the set of points with isotropy group H.

We will see that an application of this theory shows that the G-orbit of the set formed by the manifolds of T-relative equilibria tangent to the kernels of the form $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$, $\xi \in \mathfrak{t}$, with linearly independent weights is stratified by isotropy type. If $H \subset G$ is an isotropy subgroup, then the stratum of the relative equilibria with isotropy type (H) has the dimension

$$\dim G - \dim H + \dim(\mathfrak{t}')^L,$$

where $\mathfrak{t}'$ denotes the orthogonal complement of $\mathfrak{t} \cap \mathfrak{h}$ in $\mathfrak{t}$ and $L = L(H)$ is the minimal isotropy subgroup of an element of $\mathfrak{t}$ with respect to the adjoint action that contains H.

The main result to determine the isotropy subgroups asserts that for each of these manifolds there is a diffeomorphism to a neighborhood of 0 within its tangent space at the origin which preserves the isotropy groups. Thus the problem is reduced to a problem in representation theory.

This justifies the approach of [19]: The main idea is to investigate the structure of the T-relative equilibria since they are contained in every G-orbit of relative equilibria. At first glance, the method seems to have a drawback since we forget information about the G-action in between and perform a Lyapunov-Schmidt reduction with only T-invariant spaces. The results presented in this article show that anyhow the G-isotropy groups are preserved.

In the end, we will also illustrate an approach to compute the isotropy types on the Lie algebra level in the relevant subspaces of a given G-representation and discuss some examples. A striking observation is that points of the same stratum of T-relative equilibria can have different isotropy types with respect to the G-action (so the G-orbit of the stratum is not necessarily a manifold) and their isotropy groups can even be greater than the isotropy groups of the corresponding weight spaces. (If we call the weight vectors *pure states* and points with nonzero components in weight spaces corresponding to different weight *mixed states*, then the mixed states can have more symmetry than each of their pure components).

1 Preliminaries

1.1 Notation corresponding to the group action

Throughout, let G denote a compact connected Lie group and $\mathfrak{g}$ be its Lie algebra. If P is a space with a G-action and $p \in P$, then the *isotropy subgroup* of p is given by $G_p = \{g \in G \mid gp = p\}$. Its conjugacy class is the *isotropy type* of p, denoted by (G_p). The Lie algebra $\mathfrak{g}_p$ of G_p is called the *isotropy Lie algebra* of p. Two points p and q have the same *type on the Lie algebra level* if there is $g \in G$ with $\mathrm{Ad}_g(\mathfrak{g}_p) = \mathfrak{g}_q$, where Ad denotes the adjoint action of G on $\mathfrak{g}$, given by

$$\mathrm{Ad}_g\, x := \frac{\mathrm{d}}{\mathrm{d}t} g \exp(t\xi) g^{-1}\big|_{t=0}.$$

For a subgroup $H \subset G$ and a subset $M \subset P$, the set of fixed points of H within M is denoted by $M^H = \{m \in M \mid hm = m\ \forall h \in H\}$, the subset of points of isotropy type (H) by $M_{(H)} = \{m \in M \mid G_m \in (H)\}$ and the subset of points with isotropy subgroup H by $M_H = M^H \cap M_{(H)}$.

1.2 Notation and facts from representation theory

Here we provide basic definitions and facts from the representation theory of compact connected Lie groups, for details see for instance [3] or [14].

We will later on regard symplectic representations as complex representations. Thus we consider complex representations in this section.

A key idea in representation theory and as well, in our approach to find relative equilibria, is to fix a maximal torus $T \subset G$ and to consider a given finite-dimensional complex representation V as a T-representation.

Every group element is contained in a maximal torus, and all maximal tori are conjugated to T. Correspondingly, each element of the Lie algebra $\mathfrak{g}$ of G is contained in the Lie algebra of a maximal torus, and every Lie algebra of a maximal torus coincides with the image of the Lie algebra $\mathfrak{t}$ of T under the adjoint action of an element $g \in G$.

As a T-representation, V splits into the T-irreducible subrepresentations, the *weight spaces*. Each weight space is 1-dimensional and of the following form: Consider the exponential map $\exp : \mathfrak{t} \to T$. If we use the identification $T \cong \mathbb{R}^n/\mathbb{Z}^n$, then exp coincides with the projection $\mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$. An element $\alpha \in \mathfrak{t}^*$ is called an *integral form* iff α maps ker(exp) to $\mathbb{Z}$. (When we identify $\mathfrak{t}^* \cong (\mathbb{R}^n)^*$ with $\mathbb{R}^n$ via the standard inner product, then the set of integral forms coincides with $\mathbb{Z}^n$.) Every integral form α defines a T-representation on $\mathbb{C}$, which we denote by $\mathbb{C}_\alpha$, via

$$T \ni \exp(\xi) \mapsto \mathrm{e}^{2\pi \mathrm{i} \alpha(\xi)} \in \mathrm{U}(1).$$

These are exactly the irreducible complex T-representations. If $\mathbb{C}_\alpha$ is a weight space of V, then α is the corresponding *weight*. Any non-zero element of $\mathbb{C}_\alpha$ is called a *weight vector* corresponding to α. (Note that this definition of weights equates to *real infinitesimal weights* or *real weights* in [3] and [14].)

Of particular importance are the weights of the complexified adjoint action

$$\begin{aligned} G \times (\mathfrak{g} \otimes \mathbb{C}) &\to \mathfrak{g} \otimes \mathbb{C} \\ (g, \xi \otimes z &\mapsto (\mathrm{Ad}_g\, \xi) \otimes z, \end{aligned}$$

which are called *roots*. Accordingly, the weight spaces and weight vectors of $\mathfrak{g}$ are called *root spaces* and *root vectors* respectively. The roots occur in pairs $\pm\rho \in \mathfrak{t}^*$. As in [3], we denote the root space corresponding to ρ by L_ρ and the space $(L_\rho \oplus L_{-\rho}) \cap \mathfrak{g}$ corresponding to $\pm\rho$ within the real Lie algebra $\mathfrak{g}$ by $M_\rho = M_{-\rho}$. If R^+ is a subset of the set of roots R that contains exactly one element of each pair $\pm\rho$, then

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\rho \in R^+} M_\rho.$$

The kernels of the roots are hyperplanes in $\mathfrak{t}$, called *Weyl walls*. The connected components of $\mathfrak{t} \backslash \left(\bigcup U_{\rho \in R^+} \ker \rho\right)$ are called *Weyl chambers*, their closures *closed Weyl chambers*. The orthogonal reflections about the Weyl walls with respect to an adjoint-invariant inner product on $\mathfrak{g}$ restricted to $\mathfrak{t}$ generate the Weyl group $W = W(G) := {N(T)}\big/{T}$. The W-action on $\mathfrak{t}$ coincides with the one which is induced by the adjoint action restricted to $N(T) \times \mathfrak{t}$, see [3, Chapter V, (2.19)].

Each Weyl group orbit intersects every closed Weyl chamber in exactly one point. The weights of a G-representation V are W-invariant. If we fix a particular closed Weyl chamber C, each Weyl group orbit of weights of V has hence a unique element in C. We call a weight $\alpha \in C$ *higher* than a weight $\beta \in C$ iff β is contained in the convex hull of the Weyl group orbit of α. Then the irreducible G-representations are in 1-to-1-correspondence with the integral elements λ of C: For each λ there is a unique irreducible finite dimensional complex G-representation with highest weight λ. Moreover, the Weyl walls determine the adjoint isotropy subgroups of the elements of $\mathfrak{t}$:

Lemma 1.1. *Isotropy subgroups with respect to the adjoint action are connected.*

Proof. Consider $\xi \in \mathfrak{g}$. By [3, Chapter IV, (2.3)(ii)] the adjoint isotropy group G_ξ coincides with the union of the maximal tori that contain ξ. □

Thus, the adjoint isotropy groups are determined by their Lie algebras.

Lemma 1.2 (Infinitesimal formulation of [3, Chapter V, (2.3)(ii)])**.** *The adjoint isotropy Lie algebra of $\xi \in \mathfrak{t}$ is given by*

$$\mathfrak{g}_\xi = \mathfrak{t} \oplus \bigoplus_{\rho \in R^+, \xi \in \ker \rho} M_\rho.$$

In other words: The isotropy subspaces of the adjoint action intersected with $\mathfrak{t}$ are given by the intersections of Weyl walls.

Corollary 1.3. *Let K and L be isotropy subgroups of points in $\mathfrak{t}$ with respect to the adjoint action, and let $\mathfrak{t}^K$ and $\mathfrak{t}^L$ be their fixed point subspaces within $\mathfrak{t}$ respectively. Then $K \cap L$ is also an isotropy of the adjoint action and its fixed point subspace within $\mathfrak{t}$ is given by the intersection of $\mathfrak{t}$ and all Weyl walls that contain both $\mathfrak{t}^K$ and $\mathfrak{t}^L$.*

Proof. Choose $\xi_K, \xi_L \in \mathfrak{t}$ with $G_{\xi_K} = K$ and $G_{\xi_L} = L$. Choose $\varepsilon > 0$ small enough such that $\xi := \xi_K + \varepsilon \xi_L$ is not contained in any Weyl wall that does not contain ξ_K. Then ξ is contained exactly in the Weyl walls that contain ξ_K and ξ_L. Thus by Lemma 1.2 $\mathfrak{g}_\xi = \mathfrak{k} \cap \mathfrak{l}$, where $\mathfrak{k}$ and $\mathfrak{l}$ denote the Lie algebras of K and L respectively. Since G_ξ is connected, we obtain

$$G_\xi = G_\xi^\circ = (K \cap L)^\circ \subset K \cap L.$$

Conversely, we obviously have $K \cap L \subset G_\xi$. □

The roots also contain important information about the G-action on the representation V: The G-action defines a Lie algebra action of $\mathfrak{g}$ on V, that is a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(V)$, given by

$$\xi x := \frac{\mathrm{d}}{\mathrm{d}t} \exp(t\xi) x \big|_{t=0}.$$

Since V is a complex representation, this induces an action of the complexified Lie algebra $\mathfrak{g} \otimes \mathbb{C}$ by $(\xi \otimes z)x := z\xi x$.

Note that $x \in V$ is a weight vector corresponding to the weight α iff it holds for any $\xi \in \mathfrak{t}$ that

$$\xi x = (\xi \otimes 1)x = 2\pi \mathrm{i}\alpha(\xi)x.$$

Lemma 1.4. *If $Z \in \mathfrak{g} \otimes \mathbb{C}$ is a root vector corresponding to ρ and $x \in V$ is a weight vector corresponding to α, then Zx is either 0 or a weight vector with weight $\alpha + \rho$.*

Proof. If $\xi \in \mathfrak{t}$, then

$$\xi(Zx) = [\xi, Z]x + Z(\xi x) = 2\pi \mathrm{i}\rho(\xi)Zx + Z(2\pi \mathrm{i}\alpha(\xi)x) = (2\pi \mathrm{i}(\alpha + \rho)(\xi))(Zx).$$

□

Note also that $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{t} \otimes \mathbb{C} \oplus \bigoplus_{\rho \in R} L_\rho$. Thus if we consider a particular affine subspace A of $\mathfrak{t}^*$ whose underlying subspace is given by the span of the roots, then the sum of the weight spaces corresponding to the weights contained in A is $\mathfrak{g} \otimes \mathbb{C}$-invariant and hence G-invariant. We will use this fact below.

More precisely, the weight structure of an irreducible representation is as follows:

Theorem 1.5 ([14, Theorem 10.1])**.** *If V_λ is an irreducible finite-dimensional complex G-representation with highest weight λ, then an integral element $\alpha \in \mathfrak{t}^*$ is a weight of V_λ iff α is contained in the convex hull of $W\lambda$ and $\lambda - \alpha$ is an integer combination of roots of G.*

(Note that the weights can have a higher multiplicity than 1. The highest weight always has multiplicity 1. The multiplicities of the other weights are for instance determined by Kostant's multiplicity formula, see [3, Chapter VI, (3.2) and (3.3)].)

In the applications, we will also consider the coadjoint representation, the dual of the adjoint representation. Choosing an adjoint invariant inner product, we can identify both representations and accordingly $\mathfrak{t}$ and $\mathfrak{t}^*$. This defines the Weyl walls in $\mathfrak{t}^*$. Since we will often consider G_μ-invariant subrepresentations, where μ is a momentum, which is an element of $\mathfrak{t}^*$, we formulate the following results in the coadjoint version.

Lemma 1.6. *Let V be a complex G-representation. For any $\mu \in \mathfrak{t}^*$, let $(\mathfrak{t}^*)^\mu$ denote the intersection of $\mathfrak{t}^*$ and all Weyl walls in $\mathfrak{t}^*$ that contain $\mu \in \mathfrak{t}^*$. Let A be the set of weights contained in a particular affine subspace that is a shift of the orthogonal complement of $(\mathfrak{t}^*)^\mu$. Then the sum of the corresponding weight spaces $\bigoplus_{\alpha \in A} \mathbb{C}_\alpha$ is G_μ-invariant.*

Remark 1.7. Note that $(\mathfrak{t}^*)^\mu = \mathfrak{t}^*$ iff μ is not contained in any Weyl wall. In this case, affine subsets are orthogonal to $(\mathfrak{t}^*)^\mu$ iff they consist of a single point. Thus A contains at most one weight.

Proof of Lemma 1.6. We identify $\mathfrak{g}^*$ and $\mathfrak{g}$ via G-invariant product and consider μ as an element of $\mathfrak{t}$.

By Lemma 1.2, $\mathfrak{g}_\mu \otimes \mathbb{C} = \mathfrak{t} \oplus \bigoplus_{\alpha \in N} \mathbb{C}_\alpha$, with $N = \{\alpha \in R \mid \mu \in \ker \alpha\}$. Thus N coincides with the set of roots of G_μ. Since α is orthogonal to the Weyl wall

$\ker \alpha$ (with the respect to the above identification of $\mathfrak{g}$ and $\mathfrak{g}^*$), all elements of N are orthogonal to $(\mathfrak{t}^*)^\mu$.

By Theorem 1.5 the weights of any irreducible G_μ-representation differ by a sum of elements of N. Thus the weights of any irreducible G_μ-subrepresentation of V are contained in the same affine space orthogonal to $(\mathfrak{t}^*)^\mu$. □

1.3 The group $\mathrm{SU}(n)$: Maximal torus, weights, root system, Weyl group

Here we summarize the basic data about the groups $\mathrm{SU}(n)$ that we need for our examples in the end of the article. Again, we refer to [3] and [14] for details.

A maximal torus T of $\mathrm{SU}(n)$, $n \geq 2$, is given by the subgroup formed by the diagonal matrices. The diagonal entries are of the form $\mathrm{e}^{2\pi \mathrm{i} r_i}$, $r_i \in \mathbb{R}$, $i = 1, \ldots, n$. Since the product of the diagonal entries is 1, we can assume that $\sum_{i=1}^n r_i = 0$. Hence the Lie algebra $\mathfrak{t}$ can be identified with the $(n-1)$-dimensional subspace of elements of $\mathbb{R}^n$ whose entries sum up to 0. The Weyl group is given by the group S_n which acts on $\mathfrak{t}$ by permuting the n entries.

Alternatively, we fix the basis given by the $(n-1)$-vectors of the form $(0, \ldots, 0, 1, -1, 0, \ldots)$ and denote the corresponding coefficients.

For each of these two representations, the kernel of the exponential map $\exp : \mathfrak{t} \to T$ is given by the vectors with integer entries.

Both representations have their advantages: The second one is more concise, but the symmetry is easier to see in the first one.

Similarly, we obtain $\mathfrak{t}^* \cong \mathbb{R}^n / \langle (1, 1, \ldots, 1) \rangle$. We will switch between different representations of $\mathfrak{t}^*$: If n is large, we choose the representative in $\mathbb{R}^n$ whose entries sum up to 0. Bröcker and tom Dieck [3] take the representative with last entry 0, so they only need to consider $n-1$ entries, but we do not use this description here. Instead, if n is small enough to get the symmetry from the graphics, we denote the coefficients with respect to the dual base of the vectors $(0, \ldots, 0, 1, -1, 0, \ldots)$. Note that the integral forms correspond to the vectors with integer entries with respect to the second and the third representation of $\mathfrak{t}$, but not for the first one. A vector $(\alpha_1, \ldots, \alpha_n)$ with $\sum_i \alpha_i = 0$ represents an integral form iff the numbers $n\alpha_i$ are integers that are congruent modulo n.

As vectors in $\mathbb{R}^n$ with entry sum 0, the roots of $\mathrm{SU}(n)$ are given by the vectors with one entry 1, one entry -1 and $n-2$ entries 0. In the classification of root systems, this is the root system of type A_{n-1}. The root system of a product of groups is given by a product of the corresponding root systems, see [3].

1.4 Hamiltonian relative equilibria

In this article, we investigate the local structure of relative equilibria in Hamiltonian systems with G-symmetry near a given equilibrium with isotropy group G.

A point p of a G-manifold P is a *G-relative equilibrium* of a G-equivariant vector field X iff the group orbit Gp is X-invariant. Equivalently, there exists $\xi \in \mathfrak{g}$ with $X_p = \xi p$. Then ξ is called a *generator* (or *velocity*) of the relative equilibrium p. If ξ is a generator of p, then so is $\xi + \eta$ for any $\eta \in \mathfrak{g}_p$. Thus the generator is not unique in general, but it is unique regarded as an element of $\mathfrak{g}/\mathfrak{g}_p$.

Now we suppose that P is a smooth symplectic manifold with a G-invariant symplectic form ω and consider the Hamiltonian vector field X_h of a smooth G-invariant function $h : P \to \mathbb{R}$, i. e.

$$\mathrm{d}h(x) = \omega(x)(X_h(x), \cdot)$$

for every $x \in P$. Then there is an additional structure: We assume that there is a *momentum map* $\mathbf{J} : P \to \mathfrak{g}^*$ which satisfies

$$\langle \mathrm{d}J(x)v, \xi \rangle = \omega(x)(\xi_P(x), v)$$

and is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$. (In general, the momentum map exists at least locally, which suffices for our purpose, see [13, Part II, Chapter 26], also for sufficient conditions for the existence of a global momentum map.)

Thus the vector fields ξ_P are Hamiltonian: Let $\mathbf{J}^\xi : P \to \mathbb{R}$ denote the function $x \mapsto \langle \mathbf{J}(x), \xi \rangle$. Then $\mathbf{J}^\xi$ is the Hamiltonian function corresponding to ξ_P. Therefore we have another equivalent characterization of relative equilibria: $p \in P$ is a relative equilibrium with generator ξ iff p is a critical point of the function $h - \mathbf{J}^\xi$.

Note that the functions $\mathbf{J}^\xi$ are G_ξ-invariant but in general not G-invariant.

The momentum is constant along trajectories of the Hamiltonian vector field. This implies in particular that for a relative equilibrium p with generator ξ and momentum $\mu := \mathbf{J}(p)$, we have $\xi \in \mathfrak{g}_\mu$. Thus if we identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via a G-invariant product, then $[\xi, \mu] = 0$.

By the equivariant Darboux theorem (see [13, Theorem 22.2] and note also the correction by Dellnitz and Melbourne [4]), a point with isotropy group G in a symplectic manifold with a G-invariant symplectic form has a neighborhood such that there is a G-equivariant symplectomorphism to a neighborhood of 0 within a *G-symplectic representation* V, i. e. a G-representation V together with a G-invariant symplectic form. Thus we will restrict ourselves to this case. Then the functions $\mathbf{J}^\xi$ are quadratic forms on V.

1.5 Review of the results of my thesis

Let V be G-symplectic representation and $h : V \to \mathbb{R}$ be a smooth G-invariant Hamiltonian function. Then in my thesis [19], I follow the approach by Ortega and Ratiu [17] to find pairs $(x, \xi) \in V \times \mathfrak{g}$ such that x is a critical point of $h - \mathbf{J}^\xi$:

Consider the values of $\xi \in \mathfrak{g}$, such that $V_0 := \ker d^2(h - \mathbf{J}^\xi)(0)$ is non-trivial. Then perform a Lyapunov-Schmidt reduction to obtain a function $g : V_0 \times \mathfrak{g} \to \mathbb{R}$ such that for each η in a neighborhood of ξ, the critical points of $g(\cdot, \xi)$ near $0 \in V_0$ are in bijection with the critical points of $h - \mathbf{J}^\xi$ in V.

For bifurcation problems with symmetry but a trivial action on the parameter space, the Lyapunov-Schmidt-reduction can be performed in a way that the symmetry is preserved. Here, our parameter space is the Lie algebra $\mathfrak{g}$ and the G-action on $\mathfrak{g}$ is the adjoint action, which is non-trivial in general. The way out is to consider a smaller parameter space: Ortega and Ratiu [17] only search for solutions in $\mathfrak{g}^{G_\xi}$.

The approach of [19] is to consider the action of the maximal torus T and to search for the T-relative equilibria. These are exactly the G-relative equilibria that have a generator in $\mathfrak{t}$. The adjoint action of T on $\mathfrak{t}$ is trivial and hence the Lyapunov-Schmidt-reduction preserves the T-symmetry. Since every $\xi \in \mathfrak{g}$ is has an element of $\mathfrak{t}$ in its adjoint orbit and since p is a relative equilibrium with generator ξ iff gp is a relative equilibrium with generator $\mathrm{Ad}_g\,\xi$, the set of G-relative equilibria coincides with the set of G-relative equilibria.

Another important tool is the linear theory developed by Melbourne and Dellnitz [15] together with a theorem about generic bifurcation of Hamiltonian vector fields with symmetry by these two authors and Marsden [5, Theorem 3.1]. It implies that generically, there is a G-invariant inner product $\langle\cdot,\cdot\rangle$ on V such that $\omega = \langle\cdot, J\cdot\rangle$ and J defines a G-equivariant complex form on the *center space* V^c of $\mathrm{d}X_h(0)$ given by the sum of the generalized eigenspaces of $\mathrm{d}X_h(0)$ corresponding to eigenvalues with real part 0. It is easy to see that $V_0 \subset V^c$ for every V_0 of the above form. Thus we can consider V_0 as a complex T-representation if $\xi \in \mathfrak{t}$. Hence V_0 is a sum of weight spaces of the complex G-representation V^c. Each weight space $\mathbb{C}_\alpha$ is a T-symplectic representation with symplectic form $\langle\cdot, \mathrm{i}\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the real inner product described above.

The existence of a G-invariant inner product of this kind follows from the following genericity assumption:

Definition 1.8. h satisfied the *generic center space condition (GC)* iff $\mathrm{d}X_h(0)$ is non-degenerate and V^c splits into *irreducible G-symplectic* subrepresentations (G-symplectic representations with no proper non-trivial G-symplectic subrepresentations) that coincide with the eigenspaces of $\mathrm{d}^2h(0)|_{V^c}$.

Note that (GC) implies that there is a basis of weight vectors of V^c, namely a union of weight vectors of the eigenspaces of $\mathrm{d}^2h(0)|_{V^c}$, with respect to which $\mathrm{d}^2h(0)|_{V^c}$ is a diagonal matrix. Moreover, with respect to this basis $\mathrm{d}^2\mathbf{J}^\xi(0)|_{V^c}$ is also a diagonal matrix for every $\xi \in \mathfrak{t}$: If S^c denotes the set of weights of V^c (counted with multiplicities) and x_α denotes the $\mathbb{C}_\alpha$-component of $x \in V^c \cong \bigoplus_{\alpha\in S^c}\mathbb{C}_\alpha$, then $\mathbf{J}^\xi(x) = \pi\sum_{\alpha\in S^c}|x_\alpha|^2\,\alpha$. Hence $\mathrm{d}^2\mathbf{J}^\xi(0)|_{V^c}$ has the diagonal entries $2\pi\alpha(\xi)$. Thus if $\mathbb{C}_\alpha$ is a weight space of the $\mathrm{d}^2h(0)|_{V^c}$-eigenspace corresponding to the eigenvalue c_i, then for any $\xi \in \mathfrak{t}$ the space $\mathbb{C}_\alpha$ is contained in $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ iff $2\pi\alpha(\xi) = c_i$.

From now on, we assume (GC) and in addition the following genericity assumption (NR'). Together, they determine the weight structure of the subspaces that occur as non-trivial kernels of the form $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$, $\xi \in \mathfrak{t}$:

Definition 1.9. Let T be a real vector space and $S = \bigcup_{i=1}^n S_i$ be a union of subsets $S_i \subset T^*$. Then S is *full* iff for every vector $(c_1, \dots, c_n) \in \mathbb{R}^n$, there is an $x \in T$ with

$$\forall i : \forall \alpha \in S_i : \alpha(x) = c_i.$$

(For an alternative description of full sets, see [19, Remark 6.60].)

Definition 1.10. Suppose that the G-invariant Hamiltonian function h satisfies condition (GC). Then *the generalized non-resonance condition (NR')* holds for h iff for each union $S = \bigcup_i S_i$ of sets S_i of linearly independent weights of the eigenspaces U_i of $\mathrm{d}^2h(0)$ and the vector $c = (c_0, \dots c_n)$ of the corresponding eigenvalues, there is a $\xi \in \mathfrak{t}$ with

$$\forall i : \forall \alpha \in S_i : \alpha(\xi) = c_i$$

iff S is full.

Remark 1.11. If $G = T$ then (NR') is equivalent to the *condition (NR)* that for each $\xi \in \mathfrak{t}$ the kernel $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ has only linearly independent weights.

We obtain another generic requirement, which is more natural but slightly stronger, if we demand for unions $S = \bigcup_i S_i$ of sets S_i of weights of the eigenspaces U_i of $\mathrm{d}^2 h(0)$ in general that there is such a ξ iff S is full.

Remark 1.12. Note that if $S = \bigcup_{i=1}^n S_i$ is linearly independent, then S is full iff each of the sets S_i is the maximal set of weights of U_i within its affine span.

(Here the linear independence of S means in particular, that each weight occurs with multiplicity 1 in S. Similarly, the maximality of S_i within its affine span means in particular that every element of S_i occurs with multiplicity 1 in U_i).

One of the main results of [19] is that generically, if $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ has only linearly independent weights, then there is a local manifold of T-relative equilibria tangent to V_0:

Theorem 1.13 (Theorem 6.4.4. of [19].)**.** *Let V be a G-symplectic representation. Suppose that $h : V \to \mathbb{R}$ is a smooth G-invariant Hamiltonian function with critical point at* 0 *that satisfies the genericity assumptions (GC) and (NR').*

Consider (possibly empty) subsets S_i of weights of U_i such that each S_i is maximal in $\mathrm{aff}(S_i)$ *(in particular, the elements of S_i occur with multiplicity 1 in the set of weights of U_i). If $S := \bigcup_{i \in I} S_i$ is linearly independent, there is a T-invariant manifold of T-relative equilibria whose tangent space at* 0 *is given by the sum of the corresponding weight spaces of the elements of S: For $\alpha \in S_i$, we obtain the summand $\mathbb{C}_\alpha \subset U_i$.*

The sums of weight spaces as in Theorem 1.13 coincide with the kernels $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ with linearly independent weights. The local manifold of relative equilibria is given by the image of a local immersion m_{V_0} of a neighborhood of 0 in V_0 into V with $m_{V_0}(0) = 0$. If $W_0 \subset V_0$ is a T-invariant subspace, then the map m_{W_0} locally coincides with the restriction of m_{V_0} to (a neighborhood of 0 within) W_0.

Definition 1.14. We refer to the image of m_{V_0} as the *manifold that bifurcates at* V_0 or at the set X with $X = \left\{\xi \subset \mathfrak{t} \;\middle|\; V_0 \subset \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)\right\}$. Similarly, the set of points in the image of m_{V_0} that are not contained in the image of m_{W_0} for any T-invariant proper subspace W_0 of V_0 is called the *stratum that bifurcates at* V_0 (or *at* X).

The idea of the proof is based on equivariant transversality theory. The approach is very similar to an application of equivariant transversality theory to bifurcation theory worked out by Field, partly together with Richardson [11, 7, 9]. See Field's book [6] for an introduction into the theory.

We consider the function $g : V_0 \times \mathfrak{t} \to \mathbb{R}$ obtained from Lyapunov-Schmidt reduction. Up to third order, g coincides with the function

$$(x, \xi) \mapsto (h - \mathbf{J}^\xi)(x)$$

restricted to $V_0 \times \mathfrak{t}$. If $V_0 \cong \bigoplus_{i=1}^l \mathbb{C}_{\alpha_i}$, then the quadratic polynomials $p_i(x) = |x_{\alpha_i}|^2$ for $x = (x_{\alpha_1}, \ldots, x_{\alpha_l})$ generate the ring of invariant polynomials on V_0.

Hence by invariant theory, the T-invariant function g is of the form $g(p_1, \ldots, p_l, \xi)$. Thus we can decompose the function $(x, \xi) \mapsto \nabla_{V_0} g(x, \xi)$ into $g = \vartheta \circ \Gamma$, given by

$$\begin{aligned} \Gamma : V_0 \times \mathfrak{t} &\to V_0 \times \mathbb{R}^l \\ (x, \xi) &\mapsto (x, \partial_{p_1} g(x, \xi), \ldots, \partial_{p_l} g(x, \xi)) \end{aligned}$$

and

$$\begin{aligned} \vartheta : V_0 \times \mathbb{R}^l &\to V_0 \\ (x, t) &\mapsto \sum_{i=1}^{l} t_i \nabla p_i(x) = 2 \sum_{i=1}^{l} t_i x_{\alpha_i} \end{aligned}$$

(Here we choose the T-invariant real inner product $\langle x, y \rangle = \operatorname{Re} \sum_{i=1}^{l} x_{\alpha_i} \overline{y_{\alpha_i}}$ on V_0.) In terms of equivariant transversality theory, g is *G-1-jet-transverse* to $0 \in V_0$ iff Γ is transverse to the Whitney-stratified set $\Sigma := \vartheta^{-1}(0)$ in 0. As shown in [19], Γ is even transverse to $0 \in V_0 \times \mathbb{R}^l$. We then obtain for every x in a neighborhood of the origin of V_0 a generator $\xi(x) \in \mathfrak{t}$, unique modulo $\mathfrak{t}_x$, with $\nabla_{V_0} g(x, \xi) = 0$. Then $m_{V_0}(x)$ is defined as the corresponding critical point of $h - \mathbf{J}^{\xi(x)}$ in V.

In this article, we will use the decomposition $g = \vartheta \circ \Gamma$ again and consider the preimages under ϑ of isotropy subspaces of V_0 with respect to the G-action. This is one of the key arguments in the proof that – despite the fact that the space V_0 is T-invariant but in general not G-invariant – the map m_{V_0} preserves the G-isotropy subgroups. We will show this in section 2.

The knowledge of the isotropy subgroups of the T-relative equilibria is essential for the further investigation of their G-orbits. However, a first step is already done in [19]: We can classify, which of the T-relative equilibria in the images of the maps m_{V_0} are contained in the same G-orbit.

Recall that we say that the image of m_{V_0} bifurcates at

$$X = \left\{ \xi \in \mathfrak{t} \;\middle|\; V_0 \subset \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0) \right\}.$$

Suppose that X is contained in a Weyl wall. This is the case iff one of the subsets $S_i \subset S$ of weights of V_0 within an eigenspace U_i of $\mathrm{d}^2 h(0)$ contains a pair of weights $\alpha \neq w\alpha$, where $w \in W$ denotes the reflection about that Weyl wall. Then the image of m_{V_0} lies in the G-orbit of the image of m_{W_0}, where $W_0 := \bigoplus_{S \setminus \{\alpha\}} \mathbb{C}_\alpha$. Hence, we only have to consider sets of weights S as in Theorem 1.13 that do not contain such a Weyl reflection pair. Then the G-orbits of the corresponding strata coincide for sets of weights S, S' of this kind iff there is an element $w' \in W$ with $S' = w'S$. Otherwise the G-orbits are disjoint.

2 Isotropy groups: Reduction to representation theory

In the following, we will investigate the isotropy types of the relative equilibria that exist by Theorem 1.13 and the isotropy types of their momenta and their generators. All three contain important information about the structure of the set of relative equilibria: As mentioned in the introduction, Patrick and Roberts [18] have shown that if G acts freely on a symplectic manifold (P,ω) such that ω is G-invariant, then generically the set of relative equilibria is stratified by the conjugacy class K of the intersection $G_\xi \cap G_\mu$ of the isotropy subgroups of the generator ξ and the momentum μ. The dimension of the corresponding stratum is given by $\dim G + 2\dim Z(K) - \dim K$, where $Z(K)$ denotes the center of K. As pointed out in [19], for a non-free action and any isotropy subgroup H Patrick and Robert's result generically applies to the free action of the $\left(N(H)/H\right)^\circ$ on P_H, where $N(H)$ denotes the normalizer of H and $\left(N(H)/H\right)^\circ$ the identity component of $N(H)/H$. We will use this in Section 3 to compute the dimensions of the strata within the G-orbits of the T-relative equilibria that exist by Theorem 1.13.

Moreover, the investigation of the momenta, the generators and their isotropy subgroups plays a key role in the calculation of the isotropy types of the relative equilibria themselves.

In this section, we show that the local isotropy structure of the manifolds of T-relative equilibria that exist by Theorem 1.13 coincides with that of their tangent spaces at the origin. In addition, we link the isotropy groups of the momenta and generators to those of the relative equilibria of the linearized Hamiltonian vector field.

We start with an investigation of the momenta:

2.1 Momenta

Throughout, let V be a G-symplectic representation. We consider the smooth G-invariant Hamiltonian function $h : V \to 0$ with critical point at 0 and assume that the genericity conditions (GC) and (NR') hold for the center space V^c of $\mathrm{d}X_h(0)$.

Suppose that $V_0 = \bigoplus_{\alpha\in S}\mathbb{C}_\alpha$ with $S = \bigcup S_i$ as in Theorem 1.13. Recall that if any S_i contains a pair of weights $\alpha \neq w\alpha$, where w is the reflection about a Weyl wall, then the T-relative equilibria in the local manifold tangent to V_0 are contained in the G-orbit of the manifold tangent to a subspace of V_0 that has only one of the weights α and $w\alpha$. Thus we suppose in the following that no such pair is contained in S. Equivalently, the affine set $X = \left\{\xi \in \mathfrak{t} \;\middle|\; V_0 \subset \ker \mathrm{d}^2 h - \mathbf{J}^\xi(0)\right\}$ is not contained in a Weyl wall.

Let x be a point of V_0 with momentum $\mu = \mathbf{J}(x)$. Since $G_x \subset G_\mu$, the investigation of the momenta and their isotropy plays a key role in the calculation of the isotropy groups.

Lemma 2.1. *Let S be a linearly independent set of weights as in Theorem 1.13 such that the corresponding affine set $X = \left\{\xi \in \mathfrak{t} \;\middle|\; V_0 \subset \ker \mathrm{d}^2 h - \mathbf{J}^\xi(0)\right\}$ is not contained in a Weyl wall. Then there is a G-invariant splitting $\mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{c}^*$, such that the momenta of the points in the local stratum M that bifurcates at X are contained in $\mathfrak{t}^*$.*

Proof. Since X is not contained in a Weyl wall, the same holds for the set of generators of a point of M which is close to 0. Thus $x \in M$ has a generator ξ not contained in any Weyl wall. Hence $G_\xi = T$. Identify $\mathfrak{g}^*$ and $\mathfrak{g}$ via a G-invariant inner product and consider $\mu := \mathbf{J}(x)$ as an element of $\mathfrak{g}$. Then $[\mu,\xi] = 0$, thus $\mu \in \mathfrak{t}$. □

Since the momentum with respect to the T-action coincides with the projection of the momentum to $\mathfrak{t}^*$, we obtain immediately

Corollary 2.2. *Let M be as in Lemma 2.1, $x \in M$, $\mu = \mathbf{J}(x)$, and μ_T be the momentum of x with respect to the T-action. Then $G_\mu = G_{\mu_T}$.*

Corollary 2.3. *There is a local T-equivariant symplectomorphism σ from a neighborhood of 0 within V_0 to a neighborhood of 0 within M, such that $\mathbf{J}(x) = \mathbf{J}(\sigma(x))$.*

Proof. By the equivariant Darboux theorem ([13, Theorem 22.2] and [4]), there is a local T-equivariant symplectomorphism σ. Since for every x in the domain of σ and for every $\xi \in \mathfrak{t}$

$$\begin{aligned} \mathrm{d}(\mathbf{J}^\xi \circ \sigma)(x) &= \mathrm{d}\mathbf{J}^\xi(\sigma(x)) \circ \mathrm{d}\sigma(x) = \omega_{\sigma(x)}(\xi(\sigma(x)), \mathrm{d}\sigma(x)\cdot) \\ &= \omega_{\sigma(x)}(\mathrm{d}\sigma(x)\xi x, \mathrm{d}\sigma(x)\cdot) = \omega_x(\xi x, \cdot) = \mathrm{d}(\mathbf{J}^\xi)(x) \end{aligned}$$

and $\mathbf{J}^\xi \circ \sigma(0) = 0 = \mathbf{J}^\xi(0)$, we obtain $\mathbf{J}^\xi \circ \sigma = \mathbf{J}^\xi$. □

Corollary 2.2 is the main observation we need to proceed with the investigation of isotropy types of the T-relative equilibria of M and their momenta. Actually, we will only apply it to the elements of the space V_0, which form relative equilibria of the linearized Hamiltonian vector field. For V_0, we obtain:

Corollary 2.4. *Consider V_0 as in Lemma 2.1 and an isotropy subgroup $H \subset G$ of the representation V. Then for any $x \in V_0^H$ with momentum μ_T with respect to the T-action, we obtain $H \subset G_{\mu_T}$.*

2.2 Isotropy of the generators and the relative equilibria

We are now in the position to compare the isotropy groups of the elements of the local manifolds with those of their tangent spaces at the origin. So we consider $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ with $V_0 = \bigoplus_{\alpha \in S} \mathbb{C}_\alpha$ and S as in Lemma 2.1. We will call the open dense set of points of V_0 with minimal isotropy type τ with respect to the T-action the *main stratum* $(V_0)_\tau$ of V_0. It coincides with the points $x = \sum x_\alpha$ with $0 \neq x_\alpha \in \mathbb{C}_\alpha$ for every $\alpha \in S$.

We will see that V_0 is contained in a larger subspace $\tilde{V}_0$ of the form $\tilde{V}_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\eta)(0)$ with $\eta \in \mathfrak{t}^K$ for some isotropy subgroup $K \subset G$ of the adjoint action on $\mathfrak{g}$ such that K contains all isotropy groups of points the main stratum of V_0. Then the following lemma implies that locally near the origin there is a 1-to-1-correspondence between the isotropy types of the main stratum of V_0 and the stratum of T-relative equilibria that bifurcates at V_0.

We fix an inner product on $\mathfrak{g}$ which is invariant with respect to the adjoint action. Let $\mathfrak{t}'$ denote the orthogonal complement within $\mathfrak{t}$ of the isotropy Lie algebra $\mathfrak{t}_x$ of any point $x \in (V_0)_\tau$.

Lemma 2.5. *Consider $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ for some $\xi \in \mathfrak{t}$ such that the weights of V_0 are linearly independent and $X := \left\{\zeta \in \mathfrak{t} \;\middle|\; V_0 \subset \ker \mathrm{d}^2(h - \mathbf{J}^\zeta)(0)\right\}$ is not contained in any Weyl wall. Suppose that V_0 is a subspace of $\tilde{V}_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\eta)(0)$ with $\eta \in \mathfrak{t}^K$ for some isotropy subgroup $K \subset G$ of the adjoint action on $\mathfrak{g}$ that contains all the isotropy subgroups of the elements of $(V_0)_\tau$.*

Then there is a neighborhood U of $0 \in V_0$ such that for every $v_0 \in U \cap (V_0)_\tau$ the following holds: Let $L \subset K$ be the intersection of all isotropy subgroups of elements of $\mathfrak{t}$ with respect to the adjoint action that contain $H := G_{v_0}$. Then L is an isotropy subgroup of the adjoint action and the generator $\xi(v_0)$ of the relative equilibrium $m_{V_0}(v_0)$ is contained in $(\mathfrak{t}')^L$.

Proof. Corollary 1.3 yields immediately that L is an isotropy subgroup.

The local map $m_{V_0} \colon (V_0, 0) \to (V, 0)$ is given by $x \mapsto x + v_1(x, \xi(x))$, where the local map v_1 is defined in a neighborhood of $(0, \xi)$ within $V_0 \times \mathfrak{t}$ and its image is contained in the orthogonal complement V_1 of V_0 with respect to a G-invariant inner product on V. $v_1(x, \zeta)$ is defined as the unique solution to

$$\nabla_{V_1}(h - \mathbf{J}^\zeta)(x + v_1(x, \zeta)) = 0,$$

see [19, Section 2.4]. For $x \in V_0$, the generator $\xi(x)$ is uniquely defined as an element of $\mathfrak{t}'$, see [19, Section 6.4.1]. It is given by the equation

$$\begin{aligned} \nabla_{V_0} g(x, \xi(x)) &= 0, \\ \text{with} \quad g(x, \zeta) &:= \left(h - \mathbf{J}^\zeta\right)(x + v_1(x, \zeta)). \end{aligned}$$

Now the proof proceeds as follows: In a first step, we show that $\nabla_{V_0} g(x, \zeta) \in V_0^H$ if $\zeta \in \mathfrak{t}^L$. Then the main idea is to deduce the converse: In a neighborhood of $(0, \xi) \in V_0 \times \mathfrak{t}$, $\nabla_{V_0} g(x, \zeta) \in V_0^H$ implies that $\zeta \in \mathfrak{t}^L$. Since $\nabla_{V_0} g(x, \xi(x)) = 0$, we then obtain immediately that $\xi(x) \in \mathfrak{t}^L$.

The proof of the converse relies on the fact that the restriction of the map $(x, \zeta) \mapsto \nabla_{V_0} g(x, \zeta)$ to a neighborhood of $(0, \xi(0))$ intersected with $(V_0)_\tau \times \mathfrak{t}'$ is a diffeomorphism onto an open set of the subbundle of $T(V_0)_\tau$ given by the normal spaces to the T-orbits. We then simply use a dimension argument.

A way to see this diffeomorphism is to consider again the decomposition of the map $(x, \zeta) \mapsto \nabla_{V_0} g(x, \zeta)$ into the maps ϑ and Γ known from equivariant transversality theory, as it is done in [19] in the proof of Theorem 1.13. This will be done in the second step.

The third step consists of the dimension argument.

Step 1: $\nabla_{V_0} g(x, \zeta) \in V_0^H$ for $\zeta \in \mathfrak{t}^L$.

Let $\mathfrak{k}$ denote the Lie algebra of K. There is the corresponding map $\tilde{v}_1$ from a neighborhood of $(0, \eta)$ in $\tilde{V}_0 \times \mathfrak{k}$ to the orthogonal complement $\tilde{V}_1$ of $\tilde{V}_0$ such that

$$\nabla_{\tilde{V}_1}(h - \mathbf{J}^\xi)(\tilde{x} + \tilde{v}_1(\tilde{x}, \xi)) = 0.$$

By construction, the map $\tilde{v}_1$ is K-equivariant.

Obviously, the linear span S is a subspace of the linear span of $\tilde{S}$. Thus there is a subtorus $T_0 \subset T$ with $V_0 = \tilde{V}_0^{T_0}$. Moreover, note that $\mathrm{d}^2\mathbf{J}^{\xi - \eta}(0)$ vanishes on V_0 and hence $\eta - \xi$ is contained in $\mathfrak{t}_x$ for $x \in V_0$. Thus by [19, Remarks 2.10 and 2.11], $\tilde{v}_1$ can be defined on a neighborhood of $(0, \eta)$ that contains $(0, \xi)$ such that the restriction of $\tilde{v}_1$ to (a neighborhood of $(0, \eta)$ in) $V_0 \times \mathfrak{t}$ coincides with v_1.

Since $L \subset K$, the map

$$\tilde{x} \mapsto \tilde{g}(x, \zeta) := \left(h - \mathbf{J}^\zeta\right)(\tilde{x} + \tilde{v}_1(\tilde{x}, \xi))$$

is L-invariant if $\zeta \in \mathfrak{t}^L$. Thus the gradient $\nabla_{\tilde{V}_0} \tilde{g}(\cdot, \zeta)$ is L-equivariant if $\zeta \in \mathfrak{t}^L$.

Since V_0 is an isotropy subspace of $\tilde{V}_0$ with respect to the T-action,

$$\nabla_{\tilde{V}_0}\tilde{g}(x,\zeta) = \nabla_{V_0} g(x,\zeta)$$

if $x \in V_0$. Thus, if $x \subset V_0^H$ and $\zeta \in \mathfrak{t}^L$, we have $\nabla_{V_0} g(x,\zeta) \in V_0^H$.

Since $\xi - \eta \in \mathfrak{t}_{v_0}$, ξ and η are projected to the same element of $\mathfrak{t}'$. By abuse of notation, we will denote this projection by η.

Step 2: Use the decomposition $\vartheta \circ \Gamma$.

Let $S = \{\alpha_1, \dots, \alpha_l\}$ be the set of weights of V_0 and write x in the form $x = \sum_{i=1}^l x_{\alpha_i}$ with $x_{\alpha_i} \in \mathbb{C}_{\alpha_i}$. Now we use ideas from equivariant transversality theory as in the proof of Theorem 1.13. Since $g(\cdot,\zeta)$ is T-equivariant, g is of the form $g(x,\zeta) = g(p_1,\dots,p_l,\zeta)$ with $p_i = p_i(x) = |x_{\alpha_i}|^2$. We decompose the map

$$V_0 \times \mathfrak{t}' \ni (p_1,\dots,p_l,\zeta) \mapsto \nabla_{V_0} g(p_1,\dots,p_l,\zeta)$$

into $\vartheta \circ \Gamma$ with $\vartheta \colon V_0 \times \mathbb{R}^l$ defined by

$$(x,t) \mapsto \sum_{i=1}^l t_i \nabla p_i(x) = 2\sum_{i=1}^l t_i x_{\alpha_i}$$

and $\Gamma : V_0 \times \mathfrak{t}' \to V_0 \times \mathbb{R}^l$ given by

$$(x,\zeta) \mapsto (x, \partial_1 g(p_1(x),\dots,p_l(x),\zeta), \dots, \partial_l g(p_1(x),\dots,p_l(x),\zeta)).$$

Here ∂_i denotes the partial derivative of g with respect to the coordinate p_i. As calculated in [19], the derivative $\mathrm{d}\Gamma(0,\eta)$ is invertible. Thus Γ is a local diffeomorphism between a neighborhood U of $(0,\xi) \in V_0 \times \mathfrak{t}'$ and a neighborhood O of $\Gamma(0,\xi) = (0,0) \in V_0 \times \mathbb{R}^l$.

Let $Q^H \subset V_0 \times \mathbb{R}^l$ denote the preimage $\vartheta^{-1}(V_0^H)$. Then we know from above that $\Gamma(V_0^H \times (\mathfrak{t}')^L) \subset Q^H$.

Recall that $(V_0)_\tau$ denotes the main stratum of V_0 and set

$$Q^H_\tau := Q^H \cap \left((V_0)_\tau \times \mathbb{R}^l\right). \tag{1}$$

We will show in the following that actually

$$\Gamma\left(\left((V_0)^H_\tau \times (\mathfrak{t}')^L\right) \cap U\right) = Q^H_\tau \cap O$$

for an admissible choice of neighborhoods U and O of $(0,\eta)$ and $\Gamma(0,\eta)$ respectively. By definition of $\xi(x)$, we have $\vartheta \circ \Gamma(x,\xi(x)) = 0 \in V_0^H$ for $x \in V_0$. Since the map $x \mapsto \xi(x)$ is continuous, $(x,\xi(x))$ is contained in U for x small enough. Altogether this implies that $\xi(x) \in (\mathfrak{t}')^L$ for small $x \in V_0^H$.

Step 3: Proof of equation 1.

It remains to prove the claim that we can choose U and O such that equation 1 is satisfied.

We note first that for each $x \in V_0$ the map $\vartheta(x,\cdot)$ is a linear map from $\{x\}\times\mathbb{R}^l$ to V_0. Thus $Q^H \cap \left(\{x\} \times \mathbb{R}^l\right)$ is a linear subspace of $\{x\}\times\mathbb{R}^l$. Moreover, by Corollary 2.4 the space V_0^H is contained in the set of points x of V_0 with $\mathbf{J}(x) = \mathbf{J}_T(x) \in \mathfrak{t}^L$. Since $\mathbf{J}(x) = \pi \sum_{i=1}^l |x_{\alpha_i}|^2 \alpha_i$, we obtain that

$$\mathbf{J} \circ \vartheta(x,t) = 4\pi \sum_{i=1}^l t_i^2 |x_{\alpha_i}|^2 \alpha_i.$$

The isotropy subspaces of $\mathfrak{t}^*$ with respect to the coadjoint action are given by the intersection of Weyl walls. Each Weyl wall is described by a linear homogeneous equation on the space $\mathfrak{t}^*$. On the subspace spanned by the α_i (which can be considered as $(\mathfrak{t}')^*$), we obtain a linear homogeneous equation in the coefficients of the α_i. Thus the subset of $t \in \mathbb{R}^l$ such that $\vartheta(x,t)$ is contained in a particular Weyl wall is given by an equation of the form

$$\sum_{i=1}^{l} a_i t_i^2 = 0. \tag{2}$$

Here the coefficients depend on x and on the choice of the Weyl wall. If x is contained in the main stratum $(V_0)_\tau$, then $a_i = 0$ iff α_i is contained in the corresponding Weyl wall. Let S^+, S^-, and S^0 denote the sets of i with $a_i > 0$, $a_i < 0$, and $a_i = 0$ respectively. Then the solution set of equation 2 coincides with the union of the solution sets for $\lambda \geq 0$ of

$$\sum_{i\in S^+} a_i t_i^2 = \lambda = \sum_{i \in S^-} (-a_i) t_i^2.$$

Hence it is given by the product of the subspace E of $\mathbb{R}^l$ corresponding to the coordinates in S^0 and a cone over a product P of two ellipsoids. Thus each linear subspace in the solution set of equation 2 is contained in the sum of E and the linear span of a specific point in P.

Now we consider the intersection $(\mathfrak{t}^*)^L$ of Weyl walls that coincides with the isotropy subspace of L. The preimage in $\{x\} \times \mathbb{R}^l$ is given by the intersection of several products $P_j \times E_j$ as above. Consider a subspace Y in the intersection of all these products. If we choose an arbitrary point y of Y, then Y can be decomposed as $\langle y \rangle \oplus F$, where F is a subspace of E_j for every j. Thus $\dim Y \leq 1 + \dim \bigcap_j E_j$ and hence this is an upper bound for the dimension of the linear subspace

$$\left\{ t \in \mathbb{R}^l \;\middle|\; (x,t) \in Q^H \cap \left(\{x\} \times \mathbb{R}^l\right) \right\}.$$

Moreover, $\dim \bigcap E_j$ is given by the number k of the indices i such that $\alpha_i \in ((\mathfrak{t})^*)^L$.

Next, we estimate the dimension of the space $(\mathfrak{t}')^L$. First of all, we point out that via an adjoint invariant product on $\mathfrak{g}$, which is restricted to $\mathfrak{t}$, the space $\mathfrak{t}'$ can be identified with the span of the α_i, $i = 1, \ldots, n$. Obviously, the α_i with $\alpha_i \in (\mathfrak{t}')^L$ span a subspace of $(\mathfrak{t}')^L$ of dimension k. In addition, by assumption $v_0 \in (V_0)_\tau$ is a point with $(v_0)_{\alpha_i} \neq 0$ for every i and $\mu = \sum_{i=1}^{l} \left|(v_0)_{\alpha_i}\right|^2 \alpha_i$ contained in $(\mathfrak{t}^*)^L$. Thus either all the α_i are contained in $(\mathfrak{t}^*)^L$ and $\dim(\mathfrak{t}')^L = \dim(\mathfrak{t}') = l$ or $\dim(\mathfrak{t}')^L \geq k+1$. In both cases, we obtain that $\dim(\mathfrak{t}')^L \geq \dim Q^H \cap \left(\{x\} \times \mathbb{R}^l\right)$ for every $x \in V_0$.

Since Γ is a local diffeomorphism, its restriction to the set $V_0^H \times (\mathfrak{t}')^L$ is also a local diffeomorphism onto its image. As shown above, its image is contained in the space $\left(V_0^H \times \mathbb{R}^l\right) \cap Q^H$.

Γ is of the form

$$(x,\zeta) \mapsto (x, \gamma(x,\zeta))$$

with

$$\gamma(x,\zeta) := (\partial_1 g(p_1(x), \ldots, p_l(x), \zeta), \ldots, \partial_l g(p_1(x), \ldots, p_l(x), \zeta).$$

Hence for $x \in V_0^H$, the map $\zeta \mapsto \Gamma(x,\zeta)$ is an immersion from $(\mathfrak{t}')^L$ to the space $Q^H \cap \big(\{x\} \times \mathbb{R}^l\big)$, which has at most the same dimension as $(\mathfrak{t}')^L$. Thus the dimensions coincide and the map is locally surjective.

Now choose U and O in such a way, that for each $x \in V_0^H$ the intersection $O \cap \big(\{x\} \times \mathbb{R}^l\big) \cap Q^H$ is connected. (Since $\big(\{x\} \times \mathbb{R}^l\big) \cap Q^H$ is a linear subspace of $\{x\} \times \mathbb{R}^l$, we may for example take a ball for O.) Since Γ is a diffeomorphism from $\big(\{v_0\} \times (\mathfrak{t}')^L\big) \cap U$ onto its image in a space of the same dimension, its image is an open subset of $Q^H \cap \big(\{v_0\} \times \mathbb{R}^l\big) \cap O$. Moreover, $\big(\{v_0\} \times (\mathfrak{t}')^L\big) \cap U$ is closed in U. Thus its image is closed in O and hence closed in $Q^H \cap \big(\{v_0\} \times \mathbb{R}^l\big) \cap O$. By connectedness,

$$\Gamma\left(\big(\{v_0\} \times (\mathfrak{t}')^L\big) \cap U\right) = Q^H \cap \big(\{v_0\} \times \mathbb{R}^l\big) \cap O.$$

□

Corollary 2.6. *Consider V_0 and $\tilde{V}_0$ as in Lemma 2.5 that contains V_0. Then for every V_0 locally*

$$m_{V_0} \colon (V_0, 0) \to (V, 0)$$

satisfies $K_{v_0} \subset K_{m_{V_0}(v_0)}$ for $v_0 \in V_0$.

Proof. First of all we note that every point v_0 of V_0 is contained in the main stratum of some isotropy subspace of V_0 with respect to the T-action. Replacing V_0 with this subspace if necessary, we can assume that v_0 is contained $(V_0)_\tau$. Since the number of isotropy subspaces of V_0 is finite, we can find $\varepsilon > 0$ such that Lemma 2.5 applies for all $v_0 \in B_\varepsilon(0) \subset V_0$. Then $\xi(v_0) \in \mathfrak{t}^L$ with L as in Lemma 2.5. Hence $v_1(\cdot, \xi(v_0))$ is L-equivariant and thus $m(v_0) = v_0 + v_1(v_0, \xi(v_0)) \in V^H$. □

To show that $G_{v_0} \subset G_{m_{V_0}(v_0)}$ is locally true in general, we therefore have to construct $\tilde{V}_0$ such that $G_{v_0} \subset K$. To find an appropriate subgroup K with Lie algebra $\mathfrak{k}$ we introduce the notion of *orthogonal intersection*, which slightly differs from orthogonality .

Definition 2.7. Two subspaces U_1 and U_2 of an inner product space *intersect orthogonally* if their orthogonal projections to the orthogonal complement of $U_1 \cap U_2$ are orthogonal to each other.

Two affine subspaces A_1 and A_2 *intersect orthogonally* if their underlying subspaces do and in addition the intersection $A_1 \cap A_2$ is non-empty.

Lemma 2.8. *Let V_0 be as in Lemma 2.1. Consider $x \in (V_0)_\tau$. Let L denote the intersection of all coadjoint isotropy subgroups M of elements of $\mathfrak{t}^*$ with $G_x \subset M$. Then L is a coadjoint isotropy subgroup of an element of $\mathfrak{t}^*$, and $(\mathfrak{t}^*)^L$ intersects the affine span of S orthogonally.*

If $S_i \subset S$ is a subset of S that corresponds to weight spaces of a particular eigenspace of $\mathrm{d}^2h(0)$, then $(\mathfrak{t}^*)^L$ even intersects the affine span of S_i orthogonally.

Proof. Again by Corollary 1.3, L is a coadjoint isotropy subgroup of an element of $\mathfrak{t}^*$.

Since $G_x \subset L$, we have $G_x = L_x$. We split V^c into L-invariant components of the following form: First of all, consider the eigenspaces U_i of $\mathrm{d}^2h(0)|_{V^c}$, which are irreducible complex G-representations by our genericity assumption. Now we consider the orthogonal projection to $(\mathfrak{t}^*)^L$ and split each U_i further into subspaces U_i^j, $j = 1, \dots, l_i$ given by the sums of weight spaces corresponding to the weights of U_i that project to the same element of $(\mathfrak{t}^*)^L$. By Lemma 1.6 the spaces U_i^j are L-invariant.

Let $x_i^j \in U_i^j$ denote the corresponding components of $x \in (V_0)_\tau$. Then $L_x = \bigcap_{i,j} L_{x_i^j}$.

Set $\mu_i^j := \mathbf{J}(x_i^j)$. Recall that $L_{x_i^j} \subset G_{x_i^j} \subset G_{\mu_i^j}$. From $G_x \subset G_{\mu_i^j}$ together with $\mu_i^j \in \mathfrak{t}^*$ and the definition of L, we obtain that $L \subset G_{\mu_i^j}$ for every μ_i^j.

Let $S_i^j \subset S$ denote the subset of weights corresponding to weight spaces contained in U_i^j. Let x_α denote the $\mathbb{C}_\alpha$-component of x with respect to the isomorphism $V_0 \simeq \bigoplus_{\alpha \in S} \mathbb{C}_\alpha$. Then

$$\mu_i^j = \pi \sum_{\alpha \in S_i^j} |x_\alpha|^2 \, \alpha.$$

If we assume w. l. o. g. that $\sum_{\alpha \in S_i^j} |x_\alpha|^2 = 1$ (otherwise we consider a multiple of μ_i^j), we obtain that $\mu_i^j \in (\mathfrak{t}^*)^L$ is contained in the affine span of S_i^j. Then by definition of S_i^j, this implies that $(\mathfrak{t}^*)^L$ intersects the affine span of each S_j^i orthogonally. Thus this also holds for the affine spans of the sets $S_i = \bigcup_{j=1}^{l_i} S_i^j$ and $S = \bigcup_i S_i$. □

Given the affine span A of the set S of weights of V_0, we consider the intersections of Weyl walls that intersect A orthogonally. Note that the set of these intersection of Weyl walls is closed under intersections: Let U denote the underlying subspace of A. Then $A = a + U$ for some $a \in U^\perp$. A subspace I intersects A orthogonally iff its orthogonal projection to U coincides with $I \cap U$ and $a \in I$. If two subspaces I and J intersect A orthogonally, then $a \in I \cap J$ and the orthogonal projection of $I \cap J$ to U coincides with $I \cap J \cap U$.

Thus we can find a minimal element $I_{\min}$ of all intersections of Weyl walls that are orthogonal to A with respect to the partial ordering $I \leq J$ iff $I \subset J$. Then we choose K to be the isotropy subgroup of the coadjoint action such that $I_{\min} = (\mathfrak{t}^*)^K$.

Lemma 2.9. *Suppose that V^c is an irreducible G-symplectic representation and that $h : V \to \mathbb{R}$ is a smooth G-invariant function with $\mathrm{d}h(0) = 0$ that satisfies the genericity assumptions (GC) and (NR'). Consider $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0) = V_0 = \bigoplus_{\alpha \in S} \mathbb{C}_\alpha$ such that the affine span A of S does not contain the origin. Let $I_{\min}$ be the minimal element within all intersections of Weyl walls that intersect A orthogonally. Then for some isotropy group $K \subset G$ of the coadjoint action, $I_{\min}$ coincides with the isotropy subspace $(\mathfrak{t}^*)^K$. For every $x \in (V_0)_\tau$, the isotropy group G_x is contained in K. Moreover, there is an $\eta \in \mathfrak{t}^K$ with*

$$V_0 \subset \ker \mathrm{d}^2(h - \mathbf{J}^\eta)(0) =: \tilde{V}_0.$$

Proof. Since $I_{\min}$ is an intersection of Weyl walls, by Lemma 1.2 there is an isotropy group $K \subset G$ of the coadjoint action such that this intersection coincides with $(\mathfrak{t}^*)^K$. Let $I_{\min}^\perp$ be the orthogonal subspace of $I_{\min}$ within $\mathfrak{t}^*$. Extend the set S to the set $\tilde{S}$ given by the weights of V contained in

$$A + I_{\min}^{\perp} = (A \cap I_{\min}) + I_{\min}^{\perp}.$$

Let $\tilde{V}_0$ denote the sum $\bigoplus_{\alpha \in \tilde{S}} \mathbb{C}_\alpha$. Obviously, $V_0 \subset \tilde{V}_0$.

Consider $c \in \mathbb{R}$ with $\mathrm{d}^2 h(0) = 2\pi c \mathbb{1}$ on V^c.

Then ξ satisfies ($\langle \alpha, \xi \rangle = c$ iff $\alpha \in A$). Similarly, $\tilde{V}_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\eta(0))$ is equivalent to ($\langle \alpha, \eta \rangle = c$ iff $\alpha \in A + I_{\min}^{\perp}$).

Fixing a G-invariant inner product on $\mathfrak{g}$, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and $\mathfrak{t}$ with $\mathfrak{t}^*$. Let η denote the orthogonal projection of ξ to $I_{\min}$. Since $\eta - \xi \in I_{\min}^{\perp}$, the inner products of η and ξ with elements of $I_{\min}$ coincide. Thus $\langle \alpha, \eta \rangle = c$ if $\alpha \in A \cap I_{\min}$ and $\langle \alpha, \eta \rangle \neq c$ if $\alpha \in I_{\min} \setminus (A \cap I_{\min})$. Moreover $\eta \in I_{\min}$ implies that for any weight α the value $\langle \alpha, \eta \rangle$ coincides with the orthogonal projection of α to $I_{\min}$ evaluated at η. Therefore $\langle \alpha, \eta \rangle = c$ iff $\alpha \in (A \cap I_{\min}) + I_{\min}^{\perp} = A + I_{\min}^{\perp}$. By the maximality of $\tilde{S}$ within $A + I_{\min}^{\perp}$, we obtain $\tilde{V}_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\eta(0))$.

Next, we have to show that all isotropy subgroups of elements $x = \sum_{\alpha \in S} x_\alpha$ with $0 \neq x_\alpha \in \mathbb{C}_\alpha$ of V_0 are contained in K. By Lemma 2.8, there is an intersection of Weyl walls $(\mathfrak{t}^*)^L$ that intersects S orthogonally such that $G_x \subset L$. By definition of K, we have

$$(\mathfrak{t}^*)^K = I_{\min} \subset (\mathfrak{t}^*)^L$$

and hence $G_x \subset L \subset K$.

□

Now, we extend the result to the case that V is not necessarily irreducible.

Corollary 2.10. *Suppose that $h : V \to \mathbb{R}$ is a smooth G-invariant function with $\mathrm{d}h(0) = 0$ that satisfies the genericity assumptions (GC) and (NR') and consider $\ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0) = V_0 = \bigoplus_{\alpha \in S} \mathbb{C}_\alpha \subset V^c \subset V$ for some $\xi \in \mathfrak{t}$. Then there is an isotropy group $K \subset G$ of an element of $\mathfrak{t}^*$ with respect to the coadjoint action such that for any $x \in (V_0)_\tau$ the isotropy group G_x is contained in K and such that V_0 is contained in a K-invariant subspace $\tilde{V}_0$ of the form $\tilde{V}_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\eta)(0)$ for some $\eta \in \mathfrak{t}^K$.*

Proof. We split V^c into eigenspaces U_i of $\mathrm{d}^2 h(0)$, which are complex irreducible G-representations by our genericity assumption. In addition, we split V_0 into subspaces V_0^i contained in these eigenspaces. Let S_i denote the set of weights of V_0^i, and let A_i be the affine span of S_i.

Then the set of Weyl wall intersections that intersect each A_i orthogonally contains a minimal element, which coincides with $(\mathfrak{t}^*)^K$ for some isotropy subgroup $K \subset G$ of the coadjoint action. Lemma 2.8 implies that $G_x \subset K$ for every $x \in (V_0)_\tau$.

Again, let η denote the orthogonal projection of ξ to $\mathfrak{t}^K$. As above, we obtain that $V_0^i \subset \ker \mathrm{d}^2(h - \mathbf{J}^\eta)(0) =: \tilde{V}_0$ for every i and hence $V_0 \subset \tilde{V}_0$. □

Theorem 2.11. *Let V be a G-symplectic representation and $h : V \to \mathbb{R}$ be a smooth G-invariant Hamiltonian function with $\mathrm{d}h(0) = 0$ such that (GC) and (NR') hold for h. If $\xi \in \mathfrak{t}$ and $V_0 := \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ has linearly independent weights, then there is a local map m_{V_0} from a neighborhood U of 0 in V_0 to V such that $M_0 := m_{V_0}(U)$ is a manifold of T-relative equilibria with $0 \in M_0$, $T_0 M_0 = V_0$ and $G_{m_{V_0}(x)} = G_x$ for every $x \in V_0$.*

Proof. We only have to prove that $G_{m_{V_0}(x)} = G_x$ for every $x \in V_0$, if U is small enough. $G_x \subset G_{m_{V_0}(x)}$ follows from Corollary 2.10 and Corollary 2.6.

Thus it remains to prove that we can find U with $G_{m_{V_0}(x)} \subset G_x$ for every $x \in U$.

We recall that m_{V_0} is of the form $m_{V_0}(x) = x + v_1(x, \xi(x))$ with $(\mathrm{d}_x v_1)(0, \xi(0)) = 0$ and $(\mathrm{d}_\xi v_1)(0, \xi(0)) = 0$. Hence the derivative of the map $\phi\colon x \mapsto v_1(x, \xi(x))$ vanishes as well at $x = 0$.

Let $H \subset G$ be an isotropy subgroup of V. Split V_0 into the fixed point space V_0^H and its orthogonal complement C_0 within V_0. (The case $C_0 = \{0\}$ is trivial. Thus we assume $C_0 \neq \{0\}$ in the following.) If S_{C_0} denotes the unit sphere of C_0, then

$$d := \mathrm{dist}(S_{C_0}, V^H) > 0$$

and we obtain for every $c_0 \in C_0$ that $\mathrm{dist}(c_0, V^H) \geq d \|c_0\|$.

Now we suppose that for some $x = v_0^H + c_0 \in V_0$ with $v_0^H \in V_0^H$ and $c_0 \in C_0$ we have $m_{V_0}(x) \in V^H$. It is

$$m_{V_0}(v_0^H + c_0) = v_0 + c_0 + \phi(v_0^H + c_0) = v_0^H + c_0 + \phi(v_0^H) + \mathrm{d}_{V_0}\phi(v_0^H)c_0 + R(v_0^H, c_0)$$

with $\lim_{c_0 \to 0} \frac{R(v_0^H, c_0)}{\|c_0\|} = 0$.

Since $v_0^H + \phi(v_0^H) = m_{V_0}(v_0^H) \in V^H$, we deduce

$$c_0 + \mathrm{d}_{V_0}\phi(v_0^H)c_0 + R(v_0^H, c_0) \in V^H.$$

Since $\mathrm{d}_{V_0}\phi(0) = 0$ and ϕ is continuously differentiable, there is δ_1 such that

$$\left\|\mathrm{d}_{V_0}\phi(v_0^H)\right\| \leq \frac{d}{3}$$

if $\left\|v_0^H\right\| < \delta_1$.

By Hadamard's lemma, there is a smooth function f with values in $L(C_0, C_0)$ such that $R(v_0^H, c_0) = \langle c_0, f(v_0^H, c_0)c_0 \rangle$. Since f is bounded on $B_{\delta_1}(0)$, we can find $\delta_2 \leq \delta_1$ such that $R(v_0^H, c_0) \leq \frac{d}{3}\|c_0\|$ if $\left\|v_0^H\right\| < \delta_1$ and $\|c_0\| < \delta_2$. Thus if

$\|x\| < \min(\delta_1, \delta_2)$, then

$$\left\|\mathrm{d}_{V_0}\phi(v_0)c_0 + R(v_0^H, c_0)\right\| \leq \frac{2}{3} d \|c_0\|.$$

Therefore $\mathrm{d}_{V_0}\phi(v_0)c_0 + R(v_0^H, c_0) \in V^H$ implies that $c_0 = 0$ and hence $x \in V_0^H$. □

3 Dimensions of the isotropy strata

As shown by Patrick and Roberts [18], for free Hamiltonian actions of compact connected groups G the relative equilibria generically form a Whitney stratified set, whose strata are determined by the conjugacy classes of the groups $K = G_\xi \cap G_\mu$ consisting of the common elements of the isotropy subgroups of the generator ξ and the momentum μ. For a class (K) the stratum has the dimension

$$\dim G + 2 \dim Z(K) - \dim K,$$

where $Z(K)$ denotes the center of K. This can be generalized to possibly non-free actions, see [19, Section 6.3]. Generically Patrick and Robert's results apply to the subspaces of elements with the same isotropy group H and the free action of $\left({N(H)}/{H}\right)^\circ$ on that subspace. The strata are determined by the isotropy types (H) of the relative equilibria and the conjugacy classes of

$$K := \left({N(H)}/{H}\right)^\circ_\xi \cap \left({N(H)}/{H}\right)^\circ_\mu,$$

where ξ denotes the generator within the Lie algebra ${\mathfrak{n}(\mathfrak{h})}/{\mathfrak{h}}$ and μ the momentum in $\left({\mathfrak{n}(\mathfrak{h})}/{\mathfrak{h}}\right)^*$. If $H = gH'g^{-1}$ we identify the groups $\left({N(H)}/{H}\right)^\circ$ and $\left({N(H')}/{H'}\right)^\circ$ via $n \mapsto gng^{-1}$ and accordingly we identify conjugacy classes of their subgroups. Then the stratum corresponding to the types (H) and (K) for $K \subset {N(H)}/{H}$ has the dimension

$$\dim G - \dim H + 2\dim Z(K) - \dim K.$$

As shown above, the isotropy groups of the local manifold M_0 of T-relative equilibria tangent to V_0 coincide with those of V_0 if $V_0 = \bigoplus_{\alpha\in S}\mathbb{C}_\alpha$ with $S = \bigcup S_i$ as in Theorem 1.13 such that no S_i contains a Weyl reflection pair $\alpha \neq w\alpha$. To calculate the generic dimensions of the strata that intersect M_0, we need to investigate the group $\left({N(H)}/{H}\right)^\circ$ and the subgroup K.

The group $\left({N(H)}/{H}\right)^\circ$ is isomorphic to ${(C_G(H))^\circ}/{(Z(H))^\circ}$, where $C_G(H)$ denotes the centralizer of H within G, see for instance [6, Corollary 3.10.1]. Equivalently, this holds for the Lie algebras:

$${\mathfrak{n}(H)}/{\mathfrak{h}} \cong {\mathfrak{c}_G(H)}/{\mathfrak{z}(H)}\,.$$

Remark 3.1. In contrast to Field [6], we denote the Lie algebra of $C_G(H)$ by $\mathfrak{c}_G(H)$ and not by $\mathfrak{c}_G(\mathfrak{h})$ since it depends on H and not only on $\mathfrak{h}$: Consider for example a finite subgroup which is not contained in the center. Consequently, the Lie algebra $\mathfrak{n}(H)$ depends on H. Similarly, $\mathfrak{z}(H)$ depends on the group H in general: Suppose for example that $H = N(T)$ and that the Weyl group ${N(T)}/{T}$ is non-trivial.

Note also that $\left({N(H)}/{H}\right)^\circ$ and $\left({N(H^\circ)}/{H^\circ}\right)^\circ$ do not coincide in general.

Since $\mathfrak{t}_x$ coincides for all $x \in (V_0)_\tau$ and contains a point $\xi \in \mathfrak{t}$ which is not contained in any Weyl wall, we obtain

$$\mathfrak{c}_G(H) \subset \mathfrak{c}_G(\xi) = \mathfrak{g}_\xi = \mathfrak{t}.$$

Thus the Lie algebra ${\mathfrak{n}(H)}/{\mathfrak{h}}$ is Abelian and hence

$$K = Z(K) = \left({N(H)}/{H}\right)^\circ.$$

Consequently, the dimension of the stratum only depends on the isotropy type (H).

We need to determine the dimension of the group $\left(N(H) \big/ H \right)^\circ$. Again let L denote the minimal isotropy group of the adjoint action within $\mathfrak{t}$ that contains H. From $\mathfrak{c}_G(H) \subset \mathfrak{t}$, we obtain

$$\mathfrak{c}_G(H) = \mathfrak{c}_G(H) \cap \mathfrak{t} = \mathfrak{t}^H = \mathfrak{t}^L.$$

Moreover,

$$\mathfrak{z}(H) = \mathfrak{h} \cap \mathfrak{c}_G(H) = \mathfrak{h} \cap \mathfrak{t}^L = \mathfrak{t}_x \cap \mathfrak{t}^L$$

for any $x \in (V_0)_\tau$. As above, let $\mathfrak{t}'$ denote the orthogonal complement of $\mathfrak{t}_x$ within $\mathfrak{t}$. Note that we can identify $\mathfrak{t}'$ with the span of the set of weights S of V_0. Recall that $(\mathfrak{t}^*)^L$ intersects the affine span of S orthogonally, see Lemma 2.8. Hence $\mathfrak{t}^L$ intersects $\mathfrak{t}'$ orthogonally, and thus $\mathfrak{t}^L = \left(\mathfrak{t}_x \cap \mathfrak{t}^L\right) \oplus \left(\mathfrak{t}' \cap \mathfrak{t}^L\right)$. Therefore

$$\mathfrak{n}(H) \big/ \mathfrak{h} \cong (\mathfrak{t}')^L .$$

Altogether, the dimension of the stratum that contains $x \in V_0^H$ is given by

$$\dim G - \dim H + \dim (\mathfrak{t}')^L .$$

Remark 3.2. Despite the fact that $\left(N(H) \big/ H \right)^\circ$ and $\left(N(H^\circ) \big/ H^\circ \right)^\circ$ can have different dimensions, it is often sufficient to compute the Lie algebras in order to calculate the dimensions of the strata: For example, we know that $G_x \subset G_\mu$. If $\mathfrak{g}_\mu$ is the minimal isotropy Lie algebra that contains $\mathfrak{g}_x$, we can conclude that $L = G_\mu$.

Moreover, when we review the proof of Lemma 2.5, we obtain further restrictions on L: We have seen that $\dim (\mathfrak{t}')^L = k + 1$ if $k < l$, where k denotes the number of weights of V_0 contained in $(\mathfrak{t}^*)^L$ and l is the number of elements of S. Thus we only have to take the coadjoint isotropy subgroups M of $\mathfrak{t}$ into account such that the convex hull of all weights $\alpha_1, \ldots, \alpha_k$ that are not contained in the corresponding intersection $(\mathfrak{t}^*)^M$ of Weyl walls intersects I in a single point. Thus for a given point $x \in V_0$ with momentum μ we determine the minimal coadjoint isotropy group M of this type such that $M \subset G_\mu$. Then we know that $G_x \subset M$. Thus if the Lie algebra $\mathfrak{m}$ is the minimal Lie algebra of coadjoint subgroups of this type that contains $\mathfrak{g}_x$, we deduce that $L = M$.

4 Examples of non-constant isotropy within the main stratum

The T-relative equilibria that exist by Theorem 1.13 form a Whitney-stratified local set and the strata have constant isotropy type with respect to the T-action. This leads to the question if the T-relative equilibria of such a stratum have the same isotropy type with respect to the G-action. In general, this is not the case. In this section, we will discuss some examples of this phenomenon.

Therefore we do not expect in general that the G-orbit a stratum of T-relative equilibria consists of one stratum within the Whitney-stratified local set of G-relative equilibria. Nevertheless, this set is stratified by the G-isotropy type as we have shown in the last section.

Since for every kernel $V_0 = \ker \mathrm{d}^2(h - \mathbf{J}^\xi)(0)$ as above the local map m_{V_0} preserves the G-isotropy groups of the points contained in the principal stratum $(V_0)_\tau$ with respect to the T-action, the computation of the isotropy groups or the isotropy Lie algebras is pure representation theory. However, we will see that some of our considerations that come from Hamiltonian dynamics are still helpful:

It suffices to consider $V_0 \simeq \bigoplus_{\alpha \in S} \mathbb{C}_\alpha$ such that S is as in Theorem 1.13 and each S_i contains no pair $\alpha \neq w\alpha$ for any reflection w about a Weyl wall.

First of all, recall that $G_x \subset G_\mu$ if $\mu =: \mathbf{J}(x)$. Thus, we only have to investigate the G_μ-action. Moreover, we know from Lemma 2.1 that μ coincides with its projection to $\mathfrak{t}^*$. Therefore,

$$\mu = \pi \sum_{\alpha \in S} |x_\alpha|^2 \alpha,$$

when $x = \sum_{\alpha \in S} x_\alpha$ with $x_\alpha \in \mathbb{C}_\alpha$.

W. l. o. g. we assume that $x_\alpha \neq 0$ for every $\alpha \in S$. Then by Corollary 2.10 G_x is even contained in the maximal subgroup $K \subset G_\mu$ such that $\mathfrak{t}^K$ intersects the affine span of S orthogonally. To compute G_x, it suffices to consider the components of x with respect to a splitting of V into K-irreducible subspaces. Then $G_x = K_x$ is given by the intersection of the isotropy subgroups of these components.

Remark 4.1. Recall that by Lemma 1.6 the weights of each component $(V_0)_j$ of V_0 with respect to a splitting of V into K-irreducible subspaces correspond to the same equivalence class in $\mathfrak{t}^* \big/ \left(\mathfrak{t}^K\right)^\perp$. If S_j denotes the set of weights of $(V_0)_j$, then $\mathfrak{t}^K$ intersects the affine span of S_j orthogonally in exactly one point. Since it suffices to compute the isotropy groups of the points of the $(V_0)_j$ to compute the isotropy groups of the points of V_0, we will assume in the following that V_0 is contained in an irreducible K-subrepresentation, where K is the maximal isotropy subgroup of the adjoint action such that $\mathfrak{t}^K$ intersects the affine span of S orthogonally.

To investigate the isotropy groups of points in V_0, we only have to consider the unit sphere of V_0. Recall that we choose the G-invariant inner product such that for $x \in V_0 = \bigoplus_{\alpha \in S} \mathbb{C}_\alpha$ we obtain the norm $\|x\|^2 = \sum_{\alpha \in S} |x_\alpha|^2$. Thus the momenta of the elements of the unit sphere of V_0 are contained in the convex hull of S.

Now we want to construct examples of representations V^c and subspaces $V_0 \simeq \bigoplus_{\alpha \in S} \mathbb{C}_\alpha$ with S as above with the property that $(V_0)_\tau$ contains points of different isotropy types. Then there has to be a point $x \in V_0$ with $G_x \neq T_x$ and thus $G_\mu \neq T$ for $\mu = \mathbf{J}(x)$. Thus μ has to be contained in a Weyl wall.

Thus we search weights $\alpha_1, \ldots, \alpha_l$ such that there is an isotropy subspace $(\mathfrak{t}^*)^K$ such that the orthogonal projections of $\alpha_1, \ldots, \alpha_l$ to $(\mathfrak{t}^*)^K$ coincide and the intersection of $(\mathfrak{t}^*)^K$ with the interior of the convex hull of $\alpha_1, \ldots, \alpha_l$ is non-empty (and hence consists of one point). W. l. o. g we assume that K is maximal with this property. Let μ be the intersection point. Then $G_\mu = K$. By Remark 4.1, we assume that V_0 is contained in the G_μ-irreducible space $\tilde{V}_0$. [3, Chapter V, (8.1)] implies that G_μ has a connected covering group of the form $C \times Z$, where Z is the identity component of the center of G_μ and C is semi-simple. The Lie-algebra of the group Z coincides with $\mathfrak{t}^{G_\mu}$ and $\mathfrak{g}_\mu = \mathfrak{c} \oplus \mathfrak{t}^{G_\mu}$, where $\mathfrak{c}$ denotes the Lie algebra of C. We can consider $\tilde{V}_0$ as a C-representation. Since $\tilde{\mathfrak{t}} := (\mathfrak{t}^{G_\mu})^\perp \subset \mathfrak{t}$ is the Lie algebra of a maximal torus of C, the weights of the C-representation $\tilde{V}_0$ are given by their projections to the dual Lie algebra $\tilde{\mathfrak{t}}^*$. The projection of μ to $\tilde{\mathfrak{t}}^*$ is 0. Hence the projections $\bar{\alpha}_1, \ldots, \bar{\alpha}_l$ of $\alpha_1, \ldots, \alpha_l$ form a linearly dependent set $\bar{S}$, but $\bar{S}$ is still independent in the affine sense: No point is contained in the affine span of the others. Moreover, $\bar{S}$ is maximal within its affine span. Consider the two following examples of C-irreducible representations with subsets of this type:

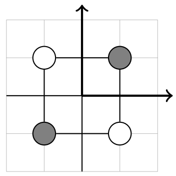

Figure 1: $C = \mathrm{SU}(2) \times \mathrm{SU}(2)$, maximal weight $(1, 1)$

Example 4.2. Suppose that $C = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and consider the C-irreducible representation with maximal weight $(1, 1)$ (with respect to the coordinate system corresponding to the basis given in Section 1.3 on each factor on the Lie algebra $\mathbb{R} \times \mathbb{R}$), see the weight diagram in Figure 1. Then the weights $(-1, 1)$ and $(1, -1)$ (depicted in white in Figure 1) contain 0 in their affine span.

Example 4.3. Let C be the group $\mathrm{SU}(3)$ and consider the C-irreducible representation with maximal weight $(2, 0)$ (again, with respect to the coordinate system corresponding to the basis given in Section 1.3), see Figure 2. Then 0 is contained in the affine span of the two weights $(2, 0)$ and $(-1, 0)$.

In both examples, the sets $\bar{S}$ have an important additional property: We call a weight a *neighbor* of another weight of an irreducible representation of K iff their difference coincides with a root of K. In our examples, $\bar{S}$ consists of two weights which have common neighbors. This makes possible that the isotropy type of points of the same stratum of T-relative equilibria differs on the Lie algebra level: If $x_\alpha \in \mathbb{C}_\alpha$, then $\mathfrak{k}x_\alpha$ is contained in the span of $\mathbb{C}_\alpha$ and the weight spaces corresponding to neighbors of α. To obtain $x = (x_\alpha, x_\beta) \in \mathbb{C}_\alpha \oplus \mathbb{C}_\beta$ and

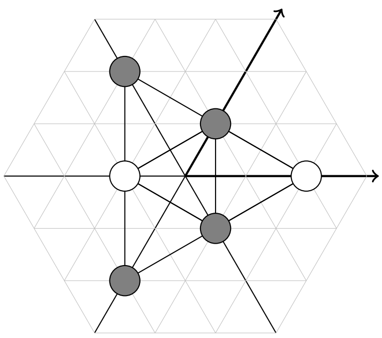

Figure 2: $C = \mathrm{SU}(3)$, maximal weight $(2, 0)$

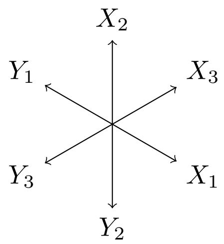

Figure 3: Action of the root vectors of SU(3)

$\xi \in \mathfrak{k}$ with $\xi x = 0$, but $\xi x_\alpha \neq 0$ and $\xi x_\beta \neq 0$, α and β hence need to have at least one common neighbor.

In the figures with weight diagrams, there is an edge between weights iff they are neighbors. The common neighbors of the weights of $\bar{S}$ are given by $(1,1)$ and $(-1,-1)$ in Example 4.2 and by $(0,1)$ and $(1,-1)$ in Example 4.3.

In the following, we will discuss some examples of representations for which we obtain strata of T-relative equilibria with different isotropy Lie algebras. Examples 4.2 and 4.3 occur as subspaces of these spaces.

We will restrict our investigation to the computation of isotropy Lie algebras. As pointed out in remark 3.2, this often suffices to determine the dimensions of the strata and indeed this will be the case for our example strata.

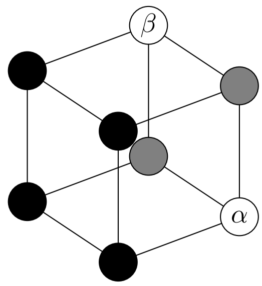

Figure 4: $G = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, maximal weight $(1,1,1)$

Example 4.4. Suppose that $G = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ and hence

$$\mathfrak{g} = \mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{su}(2).$$

Let $V_1 \simeq \mathbb{C}^2$ denote the standard representation of SU(2). Suppose that

$$V^c \simeq V_1 \otimes V_1 \otimes V_1.$$

such that the i-th factor of $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ acts on the i-th factor of the tensor product, i. e.

$$(g_1, g_2, g_3)(v_1 \otimes v_2 \otimes v_3) = (g_1 v_1) \otimes (g_2 v_2) \otimes (g_3 v_3).$$

The SU(2)-representation V_1 has the weights 1 and -1 with weight spaces spanned by e_1 and e_2 respectively (when we identify the maximal torus of SU(2) with $\mathbb{R}/\mathbb{Z}$ and its Lie algebra with $\mathbb{R}$, see Section 1.3). Hence the weights of the representation V^c are of the form $((-1)^i, (-1)^j, (-1)^k)$ with $i, j, k \in \{0, 1\}$. The Weyl walls coincide with the planes spanned by two coordinate axes.

Now we consider the space

$$V_0 = \mathbb{C}_{(1,1,-1)} \oplus \mathbb{C}_{(1,-1,1)} = \langle e_1 \otimes e_1 \otimes e_2, e_1 \otimes e_2 \otimes e_1 \rangle.$$

Since $\alpha := (1, 1, -1)$ and $\beta := (1, -1, 1)$ are linearly independent and are the maximal subset of weights within their affine span, there is a branch of relative equilibria tangent to V_0. The points (x_α, x_β) with $|x_\alpha| \neq 0 \neq |x_\beta|$ form the stratum $(V_0)_\tau$ of V_0 of minimal isotropy with respect to the T-action. Since

$$\mathbf{J}(x_\alpha, x_\beta) = \pi|x_\alpha|^2\alpha + \pi|x_\beta|^2\beta = \pi((|x_\alpha|^2 + |x_\alpha|^2, |x_\alpha|^2 - |x_\alpha|^2, |x_\beta|^2 - |x_\alpha|^2),$$

$\mathbf{J}(x_\alpha, x_\beta)$ is not contained in any Weyl wall if $|x_\alpha| \neq |x_\beta|$. In this case, $\mathfrak{g}_{\mathbf{J}(x_\alpha,x_\beta)} = \mathfrak{t}$ and hence $\mathfrak{g}_x = \mathfrak{t}_x$. If $|x_\alpha| = |x_\beta|$ in contrast, $\mathbf{J}(x_\alpha, x_\beta) =: \mu$ is contained in the intersection of two Weyl walls: the ones corresponding to the pair of roots $(0, \pm 2, 0)$ and to $(0, 0, \pm 2)$ respectively. Due to Lemma 1.2, $\mathfrak{g}_\mu = \mathfrak{t} \oplus M_{(0,2,0)} \oplus M_{(0,0,2)}$, where M_α denotes the real part of the sum of the weight spaces for $\pm\alpha$ in $\mathfrak{g} \otimes \mathbb{C}$ respectively. Thus we then obtain

$$\mathfrak{g}_\mu = \left\{ \begin{pmatrix} A & & \\ & B & \\ & & C \end{pmatrix} \middle| A = \begin{pmatrix} ai & 0 \\ 0 & -ai \end{pmatrix}, a \in \mathbb{R}, B, C \in \mathfrak{su}(2) \right\}.$$

We know that $\mathfrak{g}_x \subseteq \mathfrak{g}_\mu$. The intersection $\mathfrak{t} \cap \mathfrak{g}_x = \mathfrak{t}_x$ is constant on the stratum $(V_0)_\tau = \{(x_\alpha, x_\beta) \in V_0 \mid x_\alpha \neq 0, x_\beta \neq 0\}$: For $x \in (V_0)_\tau$, we have $\mathfrak{t}_x = \langle (0, 1, 1) \rangle$.

To compute $\mathfrak{g}_x$ in the case $|x_\alpha| = |x_\beta|$, we investigate the action of the complexification $\mathfrak{g}_\mu \otimes \mathbb{C} \simeq \mathbb{C} \times \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$. We use the basis of $\mathfrak{sl}(2, \mathbb{C})$ given in [14]:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then a basis of $\mathfrak{g}_\mu \otimes \mathbb{C} \subset \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$ is given by $H_1 := (H, 0, 0)$, $H_2 := (0, H, 0)$, $H_3 := (0, 0, H)$, $X_2 := (0, X, 0)$, $X_3 := (0, 0, X)$, $Y_2 := (0, Y, 0)$, and $Y_3 := (0, 0, Y)$. We search for elements $Z \notin \mathfrak{t} \otimes \mathbb{C} = \langle H_1, H_2, H_3 \rangle$ with $Zx = 0$ for $x \in V_0$ with $|x_\alpha| = |x_\beta| \neq 0$. While the elements of $\mathfrak{t} \otimes \mathbb{C}$ preserve the weight spaces, the elements X_2, X_3, Y_2, and Y_3 are root vectors and hence cause shifts between the weight spaces. In more detail,

$$\begin{aligned} X_2(e_1 \otimes e_1 \otimes e_2) &= 0, & X_2(e_1 \otimes e_2 \otimes e_1) &= e_1 \otimes e_1 \otimes e_1 \\ X_3(e_1 \otimes e_1 \otimes e_2) &= e_1 \otimes e_1 \otimes e_1, & X_3(e_1 \otimes e_2 \otimes e_1) &= 0 \\ Y_2(e_1 \otimes e_1 \otimes e_2) &= e_1 \otimes e_2 \otimes e_2, & Y_2(e_1 \otimes e_2 \otimes e_1) &= 0 \\ Y_3(e_1 \otimes e_1 \otimes e_2) &= 0, & Y_3(e_1 \otimes e_2 \otimes e_1) &= e_1 \otimes e_2 \otimes e_2. \end{aligned}$$

Thus the space V_0 is contained in the $\mathfrak{g}_\mu$-invariant subspace

$$\tilde{V}_0 = \langle < e_1 \otimes e_1 \otimes e_1,\ e_1 \otimes e_1 \otimes e_2\ , e_1 \otimes e_2 \otimes e_1,\ e_1 \otimes e_2 \otimes e_2 > .$$

See Figure 4 for a weight diagram of V^c. The weights of V_0 are depicted in white, the other ones of $\tilde{V}_0$ in gray.

We recognize Example 4.2: We can regard $\tilde{V}_0$ as a representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$, the complexified Lie algebra of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ identified with the second and third component of G. Then the weights of $\tilde{V}_0$ are simply $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ and V_0 is spanned by the weight spaces of $(1,-1)$ and $(-1,1)$.

Suppose that $Zx = 0$ for some $Z \in \mathfrak{g}_\mu \otimes \mathbb{C}$ and a given $x = (x_\alpha, x_\beta) \in V_0$ with $x_\alpha \neq 0$. Split Z into $Z = Z_T + Z'$ with $Z_T \in \mathfrak{t}$ and $Z' \in \langle X_2, X_3, Y_2, Y_3\rangle$. Then $Z_T V_0 \subset V_0$, while $Z'V_0$ is contained in $\langle e_1 \otimes e_1 \otimes e_1, e_1 \otimes e_2 \otimes e_2\rangle$, which is a complement of V_0 within $\tilde{V}_0$. Thus we have $Z_T x = 0$ and $Z'x = 0$. Moreover, if we split Z' further into $Z' = Z_X + Z_Y$ with $Z_X \in \langle X_2, X_3\rangle$ and $Z_Y \in \langle Y_2, Y_3\rangle$, then $Z_X x \in \langle e_1 \otimes e_1 \otimes e_1\rangle$ and $Z_Y \in \langle e_1 \otimes e_2 \otimes e_2\rangle$. Thus $Z_X x = 0$ and $Z_Y x = 0$. Suppose that $Z_X = aX_2 + bX_3$ and $Z_Y = cY_2 + dY_3$, $a, b, c, d \in \mathbb{C}$. Then

$$\begin{aligned} bx_\alpha + ax_\beta &= 0 \\ cx_\alpha + dx_\beta &= 0. \end{aligned}$$

The solutions a, b, c, d correspond to elements of the isotropy Lie subalgebra of the complexified Lie algebra $\mathfrak{g}_\mu \otimes \mathbb{C}$. To determine $\mathfrak{g}_x$, we have to identify the real solutions, i. e. the solutions in $\mathfrak{g}_\mu$. Z is contained in $\mathfrak{g}_\mu$ iff this holds for Z_T and Z'. Furthermore, $Z' \in \mathfrak{g}_\mu$ is equivalent to $c = -\bar{a}$ and $d = -\bar{b}$. Thus for a fixed $Z' \in \mathfrak{t}^\perp \subset \mathfrak{g}_\mu$ there is a non-trivial solution $x = (x_\alpha, x_\beta) \in V_0$ of $Z'x = 0$ iff

$$\det \begin{pmatrix} b & a \\ -\bar{a} & -\bar{b} \end{pmatrix} = |a|^2 - |b|^2 = 0.$$

If this condition is satisfied, the kernel is spanned by $(a, -b)^T$. Hence we obtain again the necessary condition $|x_\alpha| = |x_\beta|$ from above. Conversely, if $|x_\alpha| = |x_\beta|$, there is a non-zero Z' in the complement of $\mathfrak{t}$ within $\mathfrak{g}_\mu$ that satisfies $Z'x = 0$. Explicitly, the solutions of Z' are given by the vectors (a, b) in the complex span of $\langle x_\alpha, -x_\beta\rangle$ and the requirement $a = -\bar{c}$ and $b = -\bar{d}$. Thus, the points x with $|x_\alpha| = |x_\beta|$ have a 3-dimensional isotropy Lie algebra $\mathfrak{g}_x$ of rank 1, which is hence isomorphic to $\mathfrak{su}(2)$. In contrast, the isotropy Lie algebra of the other points of $(V_0)_\tau$ is just $\mathfrak{g}_x = \mathfrak{t}_x = \langle(0,1,1)\rangle$.

Thus the G-orbit of the stratum $m_{V_0}((V_0)_\tau)$ of T-relative equilibria consists of two strata: Since $G_x \subset T$ for every point x of $(V_0)_\tau$ with $|x_\alpha| \neq |x_\beta|$ and $(\mathfrak{t}')^T = \mathfrak{t}' = (\mathfrak{t}_x)^\perp \subset \mathfrak{t}$ is 2-dimensional, the points whose isotropy Lie algebra is conjugated to $\langle(0,1,1)\rangle$ form a manifold of dimension

$$\dim \mathfrak{g} - \dim\langle(0,1,1)\rangle + \dim \mathfrak{t}' = 9 - 1 + 2 = 10.$$

For $x \in (V_0)_\tau$ with $|x_\alpha| = |x_\beta|$, the momentum μ is contained in $\langle(1,0,0)\rangle = (\mathfrak{t}^*)^{G_\mu} \cong (\mathfrak{t}')^{G_\mu}$ and G_μ is the minimal coadjoint isotropy group which contains G_x. Thus the stratum of points with isotropy Lie algebras isomorphic to $\mathfrak{su}(2)$ has the dimension

$$\dim \mathfrak{g} - \dim \mathfrak{g}_x + \dim(\mathfrak{t}')^{G_\mu} = 9 - 3 + 1 = 7.$$

Example 4.5. Suppose that there is a point $x_0 \in V_0$ with momentum μ such that G_μ is covered by a group $\mathrm{SU}(3) \times T'$, where T' is a torus, and that V_0 is contained in an irreducible G_μ-subrepresentation $\tilde{V}_0$ of V^c such that the situation

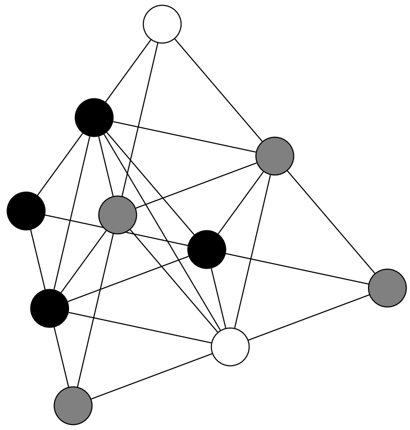

Figure 5: $G = \mathrm{SU}(4)$, maximal weight $(2,0,0)$

is as in Example 4.3. More precisely, we assume that $\tilde{V}_0$ regarded as an SU(3)-representation is irreducible with maximal weight $(2,0)$ and V_0 is given by the sum of the two weight spaces corresponding to the weights $(2,0)$ and $(-1,0)$. We will see that, as in Example 4.4, the main part of the calculation is independent of the actual embedding of $\tilde{V}_0$ into V^c. Thus we will start with the calculation of the $\mathfrak{su}(3)$-part of the isotropy Lie algebras. Afterwards, we will discuss a concrete example of a G-representation V^c with the SU(3)-subrepresentation $\tilde{V}_0$.

Again, we study the action of the complexified Lie algebra $\mathrm{SU}(3) \otimes \mathbb{C} \simeq \mathfrak{sl}(3,\mathbb{C})$. We adopt the notation of [14]: We consider the following basis of $\mathfrak{sl}(3,\mathbb{C})$:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$
$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

H_1 and H_2 form a basis of $\mathfrak{t}$ and the vectors X_i, Y_i are root vectors. If X_i corresponds to the root ρ_i, then Y_i corresponds to $-\rho_i$. With respect to the dual basis of H_1 and H_2, we have

$$\rho_1 = (2,-1), \quad \rho_2 = (-1,2), \quad \rho_3 = \rho_1 + \rho_2 = (1,1).$$

If $x \in \tilde{V}_0$ is a weight vector with the weight α, then $X_i x$ either is a weight vector corresponding to $\alpha + \rho_i$ or $\alpha + \rho_i$ is not a weight of $\tilde{V}_0$ and $X_i x = 0$. Similarly, if $\alpha - \rho_i$ is a weight of $\tilde{V}_0$, then $Y_i x$ is a weight vector corresponding to $\alpha - \rho_i$. In particular, if we consider the weight spaces of $V_0 \simeq \mathbb{C}_{(2,0)} \oplus \mathbb{C}_{(-1,0)}$, we obtain

$$\begin{aligned} X_1\left(\mathbb{C}_{(2,0)}\right) = \{0\}, \quad X_2\left(\mathbb{C}_{(2,0)}\right) = \{0\}, \quad X_3\left(\mathbb{C}_{(2,0)}\right) = \{0\}, \\ Y_1\left(\mathbb{C}_{(2,0)}\right) = \mathbb{C}_{(0,1)}, \quad Y_2\left(\mathbb{C}_{(2,0)}\right) = \{0\}, \quad Y_3\left(\mathbb{C}_{(2,0)}\right) = \mathbb{C}_{(1,-1)}, \end{aligned}$$

$$\begin{aligned} X_1\left(\mathbb{C}_{(-1,0)}\right) = \mathbb{C}_{(1,-1)}, \quad X_2\left(\mathbb{C}_{(-1,0)}\right) = \mathbb{C}_{(-2,2)}, \quad X_3\left(\mathbb{C}_{(-1,0)}\right) = \mathbb{C}_{(0,1)}, \\ Y_1\left(\mathbb{C}_{(-1,0)}\right) = \{0\}, \quad Y_2\left(\mathbb{C}_{(-1,0)}\right) = \mathbb{C}_{(0,-2)}, \quad Y_3\left(\mathbb{C}_{(-1,0)}\right) = \{0\}, \end{aligned}$$

see Figures 2 and 3.

The elements X_2, Y_2 act trivially on $\mathbb{C}_{(2,0)}$, but not on $\mathbb{C}_{(-1,0)}$ and the span of $X_2(\mathbb{C}_{(-1,0)})$ and $Y_2(\mathbb{C}_{(-1,0)})$ intersects the space $\mathfrak{sl}(3,\mathbb{C})\left(\mathbb{C}_{(2,0)}\right)$ only in 0. Hence, if $Zx = 0$ for some $x \in (V_0)_\tau$ and $Z = Z_T + Z'$ with $Z_T \in \mathfrak{t} = \langle H_1, H_2\rangle$ and $Z' \in \langle X_1, X_2, X_3, Y_1, Y_2, Y_2\rangle$, then Z' has to be a linear combination of elements X_1, X_3, Y_1, Y_3.

To determine solutions Z', x of $Z'x = 0$, we represent the linear maps X_1, X_3, Y_1, Y_3 with respect to an orthonormal basis of $\tilde{V}_0$.

The SU(3)-representation $\tilde{V}_0$ has maximal weight $(2,0)$ and is hence isomorphic to the SU(3)-representation on the space of homogeneous polynomials of degree 2 in 3 complex variables, see [12]. Thus an explicit description of the action is well-known and clear. However, we will follow another approach, which is also suitable for more complicated representations: We fix an SU(3)-invariant Hermitian inner product. Since it is in particular T-invariant, the weight spaces are orthogonal to each other. Thus we obtain an orthogonal basis of the $\mathbb{C}$-vector space $\tilde{V}_0$ if we choose one nontrivial subvector of every weight space. To obtain an orthonormal basis, we only have to determine the normalization factors.

Now we consider the 3-dimensional subspace $\mathbb{C}_{(2,0)} \oplus \mathbb{C}_{(0,1)} \oplus \mathbb{C}_{(-2,2)}$. This space is an irreducible representation of the Lie algebra generated by H_1, X_1, and Y_1, which is isomorphic to $\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{su}(2) \otimes \mathbb{C}$. Hence it can be regarded as a 3-dimensional complex representation of SU(2). Thus it is isomorphic to the one described in [14, Section 4.2, Example 4.10], which consists of the homogeneous polynomials of degree 2 on $\mathbb{C}^2$. The weight spaces are spanned by the monomials z_1^2, $z_1 z_2$, and z_2^2 with weights -2, 0, and 2 respectively. We have

$$\begin{aligned} X_1(z_1^2) = -2z_1z_2, \quad X_1(z_1z_2) = -z_2^2, \quad X_1(z_2^2) = 0, \\ Y_1(z_1^2) = 0, \quad Y_1(z_1z_2) = -z_1^2, \quad Y_1(z_2^2) = -2(z_1z_2). \end{aligned}$$

With respect to any orthonormal basis for a SU(2)-invariant inner product, the action of $X_1 - Y_1$ is represented by a skew-Hermitian matrix. For the basis z_1^2,z_1z_2, and z_2^2, we obtain

$$X_1 - Y_1 = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix}$$

Thus for an orthonormal basis $\lambda_1 z_1^2$, $\lambda_2 z_1 z_2$, and $\lambda_3 z_2^2$ with $\lambda_1, \lambda_2, \lambda_3 > 0$, the matrix is

$$X_1 - Y_1 = \begin{pmatrix} 0 & \frac{\lambda_2}{\lambda_1} & 0 \\ -2\frac{\lambda_1}{\lambda_2} & 0 & 2\frac{\lambda_3}{\lambda_2} \\ 0 & -\frac{\lambda_2}{\lambda_3} & 0 \end{pmatrix}.$$

This is skew-Hermitian for $\frac{\lambda_2}{\lambda_1} = \frac{\lambda_2}{\lambda_3} = \sqrt{2}$. In particular, $\|Y_1(z)\| = \sqrt{2}\,\|z\|$ for $z \in \mathbb{C}_{(2,0)}$.

Analogously we deduce that $\|Y_3(z)\| = \sqrt{2}\,\|z\|$ for $z \in \mathbb{C}_{(2,0)}$, $\|X_1(z)\| = \|z\|$ for $z \in \mathbb{C}_{(-1,0)}$, and $\|X_3(z)\| = \|z\|$ for $z \in \mathbb{C}_{(-1,0)}$. Since $[Y_1, Y_3] = 0$, we can choose orthonormal vectors of the form

$$e_{(2,0)} \in \mathbb{C}_{(2,0)}, \quad e_{(0,1)} = \frac{1}{\sqrt{2}} Y_1 e_{(2,0)}, \quad e_{(1,-1)} = \frac{1}{\sqrt{2}} Y_3 e_{(2,0)},$$
$$e_{(-1,0)} = Y_1 e_{(1,-1)} = Y_3 e_{(0,1)}.$$

Then a linear map $aX_1 - \bar{a}Y_1 + bX_3 - \bar{b}Y_3$, $a, b \in \mathbb{C}$ from $V_0 = \mathbb{C}_{(-1,0)} \oplus \mathbb{C}_{(2,0)}$ to $\mathbb{C}_{(0,1)} \oplus \mathbb{C}_{(1,-1)}$ is represented by the matrix

$$\begin{pmatrix} b & -\sqrt{2}\bar{a} \\ a & -\sqrt{2}\bar{b} \end{pmatrix}.$$

Since the determinant is $\sqrt{2}(|a|^2 - |b|^2)$, the kernel is non-trivial iff $|a| = |b|$. If $(x_{(-1,0)}, x_{(2,0)})$ is a kernel element, we thus have $\left|x_{(-1,0)}\right| = \sqrt{2}\left|x_{(2,0)}\right|$, which is satisfied iff the momentum of x with respect to the SU(3)-action vanishes. Vice versa, if $x = (x_{(-1,0)}, x_{(2,0)})$ satisfies this condition and $x_{(-1,0)} = \sqrt{2}\theta x_{(2,0)}$, then for any $a \in \mathbb{C}$, x is in the kernel of the above matrix iff $b := \overline{\theta a}$. Therefore $\mathfrak{su}(3)_x$ is a 3-dimensional Lie subalgebra of the real Lie algebra $\mathfrak{su}(3)$. Again, since $\mathfrak{su}(3)_x$ is the Lie algebra of the compact group $\mathrm{SU}(3)_x$ and since $\mathfrak{su}(3)_x$ is simple, $\mathfrak{su}(3)_x \simeq \mathfrak{su}(2)$.

We now give an example of a group G and a G-representation V^c such that generically, if the center space of $\mathrm{d}X_h(0)$ is isomorphic to V^c, there is a ξ in $\mathfrak{t}^K$ with $K \simeq \mathrm{SU}(3) \times T'$ for some torus T' such that $\mathrm{d}^2(h - \mathbf{J}^\xi)(0) = \tilde{V}_0$ is of the above form: Set $G = \mathrm{SU}(4)$ and consider the basis of $\mathfrak{t} \otimes \mathbb{C}$ formed by the diagonal matrices H_1, H_2, and H_3 with diagonal entries $(1, -1, 0, 0)$, $(0, 1, -1, 0)$, and $(0, 0, 1, -1)$ respectively. Let V^c be the representation with maximal weight $\lambda = (2, 0, 0)$ with respect to the dual basis of $\{H_1, H_2, H_3\}$. Now we change the basis of $\mathfrak{t}$ and replace H_3 by the diagonal matrix with diagonal entries $(1, 1, 1, -3)$, which we denote by H_3'. With respect to the dual basis of $\{H_1, H_2, H_3'\}$, λ is given by $(2, 0, 2)$.

When we consider the complexified adjoint action, H_3' is contained in the kernel of all elements of $\mathfrak{sl}(4, \mathbb{C})$ of the form

$$\left(\begin{array}{ccc|c} & & & 0 \\ & Z & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) =: \bar{Z},$$

$Z \in \mathfrak{sl}(3, \mathbb{C})$. The span of this subalgebra and $\mathfrak{t} \otimes \mathbb{C}$ generates the complexified isotropy Lie algebra of H_3' within $\mathfrak{sl}(4, \mathbb{C})$, which is isomorphic to the Lie algebra $\mathfrak{sl}(3, \mathbb{C}) \oplus \mathbb{C}$: The isomorphism is given by

$$(Z, z) \in \mathfrak{sl}(3, \mathbb{C}) \oplus \mathbb{C} \mapsto \bar{Z} + zH_3' \in \mathfrak{sl}(4, \mathbb{C})_{H_3'}.$$

The intersection of the complex Lie algebra $\mathfrak{sl}(4, \mathbb{C})_{H_3'}$ with $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(3) \oplus \mathbb{R}$ and hence the group $G_{H_3'}$ is isomorphic to $\mathrm{SU}(3) \times S^1$. (It is given by the elements

$$\left(\begin{array}{ccc|c} & & & 0 \\ & U & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & (\det U)^{-1} \end{array}\right),$$

$U \in \mathrm{U}(3)$, of $\mathrm{SU}(4)$.)

The dual vector $\nu = (0, 0, 1)$ of H_3' has the same isotropy subgroup G_ν. Moreover, the roots of G_ν are the vectors given by $(r_1, r_2, 0)$ (with respect to the dual basis of $\{H_1, H_2, H_3'\}$) such that (r_1, r_2) is a root of $\mathrm{SU}(3)$ (with respect to the dual basis of $\{H_1, H_2\}$). Thus $(\alpha_1, \alpha_2, 2)$ is a weight of V^c iff (α_1, α_2) is a weight of the $\mathrm{SU}(2)$-representation with maximal weight $(2, 0)$. Moreover, all weights of V^c occur with multiplicity 1. (This follows from the representation theory of $\mathrm{SU}(n)$ given by Young tableaux, see [12]. V^c coincides with the representation of $\mathrm{SU}(4)$ on the space of homogeneous polynomials of degree 2 in four complex variables.)

Thus the weight spaces corresponding to weights of the form $(\alpha_1, \alpha_2, 2)$ span a subspace $\tilde{V}_0$ as described above.

See Figure 5 for a weight diagram of V^c. The weights of $V_0 = \mathbb{C}_{(2,0,2)} \oplus \mathbb{C}_{(-1,0,2)}$ are depicted in white, the other weights of $\tilde{V}_0$ in gray.

For $x \in (V_0)_\tau$ with $\left|x_{(-1,0,2)}\right| = \sqrt{2}\left|x_{(2,0,2)}\right|$, the isotropy Lie algebra $\mathfrak{g}_x$ is isomorphic to $\mathfrak{su}(2)$: The affine span of the weights of $\tilde{V}_0$ is orthogonal to $\langle\nu\rangle$ and intersects $\langle\nu\rangle$ in the single point 2ν. Since $\langle\nu\rangle = ((\mathfrak{t})^*)^{G_\mu}$, all isotropy subgroups of elements of $(V_0)_\tau$ are contained in G_ν, see Lemma 2.9. We can identify the roots of $G_\nu \cong \mathrm{SU}(3) \times S^1$ with those of $\mathrm{SU}(3)$.

Again, $(\mathfrak{g}_\nu)_x$ is given by the span of $\mathfrak{t}_x$ and the space of solutions ξ of the equation $\xi x = 0$ that are contained in the real part of the span of the root vectors. The solutions ξ are determined by the above calculation. Moreover, under the identification $(\mathfrak{t})^* \cong \mathfrak{t}$ via a T-invariant inner product, the space $\mathfrak{t}_x$ is given by the orthogonal complement of the span of the weights of V_0 for any $x \in (V_0)_\tau$. Since the Lie algebra of the S^1-factor of G_ν is contained in the span of the weights of V_0, we also can identify $\mathfrak{t}_x$ with the isotropy Lie algebra of x with respect to the Lie algebra of the corresponding maximal torus of $\mathrm{SU}(3)$. Hence $(\mathfrak{g}_\nu)_x$ is isomorphic to $\mathfrak{su}(3)_x \cong \mathfrak{su}(2)$.

The stratum of points of this isotropy type has the dimension

$$\dim G - \dim G_x + (\mathfrak{t}')^L = 15 - 3 + 1 = 13,$$

since $\mathbf{J}(x) \in \langle\nu\rangle$ and $L = L(G_x) = G_\nu$.

For any $x \in (V_0)_\tau$ with $\left|x_{(-1,0,1)}\right| \neq \sqrt{2}\left|x_{(2,0,1)}\right|$, we obtain $\mathfrak{g}_x = \mathfrak{t}_x$, which is 1-dimensional. Moreover, the isotropy group of

$$\mu := \mathbf{J}(x) = \sum \left|x_{(-1,0,2)}\right|^2 (-1, 0, 2) + \left|x_{(2,0,2)}\right|^2 (-1, 0, 2)$$

is given by the connected subgroup of $G = \mathrm{SU}(4)$ whose Lie algebra is spanned by $\bar{X}_2$, $\bar{Z}_2$ and $\mathfrak{t}$. Hence $\mathbb{C}_{(-1,0,2)}$ and $\mathbb{C}_{(2,0,2)}$ are contained in different irreducible G_μ-subrepresentations of V^c and thus G_x coincides with $(G_\mu)_{(2,0,2)} \cap (G_\mu)_{(-1,0,2)}$. Hence all points of $(V_0)_\tau$ with $\left|x_{(-1,0,1)}\right| \neq \sqrt{2}\left|x_{(2,0,2)}\right|$ are of the same isotropy type and the isotropy subgroups are conjugated by elements of T. Thus $(\mathfrak{t}')^L = \mathfrak{t}'$ and we obtain

$$\dim G - \dim G_x + (\mathfrak{t}')^L = 15 - 1 + 2 = 16.$$

We also want to point out, that $G_x \subset G_\nu$ holds for all points $x \in (V_0)_\tau$ but not for all $x \in V_0$: If $x \in \mathbb{C}_{(2,0,2)}$ and $\mu := \mathbf{J}(x)$, then G_μ is a group isomorphic to $\mathrm{U}(3) \simeq \mathrm{SU}(3) \times S^1$ and not contained in G_ν. Moreover, for every root ρ of G_μ the form $(2, 0, 1) + \rho$ is not a weight of V^c and hence $G_x = G_\mu$.

Example 4.6. We now consider a configuration similar to that in Examples 4.2 and 4.4: We consider the irreducible SU(4)-representation $\tilde{V}_0$ with maximal weight $\alpha := \frac{1}{2}(1, 1, -1, -1)$, where we identify $\mathfrak{t}$ with the subspace of points (t_1, t_2, t_3, t_4) of $\mathbb{R}^4$ with $t_1 + t_2 + t_3 + t_4 = 0$. The weights of this representation are given by the Weyl group orbit of α.

We consider the two weights α and $\beta := -\alpha$, which are linearly dependent but obviously independent in the affine sense.

If $x = (x_\alpha, x_\beta) \in V_0 := \mathbb{C}_\alpha + \mathbb{C}_\beta$ and $\|x\| = 1$, then $\mathbf{J}(x) = 0$ iff $|x_\alpha| = |x_\beta| = \frac{1}{\sqrt{2}}$.

The four other weights of the representation are all common neighbors of α and β. The roots of SU(4) are given by

$$\begin{aligned}
\pm\rho_1 &:= \pm(1, -1, 0, 0), & \pm \rho_2 &:= \pm(0, 0, 1, -1),\\
\pm\rho_3 &:= \pm(0, 1, -1, 0), & \pm \rho_4 &:= \pm(1, 0, 0, -1),\\
\pm\rho_5 &:= \pm(1, 0, -1, 0), & \pm \rho_6 &:= \pm(0, 1, 0, -1).
\end{aligned}$$

Let X_i and Y_i be the root vectors given by matrices with a single non-vanishing entry 1 corresponding to ρ_i and $-\rho_i$ respectively. Then the linear maps defined by X_1, X_2, Y_1, and Y_2 vanish on $\mathbb{C}_\alpha$ and $\mathbb{C}_\beta$. If $|x_\alpha| \neq |x_\beta|$, then $\mathbf{J}(x)$ is contained exactly in the Weyl walls corresponding to $\pm\rho_1$ and $\pm\rho_2$. Thus in this case, $\mathfrak{sl}(4, \mathbb{C})_x$ is given by the sum of the 2-dimensional complex space $\mathfrak{t}_x \otimes \mathbb{C}$ and the span of X_1, X_2, Y_1, Y_2 which is isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$. Hence $\mathfrak{su}(4)_x$ is isomorphic to $\mathfrak{su}(2) \times \mathfrak{su}(2)$.

If $|x_\alpha| = |x_\beta|$, then $\mathbf{J}(x) = 0$ and hence we have to consider the action of every root of SU(4). Again, $\mathfrak{sl}(4, \mathbb{C})_x$ contains $\mathfrak{t}_x \otimes \mathbb{C}$ and the span of X_1, X_2, Y_1, Y_2, but in addition a subspace of the span of X_i, Y_i, $i \in \{3, 4, 5, 6\}$:

Since the differences of one of the weights $\pm\frac{1}{2}(1, -1, 1, -1)$ with one of the weights α, β coincide with one of the roots $\pm\rho_3, \pm\rho_4$, while the differences of one of the weights $\pm\frac{1}{2}(1, -1, -1, 1)$ with one of the weights α, β coincide with one of the roots $\pm\rho_5, \pm\rho_6$, we obtain two independent linear systems of linear equations similar to the linear system that we know from Example 4.4, one involves the coefficients of X_3, X_4, Y_3, Y_4 and the other the ones of X_5, X_6, Y_5, Y_6. Thus $\mathfrak{su}(4)_x$ is a 10-dimensional group of rank 2. Indeed, $\mathfrak{su}(4)_x$ is isomorphic to $\mathfrak{so}(5)$. (To see this, compute the roots of $\mathfrak{su}(4)_x$. The root system is of the type B_2, which is the class of the root system of $\mathfrak{so}(5)$, see [3, Chapter 5]).

When for example $G = \mathrm{SU}(5)$ and V^c is the representation with maximal weight $\frac{1}{5}(3, 3, -2, -2, -2)$, then the affine set spanned by the weights $\frac{1}{5}(3, 3, -2, -2, -2)$ and $\frac{1}{5}(-2, -2, 3, 3, -2)$ contains the point $\frac{1}{5}(1, 1, 1, 1, -4)$, which is fixed by all reflections corresponding to roots with the last entry 0. Thus its coadjoint isotropy group K is generated by the image of the embedding $\mathrm{SU}(4) \to \mathrm{SU}(5)$, given by

$$A \mapsto \left(\begin{array}{ccc|c} & & & 0\\ & A & & 0\\ & & & 0\\ \hline 0 & 0 & 0 & 1 \end{array}\right),$$

and the group of matrices $\exp(t \cdot \mathrm{diag}(1, 1, 1, 1, -4))$, $t \in \mathbb{R}$, which is isomorphic to S^1. Since $((\mathfrak{t})^*)^K$ is orthogonal to the affine span of $\bar{\alpha} = \frac{1}{5}(3, 3, -2, -2, -2)$

and $\bar{\beta} = \frac{1}{5}(-2,-2,3,3,-2)$ and intersects it in a single point, the isotropy subgroups of points in $(V_0)_\tau$ are contained in K. V_0 is contained in the space $\tilde{V}_0$ spanned by the weight spaces corresponding to weights with last entry -2. Since the weights of V^c are given by the elements of the Weyl group orbit of $\frac{1}{5}(3,3,-2,-2,-2)$, $\tilde{V}_0$ regarded as an SU(4)-representation has maximal weight $\frac{1}{2}(1,1,-1,-1)$. Thus the above computation shows that for any point $x \in (V_0)_\tau$ with $|x_\alpha| \neq |x_\beta|$, the isotropy Lie algebra is isomorphic to $\mathfrak{su}(2) \times \mathfrak{su}(2)$ and generically the corresponding stratum of relative equilibria of the same isotropy type has the dimension

$$\dim G - \dim G_x + (\mathfrak{t}')^L = 24 - 6 + 2 = 20,$$

where $L \subset G$ is given by the group with Lie algebra $\mathfrak{t} \oplus \langle X_1, X_2, Y_1, Y_2 \rangle$.

If $x \in (V_0)_\tau$ and $|x_\alpha| = |x_\beta|$, then $\mathfrak{g}_x \cong \mathfrak{so}(5)$ and the stratum an x generically corresponds to a relative equilibrium contained in an isotropy type stratum of dimension

$$\dim G - \dim G_x + (\mathfrak{t}')^{\mathrm{SU}(4)\times S^1} = 24 - 10 + 1 = 15,$$

where we identify SU(4) with the image of the above embedding $\mathrm{SU}(4) \to \mathrm{SU}(5)$ and S^1 with the group $\exp(t \cdot \mathrm{diag}(1,1,1,1,-4))$, $t \in \mathbb{R}$.

In the next example, we start again with a given group G, the G-representation V^c and a given subspace $V_0 \subset V^c$. The example illustrates the proceeding to compute the isotropy Lie algebras:

Example 4.7. Consider $G = \mathrm{SU}(6)$ and let V^c be the complexified adjoint representation of G. Again, we identify $\mathfrak{t}$ and $\mathfrak{t}^*$ with the subspace of $\mathbb{R}^6$ of the vectors whose entries sum up to 0. Let V_0 the sum of weight spaces corresponding to the three linearly independent roots $\alpha := (1,-1,0,0,0,0)$, $\beta := (0,0,1,-1,0,0)$, and $\gamma := (0,0,0,0,1,-1)$.

Note that the affine span of α, β, and γ contains the point

$$\nu := \frac{1}{3}(1,-1,1,-1,1,-1).$$

Its coadjoint isotropy subgroup is given by $G_\nu \simeq \mathrm{SU}(3) \times \mathrm{SU}(3) \times S^1$, where the first SU(3)-factor is given by the matrices whose non-vanishing entries have odd row and column numbers, the second factor is given by the matrices whose non-vanishing entries have even row and column numbers and the S^1-component is given by matrices of the form $\mathrm{diag}(\theta, \bar{\theta}, \theta, \bar{\theta}, \theta, \bar{\theta})$, $|\theta| = 1$.

Since $(\mathfrak{t}^*)^{G_\nu} = \langle \nu \rangle$ intersects the affine span of α, β and γ in the single point ν and is orthogonal to it, all points of $(V_0)_\tau$ have isotropy subgroups contained in G_ν.

For all $x \in (V_0)_\tau$ with pairwise different $|x_\alpha|$, $|x_\beta|$, and $|x_\gamma|$ the coadjoint isotropy subgroup G_μ of $\mu := \mathbf{J}(x)$ coincides with T and hence $\mathfrak{g}_x = \mathfrak{t}_x$. The corresponding stratum has the dimension

$$35 - 2 + 3 = 36.$$

To compute the isotropy Lie algebras of points of $(V_0)_\tau$ that have at least two equal values among $|x_\alpha|$, $|x_\beta|$, and $|x_\gamma|$, we consider the action of the root vectors of $G_\nu \simeq \mathrm{SU}(3) \times \mathrm{SU}(3) \times S^1$. Consider $\{X_{1,i}, Y_{1,i}, X_{2,i}, Y_{2,i}\}_{i=1,2,3}$, where $X_{1,i}, Y_{1,i}$ are the images of the corresponding root vectors X_i, Y_i of SU(3) (see Example 4.5) with respect to the embedding $A \mapsto \bar{A}_1$ of $\mathfrak{su}(3)$ into the first $\mathfrak{su}(3)$-factor of $\mathfrak{g}_\nu$ given by $(\bar{A}_1)_{2i-1,2j-1} = (A)_{i,j}$. Similarly $X_{2,i}, Y_{2,i}$ denote the image of the corresponding root vectors under the embedding of $\mathfrak{su}(3)$ the second $\mathfrak{su}(3)$-factor of $\mathfrak{g}_\nu$ given by $A \mapsto \bar{A}_2$ with $(\bar{A}_2)_{2i,2j} = (A)_{i,j}$.

Now, we consider the neighbors of the weights of V_0 with respect to the roots of G_ν, which are given by the positive roots

$$\rho_{1,1} = (1,0,-1,0,0,0), \quad \rho_{1,2} = (0,0,1,0,-1,0), \quad \rho_{1,3} = (1,0,0,0,-1,0),$$
$$\rho_{2,1} = (0,1,0,-1,0,0), \quad \rho_{2,2} = (0,0,0,1,0,-1), \quad \rho_{2,3} = (0,1,0,0,0,-1).$$

and their additive inverses. There are no common neighbors of all three of α, β, and γ. For each pair of weights of V_0 there are two weights which are neighbors of both. For example, the weights $\alpha := (1,-1,0,0,0,0)$ and $\beta := (0,0,1,-1,0,0)$ have the common neighbors $(0,-1,1,0,0,0)$ and $(1,0,0,-1,0,0)$. Here, the difference vectors are given by the roots $\pm\rho_{1,1}$ and $\pm\rho_{2,1}$, which form a subroot system isomorphic to $A_1 \times A_1$, the root system of $\mathrm{SU}(2)\times\mathrm{SU}(2)$. Each of the corresponding root vectors $X_{1,1}$, $Y_{1,1}$, $X_{2,1}$ and $Y_{2,1}$ acts trivially on the weight space $\mathbb{C}_\gamma$. Since for $x \in (V_0)_\tau$ with $|x_\alpha| = |x_\beta| \neq |x_\gamma|$ with $\mu := \mathbf{J}(x)$ these four roots form the root system of G_μ, the $M_{\rho_{1,1}} \oplus M_{\rho_{2,1}}$ part of $\mathfrak{g}_x$ is given by the solutions of a system of linear equations similar to that in Example 4.4. Thus $\mathfrak{g}_x$ contains a unique subalgebra isomorphic to $\mathfrak{su}(2)$ and the rank of $\mathfrak{g}_x$ is 2. Thus $\mathfrak{g}_x$ is isomorphic to $\mathfrak{su}(2)\oplus\mathbb{R}$. Since all $x \in (V_0)_\tau$ with $|x_\alpha| = |x_\beta|$ have the same isotropy type, we obtain that the corresponding stratum has the dimension

$$35 - 4 + 2 = 33.$$

Moreover, since the Weyl group permutes the weight spaces, the points with $|x_\alpha| = |x_\gamma| \neq |x_\beta|$ or $|x_\alpha| \neq |x_\gamma| = |x_\beta|$ have a point with $|x_\alpha| = |x_\beta| \neq |x_\gamma|$ in its G-orbit and are hence contained in the same isotropy stratum.

The last case is $|x_\alpha| = |x_\beta| = |x_\gamma|$. Then $\mathbf{J}(x) = \nu$ and we have to consider all roots of G_ν. As pointed out above, each two of the weights α, β and γ have two common neighbors, but there is no common neighbor of all three weights. In addition, $\mathfrak{g}_\nu \otimes \mathbb{C}$ is a direct sum of $\mathfrak{t}\otimes\mathbb{C}$ and the three subspaces $\mathfrak{k} := \langle X_{1,1}, Y_{1,1}, X_{2,1}, Y_{2,1}\rangle$, $\mathfrak{l} := \langle X_{1,2}, Y_{1,2}, X_{2,2}, Y_{2,2}\rangle$ and $\mathfrak{m} : \langle X_{1,3}, Y_{1,3}, X_{2,3}, Y_{2,3}\rangle$. The matrices in $\mathfrak{k}$ map the weight spaces of α and of β the weight spaces of their common neighbors and vanish on $\mathbb{C}_\gamma$. Similarly, the image of the elements of $\mathfrak{l}$ applied to V_0 is contained in the span of the weight spaces corresponding to the common neighbors of β and γ, and the image of V_0 under the action of the elements of $\mathfrak{m}$ is contained in the span of the weight spaces corresponding to the common neighbors of α and γ. Hence a vector

$$Z = Z_{\mathfrak{t}} + Z_{\mathfrak{k}} + Z_{\mathfrak{l}} + Z_{\mathfrak{m}} \in \mathfrak{t}\oplus\mathfrak{k}\oplus\mathfrak{l}\oplus\mathfrak{m}$$

satisfies $Zx = 0$ iff this holds for each of the components $Z_{\mathfrak{t}}$, $Z_{\mathfrak{k}}$, $Z_{\mathfrak{l}}$, $Z_{\mathfrak{m}}$. In addition,

$$\mathfrak{g}_\nu = \mathfrak{t}\oplus(\mathfrak{k}\cap\mathfrak{g}_\nu)\oplus(\mathfrak{l}\cap\mathfrak{g}_\nu)\oplus(\mathfrak{m}\cap\mathfrak{g}_\nu),$$

and the solution set of $\xi x = 0$ within each of the subspaces $(\mathfrak{k}\cap\mathfrak{g}_\nu)$,$(\mathfrak{l}\cap\mathfrak{g}_\nu)$, $(\mathfrak{m}\cap\mathfrak{g}_\nu)$ corresponds to a linear system similar to the one in Example 4.4. Hence $\mathfrak{g}_x$ is an 8-dimensional Lie algebra of rank 2 and contains three Lie subalgebras isomorphic to $\mathfrak{su}(2)$.

Obviously, $\mathfrak{t}_x$ is the 2-dimensional subspace of $\mathfrak{t}$ given by elements of the form (a, a, b, b, c, c), $a, b, c \in \mathbb{R}, a + b + c = 0$. Moreover, consider the positive roots $\rho_{1,1} = (1, 0, -1, 0, 0, 0)$ and $\rho_{2,1} = (0, 1, 0, -1, 0, 0)$ that correspond to the subspace $\mathfrak{k}$. If we evaluate these roots at elements of $\mathfrak{t}_x$, we obtain the result $(a-b)$ for both of them. Thus $(a, a, b, b, c, c) \mapsto (a-b)$ is a root of G_x° with a root vector contained in $\mathfrak{k}$. Similarly, the intersections $\mathfrak{l} \cap \mathfrak{g}_x$ and $\mathfrak{m} \cap \mathfrak{g}_x$ correspond to the pairs of roots that map elements of the above form to $\pm(a-c)$ and $\pm(b-c)$ respectively. Thus $\mathfrak{g}_x$ is isomorphic to $\mathfrak{su}(3)$.

The corresponding isotropy stratum has the dimension

$$35 - 8 + 2 = 29.$$

We know briefly sketch an approach to compute the isotropy types on the Lie algebra level within the branches of relative equilibria that we obtain from Theorem 1.13 in general:

We have to consider all sums of weight spaces corresponding to a linearly independent subset S of the weights of V^c such that each subset of S consisting of weights of an irreducible G-subrepresentation is maximal within its affine span. To determine the types of the isotropy Lie algebras, it is reasonable to start with the complex 1-dimensional spaces. Then we increase the complex dimension step by step. The advantage of this proceeding is that we can often reuse calculations for lower dimensional subspaces as we have seen for instance in Example 4.7.

Obviously, all non-zero points within a weight space $\mathbb{C}_\alpha$ are of the same isotropy type. They even have the same isotropy Lie algebras: For $0 \neq x \in \mathbb{C}_\alpha$, the complexified isotropy Lie algebra $\mathbb{C} \otimes \mathfrak{g}_x$ is given by the span of $\mathfrak{t} \otimes \mathbb{C}$ and the root vectors Z corresponding to roots ρ of $\mathfrak{g}_\alpha$ with $Z(\mathbb{C}_\alpha) = 0$, which holds iff $\alpha + \rho$ is not a weight of V^c. If ρ is a root of $\mathfrak{g}_\alpha$, then $\alpha + \rho$ is a weight of V^c iff this holds for $\alpha - \rho$. Recall that we denote the weight space in $\mathfrak{g} \otimes \mathbb{C}$ corresponding to a root ρ by L_ρ and the real space $(L_\rho \oplus L_{-\rho}) \cap \mathfrak{g}$ by $M_\rho = M_{-\rho}$. Then $\mathfrak{g}_x$ coincides with the sum of $\mathfrak{t}_x$ and the spaces M_ρ such that ρ is a root of $\mathfrak{g}_\alpha$ and $\alpha + \rho$ is not a weight of V^c.

To compute the isotropy algebras of the elements of $(V_0)_\tau$ for $V_0 = \sum_{\alpha \in S} \mathbb{C}_\alpha$, we have to consider all coadjoint isotropy subgroups K of elements of $\mathfrak{t}^*$ with the following property (see Remark 3.2): There is a subset $S' \subset S$ such that the convex hull of the elements of S' contains a single point of $((\mathfrak{t})^*)^K$ and all elements of $S \setminus S'$ are contained in $((\mathfrak{t})^*)^K$. Then for $x \in (V_0)_\tau$ with $\mathbf{J}(x) = \mu$, we obtain that G_x is contained in the maximal group K of this type such that $K \subset G_\mu$. To compute the Lie algebra $\mathfrak{g}_x = \mathfrak{k}_x$ we thus only need to intersect the isotropy Lie algebras of the components of x within $\sum_{\alpha \in S'} \mathbb{C}_\alpha$ and each weight space $\mathbb{C}_\alpha$ with $\alpha \neq S$. If $S' \neq S$, then these isotropy Lie algebras have been determined in previous steps.

Thus to complete the computation of the types of isotropy Lie algebras of the elements of $(V_0)\tau$, we only have to calculate the groups K of this type such that $((\mathfrak{t})^*)^K$ intersects the convex hull of S in a single point ν. If there is such a group K, then it is maximal within all coadjoint isotropy subgroups of elements of $\mathfrak{t}^*$ that intersect the affine span of S orthogonally. Thus there is at most one such group.

Moreover, the set of points in V_0 with momentum ν consists of a single T-orbit. Hence to compute the types of their isotropy Lie algebras, we can choose a single x with $\mathbf{J}(x) = \nu$ and compute $\mathfrak{g}_x$.

To compute $\mathfrak{g}_x$, we proceed similarly as in our examples: We split $\mathfrak{g}_\nu$ into $\mathfrak{t}$ and the span of spaces M_ρ for the roots ρ of G_ν. No two weights of V_0 are neighbors of each other: If the difference of two weights of V_0 was a root, then their affine span would contain a Weyl reflection pair and hence this would hold for S. Thus $(\mathfrak{g}_\nu \otimes \mathbb{C})_x$ coincides with the sum of $(\mathfrak{t} \otimes T)_x$ and the annihilator of x within the span of the spaces L_ρ for the roots ρ of G_ν. Therefore $\mathfrak{g}_x$ is given by the sum of $\mathfrak{t}_x$ and the space of solutions ξ in the span of the spaces M_ρ of the equation $\xi x = 0$. To determine the solutions ξ, we choose a basis of each space M_ρ. (Since the each of the spaces M_ρ is a real vector space of dimension 2, we have to choose 2 vectors for each root ρ of $\mathfrak{g}_\mu$. In our examples, we have chosen vectors of the form $X_i - Y_i$ and $\mathrm{i}(X_i + Y_i)$, where X_i and Y_i are root vectors for ρ and $-\rho$ respectively.) Then together, these vectors form a basis $Z_1, \ldots, Z_{2k}$ of the span of the spaces M_ρ. We then consider the matrix A with columns $(Z_i x)$. Then $\ker A \subset \mathbb{R}^{2k}$ corresponds to the space of solutions ξ of $\xi x = 0$ within the span of the spaces M_ρ.

As we have seen in Example 4.7, it can happen that the linear system decouples into subsystems that have been solved in order to determine the isotropy Lie algebras of the main stratum of a subspace of V_0. This is another reason for our proceeding.

References

[1] E. Bierstone. “General position of equivariant maps.” In: *Transactions of the American Mathematical Society* 234 (1977), pp. 447–466.

[2] E. Bierstone. “Generic Equivariant Maps.” In: *Real and Complex Singularities. Proceedings of the Nordic Summer School/NAVF Symposium in Mathematics, Oslo, August 5-25, 1976.* Ed. by Per Holm. Kluwer Academic Pub, 1977, pp. 127–161.

[3] T. Bröcker and T. tom Dieck. *Representations of Compact Lie Groups.* Graduate Texts in Mathematics 98. Springer Berlin Heidelberg, 1985.

[4] M. Dellnitz and I. Melbourne. “The Equivariant Darboux Theorem.” In: *Exploiting Symmetry in Applied and Numerical Analysis. AMS-SIAM Summer Seminar in Applied Mathematics, July 26-August 1, 1992, Colorado State University.* Ed. by E. L. Allgower, K. Georg, and R. Miranda. Lectures in applied mathematics 29. American Mathematical Soc., 1993, pp. 163–169.

[5] M. Dellnitz, I. Melbourne, and J. Marsden. “Generic bifurcation of Hamiltonian vector fields with symmetry.” In: *Nonlinearity* 5 (1992), pp. 979–996.

[6] M. Field. *Dynamics and Symmetry.* ICP advanced texts in mathematics 3. Imperial College Press, 2007.

[7] M. Field. “Local structure of equivariant dynamics.” In: *Singularity Theory and its Applications, Warwick 1989. Part II: Singularities, Bifurcations and Dynamics.* Ed. by M. Roberts and I. Stewart. Lecture Notes in Mathematics 1463. Springer Berlin Heidelberg, 1991, pp. 142–166.

[8] M. Field. “Stratifications of equivariant varieties.” In: *Bulletin of The Australian Mathematical Society* 16 (1977), pp. 279–295.

[9] M. Field. *Symmetry Breaking for Compact Lie Groups.* American Mathematical Society: Memoirs of the American Mathematical Society 574. American Mathematical Soc., 1996.

[10] M. Field. "Transversality in G-manifolds." In: *Transactions of the American Mathematical Society* 231 (1977), pp. 429–450.

[11] M. Field and R. Richardson. "Symmetry breaking and the maximal isotropy subgroup conjecture for reflection groups." In: *Archive for Rational Mechanics and Analysis* 105 (1989), pp. 61–94.

[12] W. Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry.* London Mathematical Society Student Texts 35. Cambridge University Press, 1996. DOI: 10.1017/CBO9780511626241.

[13] V. Guillemin and S. Sternberg. *Symplectic Techniques in Physics.* Cambridge University Press, 1984.

[14] B. C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction.* Graduate Texts in Mathematics 222. Springer, 2003.

[15] I. Melbourne and M. Dellnitz. "Normal forms for linear Hamiltonian vector fields commuting with the action of a compact Lie group." In: *Mathematical Proceedings of the Cambridge Philosophical Society* 114.2 (1993), pp. 235–268. DOI: 10.1017/S0305004100071577.

[16] J. Montaldi, M. Roberts, and I. Stewart. "Periodic solutions near equilibria of symmetric Hamiltonian systems." In: *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences* 325.1584 (1988), pp. 237–293. DOI: 10.1098/rsta.1988.0053.

[17] J. Ortega and T. Ratiu. "Relative equilibria near stable and unstable Hamiltonian relative equilibria." In: *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 460 (2004), pp. 1407–1431.

[18] G. Patrick and M. Roberts. "The transversal relative equilibria of a Hamiltonian system with symmetry." In: *Nonlinearity* 13 (2000), pp. 2089–2105.

[19] M. Sommerfeld. "Equivariant Transversality Theory Applied to Hamiltonian Relative Equilibria." Universität Hamburg, Fachbereich Mathematik, 2018. URL: http://ediss.sub.uni-hamburg.de/volltexte/2018/9498/.