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A modular functor from state sums for finite tensor categories and their bimodules

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Abstract

We construct a modular functor which takes its values in the bicategory of finite categories, left exact functors and natural transformations. The modular functor is defined on bordisms that are 2-framed. Accordingly we do not need to require that the finite categories appearing in our construction are semisimple, nor that the finite tensor categories that are assigned to two-dimensional strata are endowed with a pivotal structure. Our prescription can be understood as a state-sum construction. The state-sum variables are assigned to one-dimensional strata and take values in bimodule categories over finite tensor categories, whereby we also account for the presence of boundaries and defects. Our construction allows us to explicitly compute functors associated to surfaces and representations of mapping class groups acting on them.

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1 Introduction

Finite tensor categories are linear rigid monoidal categories obeying certain finiteness conditions. They arise in various contexts in representation theory, e.g. as categories of finite-dimensional representations of finite-dimensional Hopf algebras, or as representation categories of suitable vertex algebras or nets of observable algebras. They also appear naturally in rigorous approaches to low-dimensional conformal and topological quantum field theories. Fusion categories, i.e. finite tensor categories that are semisimple, form an important subclass. They occur e.g. in the classification of subfactors and in topological field theory and its applications, such as the description of local invariants of knots and links, the study of topological phases of matter, and quantum gravity, and as renormalization group fixed points of string net models.

By now a comprehensive body of mathematical results – which one may collectively refer to as "categorified representation theory" – has been built around finite tensor categories. Remarkably, many results in categorified representation theory do not require semisimplicity, but rather rely on the finiteness properties. Let us illustrate what we mean by categorified representation theory: Thinking of finite tensor categories as a categorification of rings, it is natural to study their module and bimodule categories. Direct applications of these are in the description of defects and boundaries in topological field theories. On the mathematical side, module and bimodule categories lead to a rich algebraic structure. Specifically, invertible (not necessarily semisimple) bimodule categories give rise to a (higher categorical variant) of a group, the Brauer-Picard group, which plays a central role in the construction of equivariant modular functors and thus of orbifold theories. The bicategory of module categories leads to a bicategorical variant of Morita theory.

An important structure in categorified representation theory is the Drinfeld center. For instance, Morita equivalent finite tensor categories have equivalent Drinfeld centers, and the Brauer-Picard group can be computed in terms of braided autoequivalences of the Drinfeld center [ENOM]. While finite tensor categories can be endowed with interesting additional features, e.g. with a pivotal structure, a braiding, or a ribbon structure, it is worth pointing out that there is a rich theory already without assuming any such extra features. Accordingly we take finite tensor categories without additional structure as the starting point of the present paper.

A comprehensive algebraic theory calls for an organizing principle. Indeed, modular functors should provide such a principle. This is not surprising; many algebraic theories turn out to have organizing principles that can be expressed in terms of geometric structure. For instance, when working with associative algebras it is helpful to be aware of aspects of rooted trees, while the theory of Frobenius algebras becomes very transparent in the light of two-dimensional oriented topological field theory. In a similar vein, for categorified representation theory it is commonly agreed that variants of extended three-dimensional topological field theories based on state-sum constructions should play a role. For our purposes, we have an (extended) topological field theory or – for more general input data that are not required to be semisimple – a modular functor in mind that is set up at the level of bicategories: it assigns to one-dimensional structures categories, to two-dimensional structures functors, and to elements of mapping class groups natural transformations. Such topological field theories or modular functors are of intrinsic mathematical interest and have at the same time a lot of applications, ranging from physics to computer science.

On the other hand, the standard approach to state-sum constructions, as pioneered by Turaev-Viro [TV′] and Barrett-Westbury [BaW], is unsatisfactory when it comes to "explaining" categorified algebra:

- The Turaev-Viro-Barrett-Westbury construction is based on fusion categories that are pivotal (and even spherical). In contrast, various non-trivial aspects of categorified representation theory do not require a pivotal structure, which should therefore better be treated as an additional feature. (This point of view is also advocated in [DSS].)
- Turaev-Viro theory based on a fusion category $\mathcal{A}$ assigns to a circle the Drinfeld center $\mathcal{Z}(\mathcal{A})$. The fusion category $\mathcal{A}$ itself, on the other hand, is effectively invisible, in the sense that Morita equivalent spherical fusion categories give the same extended topological field theory at the bicategorical level.
- As already pointed out, much of categorified representation theory works beyond the realm of fusion categories, for the larger class of finite tensor categories which enjoy analogous finiteness properties as fusion categories, but are not necessarily semisimple.

The central goal we achieve in this paper is a geometric framework that governs the categorified representation theory of finite tensor categories and their finite (bi)module categories. To overcome the shortcomings of conventional state-sum constructions, we work in the following setting:

- To allow for finite tensor categories that are not semisimple, we construct a modular functor rather than a 3-2-1-extended topological field theory. Specifically, we do not formulate a theory for arbitrary three-manifolds with corners, but restrict ourselves to surfaces and to actions of their mapping class groups. The idea that non-semisimple categories only allow one to deal with a restricted class of three-manifolds (or with three-manifolds having additional structure, see e.g. [BCGP, Def. 3.3]) – in our case, at least cylinders twisted by the action of an element of the mapping class group – is not new, see e.g. [DSS].
- One way to make the finite tensor category itself, rather than merely its Drinfeld center, visible in the construction, is to consider the extension of the theory to the point [DSS]. Here we expose the finite tensor category, as well as its module and bimodule categories, by instead extending the category of cobordisms to include boundaries and defects. This modification is not new either. Indeed, it is known [FuSV] that boundary conditions for a topological field theory of Reshetikhin-Turaev type based on a modular tensor category $\mathcal{C}$ correspond to Witt-trivializations, i.e. to braided equivalences $\mathcal{C} \simeq \mathcal{Z}(\mathcal{A})$ to the Drinfeld center of some fusion category $\mathcal{A}$, which, in turn, has a direct interpretation as a category of Wilson lines associated with a specific boundary condition.
  That we include defects has a further benefit: There are defects between any two topological field theories of Turaev-Viro type, and hence our construction encompasses in a single theory the Turaev-Viro theories for all choices of fusion categories.
- Finally, in order to do without a pivotal structure on the algebraic side, we supplement structure on the geometric side and work with 2-framed manifolds rather than with oriented ones. Again, this approach has been advocated before, see [DSS] as well as [Ku].

Several frameworks for addressing our goal may come to mind, such as factorization algebras, Kitaev-type state-sum models, or constructions based on fully extended topological field theories that invoke the cobordism hypothesis. The approach taken in the present paper provides an explicit state-sum construction in a purely categorical setting. This avoids the introduction of extra structure, and it does not invoke the cobordism hypothesis, but has structural similarities with constructions familiar from factorization algebras. It is tailored to the specific target bicategory of finite tensor categories with left exact (or, alternatively, right exact) functors as 1-morphisms and uses the full power of that structure. Defects are built in from the start, as the carriers of the state sum variables. Being very concrete, our approach leads directly to fully explicit computational prescriptions for specific situations of interest. Our findings are in line with the results, conjectures and expectations in other approaches, albeit the direct comparison between different frameworks is far from straightforward.

Even given the clear program outlined above, the right definitions and a full construction still turn out to be subtle. Accordingly, in a sense, our first important insight is Definition 2.9: it specifies a monoidal bicategory $\mathrm{Bord}_2^{\mathrm{def}}$ of 2-framed defect cobordisms that suits our purposes. The description of a modular functor in Definition 2.12 is then standard, and it follows from our general goal that we aim for a modular functor with values in the monoidal bicategory $\mathcal{S} = \mathcal{L}ex$, having finite categories as objects, left exact functors as morphisms, and the Deligne product as the monoidal structure. (There is also a variant that instead uses as the target category finite tensor categories with right exact functors.) Theorem 2.13, the main result of this paper, then asserts the existence of such a modular functor, i.e. of a symmetric monoidal bifunctor

$$\mathrm{T}: \quad \mathrm{Bord}_2^{\mathrm{def}} \to \mathcal{S}\,. \tag{1.1}$$

The unsuspicious words "symmetric monoidal bifunctor" specify quite a lot of structure and properties. (This partly explains the length of the present paper.) It includes, for instance – in the form of the horizontal composition of 1-morphisms – factorization of the modular functor under the gluing of surfaces, see Theorem 5.22.

Let us now summarize the main line of our arguments:

- We have to assign a finite linear category to each object of $\mathrm{Bord}_2^{\mathrm{def}}$, i.e. to certain one-manifolds with additional structure. These finite categories are suitable generalizations of Drinfeld centers. This is the topic of Section 3.
- Next we must assign a left exact functor to each 1-morphism of $\mathrm{Bord}_2^{\mathrm{def}}$, i.e. to bordisms with extra structure, which we call *defect surfaces*. This is achieved in the form of a state-sum construction and follows the standard three-step pattern of such constructions: For a surface with boundary, one first constructs a "big" vector space – actually, a linear functor. We call this functor the *pre-block functor*.

  That we work with categories that are not necessarily semisimple forces us to work systematically with natural notions from category theory. Specifically, to construct left exact functors, we use Hom functors and implement the sum over states by taking coends. Indeed we would raise the claim that the systematic use of category-theoretic concepts allows for a substantial conceptual clarification, even when dealing with semisimple categories.

  The construction of pre-block functors occupies the first part of Section 4.
- The second step in a state-sum construction consists in imposing an appropriate flatness condition. It is one of the novel insights of this paper that when making use of the 2-framing on the surfaces, one can enforce flatness of holonomies without assuming the existence of a

pivotal structure on the finite tensor categories. To be able to impose flat holonomy, the defect network of the surface must be such that each of its 2-patches has the topology of a disk; we call a surface of this type a *fine* defect surface. The solution of the flatness condition on the pre-block functor for a fine surface Σ gives another left exact functor, which assigns to Σ in a functorial way subspaces of the big vector spaces (which one might think of as 'spaces of ground states'). Constructing these functors for all fine surfaces is the second main subject of Section 4.

- A modular functor must, of course, assign a functor to *any* defect surface, not just to fine ones. In Section 5 we explain how to define such functors, which we call *block functors*, to surfaces with a defect network that is not necessarily fine. To this end we introduce the notion of a *refinement* of a defect surface. We show that refinements to fine surfaces exist, and then proceed to set up, first for disks, a system of isomorphisms for functors associated to fine refinements in such a manner that we can define the block functors for disks as limits. The block functors for general defect surfaces are then constructed from the block functors for disks. We also show that the so defined block functors obey factorization and study actions of mapping class groups.

Instead of taking finite tensor categories with left exact functors as the target bicategory of the modular functor, we could have chosen the bicategory of finite tensor categories with *right* exact functors. Each of these two bicategories is monoidal, with the Deligne product as a symmetric monoidal structure. A duality between the left and right exact functors is provided by the Eilenberg-Watts functors that were studied in [Sh, FSS2]. According to this duality, the left exact Hom functor gets replaced by the vector space dual of the Hom functor, which is right exact, and coends in the state-sum construction must be replaced by ends. Beyond this aspect, the Eilenberg-Watts calculus also plays a significant role in our approach and in interpreting our results. For instance, it makes it easy to describe how the modular functor provides, via the fusion of boundary insertions, a composition on Deligne products $\overline{\mathcal{M}} \boxtimes \mathcal{N}$ (see Proposition 4.6); for other applications, see e.g. Example 4.4 or Corollary 4.28.

Several complementary results help to make block functors computable. For instance, in Theorem 4.37 we show that the fusion of defect lines corresponds to a relative Deligne product of bimodule categories (which depends on the framings involved), while Proposition 4.22 tells us that a pair of gluing boundaries can be combined to a single one, in a way that is described by the composition of functors (see the picture (4.15) below), without changing the block functor. As a consequence, a specific simple defect surface, which we call the 'straight disk' (displayed in picture (4.7)), is of particular importance; we show that its pre-block spaces consist of natural transformations (see formula (4.11)) and that its block spaces are the corresponding module natural transformations (Corollary 4.28). The occurrence of module natural transformations is the simplest instance in which our construction contributes to the program of geometrically realizing categorified representation theory. Also, in view of the construction of block functors for general defect surfaces from those for disks, one may think of block functors as giving spaces that constitute a huge generalization of spaces of module natural transformations.

We also provide details for a few situations of specific interest, a sample being the following: The functors of braided induction (or α-induction) appear naturally in the block functor for disks with a free boundary (Example 5.9); the 'transmission functor' which was considered in [ENOM, Sect. 5.1] is obtained as the block functor for a cylinder with a circumferential defect line (Example 5.24); a variant of the twist (involving the double-dual functor) on objects of a braided monoidal category appears in the natural transformation that is obtained from a Dehn twist on the cylinder over a circle (Proposition 5.43); the block functor for a two-sphere with one circular defect line and any number of insertions (Example 4.31) has bimodule natural transformations as its values; and a pair of pants (Example 4.36) realizes, via the Eilenberg-Watts equivalences, the composition of bimodule functors as well as, in the situation that all defects involved are transparent, the tensor product in the Drinfeld center (which is the functor that also the standard Turaev-Viro construction assigns to a pair of pants). As examples for the actions of mapping class groups, we consider a braiding move on a three-punctured sphere (Proposition 5.38) and a Dehn twist (Proposition 5.43).

2 Framed defect manifolds

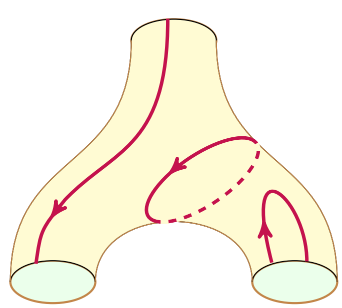
In this section we define a bicategory $\mathrm{Bord}_2^{\mathrm{def}}$ of two-dimensional 2-framed bordisms with labeled defects. To this end we introduce in a first step a geometric bicategory $\mathrm{Bord}_2^{\mathrm{def},0}$ having unlabeled defects. All manifolds considered below are assumed to be smooth and oriented.

2.1 Framed defect bordisms

We consider manifolds which can have boundaries and corners and can contain defect lines, and which in addition are endowed with a framing. Before describing the bicategory $\mathrm{Bord}_2^{\mathrm{def},0}$ of all such manifolds, we first consider a sub-bicategory $\mathrm{Bord}_{2,\mathrm{cl}}^{\mathrm{def},0}$ of manifolds without corners. We start by giving representatives for the morphisms of this sub-bicategory.

Denote by $I=[0,1]$ the standard interval and by $\mathbb{S}^1$ the standard circle; both endowed with their standard orientation. We write $I^{\sqcup n} := I \sqcup I \sqcup \ldots \sqcup I$ and $(\mathbb{S}^1)^{\sqcup n} := \mathbb{S}^1 \sqcup \mathbb{S}^1 \sqcup \ldots \sqcup \mathbb{S}^1$ for the corresponding finite disjoint unions, with $n \in \mathbb{Z}_{\geq 0}$ and with $I^{\sqcup 0}$ and $(\mathbb{S}^1)^{\sqcup 0}$ being the empty set. Let Σ be a compact oriented surface, possibly with boundary. We endow Σ with further structure that accounts for defect lines and a compatible framing. To this end we first introduce the additional datum of an embedding $\delta\colon I^{\sqcup n} \sqcup (\mathbb{S}^1)^{\sqcup m} \to \Sigma$ for some $m, n \in \mathbb{Z}_{\geq 0}$ that is subject to the following restrictions: We require that the end points of each interval are mapped to the boundary, i.e. $\delta(\{0,1\}^{\sqcup n}) \subset \partial\Sigma$; that all other points of the image of δ lie in the interior of Σ; and that each connected component of $\partial\Sigma$ must contain at least one end point of one of the intervals.

We call the image of δ in Σ and, by abuse of language, also the map δ itself, the (set of) *unlabeled defect lines* of Σ. Note that each of the defect lines inherits an orientation from the standard orientation of the interval I or the circle $\mathbb{S}^1$, respectively. As an illustration, the following picture shows a situation in which the underlying surface Σ is a sphere with three holes and in which Σ contains three unlabeled defect lines:



(2.1)

We allow only for pairs (Σ, δ) of surfaces with defect lines that can be endowed with the additional structure of a 2-*framing*, that is, with a non-zero vector field χ on Σ that along each defect line is parallel to it and whose direction matches the orientation of the defect line. Together with the orientation of Σ, the vector field χ determines a trivialization of the tangent bundle of Σ, unique up to homotopy, hence the term 2-framing for χ.

Definition 2.1.

(i) An *unlabeled defect surface without corners* is a triple (Σ, δ, χ) consisting of a compact oriented surface Σ without corners, and with unlabeled defect lines δ and a 2-framing χ on (Σ, δ).

(ii) An unlabeled defect surface (Σ, δ, χ) is called *fine* iff each connected component of $\Sigma \setminus \{\delta\}$ is topologically a disk.

(iii) A fine unlabeled defect surface (Σ, δ, χ) is called *gluable fine* iff the boundary of every connected component of $\Sigma \setminus \{\delta\}$ contains at most one connected component of the boundary of Σ.

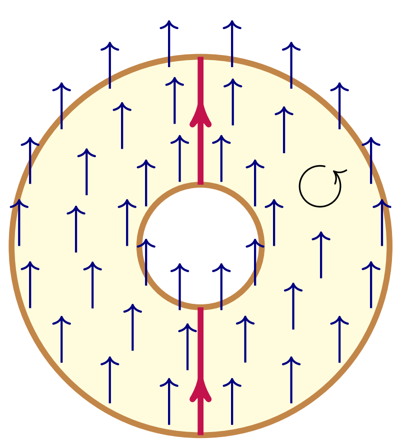
Remark 2.2.

(i) That we here augment the terminology by the qualification *unlabeled* is due to the fact that we will want to be able to distinguish between different types of defect lines and accordingly will decorate them, in Section 2.2, with suitable labels.

(ii) We do not allow for junctions of defect lines. Instead, any putative point in Σ at which defect lines would meet is realized as a boundary circle on which those defect lines end. Hereby we avoid the use of stratified manifolds which e.g. appear in the approach of [CMS, CRS].

(iii) The unlabeled defect surface resulting from the gluing of two fine unlabeled defect surfaces is not necessarily fine. In contrast, the unlabeled defect surface resulting from the gluing of two *gluable* fine unlabeled defect surfaces is again gluable fine.

As an illustration, the following picture indicates a framing for a closed defect surface whose underlying surface is an annulus and which has two defect lines (here and below, the surface is embedded in the plane and is taken to inherit the standard orientation of the plane):

$(\Sigma, \delta, \chi) \;=$  (2.2)

To introduce the *objects* of the category $\mathrm{Bord}_{2,\mathrm{cl}}^{\mathrm{def},0}$, we consider the restriction of the so defined structure to the boundary circles (together with little collars around them). A point on the boundary $\partial\Sigma$ is said to be *marked* iff it is in the image of δ, i.e. is the end point of a defect line. We label a marked point by $+1$ and call it a *positive point* if it is the image of an initial point $0 \in I$ of a defect line, and label it by -1 and call it a *negative point* if it is the image of an end point $1 \in I$. We also call a marked point on $\partial\Sigma$ – or, more generally, on a one-manifold – together with a sign ± 1 an (unlabeled) *defect point*, and the closure of the interval along a boundary circle between two neighboring defect points a *segment* s. To determine the structure induced on the boundary $\partial\Sigma$ by the 2-framing of an unlabeled defect surface, we make use of the fact that in order to glue bordisms of smooth manifolds along boundary circles, the circles need to be endowed with collars. Concretely, a connected component of $\partial\Sigma$ is to be considered with (the germ of) an embedding $\mathbb{S}^1 \times I \to \Sigma$. Thus we obtain a non-vanishing vector field on $\mathbb{S}^1 \times I$ as the pullback of a 2-framing χ on (Σ, δ). Note that at each defect point on $\partial\Sigma$ the framing vector field χ on Σ provides a non-vanishing vector; by our requirement that near any defect line δ the vector field χ is parallel to δ with matching direction, the so obtained vector at a defect point $p \in \delta \cap \partial\Sigma$ is outward-pointing iff δ is oriented towards $\partial\Sigma$, i.e. iff p is a negative point.

Further, it is natural to allow also for one-manifolds with boundary, and as a consequence admit corresponding defect surfaces (see below) as well, which then have (one-dimensional) boundaries and corners. Denoting by $\underline{\mathbb{R}}$ the trivial 1-dimensional vector bundle with oriented fiber over a given base, we then arrive at

Definition 2.3.

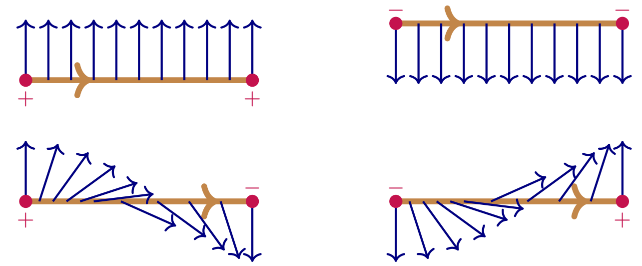
(i) An *unlabeled defect one-manifold* is a triple $(\mathrm{L}, \epsilon, \chi)$ consisting of an oriented compact one-manifold L, possibly with boundary, a finite set $\epsilon \supseteq \partial\mathrm{L}$ of defect points, and a non-zero vector field $\chi \in \Gamma(T\mathrm{L} \oplus \underline{\mathbb{R}})$.
Further, at each defect point $p \in \epsilon$ the component of $\chi(p)$ in $T\mathrm{L}$ is required to vanish, and the component of $\chi(p)$ in $\underline{\mathbb{R}}$ must be positive iff p is a positive point.

(ii) A *closed unlabeled defect one-manifold* is an unlabeled defect one-manifold with empty boundary.

(iii) A morphism of unlabeled defect one-manifolds is a homeomorphism of manifolds that preserves the non-zero vector field.

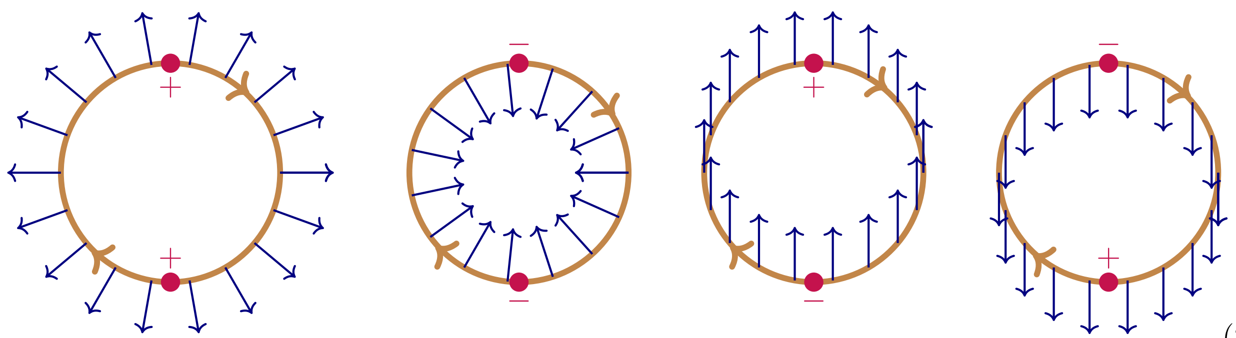
Let us illustrate a few typical situations of unlabeled defect one-manifolds by pictures. In such pictures we use the following conventions. We draw a one-manifold L as embedded in the paper plane $\mathbb{R}^2$. For the tangent space $T_p\mathrm{L}$ at any point $p \in \mathrm{L}$ we adopt the standard convention to depict it as the tangential affine line in $\mathbb{R}^2$.

To graphically represent a section in the bundle $T\mathrm{L} \oplus \underline{\mathbb{R}}$ we must in addition specify the direction of $\underline{\mathbb{R}}$; it suffices to do this for an interval and for a circle. For an *interval* embedded horizontally in the plane and oriented from left to right, we take the positive direction of $\underline{\mathbb{R}}$ to point upwards. Thus at any positive marked point on the interval the vector fields we consider point upwards, and at any negative marked point they point downwards; in particular, between a positive and a negative point the vector field is tangential in at least one point. The following picture shows examples of unlabeled defect intervals:



(2.3)

For a *circle* embedded in $\mathbb{R}^2$ we fix conventions by requiring that the trivial bundle $\underline{\mathbb{R}}$ is outward-pointing. Then the vector fields of our interest point outwards at any positive marked point and inwards at any negative one. Again, between a positive and a negative point the vector field has to be tangential in at least one point. Here are simple examples of 2-framed circles:

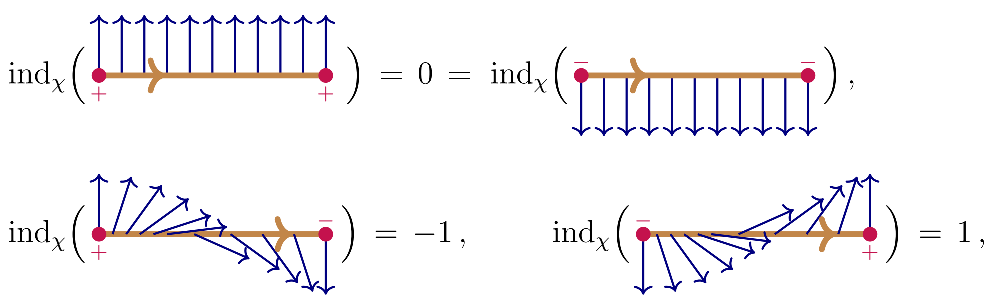


(2.4)

Up to homotopy keeping the vector field χ transversal at the defect points, χ is determined by its winding between neighboring defect points. When counted in units of π in the direction of the orientation, the winding is an integer. We call this integer the *framing index* of the segment s and denote it by

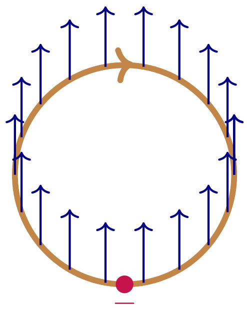
$$\mathrm{ind}_\chi(s) \in \mathbb{Z}\,. \tag{2.5}$$

The index is an even integer if the neighboring defect points have the same sign, and odd otherwise. As an illustration, the values of the index for the intervals in (2.3) are

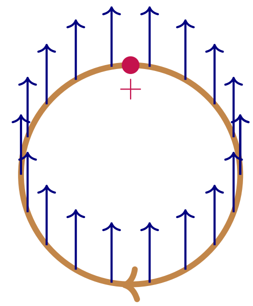


(2.6)

Similarly, the index of both segments of the first two circles in (2.4) is zero and the index of both segments of the other two circles in (2.4) is equal to 1, while the index of the single segment of each of the two circles



$$\text{and} \tag{2.7}$$



is equal to 2.

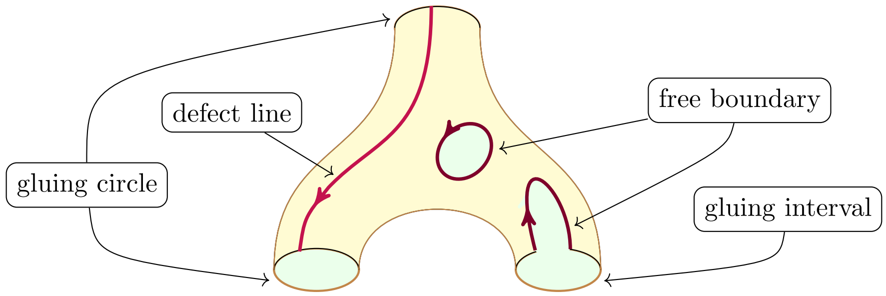
To capture the information contained in the indices of the segments of a defect one-manifold we introduce the following concept:

Definition 2.4. Let $\epsilon = (\epsilon_i)_{i=1,2,...,n}$ be an n-tuple of signs, which we consider as either linearly or cyclically ordered. An n-tuple $\kappa = (\kappa_i)_{i=1,2,...,n} \in \mathbb{Z}^n$ of integers is said to be a tuple of *framing indices* associated with a linearly ordered n-tuple ϵ of signs iff, for every $i \in \{1, 2, \dots, n-1\}$, κ_i is even if the product $\epsilon_{i+1}\,\epsilon_i$ is positive, while κ_i is odd if $\epsilon_{i+1}\,\epsilon_i$ is negative. If ϵ is considered as cyclically ordered, then we impose in addition the same rule on κ_n as a function of the product $\epsilon_n\,\epsilon_1$.

Later on, only the homotopy class of the vector field will matter; accordingly, the datum χ of an unlabeled defect one-manifold is equivalent to the datum of a tuple κ of framing indices for the signs ϵ of the defect points. (This motivates the terminology 'framing index'; compare also Remark 4.14 below.) Accordingly, we will also use the notation $(\mathrm{L}, \epsilon, \kappa)$ in place of $(\mathrm{L}, \epsilon, \chi)$ for 2-framed defect one-manifolds.

We still have to introduce the general unlabeled defect surfaces that can have general unlabeled defect one-manifolds as their boundary components. This is done as follows. Again we start with a compact oriented surface Σ, now possibly with corners. Again there is an embedding $\delta\colon I^{\sqcup n} \sqcup (\mathbb{S}^1)^{\sqcup m} \to \Sigma$ as an additional structure. But now we allow for more general embeddings than before: the image of an interval or a circle is also allowed to be contained in a connected component of the boundary $\partial\Sigma$. If this is the case, then we call the image of the interval or circle an unlabeled *free boundary*. The end points of a free boundary interval are corners of Σ; they constitute additional marked points of Σ, to which we still refer as defect points. Still at each defect point on $\partial\Sigma$ the vector field χ on Σ provides a non-vanishing vector; for δ a free boundary the so obtained vector at a defect point $p \in \delta \cap \partial_{\mathrm{glue}}\Sigma$ is outward-pointing iff δ is oriented towards $\partial_{\mathrm{glue}}\Sigma$, i.e. iff p is a negative point. If a circle is not mapped by δ to a boundary component of Σ, then its image has again to be contained in the interior of Σ.

Further, suppose that a connected component of $\partial\Sigma$ contains at least one free boundary segment. A connected component of the complement of the union of the free boundaries of that connected component is then called a *gluing interval* (see the picture (2.8) below). A connected component of $\partial\Sigma$ that does not contain any free boundary is called a *gluing circle*. If an interval is not mapped by δ to a boundary component of Σ, then its end points must be mapped to a gluing circle or gluing interval, and its interior to points in the interior of Σ. The images of the latter types of circles and intervals, which have non-empty intersection with the interior of Σ, are called *unlabeled defect lines*. As an illustration, the following picture shows a defect surface whose underlying surface is a sphere with four holes and which has one unlabeled defect line, one unlabeled free boundary interval and one unlabeled free boundary circle, and one gluing interval and two gluing circles:



(2.8)

A 2-*framing* on a general surface Σ containing defect lines δ is non-zero vector field χ on Σ that is parallel to, and has the same direction as, each defect line and each free boundary. (On the other hand, there is no such condition restricting the vector field near gluing segments.)

We can now generalize Definition 2.1(i) to

Definition 2.5. An *unlabeled defect surface* is a triple (Σ, δ, χ), where Σ is a compact oriented surface, possibly with boundary and possibly with corners, δ is the union of unlabeled defect lines in Σ and of unlabeled free boundary intervals on the boundary $\partial\Sigma$, and χ is a 2-framing on (Σ, δ).
An *isomorphism* $\varphi\colon (\Sigma, \delta, \chi) \to (\Sigma', \delta', \chi')$ *of unlabeled defect surfaces* is a diffeomorphism of the underlying manifolds that respects the orientations and the vector fields and that maps defect lines to defect lines.

A corner of Σ is necessarily one of the end points of a free boundary interval; as a consequence, the vector field at a corner is parallel to that free boundary.

Given an unlabeled defect surface (Σ, δ, χ), we split its boundary as $\partial\Sigma = \partial_{\text{glue}}\Sigma \cup \partial_{\text{free}}\Sigma$ into the two parts that consist of gluing segments and of free boundary segments, respectively. (Each of the two parts can be empty; their intersection $\partial_{\text{glue}}\Sigma \cap \partial_{\text{free}}\Sigma = \partial(\partial_{\text{free}}\Sigma)$ is the set of corners of Σ.) We refer to $\partial_{\text{glue}}\Sigma$ as the *gluing boundary* of Σ. The gluing boundary $\partial_{\text{glue}}\Sigma$ becomes in the following manner an unlabeled defect one-manifold. The embedding $\iota\colon \partial\Sigma \hookrightarrow \Sigma$ gives rise to an embedding $T_p(\partial\Sigma) \hookrightarrow \iota^*(T_p\Sigma)$ of the tangent space at every point $p \in \partial\Sigma$. Further, by using the inward-pointing normal n_p (with respect to some chosen auxiliary metric on Σ) at p one can then identify $(0, \xi) \in T_p(\partial\Sigma) \oplus \underline{\mathbb{R}}$ with $\xi n_p \in \iota^*(T_p\Sigma)$. This provides an isomorphism of the tangent bundle of Σ, restricted to the boundary, with $T(\partial\Sigma) \oplus \underline{\mathbb{R}}$. This way the 2-framing on Σ induces a 2-framing on the boundary, whereby in particular the gluing boundary $\partial_{\text{glue}}\Sigma$ is endowed with the structure of a (not necessarily connected) 2-framed defect one-manifold. We denote the so obtained unlabeled defect one-manifold by $\partial_{\text{glue}}(\Sigma, \delta, \chi)$.

Instead of using the inward-pointing normal n_p, leading to $\partial_{\text{inward}}(\Sigma, \delta, \chi) \equiv \partial_{\text{glue}}(\Sigma, \delta, \chi)$ we could as well use the outward-pointing normal $-n_p$. This would yield another unlabeled defect one-manifold $\partial_{\text{out}}(\Sigma, \delta, \chi)$ that differs from $\partial_{\text{inward}}(\Sigma, \delta, \chi)$ by replacing the vector field ψ on $\partial_{\text{glue}}(\Sigma, \delta, \chi)$ by $\psi_{\text{outward}} := \overline{\psi}$, where the overbar denotes a flip of the $\underline{\mathbb{R}}$-coordinate, i.e.

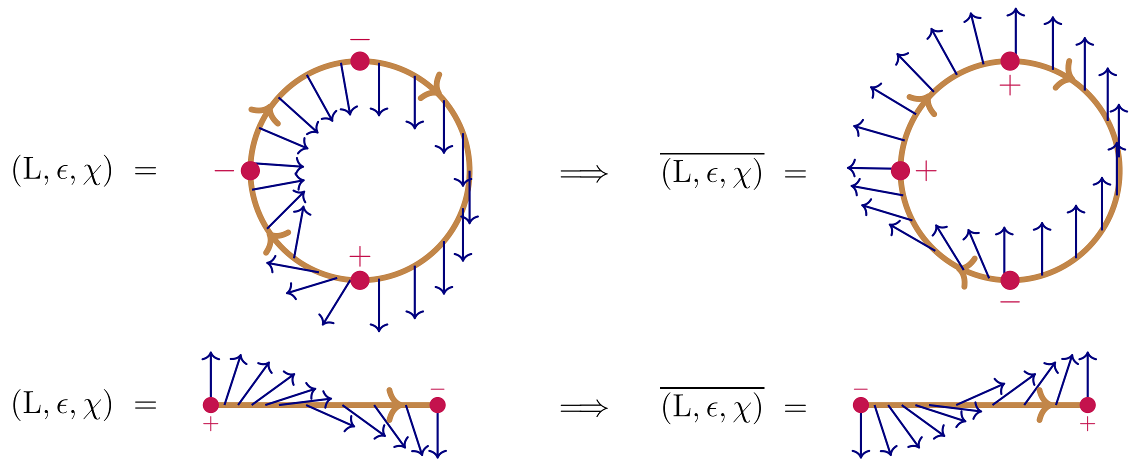
$$\overline{\psi}(p) = (\alpha, -\xi) \quad :\Longleftrightarrow \quad \psi(p) = (\alpha, \xi) \ \in T(\partial\Sigma) \oplus \underline{\mathbb{R}} . \tag{2.9}$$

This motivates the following

Definition 2.6. The *opposite* $\overline{(\mathrm{L},\epsilon,\chi)}$ of an unlabeled defect one-manifold $(\mathrm{L},\epsilon,\chi)$ consists of the manifold $\mathbb{L}$ taken with opposite orientation, which we denote by $\overline{\mathbb{L}}$, the same marked points but with flipped signs, and of the flipped vector field in the sense of (2.9), i.e.

$$\overline{(\mathrm{L},\epsilon,\chi)} := (\overline{\mathrm{L}},-\epsilon,\overline{\chi})\,. \tag{2.10}$$

The following picture shows examples of a defect circle and a defect interval together with their opposites (recall that only the homotopy class of the vector field matters):



(2.11)

To summarize, we have obtained two bicategories $\mathrm{Bord}_2^{\mathrm{def},0}$ and $\mathrm{Bord}_{2,\mathrm{cl}}^{\mathrm{def},0}$:

- Objects of $\mathrm{Bord}_2^{\mathrm{def},0}$ are unlabeled defect one-manifolds, objects of $\mathrm{Bord}_{2,\mathrm{cl}}^{\mathrm{def},0}$ are closed unlabeled defect one-manifolds. (Recall that an unlabeled defect one-manifold is oriented and endowed with a 2-framing.)
- A 1-morphism $\mathbb{L}\to\mathbb{L}'$ in $\mathrm{Bord}_2^{\mathrm{def},0}$ is an unlabeled defect surface Σ together with a *boundary parametrization*, i.e. an isomorphism $\partial_{\mathrm{glue}}\Sigma \xrightarrow{\cong} \overline{\mathrm{L}'}\sqcup \mathrm{L}$ of unlabeled defect one-manifolds that extends to a small collar over $\partial_{\mathrm{glue}}\Sigma$. The 1-morphisms of $\mathrm{Bord}_{2,\mathrm{cl}}^{\mathrm{def},0}$ are those of $\mathrm{Bord}_2^{\mathrm{def},0}$ for which the unlabeled defect surface Σ is closed.
- Composition of 1-morphisms is given by gluing an incoming boundary and an outgoing boundary which are each others' opposites. These boundaries can consist of gluing circles or gluing intervals; the gluing has to account for the parametrizations of the boundaries.
- A 2-morphism $\varphi\colon \Sigma\to\Sigma'$ in $\mathrm{Bord}_2^{\mathrm{def},0}$ between two 1-morphisms is represented by an isomorphism φ of unlabeled defect surfaces that respects the boundary parametrizations. Two isomorphisms $\varphi_0,\varphi_1\colon \Sigma\to\Sigma'$ represent the same 2-morphism iff there is an isotopy $h\colon \Sigma\times[0,1]\to\Sigma'$ with $h(-,0)=\varphi_0$ and $h(-,1)=\varphi_1$ that satisfies $\delta'=h_t(\delta)$ and $\chi'=(h_t)_*(\chi)$ for all $t\in[0,1]$. 2-morphisms in $\mathrm{Bord}_{2,\mathrm{cl}}^{\mathrm{def},0}$ are defined accordingly.
- The vertical composition of 2-morphisms is induced by composition of isomorphisms. The horizontal composition of morphisms is given by gluing of surfaces along gluing circles or gluing intervals.

These categories are symmetric monoidal, with the monoidal structure given by disjoint union. The full subcategories of $\mathrm{Bord}_2^{\mathrm{def},0}$ and $\mathrm{Bord}_{2,\mathrm{cl}}^{\mathrm{def},0}$ consisting of *fine* unlabeled defect surfaces (with and without corners, respectively) are symmetric monoidal as well.

The values of the modular functor on 2-morphisms provide us with representations of the relevant mapping class groups of defect surfaces. This will be analyzed in detail in Section 5.6.

Remark 2.7.

(i) To get a well-defined horizontal composition of 2-morphisms, we should work with collars. This is standard [Ko, Thm. 1.3.12], and nothing new happens in our context. Accordingly we suppress this issue.

(ii) By definition of the 2-morphisms, in case the unlabeled defect surface Σ does not have a boundary and does not have any defects, the 2-morphisms from Σ to itself form the framed mapping class group of Σ.

(iii) Consider a defect surface (Σ, δ, χ) with defects δ and 2-framing χ, and the same underlying surface with the same defect lines but with another 2-framing χ', together with a homotopy χ_t from χ to χ', that is, χ_t is a smooth family $\chi_t\colon \Sigma \to T\Sigma$ of framing vector fields for $t \in [0,1]$, such that $\chi_0 = \chi$ and $\chi_1 = \chi'$. Then (Σ, δ, χ) and (Σ, δ, χ') are isomorphic in $\mathrm{Bord}_2^{\mathrm{def},0}$, i.e. there exists an automorphism φ of Σ which preserves δ and satisfies $T\varphi(\chi) = \chi'$. This can be seen by considering the cylinder $\Sigma \times [0,1]$ over Σ with the vector field $\widetilde{\chi} = (\chi_t, t)$: Since $\Sigma \times [0,1]$ is compact, the vector field $\widetilde{\chi}$ has a complete flow $\varphi_t\colon \Sigma \cong (\Sigma, 0) \to (\Sigma, t) \cong \Sigma$, which is a 1-parameter family of automorphisms of Σ that preserve δ (since each χ_t is tangential to δ) with $\varphi_0 = \mathrm{id}_\Sigma$. It follows from the flow equation $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t\big|_{t=0} = \chi_t$ that $T\varphi_1(\chi_0(p)) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_1(\varphi_t(p))\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_{1+t}(p)\big|_{t=0} = \chi_1(p)$.

(iv) Since, as noted in Remark 2.2(iii), gluable fine defect surfaces compose to a gluable fine defect surface, there is a sub-bicategory $\mathrm{Bord}_2^{\mathrm{def},0,\mathrm{fine}}$ whose 1-morphisms are gluable fine defect surfaces.

2.2 Labels for defect bordisms

We are now going to assign an additional algebraic datum to each connected component of the complement of the defect lines and boundaries in a defect surface. Afterwards we also assign a corresponding datum to each defect line and to each free boundary segment. Before formulating this prescription, several further concepts need to be recalled. All algebraic categories of our interest are assumed to be finite, abelian and linear over a fixed algebraically closed field $\Bbbk$. Similarly we require functors and natural transformations to be linear, unless specified otherwise. For the notion of a *finite tensor category* see e.g. [EGNO]. We will heavily use that every object of such a category has a left and a right dual; we do not assume any relation between the two duals. Our conventions concerning dualities of a rigid category $\mathcal{C}$ are as follows. The right dual of an object c is denoted by $c^\vee$, and the right evaluation and coevaluation are morphisms

$$\mathrm{ev}_c^{\mathrm{r}} \in \mathrm{Hom}_{\mathcal{C}}(c^\vee \otimes c, \mathbf{1}) \qquad \text{and} \qquad \mathrm{coev}_c^{\mathrm{r}} \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, c \otimes c^\vee)\,, \tag{2.12}$$

while the left evaluation and coevaluation are

$$\mathrm{ev}_c^{\mathrm{l}} \in \mathrm{Hom}_{\mathcal{C}}(c \otimes {}^\vee c, \mathbf{1}) \qquad \text{and} \qquad \mathrm{coev}_c^{\mathrm{l}} \in \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, {}^\vee c \otimes c) \tag{2.13}$$

with ${}^\vee c$ the left dual of c.

Further recall that a (left) module category over a finite tensor category $\mathcal{A}$ (or, for short, an $\mathcal{A}$-module), is a finite linear category $\mathcal{M} = {}_{\mathcal{A}}\mathcal{M}$ together with a bilinear functor, exact in the first variable, from $\mathcal{A} \times \mathcal{M}$ to $\mathcal{M}$, which we call the *action* of $\mathcal{A}$ and just denote by a dot '.', as well as with natural isomorphisms μ and λ with components $\mu_{a,b,m} \in \mathrm{Hom}_{\mathcal{M}}((a \otimes b).m, a.(b.m))$ and $\lambda_m \in \mathrm{Hom}_{\mathcal{M}}(\mathbf{1}_{\mathcal{A}}.m, m)$ that satisfy pentagon and triangle relations analogous to the associator and unit constraint of a monoidal category. Right $\mathcal{A}$-modules and $\mathcal{A}$-$\mathcal{B}$-bimodules are defined analogously. For the ease of notation, we will use the symbol $\mathcal{M}$ both for module and for bimodule categories. It is natural to consider a bimodule category ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ as a 1-morphism $\mathcal{A} \to \mathcal{B}$ in the tricategory $\mathcal{F}in\mathcal{C}at_{\otimes}^{\text{l.e.}}$ that has finite tensor categories as objects, finite bimodule categories as 1-morphisms, and categories $\mathcal{L}ex_{\mathcal{A},\mathcal{B}}({}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}, {}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}})$ of left exact bimodule functors and bimodule natural transformations as 2- and 3-morphisms, respectively. (Alternatively, one could consider a tricategory with categories $\mathcal{R}ex_{\mathcal{A},\mathcal{B}}({}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}, {}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}})$ of right exact bimodule functors as 2- and 3-morphisms. In this paper we focus on the formulation with left exact functors.)

There is then an obvious notion of a *composable string* of bimodule categories, and likewise it is clear what a *cyclically composable string* of bimodule categories is. We can also allow for left and right modules, respectively, as the ends of a string of composable bimodule categories, by considering a left $\mathcal{A}$-module as a $\mathcal{A}$-vect-bimodule and a right $\mathcal{B}$-module as a vect-$\mathcal{B}$-bimodule; thus e.g. a right module category $\mathcal{M}_{\mathcal{A}}$, a bimodule category ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}}$ and a left module category ${}_{\mathcal{B}}\mathcal{K}$ form a composable string that constitutes a 1-morphism vect $\to$ vect in $\mathcal{F}in\mathcal{C}at_{\otimes}^{\text{l.e.}}$.

Let now Σ be a surface with defect lines δ. We denote by

$$\Sigma^{(1)} := \delta \cup \partial\Sigma \tag{2.14}$$

the union of the defect lines and the boundary of Σ.

Definition 2.8. By a 2-*patch* of Σ we mean a connected component of the complement of $\Sigma^{(1)}$ in Σ together with the adjacent subset of $\Sigma^{(1)}$.

A defect surface is called fine iff the underlying unlabeled defect surface is fine, i.e. iff every 2-patch is topologically a disk. A defect surface is called gluable fine iff the underlying unlabeled defect surface is gluable fine.

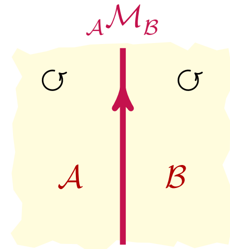
We make the following assignments, which are in line with existing literature (see e.g. Table 1 in [KK]):

- To a 2-patch we assign finite tensor category.[1]
- To a defect line that separates 2-patches labeled by finite tensor categories $\mathcal{A}$ and $\mathcal{B}$ we assign an $\mathcal{A}$-$\mathcal{B}$- or $\mathcal{B}$-$\mathcal{A}$-bimodule category, depending on the relative orientations of the defect line and the adjacent 2-patches (see the picture (2.15) below).
- Similarly, to a *free boundary* we assign a left or right module category over the monoidal category associated to the adjacent 2-patch.

The following picture fixes uniquely our convention for the bimodule categories assigned to

[1] It is worth noting that we do not require the existence of a pivotal structure on these categories. The additional geometric structure of a framing allows us to dispense with pivotal structures; compare Remark 3.1. Also note that we do not make any assumption about the topology of the 2-patches; see however the notion of a fine defect surface in Definition 2.1.

defect lines:



(2.15)

Similarly, for free boundaries our convention is fixed by the following pictures:

(2.16)

The labeling of the building blocks of Σ induces a labeling of the segments of $\partial_{\text{glue}}\Sigma$ and thereby determines an assignment of labels for defect one-manifolds: Defect points are labeled by bimodule categories, and free boundary segments as well as their end points by module categories, in such a way that, together with the orientation of the defect points, the labels form a composable string of bimodule categories.

We summarize our prescriptions in

Definition 2.9. $\mathrm{Bord}_2^{\mathrm{def}}$ is the following symmetric monoidal bicategory:

(i) Objects of $\mathrm{Bord}_2^{\mathrm{def}}$, called *defect one-manifolds*, are tuples $\mathbb{L} = \big(\mathrm{L}, \epsilon, \chi, \{\mathcal{M}_j\}\big)$ given by an unlabeled defect one-manifold together with an assignment $\{\mathcal{M}_j\}$ of labels to its marked points, consisting of a bimodule category for each defect point forming the end of a defect line, and a module category for each defect point at the end of a free boundary segment, in such a way that the finite tensor categories involved in consecutive marked points match (when taking orientations into account).

(ii) 1-morphisms $\mathbb{L} \to \mathbb{L}'$ of $\mathrm{Bord}_2^{\mathrm{def}}$, called *defect surfaces*, are tuples $\Sigma = \big(\Sigma, \delta, \chi, \{\mathcal{A}_k, \mathcal{M}_l\}\big)$ consisting of an unlabeled defect surface and an assignment of labels, together with an isomorphism $\partial_{\text{glue}}\Sigma \xrightarrow{\cong} \overline{\mathrm{L}} \sqcup \mathrm{L}'$ of defect one-manifolds, such that the labels in the interior and on the boundary of Σ match.

(iii) A 2-morphism from a 1-morphism $\mathbb{L} \to \mathbb{L}'$ given by $\Sigma = \big(\Sigma, \delta, \chi, \{\mathcal{A}_k, \mathcal{M}_l\}\big)$ to a 1-morphism given by $\Sigma' = \big(\Sigma', \delta', \chi', \{\mathcal{A}_k', \mathcal{M}_l'\}\big)$ is represented by a isomorphism $\varphi \colon \Sigma \to \Sigma'$ of unlabeled defect surfaces that preserves the labels $\{\mathcal{A}_k, \mathcal{M}_l\}$ of the various strata. We call such an isomorphism a *morphism of defect surfaces*. Two morphisms $\varphi, \varphi' \colon \Sigma \to \Sigma'$ are equivalent iff they are equivalent as morphisms of unlabeled defect surfaces.

(iv) The vertical composition of 2-morphisms is induced by composition of isomorphisms. The horizontal composition of morphisms is given by gluing of surfaces along gluing circles or gluing intervals.

(v) The monoidal structure is given by disjoint union.

Remark 2.10.

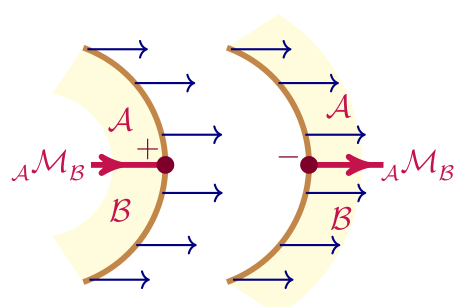
(i) We write generically $\{\mathcal{A}_i, \mathcal{M}_j\}$ etc. for the relevant sets of finite tensor and (bi)module categories. For brevity, below we will usually suppress these additional data (which are objects and 1-morphisms, respectively, of the tricategory $\mathcal{F}in\mathcal{C}at^{\mathrm{l.e.}}_{\otimes}$) in our notation.

(ii) Analogously as for unlabeled surfaces (see Remark 2.2(iii)), gluing two *gluable fine* defect surfaces gives again a gluable fine defect surface. $\mathrm{Bord}_2^{\mathrm{def}}$ has thus a full sub-bicategory $\mathrm{Bord}_2^{\mathrm{def,fine}}$ whose 1-morphisms are gluable fine defect surfaces.

The notion of opposite one-manifold extends as follows from the unlabeled to the labeled case:

Definition 2.11. The *opposite* $\overline{\mathbb{L}}$ of a (labeled) defect one-manifold $\mathbb{L}$ is the opposite $\overline{(\mathrm{L}, \epsilon, \chi)}$ of the underlying unlabeled manifold $(\mathrm{L}, \epsilon, \chi)$ together with the same assignment of labels.

Two defect surfaces Σ_1 and Σ_2 can be glued along a defect one-manifold $\mathbb{L}$ if and only if the corresponding defect one-manifold $\mathbb{L}_1 \subset \Sigma_1$ and $\mathbb{L}_2 \subset \Sigma_2$ are opposite to each other. To be able to work with smooth manifolds, gluing is actually along collars. This is demonstrated in the following picture which shows the situation locally around a defect point on a gluing segment:



(2.17)

The main result of this article is the construction of two specific modular functors.

Definition 2.12. Let $\mathcal{S}$ be a symmetric monoidal bicategory. An $\mathcal{S}$-valued *framed modular functor* (with decoration data in finite categories) is a symmetric monoidal 2-functor

$$\mathrm{T}: \quad \mathrm{Bord}_2^{\mathrm{def}} \to \mathcal{S} . \tag{2.18}$$

Given the prescriptions for labels present in our setting, a natural choice of a target bicategory $\mathcal{S}$ is $\mathcal{LEX}$, i.e. the bicategory that has as objects finite $\Bbbk$-linear categories, as 1-morphisms left exact functors and as 2-morphisms natural transformations, with monoidal structure given by the Deligne product. The restriction of functors to left exact ones is due to the fact that the Deligne product of left exact functors is defined and provides the symmetric monoidal structure at the level of 1- and 2-morphisms. The same also applies to right exact functors, and indeed we could have chosen instead the symmetric monoidal bicategory $\mathcal{REX}$ whose morphisms are right exact functors.

To summarize, we will show:

Theorem 2.13. There exists a state-sum construction that provides an explicit framed modular functor with values in $\mathcal{LEX}$, as well as a framed modular functor with values in $\mathcal{REX}$.

Several comments are in order:

Remarks 2.14.

(i) The existence of a similar functor for manifolds without defects has been shown in [DSS, Cor. 5], invoking the cobordism hypothesis. Our approach is more direct and and can be directly compared with state-sum constructions.

(ii) The definition of a modular functor implies in particular that our construction is compatible with factorization or, more specifically, with the gluing of defect surfaces along gluing intervals or gluing circles. Indeed, the composition of two 1-morphisms in $\mathrm{Bord}_2^{\mathrm{def}}$, i.e. defect surfaces Σ and Σ', is by gluing the 'outgoing' part $\partial_+\Sigma$ of the gluing boundary of Σ with the 'incoming' part $\partial_-\Sigma'$ of the gluing boundary of Σ'. For details, see Section 5.1.

(iii) Our prescription may be seen as a kind of Turaev-Viro construction. Indeed, a standard semisimple Turaev-Viro construction corresponds to specializing our prescription by using only "transparent" labelings and imposing pivotality (which allows one to eliminate the framing), see Remark 5.23. Now in the standard Turaev-Viro situation a crucial property is the independence of the choice of a triangulation. In contrast, the framed modular functor considered here is defined on arbitrary defect surfaces, without requiring the presence of a triangulation that gives fine surfaces, for which all 2-patches are contractible. However, as explained in Section 5.3, the construction of the modular functor does make use of such "refining" triangulations, all parts of which are labeled by transparent labels. As we show in Section 5.4, our construction is compatible with transparency, in the sense that the structure we have at our disposal is sufficient to define a block functor that is independent of the triangulation as a (co)limit. (Thus not only the precise position of transparently labeled defects is irrelevant – this invariance up to homotopy is valid for any topological defect – but not even their combinatorial configuration, i.e. the particular choice of refining triangulation, matters. These properties justify the qualification "transparent".)

(iv) Mapping class group elements are specific 2-morphisms in $\mathrm{Bord}_2^{\mathrm{def}}$. Our construction thus provides representations of mapping class groups. This is studied in Section 5.6.

(v) The left exact version of the functor is compatible with operations on defect labels in the following sense: As shown in Proposition 4.22, the contraction of a defect line is implemented by a composition of functors, while Theorem 4.37 implies that the fusion of two parallel defect lines corresponds to a variant of the relative Deligne product of bimodule categories, which is the composition of 1-morphisms in $\mathcal{F}in\mathcal{C}at_{\otimes}^{\mathrm{l.e.}}$. Thus in particular our construction is compatible with the identities in $\mathcal{F}in\mathcal{C}at_{\otimes}^{\mathrm{l.e.}}$. This, in turn, is implicit in the construction of the functor via refining triangulations.

3 Assigning categories to defect one-manifolds

The goal of this section is to define our modular functor on objects, that is, to associate to any defect one-manifold $\mathbb{L}$ a finite $\Bbbk$-linear category $\mathrm{T}(\mathbb{L})$. We call these categories *gluing categories*, because they are assigned to boundary segments of defect surfaces along which these can be glued together to form more complicated defect surfaces. The gluing category $\mathrm{T}(\mathbb{L})$ will be defined as the category of objects in a Deligne product, endowed with the additional structure of *balancings*. More concretely, we first take the Deligne product of all categories that are assigned to the marked points of $\mathbb{L}$. We then take objects in this Deligne product together with a balancing for each monoidal category that is assigned to a segment of $\mathbb{L}$. Such a balancing allows one to swap the action of objects in a monoidal category from one (bi)module category in a composable string to a neighboring one.

3.1 Twisted bimodule categories

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be finite tensor categories and $\mathcal{M}$ an $\mathcal{A}_1$-$\mathcal{A}_2$-bimodule category. The double left and right dual functors of a finite tensor category have a natural monoidal structure, so that we can twist the left and right actions on $\mathcal{M}$ by powers of the double left or right dual of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. To describe these twisted actions comfortably, we introduce the following notation. Given an object a of a finite tensor category $\mathcal{A}$, we use the shorthand ${}^{[\kappa]}a$, for $\kappa \in \mathbb{N}$, for the κ-fold left dual ${}^{\vee\vee\cdots\vee}a$ of a, and analogously $a^{[\kappa]}$ for the κ-fold right dual; we also write ${}^{[0]}a = a = a^{[0]}$. Thus the double left dual functor maps objects as $a \mapsto {}^{\vee\vee}a = {}^{[2]}a$. Further, in view of the canonical isomorphisms ${}^{\vee}(a^{\vee}) \cong a \cong ({}^{\vee}a)^{\vee}$ it is natural to extend these definitions by taking ${}^{[-\kappa]}a$ for $\kappa \in \mathbb{N}$ to be the κ-fold right dual, ${}^{[-\kappa]}a = a^{[\kappa]}$, and vice versa,

Now for any pair $(\kappa_1, \kappa_2) \in 2\mathbb{Z} \times 2\mathbb{Z}$ we denote by ${}^{\kappa_1}\mathcal{M}^{\kappa_2}$ the bimodule category for which the left and right actions on $\mathcal{M}$ are twisted by the κ_1- and κ_2-fold left and right dual, respectively, i.e. $\mathcal{A}_1$ acts as $m \mapsto {}^{[\kappa_1]}a_1.m$ and $\mathcal{A}_2$ as $m \mapsto m.a_2^{[\kappa_2]}$. We also abbreviate ${}^{\kappa}\mathcal{M}^0 \equiv {}^{\kappa}\mathcal{M}$ and ${}^0\mathcal{M}^{\kappa} \equiv \mathcal{M}^{\kappa}$. Similarly, for every pair of *odd* integers κ_1 and κ_2, the opposite category $\mathcal{M}^{\mathrm{opp}}$ can be endowed with the structure of an $\mathcal{A}_2$-$\mathcal{A}_1$-bimodule by setting

$$a_1 \,.\, \overline{m} := \overline{m \,.\, {}^{[\kappa_1]}a_1} \qquad \text{and} \qquad \overline{m} \,.\, a_2 := \overline{a_2^{[\kappa_2]}.m} \,. \tag{3.1}$$

Here we write $\overline{x}$ for the object $x \in \mathcal{M}$ seen as an object in $\mathcal{M}^{\mathrm{opp}}$. (The reason for the specific convention (3.1) will become clear in Remark 3.7.1.) Whenever convenient we will from now on also use the notation $\overline{\mathcal{X}}$ for the opposite category of any category $\mathcal{X}$, as well as

$$\mathcal{X}^{\epsilon} := \begin{cases} \mathcal{X} & \text{for } \epsilon = +1 \,, \\ \overline{\mathcal{X}} & \text{for } \epsilon = -1 \,. \end{cases} \tag{3.2}$$

Further, we denote the bimodule categories with actions (3.1) by ${}^{\kappa_1}\overline{\mathcal{M}}^{\kappa_2}$, and analogously for left and right modules. Note that the so obtained $\mathbb{Z}\times\mathbb{Z}$-torsor of bimodule categories does not have a natural section. Allowing also for the modules ${}^{\kappa_1}\overline{\mathcal{M}}^{\kappa_2}$, we can consider cyclically or linearly composable strings of the more general form $(\mathcal{M}_i^{\epsilon_i})_{1 \le i \le n}$.

Remark 3.1. A pivotal structure on a finite tensor category $\mathcal{A}$, if it exists, furnishes a monoidal equivalence $?^{\vee\vee} \Rightarrow \mathrm{Id}_{\mathcal{A}}$ and thus allows one to identify all the module categories that result from twists by powers of the double dual. But we do not assume the existence of a pivotal structure and accordingly use all twisted bimodule structures ${}^{\kappa_1}\mathcal{M}^{\kappa_2}$ and ${}^{\kappa_2}\overline{\mathcal{M}}^{\kappa_1}$ in our construction. However, as will be explained in Section 3.6, there is a canonical 4-periodicity in the so obtained family of bimodule categories.

3.2 Balancings for bimodule categories

A connected defect one-manifold $\mathbb{L}$ with $n>0$ marked points comes by definition with a string of bimodule categories $(\mathcal{M}_i)_{1\le i\le n}$ such that the string $(\mathcal{M}_i^{\epsilon_i})_{1\le i\le n}$ is either cyclically or linearly composable. To be able to introduce the $\Bbbk$-linear category associated to a defect one-manifold $\mathbb{L}$, one further ingredient is needed: *twisted balancings* for strings of composable bimodules categories.

Definition 3.2.

(i) Let $\mathcal{A}$ be a monoidal category and $\mathcal{M}$ an $\mathcal{A}$-bimodule. A *balancing* for an object $m\in\mathcal{M}$ is a natural family $(\sigma=(\sigma_a\colon a.m\to m.a)_{a\in\mathcal{A}})$ of morphisms in $\mathcal{M}$ such that $\sigma_1=\mathrm{id}_m$ and such that the diagram

$$\begin{array}{ccc} (a\otimes a').m & \xrightarrow{\sigma_{aa'}} & m.(a\otimes a') \\ & {\scriptstyle a.\sigma_{a'}}\searrow \quad \nearrow{\scriptstyle \sigma_a.a'} & \\ & a.m.a' & \end{array} \tag{3.3}$$

commutes for all $a,a'\in\mathcal{A}$. (For brevity we omit the constraint morphisms of the bimodule category $\mathcal{M}$.)

(ii) The *category* $\mathcal{Z}_\mathcal{A}(\mathcal{M})$ *of objects with balancings* for an $\mathcal{A}$-bimodule category $\mathcal{M}$ has as objects pairs (m,σ) consisting of an object of $\mathcal{M}$ and a balancing.

The morphisms $\mathrm{Hom}_{\mathcal{Z}_\mathcal{A}(\mathcal{M})}((m,\sigma),(m',\sigma'))$ are those morphisms $m\xrightarrow{f}m'$ in $\mathcal{M}$ for which the diagram

$$\begin{array}{ccc} a.m & \xrightarrow{\sigma} & m.a \\ {\scriptstyle a.f}\downarrow & & \downarrow{\scriptstyle f.a} \\ a.m' & \xrightarrow{\sigma'} & m'.a \end{array} \tag{3.4}$$

commutes for all $a\in\mathcal{A}$.

Lemma 3.3.
Let $\mathcal{A}$ be a monoidal category with right duals and $\mathcal{M}$ a bimodule category over $\mathcal{A}$. Then any balancing σ for an object $m\in\mathcal{M}$ is a natural *iso*morphism.

Proof. Using the tensoriality (3.3) and the naturality of σ one verifies directly that a two-sided inverse of σ_a is given by $\sigma'_a:=(a.m.\mathrm{ev}^\mathrm{r}_a)\circ(a.\sigma_{a^\vee}.a)\circ(\mathrm{coev}^\mathrm{r}_a.m.a)$. ☐

Definition 3.4.

(i) Let $(\mathcal{M}_i^{\epsilon_i})_{i=1,\dots,n}$ be a string of n cyclically composable bimodule categories, and let κ be an n-tuple of framing indices associated with ϵ. Then the *κ-framed center* (or *framed center*, for short), denoted by

$$\mathcal{M}_1^{\epsilon_1}\overset{\kappa_1}{\boxtimes}\mathcal{M}_2^{\epsilon_2}\overset{\kappa_2}{\boxtimes}\mathcal{M}_3^{\epsilon_3}\overset{\kappa_3}{\boxtimes}\cdots\overset{\kappa_{n-1}}{\boxtimes}\mathcal{M}_n^{\epsilon_n}\overset{\kappa_n}{\boxtimes}, \tag{3.5}$$

is the *category of twisted balancings*: Objects are objects of $\boxtimes_i\mathcal{M}_i^{\epsilon_i}$ (cyclically composable) together with a balancing for each action of a finite tensor category involved.

For an object of $\boxtimes_i \mathcal{M}_i^{\epsilon_i}$ of the form $m_1^{\epsilon_1} \boxtimes m_2^{\epsilon_2} \boxtimes \cdots \boxtimes m_{n-1}^{\epsilon_{n-1}} \boxtimes m_n^{\epsilon_n}$ and for the case of even values κ_i and $\epsilon_1 = \epsilon_i = \epsilon_{i+1} = \epsilon_n = 1$, the balancing consists of coherent isomorphisms

$$\begin{aligned} &m_i.a \boxtimes m_{i+1} \xrightarrow{\cong} m_i \boxtimes {}^{[\kappa_i-2]}a.m_{i+1} \qquad \text{for } \; i \in \{1,2,\dots,n{-}1\} \qquad \text{and} \\ &m_1 \boxtimes \cdots \boxtimes m_n.a \xrightarrow{\cong} {}^{[\kappa_n-2]}a.m_1 \boxtimes \cdots \boxtimes m_n \,. \end{aligned} \tag{3.6}$$

In case $\epsilon_i = \epsilon_{i+1} = -1$, the balancing is $\overline{a.m_i} \boxtimes \overline{m_{i+1}} \xrightarrow{\cong} \overline{m_i} \boxtimes \overline{m_{i+1}.{}^{[\kappa_i+2]}a}$, and analogously if $\epsilon_1 = \epsilon_n = -1$. If κ_i is odd, one deals with an object of the form $\overline{m_i} \boxtimes m_{i+1}$ and it is understood that the isomorphism is $\overline{a.m_i} \boxtimes m_{i+1} \cong \overline{m_i} \boxtimes {}^{[\kappa_i]}a.m_{i+1}$, and analogously in the case $\epsilon_i = +1$, $\epsilon_{i+1} = -1$.
Morphisms in the category are morphisms of $\boxtimes_i \mathcal{M}_i^{\epsilon_i}$ that are compatible with the balancings.

(ii) Similarly, for a collection $(\mathcal{M}_i^{\epsilon_i})_{i=1,\dots,n}$ of linearly composable bimodule categories, and for κ an n-tuple of integers that refines the signs ϵ, the κ-*framed center* is again defined as the corresponding category of twisted balancings, with only $n{-}1$ balancings involved.

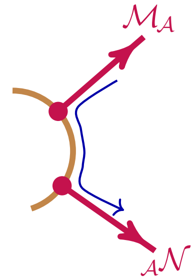
Remarks 3.5.

(i) We slightly abuse notation by omitting the bracketing for objects in multiple ordinary Deligne products. This is unproblematic because different bracketings are related by canonical coherent isomorphisms.

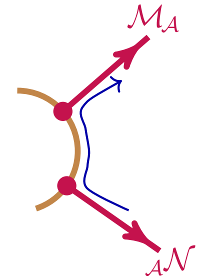
(ii) It is sufficient to specify, as done in (3.6), the balancings only for objects that are of $\boxtimes$-factorized form. Below we will often analogously use $\boxtimes$-factorized objects as placeholders for generic objects.

(iii) In the special case that $\mathcal{M} = \mathcal{A}$ is a finite tensor category, regarded as a bimodule category over itself, a balancing is a half-braiding and $\mathcal{A}\overset{2}{\boxtimes} = \mathcal{Z}(\mathcal{A})$ is the Drinfeld center of $\mathcal{A}$. This justifies the terminology "framed center". Applying the tensor product of $\mathcal{A}$ to two objects $b_i \in \mathcal{A}\overset{\kappa_i}{\boxtimes}$ gives an object in $\mathcal{A}\overset{\kappa_1+\kappa_2-2}{\boxtimes}$ In particular, only the category $\mathcal{Z}(\mathcal{A}) = \mathcal{A}\overset{2}{\boxtimes}$ is monoidal, while for general κ there are mixed tensor products $\big(\mathcal{A}\overset{\kappa_1}{\boxtimes}\big) \boxtimes \big(\mathcal{A}\overset{\kappa_2}{\boxtimes}\big) \to \mathcal{A}\overset{\kappa_1+\kappa_2-2}{\boxtimes}$.

(iv) Instead of ordering the bimodule categories as $\mathcal{M}_1^{\epsilon_1} \cdots \mathcal{M}_n^{\epsilon_n}$ we could as well order them as $\mathcal{M}_n^{\epsilon_n} \cdots \mathcal{M}_1^{\epsilon_1}$. Accordingly for each pair of consecutive bimodules we can interpret the balancing in two ways, namely as swapping the action of the relevant finite tensor category from $\mathcal{M}_i^{\epsilon_i}$ to $\mathcal{M}_{i+1}^{\epsilon_{i+1}}$ or from $\mathcal{M}_{i+1}^{\epsilon_{i+1}}$ to $\mathcal{M}_i^{\epsilon_i}$ corresponding, respectively, to the two pictures



and



(3.7)

(Here we also indicate defect lines attached at the defect points, in order to indicate how the gluing segment may appear as part of the boundary of a defect surface. This will be done analogously also in other pictures.) The orientation of the defect one-manifolds provides us with one particular ordering, e.g. we swap from $\mathcal{M}_i^{\epsilon_i}$ to $\mathcal{M}_{i+1}^{\epsilon_{i+1}}$ in the situation shown in the picture (3.21) below.

(v) By direct calculation one checks that there is an equivalence

$$(\mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{N})^{\text{opp}} \simeq \overline{\mathcal{N}} \overset{-\kappa}{\boxtimes} \overline{\mathcal{M}} \tag{3.8}$$

for any pair of a right $\mathcal{A}$-module $\mathcal{M}$ and left $\mathcal{A}$-module $\mathcal{N}$ and any index κ.

The framed center for the case of a single bimodule will be particularly relevant, so we give it a separate name:

Definition 3.6. Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ a finite $\mathcal{A}$-bimodule. For $\kappa \in 2\mathbb{Z}$, the *κ-twisted center* $\mathcal{Z}^\kappa(\mathcal{M}) \equiv \mathcal{Z}^\kappa_\mathcal{A}(\mathcal{M})$ is the category that has as objects pairs (m,σ) consisting of an object $m \in \mathcal{M}$ and a twisted balancing $\sigma = (\sigma_a)$ with $\sigma_a\colon a.m \xrightarrow{\cong} m.a^{[\kappa-2]}$, i.e. $\mathcal{Z}^\kappa(\mathcal{M}) = \mathcal{M} \overset{\kappa}{\boxtimes}$.

The forgetful functor $\mathcal{Z}^\kappa(\mathcal{M}) \to \mathcal{M}$ is an exact functor of finite categories, hence it has a left and a right adjoint. It is convenient to express the adjoints using the language of (co)ends, which we review in Appendix B.1. A right adjoint is given by the co-induction functor

$$\begin{aligned} I_{[\kappa]}:\ \ \mathcal{M} &\rightarrow \mathcal{Z}^\kappa(\mathcal{M})\,, \\ m &\mapsto \int_{a\in\mathcal{A}} a\,.\,m\,.\,a^{[\kappa-1]}\,, \end{aligned} \tag{3.9}$$

and a left adjoint by the corresponding induction functor $I^{[\kappa]}$, see Corollary B.3.

Remarks 3.7. The following statements about these categories follow directly from the definitions:

(i) For even κ we have $\mathcal{Z}^\kappa_\mathcal{A}(\mathcal{M}) = \mathcal{Z}(\mathcal{M}^\kappa)$, i.e. the twisted center is the ordinary Drinfeld center for the $\mathcal{A}$-module for which the right $\mathcal{A}$-action is twisted by the κ-fold dual.

(ii) The notation fits with the conventions for the twisted actions: for $\kappa \in 2\mathbb{Z}$ we have

$$\mathcal{N}_\mathcal{A} \overset{\kappa}{\boxtimes}{}_\mathcal{A}\mathcal{M} = \mathcal{Z}(\mathcal{N}^\kappa_\mathcal{A} \boxtimes {}_\mathcal{A}\mathcal{M}) = \mathcal{Z}(\mathcal{N}_\mathcal{A} \boxtimes {}^\kappa_\mathcal{A}\mathcal{M})\,, \tag{3.10}$$

while for two right modules $\mathcal{N}_\mathcal{A}$ and $\mathcal{M}_\mathcal{A}$ and odd κ, with the convention (3.1) one finds

$$\mathcal{N} \overset{\kappa}{\boxtimes} \overline{\mathcal{M}} = \mathcal{Z}(\mathcal{N} \boxtimes {}^\kappa\overline{\mathcal{M}}) \qquad \text{and} \qquad \overline{\mathcal{N}} \overset{\kappa}{\boxtimes} \mathcal{M} = \mathcal{Z}(\overline{\mathcal{N}}^\kappa \boxtimes \mathcal{M})\,. \tag{3.11}$$

(iii) Let $\mathcal{A}$ be a monoidal category with a right duality. Let $\mathcal{M}$ be a left $\mathcal{A}$-module and $\mathcal{N}$ a right $\mathcal{A}$-module, whereby their Deligne product $\mathcal{N} \boxtimes \mathcal{M}$ is an $\mathcal{A}$-bimodule. Then the category

$$\mathcal{N} \overset{0}{\boxtimes} \mathcal{M} \simeq \mathcal{N} \boxtimes_\mathcal{A} \mathcal{M} \tag{3.12}$$

is the relative Deligne product (see e.g. [FSS1, Sect. 2.5] for the definition). This category can also be realized as the category of modules over $\int^{a\in\mathcal{A}} a \boxtimes {}^\vee a$, which has a natural structure of a Frobenius algebra A in $\mathcal{A}\boxtimes\overline{\mathcal{A}}$. In this description the universal induction functor is the induction functor for the algebra A. Recall [FSS1] that (contrary to statements in the literature) $\mathcal{N}\boxtimes_{\mathcal{A}}\mathcal{M}$ is *not* the center of $\mathcal{N}$ and $\mathcal{M}$, but rather the twisted center $\mathcal{N}\overset{0}{\boxtimes}\mathcal{M}$, which with our conventions has objects $n\boxtimes m$ equipped with balancing $n.a\boxtimes m\cong n\boxtimes {}^{\vee\vee}a.m$.

(iv) More generally, for an $\mathcal{A}$-bimodule category $\mathcal{B}$, the category $\mathcal{Z}^2_{\mathcal{A}}(\mathcal{B})\simeq\mathcal{B}_{Z_{\mathcal{A}}}$ is the category-valued trace of the bimodule category $\mathcal{B}$, see [FSS1, Sect. 3].

(v) Just like the category-valued trace of a bimodule is defined via a universal property with respect to balanced functors [FSS1, Defs. 3.2 & 2.7], the twisted center $\mathcal{Z}^\kappa(\mathcal{M})$ of a bimodule category $\mathcal{M}$ can be characterized by the universal property that for any finite category $\mathcal{X}$, pre-composition with the co-induction $I_{[\kappa]}$ gives a distinguished equivalence

$$\begin{aligned} \mathcal{L}ex(\mathcal{Z}^\kappa(\mathcal{M}),\mathcal{X}) &\xrightarrow{\simeq} \mathcal{L}ex^\kappa(\mathcal{M},\mathcal{X}) \\ F &\longmapsto F\circ I_{[\kappa]} \end{aligned} \tag{3.13}$$

with $\mathcal{L}ex^\kappa(\mathcal{M},\mathcal{X})$ the category of κ-balanced functors, i.e. (cp. Definition B.1) functors $F\colon\mathcal{M}\to\mathcal{X}$ with coherent isomorphisms $F(c.m)\cong F(m.c^{[\kappa]})$ (see Proposition B.5 for the precise statement). A similar statement holds for the induction functor $I^{[\kappa]}$, with right exact functors that are $\kappa-2$-balanced.

(vi) For $\mathcal{M}$ a right and $\mathcal{N}$ a left $\mathcal{A}$-module, it follows from the definitions that there are distinguished equivalences

$$\mathcal{M}^\kappa\overset{\kappa'}{\boxtimes}\mathcal{N}\simeq\mathcal{M}\overset{\kappa+\kappa'}{\boxtimes}\mathcal{N}\simeq\mathcal{M}\overset{\kappa}{\boxtimes}{}^{\kappa'}\mathcal{N} \tag{3.14}$$

for any pair κ, κ' of even integers, and similar equivalences if κ or κ' are odd, for instance $\mathcal{M}^\kappa\overset{\kappa'}{\boxtimes}\mathcal{N}\simeq\overline{\mathcal{M}}\overset{\kappa+\kappa'}{\boxtimes}\mathcal{N}$ in case $\mathcal{M}$ and $\mathcal{N}$ are left modules, κ is odd and κ' even,

3.3 Balancing and (co)monads

Framed centers (which will play the role of gluing categories) were introduced in Definition 3.4 as categories of balancings. It turns out that in order to check that these categories have desirable features, such as cyclic invariance, it is convenient to express them with the help of suitable (co)monads and their (co)modules.

Recall that modules over a monad on a category $\mathcal{C}$ behave like modules over an algebra in $\mathcal{C}$ even when there is no corresponding algebra, and analogously for comodules over a comonad (for some details see Appendix B.1). Of particular interest to us is the central comonad, i.e. the endofunctor

$$Z: \quad b\longmapsto\int_{a\in\mathcal{A}} a\otimes b\otimes a^\vee \tag{3.15}$$

of a finite tensor category $\mathcal{A}$, as well as the central monad $b\mapsto\int^{a\in\mathcal{A}}a\otimes b\otimes{}^\vee a$ (these (co)ends exist, see [Sh, Thm. 3.4]). The Drinfeld center $\mathcal{Z}(\mathcal{A})$ is canonically equivalent [BV] to the category of modules over the central monad and as the category of comodules over the central comonad. This description is based on the observation that if $\mathcal{A}$ is a monoidal category with right dualities, then a natural family of isomorphisms $x.a \rightarrow a.x$ amounts to a dinatural family of morphisms $a^\vee.x.a \rightarrow x$.

The construction generalizes to $\mathcal{A}$-bimodule categories $\mathcal{M}$ as follows. Since all categories involved are finite, the end $Z_\mathcal{A}(m) := \int_{a\in\mathcal{A}} a.\,m\,.\,a^\vee$ and coend $\int^{a\in\mathcal{A}} a.\,m\,.{}^\vee a$ exist for every $m \in \mathcal{M}$ and provide endofunctors of $\mathcal{M}$ that have a natural structure of a comonad and a monad, respectively. It s natural to generalize this construction further by allowing for *twisted* balancings: Thus let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ a finite $\mathcal{A}$-bimodule category. Then for any $\kappa \in 2\mathbb{Z}$ we have a comonad on $\mathcal{M}$ given by the endofunctor

$$Z_{[\kappa]}: \quad m \longmapsto \int_{a\in\mathcal{A}} a\,.\,m\,.\,a^{[\kappa-1]}\,, \tag{3.16}$$

such that $Z_{[2]} = Z_\mathcal{A}$, and a monad given by the endofunctor

$$Z^{[\kappa]}: \quad m \longmapsto \int^{a\in\mathcal{A}} a\,.\,m\,.\,a^{[\kappa-3]}, \tag{3.17}$$

As is shown in Corollary B.3(ii) in the Appendix, the categories of $Z_{[\kappa]}$-comodules and of $Z^{[\kappa]}$-modules are both equivalent to the κ-twisted center $\mathcal{Z}^\kappa_\mathcal{A}(\mathcal{M})$ introduced in Definition 3.6. In particular, the category $\mathcal{M}^{Z_\mathcal{A}}$ of $Z_\mathcal{A}$-comodules is equivalent to the category $Z_\mathcal{A}(\mathcal{M})$ of objects with balancings.

Now on the Deligne product that arises for the gluing category assigned to a connected 2-framed one-dimensional defect manifold $\mathbb{L}$ we have to deal with *several* balancings, each of which is of the form (3.6), namely one for each consecutive pair of bimodule categories in the relevant string of composable bimodule categories. It is is not hard to see that the associated comonads (as well as the corresponding monads) *commute*, meaning that there are distributive laws, i.e. natural transformations $\ell\colon Z' \circ Z \Rightarrow Z \circ Z'$ compatible with the comonad (respectively monad) structure. (This amounts to two triangle and two pentagon identities. The corresponding structural data are either trivial or induced from the structure morphisms of the bimodule category.) Now for Z and Z' comonads, a distributive law endows the endofunctor $Z \circ Z'$ again with the structure of a comonad, with comultiplication

$$Z \circ Z' \xrightarrow{\Delta\circ\Delta'} Z \circ Z \circ Z' \circ Z' \xrightarrow{Z\circ\ell\circ Z'} Z \circ Z' \circ Z \circ Z' \tag{3.18}$$

and with counit $Z \circ Z' \xrightarrow{\varepsilon\circ\varepsilon'} \mathrm{id}$, where Δ, Δ' and ε, ε' are the coproduct and counit of Z and Z', respectively. Since the Deligne product of finite categories is symmetric, the independence of the category $\mathrm{T}(\mathbb{L},\epsilon,\sigma)$ on the linear order chosen is thus obvious.

Let us describe more explicitly how we can write the framed center as a category of comodules over commuting comonads, restricting for simplicity to the case of the framed center $\mathcal{M}_1^{\epsilon_1} \overset{\kappa_1}{\boxtimes} \mathcal{M}_2^{\epsilon_2}$ of just two categories. Denoting the (left or right) actions of $\mathcal{A}$ on these categories by $\triangleright_1\colon \mathcal{A} \rightarrow \mathrm{End}(\mathcal{M}_1)$ and $\triangleright_2\colon \mathcal{A} \rightarrow \mathrm{End}(\mathcal{M}_2)$, we have a functor

$$\triangleright_1^{\epsilon_1} \boxtimes \big(\triangleright_2 \circ {}^{[f(\kappa_1)]}(-)\big)^{\epsilon_2} : \quad \mathcal{A} \boxtimes \overline{\mathcal{A}} \rightarrow \mathrm{End}(\mathcal{M}_1^{\epsilon_1} \boxtimes \mathcal{M}_2^{\epsilon_2}) \tag{3.19}$$

with the function f given by $f(\kappa_1) = \kappa_1 + 1 - 2\epsilon_1$ for even κ_1 (i.e. for $\epsilon_1 = \epsilon_2$) and by $f(\kappa_1) = \kappa_1 + 1$ for odd κ_1. Taking the end of this functor defines the comonad $Z_{[\kappa_1]}$ on $\mathcal{M}_1^{\epsilon_1} \boxtimes \mathcal{M}_2^{\epsilon_2}$; for even κ_1 this is a special case of the comonad $Z_{[\kappa]}$ in (3.16). Applying Proposition B.2(ii) now gives

Lemma 3.8. The category of comodules over the comonad $Z_{[\kappa]}$ on $\mathcal{M}_1^{\epsilon_1} \boxtimes \mathcal{M}_2^{\epsilon_2}$ is equivalent to the framed center $\mathcal{M}_1^{\epsilon_1} \overset{\kappa}{\boxtimes} \mathcal{M}_2^{\epsilon_2}$.

In this context it is worth recalling that the κ-twisted center $\mathcal{Z}^\kappa(\mathcal{M})$ of an $\mathcal{A}$-bimodule introduced in Definition 3.6 comes with a universal functor $\mathcal{M} \to \mathcal{Z}^\kappa(\mathcal{M})$, namely the co-induction functor (3.9) which is right adjoint to the forgetful functor $\mathcal{Z}^\kappa(\mathcal{M}) \to \mathcal{M}$. The comonad associated with this adjunction is precisely $Z_{[\kappa]}$, and analogously the left adjoint corresponds to the monad $Z^{[\kappa]}$. Moreover (see part (iii) of Proposition B.5), for $\mathcal{A}$ a finite tensor category and $\mathcal{M}$ an $\mathcal{A}$-bimodule category we have an isomorphism

$$\int^{z \in \mathcal{Z}^\kappa(\mathcal{M})} \overline{z} \boxtimes z \;\cong\; \int^{m \in \mathcal{M}} \overline{m} \boxtimes Z_{[\kappa]}(m) \tag{3.20}$$

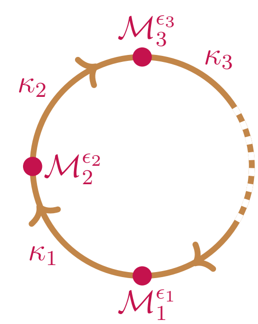
of objects in the category $\overline{\mathcal{Z}^\kappa(\mathcal{M})} \boxtimes \mathcal{Z}^\kappa(\mathcal{M})$. In particular, the object on the right hand side of (3.20) is canonically an object in this category.

3.4 Gluing categories

We have now provided all algebraic ingredients needed for introducing the gluing categories.

Definition 3.9. The *gluing category* $\mathrm{T}(\mathbb{L})$ assigned to a defect one-manifold $\mathbb{L} = (\mathrm{L}, \epsilon, \chi, \{\mathcal{M}_i\})$ is defined as follows.

(i) If the one-manifold L underlying $\mathbb{L}$ is connected, as in

$\mathbb{L} \;=$  (3.21)

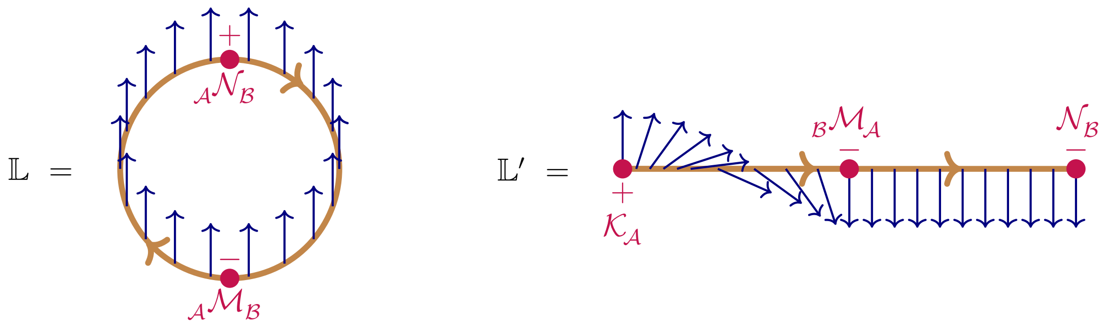
then the gluing category is the corresponding κ-framed center introduced in Definition 3.4:

$$\mathrm{T}(\mathbb{L}) := \mathcal{M}_1^{\epsilon_1} \overset{\kappa_1}{\boxtimes} \mathcal{M}_2^{\epsilon_2} \overset{\kappa_2}{\boxtimes} \mathcal{M}_3^{\epsilon_3} \overset{\kappa_3}{\boxtimes} \cdots . \tag{3.22}$$

(ii) If $\mathbb{L}$ is a disjoint union of connected defect one-manifolds $\mathbb{L}_i$, then $\mathrm{T}(\mathbb{L})$ is the Deligne product $\boxtimes_i \mathrm{T}(\mathbb{L}_i)$ of the κ-framed centers $\mathrm{T}(\mathbb{L}_i)$.

Implicitly, the prescription for the gluing categories assigned to gluing intervals is completely determined by the prescription for gluing circles: just regard left and right $\mathcal{A}$-modules as $\mathcal{A}$-vect- and as vect-$\mathcal{A}$-bimodules, respectively. In the sequel we will tacitly make this identification whenever convenient.
As an illustration, consider the following situations.

Example 3.10. Consider the following defect one-manifolds $\mathbb{L}$ and $\mathbb{L}'$:



(3.23)

Here the two intervals of $\mathbb{L}$ both have index +1. Accordingly we associate to $\mathbb{L}$ the category

$$\mathrm{T}(\mathbb{L}) = \overline{{}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}} \overset{1}{\boxtimes} {}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}} \overset{1}{\boxtimes} \ . \tag{3.24}$$

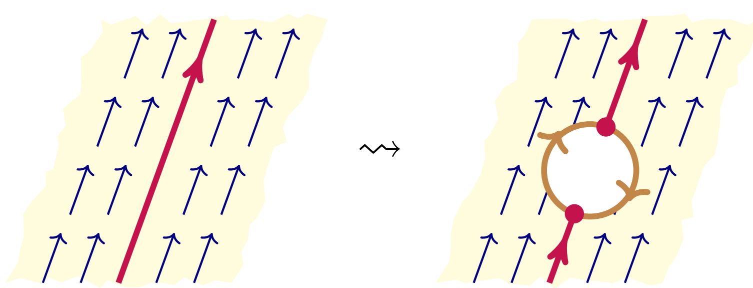
Thus we deal with the two balancings $\overline{a.m} \boxtimes n \cong \overline{m} \boxtimes {}^{\vee}a.n$ and $\overline{m} \boxtimes n.b \simeq \overline{m.{}^{\vee}b} \boxtimes n$. In the case of $\mathbb{L}'$, the first segment has index -1 and we have

$$\mathrm{T}(\mathbb{L}') = \mathcal{K}_{\mathcal{A}} \overset{-1}{\boxtimes} \overline{{}_{\mathcal{B}}\mathcal{M}_{\mathcal{A}}} \overset{0}{\boxtimes} \overline{\mathcal{N}_{\mathcal{B}}}\,. \tag{3.25}$$

In view of Lemma 3.8 our prescription for associating categories to defect one-manifolds can be concisely summarized as follows:

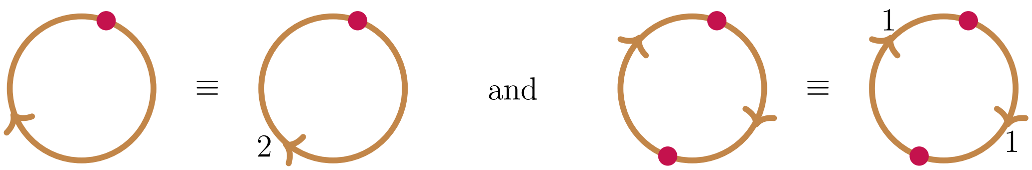
- Consider the Deligne product of the (bi)module categories involved in a composable string.
- On the so obtained category there is a comonad, specified by the values of the indices, whose comodules describe the relevant balancings. Take the category of comodules over this comonad.

The defect one-manifold $\mathbb{L}$ in Example 3.10 and its variant with reversed orientation actually play a fundamental role: they arise in particular when one modifies a defect surface locally by replacing a piece of line defect by a new gluing circle that is regarded as an incoming or outgoing boundary circle, respectively; pictorially, in the case of $\mathbb{L}$ we have



(3.26)

In this situation the new gluing circle inherits a 2-framing, and this is precisely the one of $\mathbb{L}$ in Example 3.10. Similarly, when a defect line ends at a gluing circle with a single defect point and the vector field is analogous to the one in (3.26), the 2-framing of that gluing circle (when suitably oriented) has index 2. This observation justifies the following convention, to be used in the sequel throughout: *If to a gluing circle with a single defect point no index label is attached, then it is meant to constitute a circle of index* $+2$*; any other gluing segment to which no index label is attached is meant to constitute a segment of index* $+1$. In pictures,



(3.27)

It remains to describe the category for the opposite, in the sense of Definition 2.11, of a defect one-manifold:

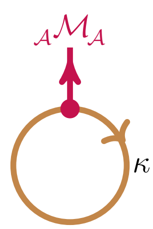
Lemma 3.11. The gluing category assigned to the opposite of a defect one-manifold is the opposite category, i.e. we have

$$\mathrm{T}(\overline{\mathbb{L}}) = \big(\mathrm{T}(\mathbb{L})\big)^{\mathrm{opp}} \tag{3.28}$$

for any defect one-manifold $\mathbb{L}$.

Proof. In view of the definition of the opposite manifold, it is sufficient to understand what happens for a single gluing segment (i.e. the situation displayed e.g. in picture (3.7)). In that situation one deals with the balanced product $\mathcal{M}_{\mathcal{A}} \overset{\kappa}{\boxtimes} {}_{\mathcal{A}}\mathcal{N}$ of just two factors, and for such a product the statement boils down to the isomorphism that was already observed in formula (3.8). ☐

Example 3.12. The κ-twisted center appears as a specific gluing category – it is the category assigned to a circle with one defect point and framing index κ, for κ even, i.e.

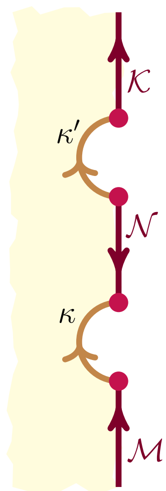
$\mathrm{T}\Big($  $\Big) = \mathcal{Z}^{\kappa}({}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}})\,.$ (3.29)

3.5 Canonical equivalences between gluing categories

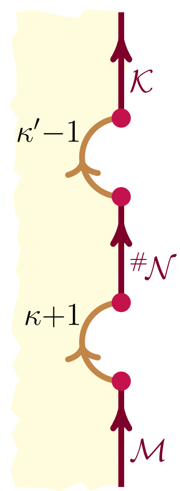
The following considerations can be used to considerably reduce the number of defect one-manifolds that we have to examine in detail: We analyze what happens when the orientation of a free boundary segment is flipped, and when two neighboring defect points are fused. Concerning the former issue we have

Proposition 3.13. Let $\mathcal{M}$ and $\mathcal{K}$ be left modules over a finite tensor category $\mathcal{A}$. Let $\mathcal{N}$ be a right $\mathcal{A}$-module, and denote by ${}^{\#}\mathcal{N} \equiv {}^{1}\overline{\mathcal{N}}$ the left $\mathcal{A}$-module with 1-twisted left action (in the convention of (3.1)). Up to canonical equivalence, the gluing category associated to a defect one-manifold does not change if the orientation of a free boundary segment labeled by

$\mathcal{N}$ is flipped, in the sense that $\mathcal{N}$ is replaced by ${}^{\#}\mathcal{N}$ and simultaneously the orientation of the boundary segment is inverted, corresponding to locally replacing the situation



by



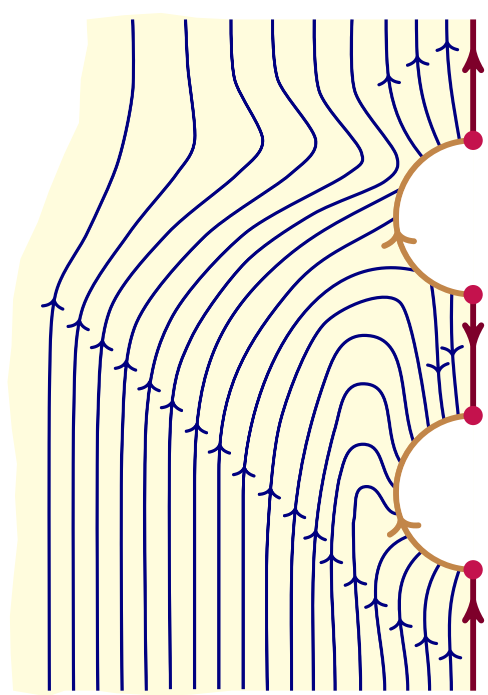
(3.30)

Proof. By definition, the action of $\mathcal{A}$ on the left $\mathcal{A}$-module ${}^{\#}\mathcal{N}$ is determined through its action on $\mathcal{N}$ by $a\,.\,\overline{n} := \overline{n.{}^{\vee}a}$. For the lower gluing segment in (3.30) we thus have an equivalence $\overline{{}_{\mathcal{A}}\mathcal{M}} \overset{\kappa}{\boxtimes} \overline{\mathcal{N}_{\mathcal{A}}} \simeq \overline{{}_{\mathcal{A}}\mathcal{M}} \overset{\kappa+1}{\boxtimes} {}^{1}_{\mathcal{A}}\overline{\mathcal{N}}$: in both categories the balancing is

$$\overline{a.x_{\mathcal{M}}} \boxtimes \overline{x_{\mathcal{N}}} \cong \overline{x_{\mathcal{M}}} \boxtimes {}^{[\kappa+1]}a.\overline{x_{\mathcal{N}}} = \overline{x_{\mathcal{M}}} \boxtimes \overline{x_{\mathcal{N}}.{}^{[\kappa+2]}a}\,. \tag{3.31}$$

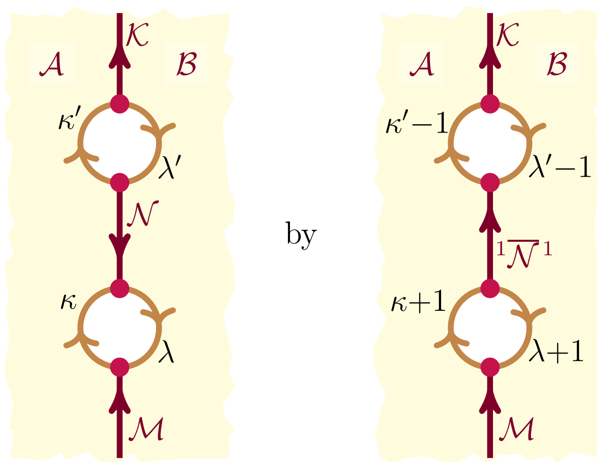
Similarly for the upper gluing segment there is an equivalence $\mathcal{N}_{\mathcal{A}} \overset{\kappa'}{\boxtimes} {}_{\mathcal{A}}\mathcal{K} \simeq \overline{{}^{1}_{\mathcal{A}}\overline{\mathcal{N}} \overset{\kappa'-1}{\boxtimes} {}_{\mathcal{A}}\mathcal{K}}$. □

As an illustration, the following picture shows the framing with indices $\kappa = 0$ and $\kappa' = 2$ on the left hand side of (3.30) that results in the straight framing on the right hand side:



(3.32)

Corollary 3.14. Let $\mathcal{A}$ and $\mathcal{B}$ be finite tensor categories. Let $\mathcal{M}$ and $\mathcal{K}$ be $\mathcal{A}$-$\mathcal{B}$-bimodules and $\mathcal{N}$ a $\mathcal{B}$-$\mathcal{A}$-bimodule. If we locally replace the situation

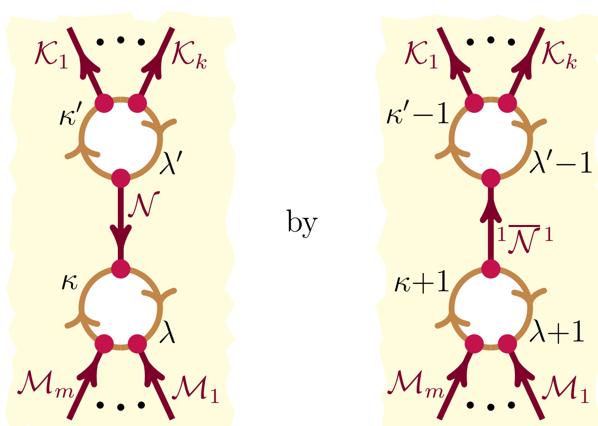


(3.33)

then up to canonical equivalence, the gluing category associated with the disjoint union of the two defect one-manifolds remains unchanged.

Proof. A statement analogous to Proposition 3.13 holds for right modules. Combining the two results for free boundary segments immediately gives the stated result for the flip of the orientation of a defect is line. □

Also note that in (3.33) it is inessential that the gluing circles have only two defect points. Indeed we can likewise replace the situation



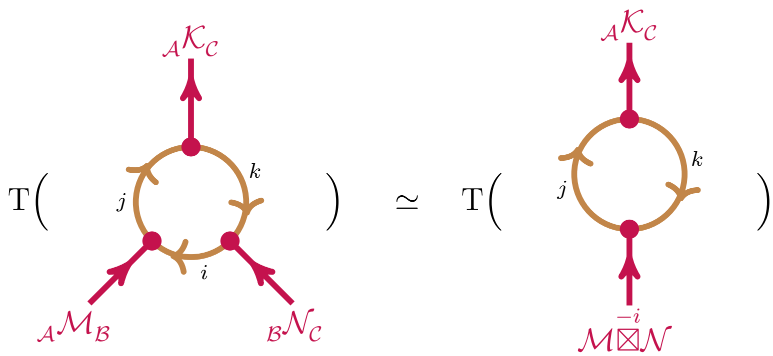
(3.34)

for any numbers m of incoming and k of outgoing defect points.

Furthermore, by applying Proposition 3.13 twice we see that the same gluing category is obtained when changing the framing in such a way that indices κ, κ' on consecutive gluing segments along a defect one-manifold (as in (3.30)) get replaced by the pair $\kappa{+}2, \kappa'{-}2$ and the category labeling the defect line gets twisted with the corresponding double dual, thus e.g. replacing $\mathcal{N}$ in (3.30) by $\mathcal{N}^{-2}$) (see also Appendix A).

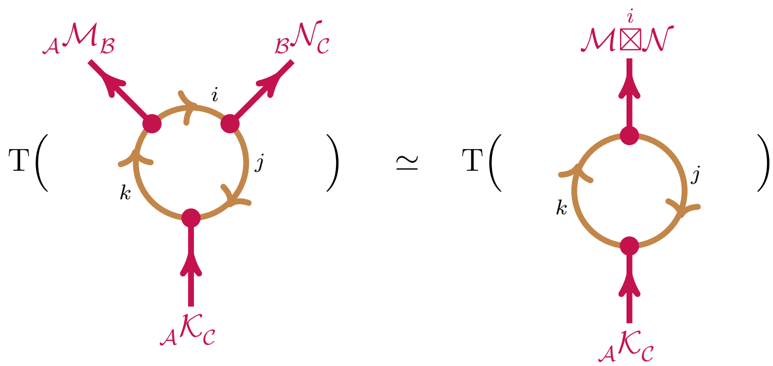
Next we describe the effect of 'fusing' two neighboring defect points on a defect one-manifold. In the pictures below we display – as we already did in the picture (3.29) – defect lines that are attached to defect points, even though for now we only deal with defect one-manifolds (this will be convenient when using our results later on to discuss the fusion of defect lines).

Proposition 3.15. For any triple (i, j, k) of framing indices there are canonical equivalences



(3.35)

and



(3.36)

of gluing categories.

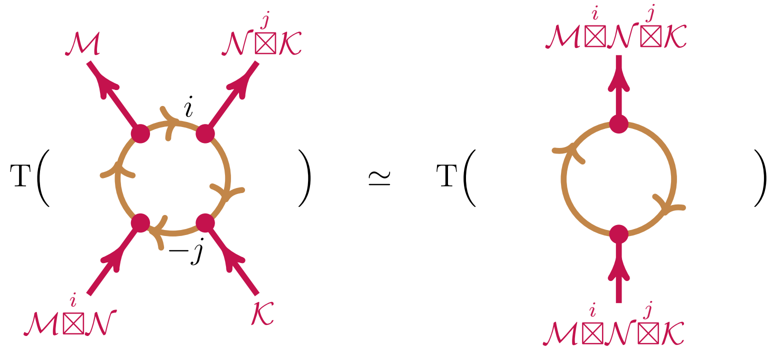
Proof. The category on the left hand side of (3.35) is $\mathcal{K}\overset{k}{\boxtimes}\overline{\mathcal{N}}\overset{i}{\boxtimes}\overline{\mathcal{M}}\overset{j}{\boxtimes}$, while the one on the right hand side is $\mathcal{K}\overset{k}{\boxtimes}[\mathcal{M}\overset{-i}{\boxtimes}\mathcal{N}]^{\mathrm{opp}}\overset{j}{\boxtimes}$. We have

$$\mathcal{K}\overset{k}{\boxtimes}\overline{\mathcal{N}}\overset{i}{\boxtimes}\overline{\mathcal{M}}\overset{j}{\boxtimes} \simeq \mathcal{K}\overset{k}{\boxtimes}(\overline{\mathcal{N}}\overset{i}{\boxtimes}\overline{\mathcal{M}})\overset{j}{\boxtimes} \simeq \mathcal{K}\overset{k}{\boxtimes}[\mathcal{M}\overset{-i}{\boxtimes}\mathcal{N}]^{\mathrm{opp}}\overset{j}{\boxtimes}, \tag{3.37}$$

where the first step uses the distributive law and the second step Lemma 3.11. Canonical functors realizing these equivalences are provided by the functors which come with the universal property (3.13) of the respective twisted centers. The proof of the equivalence (3.36) is analogous. □

These results about gluing categories tell us that the fusion of defect points provides a well defined operation for the associated categories. They also suggest that the same holds for defect *lines*, with the fusion of two defects labeled by bimodule categories ${}_{\mathcal{C}}\mathcal{M}_{\mathcal{A}}$ and ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}}$ and separated by a strip with winding i realizing a single defect labeled by the $\mathcal{C}$-$\mathcal{B}$-bimodule $\mathcal{M}\overset{-i}{\boxtimes}\mathcal{N}$. To see that this is indeed the case we will have to define the functors assigned to defect surfaces, not just the categories assigned to the gluing segments on their boundary; this will be done in Section 4.6. Analogous equivalences as in Proposition 3.15 hold for the gluing categories of circles with more than three defect points. We have for instance

Example 3.16. There is a canonical equivalence



(3.38)

Indeed, both gluing categories are equivalent to the category $\mathcal{L}ex_{\mathcal{A},\mathcal{B}}(\mathcal{M} \overset{i}{\boxtimes} \mathcal{N} \overset{j}{\boxtimes} \mathcal{K}\,, \mathcal{M} \overset{i}{\boxtimes} \mathcal{N} \overset{j}{\boxtimes} \mathcal{K})$.

As described in Appendix B.3, the Eilenberg-Watts equivalences that will be formulated in the next subsection can be lifted to twisted centers. A common feature of the gluing categories in Example 3.16 is that they are canonically equivalent to categories of endofunctors. Thereby each of these twisted centers contains a distinguished object, namely the one that corresponds to the respective identity functor. We refer to those objects as *distinguished fusion objects*.

3.6 Balanced pairings for bimodule categories

We denote, for finite categories $\mathcal{N}$ and $\mathcal{M}$, by $\mathcal{L}ex(\mathcal{N},\mathcal{M})$ and $\mathcal{R}ex(\mathcal{N},\mathcal{M})$ the finite category of left and right exact functors, respectively, from $\mathcal{N}$ to $\mathcal{M}$. It will be convenient to re-express the gluing categories associated with defect one-manifolds, which we have introduced via Deligne products, in terms of such functor categories. This is achieved with the help of a Morita invariant formulation of Eilenberg-Watts type results, which exhibits explicit equivalences from $\mathcal{L}ex(\mathcal{N},\mathcal{M})$ and $\mathcal{R}ex(\mathcal{N},\mathcal{M})$ to $\overline{\mathcal{N}} \boxtimes \mathcal{M}$. More specifically, there are (two-sided) adjoint equivalences [Sh, FSS2]

$$\begin{aligned} \Phi^{\mathrm{l}} \equiv \Phi^{\mathrm{l}}_{\mathcal{N},\mathcal{M}} :\ & \overline{\mathcal{N}} \boxtimes \mathcal{M} \xrightarrow{\simeq} \mathcal{L}ex(\mathcal{N},\mathcal{M})\,, && \overline{n} \boxtimes m \longmapsto \mathrm{Hom}_{\mathcal{N}}(n,-) \otimes m\,, \\ \Psi^{\mathrm{l}} \equiv \Psi^{\mathrm{l}}_{\mathcal{N},\mathcal{M}} :\ & \mathcal{L}ex(\mathcal{N},\mathcal{M}) \xrightarrow{\simeq} \overline{\mathcal{N}} \boxtimes \mathcal{M}\,, && F \longmapsto \int^{n\in\mathcal{N}} \overline{n} \boxtimes F(n)\,, \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} \Phi^{\mathrm{r}} \equiv \Phi^{\mathrm{r}}_{\mathcal{N},\mathcal{M}} :\ & \overline{\mathcal{N}} \boxtimes \mathcal{M} \xrightarrow{\simeq} \mathcal{R}ex(\mathcal{N},\mathcal{M})\,, && \overline{n} \boxtimes m \longmapsto \mathrm{Hom}_{\mathcal{N}}(-,n)^* \otimes m\,, \\ \Psi^{\mathrm{r}} \equiv \Psi^{\mathrm{r}}_{\mathcal{N},\mathcal{M}} :\ & \mathcal{R}ex(\mathcal{N},\mathcal{M}) \xrightarrow{\simeq} \overline{\mathcal{N}} \boxtimes \mathcal{M}\,, && G \longmapsto \int_{n\in\mathcal{N}} \overline{n} \boxtimes G(n)\,, \end{aligned} \tag{3.40}$$

where $(-)^*$ denotes the dual vector space. We refer to these pairs of adjoint functors as *Eilenberg-Watts functors*. These equivalences of categories give for any two left exact functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ a canonical isomorphism

$$\mathrm{Hom}(c, G\circ F(-)) \;\cong\; \int^{b\in\mathcal{B}} \mathrm{Hom}(c, G(b)) \otimes_{\Bbbk} \mathrm{Hom}(b, F(-))\,, \tag{3.41}$$

and analogously for right exact functors. Moreover, they are compatible with the structure of bimodule categories over finite tensor categories [FSS2, Sect. 4]. For instance, the (right exact) *Nakayama functor*

$$\mathrm{N^r} := \int^{m\in\mathcal{M}} \mathrm{Hom}_{\mathcal{M}}(-,m)^* \otimes m\,, \tag{3.42}$$

i.e. the image $\Phi^{\mathrm{r}}\circ\Psi^{\mathrm{l}}(\mathrm{id}_{\mathcal{M}})$ of the identity functor, regarded as a left exact functor, in $\mathcal{R}ex(\mathcal{M},\mathcal{M})$, is a bimodule functor from $\mathcal{M}$ to ${}^2\mathcal{M}^2$ [FSS2, Thm. 4.5], and similarly for the left exact Nakayama functor $\mathrm{N^l} = \Phi^{\mathrm{l}}\circ\Psi^{\mathrm{r}}(\mathrm{id}_{\mathcal{M}})$.

For later use we collect some properties of the left exact Nakayama functor of a finite category $\mathcal{M}$. We have

$$\int_{m\in\mathcal{M}} \overline{m}\boxtimes m \cong \Psi^{\mathrm{r}}(\mathrm{id}_{\mathcal{M}}) \cong \Psi^{\mathrm{l}}\circ\Phi^{\mathrm{l}}\circ\Psi^{\mathrm{r}}(\mathrm{id}_{\mathcal{M}}) \cong \Psi^{\mathrm{l}}(\mathrm{N^l}) = \int^{m\in\mathcal{M}} \overline{m}\boxtimes \mathrm{N^l}(m)\,. \tag{3.43}$$

Further, in case $\mathcal{M} = \mathcal{A}$ is a finite tensor category, we can express $\mathrm{N^l}$ as [FSS2, Lemma 4.10] $\mathrm{N^l}(a) = D_{\mathcal{A}} \otimes {}^{\vee\vee}a$ with $D_{\mathcal{A}}$ the distinguished invertible object [ENO, Def. 3.1] of $\mathcal{A}$, and thus

$$\int_{a\in\mathcal{A}} a^{\vee\vee}\boxtimes \overline{D_{\mathcal{A}}\otimes a} \;\cong \int^{a\in\mathcal{A}} a\boxtimes\overline{a} \tag{3.44}$$

and

$$\int_{a\in\mathcal{A}} \overline{a}\boxtimes a \;\cong \int^{a\in\mathcal{A}} \overline{a}\boxtimes D_{\mathcal{A}}\otimes {}^{\vee\vee}a\,. \tag{3.45}$$

By the cagegory-theoretic version of Radford's S^4-theorem, the invertible object $D_{\mathcal{A}}$ comes [ENO, Thm 3.3] with coherent isomorphisms $D_{\mathcal{A}}\otimes a \cong a^{\vee\vee\vee\vee}\otimes D_{\mathcal{A}}$ and can thus be regarded canonically as an object of the -2-twisted center,

$$D_{\mathcal{A}} \in \mathcal{Z}^{-2}(\mathcal{A})\,. \tag{3.46}$$

It follows that acting with $D_{\mathcal{A}}$ is an equivalence

$$D_{\mathcal{A}}\,.\,-: \quad {}^{\kappa+4}\mathcal{M} \xrightarrow{\;\simeq\;} {}^{\kappa}\mathcal{M} \tag{3.47}$$

of $\mathcal{A}$-modules for $\kappa\in 2\mathbb{Z}$. Hence there is a distinguished 4-periodicity in the family of bimodule categories ${}^{\kappa_1}\mathcal{M}^{\kappa_2}$ indexed by $(\kappa_1,\kappa_2)\in 2\mathbb{Z}\times 2\mathbb{Z}$, and similarly for odd κ. Moreover,

$$a^{[\kappa]}\otimes D_{\mathcal{A}}^{\otimes n} \cong D_{\mathcal{A}}^{\otimes n}\otimes a^{[\kappa-4n]} \tag{3.48}$$

for all $\kappa, n\in\mathbb{Z}$, and there is a canonical isomorphism $D_{\mathcal{A}}^{\vee\vee}\cong D_{\mathcal{A}}$.

The Eilenberg-Watts equivalences allow us to switch back and forth between Deligne products and categories of half-exact functors and thereby to understand features of the former type of categories in terms of the latter, and vice versa. One application that will turn out to be crucial below (see e.g. the calculation needed in Example 4.10) is the following. Every left exact module functor $F\colon \mathcal{N}_{\mathcal{A}}\to\mathcal{M}_{\mathcal{A}}$ yields a balanced pairing

$$\mathrm{Hom}_{\mathcal{M}}(-,F(-)): \quad \overline{\mathcal{M}_{\mathcal{A}}}\boxtimes\mathcal{N}_{\mathcal{A}} \to \mathrm{vect}\,, \tag{3.49}$$

with the balancing obtained by combining the module structure of F and the duality of $\mathcal{A}$: $\mathrm{Hom}(n, F(m.a)) \cong \mathrm{Hom}(n.{}^{\vee}a, F(m))$. This balancing is coherent with respect to the monoidal structure of $\mathcal{A}$. Moreover, since F ist left exact, the Eilenberg-Watts equivalence (3.39) yields a natural isomorphism $\mathrm{Hom}_{\overline{\mathcal{N}}\boxtimes\mathcal{M}}(\overline{n}\boxtimes m, \Psi^{\mathrm{l}}(F)) \cong \mathrm{Hom}_{\mathcal{N}}(n, F(m))$, i.e. the pairing (3.49) is representable by the object $\Psi^{\mathrm{l}}(F)$. The balancing of the pairing transports to a balancing of the representing object, thus yielding the following result, which one may think of as specific kinds of substitution of variables rules for coends and ends:

Lemma 3.17. For a left exact bimodule functor $F\colon \mathcal{M} \to \mathcal{N}$ between $\mathcal{B}$-$\mathcal{A}$-bimodules $\mathcal{M}$ and $\mathcal{N}$, the coend $\Psi^{\mathrm{l}}(F) = \int^{m\in\mathcal{M}} \overline{m} \boxtimes F(m) \in \overline{\mathcal{M}} \boxtimes \mathcal{N}$ comes with coherent isomorphisms

$$\begin{aligned} \int^{m\in\mathcal{M}} \overline{m} \boxtimes F(m).a &\cong \int^{m\in\mathcal{M}} \overline{m\,.{}^{\vee}a} \boxtimes F(m) \qquad \text{and} \\ \int^{m\in\mathcal{M}} \overline{m} \boxtimes b.F(m) &\cong \int^{m\in\mathcal{M}} \overline{b^{\vee}.\,m} \boxtimes F(m)\,. \end{aligned} \tag{3.50}$$

Analogously, for a right exact bimodule functor $G\colon \mathcal{M} \to \mathcal{N}$ the end $\Psi^{\mathrm{r}}(G) = \int_{m\in\mathcal{M}} \overline{m} \boxtimes G(m)$ is equipped with coherent isomorphisms

$$\begin{aligned} \int_{m\in\mathcal{M}} \overline{m} \boxtimes G(m).a &\cong \int_{m\in\mathcal{M}} \overline{m\,.\,a^{\vee}} \boxtimes G(m) \qquad \text{and} \\ \int_{m\in\mathcal{M}} \overline{m} \boxtimes b.G(m) &\cong \int_{m\in\mathcal{M}} \overline{{}^{\vee}b.\,m} \boxtimes G(m)\,. \end{aligned} \tag{3.51}$$

Proof. With the help of the representing object $\Psi^{\mathrm{l}}(F)$ we define the balancing by the requirement that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\overline{\mathcal{M}}\boxtimes\mathcal{N}}(\overline{m.a} \boxtimes n, \Psi^{\mathrm{l}}(F)) & \longrightarrow & \mathrm{Hom}_{\mathcal{N}}(n, F(m.a)) \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_{\overline{\mathcal{M}}\boxtimes\mathcal{N}}(\overline{m} \boxtimes n.{}^{\vee}a, \Psi^{\mathrm{l}}(F)) & \longrightarrow & \mathrm{Hom}_{\mathcal{N}}(n.{}^{\vee}a, F(m)) \end{array} \tag{3.52}$$

as well as the corresponding diagram for the left action commute. This produces directly the isomorphisms in (3.50). The case of a right exact bimodule functor is shown analogously via its representing property of the dual Hom functor. ☐

From this statement we obtain two types of balancings for the identity bimodule functor. We record them for later use:

$$\int_{m\in\mathcal{M}} m.a \boxtimes \overline{m} \cong \int_{m\in\mathcal{M}} m \boxtimes \overline{m.a^{\vee}} \qquad \text{and} \qquad \int_{m\in\mathcal{M}} b.m \boxtimes \overline{m} \cong \int_{m\in\mathcal{M}} m \boxtimes \overline{{}^{\vee}b.m}\,. \tag{3.53}$$

By applying the Eilenberg-Watts equivalence Φ^{l} from (3.39) to the result in Lemma 3.17 we get

Corollary 3.18. For bimodule categories ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}}$, the Eilenberg-Watts equivalences induce an equivalence

$$\overline{\mathcal{M}} \overset{1}{\boxtimes} \mathcal{N} \overset{1}{\boxtimes} \;\simeq\; \mathcal{L}ex_{\mathcal{A},\mathcal{B}}(\mathcal{M}, \mathcal{N})\,. \tag{3.54}$$

of categories.

In particular, for the regular bimodule category ${}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}$ we obtain

$$\overline{\mathcal{A}} \overset{1}{\boxtimes} \mathcal{A} \overset{1}{\boxtimes} \;\simeq\; \mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A}) \;\simeq\; \mathcal{Z}(\mathcal{A})\,. \tag{3.55}$$

Remark 3.19. Using the equivalence (3.40) instead of (3.39), we could as well have expressed framed centers through categories of right exact instead of left exact module functors. The resulting formulas would, however, be somewhat more complicated than (3.54), owing to the additional occurrence of double duals. For instance, $\overline{\mathcal{M}} \overset{1}{\boxtimes} \mathcal{N} \overset{1}{\boxtimes} \simeq \mathcal{R}ex_{\mathcal{A},\mathcal{B}}(\mathcal{M}, {}^{2}\mathcal{N}^{2})$.

Remark 3.20. The equivalences among framed centers obtained in formula (3.14) can be combined with other results so as to yield various further distinguished equivalences. Specifically, together with Corollary 3.18 we arrive at distinguished equivalences

$$\overline{\mathcal{M}} \overset{\kappa}{\boxtimes} \mathcal{N} \overset{\kappa'}{\boxtimes} \;\simeq\; \mathcal{L}ex_{\mathcal{A},\mathcal{B}}({}^{1-\kappa}\mathcal{M}^{1-\kappa'},\mathcal{N}) \;\simeq\; \mathcal{L}ex_{\mathcal{A},\mathcal{B}}(\mathcal{M}, {}^{\kappa-1}\mathcal{N}^{\kappa'-1})\,. \tag{3.56}$$

4 Assigning functors to defect surfaces

Our construction of the the framed modular functor on the level of 1-morphisms involves several different collections of left exact functors and proceeds in three steps: We first introduce, in Section 4.1, auxiliary functors $\mathrm{T}^{\mathrm{pre}}(\Sigma)$, to which we refer as *pre-block* functors. In Section 4.3, in a second step we impose constraint to construct *fine block* functors $\mathrm{T}_{\mathrm{fine}}(\Sigma)$ from the pre-block functors. As indicated by the terminology this procedure only makes sense if the surface Σ is fine in the sense of Definition 2.1. To get the actual *block* functors $\mathrm{T}(\Sigma)$ – which then furnish a modular functor in the sense of Definition 2.12 – for arbitrary, not necessarily fine, surfaces we introduce a notion of *refinement* of a surface (Section 5.3) and obtain $\mathrm{T}(\Sigma)$ as a limit over all refinements of Σ (Section 5.4).

The functors $\mathrm{T}^{\mathrm{pre}}(\Sigma)$, $\mathrm{T}_{\mathrm{fine}}(\Sigma)$ and $\mathrm{T}(\Sigma)$ are, a priori, functors from $\mathrm{T}(\partial_-\Sigma)$ to $\mathrm{T}(\partial_+\Sigma)$, with $\partial_-\Sigma \sqcup -\partial_+\Sigma = \partial_{\mathrm{glue}}\Sigma$. However, as explained in Appendix B.1 (see Equation (B.5)), for finite categories we have equivalences

$$\mathcal{L}ex(\mathcal{M},\mathcal{N}) \xrightarrow{\;\simeq\;} \mathcal{L}ex(\mathcal{M}\boxtimes\overline{\mathcal{N}},\mathrm{vect})\,. \tag{4.1}$$

Accordingly, we will focus our attention to the case that $\partial_+\Sigma = \emptyset$, in which we deal with functors from $\mathrm{T}(\partial_{\mathrm{glue}}\Sigma)$ to vect. The general case is then obtained directly by invoking the equivalences (4.1).

4.1 Pre-block functors

To obtain the pre-block functor for a defect surface Σ with $\partial_+\Sigma = \emptyset$ we start from a Hom functor whose covariant arguments come from the gluing categories for the gluing boundaries of Σ. The Hom functor pairs every such covariant variable with a contravariant one; we take a coend over each of the latter variables.

In more detail, the pre-block functors are constructed as follows. The boundary $\partial_{\mathrm{glue}}\Sigma$ is a defect one-manifold. According to (3.22) the gluing category $\mathrm{T}(\mathbb{L})$ assigned to a connected defect one-manifold $\mathbb{L}$ is a κ-framed center of the form $\mathcal{M}_1^{\epsilon_1} \overset{\kappa_1}{\boxtimes} \mathcal{M}_2^{\epsilon_2} \overset{\kappa_2}{\boxtimes} \mathcal{M}_3^{\epsilon_3} \overset{\cdots}{\boxtimes} \cdots$. We denote by $\mathrm{U}(\mathbb{L})$ the category obtained from the gluing category for a defect one-manifold $\mathbb{L}$ by forgetting the balancings of the framed center, and by $U_{\mathbb{L}}\colon \mathrm{T}(\mathbb{L}) \to \mathrm{U}(\mathbb{L})$ the corresponding forgetful functor, i.e. $\mathrm{U}(\mathbb{L}) = \mathcal{M}_1^{\epsilon_1} \boxtimes \mathcal{M}_2^{\epsilon_2} \boxtimes \cdots$. Specifically, we consider the forgetful functor $U\colon \mathrm{T}(\partial_{\mathrm{glue}}\Sigma) \longrightarrow \mathrm{U}(\partial_{\mathrm{glue}}\Sigma)$; U is an exact functor. Now note that, by construction, in the image of U each of the categories $\mathcal{M}_i$ assigned to an edge appears 'twice' – that is, once as the category itself and once as its opposite – namely once at either end of the connecting defect line. Thus we have $\mathrm{U}(\partial_{\mathrm{glue}}\Sigma) = \boxtimes_i \mathcal{M}_i \boxtimes \boxtimes_i \overline{\mathcal{M}_i}$. We can then give

Definition 4.1. The *pre-block functor* assigned to a defect surface Σ is the left exact functor

$$\mathrm{T}^{\mathrm{pre}}(\Sigma): \quad \mathrm{T}(\partial_{\mathrm{glue}}\Sigma) \to \mathrm{vect} \tag{4.2}$$

from the gluing category associated with the boundary of Σ to the category of vector spaces that is constructed in the following manner: For each factor $\mathcal{M}_i$ in $\mathrm{T}(\partial_{\mathrm{glue}}\Sigma)$ we insert an object $m_i \boxtimes \overline{m_i} \in \mathcal{M}_i \boxtimes \overline{\mathcal{M}_i}$ as a contravariant variable in a Hom functor and take the coend

$$\mathrm{T}^{\mathrm{pre}}(\Sigma)(-) := \int^{m_1,m_2,\dots,m_n} \mathrm{Hom}(m_1 \boxtimes \overline{m}_1 \boxtimes \cdots \boxtimes m_n \boxtimes \overline{m}_n \,, U(-)) \tag{4.3}$$

over these variables in the finite category of left exact functors; here the contravariant and covariant arguments of the Hom functor are matched according to the combinatorial configuration of Σ.

It follows directly from the definition that the pre-block functors depend on the defect surface Σ only via the incidence combinatorics of the gluing and free boundaries and defect lines of Σ , and that they satisfy

$$\mathrm{T}^{\mathrm{pre}}(\Sigma \sqcup \Sigma') = \mathrm{T}^{\mathrm{pre}}(\Sigma) \boxtimes \mathrm{T}^{\mathrm{pre}}(\Sigma') \,. \tag{4.4}$$

Remark 4.2. We insist on working with *left exact* functors; in particular, as explained in Appendix B.1, ends and coends are taken in categories of left exact functors (which are finite categories). Categories of left exact functors between finite tensor categories have a well-defined Deligne product, so that in particular the Deligne product on the right hand side of (4.4) exists. Alternatively, we could work with right exact functors only and the Deligne product. For a finite category $\mathcal{M}$, the Hom functor $\mathrm{Hom}\colon \overline{\mathcal{M}} \boxtimes \mathcal{M} \to \mathrm{vect}$ is left exact, while the 'dual Hom' functor

$$\begin{aligned} \widetilde{\mathrm{Hom}}: \quad \overline{\mathcal{M}} \boxtimes \mathcal{M} &\to \mathrm{vect}\,, \\ m \boxtimes n &\mapsto \mathrm{Hom}(n,m)^* \end{aligned} \tag{4.5}$$

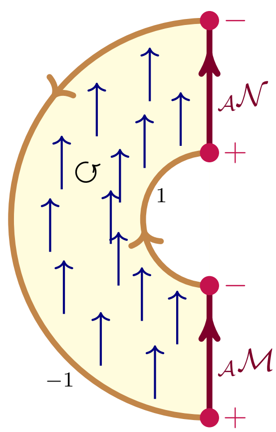
(with the star denoting the vector space dual) is right exact. Accordingly, right exact pre-block functors can be defined by the end

$$\widetilde{\mathrm{T}}^{\mathrm{pre}}(\Sigma)(-) := \int_{m_1,m_2,\dots,m_n} \widetilde{\mathrm{Hom}}(U(-), m_1 \boxtimes \overline{m}_1 \boxtimes \cdots \boxtimes m_n \boxtimes \overline{m}_n) \,. \tag{4.6}$$

Thus, in agreement with the Eilenberg-Watts equivalences (3.39) and (3.40), working with right exact functors instead of left exact ones boils down to replacing coends and Hom functors by ends and dual Hom functors.

Remark 4.3. Taking the coend makes it legitimate to refer to these variables as *state-sum variables*. Indeed, in case that the categories are finitely semisimple, this reduces to a sum over isomorphism classes of simple objects (or of "spins", in the parlance of part of the physics literature). Accordingly we refer to coends of the form appearing in (4.3) also as *state-sum coends* and regard our prescription as a state-sum construction. This fits with the fact that in state-sum models, vector spaces associated to closed surfaces, also called *block spaces* or spaces of *conformal blocks*, are constructed as subspaces of auxiliary vector spaces that need to be introduced first; we call the latter pre-block spaces.
Such a two-step procedure is also the basis of the use of state-sum models in the construction of quantum codes where, however, typically a space bigger than the pre-block space is used. In that case, the essential idea is to obtain the block space for a surface Σ as the image of the projector that the three-dimensional Turaev-Viro topological field theory assigns to the cylinder $\Sigma \times [-1, 1]$. In our construction we impose instead the condition of flat holonomy for every contractible 2-patch of a defect surface, see Section 4.3.

Example 4.4. Consider a defect surface $\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}}$ whose underlying surface is a disk, with two free boundary intervals labeled by $\mathcal{A}$-modules ${}_{\mathcal{A}}\mathcal{M}$ and ${}_{\mathcal{A}}\mathcal{N}$ for some finite tensor category $\mathcal{A}$ and two gluing intervals, and with the 2-framing given by the constant vector field pointing in the direction of the two free boundary intervals (and thus, as needed, parallel to them):

$\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}} =$  (4.7)

Here, and in similar pictures below, we draw the gluing segments as half-circles, to remind of the fact that they amount to the presence of a 'boundary insertion', which often is indicated by removing a half-disk from a two-manifold with boundary. (For instance, the picture above is 'half' of the picture (2.2). Also recall that we are allowing for smooth manifolds with corners.) Denoting the gluing interval on the right hand side of (4.7) by $\mathbb{L}_1$ and the left one by $\mathbb{L}_2$, the relevant gluing categories are

$$\mathrm{T}(\mathbb{L}_1) = \overline{\mathcal{M}} \overset{1}{\boxtimes} \mathcal{N} \qquad \text{and} \qquad \mathrm{T}(\mathbb{L}_2) = \overline{\mathcal{N}} \overset{-1}{\boxtimes} \mathcal{M} \,. \tag{4.8}$$

The resulting pre-block functor is given by

$$x \boxtimes y \;\longmapsto \int^{m\in\mathcal{M},n\in\mathcal{N}} \mathrm{Hom}_{\overline{\mathcal{M}}\boxtimes\mathcal{N}\boxtimes\overline{\mathcal{N}}\boxtimes\mathcal{M}}(\overline{m} \boxtimes n \boxtimes \overline{n} \boxtimes m, \overline{x}_m \boxtimes x_n \boxtimes \overline{y}_n \boxtimes y_m) \tag{4.9}$$

with $x = \overline{x}_m \boxtimes x_n \in \mathrm{T}(\mathbb{L}_1)$ and $y = \overline{y}_n \boxtimes y_m \in \mathrm{T}(\mathbb{L}_2)$. Using the convolution property (B.2) of the Hom functor, this can be simplified to

$$\mathrm{T}^{\mathrm{pre}}(\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}})(x \boxtimes y) = \mathrm{Hom}_{\mathcal{M}}(x_m, y_m) \otimes \mathrm{Hom}_{\mathcal{N}}(y_n, x_n)\,. \tag{4.10}$$

Also note that by Proposition B.5(ii) there is a distinguished equivalence $\mathrm{T}(\mathbb{L}_1) \simeq \mathcal{L}ex_{\mathcal{A}}(\mathcal{M},\mathcal{N})$ mapping $x \in \mathrm{T}(\mathbb{L}_1)$ to the left exact functor $\Phi^{\mathrm{l}}(x)$, while by Lemma 3.11 we have an equivalence $\mathrm{T}(\mathbb{L}_2) \simeq \mathrm{T}(\mathbb{L}_1)^{\mathrm{opp}}$. Under these equivalences the pre-block space becomes

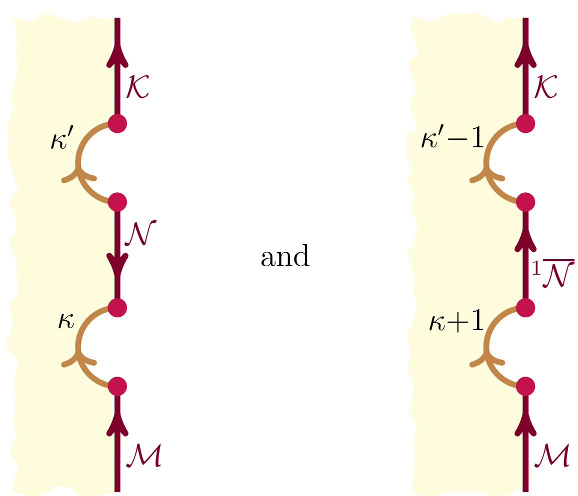
$$\mathrm{T}^{\mathrm{pre}}(\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}})(x \boxtimes y) \cong \mathrm{Nat}(\Phi^{\mathrm{l}}(\overline{y}), \Phi^{\mathrm{l}}(x))\,. \tag{4.11}$$

As will become clear in Corollary 4.23 below, the defect surface $\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}}$ (4.7) considered in Example 4.4 is indeed the most basic surface for us. In view of the particular form of the framing vector field on this surface, we will refer to $\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}}$ as the *straight disk*.

The results (4.10) and (4.11) express the pre-block functor as a Deligne product of Hom functors and as natural transformations, respectively. This is no coincidence, but is a generic feature of the construction, which is a first hint at the power of the modular functor to produce algebraically interesting quantities. To pinpoint this issue, let us also have a look at slightly more complicated surfaces.

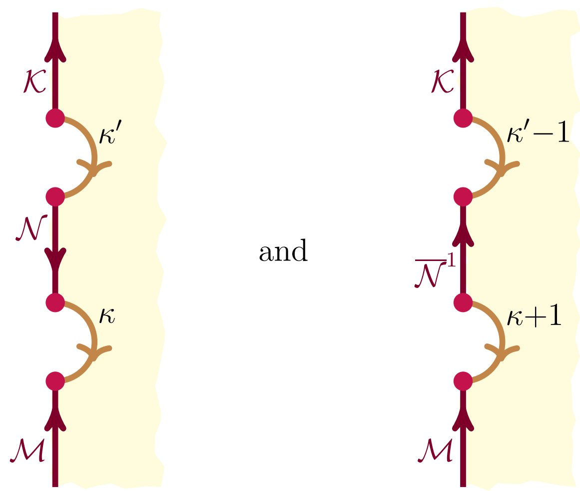
It is worth pointing out that the framing enters the definition of the pre-block only via the gluing categories. Thus, as a direct consequence of the canonical equivalence of gluing categories in Proposition 3.13, we obtain

Lemma 4.5. There is a canonical isomorphism between the pre-block functors for the configurations



(4.12)

involving a local change of the framing, as well as, more generally, for $z \in 2\mathbb{Z}+1$ an isomorphism with $\kappa+z$, $\kappa'-z$ and ${}^{2-z}\overline{\mathcal{N}}$ in place of $\kappa+1$, $\kappa'-1$ and ${}^{1}\overline{\mathcal{N}}$ on the right hand side. Similarly, there is a canonical isomorphism between the pre-block functors for the configurations



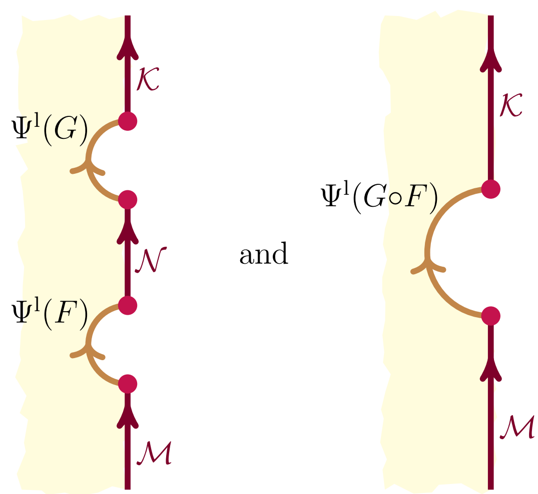
(4.13)

Our construction of pre-blocks is compatible with composition of left exact functors via the Eilenberg-Watts correspondence

$$\Psi^{\mathrm{l}}: \quad \mathcal{L}ex(\mathcal{M},\mathcal{N}) \rightarrow \overline{\mathcal{M}} \boxtimes \mathcal{N}\,. \tag{4.14}$$

Explicitly we have the following 'fusion of boundary insertions':

Proposition 4.6. There is a canonical isomorphism between the pre-block functors for the configurations



(4.15)

involving a local replacement of segments around a disk, with the framing near the segments being along the positive y-axis.

Proof. Consider left exact bimodule functors $F\colon \mathcal{M} \rightarrow \mathcal{N}$ and $G\colon \mathcal{N} \rightarrow \mathcal{K}$ for $\mathcal{B}$-$\mathcal{A}$-bimodules

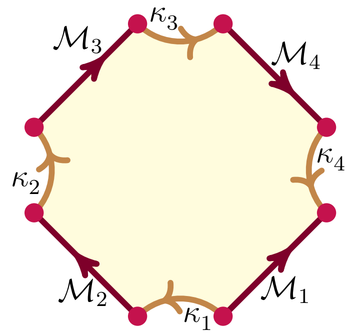
$\mathcal{M},\mathcal{N},\mathcal{K}$. The composite of the isomorphisms (see [FSS2, Cor. 3.7])

$$\begin{aligned}\int^{n\in\mathcal{N}} &\mathrm{Hom}(\overline{m}\boxtimes n\boxtimes\overline{n}\boxtimes k,\Psi^{\mathrm{l}}(F)\boxtimes\Psi^{\mathrm{l}}(G))\\ &\cong\int^{n\in\mathcal{N}}\mathrm{Hom}(\overline{m}\boxtimes n,\Psi^{\mathrm{l}}(F))\otimes\mathrm{Hom}(\overline{n}\boxtimes k,\Psi^{\mathrm{l}}(G))\\ &\cong\int^{n\in\mathcal{N}}\mathrm{Hom}(n,F(m))\otimes\mathrm{Hom}(k,G(n))\stackrel{(3.41)}{\cong}\mathrm{Hom}(k,G{\circ}F(m))\end{aligned} \tag{4.16}$$

provides the desired isomorphism between the pre-block functors for the left and right hand sides of (4.15). □

Proposition 4.6 illustrates how the modular functor realizes algebraic structures: the fusion of boundary insertions provides the composition of functors and thus, via the Eilenberg-Watts calculus, a composition on the Deligne product $\overline{\mathcal{M}}\boxtimes\mathcal{N}$. In particular, for any module category $\mathcal{M}$ it yields a distinguished object in $\overline{\mathcal{M}}\boxtimes\mathcal{M}$, namely the one that acts like a unit for the type of local replacement considered in (4.15).

Example 4.7. The generalization of Example 4.4 to a disk with any number N of gluing and free boundary segments, with the former oriented as induced by the orientation of the disk and the latter of arbitrary orientation, and with any indices is immediate. First, by invoking Proposition 3.13 we can restrict our attention to any specific choice of orientations of the free boundary segments, say one of them oriented clockwise and all others counter-clockwise, as indicated for $N=4$ in the picture



(4.17)

(with a left $\mathcal{A}$-module $\mathcal{M}_1$ and right $\mathcal{A}$-modules $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$, for some finite tensor category $\mathcal{A}$), Next we can use Proposition 4.6 to reduce the number of gluing and free boundary segments by one, at the same time composing the functors that the Eilenberg-Watts equivalence assigns to the the objects at the gluing segments. Doing so iteratively we end up with pre-blocks given by

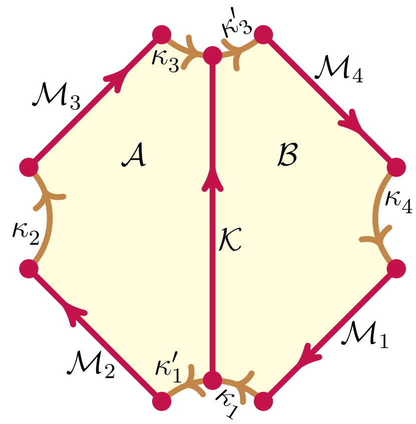
$$\mathrm{T}^{\mathrm{pre}}(x_1\boxtimes x_2\boxtimes\cdots\boxtimes x_n)\cong\mathrm{Nat}\big(\Phi^{\mathrm{l}}(\overline{x_1}),\Phi^{\mathrm{l}}(x_n)\circ\Phi^{\mathrm{l}}(x_{n-1})\circ\cdots\circ\Phi^{\mathrm{l}}(x_2)\big)\,, \tag{4.18}$$

with $x_i\in\mathrm{T}(\mathbb{L}_i)$, such that $U(x_i)\in\overline{\mathcal{M}_{i+1}}\boxtimes\mathcal{M}_i$, thereby generalizing (and using analogous notation as in) formula (4.11).

As this example indicates, by combining the Propositions and 4.6 and 3.13 one obtains

Corollary 4.8. The pre-block functor for any disk without defect lines and with an arbitrary number of free boundaries and gluing segments can be reduced to the pre-block functor for the straight disk (4.7).

Example 4.9. For $\mathcal{A}$ and $\mathcal{B}$ finite tensor categories, $\mathcal{M}_1$ and $\mathcal{M}_4$ right $\mathcal{B}$-modules, $\mathcal{M}_2$ and $\mathcal{M}_3$ right $\mathcal{A}$-modules, and $\mathcal{K}$ an $\mathcal{A}$-$\mathcal{B}$-bimodule, consider the defect surface

$\Sigma \;=$ 

(4.19)

Applied to objects

$$\begin{aligned} z_1 &= x_2 \boxtimes u \boxtimes \overline{y_1} \,\in \mathcal{M}_2 \overset{-\kappa_1'}{\boxtimes} \mathcal{K} \overset{-\kappa_1}{\boxtimes} \overline{\mathcal{M}_1}\,, \qquad & z_2 &= x_3 \boxtimes \overline{y_2} \,\in \mathcal{M}_3 \overset{-\kappa_2}{\boxtimes} \overline{\mathcal{M}_2}\,, \\ z_3 &= x_4 \boxtimes \overline{v} \boxtimes \overline{y_3} \,\in \mathcal{M}_4 \overset{-\kappa_3'}{\boxtimes} \overline{\mathcal{K}} \overset{-\kappa_3}{\boxtimes} \overline{\mathcal{M}_3}\,, \qquad & z_4 &= x_1 \boxtimes \overline{y_4} \,\in \mathcal{M}_1 \overset{-\kappa_4}{\boxtimes} \overline{\mathcal{M}_4} \end{aligned} \tag{4.20}$$

of the gluing categories for the four gluing intervals, the pre-block functor gives

$$\begin{aligned} \mathrm{T}^{\mathrm{pre}}(z_1 \boxtimes z_2 \boxtimes z_3 \boxtimes z_4) \;=& \int^{m_1\in\mathcal{M}_1, m_2\in\mathcal{M}_2, m_3\in\mathcal{M}_3, m_4\in\mathcal{M}_4, k\in\mathcal{K}} \\ & \mathrm{Hom}(m_1 \boxtimes \overline{m_1} \boxtimes m_2 \boxtimes \overline{m_2} \boxtimes m_3 \boxtimes \overline{m_3} \boxtimes m_4 \boxtimes \overline{m_4} \boxtimes k \boxtimes \overline{k}\,, \\ & \qquad x_1 \boxtimes \overline{y_1} \boxtimes x_2 \boxtimes \overline{y_2} \boxtimes x_3 \boxtimes \overline{y_3} \boxtimes x_4 \boxtimes \overline{y_4} \boxtimes u \boxtimes \overline{v})\,. \end{aligned} \tag{4.21}$$

By a multiple application of the variant (B.2) of the Yoneda lemma, this reduces to

$$\begin{aligned} \mathrm{T}^{\mathrm{pre}}(z_1 \boxtimes z_2 \boxtimes z_3 \boxtimes z_4) \;=\; & \mathrm{Hom}_{\mathcal{M}_1}(y_1, x_1) \otimes \mathrm{Hom}_{\mathcal{M}_2}(y_2, x_2) \\ & \otimes \mathrm{Hom}_{\mathcal{M}_3}(y_3, x_3) \otimes \mathrm{Hom}_{\mathcal{M}_4}(y_4, x_4) \otimes \mathrm{Hom}_{\mathcal{K}}(v, u)\,. \end{aligned} \tag{4.22}$$

This result may again be written in terms of natural transformations. There are now two distinguished ways to do so, one corresponding to fusing the $\mathcal{K}$-defect to the $\mathcal{M}_2$- and $\mathcal{M}_3$-defects (this process of fusion will be discussed in detail in Section 4.6 below), and one corresponding to fusing it to the $\mathcal{M}_1$- and $\mathcal{M}_4$-defects. Let us write out the resulting expression for the former case: one gets the natural transformation

$$\mathrm{T}^{\mathrm{pre}}(z_1 \boxtimes z_2 \boxtimes z_3 \boxtimes z_4) = \mathrm{Nat}(\Phi^{\mathrm{l}}(z_4), G(z_1, z_2, z_3)) \tag{4.23}$$

of functors in $\mathcal{L}ex(\mathcal{M}_1, \mathcal{M}_4)$, where Φ^{l} is the Eilenberg-Watts equivalence (3.39) and G is the composition $G(z_1, z_2, z_3) := \Phi^{\mathrm{l}}(z_3)\big([\Phi^{\mathrm{l}}(z_2) \boxtimes \mathrm{Id}_{\mathcal{K}}](\Phi^{\mathrm{l}}(z_1))\big)$.

In fact, the pre-block spaces for any arbitrary defect surface can be expressed as tensor products of morphism spaces, analogously as in (4.22). The so obtained expressions are, however, not particularly illuminating. Expressing them through spaces of natural transformations can be more informative, e.g. it often allows for a direct characterization of what subspaces of pre-blocks furnish the block spaces.

4.2 Holonomy

The pre-block functors do not see the framing of a defect surface and do not take the topology of the 2-patches of a defect surface into account. The proper block functors $\mathrm{T}(\Sigma)$ that we are going to introduce will, on the other hand, depend on the framing, and in their definition 2-patches with the topology of a disk will play a crucial role. To proceed from the pre-blocks to the block functors, we first define holonomy operations on pre-blocks. For doing so we will restrict our attention to the subclass of surfaces that can be patched together from disks. Recall from Definition 2.1(ii) that a defect surface Σ is called *fine* iff every 2-patch (in the sense of Definition 2.8) is contractible. From now on we assume that the defect surface Σ under consideration is fine. Then we have one holonomy operation for each 2-patch of Σ. The formal definition of these holonomy operations is notationally somewhat intricate. Instead of spelling out the details, we explain these operations through concrete examples.

Example 4.10. Let us give full details for the straight disk $\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}}$ described in Example 4.4. In this case we have two gluing intervals $\mathbb{L}_1$ and $\mathbb{L}_2$, and the pre-block functor on objects $x \in \mathrm{T}(\mathbb{L}_1)$ and $y \in \mathrm{T}(\mathbb{L}_2)$ in the gluing categories is given by formula (4.9). For $a \in \mathcal{A}$, the holonomy $\mathrm{hol}_{a,x}$ of a starting at $x = \overline{x}_m \boxtimes x_n$ is defined to be the following composite isomorphism of functors:

$$
\begin{aligned}
&\int^{m\in\mathcal{M},n\in\mathcal{N}} \mathrm{Hom}(\overline{m}\boxtimes n\boxtimes\overline{n}\boxtimes m, \overline{x}_m\boxtimes a.x_n\boxtimes\overline{y}_n\boxtimes y_m) \\
&\xrightarrow{\cong} \mathrm{Hom}(\textstyle\int_{m\in\mathcal{M},n\in\mathcal{N}} \overline{m}\boxtimes n\boxtimes\overline{n}\boxtimes m, \overline{x}_m\boxtimes a.x_n\boxtimes\overline{y}_n\boxtimes y_m) && \text{Eq. [(B.3)]} \\
&\xrightarrow{\cong} \mathrm{Hom}(\textstyle\int_{m\in\mathcal{M},n\in\mathcal{N}} \overline{m}\boxtimes a^\vee.n\boxtimes\overline{n}\boxtimes m, \overline{x}_m\boxtimes x_n\boxtimes\overline{y}_n\boxtimes y_m) && \text{[right duality]} \\
&\xrightarrow{\cong} \mathrm{Hom}(\textstyle\int_{m\in\mathcal{M},n\in\mathcal{N}} \overline{m}\boxtimes n\boxtimes\overline{a.n}\boxtimes m, \overline{x}_m\boxtimes x_n\boxtimes\overline{y}_n\boxtimes y_m) && \text{[Lemma 3.17]} \\
&\xrightarrow{\cong} \mathrm{Hom}(\textstyle\int_{m\in\mathcal{M},n\in\mathcal{N}} \overline{m}\boxtimes n\boxtimes\overline{n}\boxtimes m, \overline{x}_m\boxtimes x_n\boxtimes\overline{a^\vee.y_n}\boxtimes y_m) && \text{[right duality]} \\
&\xrightarrow{\cong} \mathrm{Hom}(\textstyle\int_{m\in\mathcal{M},n\in\mathcal{N}} \overline{m}\boxtimes n\boxtimes\overline{n}\boxtimes m, \overline{x}_m\boxtimes x_n\boxtimes\overline{y}_n\boxtimes a^{\vee\vee}.y_m) && \text{[balancing of } y\text{]} \\
&\xrightarrow{\cong} \mathrm{Hom}(\textstyle\int_{m\in\mathcal{M},n\in\mathcal{N}} \overline{m}\boxtimes n\boxtimes\overline{n}\boxtimes a^{\vee\vee\vee}.m, \overline{x}_m\boxtimes x_n\boxtimes\overline{y}_n\boxtimes y_m) && \text{[right duality]} \\
&\xrightarrow{\cong} \mathrm{Hom}(\textstyle\int_{m\in\mathcal{M},n\in\mathcal{N}} \overline{a^{\vee\vee}.m}\boxtimes n\boxtimes\overline{n}\boxtimes m, \overline{x}_m\boxtimes x_n\boxtimes\overline{y}_n\boxtimes y_m) && \text{[Lemma 3.17]} \\
&\xrightarrow{\cong} \mathrm{Hom}(\textstyle\int_{m\in\mathcal{M},n\in\mathcal{N}} \overline{m}\boxtimes n\boxtimes\overline{n}\boxtimes m, \overline{a^{\vee\vee\vee}.x_m}\boxtimes x_n\boxtimes\overline{y}_n\boxtimes y_m) && \text{[right duality]}\,.
\end{aligned}
\tag{4.24}
$$

Thus the holonomy is a distinguished isomorphism

$$
\begin{aligned}
\mathrm{hol}_{a,x}:\quad &\int^{m\in\mathcal{M},n\in\mathcal{N}} \mathrm{Hom}(\overline{m}\boxtimes n\boxtimes\overline{n}\boxtimes m, \overline{x}_m\boxtimes a.x_n\boxtimes\overline{y}_n\boxtimes y_m) \\
&\xrightarrow{\cong} \int^{m\in\mathcal{M},n\in\mathcal{N}} \mathrm{Hom}(\overline{m}\boxtimes n\boxtimes\overline{n}\boxtimes m, \overline{a^{\vee\vee\vee}.x_m}\boxtimes x_n\boxtimes\overline{y}_n\boxtimes y_m)
\end{aligned}
\tag{4.25}
$$

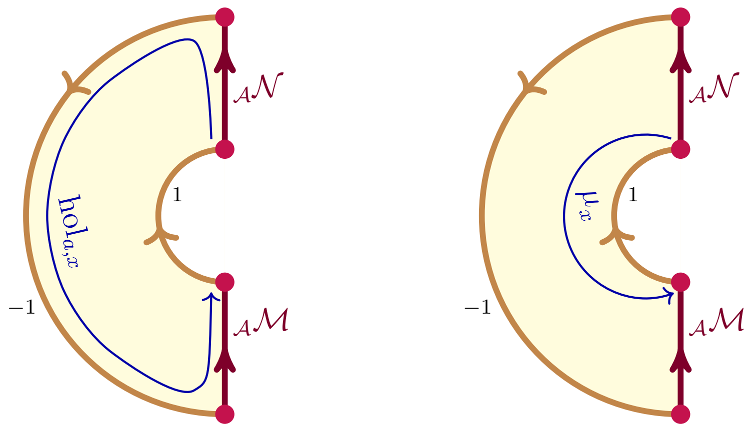
of left exact functors.

The rationale behind this prescription is simple: we proceed – counterclockwise, by convention – along the boundary of the 2-patch and use alternatingly a duality to jump between a covariant and a contravariant argument of the Hom and a balancing in one of the two arguments. In the contravariant state-sum variable, the balancing is given by Lemma 3.17 with $F = \mathrm{Id}$; in the

covariant argument coming from a gluing boundary, it is part of the structure given by the gluing categories. In the latter, the index resulting from the 2-framing enters.
On the other hand, the balancing of the variable $x \in \mathrm{T}(\mathbb{L}_1)$ gives an isomorphism

$$\begin{aligned} \mu_x : \quad \int^{m\in\mathcal{M}, n\in\mathcal{N}} & \mathrm{Hom}(\overline{m} \boxtimes n \boxtimes \overline{n} \boxtimes m, \overline{x}_m \boxtimes a.x_n \boxtimes \overline{y}_n \boxtimes y_m) \\ & \xrightarrow{\cong} \mathrm{Hom}(\textstyle\int_{m\in\mathcal{M}, n\in\mathcal{N}} \overline{m} \boxtimes n \boxtimes \overline{n} \boxtimes m, \overline{a^\vee.x_m} \boxtimes x_n \boxtimes \overline{y}_n \boxtimes y_m)\,. \end{aligned} \tag{4.26}$$

We indicate the holonomy (4.25) and the isomorphism (4.26) graphically in the following picture:

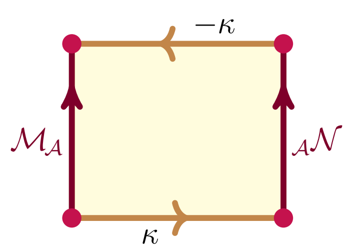


(4.27)

As explained in the Introduction, we will obtain the block functor from the pre-block functor by imposing "flatness along the disk"; this amounts to considering an equalizer of two morphisms that are built from the isomorphisms (4.25) and (4.26). The precise prescription for general fine defect surfaces will be given in Definition 4.18.

By Proposition 3.13, the defect surface of Example 4.10 gives, up to canonical equivalence, the same gluing category as a disk with two free boundary segments having opposite orientations and indices 2 and -2. The following example generalizes this situation to the case of arbitrary even indices.

Example 4.11. Consider the following disk having a framing with indices $\pm\kappa \in 2\mathbb{Z}$ along its two gluing segments, continued as a cylinder along the direction of the free boundary intervals:

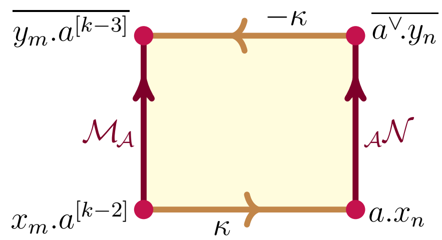


(4.28)

The gluing categories for the lower and upper segment are, respectively, $\mathrm{T}_1 := \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{N}$ and $\mathrm{T}_2 := \overline{\mathcal{N}} \overset{-\kappa}{\boxtimes} \overline{\mathcal{M}}$. Hence we have again $\mathrm{T}_2 \cong \overline{\mathrm{T}_1}$, and for objects $x_m \boxtimes x_n \in T_1$ and $\overline{y_n} \boxtimes \overline{y_m} \in T_2$ the pre-block space is

$$\int^{m\in\mathcal{M}, n\in\mathcal{N}} \mathrm{Hom}(m \boxtimes n \boxtimes \overline{n} \boxtimes \overline{m}, x_m \boxtimes x_n \boxtimes \overline{y_n} \boxtimes \overline{y_m})\,. \tag{4.29}$$

Starting the holonomy of $a \in \mathcal{A}$ at x_n counterclockwise, and suppressing the balancing of the canonical objects $\int^m \overline{m} \boxtimes m$, we arrive schematically at the situation



(4.30)

This gives the holonomy

$$\begin{aligned}
&\int^{m\in\mathcal{M},n\in\mathcal{N}} \mathrm{Hom}(m\boxtimes n\boxtimes \overline{n}\boxtimes \overline{m}, x_m\boxtimes a.x_n\boxtimes \overline{y}_n\boxtimes \overline{y}_m) \\
&\quad \xrightarrow{\cong} \int^{m\in\mathcal{M},n\in\mathcal{N}} \mathrm{Hom}(m\boxtimes n\boxtimes \overline{n}\boxtimes \overline{m}, x_m\boxtimes x_n\boxtimes \overline{a^\vee.y_n}\boxtimes \overline{y}_m) \\
&\quad \xrightarrow{\cong} \int^{m\in\mathcal{M},n\in\mathcal{N}} \mathrm{Hom}(m\boxtimes n\boxtimes \overline{n}\boxtimes \overline{m}, x_m\boxtimes x_n\boxtimes \overline{y}_n\boxtimes \overline{y_m.a^{[k-3]}}) \\
&\quad \xrightarrow{\cong} \int^{m\in\mathcal{M},n\in\mathcal{N}} \mathrm{Hom}(m\boxtimes n\boxtimes \overline{n}\boxtimes \overline{m}, x_m.a^{[k-2]}\boxtimes a.x_n\boxtimes \overline{y}_n\boxtimes \overline{y}_m)\,.
\end{aligned} \tag{4.31}$$

The powers of duals appearing in the so obtained expressions for the holonomy turn out to be significant: they will allow us to define the block functors as equalizers (Definition 4.18).

Definition 4.12. We say that the holonomy problem for a disk labeled by a finite tensor category $\mathcal{A}$ is *well-posed* iff for any $a \in \mathcal{A}$ and any object x in the gluing category associated with a gluing segment of the boundary the holonomy $\mathrm{hol}_{a,x}$ and the isomorphism σ_x provided by the balancing of x differ by a double right dual.
The holonomy problem for a fine defect surface Σ is said to be well-posed iff the holonomy problem for every 2-patch of Σ is well-posed.

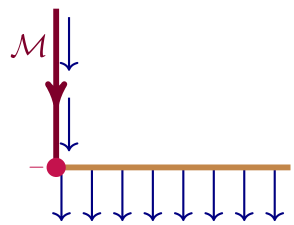
In Example 4.10, the two isomorphisms in question are given by (4.25) and (4.26), respectively. Thus, by inspection, the holonomy is well-posed in that example. Likewise, the holonomy problem in Example 4.11 is well-posed. Our next task will be to show that also in the general situation of a fine defect surface all holonomy problems are well-posed. To see this, we make use of the following statement about framing indices.

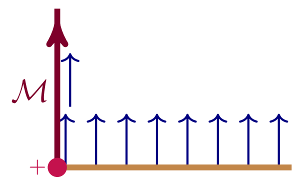
Lemma 4.13. Let $\mathbb{D}$ be a 2-framed disk with N gluing boundary segments s_i which are oriented as induced by the orientation of $\mathbb{D}$. Then

$$\sum_{i=1}^{N} \mathrm{ind}(s_i) \;=\; N-2\,. \tag{4.32}$$

Proof. Consider the standard disk $\mathbb{D}_{\rm std} \subset \mathbb{R}^2$ with marked points on its boundary that divide $\partial\mathbb{D}_{\rm std}$ into segments. That the disk has Euler characteristic 1 implies that for any non-zero continuous vector field on $\mathbb{D}_{\rm std}$ the sum of the indices of all segments on the boundary $\partial\mathbb{D}_{\rm std}$

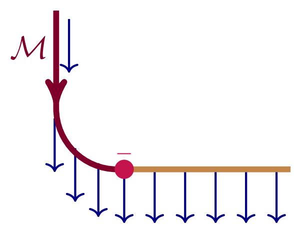
is -2 when all segments are oriented counterclockwise. To be able to apply this fact to the situation at hand, we must smoothen out the framing at the corners of the defect disk $\mathbb{D}$, at which a free boundary interval is adjacent to a gluing boundary. Thus we replace the two allowed situations

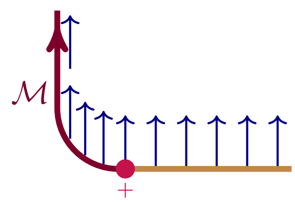


and


(4.33)

by the smoothened versions



and


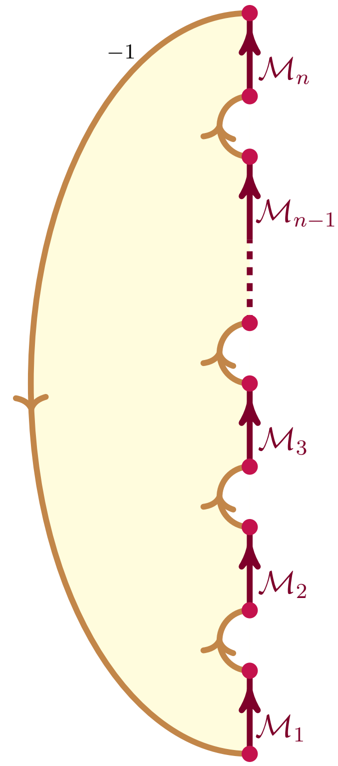
(4.34)

respectively, in which the free boundary segments are suitably deformed. Similarly, in case we deal with a defect line, labeled by $\mathcal{M}$, in the interior of $\mathbb{D}$ rather than a free boundary interval, we temporarily think of the defect line as a pair of parallel free boundary intervals and apply the procedure above to both parts. Both of the situations (4.33) and (4.34) contribute, in a counterclockwise sense, $-\frac{1}{2}$. Since there are two arcs per segment, each segment gets an additional contribution -1. This way we obtain the equality $-2 = \sum_{i=1}^{N} \big(\mathrm{ind}(s_i) - 1\big)$, thus proving (4.32). □

Remark 4.14. In fact, vector fields on $\mathbb{D}$ up to homotopy are in bijection with collections of indices that obey the relation (4.32). To see this, let $\mathbb{D}$ be a disk whose boundary is either a gluing circle (in case that $N = 0$) or consists of $N \geq 1$ free boundaries and N gluing segments s_i. All gluing segments are endowed with the induced orientation. The orientation of the free boundaries adds signs to the points at which free boundaries and gluing intervals meet. Let κ be a tuple of framing indices for these signs such that $\sum_{i=1}^{N} \kappa_i - N = -2$. Then there exists a vector field on $\mathbb{D}$ that is continuous on $\mathbb{D}$, parallel to the free boundaries and has the given index on the gluing segments, i.e. $\mathrm{ind}(s_i) = \kappa_i$ for each segment.

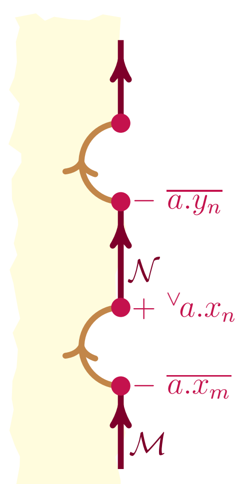
Proposition 4.15. The holonomy problem of any labeled 2-framed disk $\mathbb{D}$ is well-posed.

Proof. Consider a disk $\mathbb{D}$. We invoke the line flip of Proposition 3.13 to assume without loss of generality that all free boundary segments on $\partial\mathbb{D}$ are oriented as induced by the orientation of $\mathbb{D}$. It is straightforward to see that the holonomy problem is well-posed for one collection of indices satisfying the sum rule (4.32) iff it is well-posed for any other. It follows that we can restrict our attention to the situation



(4.35)

corresponding to a framing given by the constant vector field pointing upwards (recall that the index of all unlabeled oriented gluing intervals is +1). Now for this situation well-posedness follows from Example 4.10 together with the observation that for each $i \in \{1, 2, \dots, n-1\}$ we have (abbreviating $\mathcal{M} = \mathcal{M}_i$ and $\mathcal{N} = \mathcal{M}_{i+1}$)



(4.36)

so that no double duals arise in the part of the holonomy along the gluing segments with index 1. □

An immediate consequence is

Corollary 4.16. The holonomy problem of any disk $\mathbb{D}$ in any defect surface Σ is well-posed.

In particular, the holonomy problems for the straight disk in Example 4.10 and for the disks discussed in Example 4.11 are well-posed.

4.3 Block functors for fine surfaces

We are now in a position to set up functors that will eventually provide us with the block functors for fine defect surfaces. Since the holonomy problem for the straight disk in Example 4.10 is well-posed, the holonomy operators

$$\begin{aligned}\mathrm{T}^{\mathrm{pre}}(\overline{x}_m \boxtimes a.x_n \boxtimes \overline{y}_n \boxtimes y_m) &= \int^{m\in\mathcal{M},n\in\mathcal{N}} \mathrm{Hom}(\overline{m} \boxtimes n \boxtimes \overline{n} \boxtimes m, \overline{x}_m \boxtimes a.x_n \boxtimes \overline{y}_n \boxtimes y_m) \\ &\xrightarrow[\cong]{\mathrm{hol}_{a,x}} \int^{m\in\mathcal{M},n\in\mathcal{N}} \mathrm{Hom}(\overline{m} \boxtimes n \boxtimes \overline{n} \boxtimes m, \overline{a^{\vee\vee\vee}.\, x_m} \boxtimes x_n \boxtimes \overline{y}_n \boxtimes y_m) \\ &= \mathrm{T}^{\mathrm{pre}}(\overline{a^{\vee\vee\vee}.\, x_m} \boxtimes x_n \boxtimes \overline{y}_n \boxtimes y_m)\end{aligned} \tag{4.37}$$

allow us to formulate holonomy equations and thus to define blocks $\mathrm{T}_{\mathrm{fine}}$ for fine defect surfaces as equalizers, see Definition 4.18 below. In contrast, for non-fine surfaces this is no longer possible. As a consequence, to define the block functor T for an arbitrary defect surface Σ we will have to take a limit over suitable 'fine refinements' of Σ; we relegate this to Definition 5.19. In case Σ is already fine, the functor $\mathrm{T}_{\mathrm{fine}}(\Sigma)$ from Definition 4.18 can be taken as a distinguished representative of the limit $\mathrm{T}(\Sigma)$. This justifies an abuse of language: we refer also to the functor $\mathrm{T}_{\mathrm{fine}}(\Sigma)$ as the block functor for Σ.

The following result will later allow us to relate certain spaces of blocks to natural transformations.

Lemma 4.17. Let $T\colon \mathcal{C} \to \mathcal{C}$ be a monad on a linear category, with category $\mathcal{C}_T$ of modules, and denote by $U\colon \mathcal{C}_T \to \mathcal{C}$ the forgetful functor. Then for any pair $m = (Um, \rho_m)$ and $n = (Un, \rho_n)$ of objects of $\mathcal{C}_T$, with actions ρ_m and ρ_n, respectively, there is an equalizer diagram

$$\mathrm{Hom}_{\mathcal{C}_T}(m,n) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Um,Un) \overset{\varphi_1}{\underset{\varphi_2}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{C}}(T(Um),Un)\,, \tag{4.38}$$

where on $\gamma\colon U(m) \to U(n)$ the maps $\varphi_{1,2}$ are defined by $\varphi_1(\gamma) := \gamma \circ \rho_m$ and $\varphi_2(\gamma) := \rho_n \circ T(\gamma)$, respectively.
Similarly, for S a comonad on $\mathcal{C}$ and $x = (Ux, \delta_x)$ and $y = (Uy, \delta_y)$ objects in the category $\mathcal{C}^S$ of S-comodules, there is an equalizer diagram

$$\mathrm{Hom}_{\mathcal{C}^S}(x,y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(Ux,Uy) \overset{\psi_1}{\underset{\psi_2}{\rightrightarrows}} \mathrm{Hom}_{\mathcal{C}}(Ux,S(Uy)) \tag{4.39}$$

with $\psi_1(\gamma) := \delta_y \circ \gamma$ and $\psi_2(\gamma) := S(\gamma) \circ \delta_x$ for $\gamma\colon U(x) \to U(y)$.

Proof. It follows from the definition that $\mathrm{Hom}_{\mathcal{C}_T}(m,n)$ is the kernel of $\varphi_1 - \varphi_2$: γ is a module map if and only if $\varphi_1(\gamma) = \varphi_2(\gamma)$. The statement for the comonad follows by the same type of reasoning. □

We now consider for any $a \in \mathcal{A}$ the composite

$$\begin{aligned} \mathrm{T}^{\mathrm{pre}}(\dots \overline{x_m} \boxtimes x_n \boxtimes y) \xrightarrow{\mathrm{coev}^{\mathrm{l}}_*} & \mathrm{T}^{\mathrm{pre}}(\dots \overline{x_m} \boxtimes ({}^{\vee}a \otimes a) \,.\, x_n \boxtimes y) \\ & \xrightarrow[\cong]{\mathrm{hol}} \mathrm{T}^{\mathrm{pre}}(\dots \overline{a^{\vee\vee}.\, x_m} \boxtimes a . x_n \boxtimes y) \end{aligned} \tag{4.40}$$

of the post-composition with the left coevaluation of a and the holonomy $\mathrm{hol}_{{}^{\vee}a,a.x_n}$ of ${}^{\vee}a$ around the disk. It is important to note that these maps are dinatural in a and thus factorize over the end. We thus obtain a morphism of left exact functors

$$\mathrm{hol}_x : \quad \mathrm{T}^{\mathrm{pre}}(\dots \overline{x_m} \boxtimes x_n \boxtimes y) \longrightarrow \int_{a\in\mathcal{A}} \mathrm{T}^{\mathrm{pre}}(\dots \overline{a^{\vee\vee}.\, x_m} \boxtimes a \,.\, x_n \boxtimes y) \tag{4.41}$$

furnishing a well-posed holonomy problem. On the other hand, combining the right coevaluation and the comodule structure of $\overline{x_m} \boxtimes x_n$ that expresses the balancing provides us with a morphism $\overline{x_m} \boxtimes x_n \rightarrow \int_{a\in\mathcal{A}} \overline{a^{\vee\vee}.\, x_m} \boxtimes a \,.\, x_n$, which defines another map

$$\begin{aligned} (\mu_x)_* : \quad \mathrm{T}^{\mathrm{pre}}(\dots \overline{x_m} \boxtimes x_n \boxtimes y) \xrightarrow{\mathrm{coev}^{\mathrm{r}}_*} & \int_{a\in\mathcal{A}} \mathrm{T}^{\mathrm{pre}}(\dots \overline{x_m} \boxtimes (a \otimes a^{\vee}) \,.\, x_n \boxtimes y) \\ & \xrightarrow{\cong} \int_{a\in\mathcal{A}} \mathrm{T}^{\mathrm{pre}}(\dots \overline{a^{\vee\vee}.\, x_m} \boxtimes a \,.\, x_n \boxtimes y) \end{aligned} \tag{4.42}$$

(the end over $\mathcal{A}$ commutes with all coends in the pre-block functor, since those can be pulled in the first argument of the Hom). We can thus define the *block space* as the equalizer of the maps hol_x and $(\mu_x)_*$. More precisely, we impose one such a relation for each 2-patch (which, as the surface is assumed to be fine, is a disk) and select for each 2-patch $\mathbb{P}_p$ a starting point v_p among the defect points on $\partial\mathbb{P}_p$.

Definition 4.18. Let $\mathrm{T}^{\mathrm{pre}}$ be the pre-block functor associated with a fine defect surface. The functor $\mathrm{T}_{\mathrm{fine}}$ associated with the surface is the equalizer

$$\begin{aligned} \mathrm{T}_{\mathrm{fine}}(\dots \overline{x_m} \boxtimes x_n \boxtimes y) \longrightarrow \; & \mathrm{T}^{\mathrm{pre}}(\dots \overline{x_m} \boxtimes x_n \boxtimes y) \\ & \underset{\prod (\mu_x)_*}{\overset{\prod \mathrm{hol}_x}{\rightrightarrows}} \prod_x \int_{a\in\mathcal{A}} \mathrm{T}^{\mathrm{pre}}(\dots \overline{a^{\vee\vee}.\, x_m} \boxtimes a \,.\, x_n \boxtimes y) \,. \end{aligned} \tag{4.43}$$

As mentioned above, we call $\mathrm{T}_{\mathrm{fine}}$ the *block functor* for the defect surface, albeit a complete formulation that is valid for arbitrary defect surfaces will require to define the actual block functor T as a limit. While the pre-blocks for a defect surface Σ only depend on the incidence combinatorics of boundary segments and defect lines of Σ, the blocks also depend, via the holonomy, directly on the framing on the 2-patches of Σ.

A priori the block functor $\mathrm{T}_{\mathrm{fine}} = \mathrm{T}_{\mathrm{fine},(v)}$ for a defect surface Σ depends on the choice $(v) = \{v_i\}$ of a starting point v_i for each disk $\mathbb{D}_i$ in Σ. However, in fact it depends on these choices only up to canonical coherent natural isomorphism. To see this, we provide an alternative characterization of the equalizer: Composing Equation (4.40) with the balancing of x, we obtain a morphism

$$\widetilde{\mathrm{hol}}_{a,x} : \quad \mathrm{T}^{\mathrm{pre}}(\dots \overline{x_m} \boxtimes x_n \boxtimes y) \rightarrow \mathrm{T}^{\mathrm{pre}}(\dots \overline{x_m} \boxtimes ({}^{\vee}a \otimes a) \,.\, x_n \boxtimes y) \,, \tag{4.44}$$

which has as parallel morphism the composition with $\mathrm{coev}_a^{\mathrm{l}}$ at x_n. Again, these morphisms factorize over the end, and by composing the defining equation (4.43) with the balancing of x we see that the block space is also the equalizer

$$\mathrm{T}_{\mathrm{fine},(v)}(\dots\,\overline{x_m}\boxtimes x_n\boxtimes y) \longrightarrow \mathrm{T}^{\mathrm{pre}}(\dots\,\overline{x_m}\boxtimes x_n\boxtimes y) \xrightarrow[\prod(\mathrm{coev}^{\mathrm{l}})_*]{\prod\widetilde{\mathrm{hol}}_x} \prod_x\int_{a\in\mathcal{A}}\mathrm{T}^{\mathrm{pre}}(\dots,\overline{x_m}\boxtimes({}^\vee a\otimes a)\,.\,x_n\boxtimes y)\,. \tag{4.45}$$

We use the latter description of the block space to show

Lemma 4.19. The block functor depends on the choice of a starting point per disk only up to canonical coherent natural isomorphism.

Proof. We define canonical natural isomorphisms

$$\Gamma_{v',v}:\quad \mathrm{T}_{\mathrm{fine},(v)}(\dots,\overline{x_m}\boxtimes x_n\boxtimes y)\to\mathrm{T}_{\mathrm{fine},(v')}(\dots,\overline{x_m}\boxtimes x_n\boxtimes y)\,, \tag{4.46}$$

for each pair of starting points v,v' per disk, that satisfy the coherence relation $\Gamma_{v'',v'}\circ\Gamma_{v',v}=\Gamma_{v'',v}$. For a given disk with a choice of starting points v,v' there is an isomorphism

$$\gamma_{v',v}:\quad \mathrm{T}^{\mathrm{pre}}(\dots,\overline{x_m}\boxtimes({}^\vee a\otimes a)\,.\,x_n\boxtimes y)\xrightarrow{\cong}\mathrm{T}^{\mathrm{pre}}(\dots,\overline{x_m}\boxtimes x_n\boxtimes({}^\vee a\otimes a)\,.\,y)\,, \tag{4.47}$$

to which we refer as the *parallel transport operation* from v to v'. Here the first action of ${}^\vee a\otimes a$ is at the defect point v and the second at v', which is constructed using the balancings precisely as in the holonomy operation following a positive path along the boundary of the disk from v to v'. The latter ensures that the parallel transport operations $\gamma_{v',v}$ are coherent, and it also implies that there are two commuting triangles of isomorphisms

$$\begin{array}{ccc}
\mathrm{T}^{\mathrm{pre}}(\dots,\overline{x_m}\boxtimes x_n\boxtimes y) & \overset{\widetilde{\mathrm{hol}}_{a,v}}{\underset{\mathrm{coev}^{\mathrm{l}}}{\rightrightarrows}} & \mathrm{T}^{\mathrm{pre}}(\dots,\overline{x_m}\boxtimes({}^\vee a\otimes a)\,.\,x_n\boxtimes y) \\
 & \searrow\!\!\searrow^{\widetilde{\mathrm{hol}}_{a,v'}}_{\mathrm{coev}^{\mathrm{l}}} & \downarrow\gamma_{v',v} \\
 & & \mathrm{T}^{\mathrm{pre}}(\dots,\overline{x_m}\boxtimes x_n\boxtimes({}^\vee a\otimes a)\,.\,y
\end{array} \tag{4.48}$$

Thus we obtain $\Gamma_{v',v}$ as the universal isomorphism between the corresponding equalizers and it inherits the coherence from $\gamma_{v',v}$. □

This result justifies to disregard the dependence of $\mathrm{T}_{\mathrm{fine}}$ on the choice of starting points in the sequel. More conceptionally, one can define $\mathrm{T}_{\mathrm{fine}}(\dots,\overline{x_m}\boxtimes x_n\boxtimes y)$ as the direct limit over the isomorphisms $\Gamma_{v',v}$ defined in (4.46).

The definition also implies directly that isomorphic defect surfaces give identical block functors:

Lemma 4.20. Let $\phi\colon\Sigma\to\Sigma'$ be an isomorphism of fine defect surfaces. Then the block functors for Σ and Σ' are equal on the nose, $\mathrm{T}_{\mathrm{fine}}(\Sigma)=\mathrm{T}_{\mathrm{fine}}(\Sigma')$.

Proof. We can identify the defects and the vector fields of Σ with those of Σ' via the isomorphism ϕ. Thus the pre-block functors as well as the holonomy operations for Σ and Σ' coincide: they only depend on the incidence relations of the patches of various dimensions, and these are not changed by an isomorphism. □

Next we use Lemma 4.17 to show that block spaces produce spaces of natural transformations in specific situations. We first note that the boundary-segment-flipping lemma 4.5 for pre-blocks extends to blocks:

Proposition 4.21. There is a canonical isomorphism between the block functors for the two configurations (4.12) that appear in Lemma 4.5.

Proof. We know from Lemma 4.5 that the two pre-block functors are canonically isomorphic. Moreover, the balancings on the segments labeled by $\mathcal{N}$ and by ${}^{1}\overline{\mathcal{N}}$, respectively, agree by the definition of the respective actions. Thus the block functors are isomorphic as well. □

Similarly, our construction is compatible with composition of left exact functors via the Eilenberg-Watts correspondence and thus with the fusion of boundary insertions not only at the level of pre-blocks, but also for blocks:

Proposition 4.22. The isomorphism between the pre-block functors for the two configurations (4.15) established in Proposition 4.6 is compatible with the holonomy operators and hence induces a canonical isomorphism of the corresponding block functors.

Proof. For left exact bimodule functors $F\colon \mathcal{M}\to\mathcal{N}$ and $G\colon \mathcal{N}\to\mathcal{K}$ for $\mathcal{B}$-$\mathcal{A}$-bimodules $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{K}$, on the level of pre-blocks the isomorphism is described by the composite (4.16). By the definition of the balancings of $\Psi^{\mathrm{l}}(F)$ and $\Psi^{\mathrm{l}}(G)$ according to Equation (3.52), the first two isomorphisms in this composite are compatible with the holonomy operation. We need to show that the isomorphism

$$\begin{aligned}\int^{n\in\mathcal{N}}\mathrm{Hom}(k,G(n))\otimes\mathrm{Hom}(n,F(m)) &\cong \int^{n\in\mathcal{N}}\mathrm{Hom}(G^{\mathrm{l.a.}}(k),n)\otimes\mathrm{Hom}(n,F(m))\\ &\cong \mathrm{Hom}(G^{\mathrm{l.a.}}(k),F(m)) \cong \mathrm{Hom}(k,G\circ F(m))\end{aligned} \tag{4.49}$$

is compatible with the balancing structures. Therefore we consider the diagram

$$\begin{array}{ccccc}
\int^{n\in\mathcal{N}}\mathrm{Hom}(k,G(n))\otimes\mathrm{Hom}(n,F(m.a)) & \longrightarrow & & & \mathrm{Hom}(k,GF(m.a)) \\
\downarrow & & & \swarrow & \downarrow \\
\int^{n\in\mathcal{N}}\mathrm{Hom}(G^{\mathrm{l.a.}}(k),n)\otimes\mathrm{Hom}(n,F(m).a) & \longrightarrow & \mathrm{Hom}(G^{\mathrm{l.a.}}(k),F(m).a) & & \\
\downarrow & & \downarrow & & \\
\int^{n\in\mathcal{N}}\mathrm{Hom}(G^{\mathrm{l.a.}}(k).{}^{\vee}a,n)\otimes\mathrm{Hom}(n,F(m)) & \longrightarrow & \mathrm{Hom}(G^{\mathrm{l.a.}}(k).{}^{\vee}a,F(m)) & & \\
\downarrow & & & \searrow & \downarrow \\
\int^{n\in\mathcal{N}}\mathrm{Hom}(k,{}^{\vee}a,G(n))\otimes\mathrm{Hom}(n,F(m)) & \longrightarrow & & & \mathrm{Hom}(k.{}^{\vee}a,GF(m))
\end{array} \tag{4.50}$$

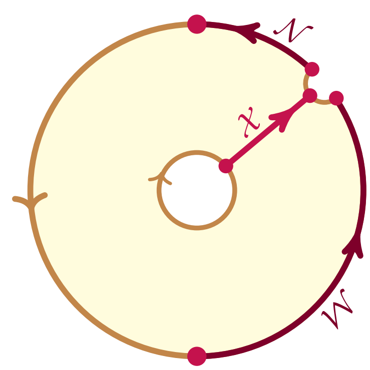
of isomorphisms. Here the horizontal arrow in the top and bottom row are variants of the isomorphism (4.49); the commutativity of the inner rectangle is the definition in Equation (3.52) of the balancing structure of the coend over $\mathcal{N}$, and all other arrows are composites of the dualities and the module structures of F and G. It thus follows directly that the whole diagram commutes and hence (4.16) is compatible with the holonomy operation. □

Analogously, the distinguished isomorphism used in Corollary 4.8 is compatible with the holonomy operators and hence induces an isomorphism of the block functors. Thus we have

Corollary 4.23. The block functor for any disk without defect points and with an arbitrary number of free and gluing segments is isomorphic, by a distinguished isomorphism, to the block functor for the straight disk (4.7).

Next we develop a conceptual formulation of the holonomy operations, which will in particular prove to be helpful later on, when we explore refinements of defect surfaces. When doing so we must account for the possibility that when performing a holonomy operation we move along one and the same defect line twice, in opposite directions. This is achieved by keeping track of normal directions, in the following manner. It suffices to consider the case of a single disk $\mathbb{D}$ in a defect surface Σ. We then consider the set $E_{\mathbb{D}}$ consisting of all defect points and all free boundary segments of $\mathbb{D}$, where the latter may result from defect lines in Σ, and where each defect point and segment is in addition equipped with the choice of a normal direction into the disk. As an illustration, consider the disk

$\mathbb{D}$ =



(4.51)

In this example the elements of $E_{\mathbb{D}}$ are, besides the defect points, the two free boundaries labeled by $\mathcal{M}$ and $\mathcal{N}$, each with a single (namely, inward) normal direction, and twice the defect line labeled by $\mathcal{X}$, with two different choices of normal direction.

Each element of $E_{\mathbb{D}}$ is a possible start and end point of one of the parallel transport operations on $\mathbb{D}$. Recall that $\int_{a\in\mathcal{A}} \overline{a} \boxtimes a \in \overline{\mathcal{A}} \boxtimes \mathcal{A}$ is a coalgebra, with comultiplication induced by the monoidal structure of $\mathcal{A}$, and with counit $\epsilon\colon \int_{a\in\mathcal{A}} \overline{a} \boxtimes a \to \overline{\mathbf{1}} \boxtimes \mathbf{1}$ given by the component at $\mathbf{1}$ of the universal dinatural transformation of the end. Recall further that forgetting the balancings provides a functor $\mathrm{T}(\partial\mathbb{D}) \to \mathrm{U}(\partial\mathbb{D})$, with $\mathrm{T}(\partial\mathbb{D})$ the Deligne product of the gluing categories for the gluing segments on $\partial\mathbb{D}$. Now for a disk $\mathbb{D}$ the pre-block functor is just the Hom functor on $\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}} \boxtimes \mathrm{U}(\partial\mathbb{D})$, and each $x \in E_{\mathbb{D}}$ corresponds to a Deligne factor in $\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}} \boxtimes \mathrm{U}(\partial\mathbb{D})$. We define a coaction of $\int_{a\in\mathcal{A}} \overline{a} \boxtimes a$ on $\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}} \boxtimes \mathrm{U}(\partial\mathbb{D})$ as follows. In $\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}}$ there are Deligne products $\mathcal{M}_s \boxtimes \overline{\mathcal{M}_s}$ and $\mathcal{M}_t \boxtimes \overline{\mathcal{M}_t}$ corresponding to x_s and x_t. There are four cases to consider, depending on whether $\mathcal{M}_s$ and $\mathcal{M}_t$ are left or right $\mathcal{A}$-modules. In each case the coaction takes place on an object $\overline{m}\boxtimes\overline{n}\in\overline{\mathcal{M}_s}\boxtimes\overline{\mathcal{M}_t}$, and depends on two integers μ and l, where $\mu=\sum_{i=1}^{l}\mu_i$ is the sum of the framing indices counted clockwise from x_s to x_t and l is the number of gluing segments along that path. We define the coaction in the four cases as

$$\overline{m}\boxtimes\overline{n} \longmapsto \begin{cases} \int_{a\in\mathcal{A}} \overline{a.m}\boxtimes\overline{a^{[\mu-l+1]}.n} & \text{if } \mathcal{M}_s \text{ and } \mathcal{M}_t \text{ are left modules,} \\ \int_{a\in\mathcal{A}} \overline{a.m}\boxtimes\overline{n.a^{[l-\nu]}} & \text{if } \mathcal{M}_s \text{ is a left and } \mathcal{M}_t \text{ a right module,} \\ \int_{a\in\mathcal{A}} \overline{m.a}\boxtimes\overline{a^{[l-\mu]}n} & \text{if } \mathcal{M}_s \text{ is a right and } \mathcal{M}_t \text{ a left module,} \\ \int_{a\in\mathcal{A}} \overline{m.a}\boxtimes\overline{n.a^{[l-\mu-1]}} & \text{if } \mathcal{M}_s \text{ and } \mathcal{M}_t \text{ are right modules,} \end{cases} \tag{4.52}$$

respectively.

Proposition 4.24. Let $\mathbb{D}$ be a disk and $\mathcal{A}$ the finite tensor category labeling its interior. Denote by $\mathrm{U}(\partial\mathbb{D})$ the category that is obtained by taking the Deligne product over the labels at all defect points of $\mathbb{D}$. Let x_s and x_t be any two elements of the set $E_\mathbb{D}$ that correspond to defect lines.

(i) The $\mathcal{A}$-coactions (4.52) yield a canonical comonad $Z_{\mathbb{D},x_s,x_t}$ on $\mathcal{L}ex(\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}}\boxtimes\mathrm{U}(\partial\mathbb{D}),\mathrm{vect})$.

(ii) The clock- and counterclockwise parallel transport operations from x_s to x_t provide two structures

$$\gamma^{\mathrm{c}}_{\mathbb{D},x_s,x_t},\gamma^{\mathrm{cc}}_{\mathbb{D},x_s,x_t}: \quad \mathrm{T}^{\mathrm{pre}}(\mathbb{D}) \longrightarrow Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\mathbb{D})) \tag{4.53}$$

of a $Z_{\mathbb{D},x_s,x_t}$-comodule on the functor $\mathrm{T}^{\mathrm{pre}}(\mathbb{D})$.

(iii) The fine block functor $\mathrm{T}_{\mathrm{fine}}(\mathbb{D})$ is the equalizer of $\gamma^{\mathrm{c}}_{\mathbb{D},x_s,x_t}$ and $\gamma^{\mathrm{cc}}_{\mathbb{D},x_s,x_t}$. It depends on the choice of $x_s,x_t\in E_\mathbb{D}$ only up to a canonical isomorphism.

In case the start and end of the parallel transport are clear from the context, we just write $\gamma^{\mathrm{c}}_{\mathbb{D}}$ for $\gamma^{\mathrm{c}}_{\mathbb{D},x_s,x_t}$, and analogously $\gamma^{\mathrm{cc}}_{\mathbb{D}}=\gamma^{\mathrm{cc}}_{\mathbb{D},x_s,x_t}$.

Proof. (i) By pre-composition (in the same way as in (B.17)) we obtain from (4.52) a coaction of the coalgebra $\int_{a\in\mathcal{A}}\overline{a}\boxtimes a$ on the functor category $\mathcal{L}ex(\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}}\boxtimes\mathrm{U}(\partial\mathbb{D}),\mathrm{vect})$. This way the coactions of $\mathcal{A}$ on the defects x_s and x_t provide canonically a comonad $Z_{\mathbb{D},x_s,x_t}$ on $\mathcal{L}ex(\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}}\boxtimes\mathrm{U}(\partial\mathbb{D}),\mathrm{vect})$.

(ii) We define the morphism $\gamma^{\mathrm{c}}_{\mathbb{D},x_s,x_t}$ as follows. The pre-block functor on $\mathbb{D}$ is given by

$$\mathrm{T}^{\mathrm{pre}}(\mathbb{D})(-) = \int^{m_s\in\mathcal{M}_s}\!\!\int^{m_t\in\mathcal{M}_t}\cdots\ \mathrm{Hom}(\overline{m_s}\boxtimes m_s\boxtimes m_t\boxtimes\overline{m_t}\boxtimes\ \ldots,-)\,, \tag{4.54}$$

where the ellipsis accounts for the additional defect lines on $\mathbb{D}$. Consider the first case in the list (4.52). Then the component $\gamma^{\mathrm{c}}_{\mathbb{D},x_s,x_t}({}^\vee a)$ of the parallel transport $\gamma^{\mathrm{c}}_{\mathbb{D},x_s,x_t}$ at ${}^\vee a\in\mathcal{A}$ is defined as the composite

$$\begin{aligned} &\gamma^{\mathrm{c}}_{\mathbb{D},x_s,x_t}({}^\vee a): \\ &\quad \mathrm{T}^{\mathrm{pre}}(\mathbb{D})(-) \xrightarrow{(\mathrm{ev}^l)^*} \int^{m_s\in\mathcal{M}_s}\!\!\int^{m_t\in\mathcal{M}_t}\cdots\ \mathrm{Hom}(\overline{m_s}\boxtimes(a\otimes{}^\vee a).m_s\boxtimes m_t\boxtimes\overline{m_t}\boxtimes\cdots,-) \\ &\qquad\xrightarrow{\gamma_{x_s,x_t}({}^\vee a)} \int^{m_s\in\mathcal{M}_s}\!\!\int^{m_t\in\mathcal{M}_t}\cdots\ \mathrm{Hom}(\overline{m_s}\boxtimes a.m_s\boxtimes a^{[\mu-l+1]}.m_t\boxtimes\overline{m_t}\boxtimes\cdots,-)\,. \end{aligned} \tag{4.55}$$

Here $\gamma^{\mathrm{c}}_{x_s,x_t}({}^\vee a)$ are (a slight generalization of) the parallel transport operations in Lemma 4.19, which are now allowed to start and end at the variables corresponding to the defect lines; we consider them in the clockwise version, i.e. the first isomorphism in $\gamma^{\mathrm{c}}_{x_s,x_t}({}^\vee a)$ is induced by the isomorphism $\int^{m_s} a.m_s \boxtimes \overline{m_s} \cong \int^{m_s} a.m_s \boxtimes \overline{a^\vee.m_s}$. Then $\gamma^{\mathrm{c}}_{\mathbb{D},x_s,x_t}$ is defined by taking the end over all $\gamma^{\mathrm{c}}_{\mathbb{D},x_s,x_t}({}^\vee a)$. $\gamma^{\mathrm{cc}}_{\mathbb{D},x_s,x_t}$ is defined analogously by using the counterclockwise parallel transport operations instead.
Together with Proposition 4.15 these prescriptions imply that, in all four cases, both $\gamma^{\mathrm{c}}_{\mathbb{D}}$ and $\gamma^{\mathrm{cc}}_{\mathbb{D}}$ are indeed natural transformations from $\mathrm{T}^{\mathrm{pre}}(\mathbb{D})$ to $Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\mathbb{D}))$.
To show that $\gamma^{\mathrm{c}}_{\mathbb{D}}$ is a comodule structure, we first note that composing it with the counit ϵ of $Z_{\mathbb{D},x_s,x_t}$ yields the identity on $\mathrm{T}^{\mathrm{pre}}(\mathbb{D})$, since the parallel transport of $\mathbf{1}$ on $\mathrm{T}^{\mathrm{pre}}(\mathbb{D})$ is the identity. Next consider, for $a,b \in \mathcal{A}$, the component $\gamma^{\mathrm{c}}_{\mathbb{D},a\otimes b}$ of the parallel transport from x_s to x_t. In each step involved in the parallel transport operation we use either adjunction morphisms or a balancing, and both of these are compatible with the monoidal structure of $\mathcal{A}$, i.e. can be split into the composite of corresponding steps for $\gamma^{\mathrm{c}}_{\mathbb{D},a}$ and $\gamma^{\mathrm{c}}_{\mathbb{D},b}$. In both cases the morphisms commute (up to changing their arguments accordingly), and after passing to the end we obtain that indeed $\gamma^{\mathrm{c}}_{\mathbb{D}}$ is a comodule structure for $\mathrm{T}^{\mathrm{pre}}(\mathbb{D})$. The proof for $\gamma^{\mathrm{cc}}_{\mathbb{D}}$ is analogous.

(iii) The statement follows from the use of the parallel transport operations, analogously as in the proof of Lemma 4.19. □

Analogous statements as in Proposition 4.24 still hold in the situation that one or both of x_s and x_t in the Lemma is a general element of the set $E_{\mathbb{D}}$: There is a comonad $Z_{\mathbb{D},x_s,x_t}$, defined again case by case, on $\mathcal{L}ex(\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}} \boxtimes \mathrm{U}(\partial\mathbb{D}), \mathrm{vect})$ such that the parallel transport operations provide comodule structures for $\mathrm{T}^{\mathrm{pre}}(\mathbb{D})$ over $Z_{\mathbb{D},x_s,x_t}$ and the block functor is the corresponding equalizer.

As in Section 4.1 we denote the Deligne product over labels for all boundary segments of a defect surface Σ by $\mathrm{U}(\partial\Sigma)$. The comonad $Z_{\mathbb{D},x_s,x_t}$ on $\mathcal{L}ex(\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}} \boxtimes \mathrm{U}(\partial\mathbb{D}), \mathrm{vect})$ induces a corresponding comonad on the functor category $\mathcal{L}ex(\mathrm{U}(\partial\Sigma)^{\mathrm{opp}} \boxtimes \mathrm{U}(\partial\Sigma), \mathrm{vect})$, which for simplicity we denote again by $Z_{\mathbb{D},x_s,x_t}$. Clearly, $\mathrm{T}^{\mathrm{pre}}(\Sigma)$ becomes a comodule over the latter comonad $Z_{\mathbb{D},x_s,x_t}$. We call the comonads $Z_{\mathbb{D},x_s,x_t}$ the *parallel transport comonads.*

4.4 Blocks as module natural transformations

In this subsection we express the block spaces for the prototypical example of the straight disk (4.7) considered in Example 4.4 as spaces of module natural transformations. Recall that the pre-block functor in this example is given by formula (4.10) or, equivalently, by the space (4.11) of natural transformations, and that the holonomy is described in (4.25).

According to Lemma 4.17, for comodules $w,x \in \mathcal{M}^T$ over a comonad T on $\mathcal{M}$, with coactions δ_w and δ_x, the vector space $\mathrm{Hom}_{\mathcal{M}^T}(w,x)$ of comodule morphisms can be described as the equalizer of the two maps

$$\psi_1, \psi_2 : \quad \mathrm{Hom}_{\mathcal{M}}(U(w), U(x)) \longrightarrow \mathrm{Hom}_{\mathcal{M}}(U(w), T(U(x))) \tag{4.56}$$

that are given by $\psi_1(f) = \delta_x \circ f$ and $\psi_2(f) = T(f) \circ \delta_w$, respectively. The properties of an adjunction readily imply

Lemma 4.25. If the comonad T on $\mathcal{M}$ is left exact, then the map ψ_2 in (4.56) equals the composite

$$\mathrm{Hom}_{\mathcal{M}}(Uw,Ux) \xrightarrow{(\delta_w^{\text{l.a.}})^*} \mathrm{Hom}_{\mathcal{M}}(T^{\text{l.a.}}(Uw),Ux) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{M}}(Uw,T(Ux)) \tag{4.57}$$

of the pre-composition with the image $\delta_w^{\text{l.a.}}$ of δ_w under the adjunction $\mathrm{Hom}_{\mathcal{M}}(Uw,T(Uw)) \cong \mathrm{Hom}_{\mathcal{M}}(T^{\text{l.a.}}(Uw),Uw)$ and the map provided by the adjunction.

In the situation of Example 4.4 we deal with the two categories (4.8), i.e. $\mathrm{T}(\mathbb{L}_1) = \overline{\mathcal{M}} \overset{1}{\boxtimes} \mathcal{N}$ and $\mathrm{T}(\mathbb{L}_2) = \overline{\mathcal{N}} \overset{-1}{\boxtimes} \mathcal{M} \simeq \mathrm{T}(\mathbb{L}_1)^{\text{opp}}$. By Lemma 3.8, $\mathrm{T}(\mathbb{L}_1)$ is equivalent to the category of comodules over the comonad $Z_{[1]}$ on $\overline{\mathcal{M}} \boxtimes \mathcal{N}$, while $\mathrm{T}(\mathbb{L}_2)$ corresponds to the comodules over the comonad $Z_{[-1]}$ on $\overline{\mathcal{N}} \boxtimes \mathcal{M}$. Now note that $(\overline{\mathcal{N}} \boxtimes \mathcal{M})^{\text{opp}} \simeq \overline{\mathcal{M}} \boxtimes \mathcal{N}$. By direct computation, using the dualities of the finite tensor category $\mathcal{A}$ and the fact that taking the opposite interchanges coend and end, we then have

Lemma 4.26. The comonads on $Z_{[1]}$ on $\overline{\mathcal{M}} \boxtimes \mathcal{N}$ and $Z_{[-1]}$ on $\overline{\mathcal{N}} \boxtimes \mathcal{M}$ are related as

$$Z_{[-1]} \;\cong\; \overline{Z_{[1]}^{\text{l.a.}}}\,, \tag{4.58}$$

where the overline indicates the opposite functor (compare Eq. (B.7)).

To highlight the power of our modular functor for producing higher algebra, we now describe the block functor for the situation of Example 4.10.

Proposition 4.27. The block functor for the straight disk $\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}}$ (4.7) is isomorphic to the Hom functor for the category of comodules over the comonad $Z_{[1]}$, i.e. we have

$$\mathrm{T}_{\text{fine}}(\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}})(x \boxtimes y) \;\cong\; \mathrm{Hom}_{Z_{[1]}}(y,x) \tag{4.59}$$

for any pair of objects $x \in \mathrm{T}(\mathbb{L}_1)$ and $y \in \mathrm{T}(\mathbb{L}_2)$. Hence the block space can be seen as a space of module natural transformations.

Proof. (i) The main idea is to invoke Lemma 4.17 which characterizes the morphisms of comodules over a comonad as an equalizer. To this end we have to match the two morphisms (4.39) in that Lemma with the two parallel arrows in the Definition 4.18 of the block functor. In the situation at hand, the block space is the equalizer

$$\mathrm{T}_{\text{fine}}(x \boxtimes y) \longrightarrow \mathrm{T}^{\text{pre}}(x \boxtimes y) \underset{(\mu_x)_*}{\overset{\mathrm{hol}_x}{\rightrightarrows}} \mathrm{T}^{\text{pre}}(Z_{[1]}(x) \boxtimes y)\,. \tag{4.60}$$

According to (4.10) the pre-blocks of our interest are given by $\mathrm{T}^{\text{pre}}(x \boxtimes y) \cong \mathrm{Hom}_{\overline{\mathcal{M}}\boxtimes\mathcal{N}}(y,x)$. In this description the map $(\mu_x)_*$ amounts to the map

$$\begin{aligned} \mathrm{Hom}(y,x) &\longrightarrow \mathrm{Hom}(y,Z_{[1]}(x))\,, \\ f &\longmapsto \mu_x \circ f. \end{aligned} \tag{4.61}$$

According to Lemma 4.26, the comodule structure δ_y on y gives the module structure $Z_{[1]}^{\text{l.a.}} y \to y$ as the image $\delta_y^{\text{l.a.}}$ of δ_y under the adjunction $\mathrm{Hom}(y, Z_{[-1]}(y)) \cong \mathrm{Hom}(Z_{[-1]}^{\text{l.a.}}(y), y)$.

(ii) To obtain (4.59), by Lemma 4.25 we are thus left with showing that the morphism hol_x corresponds to the composite

$$\mathrm{Hom}(y,x) \xrightarrow{(\delta_y^{\mathrm{l.a.}})^*} \mathrm{Hom}(Z_{[1]}^{\mathrm{l.a.}}(y),x) \xrightarrow{\cong} \mathrm{Hom}(y,Z_{[1]}(x))\,, \tag{4.62}$$

where the first map is pre-composition with $\delta_y^{\mathrm{l.a.}}$ and the second one is provided by the adjunction. To show that this is the case, we consider for $a\in\mathcal{A}$ the following two maps. First, the map

$$\begin{aligned} g_a: \quad \mathrm{T}^{\mathrm{pre}}(\overline{x_m}\boxtimes x_n\boxtimes\overline{y_n}\boxtimes y_m) &\xrightarrow{(\mathrm{ev^r})^*} \mathrm{T}^{\mathrm{pre}}(\overline{x_m}\boxtimes x_n\boxtimes\overline{(a^\vee\otimes a)\,.\,y_n}\boxtimes y_m) \\ &\xrightarrow{\cong} \mathrm{T}^{\mathrm{pre}}(\overline{x_m}\boxtimes x_n\boxtimes\overline{a^\vee.y_n}\boxtimes a^\vee.y_m)\,, \end{aligned} \tag{4.63}$$

composed from an evaluation in the contravariant argument of the pre-block functor and the balancing of y. And second, the isomorphism

$$h_a: \quad \mathrm{T}^{\mathrm{pre}}(\overline{x_m}\boxtimes x_n\boxtimes\overline{a^\vee.y_n}\boxtimes a^\vee.y_m) \xrightarrow{\cong} \mathrm{T}^{\mathrm{pre}}(\overline{a^{\vee\vee}.\,x_m}\boxtimes a\,.\,x_n\boxtimes\overline{y_n}\boxtimes y_m) \tag{4.64}$$

that is analogous to the composition of the second, third and fourth maps in the chain (4.24) of isomorphisms. Together with the component $(\mathrm{hol}_x)_a$ of the holonomy we thus have a diagram

$$\begin{array}{ccc} \mathrm{T}^{\mathrm{pre}}(\overline{x_m}\boxtimes x_n\boxtimes\overline{y_n}\boxtimes y_m) & \xrightarrow{\quad(\mathrm{hol}_x)_a\quad} & \mathrm{T}^{\mathrm{pre}}(\overline{a^{\vee\vee}.x_m}\boxtimes a.x_n\boxtimes\overline{y_n}\boxtimes y_m) \\ & {\scriptstyle g_a}\searrow \qquad \nearrow{\scriptstyle h_a} & \\ & \mathrm{T}^{\mathrm{pre}}(\overline{x_m}\boxtimes x_n\boxtimes\overline{a^\vee.y_n}\boxtimes a^\vee.y_m) & \end{array} \tag{4.65}$$

By a straightforward, albeit lengthy, computation using the definition of the holonomy hol_x, the dualities of $\mathcal{A}$ and the balancings (3.53) of the object $\int_m \overline{m}\boxtimes m$, it can be seen that this diagram commutes.

Next we take the end over $a\in\mathcal{A}$ in the diagram (4.65). Then in the top row we get the holonomy, while

$$\int_a g_a: \quad \mathrm{T}^{\mathrm{pre}}(x\boxtimes y) \longrightarrow \mathrm{T}^{\mathrm{pre}}(x\boxtimes \int_a \overline{a^\vee.y_n}\boxtimes a^\vee.y_m) \;\cong\; \mathrm{T}^{\mathrm{pre}}(x\boxtimes \int_a \overline{a.y_n}\boxtimes a.y_m)\,. \tag{4.66}$$

Moreover, by the adjunction obtained in Lemma 4.26 we have

$$\mathrm{T}^{\mathrm{pre}}(x\boxtimes \int_a \overline{a.y_n}\boxtimes a.y_m) = \mathrm{T}^{\mathrm{pre}}(x\boxtimes Z_{[-1]}(y)) \xrightarrow{\cong} \mathrm{Hom}(Z_{[1]}^{\mathrm{l.a.}}(y),x)\,. \tag{4.67}$$

It follows that we have a commuting diagram

$$\begin{array}{ccc} \mathrm{Hom}(Z_{[1]}^{\mathrm{l.a.}}(y),x) & \xrightarrow{\qquad\cong\qquad} & \mathrm{Hom}(y,Z_{[1]}(x)) \\ \big\downarrow{\scriptstyle\cong} & & \big\downarrow{\scriptstyle\cong} \\ \int_{a\in\mathcal{A}} \mathrm{T}^{\mathrm{pre}}(\overline{x_m}\boxtimes x_n\boxtimes\overline{a^\vee.y_n}\boxtimes a^\vee.y_m) & \xrightarrow{\cong} & \int_{a\in\mathcal{A}} \mathrm{T}^{\mathrm{pre}}(\overline{a^{\vee\vee}.\,x_m}\boxtimes a\,.\,x_n\boxtimes\overline{y_n}\boxtimes y_m) \end{array} \tag{4.68}$$

This concludes the proof of the first statement of the Proposition.

(iii) Due to Corollary B.7, there is an equivalence $\mathrm{T}(\mathbb{L}_1)\simeq\mathcal{L}ex_{\mathcal{A}}(\mathcal{M},\mathcal{N})$. $\square$

Combining the result (4.59) with the equivalence $\mathrm{T}(\mathbb{L}_1) \simeq \mathcal{L}ex_{\mathcal{A}}(\mathcal{M},\mathcal{N})$, we get (similarly as for the formula (4.11) for the pre-block functor):

Corollary 4.28. The block spaces for the straight disk $\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}}$ (4.7) are given by the spaces

$$\mathrm{T}_{\mathrm{fine}}(\mathbb{D}^{\uparrow}_{\mathcal{A},\mathcal{M},\mathcal{N}})(x \boxtimes y) \cong \mathrm{Nat}_{\mathcal{A}}(\Phi^{\mathrm{l}}(\overline{y}), \Phi^{\mathrm{l}}(x))\,. \tag{4.69}$$

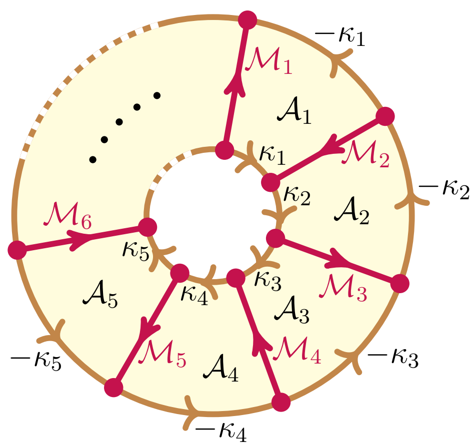
of module natural transformations between the module functors $\Phi^{\mathrm{l}}(\overline{y})$ and $\Phi^{\mathrm{l}}(x)$.

Example 4.29. Consider a disk with any number N of gluing and free boundary segments, as studied in Example 4.7, imposing in addition the sum rule $\sum_i \kappa_i = N-2$ from Lemma 4.13 on the framing indices. Then Proposition 4.27 implies that the block functor is given by

$$\mathrm{T}_{\mathrm{fine}}(x_1 \boxtimes x_2 \boxtimes \cdots \boxtimes x_n) \cong \mathrm{Nat}_{\mathcal{A}}\big(\Phi^{\mathrm{l}}(\overline{x_1}), \Phi^{\mathrm{l}}(x_n) \circ \Phi^{\mathrm{l}}(x_n) \circ \cdots \circ \Phi^{\mathrm{l}}(x_2)\big)\,, \tag{4.70}$$

where the module structure on the module functors $\Phi^{\mathrm{l}}(x_i)$ is determined by the framing indices of the adjacent gluing intervals.

In particular, the result of Proposition 4.27 for the block functor of the straight disk generalizes to the case of disks of the form (4.28). This, in turn, can be used to give the block functor for any cylinder that is built out of such disks, i.e. for all defect surfaces of the form

$\Sigma =$ 

(4.71)

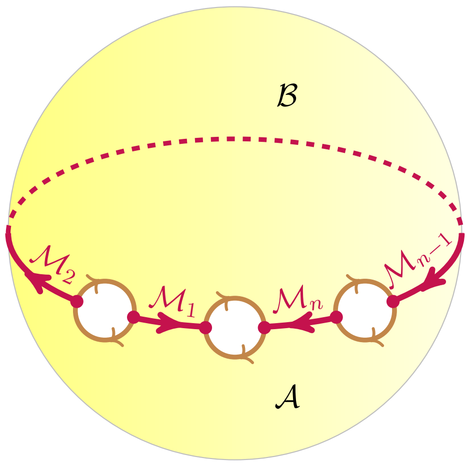
We refer to surfaces of the form (4.71), as well as to their counterparts based on a gluing interval instead of a gluing circle, as *defect cylinders*.

Corollary 4.30. Let Σ be defect cylinder as shown in picture (4.71). Denote the inner boundary circle of Σ by $\mathbb{S}_1$ and the outer one by $\mathbb{S}_2$. We have $\mathrm{T}(\mathbb{S}_1) \cong \mathrm{T}(\mathbb{S}_2)^{\mathrm{opp}}$.
Regarding Σ as a bordism from $\mathbb{S}_1 \sqcup \mathbb{S}_2$ to $\emptyset$, the block functor for Σ takes the values

$$\mathrm{T}_{\mathrm{fine}}(\Sigma)(F \boxtimes G) = \mathrm{Hom}_{\mathrm{T}(\mathbb{S}_2)}(F,G) \tag{4.72}$$

for $F \in \mathrm{T}(\mathbb{S}_1)$ and $G \in \mathrm{T}(\mathbb{S}_2)$. Regarding instead Σ as a bordism from $\mathrm{T}(\mathbb{S}_1)$ to $\mathrm{T}(\mathbb{S}_2)^{\mathrm{opp}} \cong \mathrm{T}(\mathbb{S}_1)$, $\mathrm{T}_{\mathrm{fine}}(\Sigma)$ is just the identity functor on $\mathrm{T}(\mathbb{S}_1)$.

Example 4.31. For finite tensor categories $\mathcal{A}$ and $\mathcal{B}$, an $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{M}_1$ and any finite number of $\mathcal{A}$-$\mathcal{B}$-bimodules $\mathcal{M}_i$, $i=2,3,\dots,n$, consider a sphere with one circular defect line that is interrupted by gluing circles (with arbitrary admissible framing indices, which we suppress in the picture) which cut it into intervals labeled by the bimodules $\mathcal{M}_i$:



(4.73)

This can be recognized as the defect surface Σ that is obtained when gluing a disk of the type considered in Examples 4.7 and 4.29 along its boundary to an oppositely oriented disk of the same type. Accordingly, the pre-block spaces – which only depend on the incidence relations of $\Sigma^{(1)}$ (i.e. defect lines and the boundary of Σ) – are given by the same expression (4.18) as the pre-block spaces for one of those disks, albeit with their arguments now being objects in different categories, which are assigned to gluing circles rather than gluing intervals. Imposing flat holonomy for the upper and lower disk, respectively, then amounts, in the formulation with left exact functors, to requiring that the so obtained natural transformations are module natural transformations with respect to the right $\mathcal{A}$- and left $\mathcal{B}$-action, respectively. Imposing both of these (where the order does not matter) thus yields $\mathcal{A}$-$\mathcal{B}$-*bimodule* natural transformations:

$$\mathrm{T}_{\text{fine}}(x_1 \boxtimes x_2 \boxtimes \cdots \boxtimes x_n) \cong \mathrm{Nat}_{\mathcal{A},\mathcal{B}}\big(\Phi^{\mathrm{l}}(\overline{x_1}), \Phi^{\mathrm{l}}(x_n) \circ \Phi^{\mathrm{l}}(x_{n-1}) \circ \cdots \circ \Phi^{\mathrm{l}}(x_2)\big)\,. \tag{4.74}$$

Of particular interest is the special case that all defects are transparent, i.e. that $\mathcal{B}=\mathcal{A}$ and $\mathcal{M}_i=\mathcal{A}$ for each i, and that the framing indices on the two segments of all gluing circles, except for the circle between the $\mathcal{M}_1$- and $\mathcal{M}_n$-line, are equal to 1. Then the gluing categories for all other circles are given by

$$\mathcal{A} \overset{1}{\boxtimes} \overline{\mathcal{A}} \overset{1}{\boxtimes} \overset{(3.55)}{\simeq} \mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A}) \simeq \mathcal{Z}(\mathcal{A})\,, \tag{4.75}$$

and the composition of functors in (4.74) amounts to the tensor product in the Drinfeld center $\mathcal{Z}(\mathcal{A})$.

4.5 Contraction along a defect path

Recall from Proposition 4.22 the result about the fusion of boundary insertions, i.e. that there is a canonical isomorphism between the block functors for the configurations shown in the picture (4.15). These two configurations are related by contracting a single defect line between two gluing boundaries. In this subsection we obtain a vast generalization of this procedure; we provide the behavior of the block functor under "contraction along a path of defect lines".

By a *path of defect lines* in a defect surface Σ we mean an ordered collection $\gamma = (\delta_1, \delta_2, \ldots, \delta_n)$ of defect lines such that each pair (δ_i, δ_{i+1}) shares a common gluing boundary; we denote the latter by $\mathbb{L}_{i,i+1}$. Such a path can be open or closed. In the following we first assume for simplicity that all the gluing boundaries $\mathbb{L}_{i,i+1} = \mathbb{S}_{i,i+1}$ are gluing *circles*; the situation that also gluing intervals are present will be briefly discussed afterwards. Given a path γ of defect lines, we consider a fattening of γ to a tubular neighborhood $N_\gamma \subset \Sigma$ which is sufficiently large such that its interior contains all gluing boundaries $\mathbb{S}_{i,i+1}$ on the path and sufficiently small such that it does not meet any other gluing boundaries of Σ. Paths γ of defect lines come in several types, which are distinguished by the form of the boundary ∂N_γ of their tubular neighborhood:

(i) ∂N_γ consists of a single embedded circle in Σ.

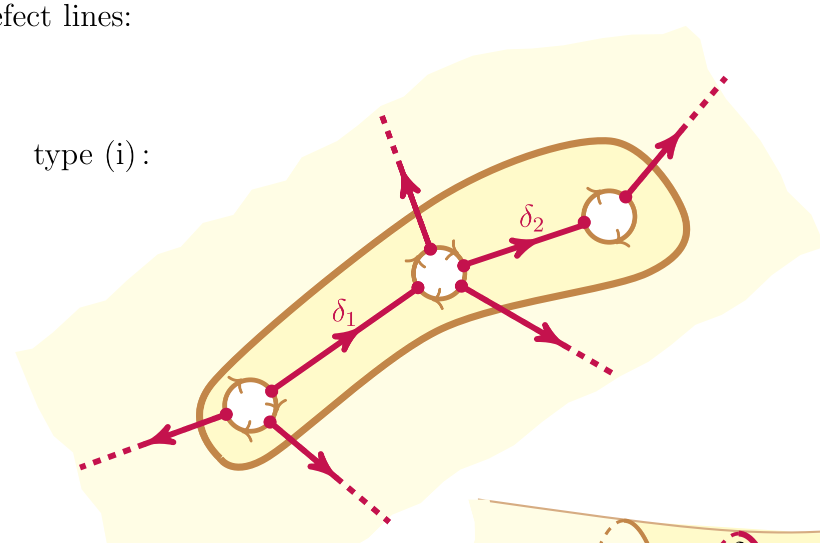
(ii) ∂N_γ is the disjoint union of of two embedded circles $\partial N_{\gamma,1}$ and $\partial N_{\gamma,2}$, both of which meet at least one defect line of Σ.

(iii) ∂N_γ is the disjoint union of of two embedded circles $\partial N_{\gamma,1}$ and $\partial N_{\gamma,2}$, and precisely one of them does not meet any defect line of Σ.

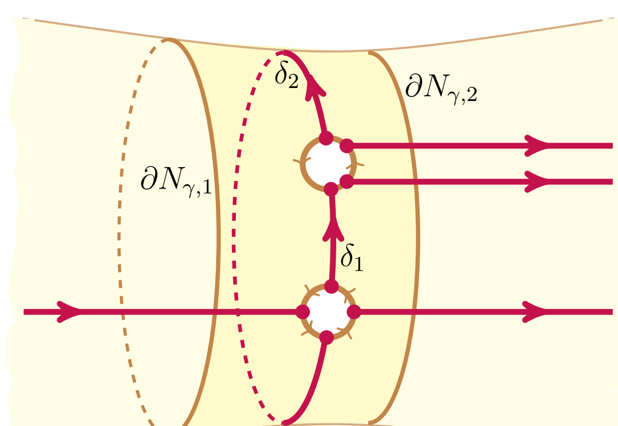
(iv) ∂N_γ is the disjoint union of of two embedded circles $\partial N_{\gamma,1}$ and $\partial N_{\gamma,2}$, none of which meets any defect line of Σ.

If Σ is fine, then in the cases (iii) and (iv) γ is a loop that encloses a disk $\mathbb{D}$ in Σ. The following pictures give examples for a path of type (i) and a path of type (ii), each consisting of two defect lines:

type (i) :



type (ii) :



(4.76)

For paths of type (i) and (ii), the boundary ∂N_γ has a natural structure of a defect one-manifold, with framing inherited from the given vector field on Σ. We denote by $\mathrm{T}(\partial N_\gamma)$ its gluing category; thus in case (ii), $\mathrm{T}(\partial N_\gamma) = \mathrm{T}(\partial N_{\gamma,1}) \boxtimes \mathrm{T}(\partial N_{\gamma,2})$. In cases (iii) and (iv), in which ∂N_γ is not a proper defect one-manifold, we still define $\mathrm{T}(\partial N_\gamma)$ to be $\mathrm{T}(\partial N_{\gamma,1}) \boxtimes \mathrm{T}(\partial N_{\gamma,2})$ where, however, now by definition we assign to a component $\partial N_{\gamma,i}$ that does not meet any defect line on Σ the category $\mathrm{T}(\partial N_{\gamma,i}) := \mathrm{vect}$.

Let now $\gamma = (\delta_1, \delta_2, \ldots, \delta_n)$ be a path of defect lines in a fine defect surface Σ, and denote the bimodule category labeling δ_i by $\mathcal{M}_i$. Denote by

$$\mathrm{T}(\mathring{N}_\gamma) := \boxtimes_{i=1}^{n-1} \mathrm{T}(\mathbb{S}_{i,i+1}) \tag{4.77}$$

the gluing category for the defect one-manifold that is the disjoint union of all gluing boundaries inside N_γ, and (analogously as at the beginning of Section 4.1)) by $\mathrm{U}(\mathring{N}_\gamma) = \boxtimes_{i=1}^{n-1} \mathrm{U}(\mathbb{S}_{i,i+1})$ the category obtained from $\mathrm{T}(\mathring{N}_\gamma)$ by forgetting all balancings of the framed centers. It follows from our conventions that

$$\mathrm{U}(\mathring{N}_\gamma) \cong \Big(\boxtimes_{i=1}^{n} \mathcal{M}_i \boxtimes \overline{\mathcal{M}_i}\Big) \boxtimes \mathrm{U}(\partial N_\gamma)\,. \tag{4.78}$$

For an object $x \in \mathrm{T}(\mathring{N}_\gamma)$ such that $x = x_\circ \boxtimes x_\partial$ with $U(x_\circ) \in \boxtimes_i \mathcal{M}_i \boxtimes \overline{\mathcal{M}_i}$ and $U(x_\partial) \in \mathrm{U}(\partial N_\gamma)$ we define

$$\widehat{x} := \int^{m_1 \in \mathcal{M}_1, \ldots, m_n \in \mathcal{M}_n} \mathrm{Hom}_{\mathcal{M}_1 \boxtimes \overline{\mathcal{M}_1} \boxtimes \cdots \boxtimes \mathcal{M}_n \boxtimes \overline{\mathcal{M}_n}} \big(m_1 \boxtimes \overline{m_1} \boxtimes \cdots \boxtimes m_n \boxtimes \overline{m_n}\,, U(x_\circ)\big) \otimes U(x_\partial)\,. \tag{4.79}$$

This is by construction an object in $\mathrm{U}(\partial N_\gamma)$.

Lemma 4.32. For $x \in \mathrm{T}(\mathring{N}_\gamma)$, the object $\widehat{x}$ defined by (4.79) comes naturally with balancings which endow it with the structure of an object in the gluing category $\mathrm{T}(\partial N_\gamma)$.

Proof. The required balancings are fully determined by the balancings of the object $x \in \mathrm{T}(\mathring{N}_\gamma)$ and the parallel transport operations (4.46) for the 2-patches in N_γ. To see this, let us write $x = \boxtimes_i x_{i,i+1}$ with $x_{i,i+1} \in \mathrm{T}(\mathbb{S}_{i,i+1})$. Consider any two neighboring defect points P_1 and P_2 on a component $\partial N_{\gamma,i}$ of ∂N_γ. If P_1 and P_2 correspond to neighboring defect points on one and the same gluing boundary $\mathbb{S}_{i,i+1}$, then the balancing on the object $\widehat{x}$ is directly provided by the balancing of $x_{i,i+1} \in \mathrm{T}(\mathbb{S}_{i,i+1})$. Otherwise P_1 and P_2 correspond to defect points that lie on different gluing boundaries which are connected by a sub-path $(\delta_j, \ldots, \delta_{j'})$ of γ. In this case the parallel transport operations (4.46) give the relevant balancing. That these balancings are indeed those for the category $\mathrm{T}(\partial N_\gamma)$ follows directly from the definitions. □

It follows that the prescription (4.79) provides a functor from $\mathrm{T}(\mathring{N}_\gamma)$ to $\mathrm{T}(\partial N_\gamma)$, provided that the path γ is of the type (i) or (ii). In case of the type (iii) or (iv), γ is a closed path, and we apply after the prescription (4.79) in addition the block equalizers (see (4.43)) for all disks that are enclosed by γ. The balancings given in the proof of Lemma 4.32 are not affected by applying these block equalizers, so that we end up again with a functor from $\mathrm{T}(\mathring{N}_\gamma)$ to $\mathrm{T}(\partial N_\gamma)$:

Definition 4.33. Let γ be a path of defect lines in a defect surface Σ.

(i) The *excision functor* for the path γ is the functor

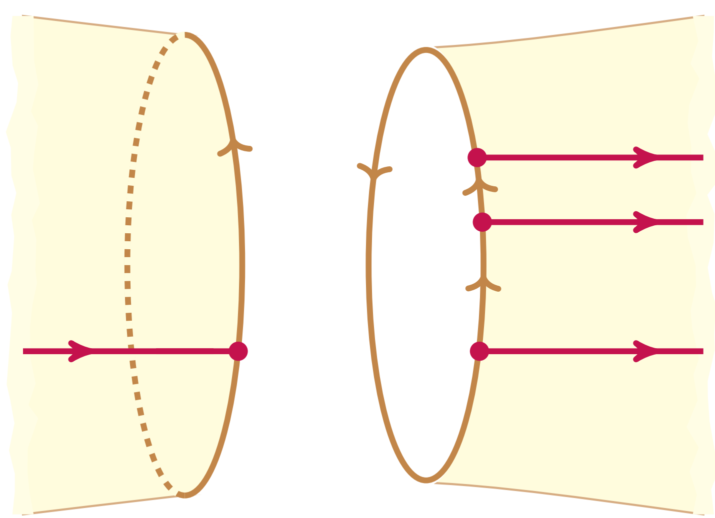
$$\mathrm{E}_\gamma : \quad \mathrm{T}(\mathring{N}_\gamma) \longrightarrow \mathrm{T}(\partial N_\gamma) \tag{4.80}$$

that is obtained by the presciption given above.

(ii) The *contraction of* Σ *along* γ, denoted as Σ_γ, is the following defect surface: For γ of the type (i) or (ii), we take $\Sigma_\gamma := \Sigma \setminus N_\gamma$ to be the complement of N_γ, with ∂N_γ as a new gluing boundary component for type (i), respectively $\partial N_{\gamma,1} \sqcup \partial N_{\gamma,2}$ as two new gluing boundaries for type (ii). If γ of the type (iii), Σ_γ is the component of $\Sigma \setminus N_\gamma$ whose boundary contains the boundary component $\partial N_{\gamma,i}$ of N_γ that meets at least one defect line of Σ. Finally, in case γ is of the type (iv) we set $\Sigma_\gamma := \emptyset$.

The termionology 'excision functor' suggests that this functor is related to locality properties of our construction. Indeed it will play a role when describing a factorization structure for the modular functor (see the proof of Theorem 5.2).

We also call the object $\mathrm{E}_\gamma(x)$ the *contraction of* x *along* γ. The following picture shows the contraction Σ_γ for the path of type (ii) that is shown in the picture (4.76):



(4.81)

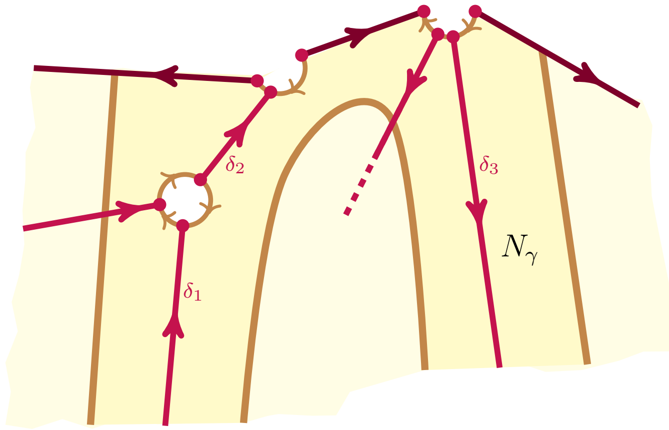
Lemma 4.34. Let γ be a path of defect lines in a defect surface Σ. There is a canonical isomorphism

$$\varphi_\gamma : \quad \mathrm{T}_{\text{fine}}(\Sigma) \xrightarrow{\cong} \mathrm{T}_{\text{fine}}(\Sigma_\gamma) \circ (\mathrm{E}_\gamma \boxtimes \mathrm{Id}) \tag{4.82}$$

of functors, where the identity functor is applied to the gluing categories for all gluing boundaries of Σ that are not met by γ (and where in the case of γ being of type (iv), the canonical equivalence $\mathrm{vect} \boxtimes \mathcal{M} \simeq \mathcal{M}$ for any finite category $\mathcal{M}$ is used implicitly on the right hand side).

Proof. For types (i) and (ii) there is even a corresponding isomorphism involving pre-block functors which holds by construction. Moreover, by definition of the balancings of the objects (4.79), all holonomy operations for Σ and for Σ_γ agree. Thus the isomorphism follows for all types (i) – (iv). $\square$

If we allow γ to contain also free boundaries, and thus some of the gluing boundaries $\mathbb{S}_{i,i+1}$ are gluing intervals, then the tubular neighborhood N_γ looks as indicated in



(4.83)

The definition of contraction of Σ along γ, as well as the statements of Lemma 4.32 and Lemma 4.34 generalize to this case accordingly.

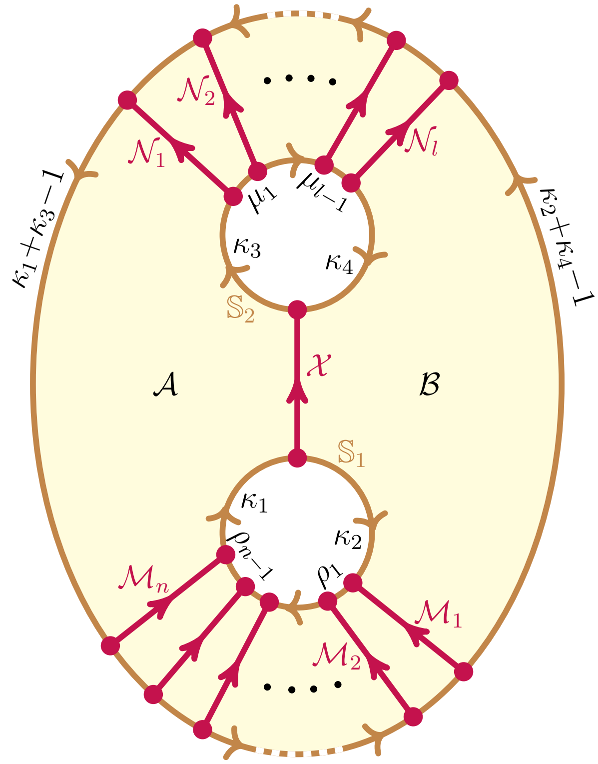
Remark 4.35. The construction can be further generalized to the case of an arbitrary graph Γ formed by defect lines of Σ. Taking a tubular neighborhood N_Γ that encloses all gluing boundaries met by Γ we obtain analogously categories $\mathrm{T}(\mathring{N}_\Gamma)$ and $\mathrm{T}(\partial N_\Gamma)$, an excision functor $\mathrm{E}_\Gamma \colon \mathrm{T}(\mathring{N}_\Gamma) \to \mathrm{T}(\partial N_\Gamma)$ between these, and a defect surface Σ_Γ together with an isomorphism $\mathrm{T}_{\mathsf{fine}}(\Sigma) \to \mathrm{T}_{\mathsf{fine}}(\Sigma_\Gamma) \circ (\mathrm{E}_\Gamma \boxtimes \mathrm{Id})$. If we take the graph Γ_{tot} formed by *all* defect lines of Σ, we obtain this way a canonical isomorphism

$$\mathrm{T}_{\mathsf{fine}}(\Sigma) \cong \mathrm{E}_{\Gamma_{\mathrm{tot}}} \tag{4.84}$$

between the block functor for Σ and the excision functor for the graph of all defect lines of Σ.

Example 4.36. We use excision to compute the block functor for the following defect surface Σ:

$$\Sigma \;=$$



(4.85)

We regard Σ as a pair of pants with $\mathbb{S}_1$ and $\mathbb{S}_2$ as incoming boundary circles. Consider first the case that $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = 1$. Then using the notation $\vec{\mathcal{M}}\overset{\vec{\rho}}{\boxtimes} := \mathcal{M}_1 \overset{\rho_1}{\boxtimes} \cdots \overset{\rho_{n-1}}{\boxtimes} \mathcal{M}_n$, by Corollary 3.18 we obtain

$$\mathrm{T}(\mathbb{S}_1) \cong \mathcal{L}ex_{\mathcal{A},\mathcal{B}}(\vec{\mathcal{M}}\overset{\vec{\rho}}{\boxtimes}, \mathcal{X}) \qquad \text{and} \qquad \mathrm{T}(\mathbb{S}_2) \cong \mathcal{L}ex_{\mathcal{A},\mathcal{B}}(\mathcal{X}, \vec{\mathcal{N}}\overset{\vec{\mu}}{\boxtimes}) \,. \tag{4.86}$$

For the path γ of defect lines in Σ that consists just of the defect line with label $\mathcal{X}$, the neighborhood N_γ encloses the gluing circles $\mathbb{S}_1$ and $\mathbb{S}_2$. For $F \in \mathrm{T}(\mathbb{S}_1)$ and $G \in \mathrm{T}(\mathbb{S}_2)$ we then compute $\mathrm{E}_\gamma(F \boxtimes G) \in \mathrm{T}(\partial N_\gamma) \cong \mathrm{T}(\mathbb{S}_3)^{\mathrm{opp}}$ by invoking the formula (4.79):

$$\begin{aligned} \mathrm{E}_\gamma(F\boxtimes G) \;&= \int^{x\in\mathcal{X}} \mathrm{Hom}\big(x \boxtimes \overline{x}\,, \big(\int^{z\in\vec{\mathcal{M}}\overset{\vec{\rho}}{\boxtimes}} \overline{z} \boxtimes F(z)\big) \boxtimes \big(\int^{y\in\mathcal{X}} \overline{y} \boxtimes G(y)\big)\big) \\ &\cong \int^{z\in\vec{\mathcal{M}}\overset{\vec{\rho}}{\boxtimes}} \int^{y\in\mathcal{X}} \int^{x\in\mathcal{X}} \mathrm{Hom}(x, F(z)) \otimes_{\Bbbk} \mathrm{Hom}(y,x) \otimes \overline{z} \boxtimes G(y) \\ &\cong \int^{z\in\vec{\mathcal{M}}\overset{\vec{\rho}}{\boxtimes}} \overline{z} \boxtimes G \circ F(z) \,. \end{aligned} \tag{4.87}$$

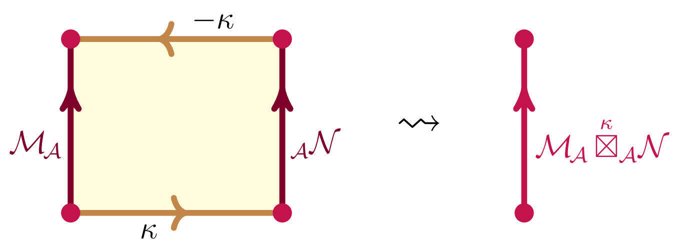
Under the Eilenberg-Watts equivalence, this amounts to the functor $G \circ F \in \mathcal{L}ex_{\mathcal{A},\mathcal{B}}(\vec{\mathcal{M}}\overset{\vec{\rho}}{\boxtimes}, \vec{\mathcal{N}}\overset{\vec{\mu}}{\boxtimes})$. Invoking Corollary 4.30 we thus we conclude that

$$\mathrm{T}_{\text{fine}}(\Sigma): \quad \mathrm{T}(\mathbb{S}_1) \boxtimes \mathrm{T}(\mathbb{S}_2) \longrightarrow \mathrm{T}(\mathbb{S}_3) \tag{4.88}$$

is the functor that corresponds to the composition of bimodule functors. (This is a simple instance of the way in which algebraic structure, here composition of left exact functors, can be extracted from our framed modular functor.) The case of general values of $\kappa_1, \kappa_2, \kappa_3$ and κ_4 is reduced to the one considered above by the equivalence obtained in Remark 3.20. When all defects involved are transparent, then via the Eilenberg-Watts isomorphisms the composition of functors gives the tensor product in the Drinfeld center (compare Example 4.31); note that this functor is the one that also the standard Turaev-Viro constructions assigns to a pair of pants.

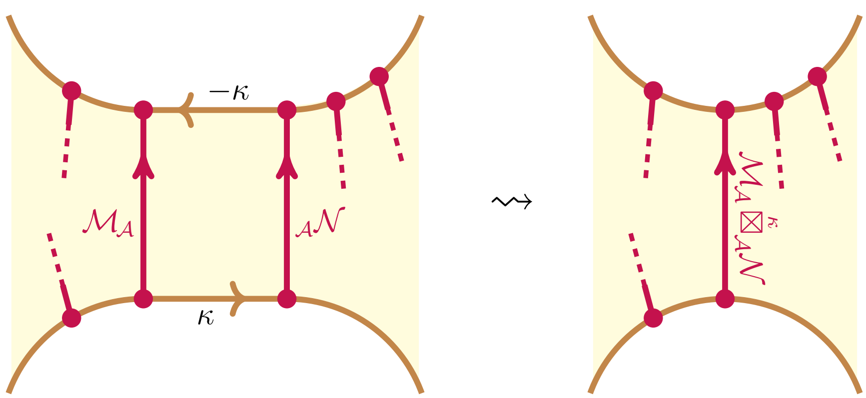
4.6 Fusion of defect lines along 2-patches

We will now show that the block functor for a defect surface remains unchanged if two parallel defect lines are *fused*, meaning that two neighboring defect lines are replaced by a single one with appropriate label. To see this, consider two parallel defect lines labeled by $\mathcal{M}_\mathcal{A}$ and ${}_\mathcal{A}\mathcal{N}$ that end on two gluing segments with framing of index $\pm\kappa$ for $\kappa \in 2\mathbb{Z}$. For our discussion only the right and left module structure, respectively, matters (and is displayed), so that we are in the situation of Example 4.11. We will see that locally we can replace this combinatorial configuration by a single defect line labeled by the framed center $\mathcal{M}_\mathcal{A} \overset{\kappa}{\boxtimes} {}_\mathcal{A}\mathcal{N}$; schematically,



(4.89)

This replacement is meant to happen locally within a generic defect surface, with all data in the parts not involved in the replacement remaining unchanged. In particular, the gluing segments on which the defect lines end will typically be part of gluing circles or gluing intervals that contain further defect points. This is indicated in the following picture:



(4.90)

We refer to this procedure as the *fusion of the defect lines* along a constantly framed 2-patch.

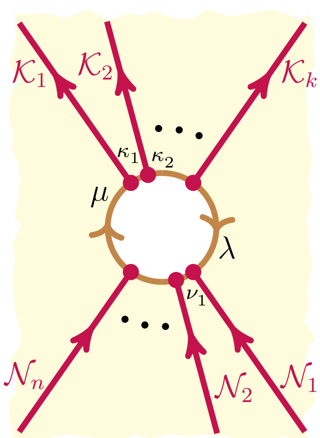
Theorem 4.37. There is a distinguished isomorphism between the block functors associated with the two combinatorial configurations (4.90).

Proof. First note that there is a canonical equivalence between the gluing categories for the gluing segments on which the two defect lines on the left hand side of (4.89) end. Consider objects $y \in \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{N}$ and $\overline{x} \in \overline{\mathcal{N}} \overset{-\kappa}{\boxtimes} \overline{\mathcal{M}}$ in the relevant gluing categories. After fusion, the disk $\mathbb{D}$ between the two lines has disappeared and there is just one defect line left; the latter gives rise to the block functor in the upper left corner of the following diagram:

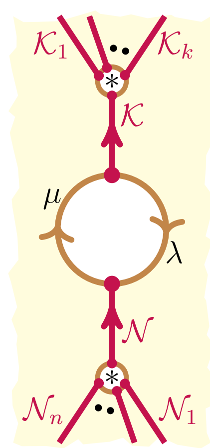
$$
\begin{array}{ccc}
\int^{z\in\mathcal{M}\overset{\kappa}{\boxtimes}\mathcal{N}} \mathrm{Hom}_{\overline{\mathcal{N}}\overset{-\kappa}{\boxtimes}\overline{\mathcal{M}}\boxtimes\mathcal{M}\overset{\kappa}{\boxtimes}\mathcal{N}}(\overline{z}\boxtimes z, \overline{x}\boxtimes y) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}\overset{\kappa}{\boxtimes}\mathcal{N}}(x,y) \\
\downarrow f & & \downarrow \\
\int^{m\in\mathcal{M}}\int^{n\in\mathcal{N}} \mathrm{Hom}_{\overline{\mathcal{N}}\boxtimes\overline{\mathcal{M}}\boxtimes\mathcal{M}\boxtimes\mathcal{N}}(\overline{n}\boxtimes\overline{m}\boxtimes m\boxtimes n, U(\overline{x})\boxtimes U(y)) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}\boxtimes\mathcal{N}}(U(x),U(y)) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\int^{m}\int^{n} \mathrm{Hom}_{\overline{\mathcal{N}}\boxtimes\overline{\mathcal{M}}\boxtimes\mathcal{M}\boxtimes\mathcal{N}}(\overline{n}\boxtimes\overline{m}\boxtimes m\boxtimes n, \int_a \overline{a.\,U(x)\,.^{[\kappa+3]}a}\boxtimes U(y)) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}\boxtimes\mathcal{N}}(Z_{[\kappa]}(U(x)),U(y))
\end{array}
\tag{4.91}
$$

In this diagram, each of the horizontal morphisms is a variant of the convolution property (B.2) of the Hom functor. The equalizer of the two morphisms on the bottom left of this diagram is by definition the block functor before fusion. The left column of the diagram is of the form of (4.39) and is thus an equalizer diagram.
This diagram (4.91) commutes: That the upper square commutes is (after using that the coend over $\mathcal{M}\boxtimes\mathcal{N}$ can be expressed as a double coend over $\mathcal{M}$ and $\mathcal{N}$, see [FSS2, Cor. 3.12]) just the definition of the morphism f, while commutativity of the two lower squares follows from the definition of the holonomy operation.
Together it follows that the right column is an equalizer diagram as well, and thus that indeed also the situation with the fused defect line describes the block functor. □

Iterating the procedure (4.90), one can analogously fuse any number of neighboring defect lines. Thus in particular there is a distinguished isomorphism between the block functors associated with the two situations



and



(4.92)

for any numbers n of incoming and k of outgoing defect points, with $\mathcal{K} = \mathcal{K}_1 \overset{\kappa_1}{\boxtimes} \cdots \overset{\kappa_{k-1}}{\boxtimes} \mathcal{K}_k$ and $\mathcal{N} = \mathcal{N}_1 \overset{\nu_1}{\boxtimes} \cdots \overset{\nu_{n-1}}{\boxtimes} \mathcal{N}_n$. Here the asterisk at the additional gluing circles on the right hand side (for which, for better readability, we omit the orientation and labelings) indicates that when calaculating a block functor they have to be evaluated on the relevant distinguished fusion objects (as introduced after Example 3.16).

5 The modular functor

5.1 Factorization

We now study how our description of block functors fits together with the gluing of defect surfaces, assuming for now that both the initial and the glued surface are fine. In the literature, this issue is often formulated as the behavior of blocks under *factorization*, and we adopt this term here. Our 2-categorical setting makes it manifest that factorization amounts to a *structure*, rather than being a property. We will show that such a structure exists, but will not investigate its uniqueness in the present paper. The general case of factorization for not necessarily fine defect surfaces will follow from the definition of the block functor of a non-fine surface as a limit, see Theorem 5.22.

According to Definition 2.9, the horizontal composition of morphisms in the bicategory $\mathrm{Bord}_2^{\mathrm{def}}$ is given by the gluing of surfaces. A 2-functor comes with additional data, implementing the compatibility of horizontal composition. The purpose of the present subsection is to exhibit these data for the block functor T. We refer to the resulting structure as the *factorization structure* of the 2-functor.

To define this factorization structure, consider the situation that the boundary of a gluable fine defect surface Σ contains gluing boundary components $\mathbb{L}_1 = \mathbb{L}$ and $\mathbb{L}_2 = \overline{\mathbb{L}}$. Denote by $\cup_{\mathbb{L}}(\Sigma)$ the (fine) defect surface that results from gluing $\mathbb{L}$ and $\overline{\mathbb{L}}$, that is, by identifying the boundary components corresponding to their parametrizations. In case that the defect surface is a disjoint sum $\Sigma = \Sigma_1 \sqcup \Sigma_2$ and that $\mathbb{L} \subseteq \partial\Sigma_1$ and $\overline{\mathbb{L}} \subseteq \partial\Sigma_2$, the surface $\cup_{\mathbb{L}}(\Sigma) = \Sigma_2 \circ \Sigma_1$ is the (partial) horizontal composition of the 1-morphisms Σ_1 and Σ_2 in $\mathrm{Bord}_2^{\mathrm{def}}$.

We will show below that there is a canonical isomorphism that expresses the block functor for the surface $\cup_{\mathbb{L}}(\Sigma)$, by taking a coend in the gluing category $\mathrm{T}(\mathbb{L})$:

$$\mathrm{T}_{\mathrm{fine}}(\cup_{\mathbb{L}}(\Sigma)) \cong \int^{z\in \mathrm{T}(\mathbb{L})} \mathrm{T}_{\mathrm{fine}}(\Sigma)(- \boxtimes z \boxtimes \overline{z}) \,. \tag{5.1}$$

Here on the right hand side we evaluate the block functor on objects z and $\overline{z}$ in the gluing categories for the gluing boundaries $\mathbb{L}$ and $\overline{\mathbb{L}}$, respectively. We select this canonical isomorphism as the definition of the factorization structure of the modular functor.

To identify this canonical isomorphism, we first recall the notion of a defect cylinder, see e.g. the picture (4.71). It follows directly from the definition that any defect cylinder Σ satisfies

$$\cup_{\mathbb{L}}(\Sigma \sqcup \Sigma) = \Sigma \,. \tag{5.2}$$

Proposition 5.1. Let Σ be a defect cylinder over a defect one-manifold. Then the endofunctor $\mathrm{T}_{\mathrm{fine}}(\Sigma)$ is a strict idempotent under horizontal composition:

$$\mathrm{T}_{\mathrm{fine}}(\Sigma) \circ \mathrm{T}_{\mathrm{fine}}(\Sigma) = \mathrm{T}_{\mathrm{fine}}(\Sigma) \,. \tag{5.3}$$

Proof. This follows directly from the Eilenberg-Watts calculus: according to Corollary 4.30, the block functor (with values in vect) for a defect cylinder is a Hom functor, which via the Eilenberg-Watts equivalences corresponds to the identity functor. When working instead with Deligne products, the isomorphism is provided by the variant (B.2) of the Yoneda lemma: Denoting the two copies of Σ by Σ_1 and Σ_2, such that $\partial\Sigma_1 = \overline{\mathbb{L}} \sqcup \mathbb{L}$ and $\partial\Sigma_2 = \mathbb{L} \sqcup \overline{\mathbb{L}}$, and fixing objects $\overline{y} \in \mathrm{T}(\overline{\mathbb{L}}) \cong \mathrm{T}(\mathbb{L})^{\mathrm{opp}}$ and $w \in \mathrm{T}(\mathbb{L})$ in the gluing categories for $\overline{\mathbb{L}} \subset \Sigma_1$ and $\mathbb{L} \subset \Sigma_2$, respectively, we have

$$\begin{aligned} \big(\mathrm{T}_{\mathrm{fine}}(\Sigma_1) \circ \mathrm{T}_{\mathrm{fine}}(\Sigma_2)\big)(\overline{y} \boxtimes w) &\overset{(3.41)}{\cong} \int^{x \in \mathrm{T}(\mathbb{L})} \mathrm{T}_{\mathrm{fine}}(\Sigma \sqcup \Sigma)(\overline{y} \boxtimes x \boxtimes \overline{x} \boxtimes w) \\ &\overset{(4.72)}{\cong} \int^{x \in \mathrm{T}(\mathbb{L})} \mathrm{Hom}_{\mathrm{T}(\mathbb{L})}(y, x) \otimes \mathrm{Hom}_{\mathrm{T}(\mathbb{L})}(x, w) \\ &\overset{(B.2)}{\cong} \mathrm{Hom}_{\mathrm{T}(\mathbb{L})}(y, w) \;=\; \mathrm{T}_{\mathrm{fine}}(\Sigma)(\overline{y} \boxtimes w)\,. \end{aligned} \tag{5.4}$$

□

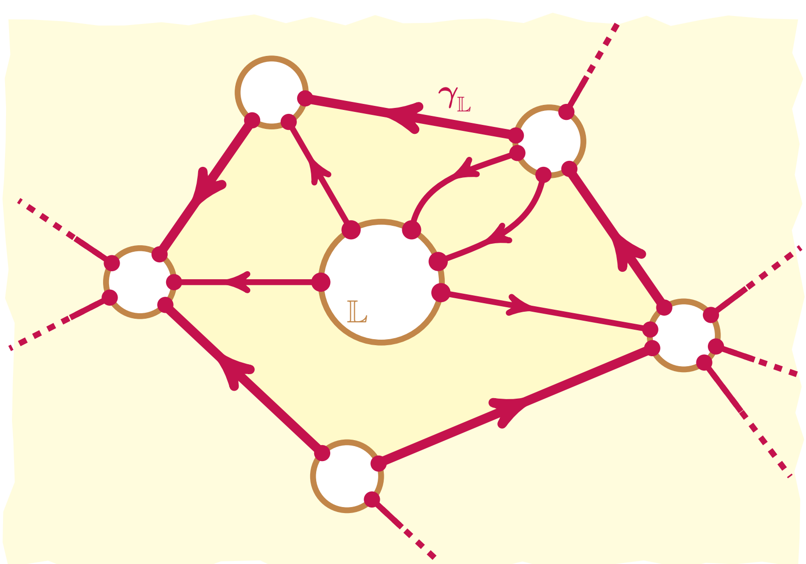
When combined with the results of Section 4.5, we obtain an analogous canonical isomorphism also in more general situations:

Theorem 5.2. Let Σ be a fine defect surface with gluing boundary components $\mathbb{L}$ and $\overline{\mathbb{L}}$ such that the glued surface $\cup_{\mathbb{L}}(\Sigma)$ is again fine. There is a canonical isomorphism

$$\int^{x \in \mathrm{T}(\mathbb{L})} \mathrm{T}_{\mathrm{fine}}(\Sigma)(- \boxtimes x \boxtimes \overline{x}) \;\cong\; \mathrm{T}_{\mathrm{fine}}(\cup_{\mathbb{L}}(\Sigma))(-) \tag{5.5}$$

of functors.

Proof. Consider the subset of defect lines of Σ, to be called the *link* $\gamma_{\mathbb{L}}$ *of the boundary component* $\mathbb{L}$, that arises as follows: Consider all 2-patches that have a common boundary with $\mathbb{L}$ and take all the defect lines on any of these 2-patches that do not have an end point on $\mathbb{L}$. The following picture shows the link $\gamma_{\mathbb{L}}$, indicated by the thickened defect lines, for a sample boundary component $\mathbb{L}$:



(5.6)

Contracting Σ along the the graph $\gamma_{\mathbb{L}}$ in the way described in Section 4.5 results in a disconnected defect surface $\Sigma_{\gamma_{\mathbb{L}}}$ such that the component of $\Sigma_{\gamma_{\mathbb{L}}}$ that contains $\mathbb{L}$ is a defect cylinder over $\mathbb{L}$. Performing the same construction also over the gluing boundary $\overline{\mathbb{L}}$ of $\Sigma_{\gamma_{\mathbb{L}}}$ then gives another defect surface $(\Sigma_{\gamma_{\mathbb{L}}})_{\gamma_{\overline{\mathbb{L}}}}$, among the components of which there are the defect cylinders over $\mathbb{L}$ and $\overline{\mathbb{L}}$, which are identical as defect surfaces (each having boundary $\mathbb{L} \sqcup \overline{\mathbb{L}}$). Applying Proposition 5.1 to the so obtained defect cylinders, it follows that

$$\begin{aligned} \int^{x\in \mathrm{T}(\mathbb{L})} \mathrm{T}_{\text{fine}}(\Sigma)(-\boxtimes x \boxtimes \overline{x}) &\cong \int^{x\in \mathrm{T}(\mathbb{L})} \mathrm{T}_{\text{fine}}\big((\Sigma_{\gamma_{\mathbb{L}}})_{\gamma_{\overline{\mathbb{L}}}}\big)(-\boxtimes x \boxtimes \overline{x}) \\ &\cong \mathrm{T}_{\text{fine}}(\cup_{\mathbb{L}}\big((\Sigma_{\gamma_{\mathbb{L}}})_{\gamma_{\overline{\mathbb{L}}}}\big))(-) \cong \mathrm{T}_{\text{fine}}(\cup_{\mathbb{L}}(\Sigma))(-)\,. \end{aligned} \tag{5.7}$$

Here the first and last isomorphisms hold by the canonical isomorphism involving the excision functor as obtained in Lemma 4.34, and we also use that $\cup_{\mathbb{L}}\big((\Sigma_{\gamma_{\mathbb{L}}})_{\gamma_{\overline{\mathbb{L}}}}\big) \cong \big((\cup_{\mathbb{L}}(\Sigma))_{\gamma_{\mathbb{L}}}\big)_{\gamma_{\overline{\mathbb{L}}}}$. □

It follows directly from the locality of the factorization in Theorem 5.2 that factorizations performed on several boundary components commute:

Corollary 5.3. Let $(\Sigma_1, \Sigma_2, \Sigma_3)$ be an ordered triple of fine defect surfaces that can composed in the given order. Then the two isomorphisms

$$\mathrm{T}_{\text{fine}}(\Sigma_3) \circ \mathrm{T}_{\text{fine}}(\Sigma_2) \circ \mathrm{T}_{\text{fine}}(\Sigma_1) \xrightarrow{\cong} \mathrm{T}_{\text{fine}}(\Sigma_3 \circ \Sigma_2 \circ \Sigma_1) \tag{5.8}$$

that correspond to either first gluing the surfaces Σ_3 and Σ_2 and then the resulting surface with Σ_1, or else first gluing Σ_2 and Σ_1 and then the result with Σ_3, are equal.

5.2 Transparent defects and fillable disks

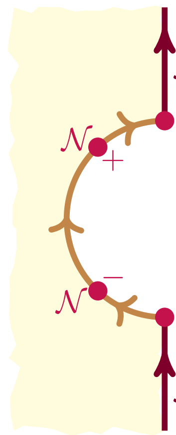
To be able to define the modular functor also on defect surfaces that are not fine, we need a suitable notion of *refinement* of surfaces. This will be introduced in Section 5.3. As a preparation, in the present subsection we provide two notions that will be convenient in that context: a subclass of defect lines that we will call transparent defects, and a subclass of defect surfaces that we will call fillable disks.

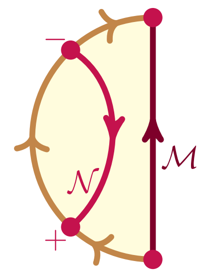
We start with

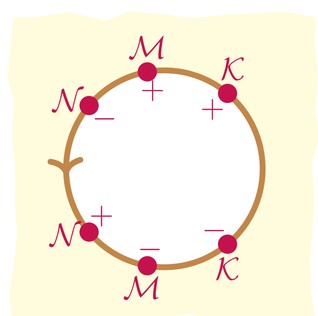
Definition 5.4. Let $\mathbb{L}$ be a gluing circle or gluing interval of a defect surface Σ, and regard Σ as a bordism with domain $\mathbb{L}$. (In case $\mathbb{L}$ is an interval, it is convenient to think of it concretely as a half-circle, as we have done before, e.g. in Proposition 4.6.) We say that $\mathbb{L}$ is *fillable* iff there exists a defect surface $\mathbb{D}_{\mathbb{L}}\colon \emptyset \to \mathbb{L}$ such that its underlying surface $D_{\mathbb{L}}$ has the topology of a disk (not containing any additional gluing circles).

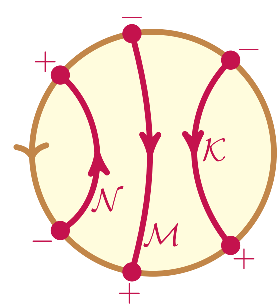
If a gluing boundary $\mathbb{L}$ is fillable in this way, we call the defect surface $\Sigma \circ \mathbb{D}_{\mathbb{L}}$ a *filling* of Σ by the disk $\mathbb{D}_{\mathbb{L}}$. The filling of a defect surface by any finite number of disks is defined analogously.

If a gluing circle or gluing interval is fillable, then it has necessarily an even number of defect points, coming in pairs which carry the same label and have opposite orientation. The following picture shows an example of a fillable circle $\mathbb{L}$ and a fillable interval $\mathbb{L}'$ and of corresponding disks $\mathbb{D}_{\mathbb{L}}$ and $\mathbb{D}_{\mathbb{L}'}$ that fill them:

$\mathbb{L} \;=$ 

$\mathbb{D}_{\mathbb{L}} \;=$ 

$\mathbb{L}' \;=$ 

$\mathbb{D}_{\mathbb{L}'} \;=$ 

(5.9)

By abuse of terminology, we apply the notion of fillability also to circles that do not contain any defect point and thus are not proper defect one-manifolds.

Next we introduce a convention which will be used later on:

Definition 5.5. For any finite tensor category $\mathcal{A}$ and any even integer κ we call the $\mathcal{A}$-bimodule

$$\mathcal{I}_\kappa := {}^{-\kappa}\!\mathcal{A} \tag{5.10}$$

the *κ-transparent defect label* (of type $\mathcal{A}$). A defect line labeled by $\mathcal{I}_\kappa$ is called a *κ-transparent defect* (of type $\mathcal{A}$).
For brevity, in case that $\kappa = 0$, we call $\mathcal{I}_0 = \mathcal{A}$ just the *transparent defect label* (of type $\mathcal{A}$), and a defect line labeled by $\mathcal{I}_0$ just a *transparently labeled* defect.

This notation and terminology is justified by the fact that $\mathcal{I}_\kappa$ behaves like a unit for the κ-twisted center: according to formula (B.27) and Lemma B.8 for an $\mathcal{A}$-bimodule category $\mathcal{M}$ there are specified equivalences $\mathcal{I}_\kappa \simeq \mathcal{A}^{-\kappa}$ and

$$\mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{I}_\kappa \;\simeq\; \mathcal{M} \;\simeq\; \mathcal{I}_\kappa \overset{\kappa}{\boxtimes} \mathcal{M} \,. \tag{5.11}$$

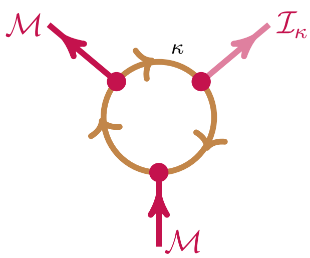
Remark 5.6. For any $\kappa, \kappa' \in 2\mathbb{Z}$ the double dual induces an equivalence ${}^{\kappa}\mathcal{I}_0^{\kappa'} \simeq \mathcal{I}_0^{\kappa+\kappa'}$, in particular

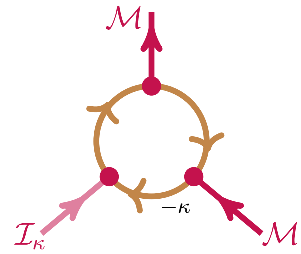
$$ {}^{\kappa}\mathcal{I}_0^{-\kappa} \;\simeq\; \mathcal{I}_0 \,. \tag{5.12}$$

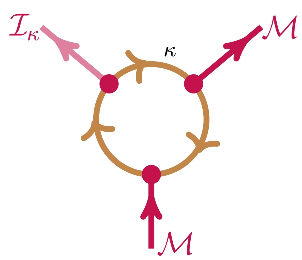
Together with formula (3.14), this implies that for any $\kappa \in 2\mathbb{Z}$ we have

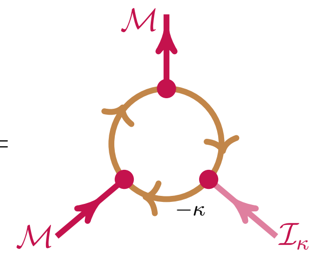
$$\mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{I}_0 \overset{-\kappa}{\boxtimes} \mathcal{N} \simeq \mathcal{M} \overset{0}{\boxtimes} {}^{\kappa}\mathcal{I}_0^{-\kappa} \overset{0}{\boxtimes} \mathcal{N} \simeq \mathcal{M} \overset{0}{\boxtimes} \mathcal{I}_0 \overset{0}{\boxtimes} \mathcal{N} \simeq \mathcal{M} \overset{0}{\boxtimes} \mathcal{N}. \tag{5.13}$$

Example 5.7. The equivalences (5.11) lead to distinguished objects in certain gluing categories: Let $\mathcal{A}$ be a finite tensor category and $\mathcal{M}$ an $\mathcal{A}$-bimodule. Via the Eilenberg-Watts equivalences (3.39) the gluing categories for the defect one-manifolds

$\mathbb{I}_\kappa^{\nearrow}(\mathcal{M}) \;:=$ 

$\mathbb{I}_\kappa^{\swarrow}(\mathcal{M}) \;:=$ 

$\mathbb{I}_\kappa^{\nwarrow}(\mathcal{M}) \;:=$ 

$\mathbb{I}_\kappa^{\searrow}(\mathcal{M}) \;:=$  (5.14)

are canonically equivalent to the category $\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{M},\mathcal{M})$ of bimodule endofunctors: we have

$$\begin{aligned} &\mathrm{T}(\mathbb{I}_\kappa^{\nearrow}(\mathcal{M})) = \overline{\mathcal{M}} \overset{1}{\boxtimes} \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{I}_\kappa \overset{1}{\boxtimes} \overset{(3.54)}{\simeq} \mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{M}, \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{I}_\kappa) \overset{(5.11)}{\simeq} \mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{M},\mathcal{M}) \qquad \text{and} \\ &\mathrm{T}(\mathbb{I}_\kappa^{\swarrow}(\mathcal{M})) = \overline{\mathcal{M}} \overset{-\kappa}{\boxtimes} \overline{\mathcal{I}_\kappa} \overset{1}{\boxtimes} \mathcal{M} \overset{1}{\boxtimes} \overset{(3.8)}{\simeq} \overline{\mathcal{I}_\kappa \overset{\kappa}{\boxtimes} \mathcal{M}} \overset{1}{\boxtimes} \mathcal{M} \overset{1}{\boxtimes} \simeq \overline{\mathcal{M}} \overset{1}{\boxtimes} \mathcal{M} \overset{1}{\boxtimes} \overset{(3.54)}{\simeq} \mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{M},\mathcal{M}), \end{aligned} \tag{5.15}$$

and similarly for the other two gluing circles. The so obtained endofunctor categories have the identity functor as a distinguished object.

Remark 5.8. The pre-images of the identity functor under the equivalences (5.15) constitute distinguished objects in the respective gluing categories. Using the precise form of the equivalences (5.11) (see the proof of Lemma B.8), one can express these distinguished objects as the subtle compounds

$$\begin{aligned} \int^{m\in\mathcal{M}} \overline{m} \boxtimes Z_{[\kappa]}(m \boxtimes \mathbf{1}) &= \int^{m\in\mathcal{M}}\!\!\int_{a\in\mathcal{A}} \overline{m} \boxtimes m.a \boxtimes {}^{[\kappa-1]}a \quad \in \mathrm{T}(\mathbb{I}_\kappa^{\nearrow}(\mathcal{M})) \qquad \text{and} \\ \int^{m\in\mathcal{M}} Z_{[\kappa]}(\overline{m} \boxtimes \overline{D_{\mathcal{A}}}) \boxtimes m &= \int^{m\in\mathcal{M}}\!\!\int_{a\in\mathcal{A}} \overline{a.m} \boxtimes \overline{D_{\mathcal{A}}.{}^{[\kappa+3]}a} \boxtimes m \\ &= \int^{a\in\mathcal{A}}\!\!\int^{m\in\mathcal{M}} \overline{m} \boxtimes \overline{a} \boxtimes {}^{[\kappa+1]}a.m \quad \in \mathrm{T}(\mathbb{I}_\kappa^{\swarrow}(\mathcal{M})) \end{aligned} \tag{5.16}$$

of ends and coends. Here the last equality follows from $\int^a a \boxtimes \overline{D_{\mathcal{A}} \otimes {}^{[\kappa+3]}a} \cong \int_a a \boxtimes \overline{{}^{[\kappa+1]}a} \in \mathcal{A} \boxtimes \overline{\mathcal{A}}$ (compare (3.44)).

Next we recall the so-called *braided induction*. Given a left $\mathcal{A}$-module $\mathcal{M}$ and a right $\mathcal{A}$-module $\mathcal{N}$, consider, for $z \in \mathcal{Z}(\mathcal{A})$, the endofunctors

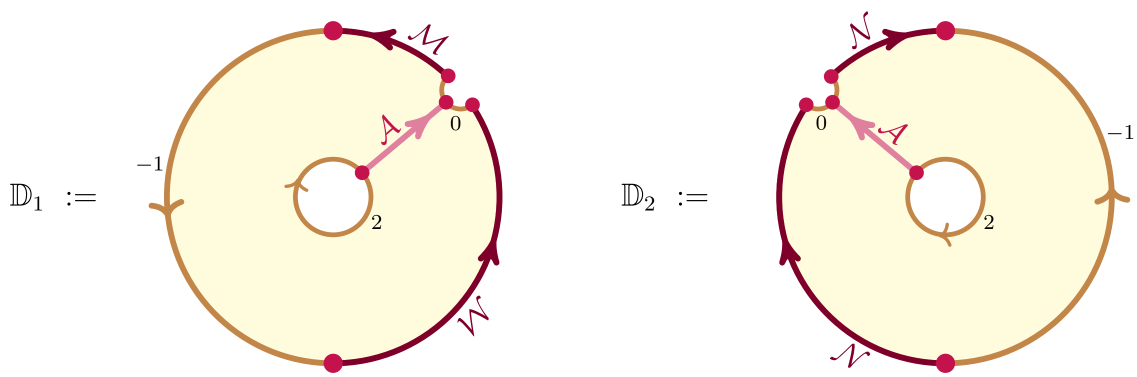
$$F_z(n) := n.z \qquad \text{and} \qquad {}_zF(m) = z.m \tag{5.17}$$

of $\mathcal{N}$ and $\mathcal{M}$, respectively. The half-braiding on z endows the functors F_z and ${}_zF$ with the structure of a module functor. By assigning to z these functors we obtain two monoidal functors

$$F_\bullet : \quad \mathcal{Z}(\mathcal{A}) \longrightarrow \mathcal{L}ex_{\mathcal{A}}(\mathcal{N},\mathcal{N}) \qquad \text{and} \qquad {}_\bullet F : \quad \mathcal{Z}(\mathcal{A}) \longrightarrow \mathcal{L}ex_{\mathcal{A}}(\mathcal{M},\mathcal{M})\,. \tag{5.18}$$

These are the functors of braided induction, also termed *α-induction*, see e.g. [Os, Sect. 5.1] for the categorical formulation used here. (Alternatively one may use the inverse half-braiding, which gives another pair of such functors.)

Example 5.9. For $\mathcal{A}$ a finite tensor category, $\mathcal{M}$ a left $\mathcal{A}$-module and $\mathcal{N}$ a right $\mathcal{A}$-module, consider the following defect surfaces $\mathbb{D}_1$ and $\mathbb{D}_2$:



(5.19)

Here, and in all pictures below, defect lines that are transparently labeled are drawn in a lighter color. Both surfaces have the same inner boundary circle $\mathbb{L}$; we denote the outer boundary circle of $\mathbb{D}_i$ by $\mathbb{L}_i$, for $i = 1, 2$. We have $\mathrm{T}(\mathbb{L}) = \mathcal{Z}(\mathcal{A})$ and $\mathrm{T}(\mathbb{L}_1) = \overline{\mathcal{M}} \overset{-1}{\boxtimes} \mathcal{M} \overset{(3.56)}{\simeq} \overline{\mathcal{L}ex_{\mathcal{A}}(\mathcal{M},\mathcal{M})}$, while $\mathrm{T}(\mathbb{L}_2) = \mathcal{N} \overset{-1}{\boxtimes} \overline{\mathcal{N}} \overset{(3.56)}{\simeq} \overline{\mathcal{L}ex_{\mathcal{A}}(\mathcal{N},\mathcal{N})}$. The block functors for the disks $\mathbb{D}_1$ and $\mathbb{D}_2$ are given by

$$\mathrm{T_{fine}}(\mathbb{D}_1)(\overline{G} \boxtimes z) = \mathcal{L}ex_{\mathcal{A}}(G, F_z) \qquad \text{and} \qquad \mathrm{T_{fine}}(\mathbb{D}_2)(\overline{G} \boxtimes z) = \mathcal{L}ex_{\mathcal{A}}(G, {}_zF)\,, \tag{5.20}$$

respectively, for $G \in \mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A})$ and $z \in \mathcal{Z}(\mathcal{A})$, and thus, by the Yoneda lemma, represent the braided inductions. In the case of $\mathbb{D}_1$ this is seen as follows (the computation of $\mathrm{T}(\mathbb{D}_2)$ is analogous). Consider the object $\widetilde{z}$ in $\overline{\mathcal{M}} \overset{-1}{\boxtimes} \mathcal{M}$ that for $z \in \mathrm{T}(\mathbb{L}_1)$ results from contracting the defect line that is connected to $\mathbb{L}_1$. Using the formulas given in Example C.6 we obtain

$$\widetilde{z} = \int^{m\in\mathcal{M}} \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a, z) \otimes \overline{m} \boxtimes a.m \cong \int^{m\in\mathcal{M}} \overline{m} \boxtimes z.m\,. \tag{5.21}$$

Under the Eilenberg-Watts correspondence this object gives the bimodule functor F_z. The claimed result for $\mathrm{T_{fine}}(\mathbb{D}_1)$ then follows from Corollary 4.28.

Combining the notion of transparent defect with the one of fillable circles and intervals allows us to introduce a particular type of defect surfaces. Topologically these specific surfaces are disks with holes, but since the holes are fillable and their gluing categories contain distinguished objects, for brevity we abuse terminology and refer to these defect surfaces just as “disks”.

Definition 5.10. Let $\mathbb{X}$ be a defect surface such that the underlying surface X is a disk and $\partial\mathbb{X}$ contains at most one free boundary segment.

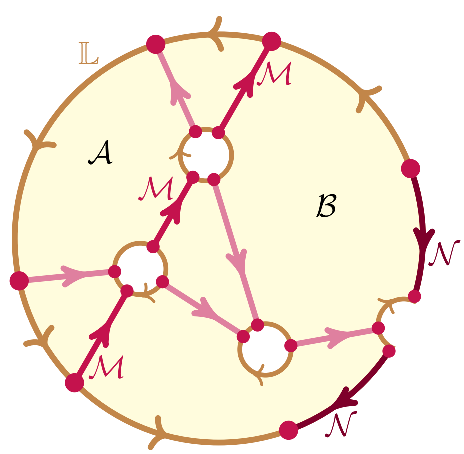
(i) A *fillable disk* $(\mathbb{D}_\mathbb{X}, \delta_{\mathrm{tr}}, \partial_{\mathrm{outer}})$ *of type* $\mathbb{X}$ is a defect surface $\mathbb{D}_\mathbb{X}$ with having following structure:

1. Every boundary circle of $\mathbb{D}_\mathbb{X}$ is oriented as induced by the orientation of the surface.
2. $\mathbb{D}_\mathbb{X}$ contains a set δ_{tr} of distinguished transparent defect lines on $\mathbb{D}_\mathbb{X}$, as well as one distinguished gluing boundary component $\partial_{\mathrm{outer}} \subseteq \partial_{\mathrm{glue}}\mathbb{D}_\mathbb{X}$, which we call the *outer boundary* of $\mathbb{D}_\mathbb{X}$.
3. Deletion of all transparent defect lines that belong to δ_{tr} results in a defect surface for which all gluing circles and gluing intervals, except for $\partial_{\mathrm{outer}}$, are fillable by a disk in the sense of Definition 5.4, and the so obtained filling is isomorphic to the defect surface $\mathbb{X}$.

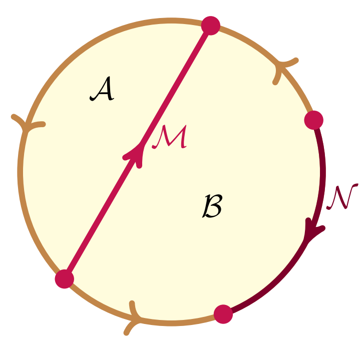
(ii) We depict a fillable disk $\mathbb{D}_\mathbb{X}$ as a subset of the plane, with the non-fillable boundary circle (or interval, in case $\mathbb{X}$ has a free boundary) forming the outer boundary, while the fillable gluing circles and intervals are referred to as *inner boundary circles* or *intervals*.

It follows that every defect line on $\mathbb{D}_\mathbb{X}$ is either a distinguished transparent defect or corresponds to a defect of $\mathbb{X}$, possibly interrupted by fillable gluing boundaries. As an illustration, the following is an example of a fillable disk (the non-fillable gluing boundary of $\mathbb{D}_\mathbb{X}$ is the interval labeled as $\mathbb{L}$, various other labels are omitted):

$\mathbb{D}_\mathbb{X}$ =



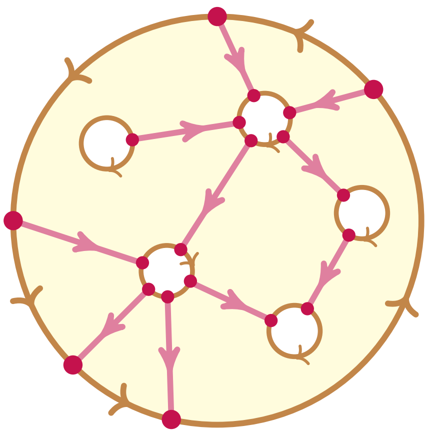
for $\mathbb{X}$ =



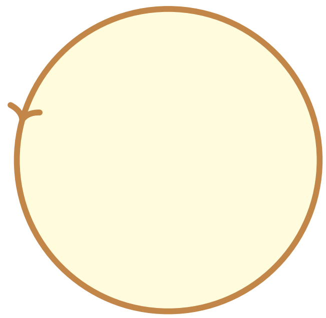
(5.22)

We also consider fillable disks for which, while they are proper defect surfaces themselves, the corresponding surface $\mathbb{X}$ is improper in the sense that it does not contain any defect lines or free boundaries. In this case we call the disk a *transparent disk* and denote it just by $\mathbb{D}_{\mathrm{tr}}$; the following is an example of a transparent disk:

$$\mathbb{D}_{\rm tr} \;=$$



$$\text{for} \quad \mathbb{X} \;=$$



(5.23)

By definition, all defect lines of a transparent disk $\mathbb{D}_{\rm tr}$ of type $\mathcal{A}$ are labeled by the transparent defect label $\mathcal{I}_0 = \mathcal{A}$ for one and the same tensor category $\mathcal{A}$. The framing of a fillable disk $\mathbb{D}_{\mathbb{X}}$ has the following property: since by definition every inner gluing segment is fillable by a disk, the indices on the outer boundary $\partial_{\rm outer}\mathbb{D}_{\mathbb{X}}$ must add up to the value 2 (thus $\partial_{\rm outer}\mathbb{D}_{\mathbb{X}}$ is not fillable). Moreover, the label of every defect line of $\mathbb{X}$ appears an even number of times as a label of a defect point on $\partial\mathbb{D}_{\mathbb{X}}$. (The latter criteria are, however, not sufficient for the existence of a fillable disk with a given boundary: If the cyclically ordered string of bimodules for the boundary $\partial\mathbb{X}$ contains a string $(\mathcal{M},\mathcal{N},\overline{\mathcal{M}},\overline{\mathcal{N}})$ with generic bimodules $\mathcal{M}$ and $\mathcal{N}$, then there does not exist a corresponding fillable disk.)

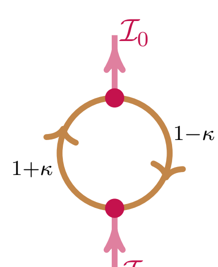
Let us also mention that the gluing categories for fillable circles and intervals can often conveniently be described as functor categories, whereby one can in particular identify certain distinguished objects in such gluing categories. Also, in Lemma C.5 in the appendix, we exhibit distinguished objects for all fillable circles and intervals.

Example 5.11.

(i) The gluing categories in (5.14) are fillable if and only if $\kappa = 0$.

(ii) Consider, for $\mathcal{A}$ a finite tensor category, the defect one-manifold

$$\mathbb{J}_\kappa \;:=$$



(5.24)

with $\kappa \in 2\mathbb{Z}$. It follows from (5.13) that there is a distinguished equivalence

$$\mathrm{T}(\mathbb{J}_\kappa) = \overline{\mathcal{I}_0} \overset{1+\kappa}{\boxtimes} \mathcal{I}_0 \overset{1-\kappa}{\boxtimes} \overset{(5.13)}{\simeq} \overline{\mathcal{I}_0} \overset{1}{\boxtimes} {}^{[\kappa]}\mathcal{I}_0^{[-\kappa]} \overset{1}{\boxtimes} \overset{(5.12)}{\simeq} \overline{\mathcal{I}_0} \overset{1}{\boxtimes} \mathcal{I}_0 \overset{1}{\boxtimes} \overset{(3.56)}{\simeq} \mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{I}_0,\mathcal{I}_0)\,. \qquad (5.25)$$

Thus in $\mathrm{T}(\mathbb{J}_\kappa)$ there is a distinguished object, given by the pre-image of the identity functor in $\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{I}_0,\mathcal{I}_0)$ under the equivalence (5.25). It is given explicitly by $\int^{a\in\mathcal{A}} \overline{a} \boxtimes {}^{[\kappa]}a \in \mathrm{T}(\mathbb{J}_\kappa)$, with balancing determined by the one of $\int^{a\in\mathcal{A}} \overline{a} \boxtimes a$.

(iii) Similarly, for the defect one-manifolds that coincide with $\mathbb{J}_\kappa$ as manifolds, but have general framings, i.e. with general indices κ and κ', the gluing category is canonically equivalent to $\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{I}_0^\kappa,\mathcal{I}_0^{\kappa'})$. If $\kappa=\kappa'$, then this functor category is equivalent to the Drinfeld center and is thus monoidal; if $\kappa-\kappa'=2$, then it is equivalent to the category of objects $x\in\mathcal{A}$ together with coherent natural isomorphisms $a\otimes x\cong x\otimes a^{\vee\vee}$ for all $a\in\mathcal{A}$, which is in general not monoidal (e.g., the monoidal unit of $\mathcal{A}$ might not have the structure of an object in this category). The categories for all other cases are equivalent to one obtained for the latter two cases, determined by $\kappa-\kappa'$ mod 4, using the distinguished invertible object in $\mathcal{A}$ as in Equation (3.47).

We take this observation as an opportunity to remark that in [DSS] the framing on a circle (without defect points) is described with the help of a corona instead of an index. For instance, the three framed manifolds shown in Table 3 of [DSS] correspond to $\kappa+\kappa'=2$, 0 and -2, respectively.

5.3 Refinement of defect surfaces

In Section 4.1 we have assigned a pre-block functor $\mathrm{T}^{\mathrm{pre}}(\Sigma)$ to any arbitrary defect surface Σ. In contrast, the block functor $\mathrm{T}(\Sigma)$ could so far be defined only for surfaces that are *fine* in the sense of Definition 2.1(ii). To complete the definition of the modular 2-functor, this restriction must be removed. As a first step towards this end we are going to construct, for any defect surface Σ, a family of fine defect surfaces Σ_{ref}, to be called *fine refinements* of Σ. In the present subsection we introduce the notion of refinement and show that fine refinements exist for any surface. Later on we will use the family of fine refinements of a given surface Σ to define the functor $\mathrm{T}(\Sigma)$; in case Σ is already fine itself, this must coincide with the functor given by Definition 4.18.

We need a precise notion of refinement of defect surfaces:

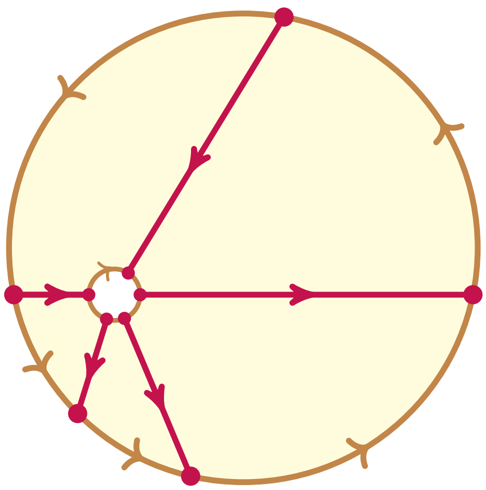
Definition 5.12. Let Σ and Σ' be two defect surfaces, both not necessarily fine. A *refinement* from Σ to Σ' is a triple $(\Sigma;\Sigma';\varphi)$, with distinguished sets $\mathbf{I}'$ of transparent defect lines, $\mathbf{S}'$ of gluing circles and $\mathbf{L}'$ of gluing intervals in Σ', such that the following holds:

1. Both end points of each of the defect lines in $\mathbf{I}'$ are contained in $\mathbf{S}'\cup\mathbf{L}'$.
2. When removing all defect lines in $\mathbf{I}'$ from Σ', each of the modified gluing circles $\mathbb{S}''_j$ that result from some $\mathbb{S}'_j\in\mathbf{S}'$ and modified gluing intervals $\mathbb{L}''_k$ that result from some $\mathbb{L}'_j\in\mathbf{L}'$ is fillable.
3. The resulting filling Σ'' of Σ' is isomorphic as a defect surface to Σ, with isomorphism $\varphi\colon\Sigma\to\Sigma''$.

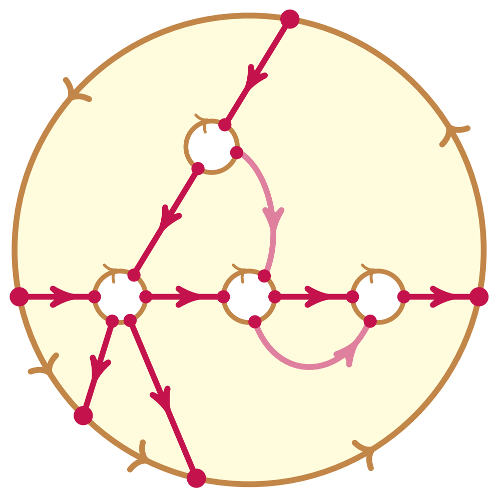
We denote the subset of the gluing boundary $\partial_{\mathrm{glue}}\Sigma_{\mathrm{ref}}$ of the surface Σ_{ref} consisting of those gluing circles and gluing intervals that are not inherited from the gluing boundary $\partial_{\mathrm{glue}}\Sigma$ of Σ by $\partial_{\mathrm{fill}}\Sigma_{\mathrm{ref}}$. We usually suppress the isomorphism φ from the notation and denote the refinement just by $(\Sigma;\Sigma')$; if $(\Sigma;\Sigma')$ is a refinement, then we also say that Σ can be *refined* to Σ', and that Σ' *refines* Σ.

The definition implies that each defect line δ of Σ corresponds either to a single defect line of Σ' with the same label as δ, or else splits into several defect lines of Σ' that all carry the same label as δ and which are interrupted by fillable gluing circles that are not present in Σ. Moreover, the gluing boundaries of Σ correspond to identical gluing boundaries of Σ'. As an illustration, the following picture shows the refinement $(\mathbb{D};\mathbb{D}')$ of a one-holed disk $\mathbb{D}$ for which $\mathbb{D}'$ has three additional gluing circles:

$\mathbb{D} =$



$\mathbb{D}' =$



(5.26)

Remark 5.13.

(i) For any refinement $(\Sigma;\Sigma_{\mathrm{ref}})$, by definition every gluing boundary of Σ_{ref} that is not a gluing boundary of Σ is fillable.

(ii) It is worth comparing the notion of refinement with Definition 5.10 of a fillable disk of type $\mathbb{X}$: It is easily seen that for every refinement $(\mathbb{X},\mathbb{X}_{\mathrm{ref}})$ with $\mathbb{X}$ a defect surface for which the underlying surface X is a disk and which has at most one free boundary segment, the refined surface $\mathbb{X}_{\mathrm{ref}}$ is a fillable disk of type $\mathbb{X}$. On the other hand, the converse is not true, because in general the outer boundary of a fillable disk of type $\mathbb{X}$ contains more defect points (transparently labeled) than the boundary of $\mathbb{X}$.

(iii) It follows directly that the gluing of any two refinements is again a refinement, i.e. that for any two refinements $(\Sigma_1;\Sigma_1')$ and $(\Sigma_2;\Sigma_2')$ for which Σ_1 and Σ_2 can be glued, the same holds for Σ_1' and Σ_2' and the pair $(\Sigma_1\circ\Sigma_2;\Sigma_1'\circ\Sigma_2')$ is a refinement as well. Moreover, if Σ_1' and Σ_2' are gluable fine refinements, then $\Sigma_1'\circ\Sigma_2'$ is gluable fine, too.

(iv) If $(\Sigma;\Sigma_{\mathrm{ref}})$ is a refinement, then for any defect surface Σ_{ref}' that is isomorphic to Σ_{ref}, $(\Sigma;\Sigma_{\mathrm{ref}}')$ is a refinement of Σ as well.

We proceed to show that fine refinements exist. From Definition 5.12 it follows immediately that any refinement Σ' of Σ can be obtained by separately refining every 2-patch of Σ, in any order. Now like any two-manifold with boundary, a 2-patch admits pair-of-pants decompositions, i.e. as a manifold it can be obtained by gluing a finite number of disks, annuli, and pairs of pants (or trinions). Since in the situation of our interest we are working with framed surfaces, we must in addition account for the framing. Moreover, unlike for manifolds without defects and boundaries we also have to consider 2-patches whose boundary contains free boundary segments. These still admit generalized pair-of-pants decompositions with a larger number of building blocks (compare [LP, Prop. 3.8]). It is not hard to see that indeed any 2-patch of any defect surface can be obtained by gluing a finite number of the following specific two-manifolds:

- a disk with boundary being a gluing circle without defect points and with framing index -2;
- an annulus with boundary components being gluing circles without defect points and with framing indices $\pm\kappa \in 2\mathbb{Z}$;
- a pair of pants with boundary components being gluing circles without defect points and with framing indices $\kappa, \kappa' \in 2\mathbb{Z}$ and $2-\kappa-\kappa'$, respectively;
- a disk whose boundary is the union of one free boundary and of one gluing interval that does not have any defect points in its interior and has framing index -1;
- a disk whose boundary is the union of three free boundaries and of three gluing intervals that do not have any defect points in their interior and have framing indices $\kappa, \kappa' \in \mathbb{Z}$ and $1-\kappa-\kappa'$, respectively;
- an *open-closed pipe*: an annulus such that one boundary component is the union of one free boundary and of one gluing interval without defect points in its interior and with framing index $\kappa \in \mathbb{Z}$, while the other boundary component is a gluing circle without defect points and with framing index $-1-\kappa$.

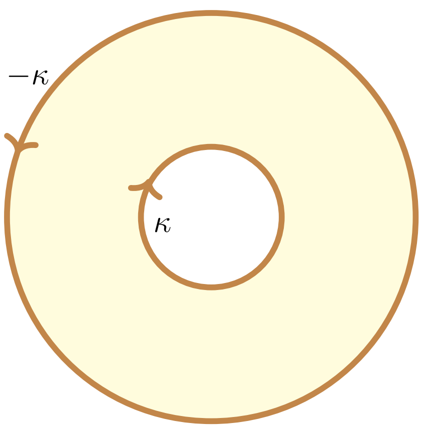
It is worth recalling that a defect one-manifold in the sense of Definition 2.9 contains at least one defect point. Thus those of the building blocks in this list which have a gluing circle without defect points as a boundary component are themselves not defect surfaces. However, our prescription implies that any fine surface that results from the refinement of any (proper) defect surface *is* again a proper defect surface.

We are now in a position to show

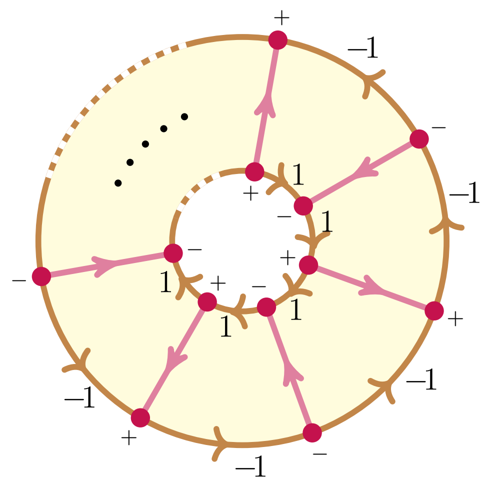
Theorem 5.14. Every defect surface can be refined to a fine defect surface.

Proof. We will construct a fine refinement for an arbitrary 2-patch; combining the refinements of all 2-patches of a defect surface Σ then provides a fine refinement of Σ. Thus consider a 2-patch $\mathbb{P}$ of a defect surface. Take any (generalized) pair-of-pants decomposition $\mathbb{P}_{\text{p.o.p.}}$ of $\mathbb{P}$, with building blocks of the form just described, making sure that the curves on Σ that define $\mathbb{P}_{\text{p.o.p.}}$ intersect all defect lines on Σ transversally (this can be achieved by applying, if necessary, a small isotopy to any chosen set of curves). The resulting building blocks come in two kinds: either they contain a gluing boundary of Σ or they do not. We first retrict our attention to building blocks of the first kind. We would now like to define standard refinements of these building blocks of $\mathbb{P}_{\text{p.o.p.}}$ that glue together to a refinement of Σ. However, to do so, we need to alter also gluing boundaries in the building blocks of $\mathbb{P}_{\text{p.o.p.}}$, namely those that do not correspond to gluing boundaries of Σ. We account for this by allowing for *generalized refinements*, which are defined in the same way as refinements, except that there is no restriction on the end points of the defect lines $\mathbb{I}'_i$. (This is not in conflict with the rationale of our construction, because the block functor is only applied after gluing such generalized refinements to genuine ones.) Thus for each of the building blocks of $\mathbb{P}_{\text{p.o.p.}}$ we now provide a standard fine generalized refinement that is chosen in such a manner that gluing together any two of the thereby obtained fine defect surfaces results in a defect surface that is fine as well.

Let us first present our prescription in much detail for the case that the building block in question is an annulus $\mathbb{A}$ (without defect points on its boundary). Denote by $\pm\kappa \in 2\mathbb{Z}$ the framing indices of the boundary circles of $\mathbb{A}$. We refine $\mathbb{A}$ in two steps. In the first step we add a single transparent defect line if $\kappa = 0$, while for $\kappa \neq 0$ we add $|\kappa|$ transparent defect lines of alternating orientation that connect the two boundary circles, in such a way that we obtain a fine defect surface $\mathbb{A}_{\text{ref},0}$ each of whose 2-patches is a straight disk. This is illustrated in the following picture:

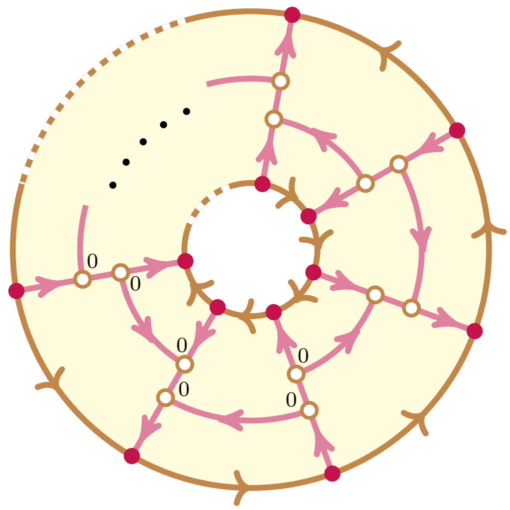


$$\rightsquigarrow$$



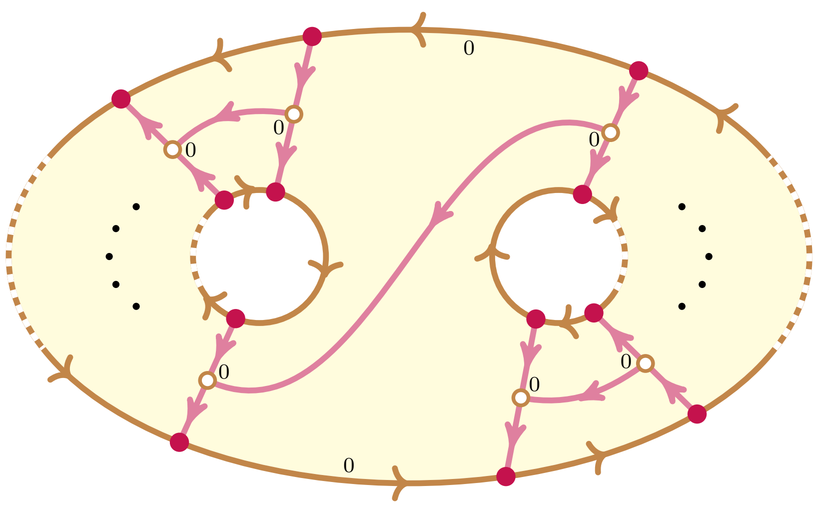
$$=: \ \mathbb{A}_{\text{ref},0} \,. \tag{5.27}$$

While the so obtained surface $\mathbb{A}_{\text{ref},0}$ is already fine, gluing it to another fine surface can still result in a defect surface that is no longer fine. The second step of the prescription eliminates this unwanted feature: We refine $\mathbb{A}_{\text{ref},0}$ further by suitably inserting additional transparent defects together with transparent gluing circles with three defect points that are of the form shown in the picture (5.14) with $\kappa = 0$ and $\mathcal{M} = \mathcal{I}_0$; thereby we obtain a fine (generalized) refinement $(\mathbb{A}; \mathbb{A}_{\text{ref}})$ of the form indicated in in the following picture (for better readability we omit the defect points and orientation of the gluing intervals of the transparent gluing circles as well as the orientation of some of the defect lines):

$$\mathbb{A}_{\mathrm{ref}} \;=$$  (5.28)

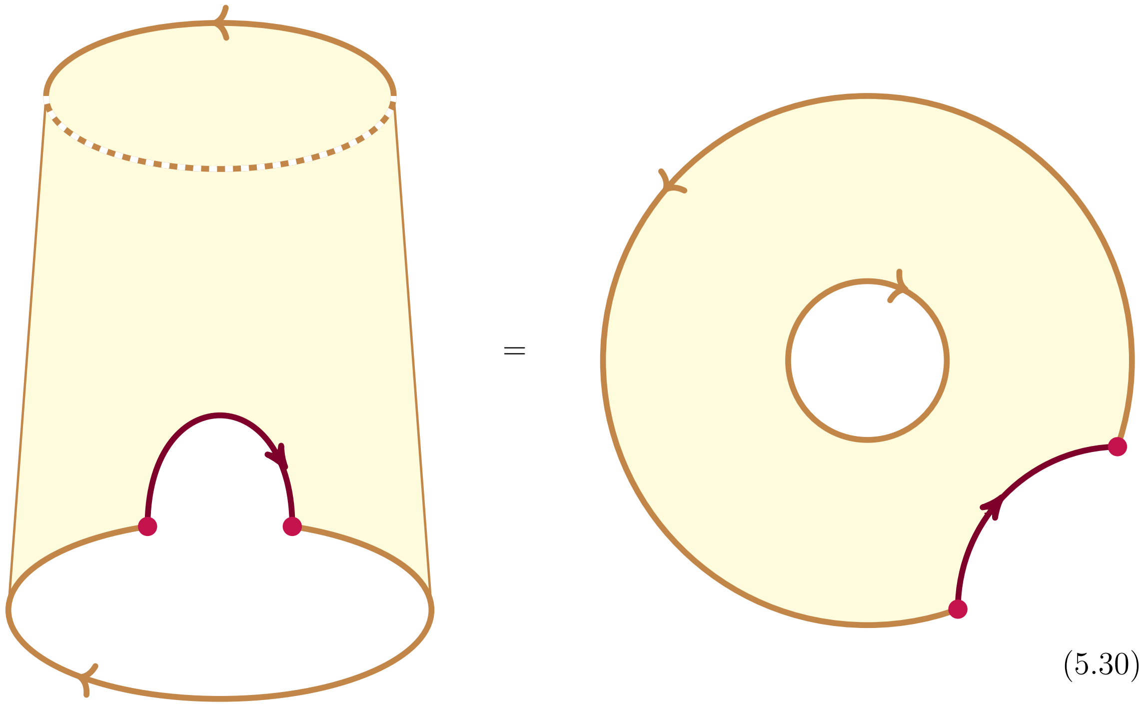
The framing on the so obtained fine surface $\mathbb{A}_{\mathrm{ref}}$ is uniquely specified by the orientation of the various defect lines together with the framing on $\mathbb{A}_{\mathrm{ref},0}$. Specifically, the framing indices of the three segments of each of the new transparent gluing circles are 0, 1 and 1. We indicate the position of some of those segments that have framing index 0 in the picture; the position of the index-0 segments for the other transparent gluing circles is analogous.

The standard fine refinements for the other building bocks of a pair-of-pants decomposition are obtained in a similar manner as in the case of the annulus $\mathbb{A}$. We content ourselves to display the resulting fine refinements $(\mathbb{B};\mathbb{B}_{\mathrm{ref}})$ and $(\mathbb{U};\mathbb{U}_{\mathrm{ref}})$ for the cases of a pair of pants $\mathbb{B}$ and of an open-closed pipe $\mathbb{U}$. The standard fine refinement for $\mathbb{B}$ looks as follows:

$$\mathbb{B}_{\mathrm{ref}} \;=$$  (5.29)

Drawing the open-closed pipe $\mathbb{U}$ as on the right hand side of

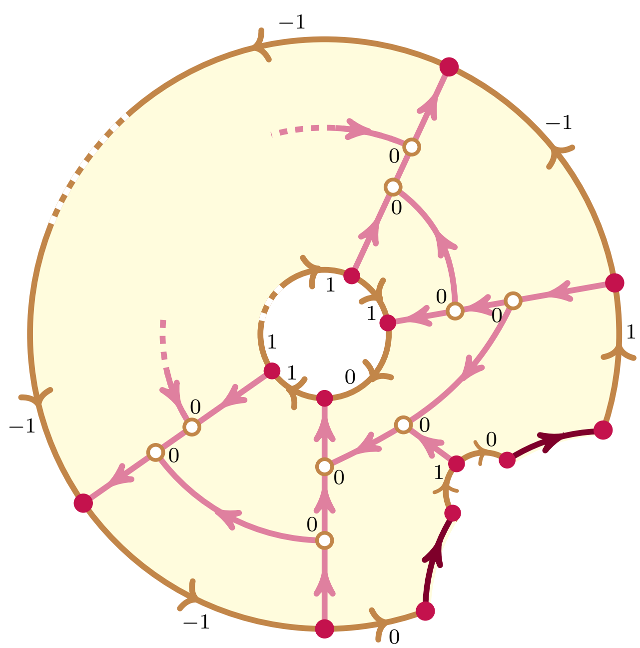
$\mathbb{U}$ =



(5.30)

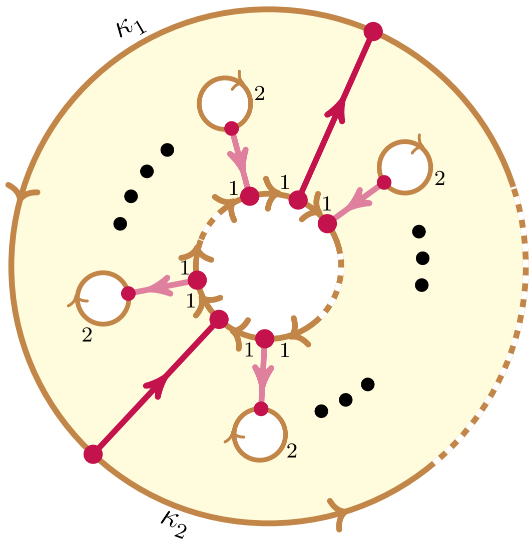
its standard fine refinement looks as follows:

$\mathbb{U}_{\mathrm{ref}}$ =



(5.31)

Finally we consider a building block $\mathbb{P}$ of the $\mathbb{P}_{\text{p.o.p.}}$ of the second kind, i.e. one that contains a gluing boundary of Σ. By picking, if necessary, a finer pair-of-pants decomposition $\mathbb{P}'_{\text{p.o.p.}}$, we can assume that $\mathbb{P}$ is a cylinder. Consider first the case that the gluing boundary of Σ in question is a circle; then we deal with a defect surface of the following form:



(5.32)

Here for each gluing segment with framing index κ_i of the outer circle there are $|\kappa_i|-1$ additional boundary circles that are connected by a single transparent defect line to the inner gluing circle in such a way that the framing index of each of the resulting new gluing segments on the inner circle is $+1$ (as shown in the picture) if κ_i is positive and -1 if κ_i is negative. This prescription is compatible with the previous treatment of building blocks that do not meet the boundary of Σ, in such a way that the resulting refinement of Σ is still fine. The case that the relevant gluing boundary of Σ is an interval is treated similarly.

To summarize: Given a defect surface Σ we choose for each 2-patch $\mathbb{P}$ of Σ any pair-of-pants decomposition $\mathbb{P}_{\text{p.o.p.}}$ and refine the building pieces of $\mathbb{P}_{\text{p.o.p.}}$ in the way described above. Gluing the so obtained refined building pieces back together gives a fine refinement $(\mathbb{P};\mathbb{P}_{\text{ref}})$ of $\mathbb{P}$. Combining the fine refinements $(\mathbb{P};\mathbb{P}_{\ell;\text{ref}})$ of all 2-patches $\mathbb{P}_\ell$ of Σ then provides a fine refinement $(\Sigma;\Sigma_{\text{ref}})$ of Σ, provided that the new transparent defect lines in neighboring 2-patches are inserted in a coordinated fashion, which is easily accomplished. Thus in short, each collection of pair-of-pants decompositions of the 2-patches of Σ defines a specific fine refinement of Σ. □

Remark 5.15. Consider the standard two-torus $\mathbb{T}=[0,1]^2/{\sim}$. It inherits a framing from the standard framing of the plane $\mathbb{R}^2$. We denote the defect surface with this standard framing by $(\mathbb{T},\chi^{\text{std}})$. Any other framing χ on $\mathbb{T}$ defines an isomorphic defect surface. Indeed, for any non-vanishing vector field on $\mathbb{T}$ there exists a pair of closed curves with zero winding number for the framing that generate the fundamental group $\pi_1(\mathbb{T})$ of the torus. If we place transparent defect lines on $\mathbb{T}$ that follow these closed curves, except that their intersection is resolved into two gluing circles each having three defect points, we arrive at a valid defect surface whose gluing circles are of type (5.14) with $\kappa=0$, and thereby at a fine refinement $(\mathbb{T};\mathbb{T}_{\text{ref}})_\chi$ of $\mathbb{T}$; identifying $(\mathbb{T};\mathbb{T}_{\text{ref}})_\chi$ with a refinement of the standard framed torus yields the desired isomorphism.

Moreover, every automorphism φ of $(\mathbb{T},\chi^{\text{std}})$ is isotopic to the identity: We can assume that φ preserves $(0,0)$. Since φ must preserve flow lines, it maps the generator corresponding to $(1,0)$ to itself, and since it must preserves angles between tangent vectors, it maps the $(0,1)$ generator to itself as well.

5.4 Block functors for non-fine surfaces

In Section 4.1 we have constructed the pre-block functor $\mathrm{T}^{\mathrm{pre}}(\Sigma)$ for any defect surface Σ, not necessarily fine. In contrast, the block functor cannot be defined for Σ as in Section 4.3 unless the surface Σ is fine, since the topology of the disk enters via the well-posedness, in the sense of Definition 4.12, for disks (see Corollary 4.16). Our goal is now to define the block functor $\mathrm{T}(\Sigma)$ for an arbitrary defect surface Σ by making use of the existence of fine refinements of Σ that was shown in Section 5.3.

Let $(\Sigma;\Sigma_{\mathrm{ref}})$ be a fine refinement from Σ to Σ_{ref}. In the first place, Σ_{ref} is just a specific fine defect surface, which comes with its own pre-block functor $\mathrm{T}^{\mathrm{pre}}(\Sigma_{\mathrm{ref}})$ and block functor $\mathrm{T}(\Sigma_{\mathrm{ref}})$. Typically $\mathrm{T}(\Sigma_{\mathrm{ref}})$ lies in a different functor category than the desired functor $\mathrm{T}(\Sigma)$. In order to obtain a functor in the correct functor category, we are going to assign to the refinement a left exact functor $\widehat{\mathrm{T}}(\Sigma;\Sigma_{\mathrm{ref}})$ that is obtained from the block functor $\mathrm{T}_{\mathrm{fine}}(\Sigma_{\mathrm{ref}})$ for the refined surface Σ_{ref} by evaluating it on a distinguished object in the gluing category for the part $\partial_{\mathrm{fill}}\Sigma_{\mathrm{ref}}$ of the gluing boundary of Σ_{ref} that is not inherited from Σ. We call this (still to be defined) distinguished object in $\mathrm{T}(\partial_{\mathrm{fill}}\Sigma_{\mathrm{ref}})$ the *silent* object of the refinement $(\Sigma;\Sigma_{\mathrm{ref}})$ and denote it by $\mho_{\Sigma;\Sigma_{\mathrm{ref}}}$, and call $\widehat{\mathrm{T}}(\Sigma;\Sigma_{\mathrm{ref}})$ the *relative block functor* for the fine refinement $(\Sigma;\Sigma_{\mathrm{ref}})$. Thus we set

$$\widehat{\mathrm{T}}(\Sigma;\Sigma_{\mathrm{ref}})(-) := \mathrm{T}(\Sigma_{\mathrm{ref}})(-\boxtimes\mho_{\Sigma;\Sigma_{\mathrm{ref}}}): \quad \mathrm{T}(\partial_{\mathrm{glue}}\Sigma_{\mathrm{ref}}\backslash\partial_{\mathrm{fill}}\Sigma_{\mathrm{ref}}) \rightarrow \mathrm{vect}\,. \tag{5.33}$$

Further, we can identify the complement of $\partial_{\mathrm{fill}}\Sigma_{\mathrm{ref}}$ in $\partial_{\mathrm{glue}}\Sigma_{\mathrm{ref}}$ with the gluing boundary of Σ, whereby the relative block functor becomes a left exact functor

$$\widehat{\mathrm{T}}(\Sigma;\Sigma_{\mathrm{ref}}): \quad \mathrm{T}(\partial_{\mathrm{glue}}\Sigma) \rightarrow \mathrm{vect}\,, \tag{5.34}$$

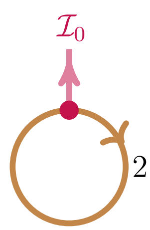
and thus an object in the same functor category in which also the block functor for Σ should be an object. Of course, $\widehat{\mathrm{T}}(\Sigma;\Sigma_{\mathrm{ref}})$ depends on both Σ and Σ_{ref}, whereas the block functor $\mathrm{T}(\Sigma)$ must only depend on Σ. We will therefore define $\mathrm{T}(\Sigma)$ as a limit of the collection $\{\widehat{\mathrm{T}}(\Sigma;\Sigma_{\mathrm{ref}})\}$ of functors over the refinements that refine the surface Σ which, as we will show, can be endowed with the structure of a diagram with values in the category of left exact functors. Addressing this issue, it still remains to specify the silent object $\mho_{\Sigma;\Sigma_{\mathrm{ref}}}$ for any refinement. Since according to Remark 5.13(i) every component of $\partial_{\mathrm{fill}}\Sigma_{\mathrm{ref}}$ is fillable by a disk, it is sufficient to define a silent object $\mho(\mathbb{L})\in\mathbb{T}(\mathbb{L})$ for every fillable gluing circle or gluing interval $\mathbb{L}$. Given those objects, we set

$$\mho_{\Sigma;\Sigma_{\mathrm{ref}}} := \boxtimes_i\, \mho(\mathbb{L}_i)\,, \tag{5.35}$$

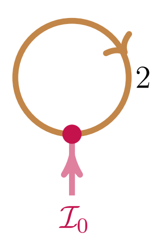
where the Deligne product is taken over all components $\mathbb{L}_i$ of $\partial_{\mathrm{fill}}\Sigma_{\mathrm{ref}}$.

To define, in turn, the objects $\mho(\mathbb{L})$ we proceed in two steps. We first consider the case that $\mathbb{L}$ is the simplest possible type of a fillable circle: for $\mathcal{A}$ a finite tensor category and $\epsilon\in\{\pm1\}$, a circle with a single defect point that is transparently labeled and has orientation ϵ. We denote such a circle by $\mathbb{Q}^{\mathcal{A}}_{\epsilon}$ and refer to it as a *tadpole circle*. Pictorially,

$\mathbb{Q}^{\mathcal{A}}_{+}$ =



$\mathbb{Q}^{\mathcal{A}}_{-}$ =



(5.36)

The relevant silent objects are obtained via a remarkable interplay between algebraic structures in finite tensor categories and the geometry of framings. The gluing categories for tadpole circles are canonically equivalent to (twisted) centers:

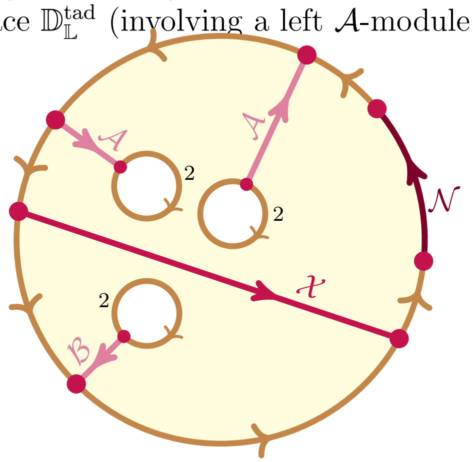
$$\mathrm{T}(\mathbb{Q}_+^{\mathcal{A}}) = \mathcal{I}_0 \overset{2}{\boxtimes} \cong \mathcal{Z}(\mathcal{A}) \qquad \text{and} \qquad \mathrm{T}(\mathbb{Q}_-^{\mathcal{A}}) = \overline{\mathcal{I}_0} \overset{2}{\boxtimes} \cong \overline{\mathcal{Z}^{-4}(\mathcal{A})} \tag{5.37}$$

(recall Definition 3.6 of the twisted center $\mathcal{Z}^{\kappa}$). Thus in particular they contain canonical objects, namely the monoidal unit in $\mathcal{Z}(\mathcal{A})$ and the distinguished invertible object (see page 32) in $\overline{\mathcal{Z}^{-4}(\mathcal{A})}$, respectively. We define the respective silent objects to be these two specific objects:

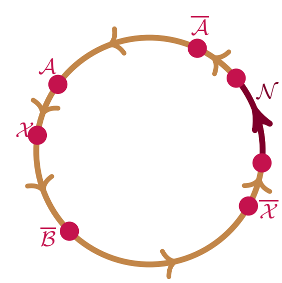
$$\mho(\mathbb{Q}_+^{\mathcal{A}}) := \mathbf{1} \; \in \mathcal{Z}(\mathcal{A}) \cong \mathrm{T}(\mathbb{Q}_+^{\mathcal{A}}) \qquad \text{and} \qquad \mho(\mathbb{Q}_-^{\mathcal{A}}) := \overline{D_{\mathcal{A}}} \; \in \overline{\mathcal{Z}^{-4}(\mathcal{A})} \cong \mathrm{T}(\mathbb{Q}_-^{\mathcal{A}}) \,. \tag{5.38}$$

In the second step we consider an arbitrary fillable gluing circle or interval $\mathbb{L}$. There exists (uniquely up to isomorphism) a defect surface $\mathbb{D}_{\mathbb{L}}^{\mathrm{tad}}$ which is a fillable disk (in the sense of Definition 5.10) such that $\mathbb{L}$ is its outer boundary and each of its inner boundary circles is a tadpole circle $\mathbb{Q}_i$ (for a suitable finite tensor category) whose single defect point is connected by a single (transparent) defect line with $\mathbb{L}$. The following picture shows an example of such a defect surface $\mathbb{D}_{\mathbb{L}}^{\mathrm{tad}}$ (involving a left $\mathcal{A}$-module $\mathcal{N}$ and an $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{X}$):

$\mathbb{D}_{\mathbb{L}}^{\mathrm{tad}} \; =$



for $\quad \mathbb{L} \; =$



(5.39)

We regard $\mathbb{D}_{\mathbb{L}}^{\mathrm{tad}}$ as a bordism $\mathbb{D}_{\mathbb{L}}^{\mathrm{tad}} \colon \bigsqcup_{i=1}^{n} \mathbb{Q}_i \to \mathbb{L}$ and define the silent object for $\mathbb{L}$ to be the image

$$\mho(\mathbb{L}) := \mathrm{T}(\mathbb{D}_{\mathbb{L}}^{\mathrm{tad}})(\mho(\mathbb{Q}_1) \boxtimes \cdots \boxtimes \mho(\mathbb{Q}_n)) \; \in \mathrm{T}(\mathbb{L}) \tag{5.40}$$

of the silent objects for the inner boundary circles under the block functor for the fine defect surface $\mathbb{D}_{\mathbb{L}}^{\mathrm{tad}}$.

To summarize: For any fine refinement $(\Sigma; \Sigma_{\mathrm{ref}})$ the prescription (5.33) provides us with a functor $\widehat{\mathrm{T}}(\Sigma; \Sigma_{\mathrm{ref}})$ in the category $\mathcal{L}ex(\mathrm{T}(\partial\Sigma), \mathrm{vect})$ in which also the pre-block functor $\mathrm{T}^{\mathrm{pre}}(\Sigma)$ is an object. We want the block functor of Σ ultimately to be a functor in $\mathcal{L}ex(\mathrm{T}(\partial\Sigma), \mathrm{vect})$ as well. By construction, the relative block functor $\widehat{\mathrm{T}}(\Sigma; \Sigma_{\mathrm{ref}})$ depends both on Σ_{ref} and on Σ. Recall from Section 2.2 that we denote by $\mathcal{LEX}$ the symmetric monoidal bicategory of finite $\Bbbk$-linear categories, left exact functors and natural transformations.

Definition 5.16.

(i) The *bicategory* Γ *of fine refinements* of defect surfaces is the following symmetric monoidal bicategory: Objects of Γ are defect one-manifolds. The only 1-morphisms of Γ are 1-endomorphisms, where a 1-endomorphism of a defect one-manifold Σ is a pair $(\Sigma; \Sigma')$ of (an isomorphism class of) a defect surface Σ and a fine defect surface Σ' that refines Σ. The only 2-morphisms of Γ are one single isomorphism $(\Sigma;\Sigma') \xrightarrow{\cong} (\Sigma;\Sigma'')$ for each pair of fine refinements that refine the same defect surface Σ. The composition of 1-morphisms is given by the gluing of surfaces and of refinements.

(ii) A *parallelization* Π for the collection of relative block functors for all fine refinements is a symmetric monoidal bifunctor

$$\Pi: \quad \Gamma \to \mathcal{LEX} \tag{5.41}$$

that maps 1-morphisms according to $\Pi((\Sigma;\Sigma')) := \widehat{\mathrm{T}}(\Sigma;\Sigma')$.

Spelled out, a parallelization Π is a collection of natural isomorphisms

$$\Pi_{\Sigma;\Sigma',\Sigma''}: \quad \widehat{\mathrm{T}}(\Sigma;\Sigma') \Longrightarrow \widehat{\mathrm{T}}(\Sigma;\Sigma'') \tag{5.42}$$

of functors that is coherent in the sense that every diagram involving three refinements commutes, $\Pi_{\Sigma;\Sigma_2,\Sigma_3} \circ \Pi_{\Sigma;\Sigma_1,\Sigma_2} = \Pi_{\Sigma;\Sigma_1,\Sigma_3}$ and that is compatible with factorization in the following sense: if $\Sigma = \Sigma' \circ \Sigma''$ is a defect surface obtained by gluing and we are given refinements $(\Sigma';\Sigma'_i)$ and $(\Sigma'';\Sigma''_i)$, for $i = 1,2$, that start at Σ' and at Σ'', respectively, then gluing the refined surfaces results in refinements $(\Sigma;\Sigma_i)$ that start at Σ, and the associated isomorphisms satisfy

$$\Pi_{\Sigma;\Sigma_1,\Sigma_2} = \Pi_{\Sigma';\Sigma'_1,\Sigma'_2} \circ \Pi_{\Sigma'';\Sigma''_1,\Sigma''_2} . \tag{5.43}$$

Our goal is now to define the block functor $\mathrm{T}(\Sigma)$ as the limit of a parallelization of Σ. For this to make sense, we have to show that for any defect surface Σ at least one parallelization exists. The rest of this subsection is devoted to the construction of such a parallelization.

Remark 5.17.

(i) The terminology ‘parallelization’ is chosen because this structure may be thought of as a bicategorical analogue of the parallelization of a tangent bundle.

(ii) A limit construction similar to ours is also used in the standard Turaev-Viro construction. In that case, parallelizations are provided by the unique natural transformations assigned to 3-manifolds that are cylinders over 2-manifolds (see e.g. [BalK]). In our framework, in which we do not have a three-dimensional topological field theory at our disposal, we are instead going to construct parallelizations in a purely two-dimensional setting.

(iii) For any diagram $\alpha: \Xi \to \mathcal{C}$ in which every morphism $\alpha(\xi \xrightarrow{g} \xi')$ is an isomorphism, the limit and colimit of α can be identified. (To see this, just note that for any representative ξ_0, $\alpha(\xi_0)$ can be endowed with the structure of a colimit by the structure morphisms $\alpha(\xi \xrightarrow{g} \xi_0)$, and with the structure of a limit by the structure morphisms $\alpha(\xi_0 \xrightarrow{g} \xi)$.) Thus the block functor $\mathrm{T}(\Sigma)$ is both a limit and a colimit of a parallelization of the collection of relative block functors.

We now outline the construction of a parallelization Π; the details of the arguments are deferred to Appendix C. We proceed in two steps. The first step addresses a local aspect of the construction: we construct a distinguished isomorphism between the block functors for any two fillable disks of the same type. This amounts to a parallelization for the subcategory of refinements $(\Sigma;\Sigma')$ for which Σ is a fillable disk. To achieve this goal, fix a defect one-manifold

$\mathbb{L} = \mathbb{L}_{\mathbb{X}}$ that can appear as the outer boundary of a fillable disk $\mathbb{D} = \mathbb{D}_{\mathbb{X}}$ of some type $\mathbb{X}$, as introduced in Definition 5.10. We have already seen that there exists a unique fillable disk $\mathbb{D}_{\mathbb{X}}^{\rm tad}$ with outer boundary $\mathbb{L}$ such that each inner boundary circle of $\mathbb{D}_{\mathbb{X}}^{\rm tad}$ is a tadpole circle (see the picture (5.39)). In Appendix C.2 we construct explicitly a canonical isomorphism

$$\varphi_{\mathbb{D}}: \quad \mathrm{T}_{\rm fine}(\mathbb{D})(\mho(\mathbb{D})) \xrightarrow{\cong} \mathrm{T}_{\rm fine}(\mathbb{D}_{\mathbb{X}}^{\rm tad})(\mho(\mathbb{D}_{\mathbb{X}}^{\rm tad}))\,. \tag{5.44}$$

Our construction of this isomorphism makes use of a combinatorial datum, namely a spanning tree for the graph $\Gamma_{\mathbb{D}}$ that has as edges those defect lines of $\mathbb{D}$ that come from $\mathbb{X}$, and as vertices the end points of these defect lines on $\partial\mathbb{D}$ together with the inner boundary circles of $\mathbb{D}$. But as we show in Lemma C.9 in the appendix, the isomorphism (5.44) does not depend on this datum.

Having obtained the isomorphisms (5.44) we can define the desired isomorphism between the block functors for any two fillable disks $\mathbb{D}$ and $\mathbb{D}'$ of the same type $\mathbb{X}$ (and thus in particular with the same silent object $\mho(\mathbb{D})$) as the vertical composition

$$\varphi_{\mathbb{D},\mathbb{D}'} := \varphi_{\mathbb{D}'}^{-1} * \varphi_{\mathbb{D}}: \quad \mathrm{T}_{\rm fine}(\mathbb{D})(-\boxtimes\mho(\mathbb{D})) \rightarrow \mathrm{T}_{\rm fine}(\mathbb{D}')(-\boxtimes\mho(\mathbb{D}))\,. \tag{5.45}$$

Clearly, the so obtained family of natural isomorphisms labeled by pairs of fillable disks of any given type $\mathbb{X}$ is coherent in the sense that

$$\varphi_{\mathbb{D},\mathbb{D}''} = \varphi_{\mathbb{D}',\mathbb{D}''} * \varphi_{\mathbb{D},\mathbb{D}'}\,. \tag{5.46}$$

We can further show (see Proposition C.11) that this family satisfies the following *factorization property*: For $\mathbb{D}$ of the form $\mathbb{D} = \mathbb{Y} \circ (\mathbb{D}_1 \sqcup \cdots \sqcup \mathbb{D}_n)$, with $\mathbb{D}_1, \ldots, \mathbb{D}_n$ non-intersecting fillable disks in $\mathbb{D}$ and $\mathbb{Y}$ the defect surface that is obtained by removing all the $\mathbb{D}_i$ from $\mathbb{D}$, for any n-tuple of replacements of the fillable disks $\mathbb{D}_i$ by fillable disks $\mathbb{D}_i'$ of the same type $\mathbb{X}_i$ and with the same outer boundary, the equality

$$\varphi_{\mathbb{D},\mathbb{D}'} = (\varphi_{\mathbb{D}_1,\mathbb{D}_1'} \boxtimes \cdots \boxtimes \varphi_{\mathbb{D}_n,\mathbb{D}_n'}) \circ \mathrm{T}_{\rm fine}(\mathbb{Y}) \tag{5.47}$$

of natural transformations holds, where $\mathbb{D}'$ is the defect surface $\mathbb{D}' := \mathbb{Y} \circ (\mathbb{D}_1' \sqcup \cdots \sqcup \mathbb{D}_n')$ and the symbol '$\circ$' stands for the horizontal composition (whiskering) of the natural transformation $\boxtimes_i \varphi_{\mathbb{D}_i,\mathbb{D}_i'}$ with the functor $\mathrm{T}_{\rm fine}(\mathbb{Y})$. (Also, here $\mathbb{Y}$ is regarded as a bordism from $\partial_{\rm g}\mathbb{D}$ to $\partial_{\rm g}\mathbb{D}_1 \sqcup \cdots \sqcup \partial_{\rm g}\mathbb{D}_n$, with $\partial_{\rm g}$ denoting the gluing part of the outer boundary, so that we deal with a functor $\mathrm{T}_{\rm fine}(\mathbb{Y})\colon \mathrm{T}(\partial\mathbb{D}) \rightarrow \boxtimes_i \mathrm{T}(\partial\mathbb{D}_i)$ and thus tacitly invoke the equivalence (4.1).)

Note that the factorization property involves the manipulation of replacing a fillable disk $\mathbb{D}_i$ inside the defect surface $\Sigma = \mathbb{Y}$ by another fillable disk $\mathbb{D}_i'$ with the same boundary. (That such a manipulation is possible rests on the factorization result of Theorem 5.22 below.) We call the corresponding manipulation for a generic defect surface Σ a *fillable-disk replacement* of $\mathbb{D}$ by $\mathbb{D}'$ in Σ and denote the resulting defect surface by $\Phi(\Sigma) \equiv \Phi_{\mathbb{D},\mathbb{D}'}(\Sigma)$. (For an example of a fillable-disk replacement, see the picture (C.46).)

By a *refinement replacement* from $(\Sigma;\Sigma')$ to $(\Sigma;\Sigma'')$ we then mean a sequence of (possibly intersecting) fillable-disk replacements $(\Phi_1, \ldots, \Phi_n)$ such that $\Phi_n(\cdots\Phi_1(\Sigma')\cdots) = \Sigma''$. Given any two refinements $(\Sigma;\Sigma_1)$ to $(\Sigma;\Sigma_2)$ one can construct a *common subrefinement* $(\Sigma;\Sigma_{1,2})$ and specific *standard* refinement replacements from $(\Sigma;\Sigma_i)$ for $i = 1, 2$ to $(\Sigma;\Sigma_{1,2})$ as a sequence of fillable-disk replacements of a very restricted type (compare Definition C.13 and the pictures

(C.44) and (C.45)). This implies in particular (see Lemma C.14) that a refinement replacement exists between any two refinements that refine the same defect surface. Moreover, it follows from the results about fillable disks in Proposition C.11 that any refinement replacement $(\Phi_1, \dots, \Phi_n)$ from $(\Sigma; \Sigma')$ to $(\Sigma; \Sigma'')$ provides a distinguished natural isomorphism

$$\varphi_{\Sigma', \Phi_n(\cdots \Phi_1(\Sigma') \cdots)} : \quad \mathrm{T}_{\text{fine}}(\Sigma')(- \boxtimes \mho') \longrightarrow \mathrm{T}_{\text{fine}}(\Sigma'')(- \boxtimes \mho'') . \tag{5.48}$$

The second step of the construction of a parallelization uses these isomorphisms to provide the natural isomorphisms (5.42). To this end we first show, with the help of a suitable notion of common subrefinement, that for any two refinements $(\Sigma; \Sigma')$ and $(\Sigma; \Sigma'')$ a refinement replacement $(\Phi_1, \dots, \Phi_n)$ from $(\Sigma; \Sigma')$ to $(\Sigma; \Sigma'')$ exists (Lemma C.14). Then we show in Lemma C.17 that for any refinement replacement $(\Phi_1, \dots, \Phi_n)$ in a fine defect surface there exists a refinement replacement $(\Phi'_1, \dots, \Phi'_{n'})$ that induces the same natural isomorphism and that involves only very specific types of fillable-disk replacements (which are introduced in Definition C.13). This finally allows us to conclude, in Proposition C.18, that any two refinement replacements $(\Phi_1, \dots, \Phi_n)$ and $(\Phi'_1, \dots, \Phi'_{n'})$ between any two given refinements $(\Sigma; \Sigma_{\text{ref}_1})$ and $(\Sigma; \Sigma_{\text{ref}_2})$ of an arbitrary defect surface Σ induce the same natural isomorphism,

$$\varphi_{\Sigma_{\text{ref}_1}, \Phi'_{n'}(\cdots \Phi'_1(\Sigma_{\text{ref}_1}) \cdots)} = \varphi_{\Sigma_{\text{ref}_1}, \Phi_n(\cdots \Phi_1(\Sigma_{\text{ref}_1}) \cdots)} . \tag{5.49}$$

We summarize our findings in the

Theorem 5.18. There exists a parallelization Π for the collection of relative block functors for all fine refinements of defect surfaces.

In fact, a parallelization not only exists, but it also satisfies a universal property with respect to parallel transport operations, and is therefore uniquely characterized. Before showing this universality, we have a look at consequences of the existence of Π. We start by giving the

Definition 5.19. The block functor $\mathrm{T}(\Sigma)$ for a (not necessarily fine) defect surface Σ is the limit

$$\mathrm{T}(\Sigma) := \lim_{\Sigma'} \Pi_{\Sigma;\Sigma'} \tag{5.50}$$

(or, equivalently, colimit) of the parallelization of Σ described above.

Remark 5.20.

(i) The proof of Proposition C.18 in Appendix C relies on the details of the parallelization only through Proposition C.11 which covers the case of fillable disks. This shows that the datum of a modular functor can be obtained from a functor $\mathrm{Bord}_2^{\text{def,fine}} \to \mathcal{L}ex$, that is, from a functor that is defined on *fine* defect surfaces, and that is equipped with a family of 'transparent' defects such that Proposition C.11 holds.

(ii) Thus the crucial input for the equivalence of refinement replacements stated in Proposition C.18 is the corresponding assertion for disks in Proposition C.11. This aspect of our construction is reminiscent of the way [Lu, Sect. 5.5] in which factorization homology allows one to extend locally constant factorization algebras from open disks to arbitrary surfaces. Accordingly one may suspect that Proposition C.11 is related to the structure of a factorization algebra on disks that is locally constant with respect to the stratification given by the defects on fillable disks. A detailed analysis of this idea is beyond the scope of the present paper. But it is reassuring that analogs of defects can be treated in the setting of locally constant factorization algebras for stratified manifolds [Gi, Sect. 6] and that factorization homology can be extended to stratifications [AFT]. Moreover, transparent defects (in our language) have appeared in this context as well, e.g. as a tool for defining a topological field theory associated with an E_n-algebra [Sc].

It follows directly from Proposition C.1 that as a particular case of the parallelization we have

Lemma 5.21. Let $(\Sigma;\Sigma_{\mathrm{ref}})$ be a refinement of a defect surface Σ such that the defect lines in $\Sigma_{\mathrm{ref}}\backslash\Sigma$ together with $\partial_{\mathrm{fill}}\Sigma_{\mathrm{ref}}$ form a tree $\gamma_{\Sigma;\Sigma_{\mathrm{ref}}}$ in Σ. Then the excision isomorphism from Lemma 4.34 applied to $\gamma_{\Sigma;\Sigma_{\mathrm{ref}}}$ provides an isomorphism

$$\mathrm{T}^{\mathrm{pre}}(\Sigma_{\mathrm{ref}})(-\boxtimes\mho_{\Sigma;\Sigma_{\mathrm{ref}}})\cong\mathrm{T}^{\mathrm{pre}}(\Sigma) \tag{5.51}$$

already for the pre-block functors, and this isomorphism induces the parallelization isomorphism for the relative block functors.

Let us now come back to the factorization issue considered in Section 5.1, which so far could be discussed only for *fine* surfaces. Our results on refinements allow us to extend the results of Section 5.1 directly to the factorization of general defect surfaces.

Theorem 5.22. *Factorization*: Let Σ be a defect surface with gluing boundary components $\mathbb{L}$ and $\overline{\mathbb{L}}$. There is a canonical isomorphism

$$\int^{x\in\mathrm{T}(\mathbb{L})}\mathrm{T}(\Sigma)(-\boxtimes x\boxtimes\overline{x})\;\cong\;\mathrm{T}(\cup_{\mathbb{L}}(\Sigma))(-) \tag{5.52}$$

of left exact functors. Moreover, the canonical isomorphisms obtained this way with any two distinct pairs $(\mathbb{L}_1,\overline{\mathbb{L}_1})$ and $(\mathbb{L}_2,\overline{\mathbb{L}_2})$ of gluing boundaries of Σ commute.

Proof. For notational simplicity we restrict our attention to the case that $\Sigma=\Sigma_1\sqcup\Sigma_2$ is the disjoint union of two defect surfaces Σ_1 and Σ_2 such that $\cup_{\mathbb{L}}(\Sigma)=\Sigma_2\circ\Sigma_1$ is the gluing of Σ_1 and Σ_2; the general case is covered by the same arguments. Let $(\Sigma_1;\Sigma_1')$ and $(\Sigma_2;\Sigma_2')$ be fine refinements that refine Σ_1 and Σ_2, respectively. Then $(\Sigma_1\circ\Sigma_2;\Sigma_1'\circ\Sigma_2')$ is a fine refinement that refines $\Sigma_1\circ\Sigma_2$. Theorem 5.2 thus provides us with an isomorphism

$$\widetilde{\varphi}_{\Sigma_2',\Sigma_1'}:\quad\widehat{\mathrm{T}}(\Sigma_2;\Sigma_2')\circ\widehat{\mathrm{T}}(\Sigma_1;\Sigma_1')\xrightarrow{\cong}\widehat{\mathrm{T}}(\Sigma_2\circ\Sigma_1;\Sigma_2'\circ\Sigma_1')\,. \tag{5.53}$$

By taking the limit over the fine refinements that refine Σ_1 and Σ_2 we then also obtain an isomorphism

$$\varphi_{\Sigma_2',\Sigma_1'}:\quad\mathrm{T}(\Sigma_2)\circ\mathrm{T}(\Sigma_1)\xrightarrow{\cong}\widehat{\mathrm{T}}(\Sigma_2\circ\Sigma_1;\Sigma_2'\circ\Sigma_1')\,. \tag{5.54}$$

Now not every refinement $(\Sigma_2\circ\Sigma_1;\Sigma')$ of $\Sigma_2\circ\Sigma_1$ is of the form $(\Sigma_2\circ\Sigma_1;\Sigma_2'\circ\Sigma_1')$, since Σ' might also refine the gluing boundary $\mathbb{L}$. However, we can compose $\varphi_{\Sigma_2',\Sigma_1'}$ with the parallelization $\Pi_{\Sigma_2\circ\Sigma_1;\Sigma',\Sigma_2'\circ\Sigma_1'}$ of $\Sigma_2\circ\Sigma_1$ so as to obtain a coherent family of isomorphisms

$$\varphi_{\Sigma'}:\quad\mathrm{T}(\Sigma_2)\circ\mathrm{T}(\Sigma_1)\xrightarrow{\cong}\widehat{\mathrm{T}}(\Sigma_2\circ\Sigma_1;\Sigma')\,. \tag{5.55}$$

Thus by the universal property of the limit we get an isomorphism $\mathrm{T}(\Sigma_2) \circ \mathrm{T}(\Sigma_1) \xrightarrow{\cong} \mathrm{T}(\Sigma_2 \circ \Sigma_1)$. Note that the block functors are a priori functors to vect; thus in 5.55 we implicitly use the Eilenberg-Watts calculus to turn the functors $\mathrm{T}(\Sigma_i)$ into functors from the gluing category of the incoming to the one of the outgoing boundary, compare the discussion around (4.1). The desired isomorphism (5.52) then follows from 5.55 by invoking the compatibility of the Eilenberg-Watts calculus with the composition of functors, as expressed by the isomorphism (3.41).

Moreover, given three composable defect surfaces $\Sigma_1, \Sigma_2, \Sigma_3$ and three corresponding fine refinements $(\Sigma_1; \Sigma_1')$, $(\Sigma_2; \Sigma_2')$ and $(\Sigma_3; \Sigma_3')$ it follows from Corollary 5.3 that the order of the factorizations does not matter. By standard arguments this property passes to the limit. □

We have thus defined the behavior of block functors under the horizontal composition of 1-morphisms in $\mathrm{Bord}_2^{\mathrm{def}}$, i.e. a factorization structure of the modular functor.

Remark 5.23. Let us comment on the relation of our construction to the standard Turaev-Viro construction. The latter takes as an input a fusion category $\mathcal{A}$ with a spherical structure, i.e. a pivotal structure such that left and right traces coincide. In its standard incarnation [BaW, TV, BalK], the Turaev-Viro construction defines a symmetric monoidal 2-functor

$$\mathrm{Bord}_{3,2,1} \longrightarrow \text{2-vect}\,, \tag{5.56}$$

where 2-vect is the symmetric monoidal bicategory of finitely semisimple $\mathbb{C}$-linear categories. $\mathrm{Bord}_{3,2,1}$ is a bicategory of bordisms, usually without defects. The manifolds in this approach are *oriented*, rather than framed.

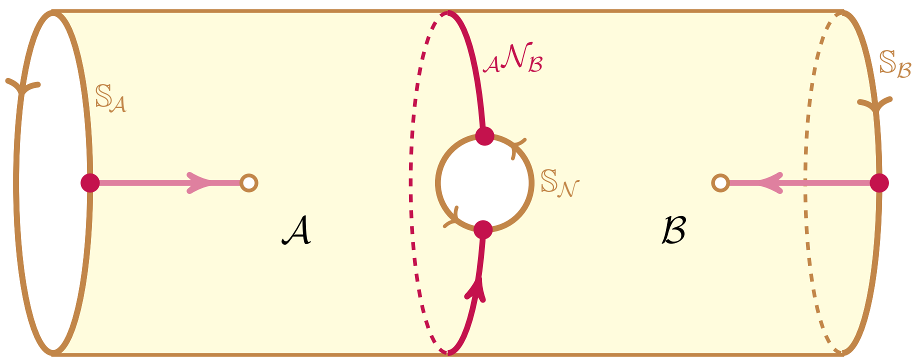
Since we do not assign natural transformations to arbitrary three-manifolds with corners, a comparison with the Turaev-Viro construction can only be made at the level of the gluing categories, the functors for two-manifolds with boundary, and the representations of mapping class groups. Concerning the level of categories, we remark that the pivotal structure allows one to canonically identify all twisted centers with the Drinfeld center $\mathcal{Z}(\mathcal{A})$; this is indeed the category associated to a circle in the standard Turaev-Viro construction.

To compare the functors assigned to surfaces without (visible) defects, we first sketch how to assign in our setting a functor to an oriented surface Σ, possibly with boundary. Again we must choose auxiliary data: a refinement $(\Sigma; \Sigma')$ of Σ as a surface together with a framing on Σ' (it is not hard to convince oneself that this can indeed be found), and then a refinement of the resulting framed surface. Given these data we get fine block functors by the prescription in Section 4. The relevant index category $\widetilde{\Gamma}$ has the same objects as Γ, but more 1-morphisms than the one in Definition 5.16, since now framings are included as auxiliary data. However, $\widetilde{\Gamma}$ also has more 2-morphisms than Γ, because it is defined to have precisely one morphisms between any two choices of framings. Thus the bicategories $\widetilde{\Gamma}$ and Γ are equivalent by construction.

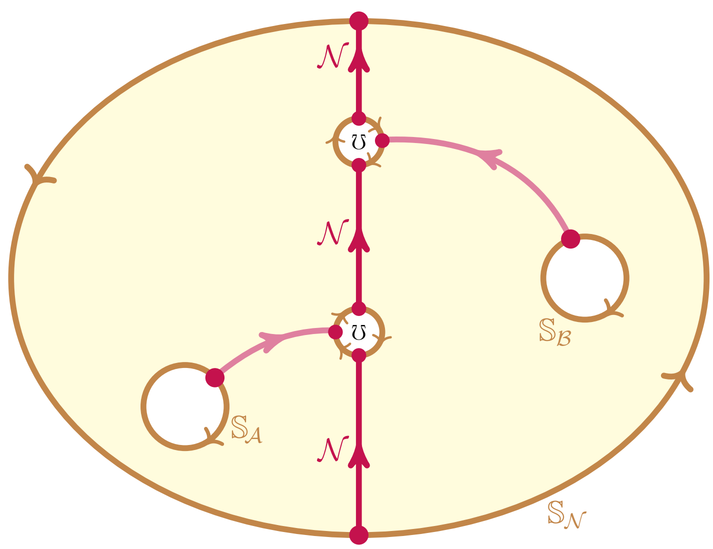
Next one observes that any two framings with the same underlying orientation differ by a suitable application of double duals. The pivotal structure should therefore give us enough natural transformations to construct a parallelization also for the index category $\widetilde{\Gamma}$. The blocks for the oriented bordism category can then again be defined, as in Definition 5.19, as (co)limits. Let us call them the *oriented blocks*.

By picking a framing we can represent an oriented block by the block obtained in our approach. (This is completely analogous to representing a block using a specific refinement.) Thus in particular we know that the oriented blocks obey factorization. As a consequence it suffices to compare the oriented conformal blocks with the standard Turaev-Viro blocks for the case of a three-punctured sphere. The standard Turaev-Viro blocks (see e.g. [BalK, Ex. 8.6]) are invariants; that the same is true for our oriented blocks is a consequence of Proposition 4.27.

Example 5.24. Consider for an $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}}$ the following defect surface Σ:

$\Sigma =$  (5.57)

We may think of Σ as describing how the situation in the region labeled by $\mathcal{A}$ is 'transmitted' to the situation in the region labeled by $\mathcal{B}$, and accordingly refer to the block functor $\mathrm{T}(\Sigma)$ as a *transmission functor*. To calculate the transmission functor, we suitably redraw Σ and pick a refinement $(\Sigma;\Sigma')$ with Σ' having two additional transparent gluing circles (to be evaluated at their respective silent objects), as shown in the following picture:

$\Sigma' =$  (5.58)

By using the isomorphisms from Lemma 5.21 to take care of the tadpole circles and applying Example 5.9 to each of the two 2-patches of Σ' we arrive at the description

$$\begin{aligned} \mathrm{T}(\Sigma'): \quad \mathrm{T}(\mathbb{S}_{\mathcal{A}}) \boxtimes \mathrm{T}(\mathbb{S}_{\mathcal{B}}) = \mathcal{Z}(\mathcal{A}) \boxtimes \mathcal{Z}(\mathcal{B}) &\longrightarrow \mathcal{L}ex_{\mathcal{A},\mathcal{B}}(\mathcal{N},\mathcal{N}) = \mathrm{T}(\mathbb{S}_{\mathcal{N}}) \,, \\ x \boxtimes y &\longmapsto F_y \circ {}_xF \end{aligned} \tag{5.59}$$

of the (fine) block functor of Σ'; where we use the braided induction functors introduced in (5.17), i.e. $\mathrm{T}(\Sigma')(x \boxtimes y)(n) = x.n.y$. In particular, when evaluating at the identity functor

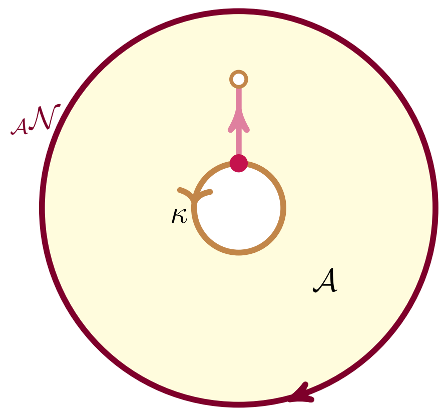
$\mathrm{Id} \in \mathcal{L}ex_{\mathcal{A},\mathcal{B}}(\mathcal{N},\mathcal{N})$ in the gluing category for the outer circle $\mathbb{S}_{\mathcal{N}}$ we obtain the functor

$$\mathcal{Z}(\mathcal{A}) \boxtimes \mathcal{Z}(\mathcal{B}) \to \mathrm{vect}\,, \qquad x \boxtimes y \mapsto \mathrm{Nat}_{\mathcal{A},\mathcal{B}}(x.(-).y, \mathrm{Id})\,. \tag{5.60}$$

Then by using the adjunctions we obtain a functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{Z}(\mathcal{B})^{\mathrm{opp}} \cong \mathcal{Z}(\mathcal{B})$. In case that the bimodule $\mathcal{N}$ is invertible, it follows from [ENOM, Sect. 5.1] that the so obtained transmission functor is an equivalence between the Drinfeld centers.

Remark 5.25. Ideally one would also like to compute the block functor for the defect surface that is obtained from the surface (5.57) by omitting the gluing circle on the defect line. This requires in particular to compute the block functor for the non-fine surface $\widehat{\Sigma}$ shown in the following picture:

$\widehat{\Sigma}$ =



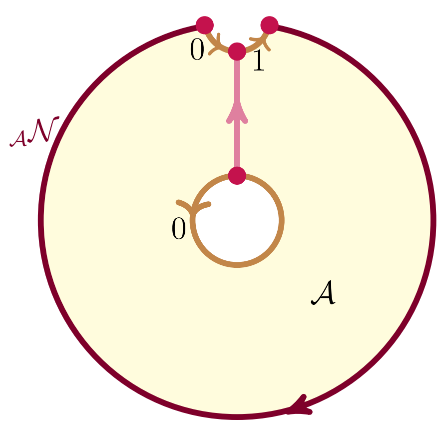
(5.61)

Here the framing index at the gluing circle is necessarily $\kappa = 0$, and the block functor is a functor

$$\mathrm{T}(\widehat{\Sigma}): \quad \mathcal{A}\overset{0}{\boxtimes} \longrightarrow \mathrm{vect}\,. \tag{5.62}$$

Now $\mathcal{A}\overset{0}{\boxtimes}$ is a *twisted* Drinfeld center, with objects consisting of an object $z \in \mathcal{A}$ together with a balancing $z \otimes a \cong {}^{\vee\vee}a \otimes z$ for $a \in \mathcal{A}$. Consider the refinement $(\Sigma; \widetilde{\Sigma}')$, with $\widetilde{\Sigma}'$ the following surface:

$\widetilde{\Sigma}'$ =



(5.63)

As pre-block functor of this surface we get $\widehat{\mathrm{T}^{\mathrm{pre}}}(\widetilde{\Sigma}')(z) = \int^{n\in\mathcal{N}} \mathrm{Hom}(n, z.n)$. After invoking the identity [FSS2, Eq. (3.52)] $\int^{n\in\mathcal{N}} \overline{n} \boxtimes n \cong \int_{n\in\mathcal{N}} \overline{n} \boxtimes \mathrm{N^r}(n)$ (with $\mathrm{N^r}$ the Nakayama functor (3.42)) this can be recognized as a space of natural transformations. The block functor is then given by the corresponding module natural transformations,

$$\mathrm{T}(\widetilde{\Sigma})(z) = \mathrm{Nat}_{\mathcal{A}}(\mathrm{N}^{\mathrm{r}}, z.-)\,, \tag{5.64}$$

where we also the fact that both the Nakayama functor N^{r} and $z.-$ are twisted module functors. This shows that, in general, when omitting the gluing circle $\mathbb{S}_{\mathcal{N}}$ from the defect surface in (5.57) one does *not* obtain the transmission functor.

5.5 Universality of the parallelization

We are now going to show that the parallelization Π whose existence was established in Theorem 5.18 can be uniquely characterized, namely by a universal property with respect to the parallel transport operations that were introduced in Section 4.2. This observation demonstrates that the choice of parallelization Π obtained by our construction is distinguished. Moreover, it will allow for concrete computations of the parallelization isomorphisms by working with pre-block functors.

Recall the parallel transport comonad $Z_{\mathbb{D},x_s,x_t}$ on $\mathcal{L}ex(\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}} \boxtimes \mathrm{U}(\partial\mathbb{D}), \mathrm{vect})$ that was considered in Proposition 4.24, for $\mathbb{D}$ a disk in a defect surface Σ and x_s, x_t elements of the set $E_{\mathbb{D}}$ defined there. Also, just as the relative block functor $\widehat{\mathrm{T}}(\Sigma;\Sigma')\colon \mathrm{T}(\Sigma) \to \mathrm{vect}$ introduced in (5.33) is the block functor for a refinement Σ' of Σ evaluated on silent objects in $\partial_{\mathrm{fill}}(\Sigma')$, we can define a relative pre-block functor $\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma;\Sigma')\colon \mathrm{T}^{\mathrm{pre}}(\Sigma) \to \mathrm{vect}$ as the pre-block functor on Σ' with silent objects inserted in $\partial_{\mathrm{fill}}(\Sigma')$, according to

$$\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma;\Sigma') := \mathrm{T}^{\mathrm{pre}}(\Sigma')(- \boxtimes \mho_{\Sigma;\Sigma'})\,. \tag{5.65}$$

The following result shows that applying the parallel transport comonad to the pre-blocks for Σ provides the pre-blocks for a refined surface Σ':

Lemma 5.26. Let $\mathbb{D}$ be a disk in a defect surface Σ and $x_s, x_t \in E_{\mathbb{D}}$ defect lines of $\mathbb{D}$.

(i) There is (unique up to isomorphism) a refinement $(\Sigma;\Sigma')$ such that Σ' differs from Σ by a new transparent defect line that connects x_s to x_t in their normal directions and such that one framing index at x_s in Σ' is 0.

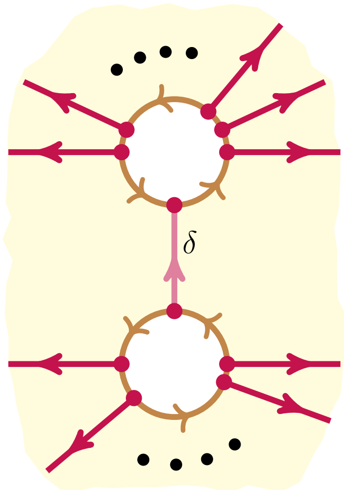
(ii) With Σ' defined this way, there is a canonical isomorphism

$$Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)) \xrightarrow{\;\cong\;} \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') \tag{5.66}$$

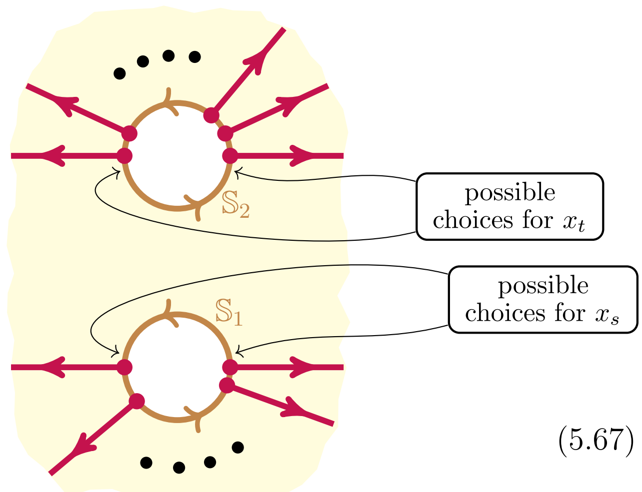
of functors.

(iii) Conversely, let $(\Sigma;\Sigma')$ be a refinement and δ a transparent defect line in Σ' that is not a defect line of Σ such that $(\Sigma;\Sigma'')$, with Σ'' the defect surface obtained by deleting δ, is still a refinement of Σ. Then for $x_s, x_t \in E_{\Sigma''}$ corresponding to (choices of) two defect points on the gluing boundaries at the start and end of δ, as indicated in

Σ' =



Σ'' =



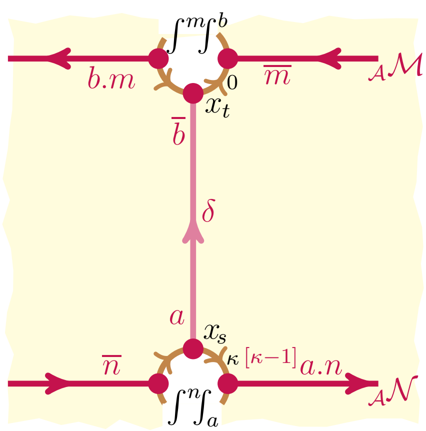
(5.67)

there is a canonical isomorphism

$$Z_{\mathbb{D},x_s,x_t}(\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma'')) \xrightarrow{\cong} \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')\,. \tag{5.68}$$

Proof. (i) $\mathbb{D}$ is a disk having x_s and x_t as parts of its boundary. Connecting x_s and x_t by an additional defect line leads to four new gluing boundaries; setting one of the corresponding new framing indices to 0 uniquely fixes the other three. On the disk any other defect line from x_s to x_t would be isotopic to the chosen one, hence all refinements that arise this way are isomorphic, compare Remark 2.7(iii).

(ii) We compute $\mathrm{T}^{\mathrm{pre}}(\Sigma')$ for the first case in the proof of Proposition 4.24: The local situation at the new defect line in Σ' is as indicated in the following picture:



(5.69)

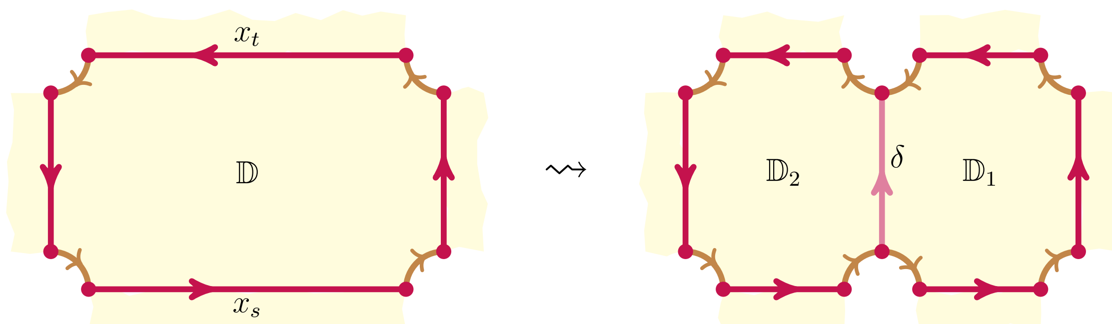
Here the labels at the defect lines together with the (co)ends indicate the relevant silent objects. In the pre-block functor we can use the Yoneda lemma for a, b, so as to obtain the relative pre-block functor

$$\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')(-) \,=\, \int^{n\in\mathcal{N}}\!\!\int^{m\in\mathcal{M}}\!\!\int_{a\in\mathcal{A}} \mathrm{Hom}(\overline{n}\boxtimes{}^{[\kappa-1]}a.n\boxtimes a.m\boxtimes\overline{m}\,\cdots, -)\,, \qquad (5.70)$$

where the ellipsis indicates the remaining defect lines of Σ'. With μ the sum of the framing indices counted clockwise from x_s to x_t and l the number of the gluing segments along that path, it follows that $\kappa = l-\mu$ and thus $\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')$ is isomorphic to $Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma))$ according to the definition of the comonad in Proposition 4.24.

(iii) Consider the silent objects $\mho(\mathbb{S}_i)$ in the gluing categories for $\mathbb{S}_1$ and $\mathbb{S}_2$ for Σ''. The only case that is not already covered by (ii) above is the situation that the relevant framings on $\mathbb{S}_i$ are not both ±1 (this can happen if the adjacent defect lines are transparent, see the case of the defect surfaces considered in Proposition 5.43). However, the transparent objects are explicitly known (see Example 5.11(ii)), so that we can proceed as in the proof of statement (ii). □

Denote, as above, the new transparent defect line on Σ' featuring in Lemma 5.26 by δ. Consider the situation shown in the following picture:



(5.71)

The defect line δ in Σ' splits the disk $\mathbb{D}$ in two disks $\mathbb{D}_1$ and $\mathbb{D}_2$, which we label in such a way that the counterclockwise path from x_s to x_t lies in $\mathbb{D}_1$. We start with the following

Lemma 5.27. Let $\mathbb{D}$, $\mathbb{D}_1$ and $\mathbb{D}_2$ be disks as in the situation shown in (5.71). Then $\gamma^{\mathrm{cc}}_{\mathbb{D}_2} = \gamma^{\mathrm{c}}_{\mathbb{D}_1}$.

Proof. We show that under the isomorphism $\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') \cong Z_{\mathbb{D}_1,x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma'))$ (with x_s and x_t the defect lines appearing in (5.71)), the morphisms $\gamma^{\mathrm{cc}}_{\mathbb{D}_2}$ and $\gamma^{\mathrm{c}}_{\mathbb{D}_1}$ both coincide with the comultiplication $Z_{\mathbb{D}_1,x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma')) \to Z^2_{\mathbb{D}_1,x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma'))$. To this end we first note that the comultiplication of the coalgebra $\int_{a\in\mathcal{A}} \overline{a}\boxtimes a \in \overline{\mathcal{A}}\boxtimes\mathcal{A}$ is given by the end over $b\in\mathcal{A}$ of the composite morphisms

$$\int_{a\in\mathcal{A}} \overline{a}\boxtimes a \xrightarrow{\;\mathrm{coev}^{\mathrm{l}}_b\;} \int_{a\in\mathcal{A}} \overline{a}\boxtimes(a\otimes{}^{\vee}b\otimes b) \xrightarrow{\;\cong\;} \int_{a\in\mathcal{A}} \overline{b\otimes a}\boxtimes(a\otimes b)\,, \qquad (5.72)$$

where the second morphism is obtained from the canonical central structure of the end. The end over these morphisms equals the end over the morphisms

$$\int_{a\in\mathcal{A}} \overline{a}\boxtimes a \xrightarrow{\;\mathrm{coev}^{\mathrm{r}}_b\;} \int_{a\in\mathcal{A}} \overline{a}\boxtimes(b\otimes b^{\vee}\otimes a) \xrightarrow{\;\cong\;} \int_{a\in\mathcal{A}} \overline{b\otimes a}\boxtimes(a\otimes b)\,. \qquad (5.73)$$

According to Proposition 4.24, the comonad $Z_{\mathbb{D}_1,x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma'))$ is given by acting with $\int_a \overline{a}\boxtimes a$ on $\mathrm{U}(\partial\mathbb{D})^{\mathrm{opp}}\boxtimes \mathrm{U}(\partial\mathbb{D})$. It follows that the two morphisms above are mapped to the morphisms $\gamma^{\mathrm{c}}_{\mathbb{D}_1}$ and $\gamma^{\mathrm{cc}}_{\mathbb{D}_1}$, respectively. Thus the statement follows. □

We now define canonical morphisms between the pre-block functors on Σ and Σ' that corresponding to the creation and deletion of δ, respectively. To this end we make uses of the following fact (which is weaker than the statement that morphisms between diagrams induce morphisms between their limits, but is still elementary):

Lemma 5.28. For $\mathcal{C}$ an abelian category, let $E_1 \xrightarrow{\iota_1} A_1$ be the equalizer of a pair of morphisms $f_1, f_2\colon A_1 \to A_2$ in $\mathcal{C}$, and $E_2 \xrightarrow{\iota_2} A_2$ the equalizer of morphisms $g_1, g_2\colon B_1 \to B_2$. Let $a\colon A_1 \to B_1$ and $b\colon A_2 \to B_2$ be such that for every $i \in \{1,2\}$ there is a $j \in \{1,2\}$ so that $g_i \circ a \circ \iota_1 = b \circ f_j \circ \iota_1$. Then by restriction a induces a unique morphism from E_1 to E_2.

Proof. It follows directly that the morphism $a\circ\iota_1$ equalizes the pair (g_1, g_2), and thus the morphism from E_1 to E_2 exists by the universal property of the equalizer E_2. In particular, the induced morphism does not depend on the choice of b that fulfills the assumption. □

Note that, in terms of diagrams, in

$$
\begin{array}{ccccc}
E_1 & \xrightarrow{\ \iota_1\ } & A_1 & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & A_2 \\
\big\downarrow & & \big\downarrow{\scriptstyle a} & & \big\downarrow{\scriptstyle b} \\
E_2 & \xrightarrow{\ \iota_2\ } & B_1 & \overset{g_1}{\underset{g_2}{\rightrightarrows}} & B_2
\end{array}
\tag{5.74}
$$

it is not required that all four right squares commute. As a special case, we see that if in the situation of the Lemma for $i \in \{1,2\}$ there is a $j \in \{1,2\}$ so that $g_i \circ a = b \circ f_j$, then there is a unique induced morphism between the equalizers.

If for a morphism $a\colon A_1 \to B_1$ as in this lemma there exists a morphism $b\colon A_2 \to B_2$ that fulfills the condition stated in the lemma, we say that a is *compatible with the equalizers*. It turns out that such a compatibility is present for the parallel transports $\gamma^{\mathrm{c}}_{\mathbb{D}}$ and $\gamma^{\mathrm{cc}}_{\mathbb{D}}$ on the pre-blocks in the situation treated in Lemma 5.26:

Proposition 5.29. Let Σ be a defect surface and $(\Sigma;\Sigma')$ the refinement described in Lemma 5.26.

(i) The clock- and counterclockwise parallel transports

$$\gamma^{\mathrm{c}}_{\mathbb{D}}, \gamma^{\mathrm{cc}}_{\mathbb{D}}: \quad \mathrm{T}^{\mathrm{pre}}(\Sigma) \to Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)) \cong \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') \tag{5.75}$$

are compatible with the equalizers to the block spaces for all parallel transport operations on Σ and on Σ'. Both morphisms induce the same morphism between the corresponding fine block functors.

(ii) Let now Σ' and Σ'' be as in Lemma 5.26(iii). The counit ϵ of the comonad $Z_{\mathbb{D},x_s,x_t}$ defines a morphism

$$\epsilon_{\mathbb{D}}: \quad \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') \cong Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)) \longrightarrow \mathrm{T}^{\mathrm{pre}}(\Sigma) \tag{5.76}$$

which is compatible with the equalizers to the block spaces for all parallel transport operations on Σ and on Σ'.

Proof. For brevity of the exposition, we identify $Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma))$ with $\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')$ via the canonical isomorphism (5.66).

(i) Consider again the situation of Figure (5.71). We treat the case of $\gamma^{\mathrm{cc}}_{\mathbb{D}} \colon \mathrm{T}^{\mathrm{pre}}(\Sigma) \to \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')$; the proof for $\gamma^{\mathrm{c}}_{\mathbb{D}}$ is analogous. We consider the parallel transports on the disks $\mathbb{D}_1$ and $\mathbb{D}_2$. For $\mathbb{D}_1$ we deal with the morphisms $\gamma^{\mathrm{c}}_{\mathbb{D}_1}$ and $\gamma^{\mathrm{cc}}_{\mathbb{D}_1}$ and find corresponding parallel transports for Σ as in Lemma 5.28. This gives rise to the diagrams

$$\begin{array}{ccc} \mathrm{T}^{\mathrm{pre}}(\Sigma) & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}}} & Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)) \\ {\scriptstyle \gamma^{\mathrm{cc}}_{\mathbb{D}}}\big\downarrow & & \big\downarrow{\scriptstyle Z_{\mathbb{D},x_s,x_t}(\gamma^{\mathrm{cc}}_{\mathbb{D}})} \\ \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}_1}} & Z_{\mathbb{D},x_s,x_t}(\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{T}^{\mathrm{pre}}(\Sigma) & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}}} & Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)) \\ {\scriptstyle \gamma^{\mathrm{cc}}_{\mathbb{D}}}\big\downarrow & & \big\downarrow{\scriptstyle Z_{\mathbb{D},x_s,x_t}(\gamma^{\mathrm{cc}}_{\mathbb{D}})} \\ \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') & \xrightarrow{\gamma^{\mathrm{c}}_{\mathbb{D}_1}} & Z_{\mathbb{D},x_s,x_t}(\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')) \end{array} \tag{5.77}$$

The first of these diagrams commutes directly. The second diagram commutes as well, owing to the fact that, according to Proposition 4.24, $\gamma^{\mathrm{cc}}_{\mathbb{D}}$ provides a comodule structure.
The corresponding diagrams for the disk $\mathbb{D}_2$ are

$$\begin{array}{ccc} \mathrm{T}^{\mathrm{pre}}(\Sigma) & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}}} & Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)) \\ {\scriptstyle \gamma^{\mathrm{cc}}_{\mathbb{D}}}\big\downarrow & & \big\downarrow{\scriptstyle Z_{\mathbb{D},x_s,x_t}(\gamma^{\mathrm{cc}}_{\mathbb{D}})} \\ \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}_2}} & Z_{\mathbb{D},x_s,x_t}(\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{T}^{\mathrm{pre}}(\Sigma) & \xrightarrow{\gamma^{\mathrm{c}}_{\mathbb{D}}} & Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)) \\ {\scriptstyle \gamma^{\mathrm{cc}}_{\mathbb{D}}}\big\downarrow & & \big\downarrow{\scriptstyle Z_{\mathbb{D},x_s,x_t}(\gamma^{\mathrm{cc}}_{\mathbb{D}})} \\ \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') & \xrightarrow{\gamma^{\mathrm{c}}_{\mathbb{D}_2}} & Z_{\mathbb{D},x_s,x_t}(\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')) \end{array} \tag{5.78}$$

Here the first diagram commutes because $\gamma^{\mathrm{cc}}_{\mathbb{D}}$ is a comodule structure, while the second diagram commutes because the parallel transports take place in different disks. It follows analogously that $\gamma^{\mathrm{c}}_{\mathbb{D}}$ is compatible with the equalizers, and this induces another morphism from $\mathrm{T}(\Sigma)$ to $\widehat{\mathrm{T}}(\Sigma')$. Since, by Proposition 4.24(iii), $\mathrm{T}(\Sigma)$ is the equalizer of both parallel transports, it follows that the two induced morphisms between the block functors agree.

(ii) We use the counit ϵ of $Z_{\mathbb{D},x_s,x_t}$, which by definition of the comonad is given by the component at $\mathbf{1}$ of the dinatural transformation of the end. Then, with Σ' as in Figure (5.71), we have $\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') \cong Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma))$, and it follows that ϵ defines a natural transformation $\epsilon_{\mathbb{D}} \colon \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') \to \mathrm{T}^{\mathrm{pre}}(\Sigma)$. For the parallel transports $\gamma^{\mathrm{c}}_{\mathbb{D}}$ and $\gamma^{\mathrm{cc}}_{\mathbb{D}}$ we need to provide corresponding parallel transports for one disk of Σ' such that the condition of Lemma 5.28 is satisfied. Consider the diagrams

$$\begin{array}{ccc} \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') & \xrightarrow{\gamma^{\mathrm{c}}_{\mathbb{D}_2}} & Z_{\mathbb{D},x_s,x_t}(\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')) \\ {\scriptstyle \epsilon_{\mathbb{D}}}\big\downarrow & & \big\downarrow{\scriptstyle Z_{\mathbb{D},x_s,x_t}(\epsilon_{\mathbb{D}})} \\ \mathrm{T}^{\mathrm{pre}}(\Sigma) & \xrightarrow{\gamma^{\mathrm{c}}_{\mathbb{D}}} & Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)) \end{array} \quad \text{and} \quad \begin{array}{ccc} \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}_2}} & Z_{\mathbb{D},x_s,x_t}(\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')) \\ {\scriptstyle \epsilon_{\mathbb{D}}}\big\downarrow & & \big\downarrow{\scriptstyle Z_{\mathbb{D},x_s,x_t}(\epsilon_{\mathbb{D}})} \\ \mathrm{T}^{\mathrm{pre}}(\Sigma) & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}}} & Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)). \end{array} \tag{5.79}$$

The left diagram commutes directly. In contrast, the right diagram does not, in general, commute. Let us compose the right diagram with the equalizer of the block functor for Σ' to obtain

$$\begin{array}{ccccc} \widehat{\mathrm{T}}(\Sigma') & & \longrightarrow & \longrightarrow & \searrow \\ & \overset{\iota}{\searrow} & & & \\ & & \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}_2}} & Z_{\mathbb{D},x_s,x_t}(\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')) \\ & \searrow & {\scriptstyle \epsilon_{\mathbb{D}}}\downarrow & & \downarrow{\scriptstyle Z_{\mathbb{D},x_s,x_t}(\epsilon_{\mathbb{D}})} \\ & & \mathrm{T}^{\mathrm{pre}}(\Sigma) & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}}} & Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)). \end{array} \tag{5.80}$$

where the unlabeled arrows are defined as the corresponding composites. Consider the outer square in this diagram. From Lemma 5.27 we conclude that $\gamma^{\mathrm{cc}}_{\mathbb{D}_2} = \gamma^{\mathrm{c}}_{\mathbb{D}_1}$, while the fact that ι is the equalizer of the parallel transports implies that $\gamma^{\mathrm{c}}_{\mathbb{D}_1} \circ \iota = \gamma^{\mathrm{cc}}_{\mathbb{D}_1} \circ \iota$. Since the diagram

$$\begin{array}{ccc} \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}_1}} & Z_{\mathbb{D},x_s,x_t}(\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')) \\ {\scriptstyle \epsilon_{\mathbb{D}}}\downarrow & & \downarrow{\scriptstyle Z_{\mathbb{D},x_s,x_t}(\epsilon_{\mathbb{D}})} \\ \mathrm{T}^{\mathrm{pre}}(\Sigma) & \xrightarrow{\gamma^{\mathrm{cc}}_{\mathbb{D}}} & Z_{\mathbb{D},x_s,x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma)) \end{array} \tag{5.81}$$

commutes directly, it follows directly that the outer square in (5.80) commutes. We can now invoke Lemma 5.28(ii) we obtain a morphism between the block functors. □

We denote the induced morphisms between the block functors from part (i) and (ii) of Proposition 5.29 by $\Gamma_\Sigma\colon \mathrm{T}(\Sigma) \to \widehat{\mathrm{T}}(\Sigma')$ and $\widehat{\epsilon}_\Sigma\colon \widehat{\mathrm{T}}(\Sigma') \to \mathrm{T}(\Sigma)$. To show that these morphisms agree with the parallelizations we require a compatibility with factorization: Let $\Sigma = \Sigma_1 \circ \Sigma_2$ be a fine defect surface that is the composite of two fine defect surfaces along a common boundary component $\mathbb{S}$. Assume that $x_s, x_t \in E_{\mathbb{D}}$ are two defect lines that intersect $\mathbb{S}$, i.e. they lie in both Σ_1 and Σ_2. We then obtain morphisms

$$\Gamma_{\Sigma_1}: \quad \mathrm{T}(\Sigma_1) \to \widehat{\mathrm{T}}(\Sigma_1') \qquad \text{and} \qquad \Gamma_{\Sigma_2}: \quad \mathrm{T}(\Sigma_2) \to \widehat{\mathrm{T}}(\Sigma_2')\,, \tag{5.82}$$

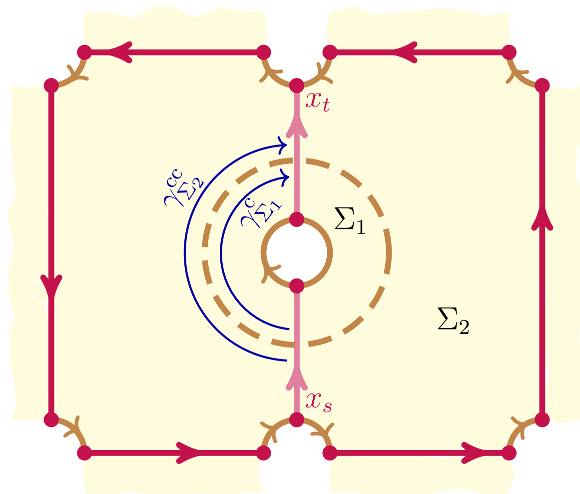
with Σ_i' the respective refinements.

Lemma 5.30. In the situation just described we have isomorphisms $\Sigma_1' \circ \Sigma_2 \xrightarrow{\cong} \Sigma_1 \circ \Sigma_2' \xrightarrow{\cong} \Sigma'$ of defect surfaces, where $(\Sigma;\Sigma')$ is the refinement of Σ that corresponds to to the pair x_s, x_t. Furthermore, the morphisms on the block functors satisfy

$$\Gamma_1 \circ \mathrm{T}(\Sigma_2) = \mathrm{T}(\Sigma_1) \circ \Gamma_2 = \Gamma_\Sigma\,. \tag{5.83}$$

Proof. Consider the following situation:

$\Sigma \;=$



(5.84)

Since by Proposition 5.29 the morphisms Γ_i do not depend on the choice of parallel transport in Σ_i, we may just consider the parallel transports $\gamma^{\mathrm{cc}}_{\Sigma_2}$ and $\gamma^{\mathrm{c}}_{\Sigma_1}$. By factorization for pre-blocks we have

$$\mathrm{T}^{\mathrm{pre}}(\Sigma_1) \circ \mathrm{T}^{\mathrm{pre}}(\Sigma_2) \;\cong\; \mathrm{T}^{\mathrm{pre}}(\Sigma_1) \boxtimes \mathrm{T}^{\mathrm{pre}}(\Sigma_2) \big(\int^{z \in \mathrm{T}(\mathbb{S})} \overline{z} \boxtimes z \big), \tag{5.85}$$

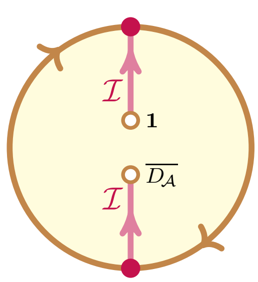
and the parallel transports $\gamma^{\mathrm{cc}}_{\Sigma_2}$ and $\gamma^{\mathrm{c}}_{\Sigma_1}$ correspond to post-composition with the canonical morphism $\int^{z \in \mathrm{T}(\mathbb{S})} \overline{z} \boxtimes z \to \int^{z \in \mathrm{T}(\mathbb{S})} \int_{a \in \mathcal{A}} \overline{a.z} \boxtimes a.z$. It follows that $\Gamma_1 \circ \mathrm{T}(\Sigma_2) = \mathrm{T}(\Sigma_1) \circ \Gamma_2$. Using again that Γ_Σ is independent of the choice of clock- or counterclockwise parallel transport, it follows that both of these are also equal to Γ_Σ. □

Next we show that the induced morphisms between the block functors are identical to the parallelization morphisms from Theorem 5.18, which implies in particular that the morphisms Γ_Σ and $\widehat{\epsilon}_\Sigma$ are indeed isomorphisms.

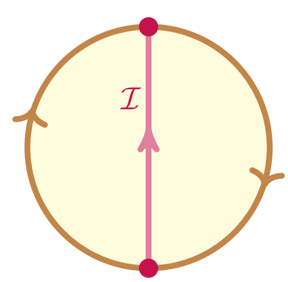
Proposition 5.31. The morphisms $\Gamma_\Sigma \colon \mathrm{T}_{\mathrm{fine}}(\Sigma) \to \mathrm{T}_{\mathrm{fine}}(\Sigma')$ and $\widehat{\epsilon}_\Sigma \colon \mathrm{T}_{\mathrm{fine}}(\Sigma') \to \mathrm{T}_{\mathrm{fine}}(\Sigma)$ that are induced by the morphisms (5.75) and (5.76) are the same as the parallelization isomorphisms.

Proof. Consider first the situation that the defect surfaces Σ_1 and Σ_1' are as in the following picture:

$\Sigma_1 \;=$



$\Sigma_1' \;=$

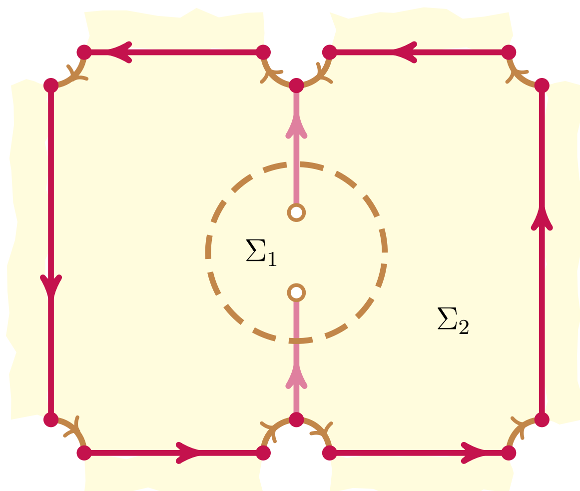


(5.86)

As start x_s and end x_t of the parallel transport we select the two vertical defect lines. Then by Lemma 5.26 we identify $Z_{\Sigma_1, x_s, x_t}(\mathrm{T}^{\mathrm{pre}}(\Sigma_1))$ with $\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma_1')$. Then the claim follows by observing that the diagram

$$\begin{array}{ccc} \mathrm{T}(\Sigma_1) & \longrightarrow & \mathrm{T}^{\mathrm{pre}}(\Sigma_1) \\ {\scriptstyle \Pi_{\Sigma_1;\Sigma_1,\Sigma_1'}}\big\downarrow & & \big\downarrow{\scriptstyle \gamma^{\mathrm{c}}_{\Sigma_1}(x_s,x_t)} \\ \widehat{\mathrm{T}}(\Sigma_1') & \longrightarrow & \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma_1') \end{array} \tag{5.87}$$

commutes by Lemma C.3.
For the case of a general defect surface Σ, with start x_s and end x_t of the parallel transport, take the defect surface Σ' as on the right hand side of (5.71), and the defect surface $\widetilde{\Sigma}$, with factorization into Σ_1 and Σ_2, as indicated in

$\widetilde{\Sigma} =$ 

(5.88)

Then consider the diagram

$$\begin{array}{ccc} \mathrm{T}(\Sigma) & \longrightarrow & \mathrm{T}^{\mathrm{pre}}(\Sigma) \\ \big\downarrow & & \big\downarrow \\ \mathrm{T}(\widetilde{\Sigma}) & \longrightarrow & \mathrm{T}^{\mathrm{pre}}(\widetilde{\Sigma}) \\ {\scriptstyle \Pi_{\Sigma_1;\Sigma_1,\Sigma_1'}}\big\downarrow & & \big\downarrow{\scriptstyle \gamma^{\mathrm{c}}_{\Sigma_1}} \\ \widehat{\mathrm{T}}(\Sigma') & \longrightarrow & \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma') \end{array} \qquad \text{(left curved arrow: } \Pi_{\Sigma;\Sigma,\Sigma'}\text{, right curved arrow: } \gamma^{\mathrm{cc}}_{\Sigma}) \tag{5.89}$$

where all horizontal arrows are the canonical morphisms from the block to the pre-block functor, the vertical arrows in the upper square are obtained from Lemma 5.21. This diagram commutes: Commutativity of the upper square follows from excision, while commutativity of the lower square is a consequence of factorization and commutativity of the diagram 5.87 above. The subdiagram to the left commutes by definition of the parallelization, and the subdiagram to the right by Lemma 5.30. It follows that $\Gamma_\Sigma = \Pi_{\Sigma;\Sigma,\Sigma'}$ is in particular invertible.
The morphism $\widehat{\epsilon}_\Sigma$ is by construction right inverse to $\gamma^{\mathrm{cc}}_\Sigma(x_s, x_t)$: For the relative pre-preblock functor $\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma')$, the equality $\gamma^{\mathrm{cc}}_\Sigma(x_s, x_t) \circ \epsilon = \mathrm{id}$ holds by construction. Thus it follows that $\Gamma_\Sigma \circ \widehat{\epsilon} = \mathrm{id}$, and since Γ_Σ is invertible, $\widehat{\epsilon}$ its two-sided inverse, which thus agrees with the inverse of the parallelization morphism. □

Remark 5.32. Proposition 5.31 provides a more conceptual understanding of the parallelization Π. Our specific construction of Π has, in contrast, the virtue that it is more local and thereby allows one to establish the coherence properties of the parallelization.

5.6 Actions of mapping class groups

The structures we have defined provide us directly with a representation of the mapping class group of a defect surface Σ by isomorphisms of the block functor $\mathrm{T}(\Sigma)$. To make this precise we define the value of the block functor on the 2-morphisms of the bicategory $\mathrm{Bord}_2^{\mathrm{def}}$ (recall the description of the latter in Definition 2.9).

For the present purposes it is convenient to record all the structure of a defect surface Σ in the notation. Thus we write $\Sigma = (\mathit{\Sigma}, \rho, \delta, \chi)$, where $\mathit{\Sigma}$ is the underlying surface, whose boundary is parametrized according to the parametrization ρ, δ is the set of defect lines on Σ, and χ is the framing. Recall from Section 2.1 that a 2-morphism

$$\varphi: \quad (\mathit{\Sigma}_1, \rho_1, \delta_1, \chi_1) \longrightarrow (\mathit{\Sigma}_2, \rho_2, \delta_2, \chi_2) \tag{5.90}$$

in $\mathrm{Bord}_2^{\mathrm{def}}$ is an isotopy class of isomorphisms from $\mathit{\Sigma}_1$ to $\mathit{\Sigma}_2$ preserving all the structure, i.e. it is a diffeomorphism relative to the boundary parametrizations, it respects the defect lines, and the push-forward vector field $\varphi_*\chi_1$ is equal to χ_2. In particular, given an isomorphism $\varphi\colon \Sigma_1 \to \Sigma_2$, with $(\mathit{\Sigma}_1, \rho_1, \delta_1, \chi_1)$ a defect surface, there is the induced structure $(\mathit{\Sigma}_2, \varphi(\rho_1), \varphi_*(\delta_1), \varphi_*(\chi_1))$ of a defect surface on Σ_2 such that φ represents a morphism of defect surfaces. In case that $\varphi\colon \mathit{\Sigma} \to \mathit{\Sigma}$ is an automorphism of the underlying surface $\mathit{\Sigma}$ of $\Sigma = (\mathit{\Sigma}, \rho, \delta, \chi)$, it can happen that $(\mathit{\Sigma}, \varphi(\rho), \varphi_*(\delta), \varphi_*(\chi))$ is not the same object as Σ in $\mathrm{Bord}_2^{\mathrm{def}}$: the boundary parametrization might have changed and/or the induced vector field may not be homotopic to the original one. We will exhibit examples of both phenomena later.

Based on the parallelization and on the definition of the block functor T as a limit (see Definition 5.19) we can directly specify the value of T on the 2-morphisms in $\mathrm{Bord}_2^{\mathrm{def}}$:

Lemma 5.33. Let $\varphi\colon (\Sigma_1, \rho_1, \delta_1, \chi_1) \to (\Sigma_2, \rho_2, \delta_2, \chi_2)$ be a morphism of defect surfaces. For every fine refinement $(\Sigma_1; \Sigma_1')$, the image $\varphi_*(\delta_1')$ of the defects in Σ_1' provides a fine refinement of Σ_2. As a consequence, φ induces an isomorphism

$$\mathrm{T}(\varphi): \quad \lim \Pi_{\Sigma_1;\Sigma_1'} \xrightarrow{\cong} \lim \Pi_{\Sigma_2;\Sigma_2'} \tag{5.91}$$

of block functors.

Proof. The first statement is geometrically obvious. As in Definition 5.16 we denote by $\Gamma(\Sigma_i)$ the category of fine refinements of the surface Σ_i, so that we have $\mathrm{T}(\Sigma_i) = \lim_{\Sigma_i'} \Pi_{\Sigma_i}(\Sigma_i')$, where $\Pi_{\Sigma_i}\colon \Gamma(\Sigma_i) \to \mathcal{L}ex(\mathrm{T}(\partial\Sigma_i), \mathrm{vect})$ with $\Pi_{\Sigma_i}(\Sigma') = \Pi_{\Sigma_i;\Sigma'}$ the parallelization on Σ_i. The push-forward along φ provides an equivalence $\varphi_*\colon \Gamma(\Sigma_1) \xrightarrow{\simeq} \Gamma(\Sigma_2)$. Since the boundaries agree, we have $\mathrm{T}(\partial\Sigma_1) = \mathrm{T}(\partial\Sigma_2)$. Next we will show that there is a strict equality $\Pi_{\Sigma_1} = \Pi_{\Sigma_2} \circ \varphi_*\colon \Gamma(\Sigma_1) \to \mathcal{L}ex(\mathrm{T}(\partial\Sigma_1), \mathrm{vect})$ of functors. It then follows by standard arguments for limits that there is an induced isomorphism $\mathrm{T}(\varphi)$ as in 5.91.

To verify our claim, consider a refinement $(\Sigma_1, \rho_1, \chi_1; \Sigma_1')$ of Σ_1, i.e. an object in $\Gamma(\Sigma_1)$. Since the block functor is determined entirely by the combinatorial data of the framing indices and the incidence relations of the defect lines, and since these data agree on the two defect surfaces $(\Sigma_2, \rho_2, \chi_2; \varphi(\Sigma_1'))$ and $(\Sigma_1, \rho_1, \chi_1; \Sigma_1')$, it follows that the functors $\Pi_{\Sigma_1}(\Sigma_1; \Sigma_1')$ and $\Pi_{\Sigma_2}(\Sigma_2; \varphi(\Sigma_1'))$ are equal. Thus the functors Π_{Σ_1} and $\Pi_{\Sigma_2} \circ \varphi_*$ agree on objects. Moreover, every local fillable-disk replacement on Σ_1' corresponds to a local fillable-disk replacement under φ_*. Therefore the functors agree on morphisms as well. $\square$

Definition 5.34. The value of the modular functor T on a 2-morphism φ in $\mathrm{Bord}_2^{\mathrm{def}}$ is the isomorphism (5.91) described in Lemma 5.33.

To determine the isomorphism $\mathrm{T}(\varphi)$ in practice, one picks representatives for the block functors on Σ_1 and Σ_2 by choosing fine refinements $(\Sigma_1; \Sigma_1')$ and $(\Sigma_2; \Sigma_2')$ and uses the cone isomorphisms $\mathrm{T}(\Sigma_1) \cong \widehat{\mathrm{T}}(\Sigma_1, \rho_1, \delta_1, \chi_1; \Sigma_1')$ and $\mathrm{T}(\Sigma_2) \cong \widehat{\mathrm{T}}(\Sigma_2, \rho_2, \delta_2, \chi_2; \Sigma_2')$ to identify the block functors with the latter representatives. Then $\mathrm{T}(\varphi)$ can be computed as follows:

Lemma 5.35. The functor $\mathrm{T}(\Sigma_2, \rho_2, \chi_2; \varphi(\Sigma_1'))$ is equal to the functor $\mathrm{T}(\Sigma_1, \rho_1, \chi_1; \Sigma_1')$. The isomorphism $\mathrm{T}(\varphi)$ is given by

$$\begin{aligned} \mathrm{T}(\varphi) = \Pi_{\Sigma_2; \varphi(\Sigma_1'), \Sigma_2'} : \\ \widehat{\mathrm{T}}(\Sigma_1, \rho_1, \delta_1, \chi_1; \Sigma_1') = \widehat{\mathrm{T}}(\Sigma_2, \rho_2, \chi_2; \varphi(\Sigma_1')) \longrightarrow \widehat{\mathrm{T}}(\Sigma_2, \rho_2, \chi_2; \Sigma_2') \end{aligned} \tag{5.92}$$

on the chosen representatives for the block functors.

Proof. The first statement follows from the proof of Lemma 5.33. The second statement follows from the definition of $\mathrm{T}(\varphi)$ as the universal arrow between the limits, given that we chose representatives for the limits. $\square$

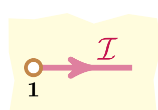
For the computation of mapping class group representations we need to derive explicit descriptions of the parallelization isomorphisms. As a preparation we introduce the following graphical conventions. First, a tadpole circle $\mathbb{Q}_\pm$ inside a defect surface is drawn as an unlabeled small circle, according to

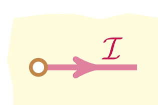
$\mathbb{Q}_+ \;=$ 

and $\mathbb{Q}_- \;=$ 

(5.93)

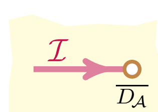
respectively, whenever this facilitates the description. Further, in case we deal with a block functor for some surface Σ that is to be evaluated at the silent object $\mho(\mathbb{Q}_+) = \mathbf{1} \in \mathcal{Z}(\mathcal{A}) \cong \mathrm{T}(\mathbb{Q}_+)$ or $\mho(\mathbb{Q}_-) = \overline{D_{\mathcal{A}}} \in \overline{\mathcal{Z}^{-4}(\mathcal{A})} \cong \mathrm{T}(\mathbb{Q}_-)$ of a tadpole circle, we introduce a further convention that allows us to omit the argument $\mho(\mathbb{Q}_\pm)$ of the block functor: we attach instead the silent object as a label to that tadpole circle:

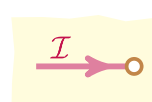
$\mathrm{T}\Big($ 

$\Big) \;:=\; \mathrm{T}\Big($ 

$\Big)(- \boxtimes \mho(\mathbb{Q}_+))$

(5.94)

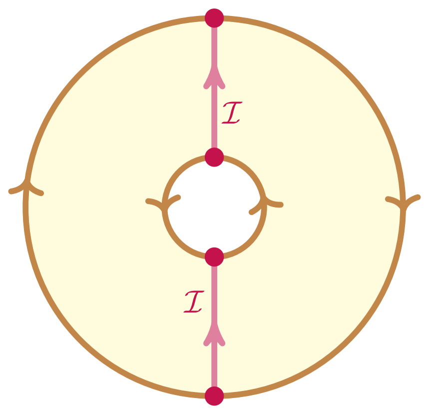
and $\mathrm{T}\Big($ 

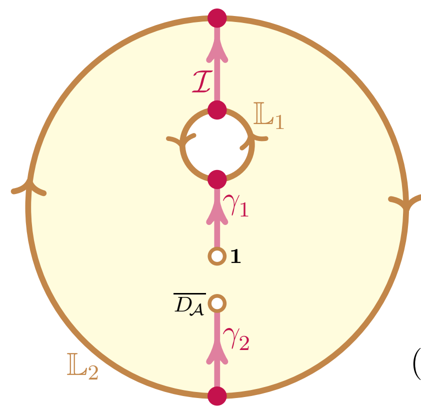
$\Big) \;:=\; \mathrm{T}\Big($ 

$\Big)(- \boxtimes \mho(\mathbb{Q}_-))$

(Moreover, later on we will sometimes want to refrain from specifying whether a tadpole circle is of the form $\mathbb{Q}_+$ or $\mathbb{Q}_-$. We then just use $\mho$ as a generic symbol for the silent object of such a circle. Likewise we treat the appearance of silent objects for other gluing circles, such as for the trivalent gluing circles in the picture (5.100) below.)

In order to discuss the braid group representation, we now consider the following two situations:

$\mathbb{D}_1 \;:=$ 

$\mathbb{D}_2 \;:=$ 

(5.95)

As indicated in the picture for $\mathbb{D}_2$, we denote by $\mathbb{L}_1$ and $\mathbb{L}_2$, respectively, the inner and outer gluing circle (which are the same for the two disks $\mathbb{D}_1$ and $\mathbb{D}_2$), and by γ_i, for $i=1,2$, the defect line that connects $\mathbb{L}_i$ to a tadpole disk. Invoking Corollary 4.30 we see that the block functor for $\mathbb{D}_1$ is given by $\mathrm{T}(\mathbb{D}_1)(\overline{G}\boxtimes F)=\mathrm{Nat}_{\mathcal{A},\mathcal{A}}(G,F)$ for $F\in\mathrm{T}(\mathbb{L}_1)\simeq\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A})$ and $\overline{G}\in\mathrm{T}(\mathbb{L}_2)\simeq\overline{\mathrm{T}(\mathbb{L}_1)}$. Recall that for $F\in\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A})$, the object $F(\mathbf{1})\in\mathcal{A}$ carries a canonical structure of an object in $\mathcal{Z}(\mathcal{A})$, using the structure of F as a bimodule functor. This structure is used in

Lemma 5.36.

(i) The block functor $\mathrm{T}(\mathbb{D}_2)$ is canonically isomorphic to the functor

$$\mathrm{T}(\mathbb{L}_1)\boxtimes\mathrm{T}(\mathbb{L}_2)\ni\; F\boxtimes\overline{G}\longmapsto \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(G(\mathbf{1}),F(\mathbf{1}))\;\in\mathrm{vect}\,. \tag{5.96}$$

(ii) The parallelization isomorphism $\Pi_{\mathbb{D}_1;\mathbb{D}_1,\mathbb{D}_2}$ is given by

$$\begin{aligned}\mathrm{Nat}_{\mathcal{A},\mathcal{A}}(G,F)&\longrightarrow\mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(G(\mathbf{1}),F(\mathbf{1}))\\ \mu&\longmapsto\mu_{\mathbf{1}}\,,\end{aligned} \tag{5.97}$$

and its inverse is provided by applying the braided induction (5.17) to morphisms.

Proof. (i) We denote by $\mathrm{E}_{\gamma_1}\colon\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A})\to\mathcal{Z}(\mathcal{A})$ and by $\mathrm{E}_{\gamma_2}\colon\overline{\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A})}\to\overline{\mathcal{Z}(\mathcal{A})}$ the excision functors (as introduced in Definition 4.33) that are associated to the defect lines γ_1 and γ_2 in the disk $\mathbb{D}_2$, respectively. Lemma 4.34 provides us with a canonical isomorphism $\mathrm{T}(\mathbb{D}_2)(\overline{G}\boxtimes F)\xrightarrow{\cong}\mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(\mathrm{E}_{\gamma_2}(G),\mathrm{E}_{\gamma_1}(F))$. Moreover, by the Eilenberg-Watts equivalences we have $\mathrm{E}_{\gamma_1}(F)=\int^{a\in\mathcal{A}}\mathrm{Hom}_\mathcal{A}(a,\mathbf{1})\boxtimes F(a)\cong F(\mathbf{1})$, while for E_{γ_2} we can use that $\overline{\int^{a\in\mathcal{A}}G(a)\boxtimes\overline{a}}$ $=\int_{a\in\mathcal{A}}\overline{G(a)}\boxtimes a$ in $\mathrm{T}(\mathbb{L}_2)$ and obtain

$$\begin{aligned}\mathrm{E}_{\gamma_2}\Big(\int_{a\in\mathcal{A}}\overline{G(a)}\boxtimes a\Big)&=\int_{a\in\mathcal{A}}\mathrm{Hom}_\mathcal{A}(D,a)\otimes\overline{G(a)}\;\cong\int^{a\in\mathcal{A}}\mathrm{Hom}_\mathcal{A}(D,D\otimes{}^{\vee\vee}a)\otimes\overline{G(a)}\\ &\cong\int^{a\in\mathcal{A}}\mathrm{Hom}_\mathcal{A}(\mathbf{1},{}^{\vee\vee}a)\otimes\overline{G(a)}\;\cong\int^{a\in\mathcal{A}}\mathrm{Hom}_\mathcal{A}(\mathbf{1},a)\otimes\overline{G(a)}\;\cong\;\overline{G(\mathbf{1})}\,.\end{aligned} \tag{5.98}$$

Here we use in the first step that G is exact and then make use of the isomorphism (3.45).

(ii) The prescription (5.97) indeed defines an isomorphism. The local fillable-disk replacement from $\mathbb{D}_1$ to $\mathbb{D}_2$ is covered by the situation analyzed in Lemma C.3: We find sub-disks of $\mathbb{D}_1$ and $\mathbb{D}_2$ which are of the same type as the ones on the left and on the right of 5.100. Thus in view of Lemma C.3, what we need to show is that (5.97) comes from the corresponding morphism between the pre-block functors. The latter is given by the dinatural morphism from an end and in our case is given by

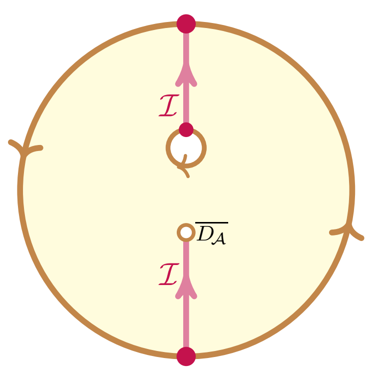
$$\mathrm{Nat}(G,F) \cong \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(G(a),F(a)) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(G(\mathbf{1}),F(\mathbf{1}))\,. \tag{5.99}$$

This proves the claim. □

We now consider in detail two specific types of isomorphisms φ: the exchange of two boundary circles on a three-holed sphere and the Dehn twist on a cylinder over a circle. These are of particular interest, because Dehn twists generate the mapping class groups, while manipulations analogous to the one for the three-holed sphere generate a braid group, which is a prominent example of (a subgroup of) a mapping class group.

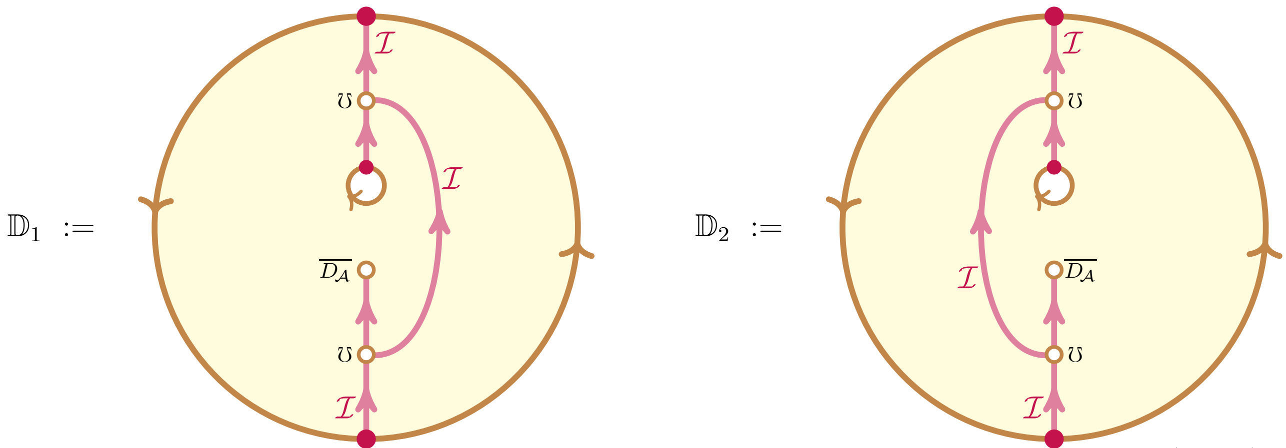
To deduce the action of the first of these two types of isomorphisms we start with the following fillable disk $\mathbb{D}$:

$\mathbb{D}$:=



(5.100)

Now consider the two refinements $(\mathbb{D};\mathbb{D}_1)$ and $(\mathbb{D};\mathbb{D}_2)$ with



(5.101)

We denote the outer and inner gluing circles (of both $\mathbb{D}_1$ and $\mathbb{D}_2$) by $\mathbb{L}$ and $\mathbb{L}'$, respectively, and regard the block functors for $\mathbb{D}_1$ and $\mathbb{D}_2$ as functors from $\mathrm{T}(\overline{\mathbb{L}})\boxtimes\mathrm{T}(\mathbb{L}')$ to vect.

Lemma 5.37.

(i) The block functors for the defect surfaces (5.101) are canonically isomorphic to

$$\mathrm{T}_{\text{fine}}(\mathbb{D}_1)(\overline{G}\boxtimes z) = \mathcal{L}ex_{\mathcal{A}}(G,F_z) \qquad \text{and} \qquad \mathrm{T}_{\text{fine}}(\mathbb{D}_2)(\overline{G}\boxtimes z) = \mathcal{L}ex_{\mathcal{A}}(G,{}_zF)\,, \tag{5.102}$$

respectively, for $G\in\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A})=\mathrm{T}_{\text{fine}}(\mathbb{L})$ and $z\in\mathcal{Z}(\mathcal{A})=\mathrm{T}_{\text{fine}}(\mathbb{L}')$, with F_z and ${}_zF$ the braided-induced functors in $\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A})$, see (5.17).

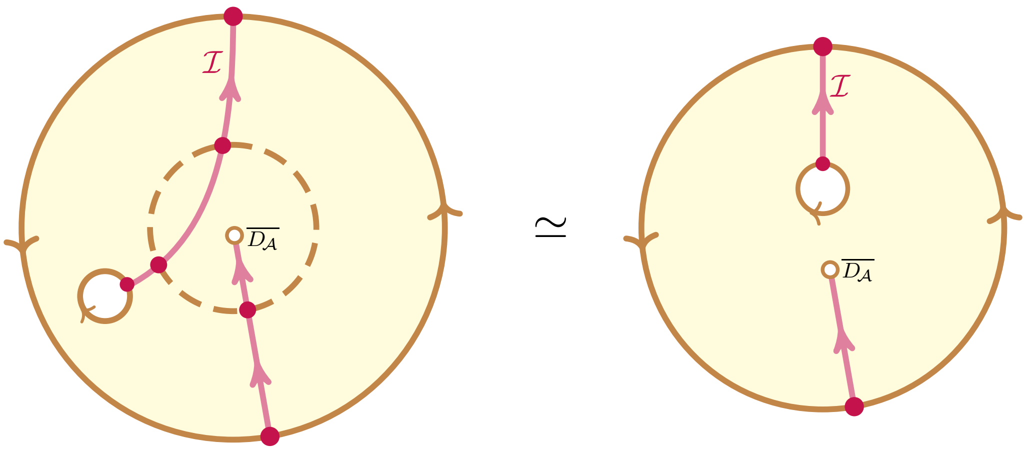
(ii) The braiding $F_z(y)=y\otimes z\xrightarrow{c_{y,z}} z\otimes y$ provides a bimodule isomorphism $F_z\cong{}_zF$ and the parallelization isomorphism $\Pi_{\mathbb{D};\mathbb{D}_1,\mathbb{D}_2}$ is given by post-composing with this isomorphism.

Proof. (i) Similarly to Example 5.9 we use for $z\in\mathcal{Z}(\mathcal{A})=\mathrm{T}_{\text{fine}}(\mathbb{L}')$ the excision functors to obtain the situation of the left hand side of (5.95) with object F_z in the inner boundary. Thus the block functors are given by

$$\mathrm{T}(\mathbb{D}_1)(\overline{G}\boxtimes z) = \mathcal{L}ex_{\mathcal{A},\mathcal{A}}(G,F_z) \qquad \text{and} \qquad \mathrm{T}(\mathbb{D}_2)(\overline{G}\boxtimes z) = \mathcal{L}ex_{\mathcal{A},\mathcal{A}}(G,{}_zF) \tag{5.103}$$

for $G\in\mathrm{T}(\mathbb{L})=\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(\mathcal{A},\mathcal{A})$. and $z\in\mathrm{T}(\mathbb{L}')=\mathcal{Z}(\mathcal{A})$.

(ii) To compute the natural isomorphism $\Pi_{\mathbb{D};\mathbb{D}_1,\mathbb{D}_2}\colon\mathrm{T}(\mathbb{D}_1)\to\mathrm{T}(\mathbb{D}_2)$, we factor it through the disk $\mathbb{D}$, which is possible because $\mathbb{D}$ is already fine. This is achieved by the fillable-disk replacement that replaces the disk $\mathbb{D}_1$ by the left hand side of



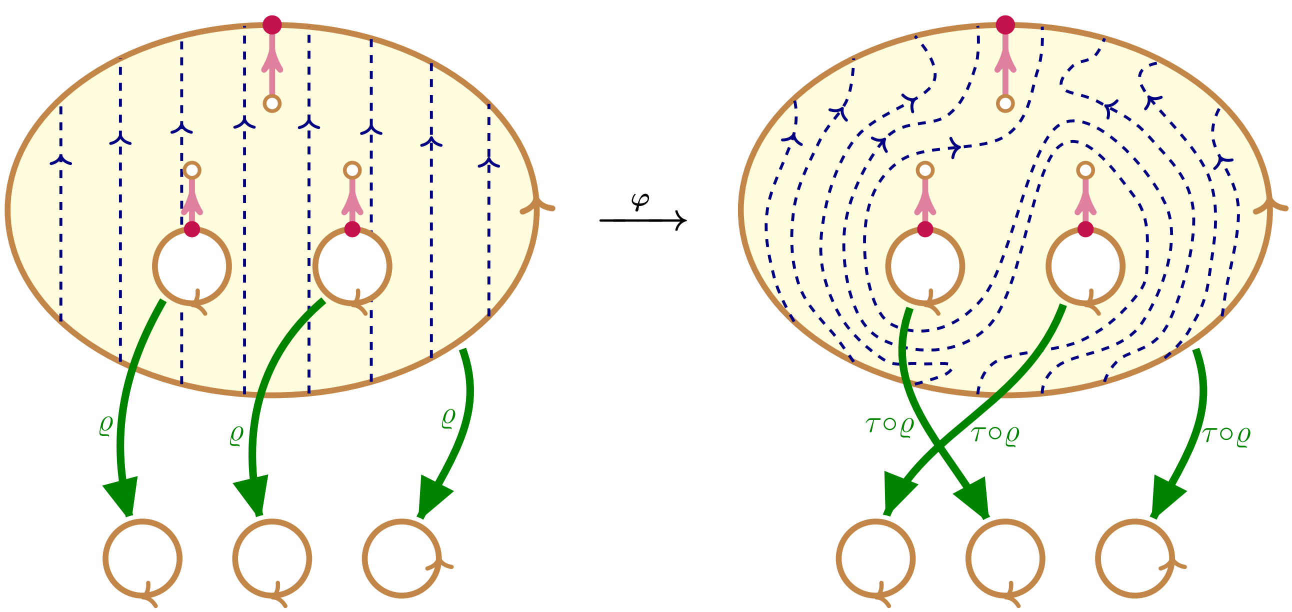
$$= \mathbb{D}\,. \tag{5.104}$$

By Lemma 5.36 there is a canonical isomorphism $\mathrm{T}(\mathbb{D})(\overline{G}\boxtimes z) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(G(\mathbf{1}), z)$ and the parallelization isomorphism $\Pi_{\mathbb{D};\mathbb{D}_1,\mathbb{D}}\colon \mathrm{T}(\mathbb{D}_1) \to \mathrm{T}(\mathbb{D})$ is given by

$$\mathcal{L}ex_{\mathcal{A},\mathcal{A}}(G, F_z) \ni \mu \longmapsto \mu_{\mathbf{1}} \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(G(\mathbf{1}), z)\,. \tag{5.105}$$

Analogously $\Pi_{\mathbb{D};\mathbb{D}_2,\mathbb{D}}\colon \mathrm{T}(\mathbb{D}_2) \xrightarrow{\cong} \mathrm{T}(\mathbb{D})$ is given by evaluation on $\mathbf{1}\in\mathcal{A}$ which has the braided induction ${}_zF$ as a quasi-inverse functor. The claimed expression for $\Pi_{\mathbb{D};\mathbb{D}_1,\mathbb{D}_2}$ now follows directly from $\Pi_{\mathbb{D};\mathbb{D}_1,\mathbb{D}_2} = \Pi^{-1}_{\mathbb{D};\mathbb{D}_2,\mathbb{D}} \circ \Pi_{\mathbb{D};\mathbb{D}_1,\mathbb{D}}$. □

Consider now a braiding move on a three-punctured sphere (Σ, ϱ) with boundary parametrization ϱ, i.e. a diffeomorphism $\varphi\colon (\Sigma,\varrho,\chi) \to (\Sigma, \tau\circ\varrho, \chi)$ that changes the boundary parametrization to $\tau\circ\varrho$, where τ is the symmetric monoidal braiding on $\mathrm{Bord}_2^{\mathrm{def}}$ as indicated in

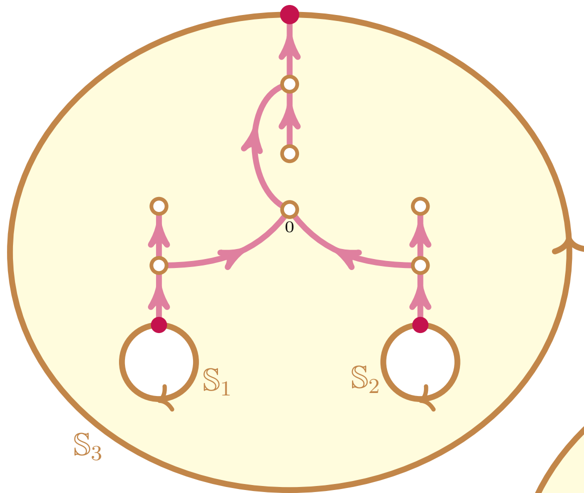


(5.106)

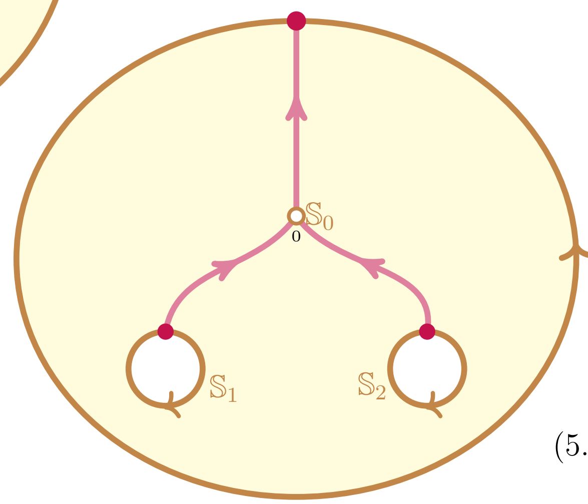
Note that the framing $\varphi_*(\chi)$ is homotopic to χ. Thus we can consider the two refinements of

$(\Sigma, \varrho, \chi; \Sigma_1)$ and $(\Sigma, \tau \circ \varrho, \chi; \Sigma_2)$ depicted in

$\Sigma_1 \;=$



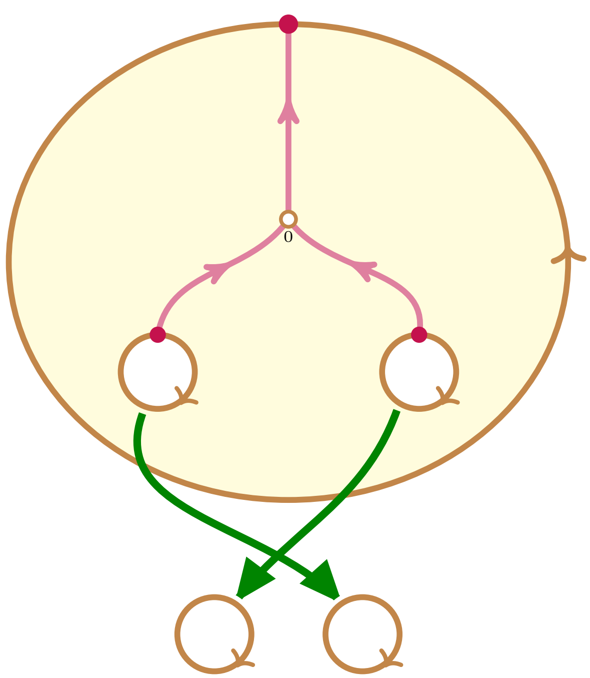
$\simeq$



(5.107)

and

$\Sigma_2 \;=$



(5.108)

respectively. The gluing categories for the inner gluing circles $\mathbb{S}_1$ and $\mathbb{S}_2$ are $\mathcal{Z}(\mathcal{A})$, while the one for the outer gluing circle $\mathbb{S}_3$ is $\overline{\mathcal{Z}(\mathcal{A})}$.

Proposition 5.38. The block functors for Σ_1 and Σ_2 evaluated at the objects $x \in \mathrm{T}(\mathbb{S}_1) = \mathcal{Z}(\mathcal{A})$, $y \in \mathrm{T}(\mathbb{S}_2) = \mathcal{Z}(\mathcal{A})$ and $\overline{z} \in \mathrm{T}(\mathbb{S}_3) = \overline{\mathcal{Z}(\mathcal{A})}$ are

$$\begin{aligned} \mathrm{T}(\Sigma_1)(x \boxtimes y \boxtimes \overline{z}) &= \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(z, x \otimes y) \qquad \text{and} \\ \mathrm{T}(\Sigma_2)(x \boxtimes y \boxtimes \overline{z}) &= \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(z, y \otimes x)\,, \end{aligned} \tag{5.109}$$

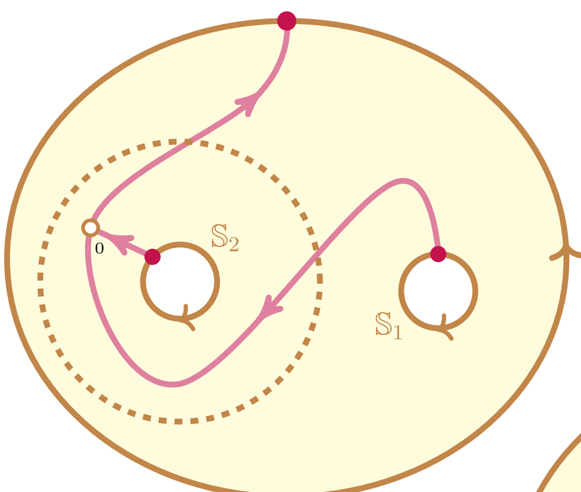
respectively. The isomorphism corresponding to the value $\mathrm{T}(\varphi)$ of the block functor on the diffeomorphism φ is given by composition with the braiding of x and y.

Proof. First we use again the excision functors to reduce the situation to the one on the right hand side of Equation 5.107. We then contract the objects in the gluing categories for the circles $\mathbb{S}_1$ and $\mathbb{S}_2$ in Σ_1 with the silent object for the transparent gluing circle $\mathbb{S}_0$ in the diagram on the right hand side of (5.107); the latter circle is the one denoted by $\mathbb{I}_0^{\swarrow}(\mathcal{M})$, with $\mathcal{M}$ set to $\mathcal{A}$, in (5.14), and its silent object is recorded in Equation (C.26). Hereby we obtain the object

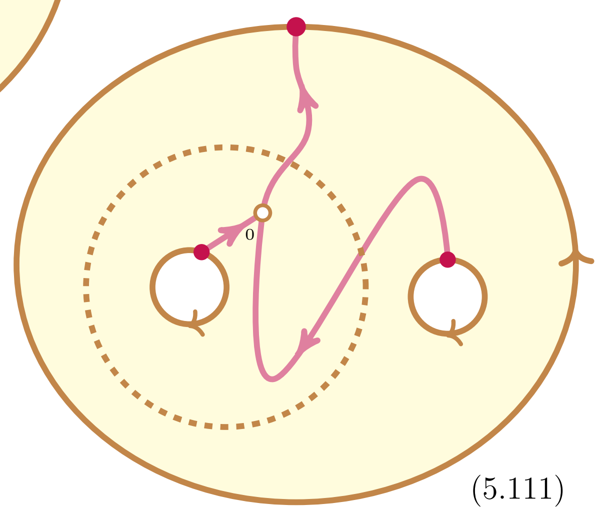
$$\int^{b\in\mathcal{A}}\!\!\int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,x) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(b,y) \otimes a.b \;\cong\; x \otimes y \;\in \mathcal{Z}(\mathcal{A})\,. \tag{5.110}$$

This gives the claimed expression for the block functor on Σ_1. Analogously we compute the block functor for Σ_2. Next consider the result of applying the diffeomorphism φ to the refined surface Σ_1, which results in the refinement $\varphi(\Sigma_1)$ in the first of the following pictures:

$\varphi(\Sigma_1)\; =$



$\simeq$



(5.111)

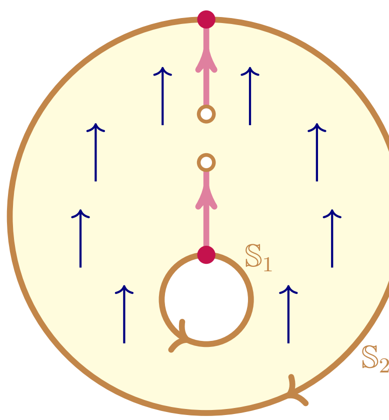
We factor the computation of $\varphi(\Sigma_1)$ through the disk $\mathbb{D}$ to obtain the isomorphism (compare the proof of Lemma 5.33)

$$\mathrm{T}(\varphi(\Sigma_1))(x \boxtimes y \boxtimes \overline{z}) = \mathrm{T}(\Sigma_1)(x \boxtimes y \boxtimes \overline{z}) = \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(z, x \otimes y) \cong \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(z, {}_xF(y))\,. \tag{5.112}$$

Next we perform the refinement change from the first to the second picture in (5.111) and apply Lemma 5.37 so as to obtain the isomorphism ${}_xF(y) \cong F_x(y)$, which is given by the braiding of x and y. From the pictures it is evident that the resulting refinement is isotopic to Σ_2. Thus the functor we have computed is indeed $\mathrm{T}(\varphi)$. □

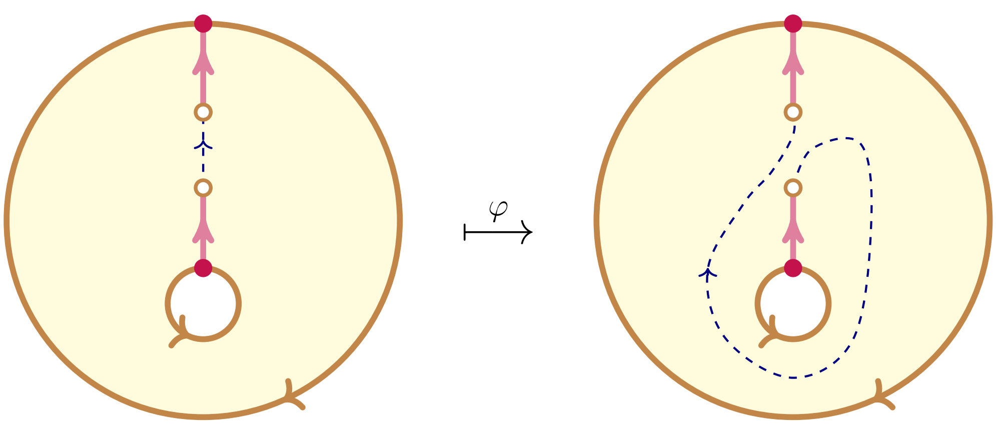
Remark 5.39. Recall the comparison with the standard Turaev-Viro construction in Remark 5.23. It remains to compare the representations of the mapping class groups. To this end we observe that an oriented modular functor is completely determined by a Lego-Teichmüller game (compare e.g. [BakK, FuS1]). As a consequence it suffices to compare the modular functors on n-holed spheres, which has been discussed in Remark 5.23, and on 5 elementary moves between specific surfaces. A detailed comparison of all these moves in the present and standard Turaev-Viro approaches is beyond the scope of this paper. But note that the braiding move is covered by Proposition 5.38. We leave a detailed analysis of all moves, in particular of the S-move, which involves surfaces of genus one, to future work.

Instead, let us now compute, besides the braiding move, also the value of the modular functor on a Dehn twist. In this case we start with a non-fine defect surface (Σ, χ) given by a two-punctured sphere, as shown in the following picture:



(5.113)

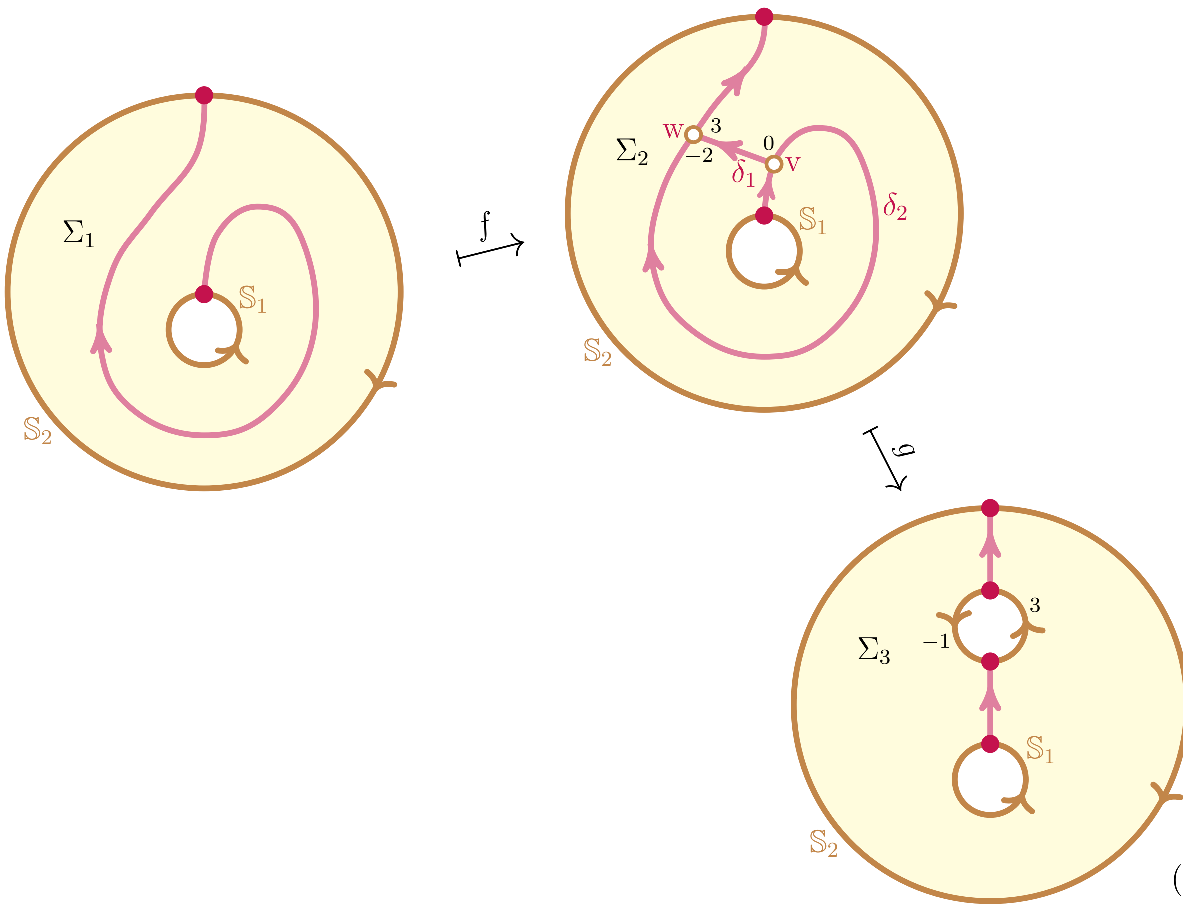
Here the framing is the one that corresponds to a straight cylinder. The gluing categories are $\mathrm{T}(\mathbb{S}_1) = \mathcal{Z}(\mathcal{A})$ and $\mathrm{T}(\mathbb{S}_2) = \overline{\mathcal{Z}(\mathcal{A})}$. Consider the diffeomorphism $\varphi\colon (\Sigma, \chi) \to (\Sigma, \varphi_*(\chi))$ indicated in



(5.114)

The framings χ and $\varphi_*(\chi)$ are non-homotopic, as can e.g. be seen by realizing that the the winding number along the dashed line on the right hand side of (5.114) is non-zero.

To compute the value of T on φ, we make use of the universal property of the parallelization that were developed in Proposition 5.29, which allows us to work entirely on the level of pre-block spaces. Consider the refinements $(\Sigma;\Sigma_i)$, with $i\in\{1,2,3\}$, of $\varphi_*(\Sigma)$ shown in the following picture:

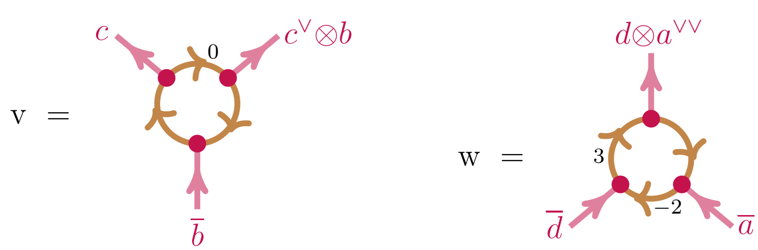


(5.115)

The change $(\Sigma;\Sigma_1)\xmapsto{f}(\Sigma;\Sigma_2)$ of refinements consists of adding the defect line δ_1, while the refinement change $(\Sigma;\Sigma_2)\xmapsto{g}(\Sigma;\Sigma_3)$ is the deletion of δ_2. For obtaining a representative for the value of the block functor on $\varphi_*(\Sigma)$ we pick the refinement $(\Sigma;\Sigma_3)$. With the help of Lemma 5.33 we then get

$$\mathrm{T}(\varphi)=\Pi_{\Sigma;\varphi(\Sigma),\Sigma_3}:\quad \widehat{\mathrm{T}}(\Sigma)\longrightarrow\widehat{\mathrm{T}}(\varphi_*(\Sigma);\Sigma_3)\,. \tag{5.116}$$

We first consider the pre-block spaces. The proof of the following statements follows directly by combining the canonical isomorphisms from the Yoneda lemma with the explicit expressions for the silent objects of the two relevant gluing circles. As in (5.115) we denote the latter circles by v and w, and choose the conventions



(5.117)

for the labels of these defect one-manifolds; hereby their silent objects take the form

$$\mho_{\mathrm{v}} = \int^{b\in\mathcal{A}}\!\!\int_{c\in\mathcal{A}} c\boxtimes c^{\vee}\otimes b\boxtimes \overline{b} \qquad\text{and}\qquad \mho_{\mathrm{w}} = \int^{a\in\mathcal{A}}\!\!\int^{m\in\mathcal{A}} m\otimes a^{\vee\vee}\boxtimes\overline{a}\boxtimes\overline{m}\,. \tag{5.118}$$

Lemma 5.40. The (relative) pre-block spaces for the defect surfaces Σ_i shown in (5.115) are

$$\begin{aligned}
\mathrm{T}^{\mathrm{pre}}(\Sigma_1)(\overline{y}\boxtimes x) \;&=\; \int^{a\in\mathcal{A}} \mathrm{Hom}(y,a)\otimes_{\Bbbk}\mathrm{Hom}(a,x) \;\cong\; \mathrm{Hom}(y,x)\,,\\
\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma_2)(\overline{y}\boxtimes x) \;&=\; \int^{b\in\mathcal{A}}\!\!\int_{c\in\mathcal{A}}\int^{a\in\mathcal{A}}\!\!\int^{d\in\mathcal{A}} \mathrm{Hom}(b,x)\otimes_{\Bbbk}\mathrm{Hom}(d,c^{\vee}\otimes b)\\
&\qquad\qquad\qquad\otimes_{\Bbbk}\mathrm{Hom}(a,c)\otimes_{\Bbbk}\mathrm{Hom}(y,d\otimes a^{\vee\vee})\\
&\cong\; \int^{d\in\mathcal{A}}\!\!\int_{c\in\mathcal{A}} \mathrm{Hom}(y,d\otimes c^{\vee\vee})\otimes_{\Bbbk}\mathrm{Hom}(c^{\vee\vee}\otimes d,x)\\
&\cong\; \int^{d\in\mathcal{A}}\!\!\int_{c\in\mathcal{A}} \mathrm{Hom}(y,d\otimes c)\otimes_{\Bbbk}\mathrm{Hom}(c\otimes d,x)\,,\\
\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma_3)(\overline{y}\boxtimes x) \;&=\; \int^{d\in\mathcal{A}} \mathrm{Hom}(y,d^{\vee\vee})\otimes_{\Bbbk}\mathrm{Hom}(d,x) \;\cong\; \mathrm{Hom}(y,x^{\vee\vee})
\end{aligned} \tag{5.119}$$

for $x\in\mathrm{T}(\mathbb{S}_1)$ and $y\in\mathrm{T}(\mathbb{S}_2)$ (here all Hom spaces are morphism spaces in $\mathcal{A}$).

With the results on the block functors given in Corollary 4.30 we then obtain

Lemma 5.41. The block functors for the defect surfaces Σ_1 and Σ_3 are, up to canonical isomorphism, given by

$$\mathrm{T}(\Sigma_1)(\overline{y}\boxtimes x) = \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(y,x) \qquad\text{and}\qquad \mathrm{T}(\Sigma_3)(\overline{y}\boxtimes x) = \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(y,x^{\vee\vee}), \tag{5.120}$$

respectively.

We will not need an explicit expression for the block functor on Σ_2.

The following notion is a direct generalization of the one of a ribbon twist in a ribbon category:

Definition 5.42. The *twist* on an object x in a braided monoidal category $\mathcal{C}$ with right duality is the isomorphism $x \xrightarrow{\cong} x^{\vee\vee}$ given by

$$x \xrightarrow{\mathrm{coev}^{\mathrm{r}}_{x^{\vee}}\otimes\mathrm{id}_x} x^{\vee}\otimes x^{\vee\vee}\otimes x \xrightarrow{\mathrm{id}_{x^{\vee}}\otimes c_{x^{\vee\vee},x}} x^{\vee}\otimes x\otimes x^{\vee\vee} \xrightarrow{\mathrm{ev}^{\mathrm{r}}_{x}\otimes\mathrm{id}_{x^{\vee\vee}}} x^{\vee\vee}\,, \tag{5.121}$$

with c the braiding in $\mathcal{C}$.

In case $\mathcal{C}$ is actually ribbon, it is in particular pivotal, and the ribbon twist is the composition of the twist (5.121) and the pivotal structure [BakK, Eq. (2.2.27)]. In the situation at hand, we deal with the case that $\mathcal{C}=\mathcal{Z}(\mathcal{A})$, with $\mathcal{A}$ not assumed to be pivotal.

We proceed to compute $\mathrm{T}(\varphi)$.

Proposition 5.43. The value of T on the diffeomorphism φ from (5.114) is the natural isomorphism

$$\mathrm{T}(\varphi)(\overline{y}\boxtimes x):\quad \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(y,x)\to\mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(y,x^{\vee\vee}) \tag{5.122}$$

that is given by post-composition with the twist (5.121) on x.

Proof. We are going to show that the morphism between the pre-block spaces for Σ_1 and Σ_3 that is induced by the refinement changes in (5.115) is given by post-composition with the twist on x. By the universal property of $\Pi_{\Sigma;\varphi(\Sigma),\Sigma_3} = \mathrm{T}(\varphi)$ obtained in Proposition 5.29 we can then conclude that $\mathrm{T}(\varphi)$ is the post-composition with the twist as well.

We first compute the morphism $f^{\mathrm{pre}} \colon \mathrm{T}^{\mathrm{pre}}(\Sigma_1) \to \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma_2)$ that corresponds to the first refinement change in (5.115). With the help of the parallel transport operations on Σ_2 we see that f^{pre} is the following composite:

$$
\begin{aligned}
\int^{a\in\mathcal{A}} \mathrm{Hom}(y,a) \otimes_{\Bbbk} \mathrm{Hom}(a,x) \;\longrightarrow\; & \int_{c\in\mathcal{A}} \int^{a\in\mathcal{A}} \mathrm{Hom}(y, {}^{\vee\vee}c \otimes a) \otimes_{\Bbbk} \mathrm{Hom}(c\otimes a, x) \\
\xrightarrow{\cong} & \int_{c\in\mathcal{A}} \int^{a\in\mathcal{A}} \mathrm{Hom}({}^{\vee}c \otimes y, a) \otimes_{\Bbbk} \mathrm{Hom}(c\otimes a, x) \\
\xrightarrow{\cong} & \int^{a\in\mathcal{A}} \int_{c\in\mathcal{A}} \mathrm{Hom}(y \otimes {}^{\vee}c, a) \otimes_{\Bbbk} \mathrm{Hom}(c\otimes a, x) \\
\xrightarrow{\cong} & \int^{d\in\mathcal{A}} \int_{c\in\mathcal{A}} \mathrm{Hom}(y, d\otimes c) \otimes_{\Bbbk} \mathrm{Hom}(c\otimes d, x) \;\xrightarrow{\cong}\; \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma_2)(\overline{y} \boxtimes x)\,.
\end{aligned}
\tag{5.123}
$$

Here the first morphism is the comonad structure of $\int^{m\in\mathcal{M}} \overline{m} \boxtimes m \in \overline{\mathcal{M}} \overset{1}{\boxtimes} \mathcal{M} \overset{1}{\boxtimes}$, the second and fourth are adjunction isomorphisms, and the third is obtained from the braiding of $y \in \mathcal{Z}(\mathcal{A})$; the last isomorphism holds by Lemma 5.40.

Next we compute the morphism $g^{\mathrm{pre}} \colon \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma_2) \to \widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma_3)$ that corresponds to the second refinement change in (5.115), given by deletion of the defect line δ_2. To this end we would like to make use of the dinatural component at $\mathbf{1} \in \mathcal{A}$ of the end. Before we can do so we must, however, first use the isomorphism

$$
\int^{b\in\mathcal{A}} \int_{c\in\mathcal{A}} c \boxtimes c^{\vee} \otimes b \boxtimes \overline{b} \;\cong\; \int^{b\in\mathcal{A}} \int_{c\in\mathcal{A}} b \otimes c \boxtimes c^{\vee} \boxtimes \overline{b}
\tag{5.124}
$$

for the silent object at the gluing circle v in (5.115), which arises from the canonical isomorphism $\int_{c\in\mathcal{A}} c^{\vee} \otimes b \boxtimes c \cong \int_{c\in\mathcal{A}} c^{\vee} \boxtimes b \otimes c$. When using the first expression for $\widehat{\mathrm{T}^{\mathrm{pre}}}(\Sigma_2)$ in (5.119) we obtain the first isomorphism in the following composite, which is g^{pre}:

$$
\begin{aligned}
& \int^{b\in\mathcal{A}} \int_{c\in\mathcal{A}} \int^{a\in\mathcal{A}} \int^{d\in\mathcal{A}} \mathrm{Hom}(b,x) \otimes_{\Bbbk} \mathrm{Hom}(d, c^{\vee} \otimes b) \otimes_{\Bbbk} \mathrm{Hom}(a,c) \otimes_{\Bbbk} \mathrm{Hom}(y, d\otimes a^{\vee\vee}) \\
& \quad \xrightarrow{\cong} \int^{b} \int_{c} \int^{a} \int^{d} \mathrm{Hom}(b,x) \otimes_{\Bbbk} \mathrm{Hom}(d, c^{\vee}) \otimes_{\Bbbk} \mathrm{Hom}(a, b\otimes c) \otimes_{\Bbbk} \mathrm{Hom}(y, d\otimes a^{\vee\vee}) \\
& \qquad \xrightarrow{\cong} \int_{c\in\mathcal{A}} \mathrm{Hom}(y, c^{\vee} \otimes (x\otimes c)^{\vee\vee}) \;\longrightarrow\; \mathrm{Hom}(y, x^{\vee\vee})\,.
\end{aligned}
\tag{5.125}
$$

Here the last morphism is the dinatural component of the end at $\mathbf{1}$.

Using repeatedly the Yoneda isomorphism, one can express the morphism $g^{\mathrm{pre}} \circ f^{\mathrm{pre}}$ as the composite along the upper-right path from $\mathrm{Hom}(y,x)$ to $\mathrm{Hom}(y\otimes x^\vee,\mathbf{1})$ in the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(y,x) & \longrightarrow & \int_c \mathrm{Hom}(y, {}^\vee c\otimes c\otimes x) & \xrightarrow{\cong} & \int_c \mathrm{Hom}(c\otimes y, c\otimes x) \\
\Big\downarrow & & & & \Big\downarrow\cong \\
 & & & & \int_c \mathrm{Hom}(y\otimes c, c\otimes x) \\
 & & & & \Big\downarrow\cong \\
 & & & & \int_c \mathrm{Hom}(y\otimes c\otimes x^\vee, c) \\
 & & & & \Big\downarrow \\
\mathrm{Hom}(x^\vee\otimes y,\mathbf{1}) & & \longrightarrow & & \mathrm{Hom}(y\otimes x^\vee,\mathbf{1})
\end{array} \tag{5.126}$$

Here the first morphism in the upper row is given by the canonical morphism $\mathbf{1}\to\int_{c\in\mathcal{A}}{}^\vee c\otimes c$ that is obtained from the coevaluation of $\mathcal{A}$. The subsequent morphisms are obtained from the adjunction and the braiding of y, followed by the morphism induced from $\int_c \overline{c}\boxtimes c\otimes x\cong\int_c \overline{c\otimes x^\vee}\boxtimes c$, while the final vertical morphism in this path is again the dinatural component at $\mathbf{1}$. The left-lower path from $\mathrm{Hom}(y,x)$ to $\mathrm{Hom}(y\otimes x^\vee,\mathbf{1})$ is provided directly by the adjunction and the braiding of y. It follows from the dinaturality of the involved morphisms that the diagram commutes. Further, the naturality of the braiding implies that composing the left-lower path with the adjunction isomorphism $\mathrm{Hom}(y\otimes x^\vee,\mathbf{1})\cong\mathrm{Hom}(y,x^{\vee\vee})$ is the same as post-composition with the twist on x. Thus we have shown that $g^{\mathrm{pre}}\circ f^{\mathrm{pre}}$ is given by post-composition with the twist on x. Now post-composing with the twist is even an isomorphism $\mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(y,x)\xrightarrow{\cong}\mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(y,x^{\vee\vee})$, and thus we have a commuting diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(y,x) & \longrightarrow & \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(y,x^{\vee\vee}) \\
U\Big\downarrow & & \Big\downarrow U \\
\mathrm{Hom}_{\mathcal{A}}(y,x) & \longrightarrow & \mathrm{Hom}_{\mathcal{A}}(y,x^{\vee\vee})
\end{array} \tag{5.127}$$

where the horizontal arrows are post-composition with the twist on x and the vertical arrows forget the half-braidings. By the universal property of $\mathrm{T}(\varphi)$, as obtained in Proposition 5.29, we conclude that $\mathrm{T}(\varphi)$ is the post-composition with the twist. □

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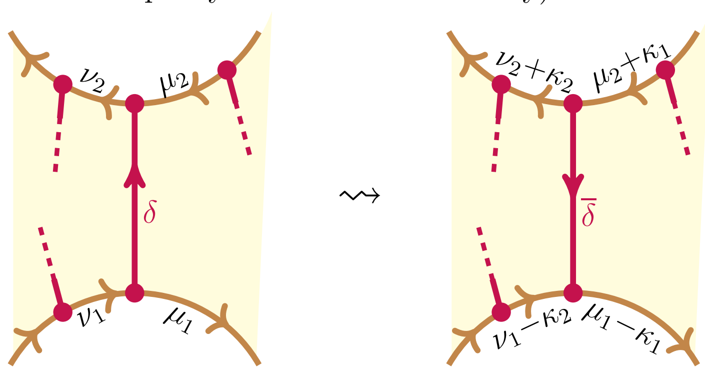
A Framing shifts

In this appendix we examine the operation of shifting the framing of a defect surface along a defect line. As we will see, performing this operation simultaneously for all defect lines yields canonical autoequivalences of the bicategory of unlabeled defect surfaces.

Recall the bicategory $\mathrm{Bord}_2^{\mathrm{def},0}$ of unlabeled defect surface defined in Section 2.1. A homotopy χ_t between two framings χ_1 and χ_2 on an unlabeled defect surface is by definition required to be in particular a framing for each t, so the corresponding vector field has to be parallel to the defect lines of Σ for all t. By Remark 2.7(iii) such homotopies yield isomorphic defect surfaces, $(\Sigma,\chi_1)\cong(\Sigma,\chi_2)$. If one would relax the condition, so that χ_t is only required to be a non-zero vector field for all t, but not necessarily parallel to the defect lines for $t\in(0,1)$, many more framings would be related, but the resulting 1-morphisms in $\mathrm{Bord}_2^{\mathrm{def},0}$ would, in general, be non-isomorphic.

There is a local prototype for this kind of more general homotopy, which allows one to relate different framings on the same unlabeled defect surface Σ: Consider a defect line δ on a defect surface Σ with a local neigbourhood that looks like the left hand side of the following picture (using the framing index to specify the vector field locally):



(A.1)

For any $(\kappa_1,\kappa_2)\in(2\mathbb{Z}{+}1)^2$ the surface shown on the right hand side of (A.1) has a unique framing that differs only locally from the framing on the left hand side. An instance of this operation in the analogous situation of a free boundary occurs in Proposition 3.13. We call such operations on an unlabeled defect surface Σ the *odd framing shifts* at δ_i on Σ and denote them by $S_{\kappa_1,\kappa_2}(\delta_i)(\Sigma)$. Analogously there are *even framing shifts* $S_{\kappa_1,\kappa_2}(\delta_i)(\Sigma)$ at δ_i for $(\kappa_1,\kappa_2)\in(2\mathbb{Z})^2$, for which the orientation of δ_i does not change.

The so defined framing shifts amount to the following modification of the framings of unlabeled defect one-manifolds $\mathbb{L}$: For p a defect point on $\mathbb{L}$ and $(\kappa_1,\kappa_2)\in(2\mathbb{Z}{+}1)^2$, the one-manifold $S_{\kappa_1,\kappa_2}(p)(\mathbb{L})$ is the unlabeled defect one-manifold for which the sign of p is flipped and, if p is positive, the framing index on the segment preceding p is decreased by κ_2 while the one after p is decreased by κ_1; if instead p is negative, the framing index on the segment preceding p is increased by κ_1 and the one after p is increased by κ_2. If $(\kappa_1,\kappa_2)\in(2\mathbb{Z})^2$, the orientation of p is kept and the change on the framing indices is, with the same notation as in Figure (A.1),

$$\nu_1\mapsto\nu_1-\kappa_2\,,\qquad \mu_1\mapsto\mu_1-\kappa_1\,,\qquad \nu_2\mapsto\nu_2+\kappa_2\qquad\text{and}\qquad \mu_2\mapsto\mu_2+\kappa_1\,. \tag{A.2}$$

For $(\kappa_1,\kappa_2)\in(2\mathbb{Z})^2$ consider, for a given an unlabeled defect surface Σ, the unlabeled defect surface

$$S_{\kappa_1,\kappa_2}(\Sigma):=\prod_i S_{\kappa_1,\kappa_2}(\delta_i)(\Sigma)\,, \tag{A.3}$$

where the product ranges over all defect lines of Σ. For an unlabeled defect one-manifold $\mathbb{L}$ we define analogously $S_{\kappa_1,\kappa_2}(\mathbb{L}) := \prod_i S_{\kappa_1,\kappa_2}(p_i)(\mathbb{L})$, with product over all defect points of $\mathbb{L}$. We tacitly extend these prescriptions for defect lines and defect points to apply also to free boundary segments and their end points, in which case only one instead of two framing indices are shifted, without modifying the notation. The following statements then follow directly from the definitions.

Lemma A.1. Let $(\kappa_1, \kappa_2) \in (2\mathbb{Z})^2$. The assignments $\mathbb{L} \mapsto S_{\kappa_1,\kappa_2}(\mathbb{L})$ and $\Sigma \mapsto S_{\kappa_1,\kappa_2}(\Sigma)$ for unlabeled defect one-manifolds $\mathbb{L}$ and unlabeled defect surfaces Σ define a symmetric monoidal functor $S_{\kappa_1,\kappa_2} \colon \mathrm{Bord}_2^{\mathrm{def},0} \to \mathrm{Bord}_2^{\mathrm{def},0}$. This functor is an autoequivalence.

We call the symmetric monoidal functor $S_{\kappa_1,\kappa_2} \colon \mathrm{Bord}_2^{\mathrm{def},0} \to \mathrm{Bord}_2^{\mathrm{def},0}$ the (κ_1, κ_2)-*framing shift functor*. For a defect surface Σ with labels we define $S_{\kappa_1,\kappa_2}(\Sigma)$ by applying S_{κ_1,κ_2} to the underlying unlabeled defect surface and keeping the labels, and analogously for labeled defect one-manifolds. Thereby we obtain, by composing with framing shifts, a whole family of modular functors from a given one:

Proposition A.2. Let $\mathrm{T} \colon \mathrm{Bord}_2^{\mathrm{def}} \to \mathcal{S}$ be a modular functor. Then for every pair $(\kappa_1, \kappa_2) \in (2\mathbb{Z})^2$, the functor

$$ {}^{\kappa_1}\mathrm{T}^{\kappa_2} := \mathrm{T} \circ S_{\kappa_1,\kappa_2} \tag{A.4} $$

is a modular functor as well.

We call the so obtained functor ${}^{\kappa_1}\mathrm{T}^{\kappa_2}$ the (κ_1, κ_2)-*shift of* T. This functor can also be described in terms of T by shifting the labels instead of the framings: For each $(\kappa_1, \kappa_2) \in (2\mathbb{Z})^2$ and bimodule $\mathcal{M}_i$ we define the bimodule

$$ \widetilde{S}_{\kappa_1,\kappa_2}(\mathcal{M}_i) := {}^{-\kappa_1}\mathcal{M}^{-\kappa_2}. \tag{A.5} $$

For a defect surface Σ the defect surface $\widetilde{S}_{\kappa_1,\kappa_2}(\Sigma)$ is now defined as the *same* unlabeled defect surface Σ, but with $\widetilde{S}_{\kappa_1,\kappa_2}$ applied to the labels of each defect line of Σ. It follows that $\widetilde{S}_{\kappa_1,\kappa_2}(\Sigma)$ is again a defect surface, and extending the operation $\widetilde{S}_{\kappa_1,\kappa_2}$ in the obvious way to defect one-manifolds yields a symmetric monoidal functor

$$ \widetilde{S}_{\kappa_1,\kappa_2} : \quad \mathrm{Bord}_2^{\mathrm{def}} \longrightarrow \mathrm{Bord}_2^{\mathrm{def}}. \tag{A.6} $$

It is straightforward to see that the analogue of Proposition 3.13 for defect lines holds. Thus we conclude that

Proposition A.3. Let $\mathrm{T} \colon \mathrm{Bord}_2^{\mathrm{def}} \to \mathcal{S}$ be a modular functor. For any pair $(\kappa_1, \kappa_2) \in (2\mathbb{Z})^2$ the modular functors ${}^{\kappa_1}\mathrm{T}^{\kappa_2} = \mathrm{T} \circ S_{\kappa_1,\kappa_2}$ and $\mathrm{T} \circ \widetilde{S}_{\kappa_1,\kappa_2}$ are canonically isomorphic.

B Categorical constructions

In this appendix we briefly mention pertinent concepts and constructions on categories, such as (co)monads, (co)ends and the Eilenberg-Watts calculus on finite linear categories.

B.1 General concepts

Finite tensor categories. Throughout this paper, all categories appearing as labels for defect manifolds are assumed to be enriched over finite-dimensional $\Bbbk$-vector spaces, with $\Bbbk$ a fixed algebraically closed field. A ($\Bbbk$-linear) *finite category* is an abelian category for which the set of isomorphism classes of simple objects is finite and for which every object has finite length and a projective cover. A ($\Bbbk$-linear) *finite tensor category* is a finite $\Bbbk$-linear monoidal category with simple monoidal unit and with a left and a right duality. For $\mathcal{A}$ and $\mathcal{B}$ finite categories we denote by $\mathcal{L}ex(\mathcal{A},\mathcal{B})$ and $\mathcal{R}ex(\mathcal{A},\mathcal{B})$ the finite categories of ($\Bbbk$-linear) left exact and right exact functors from $\mathcal{A}$ to $\mathcal{B}$, respectively.

Module categories. A right *module category* over a monoidal category $\mathcal{A}$, or right $\mathcal{A}$-*module*, for short, is a category $\mathcal{M}$ together with a bilinear bifunctor $\mathcal{M}\times\mathcal{A}\to\mathcal{A}$ (which we denote by ".") that is exact in each argument, and with a natural family of isomorphisms $(m.a).b \to m.(a\otimes b)$ for $a,b\in\mathcal{A}$ and $m\in\mathcal{M}$ obeying obvious coherence axioms. Left $\mathcal{A}$-modules are defined analogously. For $\mathcal{A}$ and $\mathcal{B}$ monoidal categories, a $\mathcal{A}$-$\mathcal{B}$-*bimodule* is a category that is a left $\mathcal{A}$-module and a right $\mathcal{B}$-module together with natural coherence isomorphisms connecting the left and right actions.

A (right) $\mathcal{A}$-*module functor* (F,ϕ) between right $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$ is a linear functor $F\colon \mathcal{M}\to\mathcal{N}$ together with a natural family ϕ of isomorphisms $F(m.a)\to F(m).a$, for $a\in\mathcal{A}$ and $m\in\mathcal{M}$, such that the obvious pentagon and triangle diagrams commute. A *module natural transformation* η between right $\mathcal{A}$-module functors (F,ϕ^F) and (G,ϕ^G) is a natural transformation $\eta\colon F\Rightarrow G$ such that $\phi^G_{m,a}\circ\eta_{m.a} = (\eta_m.\mathrm{id}_a)\circ\phi^F_{m,a}$ for all $a\in\mathcal{A}$ and $m\in\mathcal{M}$,

Ends and coends. A *dinatural transformation* from a functor $F\colon\ \mathcal{C}\times\mathcal{C}^{\mathrm{opp}}\to\mathcal{D}$ to an object $d\in\mathcal{D}$ is a family of morphisms $\varphi_c\colon F(c,c)\to d$ for $c\in\mathcal{C}$ such that $\varphi_{c'}\circ F(g,c') = \varphi_c\circ F(c,g)$ for all $g\in\mathrm{Hom}_\mathcal{C}(c,c')$. A *coend* (C,ι) for $F\colon\ \mathcal{C}\times\mathcal{C}^{\mathrm{opp}}\to\mathcal{D}$ is an object $C\in\mathcal{D}$ together with a dinatural transformation $\iota\colon F\to C$ that is universal among all dinatural transformations φ from F to an object of $\mathcal{D}$, i.e. for any such dinatural transformation there is a unique morphism ϖ obeying $\varphi_c=\varpi\circ\iota_c$ for all $c\in\mathcal{C}$. The coend (C,ι), as well as the underlying object C, is denoted by $\int^{c\in\mathcal{C}}F(c,c)$. Dually, an *end* $(E,\jmath)=\int_{c\in\mathcal{C}}F(c,c)$ for F is a universal dinatural transformation from a constant to F. If a coend or end exists, then it is unique up to unique isomorphism. When the categories in question are functor categories, one often wants specific properties of the functors to be preserved under taking ends and coends. In the case of interest to us, the relevant property is representability. Accordingly, we are considering left exact functors and impose the universal property for ends and coends within categories of left exact functors. (Unlike e.g. in [Ly], we do not use a separate symbol $\oint$ for such 'left exact (co)ends'.) For more information on ends and coends see e.g. [FuS2] and the literature cited there.

The Hom functor preserves and reverses ends in the sense that

$$\begin{aligned}\mathrm{Hom}_\mathcal{D}(-,\textstyle\int_{c\in\mathcal{C}}F(c,c)) &\cong \displaystyle\int_{c\in\mathcal{C}}\mathrm{Hom}_\mathcal{D}(-,F(c,c)) \qquad \text{and}\\ \mathrm{Hom}_\mathcal{D}(\textstyle\int^{c\in\mathcal{C}}F(c,c),-) &\cong \displaystyle\int_{c\in\mathcal{C}}\mathrm{Hom}_\mathcal{D}(F(c,c),-)\end{aligned} \tag{B.1}$$

for any functor $F\colon\ \mathcal{C}\times\mathcal{C}^{\mathrm{opp}}\to\mathcal{D}$. Further, it follows from the (co-)Yoneda lemma (see e.g. [FSS2, Prop. 2.7]) that for any linear functor F between finite linear categories $\mathcal{A}$ and $\mathcal{A}'$ there are natural isomorphisms

$$\int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(a,-)\otimes F(a) \;\cong\; F \;\cong\; \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(-,a)^*\otimes F(a) \tag{B.2}$$

of linear functors. When dealing with Deligne products of finite linear categories one has in addition

$$\mathrm{Hom}_{\overline{\mathcal{C}}\boxtimes\mathcal{D}}\big(-\,,\textstyle\int^{c\in\mathcal{C}} \overline{c}\boxtimes F(c)\big) \;\cong\; \displaystyle\int^{c\in\mathcal{C}} \mathrm{Hom}_{\overline{\mathcal{C}}\boxtimes\mathcal{D}}\big(-\,,\overline{c}\boxtimes F(c)\big) \tag{B.3}$$

for any *left exact* functor $F\in\mathcal{L}ex(\mathcal{C},\mathcal{D})$, and a similar reversed isomorphism for right exact functors [FSS2, Prop. 3.4]. These isomorphisms can e.g. be used, in conjunction with the Eilenberg-Watts equivalences (3.39), to show that for any two finite categories $\mathcal{M}$ and $\mathcal{N}$ the mapping

$$F \longmapsto \mathrm{Hom}_{\mathcal{N}}(-,F(-)) \tag{B.4}$$

defines an equivalence

$$\mathcal{L}ex(\mathcal{M},\mathcal{N}) \xrightarrow{\;\simeq\;} \mathcal{L}ex(\mathcal{M}\boxtimes\overline{\mathcal{N}},\mathrm{vect})\,. \tag{B.5}$$

An inverse equivalence $\mathcal{L}ex(\mathcal{M}\boxtimes\overline{\mathcal{N}},\mathrm{vect})\xrightarrow{\simeq}\mathcal{L}ex(\mathcal{M},\mathcal{N})$ is given by

$$G \longmapsto \int^{n\in\mathcal{N}} G(-\boxtimes\overline{n})\otimes n\,. \tag{B.6}$$

Monads and comonads. A *monad* $M=(M,\mu,\eta)$ on a category $\mathcal{C}$ is an algebra (or monoid) in the monoidal category of endofunctors of $\mathcal{C}$, that is, an endofunctor M together with natural transformations $\mu=(\mu_c)_{c\in\mathcal{C}}\colon M\circ M\Rightarrow M$ (product) and $\eta=(\eta_c)_{c\in\mathcal{C}}\colon \mathrm{id}_{\mathcal{C}}\Rightarrow M$ (unit) which satisfy associativity and unit properties, i.e. $\mu_c\circ T(\mu_c)=\mu_c\circ\mu_{T(c)}$ and $\mu_c\circ T(\eta_c)=\mathrm{id}_{T(c)}=\mu_c\circ\eta_{T(c)}$. Analogously, a *comonad* on $\mathcal{C}$ is a coalgebra in the category of endofunctors of $\mathcal{C}$, i.e. an endofunctor W together with a coproduct $W\Rightarrow W\circ W$ and counit $W\Rightarrow \mathrm{id}_{\mathcal{C}}$ satisfying coassociativity and counit properties. Every adjoint pair of functors F and G (G right adjoint to F) gives rise to a monad structure on the endofunctor $G\circ F$ and a comonad structure on $F\circ G$.

A (left) *comodule* over a comonad W on $\mathcal{C}$ (also called a W-coalgebra) is an object $x\in\mathcal{C}$ together with a morphism $\delta\colon x\to W(x)$ satisfying analogous compatibility conditions with the coproduct Δ and counit ε of W as a comodule over a comonoid, i.e. $\Delta_x\circ\delta=W(\delta)\circ\delta$ and $\varepsilon_x\circ\delta=\mathrm{id}_x$. Right comodules and left and right modules over a monad are defined analogously.

If W is a comonad on a category $\mathcal{C}$, then the opposite functor

$$\overline{W}:\quad \overline{\mathcal{C}}\to\overline{\mathcal{C}} \tag{B.7}$$

is a monad on $\overline{\mathcal{C}}$. Further, if the adjoint functors $W^{\mathrm{l.a.}}$ and $W^{\mathrm{r.a.}}$ exist, then they are monads on $\mathcal{C}$, while the functors $\overline{W^{\mathrm{l.a.}}},\overline{W^{\mathrm{r.a.}}}\colon\overline{\mathcal{C}}\to\overline{\mathcal{C}}$ are comonads on $\overline{\mathcal{C}}$. Moreover, if $x\in\mathcal{C}$ is a comodule over W, then it is also naturally a module over the monads $W^{\mathrm{l.a.}}$ and $W^{\mathrm{r.a.}}$, while the object $\overline{x}\in\overline{\mathcal{C}}$ is a module over the monad $\overline{W}$ and a comodule over the comonads $\overline{W^{\mathrm{l.a.}}}$ and $\overline{W^{\mathrm{r.a.}}}$. Analogous statements hold for monads on $\mathcal{C}$.

B.2 The canonical κ-twisted (co)monads and their (co)modules

Here we define various versions of κ-balanced functors and exhibit the framed center as a universal category for κ-balanced functors, making use of corresponding κ-twisted (co)monads.

We consider a cyclically composable string $(\mathcal{M}_1^{\epsilon_1}, \mathcal{M}_2^{\epsilon_2}, \dots, \mathcal{M}_n^{\epsilon_n})$ of bimodules with corresponding balancings κ_i between $\mathcal{M}_i$ and $\mathcal{M}_{i+1}$. There are four different situations to be distinguished, indexed by the sequence $\{(\epsilon_i, \epsilon_{i+1})\}$. The following definition captures all cases:

Definition B.1. Let $(\mathcal{M}_1^-, \mathcal{M}_2^-, \mathcal{M}_3^+, \mathcal{M}_4^+)$ be a cyclically composable string of bimodules with corresponding sequence $\vec\kappa = (\kappa_1, \kappa_2, \kappa_3, \kappa_4)$ of balancings κ_i between $\mathcal{M}_i$ and $\mathcal{M}_{i+1}$. The category $\mathcal{L}ex_{\vec\kappa}^{\mathrm{bal}}((\mathcal{M}_1^-, \mathcal{M}_2^-, \mathcal{M}_3^+, \mathcal{M}_4^+), \mathcal{X})$ of κ-balanced functors to a finite category $\mathcal{X}$ consists of functors $F\colon \mathcal{M}_1^- \boxtimes \cdots \boxtimes \mathcal{M}_4^+ \to \mathcal{X}$ with coherent isomorphisms

$$\begin{aligned}
&F(\overline{a.m_1} \boxtimes \overline{m_2} \boxtimes m_3 \boxtimes m_4) \xrightarrow{\cong} F(\overline{m_1} \boxtimes \overline{m_2.^{[\kappa_1+4]}a} \boxtimes m_3 \boxtimes m_4)\,,\\
&F(\overline{m_1} \boxtimes \overline{b.m_2} \boxtimes m_3 \boxtimes m_4) \xrightarrow{\cong} F(\overline{m_1} \boxtimes \overline{m_2} \boxtimes {}^{[\kappa_2+2]}b.m_3 \boxtimes m_4)\,,\\
&F(\overline{m_1} \boxtimes \overline{m_2} \boxtimes m_3.c \boxtimes m_4) \xrightarrow{\cong} F(\overline{m_1} \boxtimes \overline{m_2} \boxtimes m_3 \boxtimes {}^{[\kappa_3]}c.m_4) \qquad \text{and}\\
&F(\overline{m_1} \boxtimes \overline{m_2} \boxtimes m_3 \boxtimes m_4.d) \xrightarrow{\cong} F(\overline{m_1.^{[\kappa_4+2]}d} \boxtimes \overline{m_2} \boxtimes m_3 \boxtimes m_4)\,.
\end{aligned} \tag{B.8}$$

In accordance with the various situations, for each value of i there are four (related) comonads $Z_{[\kappa_i]}(\mathcal{M}_i^{\epsilon_i} \boxtimes \mathcal{M}_{i+1}^{\epsilon_{i+1}})$ on the category $\mathcal{M}_i^{\epsilon_i} \boxtimes \mathcal{M}_{i+1}^{\epsilon_{i+1}}$, defined by

$$\begin{aligned}
&Z_{[\kappa_1]}(\mathcal{M}_1^- \boxtimes \mathcal{M}_2^-)(\overline{m_1} \boxtimes \overline{m_2}) := \int_{a\in\mathcal{A}} \overline{a.m_1} \boxtimes \overline{m_2.^{[\kappa_1+3]}a}\\
&Z_{[\kappa_2]}(\mathcal{M}_2^- \boxtimes \mathcal{M}_3)(\overline{m_2} \boxtimes m_3) := \int_{b\in\mathcal{B}} \overline{b.m_2} \boxtimes {}^{[\kappa_2+1]}b.m_3\\
&Z_{[\kappa_3]}(\mathcal{M}_3 \boxtimes \mathcal{M}_4)(m_3 \boxtimes m_4) := \int_{c\in\mathcal{C}} \overline{m_3.c} \boxtimes {}^{[\kappa_3-1]}c.m_4 \qquad \text{and}\\
&Z_{[\kappa_4]}(\mathcal{M}_4 \boxtimes \mathcal{M}_1^-)(m_4 \boxtimes \overline{m_1}) := \int_{d\in\mathcal{D}} \overline{m_4.d} \boxtimes \overline{m_1.^{[\kappa_4+1]}d}\,,
\end{aligned} \tag{B.9}$$

respectively. These comonads can be combined to a comonad on $\vec{\mathcal{M}} = (\mathcal{M}_1^-, \mathcal{M}_2^-, \mathcal{M}_3^+, \mathcal{M}_4^+)$. We denote this comonad by $Z_{\vec\kappa} = Z_{\vec\kappa}(\vec{\mathcal{M}})$ and refer to it as the *canonical $\vec\kappa$-twisted comonad* on $\vec{\mathcal{M}}$. This terminology is justified by the following result:

Proposition B.2. Let $\vec{\mathcal{M}} = (\mathcal{M}_1^{\epsilon_1}, \dots, \mathcal{M}_n^{\epsilon_n})$ be a string of cyclically composable bimodules with balancings $\vec\kappa$. Denote by $\vec{\mathcal{M}}\boxtimes := \mathcal{M}_1^{\epsilon_1} \boxtimes \cdots \boxtimes \mathcal{M}_n^{\epsilon_n}$ the Deligne product of the bimodules and by $\vec{\mathcal{M}}\overset{\vec\kappa}{\boxtimes} := \mathcal{M}_1^{\epsilon_1} \overset{\kappa_1}{\boxtimes} \mathcal{M}_2^{\epsilon_2} \overset{\kappa_2}{\boxtimes} \mathcal{M}_3^{\epsilon_3} \overset{\kappa_3}{\boxtimes} \cdots \overset{\kappa_{n-1}}{\boxtimes} \mathcal{M}_n^{\epsilon_n} \overset{\kappa_n}{\boxtimes}$ the corresponding framed center, as in Definition 3.4.

(i) For any object $x \in \vec{\mathcal{M}}\boxtimes$ the object $Z_{[\vec\kappa]}(x)$ has a canonical structure of an object in $\vec{\mathcal{M}}\overset{\vec\kappa}{\boxtimes}$, to be denoted by $I_{[\vec\kappa]}(x)$. This naturally defines a functor $I_{[\vec\kappa]} \in \mathcal{L}ex_{\vec\kappa}^{\mathrm{bal}}(\vec{\mathcal{M}}\boxtimes, \vec{\mathcal{M}}\overset{\vec\kappa}{\boxtimes})$ such that $Z_{[\vec\kappa]} = U \circ I_{[\vec\kappa]}$, with $U\colon \vec{\mathcal{M}}\overset{\vec\kappa}{\boxtimes} \to \vec{\mathcal{M}}$ the functor that forgets the balancing..

(ii) The framed center $\vec{\mathcal{M}}\overset{\vec{\kappa}}{\boxtimes}$ together with the $\vec{\kappa}$-balanced functor $I_{[\vec{\kappa}]}$ is universal for $\vec{\kappa}$-balanced functors: For any finite category $\mathcal{X}$, pre-composition with $I_{[\vec{\kappa}]}$ is an equivalence

$$\mathcal{L}ex(\vec{\mathcal{M}}\overset{\vec{\kappa}}{\boxtimes}, \mathcal{X}) \xrightarrow{\simeq} \mathcal{L}ex^{\mathrm{bal}}_{\vec{\kappa}}(\vec{\mathcal{M}}\boxtimes, \mathcal{X})\,. \tag{B.10}$$

Proof. Consider part (i) in the special case of a single bimodule $\mathcal{M}$. For $\kappa \in 2\mathbb{Z}$ the comonad $Z_{[\kappa]}$ on $\mathcal{M}$ is given by the endofunctor

$$Z_{[\kappa]}: \quad m \longmapsto \int_{a\in\mathcal{A}} a\,.\,m\,.\,a^{[\kappa-1]}. \tag{B.11}$$

$Z_{[\kappa]}$ can be viewed as acting with the object $\int_{a\in\mathcal{A}} a\boxtimes\overline{a} \in \mathcal{A}\boxtimes\overline{\mathcal{A}} \cong \Psi^{\mathrm{r}}(\mathrm{id}_{\mathcal{A}})$ on $\mathcal{M}$, after applying a suitable power of the double dual functor to it. The balancings of $I_{[\kappa]}(m) \in \vec{\mathcal{M}}\overset{\kappa}{\boxtimes}$ with underlying object $Z_{[\kappa]}(m)$ are determined by invoking the coherent isomorphisms

$$\int_{a\in\mathcal{A}} b\otimes a\boxtimes\overline{a} \cong \int_{a\in\mathcal{A}} a\boxtimes\overline{{}^{\vee}b\otimes a} \qquad \text{and} \qquad \int_{a\in\mathcal{A}} a\otimes b\boxtimes\overline{a} \cong \int_{a\in\mathcal{A}} a\boxtimes\overline{a\otimes b^{\vee}} \tag{B.12}$$

for $b\in\mathcal{A}$; these are obtained by setting $\mathcal{M}=\mathcal{A}$ in the isomorphisms (3.53). It follows that this way we have defined a functor $I_{[\kappa]}\colon \mathcal{M}\to\vec{\mathcal{M}}\overset{\kappa}{\boxtimes}$ that satisfies $U\circ I_{[\kappa]} = Z_{[\kappa]}$. The proof of the general case follows by the same reasoning.

(ii) We first show that the framed center is equivalent to the category of comodules over $Z_{\vec{\kappa}}$: This follows in the case of a single bimodule $\mathcal{M}$ with the help of the linear isomorphisms

$$\mathrm{Hom}_{\mathcal{M}}(m.a, {}^{[\kappa-2]}a.m) \;\cong\; \mathrm{Hom}_{\mathcal{M}}(m, {}^{[\kappa-2]}a.m.a^{\vee}) \tag{B.13}$$

after taking the end. The general case is treated analogously. The universal property of the framed center now follows, analogously as the universal property of the center of a bimodule category [GeNN], by observing that the left adjoint of a κ-balanced functor takes values in the framed center. □

Analogous considerations apply to monads: On a bimodule with framing κ we define the monad $Z^{[\kappa]}$ by $Z^{[\kappa]}(m) := \int^{a\in\mathcal{A}} a\,.\,m\,.\,a^{[\kappa-3]}$. Similarly there are monads in each of the four types of situations for the framed center; explicit expressions for these are obtained by replacing in the formulas (B.9) the end by an coend and κ by $\kappa-2$. The monads define corresponding induction functors $I^{\vec{\kappa}}$; these are the universal functors for $\kappa-2$-balanced right exact functors.

Corollary B.3.

(i) The forgetful functor $U\colon \mathcal{Z}^{\vec{\kappa}}(\vec{\mathcal{M}}) \to \vec{\mathcal{M}}$ is left adjoint to the co-induction functor $I_{[\vec{\kappa}]}$ and right adjoint to the induction functor $I^{[\vec{\kappa}]}$ that correspond to the endofunctors $Z_{[\vec{\kappa}]}$ and $Z^{[\vec{\kappa}]}$, respectively.

(ii) The category $\mathcal{Z}^{\kappa}(\mathcal{M})$ is equivalent to the category of modules over the monad $Z^{[\kappa]}$, and equivalent to the category of comodules over the comonad $Z_{[\kappa]}$.

B.3 Extensions of the Eilenberg-Watts calculus

We now collect a few useful results which extend the Eilenberg-Watts calculus of [FSS2] that is recapitulated in Section 3.6. We first present a mild generalization of the Eilenberg-Watts equivalences (3.40) and (3.39):

Lemma B.4. For finite categories $\mathcal{M}, \mathcal{K}$ and $\mathcal{N}$ there are adjoint equivalences

$$\mathcal{L}ex(\mathcal{M} \boxtimes \mathcal{K}, \mathcal{N}) \simeq \mathcal{L}ex(\mathcal{M}, \mathcal{N} \boxtimes \overline{\mathcal{K}}) \tag{B.14}$$

and

$$\mathcal{R}ex(\mathcal{M} \boxtimes \mathcal{K}, \mathcal{N}) \simeq \mathcal{R}ex(\mathcal{M}, \mathcal{N} \boxtimes \overline{\mathcal{K}})\,. \tag{B.15}$$

Proof. For $F \in \mathcal{L}ex(\mathcal{M} \boxtimes \mathcal{K}, \mathcal{N})$ we set $\widehat{\Psi}^{\mathrm{l}}(F) := \Psi^{\mathrm{l}}(\widehat{F})$, where $\widehat{F}\colon \mathcal{M} \to \mathcal{L}ex(\mathcal{K}, \mathcal{N})$ is defined by $\widehat{F}(m) := F(m \boxtimes -)$, and for $G \in \mathcal{L}ex(\mathcal{M}, \mathcal{N} \boxtimes \overline{\mathcal{K}})$ we set $\widehat{\Phi}^{\mathrm{l}}(G)(m \boxtimes k) := \Phi^{\mathrm{l}}(G(m))(k)$. The so obtained functors

$$\widehat{\Psi}^{\mathrm{l}}\colon \mathcal{L}ex(\mathcal{M} \boxtimes \mathcal{K}, \mathcal{N}) \to \mathcal{L}ex(\mathcal{M}, \mathcal{N} \boxtimes \overline{\mathcal{K}}) \qquad \text{and} \qquad \widehat{\Phi}^{\mathrm{l}}\colon \mathcal{L}ex(\mathcal{M}, \mathcal{N} \boxtimes \overline{\mathcal{K}}) \to \mathcal{L}ex(\mathcal{M} \boxtimes \mathcal{K}, \mathcal{N}) \tag{B.16}$$

constitute a parameter version of the Eilenberg-Watts equivalences. Accordingly an argument parallel to the one that proves the ordinary Eilenberg-Watts equivalences shows that they are quasi-inverses of each other.
The case of right exact functors is treated dually. □

The ordinary Eilenberg-Watts equivalences are recovered from this statement by taking $\mathcal{M}$ to be vect.

We next collect without proof a few statements that involve a lift of the Eilenberg-Watts calculus to categories of (co)modules over a (co)monad on a functor category. A proof of these statements is given in [FSS3], where a module Eilenberg-Watts calculus is set up which e.g. allows for a novel perspective on the center of module categories. The proof in [FSS3] makes use of the fact that for $\Phi\colon\ \mathcal{X} \leftrightarrows \mathcal{Y} : \Psi$ an adjoint equivalence between categories and $T^{\mathcal{X}}$ a (co)monad on $\mathcal{X}$, the functor $\Phi \circ T^{\mathcal{X}} \circ \Psi =: T^{\mathcal{Y}}$ is canonically a (co)monad on $\mathcal{Y}$. Also note that given a (co)monad $T\colon \mathcal{M} \to \mathcal{M}$ on a category $\mathcal{M}$, for any category $\mathcal{X}$ the functor category $\mathcal{F}un(\mathcal{M}, \mathcal{X})$ inherits a (co)monad T^* by pre-composition with T, i.e.

$$T^*(F) = F \circ T \tag{B.17}$$

for $F \in \mathcal{F}un(\mathcal{M}, \mathcal{X})$.

Proposition B.5. Let $\mathcal{M} = {}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ be a finite bimodule category over a finite tensor category $\mathcal{A}$ and $\mathcal{Z}^\kappa(\mathcal{M})$ its κ-twisted center, and let $\mathcal{X}$ be a finite linear category.

(i) The Eilenberg-Watts calculus provides explicit equivalences

$$\begin{array}{ccc} \mathcal{L}ex^\kappa(\mathcal{M}, \mathcal{X}) & \xrightarrow{\ \simeq\ } & \mathcal{L}ex(\mathcal{Z}^\kappa(\mathcal{M}), \mathcal{X}) \\ & {\scriptstyle\Psi^{\mathrm{l}}}\searrow \qquad \nearrow{\scriptstyle\Phi^{\mathrm{l}}} & \\ & \overline{\mathcal{Z}^\kappa(\mathcal{M})} \boxtimes \mathcal{X} & \end{array} \tag{B.18}$$

of linear categories. Moreover, the category $\mathcal{L}ex^\kappa(\mathcal{M}, \mathcal{X})$ is equivalent to the category of comodules over the comonad $(Z_{[\kappa]})_*$ on $\mathcal{L}ex(\mathcal{M}, \mathcal{X})$,

$$\mathcal{L}ex^\kappa(\mathcal{M}, \mathcal{X}) \,\simeq\, (Z_{[\kappa]})_*\text{-comod}(\mathcal{L}ex(\mathcal{M}, \mathcal{X}))\,. \tag{B.19}$$

(ii) For any left exact κ+2-balanced functor $F\colon \mathcal{M} \to \mathcal{X}$ there is an isomorphism

$$\int^{z\in\mathcal{Z}^\kappa(\mathcal{M})} \overline{z} \boxtimes \widehat{F}(z) \;\cong\; \int^{m\in\mathcal{M}} \overline{m} \boxtimes F(m) \tag{B.20}$$

of objects in $\overline{\mathcal{Z}^\kappa(\mathcal{M})} \boxtimes \mathcal{X}$, where $\widehat{F} := \Phi^{\mathrm{l}} \circ \Psi^{\mathrm{l}}(F)$.

(iii) Specifically, for the co-induction functor $I_{[\kappa]}\colon \mathcal{M} \to \mathcal{Z}^\kappa(\mathcal{M})$ that corresponds to the comonad $Z_{[\kappa]}$, the corresponding functor $\widehat{I_{[\kappa]}}\colon \mathcal{Z}^\kappa(\mathcal{M}) \to \mathcal{Z}^\kappa(\mathcal{M})$ is the identity functor, whereby the isomorphism (B.20) reduces to

$$\int^{z\in\mathcal{Z}^\kappa(\mathcal{M})} \overline{z} \boxtimes \mathrm{Id}(z) \;\cong\; \int^{m\in\mathcal{M}} \overline{m} \boxtimes Z_{[\kappa]}(m) \;\cong\; \int^{m\in\mathcal{M}} \overline{m} \boxtimes \int_{a\in\mathcal{A}} a.m.a^{[\kappa-1]}. \tag{B.21}$$

An analogous isomorphism holds for ends: $\int_{z\in\mathcal{Z}^\kappa(\mathcal{M})} \overline{z} \boxtimes z \cong \int^{m\in\mathcal{M}} \overline{m} \boxtimes Z^{[\kappa]}(m)$.

Combining these assertions with Lemma B.4 we arrive at

Lemma B.6. The equivalences (B.14) induce equivalences

$$\mathcal{L}ex(\mathcal{N}_\mathcal{A} \overset{\kappa}{\boxtimes}{}_\mathcal{A}\mathcal{M}, \mathcal{K}) \cong \mathcal{L}ex_\mathcal{A}(\mathcal{N}_\mathcal{A}, \mathcal{K} \boxtimes \overline{\mathcal{M}}^{-\kappa+1}) \tag{B.22}$$

for any $\kappa \in 2\mathbb{Z}$.

Proof. Using that $\mathcal{N}_\mathcal{A} \overset{\kappa}{\boxtimes}{}_\mathcal{A}\mathcal{M} \cong \mathrm{Comod}_{\mathcal{Z}_{[\kappa]}}(\mathcal{N} \boxtimes \mathcal{M})$, Proposition B.5(i) implies that

$$\begin{aligned} \mathcal{L}ex(\mathcal{N}_\mathcal{A} \overset{\kappa}{\boxtimes}{}_\mathcal{A}\mathcal{M}, \mathcal{K}) &\cong \mathcal{L}ex(\mathrm{Comod}_{\mathcal{Z}_{[\kappa]}}(\mathcal{N} \boxtimes \mathcal{M}), \mathcal{K}) \\ &\cong \mathrm{Comod}_{(\mathcal{Z}_{[\kappa]})^*} \mathcal{L}ex(\mathcal{N} \boxtimes \mathcal{M}, \mathcal{K}) \cong \mathrm{Comod}_{\widetilde{\mathcal{Z}_{[\kappa]}}} \mathcal{L}ex(\mathcal{N}, \mathcal{K} \boxtimes \overline{\mathcal{M}})\,, \end{aligned} \tag{B.23}$$

where $\widetilde{\mathcal{Z}_{[\kappa]}}$ is the comonad on $\mathcal{L}ex(\mathcal{N}, \mathcal{K} \boxtimes \overline{\mathcal{M}})$ that is induced by the equivalence from Lemma B.4. We proceed to compute $\widetilde{\mathcal{Z}_{[\kappa]}}$. For $F \in \mathcal{L}ex(\mathcal{N}, \mathcal{K} \boxtimes \overline{\mathcal{M}})$ and $n \in \mathcal{N}$ we have

$$\begin{aligned} \widetilde{\mathcal{Z}_{[\kappa]}}(F)(n) &= \big(\widehat{\Psi}^{\mathrm{l}}(\mathcal{Z}_{[\kappa]})^*(\widehat{\Phi}^{\mathrm{l}}(F))\big)(n) = \int^{m\in\mathcal{M}}\!\!\int_{a\in\mathcal{A}} \mathrm{Hom}_{\overline{\mathcal{M}}}(a.m, F(n.a^{[\kappa+1]})) \boxtimes \overline{m} \\ &= \int^{m\in\mathcal{M}}\!\!\int_{a\in\mathcal{A}} \mathrm{Hom}_{\overline{\mathcal{M}}}(m, F(n.a^{[\kappa+1]})) \boxtimes \overline{a^\vee.m} \\ &\cong \int_{a\in\mathcal{A}}\int^{m\in\mathcal{M}} \mathrm{Hom}_{\overline{\mathcal{M}}}(m, F(n.a^{[\kappa+1]})) \boxtimes \overline{a^\vee.m} \\ &\cong \int_{a\in\mathcal{A}} a^\vee.\, F(n.a^{[\kappa+1]}) \;\cong\; \int_{a\in\mathcal{A}} a\,.\,F(n.a^{[\kappa]})\,. \end{aligned} \tag{B.24}$$

As a consequence, a $\widetilde{\mathcal{Z}_{[\kappa]}}$-comodule structure on F consists of a coherent family of morphisms $F(n) \to a.F(n.a^{[\kappa]})$ for $n \in \mathcal{N}$. By adjunction, this is equivalent to a family of coherent natural isomorphisms $F(n.a) \xrightarrow{\cong} a^{[\kappa-1]}.F(n)$ which, in turn, is equivalent to F belonging to the category $\mathcal{L}ex_\mathcal{A}(\mathcal{N}_\mathcal{A}, \mathcal{K} \boxtimes \overline{\mathcal{M}}^{-\kappa+1})$. ☐

As a particular case we obtain

Corollary B.7. For bimodule categories ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}$ and ${}_{\mathcal{A}}\mathcal{N}_{\mathcal{B}}$, the Eilenberg-Watts equivalences induce an equivalence

$$\overline{\mathcal{M}} \overset{1}{\boxtimes} \mathcal{N} \overset{1}{\boxtimes} \;\simeq\; \mathcal{L}ex_{\mathcal{A},\mathcal{B}}(\mathcal{M},\mathcal{N}) \tag{B.25}$$

of categories.

Proof. For the $\mathcal{A}$-action we have

$$\begin{aligned} {}_{\mathcal{A}}\mathcal{N} \overset{1}{\boxtimes} {}_{\mathcal{A}}\overline{\mathcal{M}} \;&\simeq \mathcal{L}ex\big(\overline{\mathcal{N} \overset{1}{\boxtimes} \overline{\mathcal{M}}}, \mathrm{vect}\big) \\ &\simeq\; \mathcal{L}ex(\mathcal{M} \overset{-1}{\boxtimes} \overline{\mathcal{N}}, \mathrm{vect}) \;=\; \mathcal{L}ex(\mathcal{M} \overset{0}{\boxtimes} {}^{[-1]}\mathcal{N}, \mathrm{vect}) \;\simeq\; \mathcal{L}ex_{\mathcal{A}}(\mathcal{M}_{\mathcal{A}}, \mathcal{N}_{\mathcal{A}})\,, \end{aligned} \tag{B.26}$$

where the last step uses Lemma B.6 as well as the canonical equivalence $\overline{{}^{[-1]}\overline{\mathcal{N}}}^{[1]} \simeq \mathcal{N}_{\mathcal{A}}$. The $\mathcal{B}$-action is treated analogously. □

B.4 Twisted identity bimodules

Recall from Section 3.1 the twisted variant ${}^{\kappa_1}\mathcal{M}^{\kappa_2}$ of a bimodule $\mathcal{M}$, for any pair κ_1, κ_2 of even integers. If $\mathcal{M} = \mathcal{A}$ is the regular $\mathcal{A}$-bimodule, the double duality functor $(-)^{\vee\vee}$ (which is monoidal) provides a distinguished equivalence

$${}^{\kappa_1}\mathcal{A}^{\kappa_2} \simeq {}^{\kappa_1-2}\mathcal{A}^{\kappa_2+2}, \tag{B.27}$$

of bimodules, and thus by iteration we get in particular ${}^{\kappa}\mathcal{M}^0 \simeq {}^0\mathcal{M}^{\kappa}$, i.e. ${}^{\kappa}\mathcal{A} \simeq \mathcal{A}^{\kappa}$. Furthermore, we have

Lemma B.8. Let $\mathcal{A}$ be a finite tensor category, $\mathcal{M}$ a right and $\mathcal{N}$ a left $\mathcal{A}$-module, and let $\kappa \in 2\mathbb{Z}$. The functors

$$\begin{aligned} \rho: \quad \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{A} \;&\longrightarrow \mathcal{M}\,, \\ m \boxtimes a \;&\longmapsto \mathrm{Hom}_{\mathcal{A}}(D_{\mathcal{A}}, a) \otimes_{\Bbbk} m \end{aligned} \tag{B.28}$$

and

$$\begin{aligned} \lambda: \quad {}^{-\kappa}\mathcal{A} \boxtimes \mathcal{N} \;&\longrightarrow\; \mathcal{N}\,, \\ a \boxtimes n \;&\longmapsto\; a^{[-\kappa]}.\, n \end{aligned} \tag{B.29}$$

furnish distinguished equivalences

$$\mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{A}^{-\kappa} \xrightarrow{\;\simeq\;} \mathcal{M} \qquad \text{and} \qquad {}^{-\kappa}\mathcal{A} \overset{\kappa}{\boxtimes} \mathcal{N} \xrightarrow{\;\simeq\;} \mathcal{N} \tag{B.30}$$

of module categories.

Proof. The linear functor $\widetilde{\rho}\colon \mathcal{M} \boxtimes \mathcal{A}^{-\kappa} \to \mathcal{M}$ given by $\widetilde{\rho}(m \boxtimes a) = m.a^{[\kappa]}$ has a natural structure of a κ-balanced module functor, i.e. we have $\widetilde{\rho}(m.x \boxtimes a) \cong \widetilde{\rho}(m \boxtimes {}^{[\kappa]}x.a)$. By the universal property of $Z_{[\kappa]}$ from Proposition B.2, $\widetilde{\rho}$ therefore induces a module functor $\rho'\colon \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{A}^{-\kappa} \to \mathcal{M}$;

we claim that $\rho' = \rho$ is the functor defined in (B.28). To this end, set $\widetilde{\rho}^{-1}(m) := m \boxtimes \mathbf{1}$ and $\rho^{-1} := Z_{[\kappa]} \circ \widetilde{\rho}^{-1}$ and consider the diagram

$$\begin{array}{ccccc} \mathcal{M} \boxtimes \mathcal{A} & & \xrightarrow{\widetilde{\rho}^{-1} \circ \widetilde{\rho}} & & \mathcal{M} \boxtimes \mathcal{A} \\ {\scriptstyle Z_{[\kappa]}} \downarrow & \searrow^{\widetilde{\rho}} & & \nearrow^{\widetilde{\rho}^{-1}} & \downarrow {\scriptstyle Z_{[\kappa]}} \\ \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{A}^{-\kappa} & \xrightarrow[\rho']{} & \mathcal{M} & \xrightarrow[\rho^{-1}]{} & \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{A}^{-\kappa} \end{array} \tag{B.31}$$

Since $Z_{[\kappa]}$ is κ-balanced, the functor

$$\begin{aligned} \rho^{-1} : \quad \mathcal{M} &\longrightarrow \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{A} \\ m &\longmapsto Z_{[\kappa]}(m \boxtimes \mathbf{1}) = \int_{a \in \mathcal{A}} m.a^{[\kappa-1]} \boxtimes a \end{aligned} \tag{B.32}$$

satisfies

$$m.a \longmapsto Z_{[\kappa]}(m.a \boxtimes \mathbf{1}) \cong Z_{[\kappa]}(m \boxtimes a^{[-\kappa]}) = Z_{[\kappa]}(m \boxtimes \mathbf{1}) \otimes a^{[-\kappa]} \tag{B.33}$$

and is thus a module functor. Moreover, the functors ρ' and ρ^{-1} are quasi-inverse, hence ρ' is an equivalence: By the balancing of $Z_{[\kappa]}$, the functor $Z_{[\kappa]} \circ \widetilde{\rho}^{-1} \circ \widetilde{\rho}$ is isomorphic to $Z_{[\kappa]}$ as a right $\mathcal{A}$-module functor, and thus by the universal property of $Z_{[\kappa]}$ we have $\rho^{-1} \circ \rho' \cong \mathrm{Id}$ as module functors. It is even more direct to see that $\widetilde{\rho} \circ \widetilde{\rho}^{-1} \cong \mathrm{id}_{\mathcal{M}}$ as module functors, so that we also have $\rho' \circ \rho^{-1} \cong \mathrm{Id}$.

We now compute the functor ρ' explicitly. Denote by $U : \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{A} \to \mathcal{M} \boxtimes \mathcal{A}$ the forgetful functor and consider the diagram

$$\begin{array}{ccc} \mathcal{M} \boxtimes \mathcal{A}^{\kappa} & \xrightarrow{\widetilde{\rho}} & \mathcal{M} \\ {\scriptstyle Z_{[\kappa]}} \downarrow & & \uparrow {\scriptstyle \mathrm{Hom}_{\mathcal{A}}(D_{\mathcal{A}}, ?) \otimes_{\Bbbk} ?} \\ \mathcal{M} \overset{\kappa}{\boxtimes} \mathcal{A} & \xrightarrow{U} & \mathcal{M} \boxtimes \mathcal{A} \end{array} \tag{B.34}$$

This diagram commutes up to a module natural isomorphism: Using that there is a distinguished isomorphism $\int_{a \in \mathcal{A}} \overline{a^{\vee\vee} \otimes D_{\mathcal{A}}} \boxtimes a \cong \int^{a \in \mathcal{A}} \overline{a} \boxtimes a$, which is equivalent to (3.45), we obtain a distinguished isomorphism

$$\begin{aligned} \big(\mathrm{Hom}(D_{\mathcal{A}}, -) \otimes -\big) \circ U \circ Z_{[\kappa]}(m \boxtimes b) &= \int_{a \in \mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(D_{\mathcal{A}}, a \otimes b) \otimes_{\Bbbk} m.a^{[\kappa-1]} \\ &\cong \int_{a \in \mathcal{A}} \mathrm{Hom}(a^{\vee\vee} \otimes D_{\mathcal{A}}, b) \otimes_{\Bbbk} m.a^{[\kappa]} \\ &\cong \int^{a \in \mathcal{A}} \mathrm{Hom}(a, b) \otimes_{\Bbbk} a^{[\kappa]} \cong m.b^{[\kappa]} \end{aligned} \tag{B.35}$$

for any $m \boxtimes b \in \mathcal{M} \boxtimes \mathcal{A}$. This shows that $\rho' = \mathrm{Hom}_{\mathcal{A}}(D_{\mathcal{A}}, ?) \otimes_{\Bbbk} ? \equiv \rho$, as claimed.

To show the second of the equivalences (B.30), consider the functor

$$\begin{aligned} \mathcal{N} &\longrightarrow {}^{-\kappa}\!\mathcal{A} \overset{\kappa}{\boxtimes} \mathcal{N} , \\ n &\longmapsto Z_{[\kappa]}(\mathbf{1} \boxtimes n) = \int_{a \in \mathcal{A}} a^{[\kappa-1]} \boxtimes a.n . \end{aligned} \tag{B.36}$$

This is an $\mathcal{A}$-module functor, and it is straightforward to check that it is an equivalence with a quasi-inverse given by the functor that is induced by the functor λ as given in (B.29), which is κ-balanced and is an $\mathcal{A}$-module functor. $\square$

Next we consider the case that κ is odd. Recall that $\mathcal{I}_0 = \mathcal{A}$. We will construct an equivalence $\mathcal{M} \overset{\kappa}{\boxtimes} \overline{\mathcal{I}_0} \simeq \mathcal{M}$, where $\mathcal{M}_{\mathcal{A}}$ is a right module as in the case of even κ, but now the action on $\mathcal{M}$ gets balanced with the right action on $\overline{\mathcal{I}_0}$, whereby the remaining action on $\mathcal{M} \overset{\kappa}{\boxtimes} \overline{\mathcal{I}_0}$ is a left $\overline{\mathcal{A}}$-action. Accordingly also on the right hand side we need to work with a left $\overline{\mathcal{A}}$-action, which we obtain in the form $\overline{a}.m := m.a^{[\kappa]}$. In accordance with the notation in (3.1) we denote the resulting module by ${}^{[-\kappa]}_{\;\;\overline{\mathcal{A}}}\mathcal{M}$. The analogous notation for left modules ${}_{\mathcal{A}}\mathcal{N}$ is $\mathcal{N}^{[-\kappa]}_{\overline{\mathcal{A}}}$ with $n.\overline{a} = {}^{[\kappa]}a.n$.

Lemma B.9. Let $\mathcal{A}$ be a finite tensor category, $\mathcal{M}$ a right and $\mathcal{N}$ a left $\mathcal{A}$-module, and let $\kappa \in 2\mathbb{Z}+1$. There are canonical equivalences

$$\mathcal{M} \overset{\kappa}{\boxtimes} \overline{\mathcal{I}_0} \simeq {}^{[-\kappa]}_{\;\;\overline{\mathcal{A}}}\mathcal{M} \qquad \text{and} \qquad \overline{\mathcal{I}_0} \overset{\kappa}{\boxtimes} \mathcal{N} \simeq \mathcal{N}^{[\kappa-2]}_{\overline{\mathcal{A}}} \tag{B.37}$$

of module categories.

Proof. We treat explicitly the case of a right module $\mathcal{M}$, which is analogous to the proof of Lemma B.8. Again we define a diagram

$$\begin{array}{ccccc}
\mathcal{M} \boxtimes \overline{\mathcal{I}_0} & & \xrightarrow{\;\widetilde{\rho}^{-1} \circ \widetilde{\rho}\;} & & \mathcal{M} \boxtimes \overline{\mathcal{I}_0} \\
\downarrow{\scriptstyle Z_{[\kappa]}} & \searrow^{\widetilde{\rho}} & & \nearrow^{\widetilde{\rho}^{-1}} & \downarrow{\scriptstyle Z_{[\kappa]}} \\
\mathcal{M} \overset{\kappa}{\boxtimes} \overline{\mathcal{I}_0} & \xrightarrow[\rho]{} & \mathcal{M} & \xrightarrow[\rho^{-1}]{} & \mathcal{M} \overset{\kappa}{\boxtimes} \overline{\mathcal{I}_0}
\end{array} \tag{B.38}$$

of functors, where this time we set $\widetilde{\rho}(m \boxtimes \overline{a}) := m.(D_{\mathcal{A}} \otimes a^{[\kappa-2]})$. The functor $\widetilde{\rho}$ is κ+2-balanced:

$$\widetilde{\rho}(m.b \boxtimes \overline{a}) \cong m.(b \otimes D_{\mathcal{A}} \otimes a^{[\kappa-2}]) \cong m.(D_{\mathcal{A}} \otimes {}^{[4]}b \otimes a^{[\kappa-2]}) \cong \widetilde{\rho}(m \boxtimes \overline{a.{}^{[\kappa+2]}b}) \,. \tag{B.39}$$

The functor $\widetilde{\rho}$ also obeys $\widetilde{\rho}(m \boxtimes \overline{b.a}) \cong \widetilde{\rho}(m \boxtimes a).b^{[\kappa-2]}$ and thus is a left $\overline{\mathcal{A}}$-module functor $\widetilde{\rho}\colon \mathcal{M} \overset{\kappa}{\boxtimes} \overline{\mathcal{I}_0} \to {}^{[\kappa-2]}_{\;\;\overline{\mathcal{A}}}\mathcal{M}$. Hence by the universal property of the twisted center, ρ is a well defined module functor.
Further, we define $\widetilde{\rho}^{-1}(m) := m \boxtimes \overline{D_{\mathcal{A}}} \in \mathcal{M} \boxtimes \overline{\mathcal{I}_0}$ and set $\rho^{-1} := Z_{[\kappa]} \circ \widetilde{\rho}^{-1}$. This is indeed a module functor, as

$$\begin{aligned}
\rho^{-1}(m.x) \;&=\; \int_{a\in\mathcal{A}} m.x.a \boxtimes \overline{D_{\mathcal{A}} \otimes {}^{[\kappa+1]}a} \;\cong\; \int_{a\in\mathcal{A}} m.a \boxtimes \overline{D_{\mathcal{A}} \otimes {}^{[\kappa+1]}({}^{\vee}x \otimes a)} \\
&\cong\; \int_{a\in\mathcal{A}} m.a \boxtimes \overline{{}^{[\kappa-2]}x \otimes D_{\mathcal{A}} \otimes {}^{[\kappa+1]}a} \;=\; {}^{[\kappa-2]}x.\rho^{-1}(m) \,.
\end{aligned} \tag{B.40}$$

Analogously as in the proof of Lemma B.8 it then follows that ρ and ρ^{-1} furnish an equivalence of module categories, as required. Moreover, again analogously as above we see that ρ is given explicitly by

$$\begin{aligned}
\rho\colon \quad \mathcal{M} \overset{\kappa}{\boxtimes} \overline{\mathcal{I}_0} &\longrightarrow \mathcal{M} \\
m \boxtimes \overline{a} &\longmapsto \mathrm{Hom}_{\mathcal{A}}(a, \mathbf{1}) \otimes m \,.
\end{aligned} \tag{B.41}$$

The case of a left module follows directly by taking opposite categories. $\square$

C Construction of a parallelization

In this appendix we provide details of the construction of a parallelization Π for the collection of relative block functors for all fine refinements, as defined in Definition 5.16(ii).

C.1 Isomorphisms among block functors of fillable disks

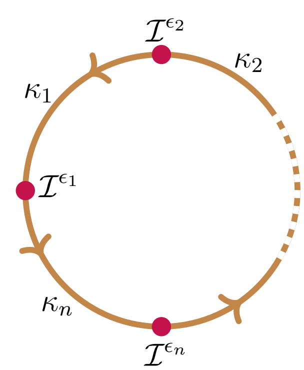
As a preparatory step we restrict our attention to a specific class of defect surfaces, namely *fillable disks*, and construct a distinguished isomorphism between the block functors for any two such disks of the same type. Recall from Definition 5.10 that a fillable disk of type $\mathbb{X}$ is a defect surface $\mathbb{D}_\mathbb{X}$ together with a set δ_{tr} of transparent defect lines and with a distinguished boundary segment ∂_{outer}, called the *outer boundary* of $\mathbb{D}_\mathbb{X}$. The removal of δ_{tr} from $\mathbb{D}_\mathbb{X}$ gives a defect surface for which every gluing circle and gluing interval except for ∂_{outer} is fillable by a disk in the sense of Definition 5.4, and the corresponding filling of $\mathbb{D}_\mathbb{X} \backslash \delta_{\text{tr}}$ yields a defect surface $\mathbb{X}$ with underlying surface being a disk and with $\partial\mathbb{X}$ containing at most one free boundary segment.

For a fillable disk $\mathbb{D} = \mathbb{D}_\mathbb{X}$ of arbitrary type $\mathbb{X}$, denote by $\mathbb{L} := \partial_{\text{glue}}\mathbb{D} \backslash \partial_{\text{fill}}\mathbb{D}$ the gluing part of the outer boundary of $\mathbb{D}$. The prescription (5.40) provides a distinguished object

$$\mho(\mathbb{L}) \in \mathrm{T}(\mathbb{L}) \,, \tag{C.1}$$

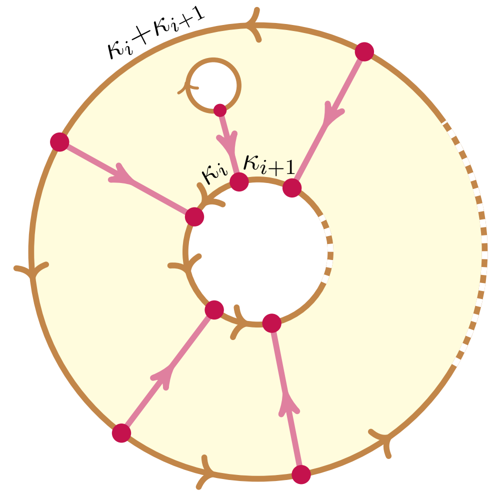
called the *silent* object for $\mathbb{L}$, in the gluing category for $\mathbb{L}$. We will use these silent objects to specify particular isomorphisms between functors associated to fillable disks. For the construction of these isomorphisms we restrict our attention temporarily to the situation that the outer boundary of $\mathbb{D}$ is a gluing circle $\mathbb{S}$ all of whose defect points are transparently labeled. The assumption that $\mathbb{D}$ is fillable means that $\mathbb{S}$ is fillable in the sense of Definition 5.4; also, each of its defect points is labeled either by $\mathcal{I}$ for one and the same finite tensor category $\mathcal{A}$ or by $\overline{\mathcal{A}}$. In the sequel we write $\mathcal{I}$ for $\mathcal{A}$ in order to remind us that it appears as a transparent label.

Denote by $\mathbb{S}_{n,\kappa}$ such a gluing circle with $n > 1$ defect points and n-tuple $\kappa = (\kappa_n, \kappa_{n-1}, \dots, \kappa_1)$ of framing indices (and with corresponding orientations $\epsilon_i \in \{1, -1\}$ of the defect points), i.e.

$\mathbb{S}_{n,\kappa} \;=$  , (C.2)

Further, denote by $\mathbb{S}_{n,\kappa}^{(i)}$, for $i \in \{1, 2, \dots, n\}$, the fillable circle that is obtained by removing the ith defect point from the circle $\mathbb{S}_{n,\kappa}$. Thus $\mathbb{S}_{n,\kappa}^{(i)}$ is a circle of type $\mathbb{S}_{n-1,\kappa'}$ with framing indices $\kappa' = (\kappa_n, \dots, \kappa_{i+2}, \kappa_{i+1}{+}\kappa_i, \kappa_{i-1}, \dots, \kappa_1)$. Then for any possible choice of n, κ and i we consider the two defect surfaces

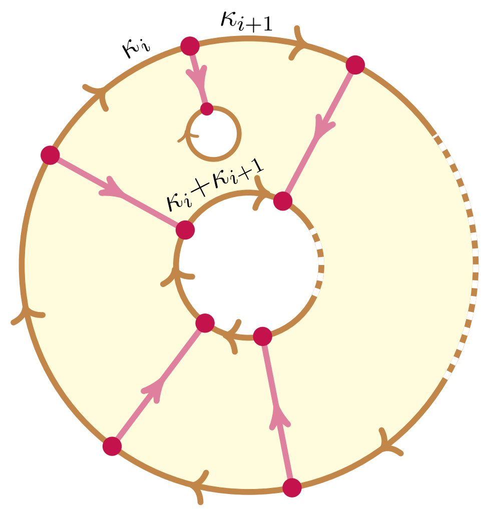
$$\Sigma^{(i)}_{n,\kappa} \;:=$$



(C.3)

and

$$\widetilde{\Sigma}^{(i)}_{n,\kappa} \;:=$$



(C.4)

The boundary of $\Sigma^{(i)}_{n,\kappa}$ and of $\widetilde{\Sigma}^{(i)}_{n,\kappa}$ is the union of the circles $\mathbb{S}_{n,\kappa}$ and $\mathbb{S}^{(i)}_{n,\kappa}$ (respectively their opposites) and of a *tadpole circle* $\mathbb{Q}_{\pm}$, i.e. a fillable circle with a single transparently labeled defect point, as indicated in (5.36). We regard these surfaces as bordisms

$$\Sigma^{(i)}_{n,\kappa}: \quad \mathbb{S}_{n,\kappa} \sqcup \mathbb{Q}_{+} \to \mathbb{S}^{(i)}_{n,\kappa} \qquad \text{and} \qquad \widetilde{\Sigma}^{(i)}_{n,\kappa}: \quad \mathbb{S}^{(i)}_{n,\kappa} \sqcup \mathbb{Q}_{-} \to \Sigma^{(i)}_{n,\kappa}, \tag{C.5}$$

respectively. We then consider the functors

$$\begin{aligned} G^{(i)}_{n,\kappa} &:= \mathrm{T}(\Sigma^{(i)}_{n,\kappa})(-\boxtimes \mho(\mathbb{Q}_{-\epsilon_i})): \quad \mathrm{T}(\mathbb{S}_{n,\kappa}) \to \mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa}) \qquad \text{and} \\ \widetilde{G}^{(i)}_{n,\kappa} &:= \mathrm{T}(\widetilde{\Sigma}^{(i)}_{n,\kappa})(-\boxtimes \mho(\mathbb{Q}_{\epsilon_i})): \quad \mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa}) \to \mathrm{T}(\mathbb{S}_{n,\kappa}), \end{aligned} \tag{C.6}$$

respectively, where $\mho(\mathbb{Q}_{\pm})$ are the silent objects (5.38) for the tadpole circles. These functors which may be viewed as relative block functors of the form described in (5.33) for two fine refinements $(\Sigma;\Sigma')$ for which Σ is a cylinder over a circle $\mathbb{S}_{n-1,\kappa'}$.

Let us describe the functor $G^{(i)}_{n,\kappa}$ in detail. First note that for $z_1^{\epsilon_1}\boxtimes\cdots\boxtimes z_n^{\epsilon_n}\in \mathrm{T}(\mathbb{S}_{n,\kappa})$ and $\epsilon_i=-1$ the object

$$\mathrm{Hom}_{\mathcal{A}}(z_i,\mathbf{1})\otimes z_1^{\epsilon_1}\boxtimes\cdots\widehat{z_i^{\epsilon_i}}\cdots\boxtimes z_n^{\epsilon_n}\ \in \mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa}) \tag{C.7}$$

(with the symbol $\widehat{z}$ indicating that the factor z is to be removed from the expression) comes canonically with the following balancings: For $a\in\mathcal{A}$ the balancing between z_{i-1} and z_{i+1} is, in case the orientations are as in the picture (C.3),

$$\begin{aligned}\mathrm{Hom}_{\mathcal{A}}(z_i,\mathbf{1})\otimes z_{i-1}.a\boxtimes\overline{z_{i+1}} &= \mathrm{Hom}_{\mathcal{A}}(z_i,\mathbf{1})\otimes(z_{i-1}\otimes a)\boxtimes\overline{z_{i+1}}\\ &\cong \mathrm{Hom}_{\mathcal{A}}(z_i\otimes{}^{[\kappa_i]}a,\mathbf{1})\otimes z_{i-1}\boxtimes\overline{z_{i+1}}\cong \mathrm{Hom}_{\mathcal{A}}(z_i,{}^{[\kappa_i-1]}a)\otimes z_{i-1}\boxtimes\overline{z_{i+1}}\\ &\cong \mathrm{Hom}_{\mathcal{A}}({}^{[\kappa_i-2]}a\otimes z_i,\mathbf{1})\otimes z_{i-1}\boxtimes\overline{z_{i+1}}\cong \mathrm{Hom}_{\mathcal{A}}(z_i,\mathbf{1})\otimes z_{i-1}\boxtimes\overline{z_{i+1}.{}^{[\kappa_i+\kappa_{i+1}]}a}\,,\end{aligned} \tag{C.8}$$

and similarly for other combinations of orientations. In these expressions we display only the relevant part of the object, and we use the balancings of the object $z_1^{\epsilon_1}\boxtimes\cdots\boxtimes z_n^{\epsilon_n}$ in steps two and five, and the definition of the module structures on $\mathcal{I}$ in steps one and five.

Proposition C.1.

(i) The relative block functor $G^{(i)}_{n,\kappa}$ assigned to the defect surface (C.3) is an equivalence

$$G^{(i)}_{n,\kappa}:\quad \mathrm{T}(\mathbb{S}_{n,\kappa})\xrightarrow{\simeq}\mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa})\,. \tag{C.9}$$

For $\epsilon_i=-1$ this functor is given explicitly by

$$G^{(i)}_{n,\kappa}(z_1^{\epsilon_1}\boxtimes\cdots\boxtimes z_n^{\epsilon_n})=\mathrm{Hom}_{\mathcal{A}}(z_i,\mathbf{1})\otimes z_1^{\epsilon_1}\boxtimes\cdots\widehat{z_i^{\epsilon_i}}\cdots\boxtimes z_n^{\epsilon_n} \tag{C.10}$$

for $z_1^{\epsilon_1}\boxtimes\cdots\boxtimes z_n^{\epsilon_n}\in\mathrm{T}(\mathbb{S}_{n,\kappa})$ (with balancings as described in (C.8)), while for $\epsilon_i=1$ it is

$$G^{(i)}_{n,\kappa}(z_1^{\epsilon_1}\boxtimes\cdots\boxtimes z_n^{\epsilon_n})=\mathrm{Hom}_{\mathcal{A}}(D_{\mathcal{A}},z_i)\otimes z_1^{\epsilon_1}\boxtimes\cdots\widehat{z_i^{\epsilon_i}}\cdots\boxtimes z_n^{\epsilon_n}. \tag{C.11}$$

(ii) Similarly, the functor $\widetilde{G}^{(i)}_{n,\kappa}$ is an equivalence as well, and for $\epsilon_i=1$ it is given by

$$\widetilde{G}^{(i)}_{n,\kappa}(z_1^{\epsilon_1}\boxtimes\cdots\boxtimes z_n^{\epsilon_n})=z_1\boxtimes\cdots Z_{\kappa_i}(z_i^{\epsilon_1}\boxtimes\mathbf{1})\boxtimes\cdots\boxtimes z_n^{\epsilon_n}, \tag{C.12}$$

while for $\epsilon_i=-1$ it is

$$\widetilde{G}^{(i)}_{n,\kappa}(z_1^{\epsilon_1}\boxtimes\cdots\boxtimes z_n^{\epsilon_n})=z_1\boxtimes\cdots Z_{\kappa_i}(z_i^{\epsilon_1}\boxtimes\overline{D_{\mathcal{A}}})\boxtimes\cdots\boxtimes z_n^{\epsilon_n}. \tag{C.13}$$

Proof. (i) Clearly the object (C.7) with balancings (C.8) is an object in the gluing category $\mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa})$, and prescribing $G^{(i)}_{n,\kappa}(z_1^{\epsilon_1}\boxtimes\cdots\boxtimes z_n^{\epsilon_n})\in\mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa})$ as in (C.10) defines a functor $H^{(i)}_{n,\kappa}\colon\mathrm{T}(\mathbb{S}_{n,\kappa})\to\mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa})$. That $H^{(i)}_{n,\kappa}$ indeed coincides with $G^{(i)}_{n,\kappa}$ as defined in (C.6) is seen as follows. Recall that we denote by $U\colon\mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa})\to\mathrm{U}(\mathbb{S}^{(i)}_{n,\kappa})=(\mathcal{I}^{\epsilon_1})_1\boxtimes(\mathcal{I}^{\epsilon_2})_2\boxtimes\cdots\widehat{(\mathcal{I}^{\epsilon_i})_i}\cdots\boxtimes(\mathcal{I}^{\epsilon_n})_n$ the forgetful functor to the category that is obtained from the gluing category for $\mathbb{S}^{(i)}_{n,\kappa}$ by ignoring the balancings. With the help of U the pre-block functor for $\Sigma^{(i)}_{n,\kappa}$, when evaluated on the silent object of $\mathbb{Q}_+$, can be written as $\mathrm{T}^{\mathrm{pre}}(\Sigma^{(i)}_{n,\kappa})(z\boxtimes\mho(\mathbb{Q}_+))=\mathrm{Hom}_{\mathrm{U}(\mathbb{S}^{(i)}_{n,\kappa})}(U(-),UH^{(i)}_{n,\kappa}(z))$ for $z\in\mathrm{T}(\mathbb{S}_{n,\kappa})$. Moreover, it follows directly from the definition of the balancing of $H^{(i)}_{n,\kappa}(z)$ that the forgetful functor provides the equalizer $\mathrm{Hom}_{\mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa})}(-,H^{(i)}_{n,\kappa}(z))\to\mathrm{Hom}_{\mathrm{U}(\mathbb{S}^{(i)}_{n,\kappa})}(U(-),UG^{(i)}_{n,\kappa}(z))$

for the parallel transport equations on $\Sigma^{(i)}_{n,\kappa}$. As a consequence we also have $G^{(i)}_{n,\kappa}(z) = \mathrm{Hom}_{\mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa})}(-, H^{(i)}_{n,\kappa}(z))$, and thus $G^{(i)}_{n,\kappa} = H^{(i)}_{n,\kappa}$, as claimed.
Further, the explicit form of the functor ρ in (B.41) (applied here to the case that $\mathcal{M} = \mathcal{I}$) tells us that $G^{(i)}_{n,\kappa}$ can also be seen as coming from the equivalence $\mathcal{M} \overset{\kappa}{\boxtimes} \overline{\mathcal{I}} \to \mathcal{M}$, with $\mathcal{M}$ the bimodule labeling the defect point that is adjacent to the one labeled by $\mathcal{I}$ on $\mathbb{S}_{n,\kappa}$; thus in particular $G^{(i)}_{n,\kappa}$ is an equivalence.
The case $\epsilon_i = 1$ is treated analogously.
(ii) For analyzing the functor $\widetilde{G}^{(i)}_{n,\kappa}$ in the case $\epsilon_i = 1$, we consider an object of the form $z = z_1^{\epsilon_1} \boxtimes \cdots z_{i-1}^{\epsilon_{i-1}} \boxtimes z_{i+1}^{\epsilon_{i+1}} \cdots \boxtimes z_n^{\epsilon_n} \in \mathrm{T}(\mathbb{S}^{(i)}_{n,\kappa})$. It is straightforward to see that via the balancing of the comonad $Z_{[\kappa]}$ the object $\widetilde{G}^{(i)}_{n,\kappa}(z)$ has a canonical structure of an object in $\mathrm{T}(\mathbb{S}_{n,\kappa})$. The pre-block functor for $\widetilde{\Sigma}^{(i)}_{n,\kappa}$ takes the values

$$\mathrm{T}^{\mathrm{pre}}(\widetilde{\Sigma}^{(i)}_{n,\kappa})(x, z \boxtimes \mho(\mathbb{Q}_-)) = \mathrm{Hom}_{\mathrm{U}(\mathbb{S}_{n,\kappa})}(U(x), z_1^{\epsilon_1} \boxtimes \cdots \boxtimes z_{i-1}^{\epsilon_{i-1}} \boxtimes \mathbf{1} \boxtimes z_{i+1}^{\epsilon_{i+1}} \boxtimes \cdots \boxtimes z_n^{\epsilon_n}) \qquad \text{(C.14)}$$

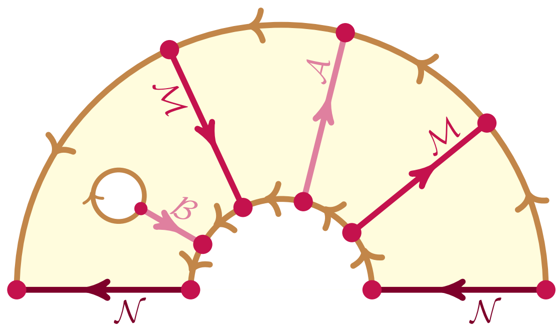
on objects $x \in \mathrm{T}(\mathbb{S}_{n,\kappa})$, where $\mathrm{U}(\mathbb{S}_{n,\kappa}) = (\mathcal{I}^{\epsilon_1})_1 \boxtimes (\mathcal{I}^{\epsilon_2})_2 \boxtimes \cdots \boxtimes (\mathcal{I}^{\epsilon_n})_n$. Consider now the category $\widetilde{\mathrm{U}}(\mathbb{S}_{n,\kappa}) := (\mathcal{I}^{\epsilon_1})_1 \boxtimes \cdots \boxtimes \big((\mathcal{I}^{\epsilon_{i-1}})_{i-1} \overset{\kappa}{\boxtimes} (\mathcal{I}^{\epsilon_i})_i\big) \boxtimes (\mathcal{I}^{\epsilon_{i+1}})_{i+1} \boxtimes \cdots \boxtimes (\mathcal{I}^{\epsilon_n})_n$ with corresponding forgetful functor $\widetilde{U}\colon \mathrm{T}(\mathbb{S}_{n,\kappa}) \to \widetilde{\mathrm{U}}(\mathbb{S}_{n,\kappa})$. Using co-induction gives an isomorphism

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{U}(\mathbb{S}_{n,\kappa})}(U(x), z_1^{\epsilon_1} \boxtimes \cdots \boxtimes z_{i-1}^{\epsilon_{i-1}} \boxtimes \mathbf{1} \boxtimes \cdots z_n^{\epsilon_n}) \\ &\qquad \cong \mathrm{Hom}_{\widetilde{\mathrm{U}}(\mathbb{S}_{n,\kappa})}(\widetilde{U}(x), z_1^{\epsilon_1} \boxtimes \cdots \boxtimes Z_{[\kappa_i]}(z_{i-1}^{\epsilon_{i-1}} \boxtimes \mathbf{1}) \boxtimes \cdots \boxtimes z_n^{\epsilon_n}) \,. \end{aligned} \qquad \text{(C.15)}$$

Next note that the forgetful functor $\widetilde{U}$ may be regarded as a composition of forgetful functors $\widetilde{U}_{j,j+1}$, each of which is applied to the twisted center associated with two adjacent defect points $(j, j+1)$ on $\mathbb{S}^{(i)}_{n,\kappa}$; thus for every 2-patch of $\widetilde{\Sigma}^{(i)}_{n,\kappa}$ there is a corresponding pair of forgetful functors in the Hom functor on the right hand side of (C.15). It therefore follows in the same way as in the case of $G^{(i)}_{n,\kappa}$ that the block functor is given by $\mathrm{T}(\widetilde{\Sigma}^{(i)}_{n,\kappa})(x, z \boxtimes \mho(\mathbb{Q}_-)) = \mathrm{Hom}_{\mathrm{T}(\mathbb{S}_{n,\kappa})}(x, \widetilde{G}^{(i)}_{n,\kappa}(z))$, thus proving the explicit form of $\widetilde{G}^{(i)}_{n,\kappa}$ given in (C.12). Moreover, again as in the case of $G^{(i)}_{n,\kappa}(z)$, we see from the expression (B.32) for the functor ρ^{-1} that $\widetilde{G}^{(i)}_{n,\kappa}$ is an equivalence.
The case $\epsilon_i = -1$ is treated analogously. □

The boundary circles $\mathbb{S}_{n,\kappa}$ and $\mathbb{S}^{(i)}_{n,\kappa}$ of the defect surfaces $\Sigma^{(i)}_{n,\kappa}$ and $\widetilde{\Sigma}^{(i)}_{n,\kappa}$, for which all defect points are transparently labeled, can play the role of the outer boundary of a *transparent disk*, i.e. a fillable disk of the type shown in (5.23). We now allow for general fillable circles as well as for fillable intervals, which can play the role of the outer boundary of a fillable disk of arbitrary type $\mathbb{X}$, like e.g. the one shown in 5.22. There are then obvious analogues $\Sigma^{(i)}_{\mathbb{X}}$ and $\widetilde{\Sigma}^{(i)}_{\mathbb{X}}$ of the surfaces $\Sigma^{(i)}_{n,\kappa}$ and $\widetilde{\Sigma}^{(i)}_{n,\kappa}$. For instance, for $\mathbb{X}$ as in (5.22), an example of a defect surface

$\Sigma_{\mathbb{X}} \colon \mathbb{L}_{\mathbb{X}} \to \mathbb{L}_{\mathbb{X}}^{(i)}$ is given by



(C.16)

where $\mathbb{L}_{\mathbb{X}}$ appears as the inner and $\mathbb{L}_{\mathbb{X}}^{(i)}$ as the outer boundary interval.

The obvious analogue of Proposition C.1 holds in this generic case, too. That is, the functors

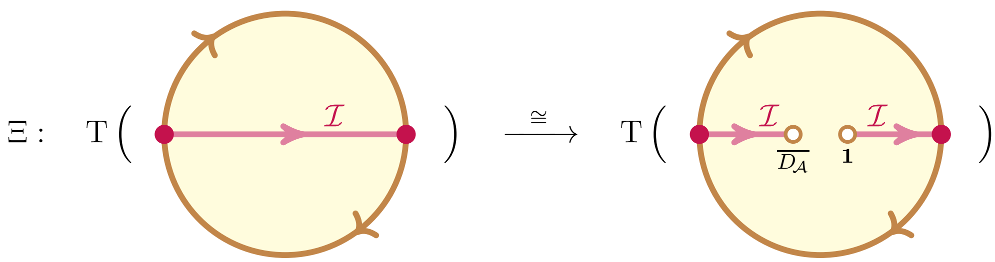
$$G_{\mathbb{X}}^{(i)} := \mathrm{T}(\Sigma_{\mathbb{X}}^{(i)})(- \boxtimes \mho(\mathbb{Q}_{-\epsilon_i})) \qquad \text{and} \qquad \widetilde{G}_{\mathbb{X}}^{(i)} := \mathrm{T}(\widetilde{\Sigma}_{\mathbb{X}}^{(i)})(- \boxtimes \mho(\mathbb{Q}_{\epsilon_i})) \tag{C.17}$$

are equivalences and have similar expressions as in the transparent case. For example, for an object $z = \overline{x_{\mathcal{N}}} \boxtimes \overline{a} \boxtimes \overline{x_{\mathcal{M}}} \boxtimes b \boxtimes y_{\mathcal{M}} \boxtimes y_{\mathcal{N}}$ in the gluing category $\mathrm{T}(\mathbb{S}_{\mathbb{X}}) = \overline{\mathcal{N}} \boxtimes \overline{\mathcal{A}} \boxtimes \overline{\mathcal{M}} \boxtimes \mathcal{A} \boxtimes \mathcal{M} \boxtimes \mathcal{N}$ for the inner boundary interval of the surface (C.16) one has

$$G_{\mathbb{X}}^{(2)}(z) = \mathrm{Hom}_{\mathcal{A}}(a, \mathbf{1}) \otimes \overline{x_{\mathcal{N}}} \boxtimes \overline{x_{\mathcal{M}}} \boxtimes b \boxtimes y_{\mathcal{M}} \boxtimes y_{\mathcal{N}} \,. \tag{C.18}$$

We now establish further properties of the functors (C.17). Recall the convention (5.94) for block functors evaluated at silent objects. We use this convention in the following statement:

Lemma C.2. There is a canonical isomorphism

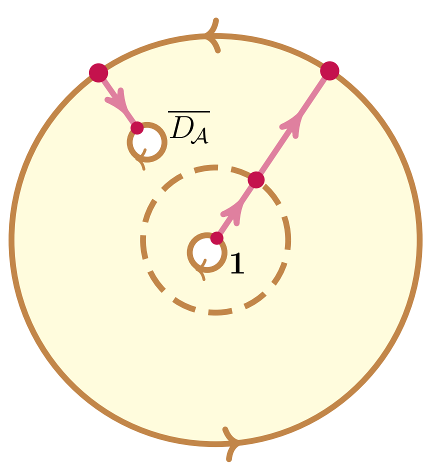


(C.19)

of functors.

Proof. Let us factorize the disk $\mathbb{D}$ that appears on the right hand side of (C.19) in the way indicated by the dashed circle in

$\mathbb{D}$ =



(C.20)

With the help of the description of the functors $G^{(i)}$ and $\widetilde{G}^{(i)}$ in Proposition C.1 we see that $\mathrm{T}(\mathbb{D}) = Z_{[1]}(\mathbf{1} \boxtimes \overline{D_{\mathcal{A}}}) = \int_{a\in\mathcal{A}} a \boxtimes \overline{D_{\mathcal{A}}.^{[2]}a} \cong \int^{a\in\mathcal{A}} a \boxtimes \overline{a} \in \mathrm{T}(\partial\mathbb{D})$, using the canonical isomorphism (3.44) of objects in $\mathrm{T}(\partial\mathbb{D})$. Noticing that $\int^a a \boxtimes \overline{a}$ is the value of the block functor on the left hand side of (C.19) then establishes the isomorphism Ξ. □

The so obtained isomorphism is a universal morphism in the following sense. Denote by $\mathbb{D}_1$ and $\mathbb{D}_2$ the defect surfaces on the left and right hand sides of (C.19), respectively, and by $\mathbb{S}$ their common gluing boundary, with gluing category $\mathrm{T}(\mathbb{S}) = \overline{\mathcal{A}}\overset{-1}{\boxtimes}\mathcal{A}\overset{-1}{\boxtimes}$. Then the pre-block functors are given by

$$\begin{aligned} \mathrm{T}^{\mathrm{pre}}(\mathbb{D}_1)(G) &= \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(G_1, a) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(a, G_2) \qquad \text{and} \\ \mathrm{T}^{\mathrm{pre}}(\mathbb{D}_2)(G) &= \mathrm{Hom}_{\mathcal{A}}(G_2, \mathbf{1}) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(D_{\mathcal{A}}, G_{\mathbf{1}})\,, \end{aligned} \tag{C.21}$$

respectively, for $G = \overline{G_1} \boxtimes G_2 \in \mathrm{T}(\mathbb{S})$. Using first the isomorphism between coend and end that follows from the isomorphism (3.45), and then the dinatural transformation of the end, we obtain a canonical morphism

$$\begin{aligned} \mathrm{T}^{\mathrm{pre}}(\mathbb{D}_1)(G) \;&=\; \int^{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(G_1, a) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(a, G_2) \\ &\cong\; \int_{a\in\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(G_1, a^{\vee\vee}) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(D_{\mathcal{A}} \otimes a, G_2) \\ &\qquad \longrightarrow \mathrm{Hom}_{\mathcal{A}}(G_2, \mathbf{1}) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{A}}(D_{\mathcal{A}}, G_1) \;=\; \mathrm{T}^{\mathrm{pre}}(\mathbb{D}_2)(G)\,. \end{aligned} \tag{C.22}$$

Lemma C.3.

1. The canonical morphism (C.22) from $\mathrm{T}^{\mathrm{pre}}(\mathbb{D}_1)$ to $\mathrm{T}^{\mathrm{pre}}(\mathbb{D}_2)$ is compatible with all parallel transport operations on $\mathbb{D}_1$ and $\mathbb{D}_2$, i.e. it commutes with $\mathrm{hol}_{a,x}$ for all objects $a \in \mathcal{A}$ and starting points x.
2. The morphism (C.22) induces a morphism between the parallel transport equalizers, i.e. between the corresponding block functors. The so obtained morphism between the block functors is the isomorphism (C.19) in Lemma C.2.

Proof. (i) The compatibility with the parallel transport operations follows by direct computation.
(ii) We obtain a morphism between the block functors by the universal property of the equalizer. It is straightforward to check that applying the forgetful functor from blocks to pre-blocks to the morphism (C.19) reproduces the morphism (C.22). □

As a consequence of Lemma C.2 we have

Lemma C.4. The isomorphism Ξ in (C.19) provides a distinguished adjoint equivalence between the functors $G^{(i)}_{\mathbb{X}}$ and $\widetilde{G}^{(i)}_{\mathbb{X}}$ for any type $\mathbb{X}$.

Proof. We factorize the bordism $\widetilde{\Sigma}^{(i)}_{\mathbb{X}} \circ \Sigma^{(i)}_{\mathbb{X}}$ in such a way that one of the factors is the disk $\mathbb{D}$ on the right hand side of (C.19). The isomorphism Ξ can then be used to define a natural isomorphism $\widetilde{G}^{(i)}_{\mathbb{X}} \circ G^{(i)}_{\mathbb{X}} \cong \mathrm{id}_{\mathbb{S}_{\mathbb{X}}}$. By Lemma B.8, the two functors are inverse equivalences; as a consequence there is a unique way to define the isomorphism $G^{(i)}_{\mathbb{X}} \circ \widetilde{G}^{(i)}_{\mathbb{X}} \xrightarrow{\cong} \mathrm{id}_{\mathbb{S}^{(i)}_{\mathbb{X}}}$ in such a way that the equivalence is an adjoint equivalence. □

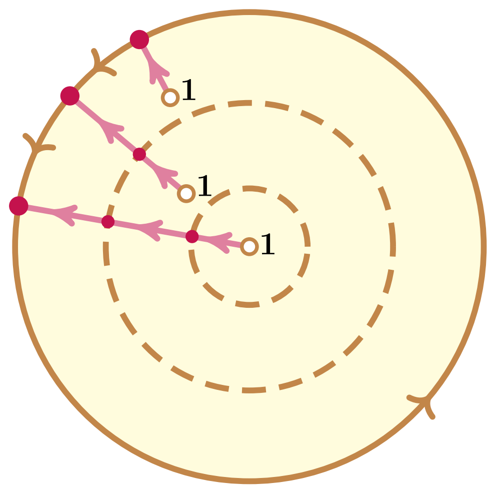
Next we note that by successively applying the functors $\widetilde{G}^{(i)}_{n,\kappa}$ with all possible values of i to the transparently labeled gluing circle $\mathbb{S}_{n,\kappa}$ (C.2) we obtain a fillable disk $\mathbb{D}_{n,\kappa}$ each of whose inner boundaries is a tadpole circle. This allows for the following description of the silent object $\mho(\mathbb{S}_{n,\kappa})$, as defined according to (5.40):

Lemma C.5. The silent object $\mho_{n,\kappa} := \mho(\mathbb{S}_{n,\kappa})$ for any transparent gluing circle $\mathbb{S}_{n,\kappa}$ can be recovered from the functors $\widetilde{G}^{(i_\ell)}_{\ell,\kappa}$ with $\ell = 2, 3, \dots, n$ and with $(i_1, i_2, \dots, i_n)$ any permutation of $(1, 2, \dots, n)$ as follows (for brevity we abuse notation by writing the same generic label κ for all the tuples of framing indices involved): there is a canonical isomorphism $\rho\colon \mho_{n,\kappa} \xrightarrow{\cong} \widetilde{G}(\mho(\mathbb{Q}_{\epsilon_{i_1}}))$ with $\widetilde{G}$ the composite

$$\mathrm{T}(\mathbb{Q}_{\epsilon_{i_1}}) \xrightarrow{\widetilde{G}^{(i_2)}_{2,\kappa}} \mathrm{T}(\mathbb{S}_{2,\kappa}) \xrightarrow{\widetilde{G}^{(i_3)}_{3,\kappa}} \mathrm{T}(\mathbb{S}_{3,\kappa}) \xrightarrow{\widetilde{G}^{(i_4)}_{4,\kappa}} \cdots\cdots \xrightarrow{\widetilde{G}^{(i_n)}_{n,\kappa}} \mathrm{T}(\mathbb{S}_{n,\kappa})\,. \tag{C.23}$$

An analogous statement holds for the silent object for the outer boundary of any fillable disk of arbitrary type $\mathbb{X}$.

Proof. This statement follows directly by applying the block functor to a situation involving consecutive gluings each of which involves a single tadpole circle, as indicated in the following picture



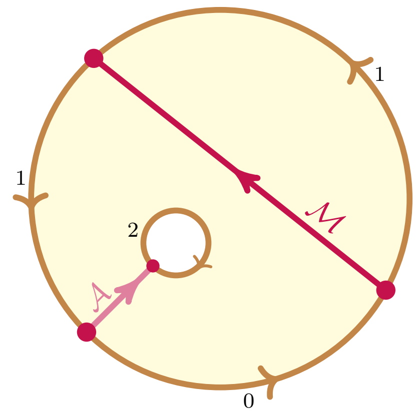
(C.24)

(in the situation shown we have $n = 3$ and $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$). □

It follows that explicit expressions for the silent object $\mho_{\mathbb{D}}$ for the outer boundary $\partial\mathbb{D}$ of a fillable disk $\mathbb{D}$ of type $\mathbb{X}$ can be obtained by the following procedure: For each defect line δ_i in $\mathbb{X}$ take a pair of variables $(m_i, \overline{m_i})$ in the categories $\mathcal{M}_i$ and $\overline{\mathcal{M}}_i$ labeling the two defect points on $\partial\mathbb{D}$ at the ends of the defect, together with the relevant silent objects $\mathbf{1}_{\mathcal{A}_j}$ and $\overline{D_{\mathcal{A}_j}}$, respectively, for the tadpole circles in $\mathbb{D}$. Build the Deligne product of these objects and take the coend over the variables m_i. Finally apply for every 2-patch of $\mathbb{D}$, having n gluing segments on $\partial\mathbb{D}$ with framings $\{\kappa_j\}$, the corresponding comonads $\mathrm{T}_{[\kappa_j]}$ for $n-1$ of the gluing segments (up to canonical isomorphism it does not matter which one of the n gluing segments is omitted). The following example illustrates this procedure:

Example C.6. Consider the fillable disk

$$\mathbb{D} \;=$$



(C.25)

Using Equation (3.45) and Lemma 3.17 we obtain

$$\begin{aligned}\mho_{\mathbb{D}} &= \int^{m\in\mathcal{M}} \mathrm{T}_{[0]}(\overline{m}\boxtimes\overline{D_{\mathcal{A}}})\boxtimes m = \int^{m\in\mathcal{M}}\!\!\int_{a\in\mathcal{A}} \overline{a^\vee.m}\boxtimes\overline{D.{}^{[3]}a}\boxtimes m \\ &\cong \int^{m\in\mathcal{M}}\!\!\int^{a\in\mathcal{A}} \overline{a^\vee.m}\boxtimes\overline{a}\boxtimes m \cong \int^{m\in\mathcal{M}}\!\!\int^{a\in\mathcal{A}} \overline{m}\boxtimes\overline{a}\boxtimes a.m\,.\end{aligned} \tag{C.26}$$

We have actually already encountered the defect one-manifold that constitutes the outer boundary of the disk $\mathbb{D}$: it is the defect circle $\mathbb{I}^{\swarrow}_{\kappa}(\mathcal{M})$ in (5.14) with framing index $\kappa=0$. Similarly we obtain the following list of defect one-manifolds and silent objects for all other transparent disks with outer boundaries given by one of the circles (5.14):

$$\begin{aligned}
&\mathbb{I}^{\nearrow}_{\kappa}(\mathcal{M}): && \mho = \int^{m\in\mathcal{M}}\!\!\int_{a\in\mathcal{A}} \overline{m}\boxtimes m\,.\,a^{[\kappa-1]}\boxtimes a = \int^{m\in\mathcal{M}}\!\!\int_{a\in\mathcal{A}} \overline{m}\boxtimes m\,.\,a\boxtimes{}^{[\kappa-1]}a\,,\\
&\mathbb{I}^{\swarrow}_{-\kappa}(\mathcal{M}): && \mho = \int^{m\in\mathcal{M}}\!\!\int^{a\in\mathcal{A}} \overline{m}\boxtimes\overline{a}\boxtimes a^{[\kappa]}.\,m\,,\\
&\mathbb{I}^{\nwarrow}_{\kappa}(\mathcal{M}): && \mho = \int^{m\in\mathcal{M}}\!\!\int_{a\in\mathcal{A}} \overline{m}\boxtimes a\boxtimes{}^{[\kappa-1]}a\,.\,m\,,\\
&\mathbb{I}^{\searrow}_{-\kappa}(\mathcal{M}): && \mho = \int^{m\in\mathcal{M}}\!\!\int^{a\in\mathcal{A}} \overline{a}\boxtimes\overline{m}\boxtimes m\,.{}^{[\kappa]}a\,.
\end{aligned} \tag{C.27}$$

Next we show

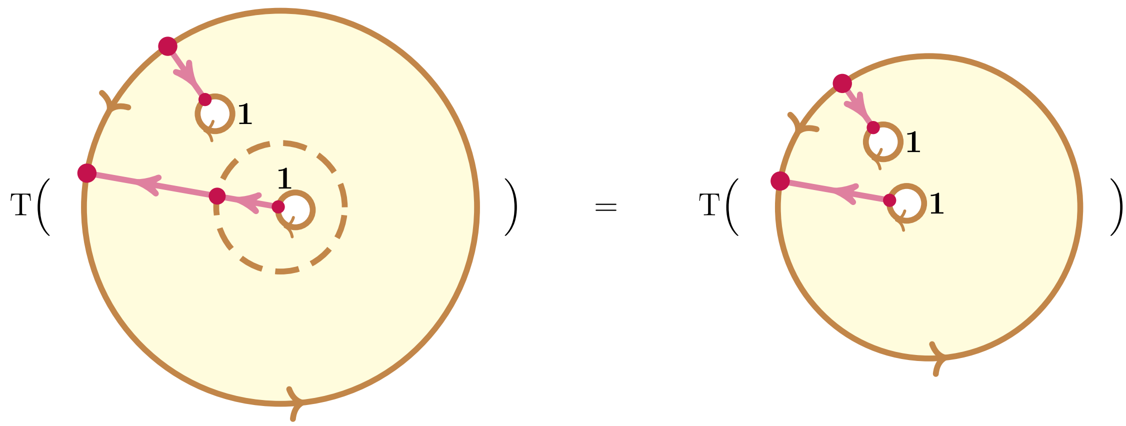
Lemma C.7. The functors $G^{(i)}\equiv G^{(i)}_{\mathbb{X}}\colon \mathrm{T}(\mathbb{L}_{\mathbb{X}})\to\mathrm{T}(\mathbb{L}^{(i)}_{\mathbb{X}})$ for fixed type $\mathbb{X}$ and different values of i commute up to canonical natural isomorphism, i.e. for any pair i,j with $i\neq j$ there is a canonical isomorphism $\gamma^{(i,j)}\colon G^{(i,j)}\circ G^{(i)}\cong G^{(j,i)}\circ G^{(j)}$ (with obvious notation).
Moreover, these isomorphisms are compatible with the silent objects in the following sense: For every i there is a canonical isomorphism $\rho^{(i)}\colon G^{(i)}_{\mathbb{X}}(\mho(\mathbb{L}_{\mathbb{X}}))\xrightarrow{\cong}\mho^{(i)}:=\mho(\mathbb{L}^{(i)}_{\mathbb{X}})$, and analogous isomorphisms relating the silent objects for the gluing boundaries $\mathbb{L}^{(i)}_{\mathbb{X}}$ and $\mathbb{L}^{(i,j)}_{\mathbb{X}}$ etc., such that

the diagram

$$\begin{array}{ccccc} G^{(i,j)}\circ G^{(i)}(\mho(\mathbb{L}_{\mathbb{X}})) & \xrightarrow{G^{(i,j)}(\rho^{(i)})} & G^{(i,j)}(\mho^{(i)}) & \searrow^{\rho^{(i,j)}} & \\ \downarrow{\scriptstyle \gamma^{(i,j)}} & & & & \mho^{(i,j)} \\ G^{(j,i)}\circ G^{(j)}(\mho(\mathbb{L}_{\mathbb{X}})) & \xrightarrow{G^{(j,i)}(\rho^{(j)})} & G^{(j,i)}(\mho^{(j)}) & \nearrow_{\rho^{(j,i)}} & \end{array} \tag{C.28}$$

commutes.

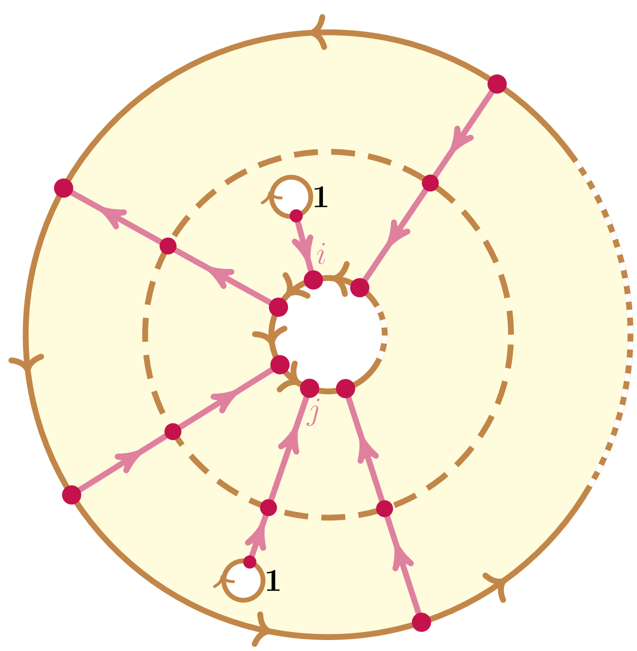
Proof. That the functors $\widetilde{G}^{(i)}_{n,\kappa}$ respect the silent objects is easily seen graphically. In the transparent case, the relevant situation is



(C.29)

The case of generic type $\mathbb{X}$ is analogous.
The isomorphism $\gamma^{(i,j)}$ and the commutativity of (C.28) are seen graphically, by comparing the following two situations, for which we clearly have $\mathrm{T}(\Sigma^{(ij)}) = \mathrm{T}(\Sigma^{(ji)})$:

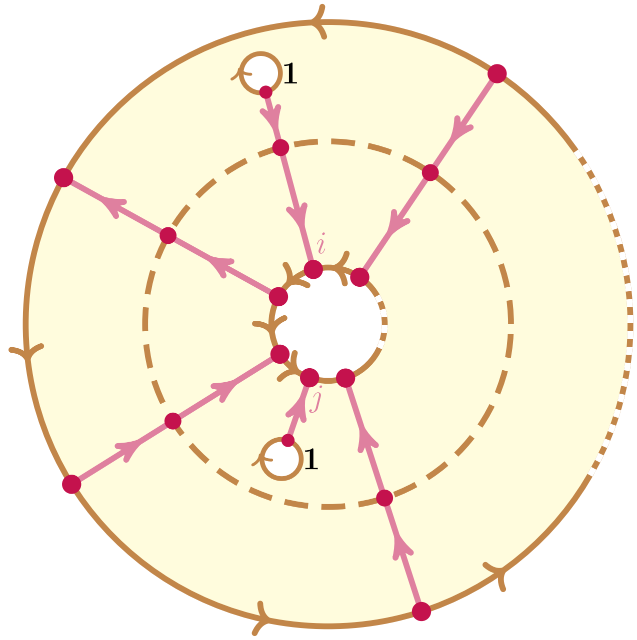
$\Sigma^{(ij)} :=$



(C.30)

and

$$\Sigma^{(ji)} \;:=$$



(C.31)

In these pictures, all but the most relevant labels are omitted, while also the circle along which the two bordisms are glued (including its defect points) is indicated as a dashed circle. □

C.2 Changing refinements

We now show that to a change of refinement from $(\Sigma;\Sigma_{\text{ref}})$ to $(\Sigma;\Sigma'_{\text{ref}})$ there is associated a canonical isomorphism between the respective relative block functors $\widehat{\mathrm{T}}(\Sigma;\Sigma_{\text{ref}})$ and $\widehat{\mathrm{T}}(\Sigma;\Sigma'_{\text{ref}})$. This is achieved in two steps: first we consider refinements of fillable disks, and afterwards refinements of arbitrary defect surfaces. The following terminology will be convenient:

Definition C.8. By a *fillable-disk replacement* $\Phi_{\mathbb{D},\mathbb{D}'}$ from $\mathbb{D}$ to $\mathbb{D}'$ we mean the operation of replacing in a defect surface Σ a fillable disk $\mathbb{D}\subset\Sigma$ of some type $\mathbb{X}$ by a fillable disk $\mathbb{D}'$ of the same type with the same outer boundary.

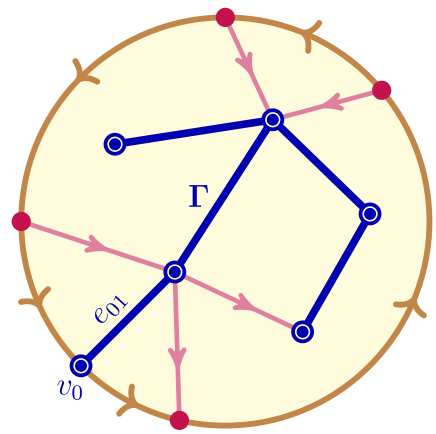
Owing to the factorization result for fine defect surfaces in Theorem 5.2, such a manipulation is completely under control by canonical isomorphisms.

We start by recalling that for each fillable disk $\mathbb{D}=\mathbb{D}_{\mathbb{X}}$ there exists a disk $\mathbb{D}^{\text{tad}}=\mathbb{D}^{\text{tad}}_{\mathbb{X}}$ with the same outer boundary as $\mathbb{D}$ and with all inner boundaries being tadpole circles (compare the example shown in (5.39)). We abbreviate by $\mathbb{L}_{\mathbb{D}}$ the gluing part of the outer boundary of $\mathbb{D}$.

We want to construct an isomorphism

$$\varphi_{\mathbb{D}}(\Gamma): \quad \mathrm{T}(\mathbb{D})(\mho(\mathbb{D})) \xrightarrow{\;\cong\;} \mathrm{T}(\mathbb{D}^{\text{tad}})(\mho(\mathbb{D}^{\text{tad}})) \tag{C.32}$$

of functors from $\mathbb{T}(\mathbb{L}_{\mathbb{D}})$ to vect. The construction given below will a priori depend on a combinatorial datum Γ that is defined as follows: Considering the inner boundary circles of $\mathbb{D}$ as (fattened) vertices, the set of inner boundary circles together with the defect lines of $\mathbb{D}$ and their end points on the outer boundary circle or on generic defects of $\mathbb{D}$ form a graph $\Gamma_{\mathbb{D}}$. Without loss of generality we assume that this graph is connected (otherwise the arguments below are to be applied to every connected component separately, and the order in which this is done is irrelevant). Then we select a subgraph $\Gamma \subset \Gamma_{\mathbb{D}}$ that is a *spanning tree* in $\Gamma_{\mathbb{D}}$, i.e. a rooted tree Γ with a minimal number of edges such that every vertex of $\Gamma_{\mathbb{D}}$ is met by Γ. (It is well known that a spanning tree exists for every graph.) We can also take the root v_0 of Γ to be lie on the outer boundary $\mathbb{L}_{\mathbb{D}}$. As an example, the following picture shows such a spanning tree Γ for the transparent disk $\mathbb{D}_{\text{tr}}$ shown in (5.23):



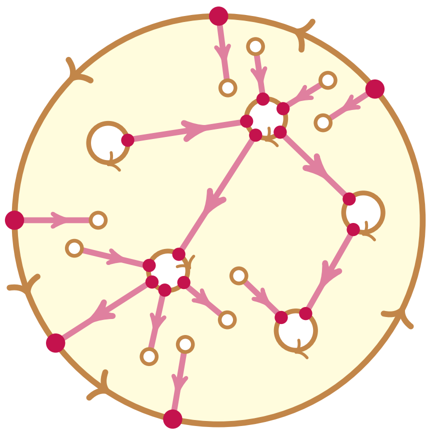
(C.33)

Here for clarity the vertices of Γ are drawn as encircled points, and also the remaining defect lines that do not give rise to edges of Γ are indicated.

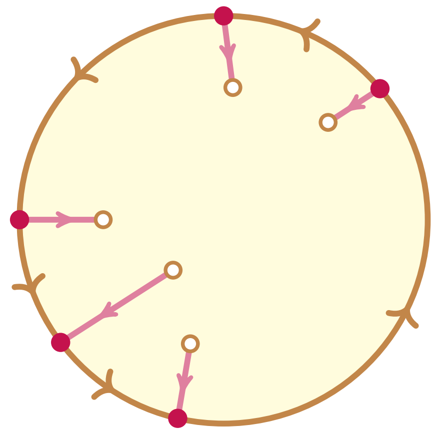
By the *length* of a path in a graph we mean the number of its edges. Then the *depth* of a vertex v of Γ is defined as the length of the (unique) path from v to the root of Γ; in particular the root has depth 0.

For any choice of spanning tree $\Gamma \subset \mathbb{D}$, an isomorphism (C.32) is obtained by the following prescription: Apply the canonical isomorphism Ξ from Lemma C.2 for every edge $e \in \mathbb{D} \setminus \Gamma$; this results in a transparent disk with two new tadpole vertices for each defect line not covered by Γ, which we denote by $\widetilde{\mathbb{D}}(\Gamma)$. Next apply the canonical isomorphisms $\rho^{(i)}$ from Lemma C.7 for every vertex and every edge of this disk $\widetilde{\mathbb{D}}(\Gamma)$ (in arbitrary order), whereby we end up with a tadpole disk $\mathbb{D}^{\text{tad}}(\Gamma)$. As an illustration, in the case of the spanning tree chosen in (C.33), the disks $\widetilde{\mathbb{D}}(\Gamma)$ and $\mathbb{D}^{\text{tad}}(\Gamma)$ look as follows:

$\widetilde{\mathbb{D}}(\Gamma)\ =$



$\mathbb{D}^{\text{tad}}_{\mathbb{X}}(\Gamma)\ =$



(C.34)

Altogether this defines canonically an isomorphism $\varphi_{\mathbb{D}}(\Gamma)$ of the form (C.32). Next we show that this isomorphism does in fact not depend on the choices made in its construction:

Lemma C.9.

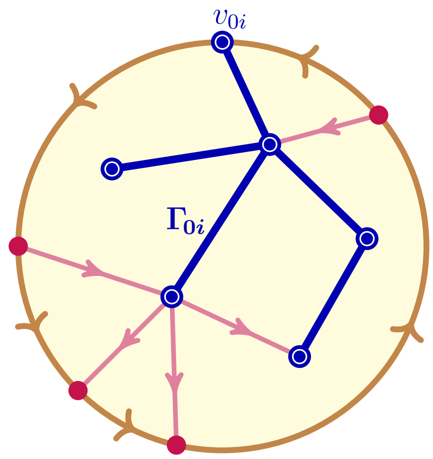
(i) The isomorphism $\varphi_{\mathbb{D}}(\Gamma)\colon \mathrm{T}(\mathbb{D})(-\boxtimes \mho(\mathbb{D})) \xrightarrow{\cong} \mathrm{T}(\mathbb{D}^{\text{tad}})(-\boxtimes \mho(\mathbb{D}^{\text{tad}}))$ does not depend on the order in which the isomorphisms $\rho^{(i)}$ are applied.

(ii) Let Γ and Γ' be two spanning trees for $\mathbb{D}$. Then $\varphi_{\mathbb{D}}(\Gamma) = \varphi_{\mathbb{D}}(\Gamma')$.

Proof. (i) Obviously, any two isomorphisms $\rho^{(i)}$ commute if they are applied on two different vertices. If they are applied on one and the same vertex, the statement follows directly from Lemma C.7.

(ii) As above we assume without loss of generality that the graph Γ_{tot} on $\mathbb{D}$ that is formed by the defect lines is connected and fix a spanning tree Γ for Γ_{tot}, with root vertex v_0. Denote by E_0 the set of all edges of Γ_{tot} that have one of their ends on the outer boundary $\mathbb{L}_{\mathbb{D}}$. By construction, exactly one edge $e_{01} \in E_0$ (as indicated in the picture (C.33)) belongs to the spanning tree Γ. Removing e_{01} from Γ and replacing it by any other edge $e_{0i} \in E_0$ gives another spanning tree, with different root v_{0i}, which we denote by Γ_{0i}. For instance, the following spanning tree Γ_{0i}

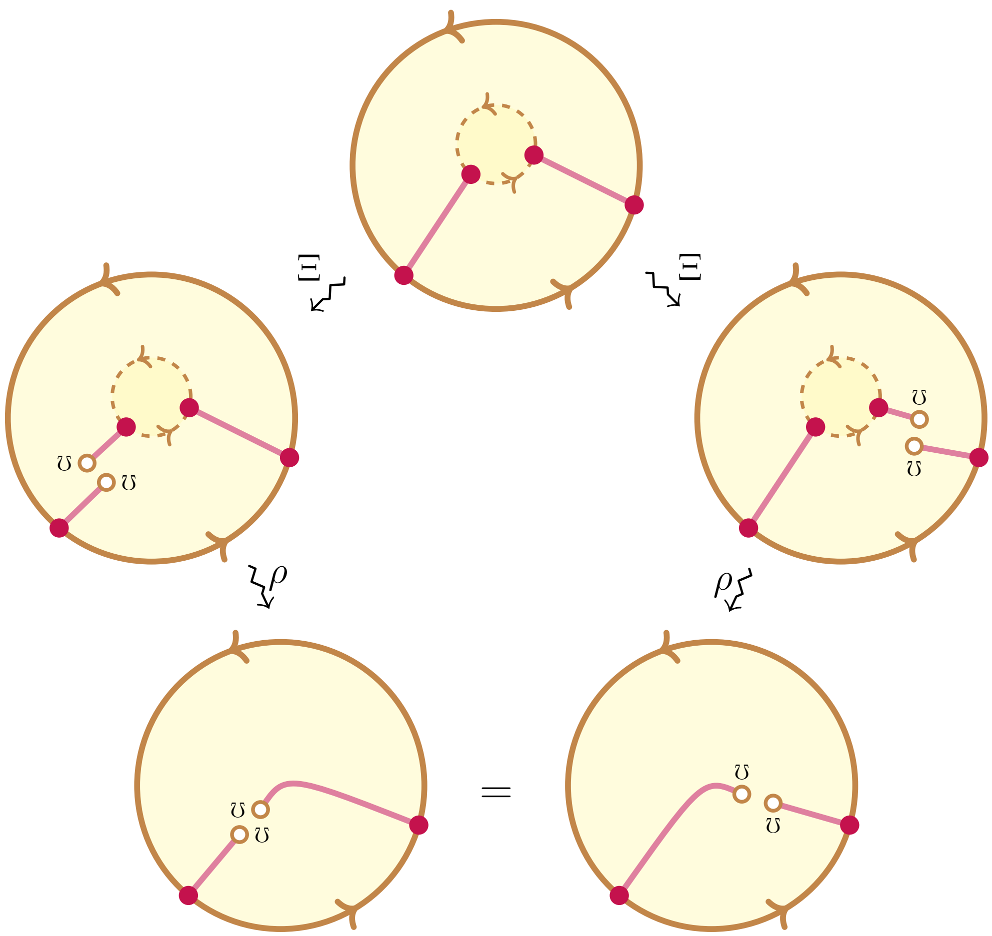
for the transparent disk (5.22) arises this way from the spanning tree shown in (C.33):



(C.35)

We are now going to show that $\varphi_{\mathbb{D}}(\Gamma) = \varphi_{\mathbb{D}}(\Gamma_{0i})$. It is enough to assume that E_0 has precisely two elements. In this case the statement is implied by the following result:

Lemma C.10. The natural isomorphisms between block functors that are indicated in the following picture commute, for any choice of orientations of the (suppressed) transparent defect lines in the fillable disk that is present in the two upper rows of the picture:

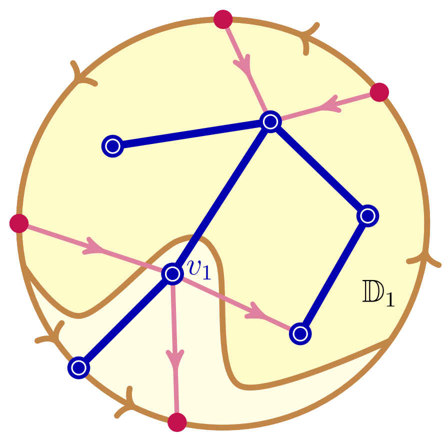


(C.36)

(For concreteness, the picture shows the case that the outer boundary $\mathbb{L}_{\mathbb{D}}$ is a gluing circle, but the statement applies to fillable disks of arbitrary type $\mathbb{X}$.)

Proof. The functors ρ and Ξ are both defined using the adjoint equivalence between the functors $G_{\mathbb{X}}$ and $\widetilde{G}_{\mathbb{X}}$ that follow from Lemma C.2. If we factorize the block functors into corresponding composites of $G_{\mathbb{X}}$ and $\widetilde{G}_{\mathbb{X}}$, then the statement reduces to the zigzag identity for this adjoint equivalence. □

We continue the proof of Lemma C.9 by induction on the depth of the vertices of Γ. Consider the edges of Γ from the single depth-1 vertex v_1 to the depth-2 vertices. Pick a slightly smaller disk $\mathbb{D}_1 \subset \mathbb{D}$ that does not contain the vertices v_0 and v_1, but contains all other vertices of Γ of depth larger than 1, as indicated in the picture



(C.37)

Define the graph Γ_1 as the graph obtained by erasing from $\Gamma \cap \mathbb{D}_1$ the edges E_0. This graph has, in general, several components. In the sequel we assume for simplicity that Γ_1 is connected – if it is not, then each of its (finitely many) components is to be treated analogously. With this assumption, Γ_1 furnishes a spanning tree for the disk $\mathbb{D}_1$, with root v_1' at the intersection of $\mathbb{D}_1$ and the edge of Γ that connects v_1 with the (by the assumption just made, unique) depth-2 vertex. Repeating the previous argument we see that the corresponding isomorphism $\varphi_{\mathbb{D}_1}$ remains unchanged if we replace the edge containing v_1' on the spanning tree Γ_1 by a different edge. By iterating this process we can reach any spanning tree Γ'. We can thus conclude that $\varphi_{\mathbb{D}}(\Gamma) = \varphi_{\mathbb{D}}(\Gamma')$ for all spanning trees Γ and Γ' for $\mathbb{D}$. □

In view of this result from now on we just write $\varphi_{\mathbb{D}}$ for the isomorphism $\varphi_{\mathbb{D}}(\Gamma)$, for any choice of spanning tree Γ. Next we observe that our construction is local, in the following sense:

Proposition C.11. Let $\mathbb{D}$ and $\mathbb{D}'$ be two fillable disks of the same type. There is a distinguished family

$$\varphi_{\mathbb{D},\mathbb{D}'}: \quad \mathrm{T}(\mathbb{D})(- \boxtimes \mho(\mathbb{D})) \longrightarrow \mathrm{T}(\mathbb{D}')(- \boxtimes \mho(\mathbb{D})) \tag{C.38}$$

of natural isomorphisms, one for each fillable-disk replacement $\Phi_{\mathbb{D},\mathbb{D}'}$ with the following properties:

1. (*Coherence*): For any triple $\mathbb{D}, \mathbb{D}', \mathbb{D}''$ of fillable disks of the same type the vertical composition of the natural transformations $\varphi_{\mathbb{D},\mathbb{D}'}$ and $\varphi_{\mathbb{D}',\mathbb{D}''}$ is given by

$$\varphi_{\mathbb{D}',\mathbb{D}''} * \varphi_{\mathbb{D},\mathbb{D}'} = \varphi_{\mathbb{D},\mathbb{D}''} . \tag{C.39}$$

2. (*Factorization*): Given a fillable disk of the form $\mathbb{D} = \mathbb{Y} \circ (\mathbb{D}_1 \sqcup \cdots \sqcup \mathbb{D}_n)$, with $\mathbb{D}_1, \mathbb{D}_2, \ldots, \mathbb{D}_n$ non-intersecting fillable disks in $\mathbb{D}$ (of types $\mathbb{X}_i$) and $\mathbb{Y}$ the defect surface that results from removing all the disks $\mathbb{D}_i$, $i = 1, 2, \ldots, n$, from $\mathbb{D}$, we have, for any n-tuple of fillable-disk replacements $\big(\Phi_{\mathbb{D}_i,\mathbb{D}'_i}\big)_{i=1,\ldots,n}$ that do not change the outer boundaries $\partial_{\text{outer}}\mathbb{D}_i$, the equality

$$\varphi_{\mathbb{D},\mathbb{D}'} = (\varphi_{\mathbb{D}_1,\mathbb{D}'_1} \boxtimes \cdots \boxtimes \varphi_{\mathbb{D}_n,\mathbb{D}'_n}) \circ \mathrm{T}(\mathbb{Y}) \tag{C.40}$$

of natural transformations, where $\mathbb{D}'$ is the defect surface $\mathbb{D}' = \mathbb{Y} \circ (\mathbb{D}'_1 \sqcup \cdots \sqcup \mathbb{D}'_n)$ and '$\circ$' is the horizontal composition of natural transformations.

Proof. For any pair $\mathbb{D}$ and $\mathbb{D}'$ of fillable disks of the same type $\mathbb{X}$, we define the isomorphism $\varphi_{\mathbb{D},\mathbb{D}'}$ by $\varphi_{\mathbb{D},\mathbb{D}'} := \varphi_{\mathbb{D}'}^{-1} \circ \varphi_{\mathbb{D}}$, with $\varphi_{\mathbb{D}} \colon \mathrm{T}(\mathbb{D})(- \boxtimes \mho(\mathbb{D})) \to \mathrm{T}(\mathbb{D}^{\text{tad}})(- \boxtimes \mho(\mathbb{D}^{\text{tad}})$ the isomorphism constructed above. It follows directly from this definition that $\varphi_{\mathbb{D},\mathbb{D}'}$ satisfies coherence. To establish factorization, we observe that there is a spanning tree Γ' for $\mathbb{Y}$ in $\mathbb{D} = \mathbb{Y} \circ (\mathbb{D}_1 \sqcup \cdots \sqcup \mathbb{D}_n)$ that has exactly one vertex on each boundary component of $\mathbb{Y}$. We can complete this graph Γ' to a spanning tree Γ of $\mathbb{D}$ in such a way that $\Gamma_i := \Gamma \cap \mathbb{D}_i$ is a spanning tree for $\mathbb{D}_i$ for every $i \in \{1, 2, \ldots, n\}$. Since the order in which we apply the isomorphisms ρ in the definition of $\varphi_{\mathbb{D}}$ is irrelevant, we readily see that the equality (C.40) indeed holds. ☐

Now recall the notion of a fillable-disk replacement $\Phi_{\mathbb{D},\mathbb{D}'}$ inside a defect surface Σ (which is e.g. implicit in the factorization property (C.40)). We denote the resulting defect surface by $\Phi(\Sigma) \equiv \Phi_{\mathbb{D},\mathbb{D}'}(\Sigma)$. Let Σ be an arbitrary defect surface and $(\Sigma; \Sigma_{\text{ref}_1})$ and $(\Sigma; \Sigma_{\text{ref}_2})$ be any two refinements of Σ.

Definition C.12.

(i) A *refinement replacement* from $(\Sigma; \Sigma_{\text{ref}_1})$ to $(\Sigma; \Sigma_{\text{ref}_2})$ is a sequence of (possibly intersecting) fillable-disk replacements $(\Phi_1, \Phi_2, \ldots, \Phi_n)$ such that

$$\Phi_n(\cdots \Phi_1(\Sigma_{\text{ref}_1}) \cdots) = \Sigma_{\text{ref}_2} . \tag{C.41}$$

(ii) We call two refinement replacements $(\Phi_1, \ldots, \Phi_n)$ and $(\Phi'_1, \ldots, \Phi'_{n'})$ from $(\Sigma; \Sigma_{\text{ref}_1})$ to $(\Sigma; \Sigma_{\text{ref}_2})$ *equivalent* iff the induced natural isomorphisms agree.

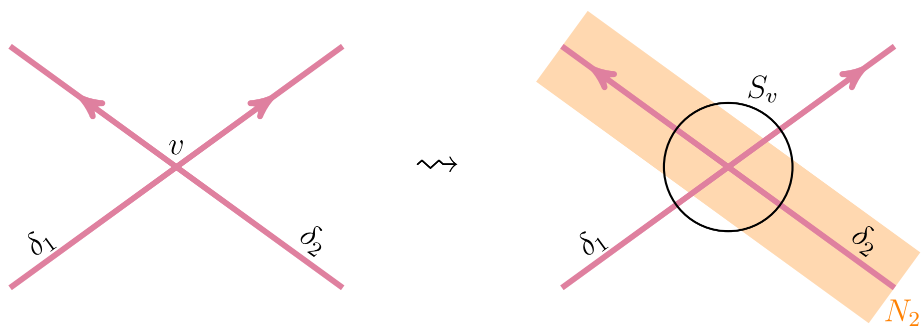
According to Proposition C.11, any fillable-disk replacement Φ in Σ_{ref_1} provides us with an isomorphism $\varphi_{\Sigma,\Phi(\Sigma)} \colon \mathrm{T}(\Sigma_{\text{ref}_1})(- \boxtimes \mho_1) \to \mathrm{T}(\Phi(\Sigma_{\text{ref}_1})))(- \boxtimes \mho'_1)$, with $\mho_1$ and $\mho'_1$ the silent objects for the respective fillable disks involved. Hence a refinement replacement $(\Phi_1, \ldots, \Phi_n)$ from $(\Sigma; \Sigma_{\text{ref}_1})$ to $(\Sigma; \Sigma_{\text{ref}_2})$ gives an isomorphism

$$\varphi_{\Sigma_{\text{ref}_1},\Phi_n(\cdots\Phi_1(\Sigma_{\text{ref}_1})\cdots)} : \quad \mathrm{T}(\Sigma_{\text{ref}_1})(- \boxtimes \mho_1) \longrightarrow \mathrm{T}(\Sigma_{\text{ref}_2})(- \boxtimes \mho_2) . \tag{C.42}$$

As we will see in Lemma C.14 below, a refinement replacement exists between any two fine refinements that refine a given defect surface.

To proceed we introduce the notion of *common subrefinement*. Let $(\Sigma;\Sigma_1)$ and $(\Sigma;\Sigma_2)$ be refinements that refine the same defect surface Σ. Then the common subrefinement $(\Sigma;\Sigma_{1,2})$ of $(\Sigma;\Sigma_1)$ and $(\Sigma;\Sigma_2)$ is constructed by combining all transparent defects from Σ_1 and from Σ_2 in the following manner: First take the collection of all transparent defects δ_2 of Σ_2 that are not part of Σ_1. We can use the embedding of the defects δ_2 in Σ_2 to embed δ_2 in the surface Σ_1 in such a way that any resulting intersections of transparent defects are generic (if necessary, deform the defects slightly to achieve this, see Remark 5.13 (iv)). Denote the so obtained surface with defects by $\mathring{\Sigma}_{1,2}$.

The following prescription makes $\mathring{\Sigma}_{1,2}$ into a defect surface $\Sigma_{1,2}$ endowed with a vector field that (just like the representatives of the framings χ_1 on Σ_1 and χ_2 on Σ_2) is homotopic to the one of Σ: Consider a tubular neighborhood N_2 of all defects δ_2 in Σ_2, and take, for each intersection $v\in\mathring{\Sigma}_{1,2}$ of (the images of) a defect δ_2 of Σ_2 with a defect δ_1 of Σ_1, a small circle S_v around v that intersects δ_1 outside the image of N_2, as indicated in

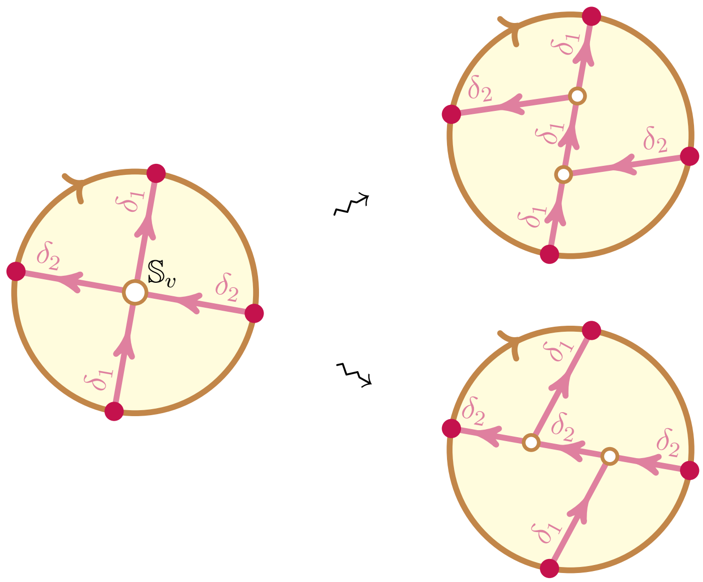


(C.43)

Since Σ_1 and Σ_2 both refine Σ, there is a homotopy $h_t\colon T_p\varSigma\to T_p\varSigma$ for $t\in[0,1]$ and all $p\in\varSigma$ satisfying $h_0=\mathrm{id}$ and $h_1(\chi_2)=\chi_1$. Now let $N_2'\subset N_2$ be a smaller tubular neighborhood of the defects δ_2 and $b\colon \varSigma\to[0,1]$ a smooth monotonous function that is 0 on N_2' and 1 on $\varSigma\setminus N_2$. Then by setting $\chi_{1,2}(p):=h_{b(p)}(\chi_2(p))$ for $p\in\varSigma$ we obtain a vector field $\chi_{1,2}$ on $\varSigma$ that looks like the framing of Σ_i, for $i\in\{1,2\}$, around the defects of $\mathring{\Sigma}_{1,2}$ that correspond to the defects of Σ_i and thus defines a framing on $\mathring{\Sigma}_{1,2}$ of the desired form.

To obtain a proper defect surface we still have to get rid of the intersections v between defect lines. To this end we remove for each such point v the interior of the disk bounded by S_v from $\mathring{\Sigma}_{1,2}$ and replace S_v by a gluing circle, with appropriate defect points at the intersection of S_v with δ_1 and δ_2. Now notice that the vector field $\chi_{1,2}$ is such that all the thus obtained gluing circles $\mathbb{S}_v$ are fillable. This means that after forgetting all transparent defects the framing of $\Sigma_{1,2}$ is by construction homotopic to the framing of Σ; hence we have indeed constructed a refinement $(\Sigma;\Sigma_{1,2})$ of Σ. Also, if Σ_1 and Σ_2 are fine, then so is $\Sigma_{1,2}$.

Moreover, each of the resulting ‘four-valent’ fillable gluing circles $\mathbb{S}_v$ in the common subrefinement $(\Sigma;\Sigma_{1,2})$ can be ‘resolved’ to a pair of two three-valent fillable gluing circles. This can be done in two specific ways, as indicated in



(C.44)

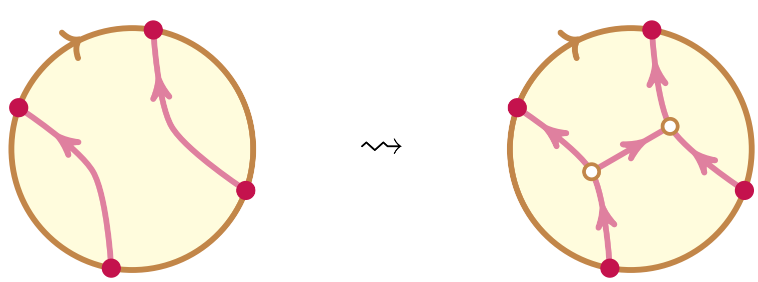
It will be convenient to have separate terminology for specific manipulations of defect networks:

Definition C.13. Let $(\Sigma;\Sigma_1)$ and $(\Sigma;\Sigma_2)$ be refinements refining the same defect surface Σ, and $(\Sigma;\Sigma_{1,2})$ a common subrefinement.

(i) We call the change of defect mesh shown in (C.44) the *resolvement* of the four-valent gluing circle $\mathbb{S}_v$ to Σ_1 and to Σ_2, respectively.

(ii) We call a refinement replacement $\Phi_{\mathbb{D},\widetilde{\mathbb{D}}}$ of *creation type*, respectively of *annihilation type*, iff the defects on $\widetilde{\mathbb{D}}$ are obtained by adding defects to, respectively deleting transparent defects from, the disk $\mathbb{D}$.

Given any two refinements $(\Sigma;\Sigma_1)$ and $(\Sigma;\Sigma_2)$ and a common subrefinement $(\Sigma;\Sigma_{1,2})$, we obtain a specific refinement replacement from $(\Sigma;\Sigma_1)$ to $(\Sigma;\Sigma_2)$ by the following two steps: First, perform resolvements, in the sense of Definition C.13, of each four-valent gluing circle that arises in the construction of $\Sigma_{1,2}$ to Σ_1. Next perform local creation-type fillable-disk replacements by adding single Σ_2-defects to Σ_1, as indicated in the following picture which shows disks that arise from a tubular neighborhood of the defect line in Σ_2:



(C.45)

Finally use replacements of annihilation type to resolve back to the four-valent gluing circles of $\Sigma_{1,2}$. We refer to this procedure as a *standard refinement replacement* from $(\Sigma;\Sigma_1)$ to $(\Sigma;\Sigma_{1,2})$.

Lemma C.14. For any two refinements $(\Sigma;\Sigma_1)$ and $(\Sigma;\Sigma_2)$ that refine the same defect surface Σ there exists a refinement replacement $(\Phi_1,\dots,\Phi_n)$ from $(\Sigma;\Sigma_1)$ to $(\Sigma;\Sigma_2)$.

Proof. Choose any common subrefinement $(\Sigma;\Sigma_{1,2})$ of $(\Sigma;\Sigma_1)$ and $(\Sigma;\Sigma_2)$. Composing the standard refinement replacement from $(\Sigma;\Sigma_1)$ to $(\Sigma;\Sigma_{1,2})$ with the inverse of the standard replacement from $(\Sigma;\Sigma_2)$ to $(\Sigma;\Sigma_{1,2})$ gives a refinement replacement from $(\Sigma;\Sigma_1)$ to $(\Sigma;\Sigma_2)$ that factors through $(\Sigma;\Sigma_{1,2})$. □

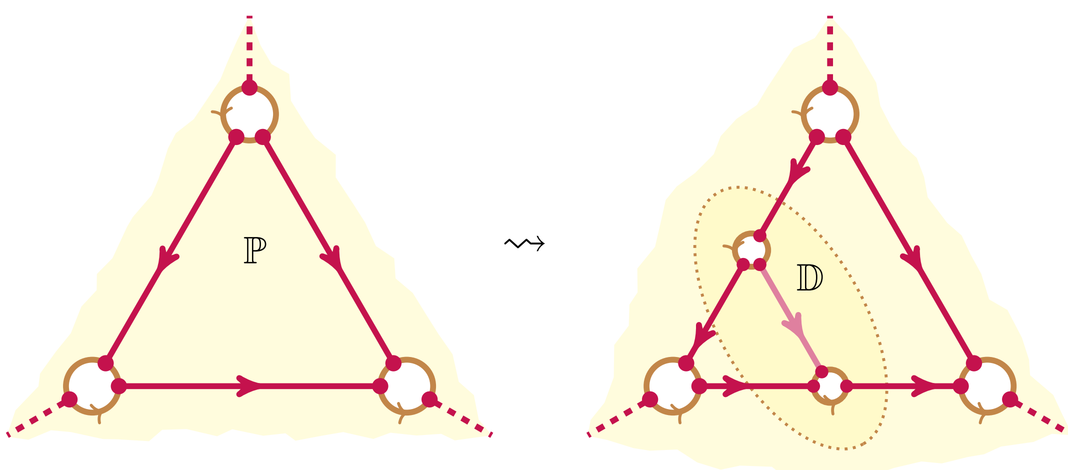
We are now almost ready to show that any two refinement replacements are equivalent. Before giving the proof we just introduce some further convenient terminology.

Definition C.15. Let Σ be a defect surface.

(i) Let δ be a set of defect lines on Σ. We say that a fillable-disk replacement $\Phi_{\mathbb{D},\mathbb{D}'}$ on Σ *keeps the defects* δ iff each of the defects in δ corresponds to a defect on $\Phi_{\mathbb{D},\mathbb{D}'}(\Sigma)$, possibly interrupted by gluing circles that are not present in Σ (like e.g. in the refinement shown in the picture (5.26)).
Analogously we say that a sequence of fillable-disk replacements keeps δ iff each of its members keeps δ.

(ii) Let $\mathbb{P}$ be a 2-patch (in the sense of Definition 2.8) of Σ. A disk $\mathbb{D}$ on Σ is said to be *local with respect to* $\mathbb{P}$, or $\mathbb{P}$-*local*, for short, iff $\mathbb{D}$ does not meet a gluing circle on $\partial\mathbb{P}$ and there are no defects in $\mathbb{D}\setminus\mathbb{P}$. A fillable-disk replacement $\Phi_{\mathbb{D},\mathbb{D}'}$ on a $\mathbb{P}$-local disk $\mathbb{D}$ is said to be $\mathbb{P}$-*local* iff $\Phi_{\mathbb{D},\mathbb{D}'}$ keeps the defects on $\partial\mathbb{P}$ and $\mathbb{D}'$ is $\mathbb{P}$-local as well (that is, no defects are created in $\mathbb{D}\setminus\mathbb{P}$).
Analogously we say that a sequence of fillable-disk replacements on Σ is $\mathbb{P}$-*local* iff each of its members is $\mathbb{P}$-local.

The following picture gives an example of a $\mathbb{P}$-local fillable-disk replacement:



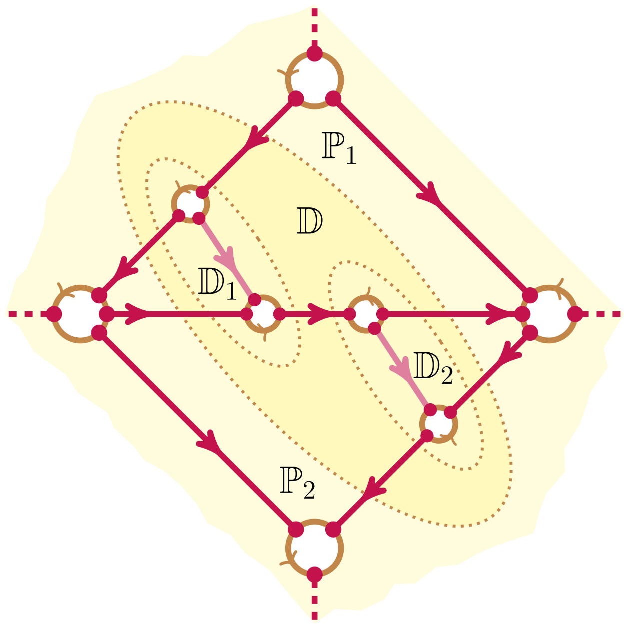
(C.46)

Lemma C.16. Let Σ be a fine defect surface and $(\Sigma;\Sigma_{\mathrm{ref}})$ be a refinement that refines Σ.

(i) Any sequence $(\Phi_1,\dots,\Phi_n)$ of fillable-disk replacements from Σ to Σ_{ref} that keeps the defects of Σ is equivalent to a sequence that is local with respect to all 2-patches of Σ.

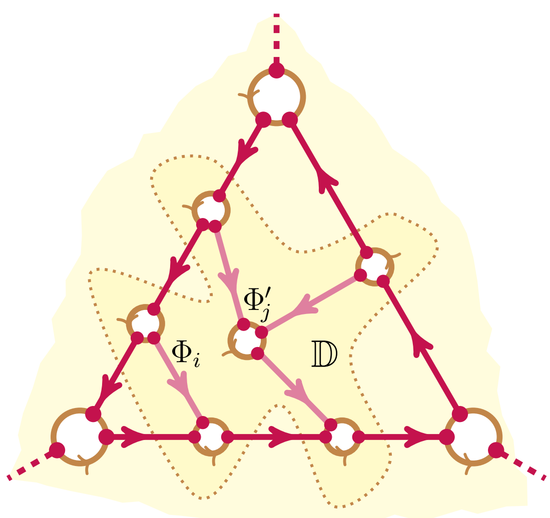
(ii) Any two sequences $(\Phi_1,\dots,\Phi_n)$ and $(\Phi'_1,\dots,\Phi'_{n'})$ of fillable-disk replacements from Σ to Σ_{ref} keeping the defects of Σ are equivalent.

Proof. (i) We "localize" $(\Phi_1, \dots, \Phi_n)$ as follows with respect to the 2-patches of Σ. Consider any of the fillable-disk replacements Φ_j. Since, by assumption, Φ_j keeps the defects on Σ, there is a sequence $\{\Phi_{j,s}\}$ of fillable-disk replacements that are local with respect to the 2-patches of Σ, such that $\{\Phi_{j,s}\}$ is equivalent to Φ_j by the factorization property of Proposition C.11. An illustration of this localization procedure is given in the following picture, in which the disks $\mathbb{D}_1$ and $\mathbb{D}_2$ contain a pair of neighboring defect circles that result from the resolvement of a circle with four defect points, with $\mathbb{D}_1$ being $\mathbb{P}_1$-local and $\mathbb{D}_2$ being $\mathbb{P}_2$-local, while $\mathbb{D}$ is neither $\mathbb{P}_1$- nor $\mathbb{P}_2$-local:



(C.47)

(ii) Owing to (i) we can without loss of generality assume that both sequences are local with respect to the defects in Σ. It is then enough to consider a single 2-patch $\mathbb{P}$ of Σ. Let $(\Phi_1, \dots, \Phi_p)$ and $(\Phi'_1, \dots, \Phi'_{p'})$ be two $\mathbb{P}$-local sequences of disk replacements. Since by assumption Σ is fine, $\mathbb{P}$ is a disk which, in turn, implies that there is a disk $\mathbb{D}$ on Σ such that both sequences lie entirely in $\mathbb{D}$. Thus the two sequences are equivalent by Proposition C.11. Again we give an illustrative example:

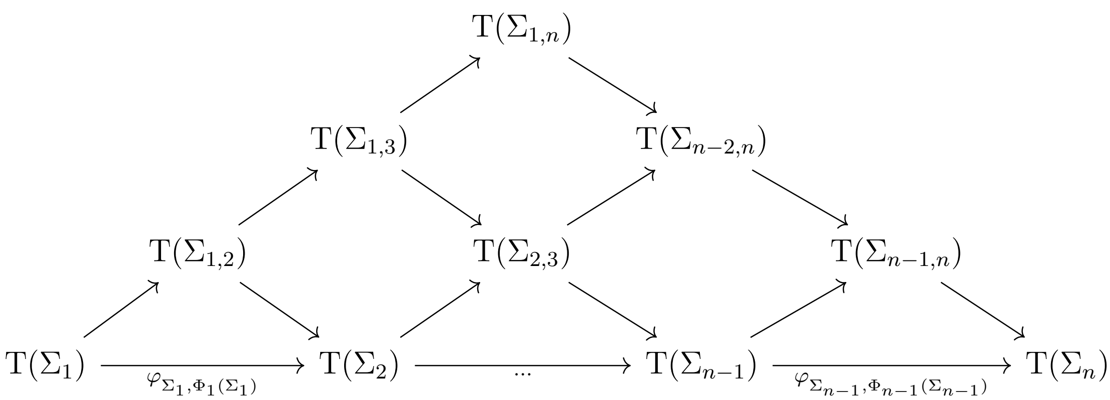


(C.48)

This picture shows the 2-patch $\mathbb{P}$ and indicates the disk $\mathbb{D}$ that encloses both $\mathbb{P}$-local fillable-disk replacements Φ_i and Φ'_j, which consist of one and three transparently labeled defect lines, respectively. □

Lemma C.17. Let Σ be a fine defect surface. Any sequence $(\Phi_1, \Phi_2, \dots, \Phi_n)$ of fine fillable-disk replacements on disks $\{\mathbb{D}_i\}$ in Σ is equivalent to a sequence $(\Phi'_1, \dots, \Phi'_{n'})$ of fillable-disk replacements on $\{\mathbb{D}_i\}$ such that, for some $1 \le p < q \le n'$ the fillable-disk replacements $\Phi'_1, \dots, \Phi'_p$ are replacements of creation type, $\Phi'_q, \dots, \Phi'_{n'}$ are of annihilation type, and $\Phi'_{p+1}, \dots, \Phi'_{q-1}$ are resolvements of vertices.

Proof. We consider iteratively common subrefinements. We then need to show commutativity of a diagram of the following form, in which the bottom row consists of the original sequence $(\Phi_1, \dots, \Phi_n)$ (depicted for the case $n = 4$):



(C.49)

(In this diagram and in the rest of the proof, to save space we abuse notation by just writing $\mathrm{T}(\Sigma)$ in place of $\mathrm{T}(\Sigma)(-\boxtimes\mho)$.) We construct the diagram by proceeding from bottom to top. First, the triangle above the arrow labeled by $\varphi_{\Sigma_i,\Phi_i(\Sigma_i)}$ is obtained by standard refinement replacements on Φ_i: The arrow from $\mathrm{T}(\Sigma_i)$ to $\mathrm{T}(\Sigma_{i,i+1})$ is a replacement of creation type to the common subrefinement $\Sigma_{i,i+1}$ of Σ_i and Σ_{i+1}, while the arrow from $\mathrm{T}(\Sigma_{i,i+1})$ to $\mathrm{T}(\Sigma_{i+1})$ is an analogous replacement of annihilation type. All three arrows in the so obtained triangle are replacements inside one and the same disk, and hence the triangle commutes by Proposition C.11.

Next consider a square above two consecutive triangles. It involves, besides Σ_i and the subrefinements $\Sigma_{i-1,i}$ and $\Sigma_{i,i+1}$, the common standard subrefinement $\Sigma_{i-1,i+1}$ of $\Sigma_{i-1,i}$ and $\Sigma_{i,i+1}$. The arrow from $\mathrm{T}(\Sigma_i)$ to $\mathrm{T}(\Sigma_{i,i+1})$ keeps the defects from Σ_i, and likewise the composite of the other three arrows (with the first of them to be inverted) is a sequence of fillable-disk replacements from Σ_i to $\Sigma_{i,i+1}$ that keeps the defects from Σ_i. Since Σ_i is by assumption fine, it follows from Lemma C.16 that the square commutes. For any of the squares 'higher up" in the diagram, we can likewise use the defects of the fine surface at the bottom of the square to invoke Lemma C.16.

We have thus shown that all triangles and all squares in the diagram (C.49) commute, and hence the whole diagram commutes. Moreover, by construction the diagram is of the required type. □

We are now finally in a position to state

Proposition C.18. Let $(\Sigma;\Sigma_{\mathrm{ref}_1})$ and $(\Sigma;\Sigma_{\mathrm{ref}_2})$ be two refinements that refine the same defect surface Σ. Any two refinement replacements $(\Phi'_1,\dots,\Phi'_{n'})$ and $(\Phi''_1,\dots,\Phi''_{n''})$ from Σ_{ref_1} to Σ_{ref_2} are equivalent, i.e. they satisfy

$$\varphi_{\Sigma_{\mathrm{ref}_1},\Phi''_{n''}(\cdots\Phi''_1(\Sigma_{\mathrm{ref}_1})\cdots)} = \varphi_{\Sigma_{\mathrm{ref}_1},\Phi'_{n'}(\cdots\Phi'_1(\Sigma_{\mathrm{ref}_1})\cdots)}\,. \tag{C.50}$$

Proof. We show that any sequence of refinement replacements $(\Phi'_1,\dots,\Phi'_{n'})$ is equivalent to the standard refinement replacement $(\Phi_1,\dots,\Phi_n)$, see Lemma C.14. By Lemma C.17 we can assume that $(\Phi'_1,\dots,\Phi'_{n'})$ consists first of replacements $(\Phi'_1,\dots,\Phi'_k)$, for $k\le n'$, of creation type to a refinement $(\Sigma;\Sigma'_{\mathrm{ref}_{12}})$, then of annihilation type refinements $(\Phi'_{l+1},\dots,\Phi'_{n'})$ $k\le l\le n'$, and in between of resolvements of vertices. Likewise, the standard refinement replacement consists first of creation type replacements $(\Phi_1,\dots,\Phi_p)$, for $p\le n$, to the common subrefinement $\Sigma_{\mathrm{ref}_{12}}$ of Σ_{ref_1} and Σ_{ref_2} and then of annihilation type replacements $(\Phi_{p+1},\dots,\Phi_n)$ to Σ_{ref_2}. The refinement $\Sigma'_{\mathrm{ref}_{12}}$ is necessarily a subrefinement of $\Sigma_{\mathrm{ref}_{12}}$, thus there exists a sequence of refinement replacements $(\Phi_{q_1},\dots,\Phi_{q_s})$ from $\Sigma_{\mathrm{ref}_{12}}$ to $\Sigma'_{\mathrm{ref}_{12}}$ keeping the defects from $\Sigma_{\mathrm{ref}_{12}}$. Consider then the sequence

$$(\Phi_1,\dots,\Phi_p,\Phi_{q_1},\dots,\Phi_{q_s},\Phi_{q_s}^{-1},\dots,\Phi_{q_1}^{-1},\Phi_{p+1},\dots,\Phi_n) \tag{C.51}$$

of replacements from Σ_{ref_1} to Σ_{ref_2}. This sequence is clearly equivalent to the standard refinement replacement, and $(\Phi_1,\dots,\Phi_p,\Phi_{q_1},\dots\Phi_{q_s})$ is a sequence of fillable-disk replacements from Σ_{ref_1} to $\Sigma'_{\mathrm{ref}_{12}}$ that keeps the defects of Σ_{ref_1}, just like $(\Phi'_1,\dots,\Phi'_k)$ is. Thus by Lemma C.16 they are equivalent. In the same way, the sequences $(\Phi_{q_s}^{-1},\dots,\Phi_{q_1}^{-1},\Phi_{p+1},\dots,\Phi_n)$ and $(\Phi'_{k+1},\dots,\Phi'_{n'})$ both keep the defects from Σ_{ref_2} and are thus equivalent as well (apply Lemma C.16 to the inverses of the sequences). Thus the statement follows. □

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