

QUICKLY PROVING DIESTEL'S NORMAL SPANNING TREE CRITERION

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ABSTRACT. We present two short proofs for Diestel's criterion that a connected graph has a normal spanning tree provided it contains no subdivision of a countable clique in which every edge has been replaced by uncountably many parallel edges.

§1. OVERVIEW

This paper continues a line of inquiry started in [7] with the aim to find efficient algorithms for constructing normal spanning trees in infinite graphs. A rooted spanning tree T of a graph G is called *normal* if the end vertices of any edge of G are comparable in the natural tree order of T . Intuitively, all the edges of G run ‘parallel’ to branches of T , but never ‘across’.

Every countable connected graph has a normal spanning tree, but uncountable graphs might not, as demonstrated by complete graphs on uncountably many vertices. While exact characterisations of graphs with normal spanning trees exist, see e.g. [5, 6], these may be hard to verify in practice. The most applied sufficient condition for normal spanning trees is the following criterion due to Halin [4], and its strengthening due to Diestel [2], see also [6, §6] for an updated proof.

Theorem 1 (Halin, 1978). *Every connected graph without a TK^{\aleph_0} has a normal spanning tree.*

Theorem 2 (Diestel, 2016). *Every connected graph without fat TK^{\aleph_0} has a normal spanning tree.*

Here, a TK^{\aleph_0} is any subdivision of the countable clique K^{\aleph_0} , and a *fat* TK^{\aleph_0} is any subdivision of the multigraph obtained from a K^{\aleph_0} by replacing every edge with \aleph_1 parallel edges.

Until recently, only fairly involved proofs of these results were available: Halin's original proof employing his theory of simplicial decompositions [4], and Diestel's proof strategy building on the forbidden minor characterisation for normal spanning trees [2, 6].

In [7], however, the present author found a simple greedy algorithm which constructs the desired normal spanning tree in Halin's Theorem 1 in just ω many steps. The purpose of this note is to provide two simple proofs also for Theorem 2, one of them again an ω -length algorithm.

Notably, this algorithm also yields a new, local version of Theorem 2: Given a set of vertices U of a connected graph G , there exists a normal tree of G containing U if and only if every fat TK^{\aleph_0} in G can be separated from U by a finite set of vertices, see Theorem 3 below.

§2. TREE ORDERS AND NORMAL TREES

We follow the notation in [1]. The *tree-order* \leq_T of a tree T with root r is defined by setting $u \leq_T v$ if u lies on the unique path from r to v in T . For a vertex v of T , let $[v] := \{t \in T : t \leq_T v\}$.

For rooted trees that are not necessarily spanning, one generalises the notion of normality as follows: A rooted tree $T \subseteq G$ is *normal (in G)* if the end vertices of any T -path in G (a path in G with end vertices in T but all edges and inner vertices outside of T) are comparable in the tree order of T . If T is spanning, this clearly reduces to the definition given in the introduction. If $T \subseteq G$ is normal, then the set of neighbours $N(D)$ of any component D of $G - T$ forms a chain in T , i.e. all vertices of $N(D)$ are comparable in \leq_T . Moreover, incomparable nodes v, w of any normal tree $T \subseteq G$ are separated in G by $[v] \cap [w]$.

Fact 1 (Jung [5, Satz 6]). *Let G be a graph with a normal spanning tree. Then for every connected subgraph $C \subseteq G$ and every $r \in C$ there is a normal spanning tree of C with root r .*

For distinct vertices v, w of G we denote by $\kappa(v, w) = \kappa_G(v, w)$ the connectivity between v and w in G , i.e. the largest size of a family of independent $v - w$ paths. If v and w are non-adjacent, this is by Menger's theorem equivalent to the minimal size of a $v - w$ separator in G .

Fact 2 (Halin, [3, (15)]). *Let U be a countable set of vertices in G . There is a fat TK^{\aleph_0} with branch vertices U if and only if $\kappa(u, v)$ is uncountable for all $u \neq v \in U$.*

§3. THE FIRST PROOF

First proof of Theorem 2. By induction on $|G|$. We may assume that $|G|$ is uncountable. Suppose we have a continuous increasing ordinal-indexed sequence $(G_i : i < \sigma)$ of induced subgraphs all of size less than $|G|$ with $G = \bigcup_{i < \sigma} G_i$ such that

- (i) the end vertices of any G_i -path in G have infinite connectivity in G_i , and
- (ii) the end vertices of any G_i -path in G have uncountable connectivity in G .

Then we can construct normal spanning trees T_i of G_i extending each other all with the same root by (transfinite) recursion on i . If $\ell < \sigma$ is a limit, we may simply define $T_\ell = \bigcup_{i < \ell} T_i$. For the successor case, suppose that T_i is already defined. By (ii), the neighbourhood $N(C)$ is finite for every component C of $G_{i+1} - G_i$ (otherwise we get a fat TK^{\aleph_0} by Fact 2), and by (i), $N(C)$ lies on a chain of T_i (as incomparable vertices in T_i are separated in G_i by the intersection of their finite down-closures). Let $t_C \in N(C)$ be maximal in the tree order of T_i , and let r_C be a neighbour of t_C in C . By the induction hypothesis and Fact 1, C has a normal spanning tree T_C with root r_C . Then T_i together with all T_C and edges $t_C r_C$ is a normal spanning tree T_{i+1} of G_{i+1} . Once the recursion is complete, $T = \bigcup_{i < \sigma} T_i$ is the desired normal spanning tree of G .

It remains to construct a sequence $(G_i : i < \sigma)$ with (i) and (ii). This can be done, for example, by taking a continuous increasing chain $(M_i : i < \sigma)$ with $\sigma = cf(|G|)$ of $<|G|$ -sized elementary

submodels M_i of a large enough fragment of ZFC with $G \in M_i$, such that $G \subseteq \bigcup_{i < \sigma} M_i$, see [8]. Then $G_i = G \cap M_i$ is as required.

Alternatively, use a countable closure argument to construct G_i such that for every pair $v, w \in V(G_i)$ with $\kappa_G(v, w) \leq \aleph_0$, the graph G_i contains a maximal family of independent $v-w$ paths in G (this will guarantee (ii)), and for all other pairs, G_i contains at least countably many independent $v-w$ paths (this will guarantee (i)); and note that properties (i) and (ii) are preserved under increasing unions. \square

§4. THE SECOND PROOF

Our second proof extracts the closure properties (i) and (ii) in the previous construction, and combines them into a single algorithm constructing the normal spanning tree in ω many steps, avoiding ordinals and transfinite constructions altogether.

Second proof of Theorem 2. For every pair of distinct vertices v and w of G with $\kappa(v, w)$ at most countable, fix a maximal collection $\mathcal{P}_{v,w} = \{P_{v,w}^1, P_{v,w}^2, \dots\}$ of independent $v-w$ paths in G .

Construct a countable chain $T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$ of rayless normal trees in G with the same root $r \in V(G)$ as follows: Put $T_0 = \{r\}$, and suppose T_n has already been defined. Since T_n is a rayless normal tree, any component D of $G - T_n$ has a finite neighbourhood $N(D)$ in T . For each pair $v \neq w \in N(D)$ with countable connectivity select the path $P_{v,w}^D$ with least index $\mathcal{P}_{v,w}$ intersecting D . By [1, Proposition 1.5.6], we may extend T_n finitely into every such component D as to cover $P_{v,w}^D \cap D$ for all $v \neq w \in N(D)$ (or at least one arbitrarily chosen vertex making the extension into D is non-trivial), so that the extension $T_{n+1} \supseteq T_n$ is a rayless normal tree with root r . This completes the construction.

The union $T = \bigcup_{n \in \mathbb{N}} T_n$ with root r is a normal tree in G . We claim that T is spanning unless G contains a fat TK^{\aleph_0} . If T is not spanning, consider a component C of $G - T$. Then $N(C) \subseteq T$ is infinite: otherwise, $N(C) \subseteq T_n$ for some $n \in \mathbb{N}$ but then we would have extended T_n into C , a contradiction. For every n , let D_n be the unique component of $G - T_n$ containing C .

By Fact 2, it suffices to show that $\kappa(v, w)$ is uncountable for every $v \neq w \in N(C)$. Consider a T -path P from v to w with $\dot{P} \subseteq C$. If $\kappa(v, w)$ was countable, then by maximality of $\mathcal{P}_{v,w}$ there is $P_{v,w}^k \in \mathcal{P}_{v,w}$ with say $P_{v,w}^k \cap \dot{P} \ni x$. Let m be minimal with $v, w \in T_m$. Since the $P_{v,w}^{D_n}$ are pairwise distinct, the path $P_{v,w}^k$ was selected as $P_{v,w}^{D_n}$ for some n with $m \leq n \leq m+k$. But then $x \in P_{v,w}^k \cap \dot{P} \subseteq P_{v,w}^{D_n} \cap D_n \subseteq T_{n+1} \subseteq T$ contradicts that P is a T -path. \square

§5. LOCAL VERSIONS OF DIESTEL'S CRITERION

By a slight modification of this ω -length algorithm, one readily obtains a proof of the following results, which answer [6, Problem 3].

Theorem 3. *A set of vertices U in a connected graph G is contained in a normal tree of G provided every fat TK^{\aleph_0} in G can be separated from U by a finite set of vertices.*

Proof. Let U be a set of vertices such that every fat TK^{\aleph_0} in G can be separated from U by a finite set of vertices. Use the algorithm from Section 4, but only extend T_n into a component D of $G - T_n$ with $U \cap D \neq \emptyset$. Additionally, make sure to cover at least one vertex from $U \cap D$.

It remains to argue that U is contained in $T = \bigcup T_n$. Otherwise, there is a component C of $G - T$ containing a vertex from U . As in Section 4, this gives us a fat TK^{\aleph_0} in G which furthermore cannot be separated from U by a finite set of vertices, cf. [7]. \square

Theorem 4. *A connected graph has a normal spanning tree if and only if its vertex set is a countable union of sets each separated from any fat TK^{\aleph_0} by a finite set of vertices.*

Proof. For the forward implication, recall that the levels of any normal spanning tree can be separated by a finite set of vertices from any ray, and hence in particular from any fat TK^{\aleph_0} . Conversely, let $\{V_n : n \in \mathbb{N}\}$ be a collection of fat TK^{\aleph_0} -dispersed sets in G with $V(G) = \bigcup_{n \in \mathbb{N}} V_n$. Adapt the algorithm from Section 4, so that when extending T_n into a component D of $G - T_n$, we additionally cover a vertex $v_D \in D \cap V_{n_D}$ where n_D minimal such that $V_{n_D} \cap D \neq \emptyset$. The proof then proceeds as in [7]. \square

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