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CFT correlators for Cardy bulk fields via string-net models

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Abstract

We show that string-net models provide an explicit way to construct invariants of mapping class group actions. Concretely, we consider string-net models for a modular tensor category $\mathcal{C}$. We show that the datum of a specific commutative symmetric Frobenius algebra in the Drinfeld center $Z(\mathcal{C})$ gives rise to invariant string-nets. The Frobenius algebra has the interpretation of the algebra of bulk fields of a two-dimensional rational conformal field theory in the Cardy case.

Contents

1 Introduction

Two-dimensional conformal field theories, apart from their intrinsic physical interest, are quantum field theories that are amenable to a precise mathematical study. In this paper, we use string-net models to study consistent systems of bulk field correlators in a class of such models.

A consistent system of correlators in a two-dimensional conformal field theory is obtained by specifying elements in spaces of conformal blocks, subject to certain consistency conditions. For a conformal field theory with the monodromy data given by a braided monoidal category $\mathcal{D}$, the spaces of conformal blocks can be constructed as morphism spaces in $\mathcal{D}$. They are endowed with projective actions of mapping class groups given by the structures of $\mathcal{D}$. For a rational conformal field theory, the category $\mathcal{D}$ is a (semisimple) modular tensor category and the spaces of conformal blocks are provided by the state spaces of a three-dimensional topological field theory, namely the Reshetikhin-Turaev TFT based on $\mathcal{D}$. In this framework, the task of finding a consistent system of correlators is equivalent to finding for each surface Σ a vector in the space of conformal blocks on the double $\widehat{\Sigma}$. This element has to be invariant under the action of mapping class group and the set of elements has to be consistent under sewing of the surfaces [FRS02, FRS04a, FRS04b, FRS05, FFRS06a].

In this article, we only consider bulk fields on oriented surfaces. Instead of taking the doubles of the surfaces one can take the enveloping category $\mathcal{C}^{rev} \boxtimes \mathcal{C}$ of a modular tensor category as the category $\mathcal{D}$. Modularity implies that we have a braided equivalence: $\mathcal{C}^{rev} \boxtimes \mathcal{C} \simeq Z(\mathcal{C})$, where $Z(\mathcal{C})$ is the Drinfeld center of $\mathcal{C}$, see e.g. [Shi19] for a statement that includes non-semisimple categories as well. The Reshetikhin-Turaev construction for $Z(\mathcal{C})$ is equivalent to the extended Turaev-Viro-Barrett-Westbury state-sum construction based on $\mathcal{C}$ [TV10, KJB10, Bal10a, Bal10b].

The string-net model arises in the study of topological order in condensed matter physics [LW05], and has recently been shown to be equivalent to the Turaev-Viro-Barrett-Westbury state-sum construction as a 3-2-1 extended TFT [KJ11, Goo18]. The string-net model has two advantages that are attractive in our context: a vector in the the space of conformal blocks can be described by a string-net, and the action of the mapping class group, when expressed in terms of such vectors, is completely geometrical.

In this paper, we restrict to a specific type of local rational conformal field theory: we assume that the object $F \in Z(\mathcal{C})$ that describes the bulk fields is the object

$$L = \bigoplus_{i \in \mathcal{I}(\mathcal{C})} X_i^\vee \otimes X_i \in \mathcal{C}$$

along with a certain half-braiding (see subsection 3.2). This is usually referred to as the bulk algebra in the Cardy case. Indeed, this object comes with a natural structure of a commutative symmetric Frobenius algebra in $Z(\mathcal{C})$, as befits an algebra of bulk fields, see theorem 3.6 for details. Our main result can now be described as follows: a consistent systems of correlators can be obtained by assigning to each surface Σ, possibly with non-empty boundary, a string-net given by decomposing the surface into pairs of pants and placing the appropriate elementary string-nets that are constructed from the structural morphisms of the Frobenius algebra F on each component. For instance, for a surface of genus one with one ingoing and two outgoing boundary components, we have the following string-net:

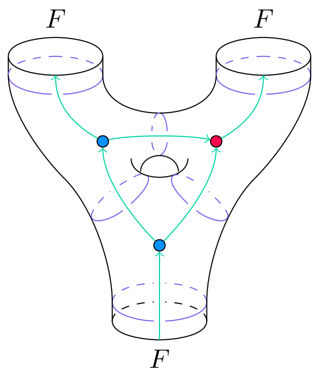

Figure 1.1: The string-net assigned to the extended surface of genus one with one ingoing and two outgoing boundary circles according to a certain pairs of pants decomposition.

Here the red and blue circular coupons stand for the multiplication and co-multiplication of the Frobenius algebra F respectively, while the purple circles stand for the boundary projectors (introduced in remark 2.8) that account for the half-braiding of F. We show that this assignment is independent of the choice of pairs of pants decomposition for each surface hence well defined, and it gives rise to a consistent system of bulk field correlators (theorem 3.18).

This paper is organized as follows: in Section 2, we briefly review string-net models, following [KJ11]. We next recall some facts about modular tensor categories in subsection 3.1, review the Cardy bulk algebra F in subsection 3.2 and the notion of a consistent system of bulk field correlators in subsection 3.3. The proofs of our main results are contained in sections 3.4 - 3.7. They are based on a remarkable interplay of the algebraic properties of F and the moves in the Lego-Teichmüller game.

This interplay calls for a deeper explanation which is the subject of subsection 3.8. We notice that the Frobenius algebra is isomorphic to a different Frobenius algebra, see theorem 3.19. When expressed in terms of this Frobenius algebra, our string nets become manifestly invariant under the mapping class group, cf. theorem 3.21. After translating these string-nets back to the string-nets labeled by the original Frobenius algebra F, we obtain the simplified form of the string-nets of our construction which exhibit manifest invariance under the mapping class group as well, see theorem 3.22. This implies, for instance, that the colored graph shown in figure 1.1 is, in fact, equivalent to the following colored graph and that the two graphs thus define the same string-net:

$$\sum_{i,j,k,l,m,n\in\mathcal{I}(\mathcal{C})} \frac{d_j d_k d_l d_m d_n}{D^6}$$

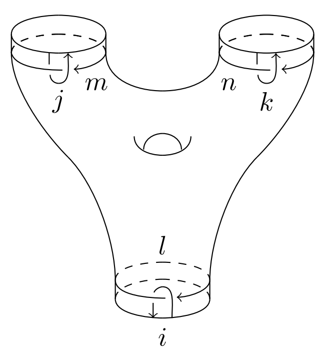

Figure 1.2: The simplified form of the string-net assigned to the extended surface $\Sigma^1_{1|2}$.

We expect that our results can be generalized in several directions: beyond the Cardy case and to correlators including also boundary and defect fields. A generalization of the string net construction to non-semisimple finite tensor categories remains, at the moment, a challenge. It would allow to address correlators of logarithmic conformal field theories as well.

ACKNOWLEDGEMENTS:

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2 String-net models

2.1 Spherical fusion categories

The string-net models are usually defined for spherical fusion categories. We first review a few relevant categorical concepts and fix our notations for the graphical calculus.

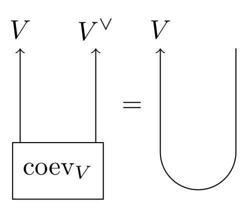
Recall that a *right dual* of an object V in a strict monoidal category $\mathcal{C}$ is an object $V^\vee$ together with morphisms $\mathrm{coev}_V \in \mathrm{Hom}_\mathcal{C}(\mathbb{I}, V \otimes V^\vee)$ and $\mathrm{ev}_V \in \mathrm{Hom}_\mathcal{C}(V^\vee \otimes V, \mathbb{I})$ satisfying

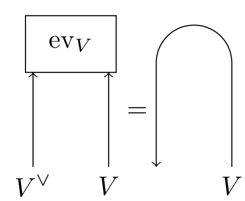
$$(\mathrm{id}_V \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes \mathrm{id}_V) = \mathrm{id}_V$$

and

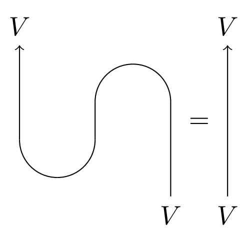
$$(\mathrm{ev}_V \otimes \mathrm{id}_{V^\vee}) \circ (\mathrm{id}_{V^\vee} \otimes \mathrm{coev}_V) = \mathrm{id}_{V^\vee}.$$

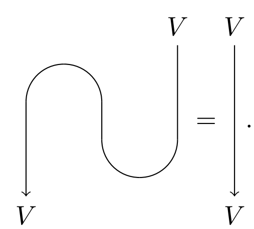
Graphically we have:





and





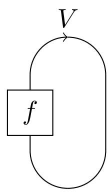
Here we replaced $V^\vee$ by V upon reversing the direction of the arrow. Left duality is defined similarly by reversing the arrows in the graphical notation.

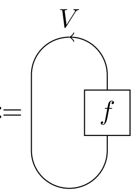
A *pivotal structure* on a rigid monoidal category is a monoidal natural isomorphism

$$\omega \colon \mathrm{id}_\mathcal{C} \Rightarrow (-)^{\vee\vee}.$$

A pivotal structure is called strict if $\mathrm{id}_{\mathcal{C}} = (-)^{\vee\vee}$ and $\omega = \mathrm{id}_{\mathrm{id}_{\mathcal{C}}}$. It is known that every pivotal category is pivotally equivalent to a pivotal category with strict pivotal structure [NS07, Theorem 2.2], hence we will assume the pivotal structure to be strict in the following without loss of generality. For a strict pivotal category, the left and right duality strictly coincide as functors.

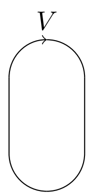
In a pivotal category we have the notions of *right and left traces* for any $f \in \mathrm{End}_{\mathcal{C}}(V)$. Graphically:

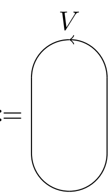
$\mathrm{tr}_{\mathrm{r}}(f) \coloneqq$ 

$\in \mathrm{End}_{\mathcal{C}}(\mathbb{I})$ $\qquad$ $\mathrm{tr}_{\mathrm{l}}(f) \coloneqq$ 

$\in \mathrm{End}_{\mathcal{C}}(\mathbb{I})$.

When applied to $\mathrm{id}_V \in \mathrm{End}_{\mathcal{C}}(V)$, we get the definitions of the *left and right categorical dimension* of the object $V \in \mathcal{C}$:

$\dim_{\mathrm{r}}(V) \coloneqq$ 

$\in \mathrm{End}_{\mathcal{C}}(\mathbb{I})$ $\qquad$ $\dim_{\mathrm{l}}(V) \coloneqq$ 

$\in \mathrm{End}_{\mathcal{C}}(\mathbb{I})$.

A pivotal category is called *spherical* if the left and right traces coincide, i.e. $\mathrm{tr}(f) \coloneqq \mathrm{tr}_{\mathrm{r}}(f) = \mathrm{tr}_{\mathrm{l}}(f)$ and $\dim(V) = \dim_{\mathrm{r}}(V) = \dim_{\mathrm{l}}(V)$.

From now on, we fix an algebraically closed field $\mathbb{K}$ of characteristic 0.

Definition 2.1. A *fusion category* over $\mathbb{K}$ is a rigid $\mathbb{K}$-linear monoidal category $\mathcal{C}$ that is finitely semisimple, with the monoidal unit $\mathbb{I}$ being simple.

Here being $\mathbb{K}$-linear means that the sets of morphisms are $\mathbb{K}$-vector spaces and the composition as well as the monoidal product are bilinear. Being finitely semisimple means that there are finitely many isomorphism classes of simple objects (objects with no non-trivial subobject) and every object is a direct sum of finitely many simple objects. Note that $\mathbb{K}$-linearity and finitely-semisimplicity together imply that the morphism spaces are finite dimensional.

Definition 2.2. A *spherical fusion category* over $\mathbb{K}$ is a spherical category $\mathcal{C}$ that is also a fusion category over $\mathbb{K}$.

Let us denote the set of isomorphism classes of simple objects by $\mathcal{I}(\mathcal{C})$, and fix a representative X_i for each $i \in \mathcal{I}(\mathcal{C})$. In addition, we require $0 \in \mathcal{I}(\mathcal{C})$ and $X_0 = \mathbb{I}$. Duality furnishes a involution on $\mathcal{I}(\mathcal{C})$, i.e. $i \mapsto \bar{i} \coloneqq [X_i^\vee]$. We require that $X_{\bar{i}} = X_i^\vee$ whenever $i \neq \bar{i}$. Since $\mathbb{K}$ is assumed to be algebraically closed, the only finite dimensional division algebra over $\mathbb{K}$ is $\mathbb{K}$ itself. Thus we have the Schur's lemma:

$$\mathrm{Hom}_{\mathcal{C}}(X_i, X_j) \cong \delta_{i,j}\mathbb{K}.$$

In particular, $d_X \coloneqq \dim(X) \in \mathrm{End}_{\mathcal{C}}(\mathbb{I}) \cong \mathbb{K}$. Define the global dimension of the spherical fusion category $\mathcal{C}$ to be

$$D^2 \coloneqq \sum_{i \in \mathcal{I}(\mathcal{C})} d_i^2.$$

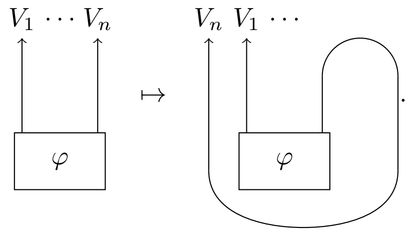
By [ENO05, Theorem 2.3], $D^2 \neq 0$.

We define the functor $\underbrace{\mathcal{C} \boxtimes \ldots \boxtimes \mathcal{C}}_{n} \to \mathcal{V}\mathrm{ect}_{\mathbb{K}}$ by

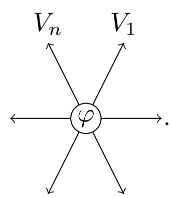
$$V_1 \boxtimes \ldots \boxtimes V_n \mapsto \langle V_1, \ldots, V_n \rangle \coloneqq \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, V_1 \otimes \ldots \otimes V_n).$$

The pivotal structure furnishes a natural isomorphism by

$$z_{V_1 \boxtimes \ldots \boxtimes V_n} \colon \langle V_1, \ldots, V_n \rangle \xrightarrow{\cong} \langle V_n, V_1, \ldots, V_{n-1} \rangle$$



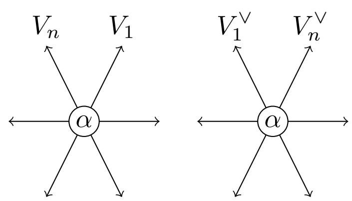
It can be seen from the graphical calculus that $z^n = \mathrm{id}$. Thus, up to a natural isomorphism, $\langle V_1, \ldots, V_n \rangle$ depends only on the cyclic order of $V_1, \ldots, V_n$. This allows us to represent an element $\varphi \in \langle V_1, \ldots, V_n \rangle$ by a round coupon with n outgoing legs colored by $V_1, \ldots, V_n$ in clockwise order:



We are able to connect legs with dual labels: define the composition map

$$\begin{aligned} \langle V_1, \ldots, V_n, X^\vee \rangle \otimes_{\mathbb{K}} \langle X, W_1, \ldots, W_m \rangle &\to \langle V_1, \ldots, V_n, W_1, \ldots, W_m \rangle \\ \varphi \otimes_{\mathbb{K}} \psi &\mapsto \varphi \circ_X \psi \coloneqq \mathrm{ev}_X \circ (\varphi \otimes_{\mathbb{K}} \psi). \end{aligned}$$

This gives rise to a pairing: $\langle V_1, \ldots, V_n \rangle \otimes_{\mathbb{K}} \langle V_n^\vee, \ldots, V_1^\vee \rangle \to \mathbb{K}$. It is nondegenerate due to the nondegeneracy of the evaluation maps. Hence for any choice of bases $\{\varphi_\alpha\}_{\alpha \in A}$ of $\langle V_1, \ldots, V_n \rangle$, we define the dual bases $\{\varphi^\alpha\}_{\alpha \in A}$ with respect to this nondegenerate pairing. In the following we will use the summation convention:

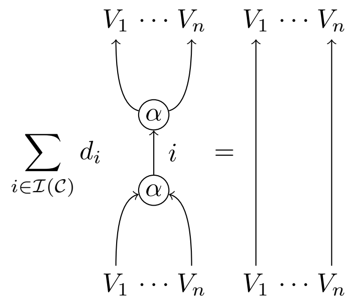


$$\coloneqq \sum_{\alpha \in A} \varphi^\alpha \otimes_{\mathbb{K}} \varphi_\alpha.$$

Such expressions are independent of the choice of bases.

We now introduce a very useful relation:

Proposition 2.3. *For any* $V_1, \ldots, V_n \in \mathcal{C}$, we have



We call this a *completeness relation*.

2.2 String-net spaces

We now give a brief introduction to the string-net construction. We refer to [KJ11] for more details, and to [LW05] for motivations from physics.

Let's consider *finite graphs* (i.e. the sets of the vertices and the edges are both finite) embedded in an oriented surface Σ , which is not required to be compact and may have non-empty boundary. For such a graph Γ, denote by $\mathcal{E}^{or}(\Gamma)$ the set of its *oriented edges* and $\mathcal{V}(\Gamma)$ the set of its *vertices*. One-valent vertices are called *endings*. We denote the set of endings of Γ by $\mathcal{V}^{en}(\Gamma)$, and define $\mathcal{V}^{in}(\Gamma) := \mathcal{V}(\Gamma) \setminus \mathcal{V}^{en}(\Gamma)$. We require $\Gamma \cap \partial\Sigma = \mathcal{V}^{en}(\Gamma)$. We will call the edges terminating at endings *legs*. Note that we don't make a choice of orientations for the edges of the finite graphs.

Definition 2.4. Let Σ and Γ be as defined above. A $\mathcal{C}$*-coloring* (or simply *coloring* when there is no ambiguity) of Γ is given by the following data:

- A map $V \colon \mathcal{E}^{or}(\Gamma) \to \mathrm{Obj}(\mathcal{C})$ such that for every $e \in \mathcal{E}^{or}(\Gamma)$, we have $V(e) = V(\overline{e})^*$, where $\overline{e}$ is the edge with opposite orientation of e.
- A choice of a vector $\varphi(v) \in \langle V(e_1), \dots, V(e_n)\rangle$ for every $v \in \mathcal{V}^{in}(\Gamma)$, where $e_1, \dots, e_n$ are incident to v, taken in clockwise order and with outward orientation.

An *isomorphism* f of two colorings (V, φ) and (V', φ') is a collection of isomorphisms $f_e \colon V(e) \xrightarrow{\cong} V'(e)$ that is compatible with $V(e) = V(\overline{e})^*$ and such that $\varphi'(v) = f \circ \varphi(v)$.

Let $B \subset \partial\Sigma$ be a finite collection of points on $\partial\Sigma$ and $\mathbf{V} \colon B \to \mathrm{Obj}(\mathcal{C})$ a map. A $\mathcal{C}$-colored graph Γ with *boundary value* $\mathbf{V}$ is a colored graph such that $\mathcal{V}^{en}(\Gamma) = B$ and $V(e_b) = \mathbf{V}(b)$, where $b \in B$ and e_b is the edge incident to b with outgoing orientation. We define $\mathrm{Graph}(\Sigma, \mathbf{V})$ to be the set of $\mathcal{C}$-colored graphs in Σ with boundary value $\mathbf{V}$, and $\mathrm{VGraph}(\Sigma, \mathbf{V})$ to be the $\mathbb{K}$-vector space freely generated by $\mathrm{Graph}(\Sigma, \mathbf{V})$.

This would be typically an infinite-dimensional vector space since unless $\Sigma = \emptyset$, the set $\mathrm{Graph}(\Sigma, \mathbf{V})$ has infinitely many elements.

When Σ happens to be a disc $D \subset \mathbb{R}^2$, a colored graph $\Gamma \in \mathrm{Graph}(D, \mathbf{V})$ can be naturally viewed as the graphical representation of some morphism in $\mathcal{C}$. Indeed, there is a canonical linear surjection [KJ11, Theorem 2.3]

$$\langle - \rangle_D : \mathrm{VGraph}(\Sigma, \mathbf{V}) \to \langle V(e_1), \dots, V(e_n)\rangle \, ,$$

where $B = \{b_1, \dots, b_n\}$ and $e_1, \dots, e_n$ are the corresponding outgoing legs, taken in the clockwise order.

The finite dimensional vector space $\langle V(e_1), \dots, V(e_n)\rangle \cong \mathrm{VGraph}(D, \mathbf{V})/\ker \langle - \rangle_D$ can be viewed as the space of linear combinations of $\mathcal{C}$-colored graphs with a fixed boundary value, where two combinations are identified if they represent the same morphism in $\mathcal{C}$ according to the graphical calculus. The identification in turn allows us to perform graphical calculus in this space. This inspires us to use $\mathrm{VGraph}(D, \mathbf{V})/\ker \langle - \rangle_D$ as a local model to define a vector space for an arbitrary oriented surface Σ with a prescribed boundary value $\mathbf{V}$, so that we can perform graphical calculus locally.

Definition 2.5. Let $D \subset \Sigma$ be an embedded disc, a *null graph with respect to D* is a linear combination of colored graphs $\mathbf{\Gamma} = c_1\Gamma_1 + \ldots + c_n\Gamma_n \in \mathrm{VGraph}(\Sigma, \mathbf{V})$ such that

- $\mathbf{\Gamma}$ is transversal to ∂D (i.e. no vertex of Γ_i is on ∂D and the edges of each Γ_i intersect with ∂D transversally).
- All Γ_i coincide outside of D.
- $\langle \mathbf{\Gamma} \rangle_D = \sum_i c_i \langle \Gamma_i \cap D \rangle_D = 0$.

Denote by $\mathrm{N}(\Sigma, \mathbf{V}) \subset \mathrm{VGraph}(\Sigma, \mathbf{V})$ the subspace spanned by null graphs for all possible embedded disks $D \subset \Sigma$.

Definition 2.6. Let Σ be an oriented surface and let $\mathbf{V} \colon B \to \mathrm{Obj}(\mathcal{C})$ be a boundary value. Define the *string-net space* for $(\Sigma, \mathbf{V})$ to be the quotient space

$$Z_{SN,\mathcal{C}}(\Sigma, \mathbf{V}) \coloneqq \mathrm{VGraph}(\Sigma, \mathbf{V})/\mathrm{N}(\Sigma, \mathbf{V}).$$

As before, we have a linear surjection

$$\langle - \rangle_\Sigma : \mathrm{VGraph}(\Sigma, \mathbf{V}) \to Z_{SN,\mathcal{C}}(\Sigma, \mathbf{V}).$$

The map has several nice properties. For instance, it is linear in the colors of vertices and additive with respect to direct sums, isotopic graphs and graphs with isomorphic colorings have the same image. But most importantly, it allows us to replace graphs that only differ by local relations. That is to say, all equations from the graphical calculus for the spherical fusion category $\mathcal{C}$, e.g. the one from proposition 2.3, still holds true inside any embedded disc on the surface.

2.3 Drinfeld center and the extended string-net spaces

It is standard that one can associate to any monoidal category $\mathcal{C}$ a braided monoidal category $Z(\mathcal{C})$, called the Drinfeld center of $\mathcal{C}$. We give the definition of the Drinfeld center of a monoidal category $\mathcal{C}$ here to fix our notations.

Definition 2.7. The *Drinfeld center* $Z(\mathcal{C})$ of a monoidal category $\mathcal{C}$ is a braided monoidal category with

- objects given by the pairs (U, γ_U), where $U \in \mathcal{C}$ is an object and $\gamma_U \colon U \otimes - \Rightarrow - \otimes U$ a natural isomorphism called the *half-braiding* that satisfies
$$\gamma_{U;V\otimes W} = (\mathrm{id}_V \otimes \gamma_{U;W}) \circ (\gamma_{U;V} \otimes \mathrm{id}_W)$$
for all $V, W \in \mathcal{C}$;
- morphisms $(U, \gamma_U) \to (V, \gamma_V)$ given by all $f \in \mathrm{Hom}_{\mathcal{C}}(U, V)$ that satisfy
$$(\mathrm{id}_W \otimes f) \circ \gamma_{U;W} = \gamma_{V;W} \circ (f \otimes \mathrm{id}_W)$$
for all $W \in \mathcal{C}$;

- monoidal product defined by $(U,\gamma_U)\otimes(V,\gamma_V) \coloneqq (U\otimes V, \gamma_{U\otimes V})$, where

$$\gamma_{U\otimes V;W} \coloneqq (\gamma_{U;W}\otimes \mathrm{id}_V)\circ(\mathrm{id}_U\otimes\gamma_{V;W})$$

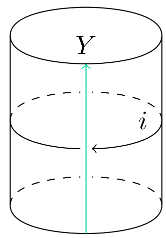
  for all $W\in\mathcal{C}$;

- braiding defined by $\beta^{Z(\mathcal{C})}_{(U,\gamma_U),(V,\gamma_V)} \coloneqq \gamma_{U;V}$.

The definition of string-net spaces can be modified so that one assign to each boundary circle an object in the Drinfeld center $Z(\mathcal{C})$. We now give a working description of the extended string-net spaces that are relevant to our construction of CFT correlators and refer to [KJ11, Section 6, 7] for details.

Remark 2.8. For all $Y\in Z(\mathcal{C})$, the following string-net on a cylinder is a projector with respect to sewing the string-nets on cylinders:

$$P_Y \coloneqq \sum_{i\in\mathcal{I}(\mathcal{C})}\frac{d_i}{D^2}$$



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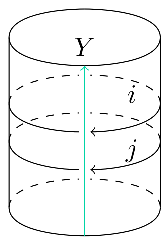
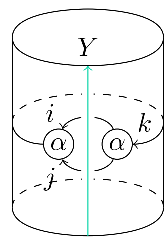
Here the crossing is given by the half-braiding of Y. By using proposition 2.3 and the naturality of the half-braiding, we see that P is indeed a projector:

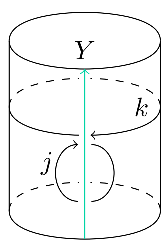
$$P_Y^2 = \sum_{i,j\in\mathcal{I}(\mathcal{C})}\frac{d_i d_j}{D^4}$$



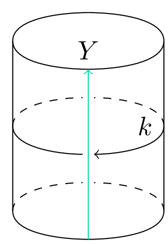
$$= \sum_{i,j,k\in\mathcal{I}(\mathcal{C})}\frac{d_i d_j d_k}{D^4}$$



$$= \sum_{j,k\in\mathcal{I}(\mathcal{C})}\frac{d_j d_k}{D^4}$$



$$= \sum_{k\in\mathcal{I}(\mathcal{C})}\frac{d_k}{D^2}$$



$= P.$

We are interested in the case where $\Sigma\cong\Sigma^g_n$, here Σ^g_n means a compact oriented surface of genus g with n boundary components. Denote by $(\Sigma^g_n, Y_1,\dots,Y_n)$ a $Z(\mathcal{C})$-*marked surface*, i.e. Σ^g_n together with

- a numbering of $\pi_0(\partial\Sigma)$ with $1,\dots,n$,
- a choice of a point in each connected component of $\partial\Sigma$,
- a choice of n objects $Y_1,\dots,Y_n\in Z(\mathcal{C})$.

We denote the extended string-net space for the $Z(\mathcal{C})$-marked surface Σ_n^g with this boundary value by $Z_{SN,\mathcal{C}}(\Sigma_n^g, Y_1, \ldots, Y_n)$. This is defined to be a subspace of the (unextended) string-net spaces of Σ_n^g with boundary value given by the underlying objects of $Y_1, \ldots, Y_n$ in $\mathcal{C}$, which is spanned by all the string-nets with additional projectors (introduced in remark 2.8) placed near the corresponding boundary circles. For instance, a generic vector in $Z_{SN,\mathcal{C}}(\Sigma_3^1, Y_1, Y_2, Y_3)$ can be defined by a linear combination of equivalence classes of colored graphs on Σ_3^1 such as:

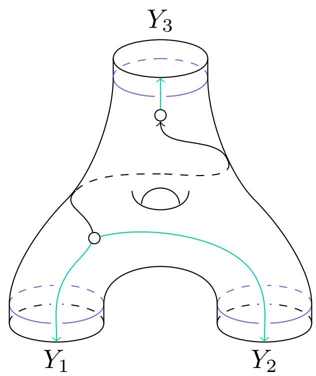


Figure 2.1: A generic string-net in $Z_{SN,\mathcal{C}}(\Sigma_3^1, Y_1, Y_2, Y_3)$.

Here the purple circles stand for the projectors $P_{Y_1}, P_{Y_2}, P_{Y_3}$ introduced in remark 2.8.

By the result of [KJ11], there is a canonical isomorphism $Z_{SN,\mathcal{C}}(\Sigma_n^g, Y_1, \ldots, Y_n) \cong Z_{TV,\mathcal{C}}(\Sigma_n^g, Y_1, \ldots, Y_n)$, where $Z_{TV,\mathcal{C}}(\Sigma_n^g, Y_1, \ldots, Y_n)$ is the state space for $(\Sigma_n^g, Y_1, \ldots, Y_n)$ in the extended Turaev-Viro-Barrett-Westbury topological field theory. Hence:

Proposition 2.9. *There are isomorphisms*

$$Z_{SN,\mathcal{C}}(\Sigma_n^g, Y_1, \ldots, Y_n) \cong Z_{TV,\mathcal{C}}(\Sigma_n^g, Y_1, \ldots, Y_n) \cong \mathrm{Hom}_{Z(\mathcal{C})}(\mathbb{I}_{Z(\mathcal{C})}, Y_1 \otimes \ldots \otimes Y_n \otimes L_{Z(\mathcal{C})}^{\otimes g})$$

that are functorial with respect to the morphisms in $Z(\mathcal{C})$, where

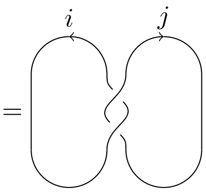
$$L_{Z(\mathcal{C})} := \bigoplus_{i \in \mathcal{I}(Z(\mathcal{C}))} Z_i^\vee \otimes Z_i.$$

3 Consistent systems of correlators

3.1 Modular tensor categories

The categorical ingredient of the string-net construction is a spherical fusion category $\mathcal{C}$, which is not necessarily braided. However, for the application to the conformal field theories, we need a category with the structure of a ribbon fusion category over $\mathbb{C}$ with an additional nondegeneracy property:

Definition 3.1. A *modular tensor category* $\mathcal{C}$ is a ribbon fusion category over $\mathbb{C}$ with the braiding being nondegenerate in the sense that the matrix $(s_{i,j})_{i,j \in \mathcal{I}(\mathcal{C})}$ is invertible, where

$$s_{i,j} := \mathrm{tr}(\beta_{j,i} \circ \beta_{i,j}) =$$ 
$$\in \mathrm{End}_{\mathcal{C}}(\mathbb{I}) \cong \mathbb{C}.$$

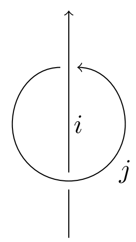
It can be seen from the cyclic symmetry of the categorical trace that $s_{i,j} = s_{j,i}$. Moreover, one can show that (see e.g. [BK01, Theorem 3.7.1])

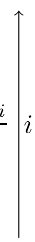
$$\sum_{k \in \mathcal{I}(\mathcal{C})} s_{i,k} s_{k,j} = D^2 \delta_{i,\bar{j}}.$$

For a spherical fusion category $\mathcal{C}$, the Drinfeld center $Z(\mathcal{C})$ is a modular tensor category.

We are going to use the following equations:

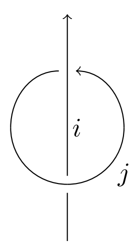
Lemma 3.2. *For all $i \in \mathcal{I}(\mathcal{C})$, we have*

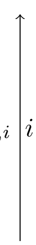


$$= \frac{s_{j,i}}{d_i}$$  $i.$

Proof. Simply compare the traces of the both sides. ☐

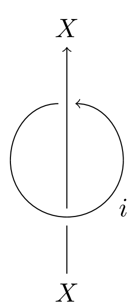
Lemma 3.3. *For all $i \in \mathcal{I}(\mathcal{C})$, we have*

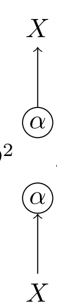
$$\sum_{j \in \mathcal{I}(\mathcal{C})} d_j$$ 

$$= D^2 \delta_{0,i}$$  $i.$

Proof. The trace of the left-hand side $= \sum_{j \in \mathcal{I}(\mathcal{C})} d_j s_{j,i} = \sum_{j \in \mathcal{I}(\mathcal{C})} s_{0,j} s_{j,i} = D^2 \delta_{\bar{0},i} = D^2 \delta_{0,i} =$ the trace of the right-hand side. ☐

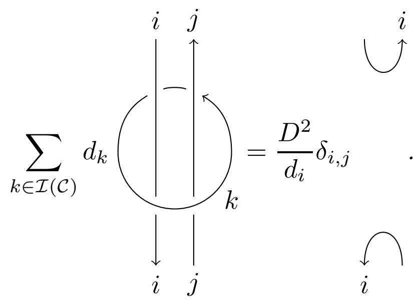
Lemma 3.4. *For all $X \in \mathcal{C}$, we have*

$$\sum_{i \in \mathcal{I}(\mathcal{C})} d_i$$ 

$$= D^2$$ 

.

In particular, for all $i, j \in \mathcal{I}(\mathcal{C})$, we have



Proof. Using proposition 2.3 and lemma 3.3. ☐

For a ribbon category $\mathcal{C}$, we denote by $\mathcal{C}^{rev}$ its reverse category, i.e. the same monoidal category with inverse braiding and twist. There is a canonical braided functor

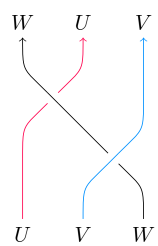
$$\begin{aligned} \Xi \colon \mathcal{C}^{rev} \boxtimes \mathcal{C} &\to Z(\mathcal{C}) \\ U \boxtimes V &\mapsto (U \otimes V, \gamma_{U \otimes V}), \end{aligned}$$

where

$$\gamma_{U \otimes V; W} \coloneqq (\beta^{-1}_{W,U} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes \beta_{V,W})$$

or graphically:

$\gamma_{U \otimes V; W} \coloneqq$



In fact, modularity could be formulated in terms of the functor Ξ, see e.g. [Shi19]:

Theorem 3.5. *A ribbon fusion category $\mathcal{C}$ is a modular tensor category if and only if the canonical functor*

$$\Xi \colon \mathcal{C}^{rev} \boxtimes \mathcal{C} \to Z(\mathcal{C})$$

is a braided equivalence.

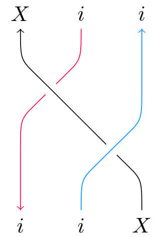
3.2 The Cardy bulk algebra

The following results are in principle known. We collect them for the convenience of the reader.

We define the object

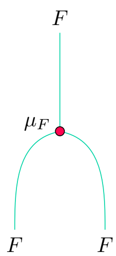
$$L \coloneqq \bigoplus_{i \in \mathcal{I}(\mathcal{C})} X_i^\vee \otimes X_i$$

and equip it with the half-braiding which we call the *dolphin half-braiding*

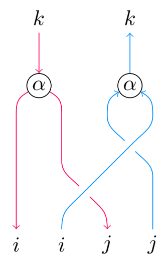
$$\gamma^{dol}_{L;X} := \bigoplus_{i \in \mathcal{I}(\mathcal{C})}$$  .

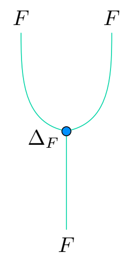
We next show that it has a natural Frobenius algebra structure in $Z(\mathcal{C})$. We denote $F := (L, \gamma^{dol}_L) \in Z(\mathcal{C})$ in the following.

Theorem 3.6. *$(F, \mu_F, \eta_F, \Delta_F, \varepsilon_F)$ is a Frobenius algebra in $Z(\mathcal{C})$, where*

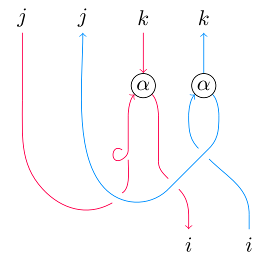


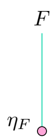
$$:= \bigoplus_{i,j,k \in \mathcal{I}(\mathcal{C})} d_k$$



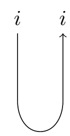


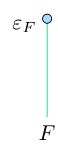
$$:= \bigoplus_{i,j,k \in \mathcal{I}(\mathcal{C})} \frac{d_j d_k}{D^2}$$



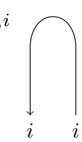


$$:= \bigoplus_{i \in \mathcal{I}(\mathcal{C})} \delta_{0,i}$$

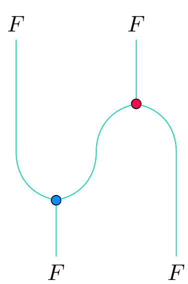




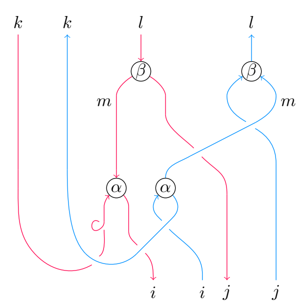
$$:= \bigoplus_{i \in \mathcal{I}(\mathcal{C})} D^2 \delta_{0,i}$$

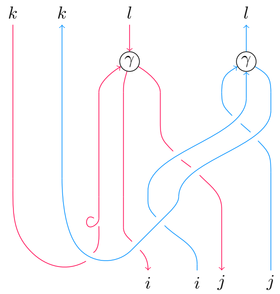
 .

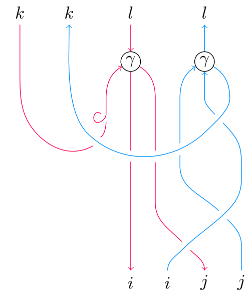
Proof. Note that, all the morphisms here are indeed morphisms of the Drinfeld center $Z(\mathcal{C})$. We only show one of the Frobenius property, the other one can be shown in a similar way:

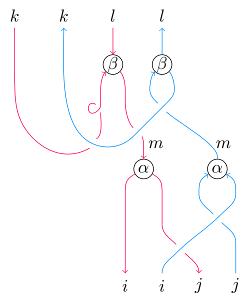


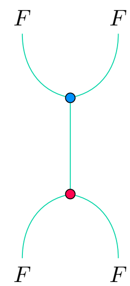
$$= \bigoplus_{i,j,k,l,m \in \mathcal{I}(\mathcal{C})} \frac{d_k d_l d_m}{D^2}$$



$$= \bigoplus_{i,j,k,l\in\mathcal{I}(\mathcal{C})} \frac{d_k d_l}{D^2}$$ 

$$= \bigoplus_{i,j,k,l\in\mathcal{I}(\mathcal{C})} \frac{d_k d_l}{D^2}$$ 

$$= \bigoplus_{i,j,k,l,m\in\mathcal{I}(\mathcal{C})} \frac{d_k d_l d_m}{D^2}$$ 

$=$ 

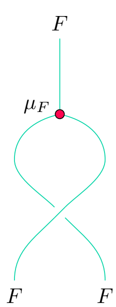
.

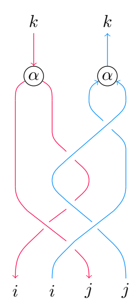
□

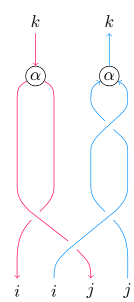
Remark 3.7. The object $L = \bigoplus_{i\in\mathcal{I}(\mathcal{C})} X_i^\vee \otimes X_i$ is actually an explicit expression for the coend $\int^{X\in\mathcal{C}} X^\vee \otimes X$ in the case of fusion categories. The coend makes sense in the more general case of finite tensor categories that are not necessarily semisimple. It carries a canonical Hopf algebra structure with a non-degenerate Hopf pairing [Lyu95]. The Frobenius algebra structure in theorem 3.6 is obtained by changing the co-multiplication and the counit of the Hopf algebra.

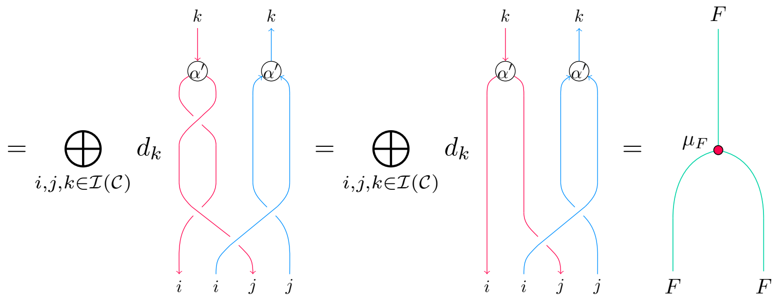
Theorem 3.8. *The Frobenius algebra* $(F, \mu_F, \eta_F, \Delta_F, \varepsilon_F)$ *is commutative, symmetric, and cocommutative.*

Proof. We first prove the commutativity:



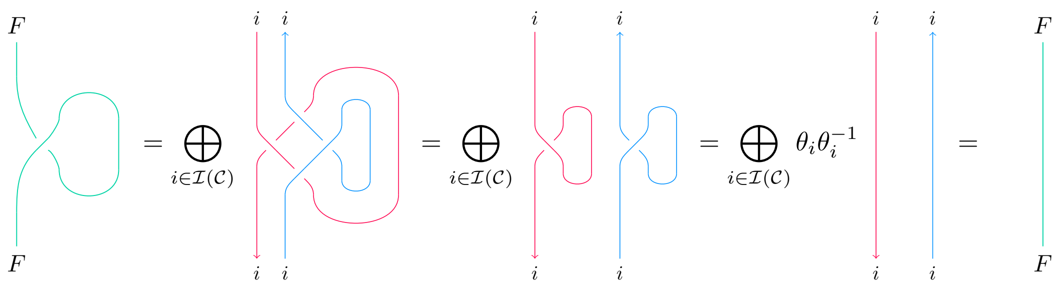
$$= \bigoplus_{i,j,k\in\mathcal{I}(\mathcal{C})} d_k$$ 

$$= \bigoplus_{i,j,k\in\mathcal{I}(\mathcal{C})} d_k$$ 



We always represent by an over-crossing of a green line the corresponding half-braiding of its label in $Z(\mathcal{C})$.

Then we show that F has trivial twist:



Here we have used the fact that for $i \in \mathcal{I}(\mathcal{C})$, $\theta_i := \theta_{X_i} \in \mathrm{End}_{\mathcal{C}}(X_i) \cong \mathbb{C}$. By a result for Frobenius algebras in general ribbon categories [FFRS06b, Proposition 2.25], F is also symmetric and cocommutative. □

3.3 Consistent systems of bulk field correlators

We now give a summary of the concept of consistent systems of CFT bulk field correlators as introduced in [FS17].

An *extended surface* Σ is an oriented surface with a partition of the boundary components into ingoing and outgoing parts, i.e. $\partial\Sigma = \partial_{in}\Sigma \sqcup \partial_{out}\Sigma$ and a marked point for each boundary component. We denote by $\Sigma^g_{p|q}$ an extended surface of genus g with p ingoing boundary components and q outgoing boundary components.

Definition 3.9. The *mapping class group* $\mathrm{Map}(\Sigma)$ of an extended surface Σ is the group of homotopy classes of orientation preserving homeomorphisms $\Sigma \to \Sigma$ that map $\partial_{in}\Sigma$ to itself (hence also $\partial_{out}\Sigma$ to itself) and map marked points to marked points.

Along with the action of the mapping class group on an extended surface, we will also consider the *sewing* of the surface: A sewing $s_{\alpha,\beta}$ along $(\alpha, \beta) \in \pi_0(\partial_{in}\Sigma) \times \pi_0(\partial_{out}\Sigma)$ gives us a new extended surface $s_{\alpha,\beta}(\Sigma) := \cup_{\alpha,\beta}\Sigma$ by identifying the boundary component $\partial_\alpha\Sigma$ with $\partial_\beta\Sigma$ via an orientation preserving homeomorphism $f\colon \partial_\alpha\Sigma \to \partial_\beta\Sigma$ that maps the marked point on $\partial_\alpha\Sigma$ to the marked point on $\partial_\beta\Sigma$. The resulted surface is independent of f up to homeomorphisms.

Definition 3.10. The category $\mathcal{S}\mathrm{urf}$ is the symmetric monoidal category having extended surfaces Σ as objects and the pairs $(\varphi, s_{\alpha,\beta})$ as morphisms $\Sigma \to \cup_{\alpha,\beta}\Sigma$, where $\varphi \in \mathrm{Map}(\Sigma)$ is a mapping class and $s_{\alpha,\beta}$ a sewing. The monoidal product is given by the disjoint union.

In order to describe the composition of the morphisms in the category $\mathcal{S}\mathrm{urf}$, we need the relations among the pairs of mapping classes and sewing. Such relations are discussed in detail in [HLS00].

In a two-dimensional local conformal field theory (to which we refer as a CFT in the following), one assigns to each extended surface Σ a $\mathbb{C}$-vector space called the space of *conformal blocks* on Σ that carries a projective representation of the mapping class group $\mathrm{Map}(\Sigma)$. The spaces of conformal blocks can be viewed as the spaces of solutions to the differential equations imposed by the conformal symmetries called the *Ward identities*. The *bulk field correlators* are elements in the spaces of conformal blocks that satisfies certain consistency conditions regarding the action of mapping class groups and sewing. The spaces of conformal blocks can be constructed as the morphism spaces in a braided monoidal category $\mathcal{D}$ involving an object $F \in \mathcal{D}$. The braided monoidal category $\mathcal{D}$ should be imagined as the representation category of a conformal vertex operator algebra, or a conformal net of observables, and the object $F \in \mathcal{D}$ should be considered as the space of bulk fields. We say that the CFT has the *monodromy data* based on $\mathcal{D}$ and the *bulk object* F.

Since we are interested in correlators of bulk fields, we consider conformal blocks that are based on the modular tensor category $\mathcal{D} = \mathcal{C}^{rev} \boxtimes \mathcal{C}$ for a modular tensor category $\mathcal{C}$ (correlators of bulk fields are obtained by combining conformal blocks for left movers with those for right movers). As mentioned in theorem 3.5, modularity is equivalent to the canonical functor $\Xi\colon \mathcal{C}^{rev} \boxtimes \mathcal{C} \to Z(\mathcal{C})$ being a braided equivalence. Hence, by replacing $\mathcal{D} = \mathcal{C}^{rev} \boxtimes \mathcal{C}$ with the Drinfeld center $Z(\mathcal{C})$, we can apply the Turaev-Viro-Barrett-Westbury state-sum construction, or equivalently, the string-net model described in section 2.

In the so called *Cardy case* rational CFT, we take the object $F = (L, \gamma_L^{dol})$ as the bulk object for our CFT. Define the *pinned block functor*

$$\mathrm{Bl}^F\colon \mathcal{S}\mathrm{urf} \to \mathcal{V}\mathrm{ect}_{\mathbb{C}}$$

by assigning to the extended surface $\Sigma^g_{p|q}$ the finite dimensional vector space

$$\mathrm{Bl}^F(\Sigma^g_{p|q}) \coloneqq Z_{SN,\mathcal{C}}(\Sigma^g_{p+q}, \underbrace{F^\vee, \ldots, F^\vee}_{p}, \underbrace{F, \ldots, F}_{q}) \cong \mathrm{Hom}_{Z(\mathcal{C})}(\mathbb{I}_{Z(\mathcal{C})}, (F^\vee)^{\otimes p} \otimes F^{\otimes q} \otimes L_{Z(\mathcal{C})}^{\otimes g}),$$

and to a morphism (φ, s) between extended surfaces the natural action of the mapping class φ followed by the concatenation of the string-net induced by the sewing s.

Define the *trivial block functor* $\Delta_{\mathbb{C}}\colon \mathcal{S}\mathrm{urf} \to \mathcal{V}\mathrm{ect}_{\mathbb{C}}$ by assigning to every extended surface the vector space $\mathbb{C}$ and to every morphism the identity $\mathrm{id}_{\mathbb{C}}$. *A consistent system of bulk field correlators* is then a monoidal natural transformation

$$v_F\colon \Delta_{\mathbb{C}} \to \mathrm{Bl}^F$$

such that $v_F(\Sigma^0_{1|1}) \coloneqq (v_F)_{\Sigma^0_{1|1}}(1) \in \mathrm{Bl}^F(\Sigma^0_{1|1}) \cong \mathrm{End}_{Z(\mathcal{C})}(F)$ is invertible.

Unraveling the rather compact definition above, we see that the so defined consistent system of bulk field correlators amounts to a choice of a vector

$$v_F(\Sigma^g_{p|q}) \coloneqq (v_F)_{\Sigma^g_{p|q}}(1) \in \mathrm{Bl}^F(\Sigma^g_{p|q})$$

for each extended surface $\Sigma^g_{p|q}$ that is invariant under the action of the mapping class group $\mathrm{Map}(\Sigma^g_{p|q})$, such that the linear map induced by a sewing maps the chosen vector to the chosen vector for the sewn surface.

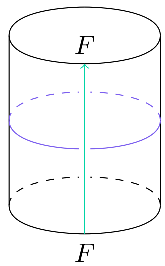
It is shown [FS17, Theorem 4.10] that for a (not necessarily semisimple) modular finite category $\mathcal{D}$, the consistent systems of bulk field correlators with monodromy data based on $\mathcal{D}$ and with bulk object $F \in \mathcal{D}$ are in bijection with structures of a *modular Frobenius algebra* [FS17, Definition 4.9] on F.

Since the bulk object F we are considering does carry the structure of a modular Frobenius algebra [KR09, Section 3] that is given in theorem 3.6, the existence of a consistent set of correlators is guaranteed. We now propose an explicit construction of the bulk field correlators using the string-net model.

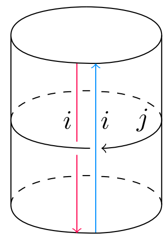
3.4 Invariance under Dehn twists

We start our construction by proposing that the correlator $v_F(\Sigma^0_{1|1})$ is given by the following string-net on a cylinder:

$$v^0_{1|1} \coloneqq$$



$$= \sum_{i,j\in\mathcal{I}(\mathcal{C})} \frac{d_j}{D^2}$$



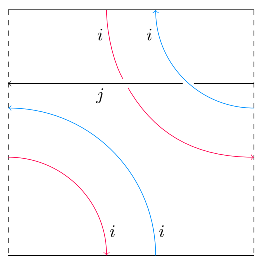
.

In fact, this is the only possible choice since it needs to be an invertible idempotent, hence an identity in $Z_{SN,\mathcal{C}}(\Sigma^0_2, F^\vee, F) \cong \mathrm{End}_{Z(\mathcal{C})}(F)$.

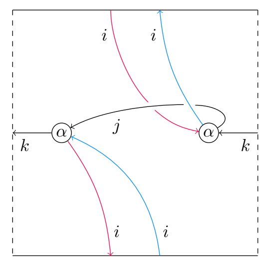
Lemma 3.11. *The string-net $v^0_{1|1}$ is invariant under the action of the mapping class group* $\mathrm{Map}(\Sigma^0_{1|1})$.

Proof. Since $\mathrm{Map}(\Sigma^0_{1|1})$ is generated by a Dehn twist T, we only have to show the invariance under T. Indeed:

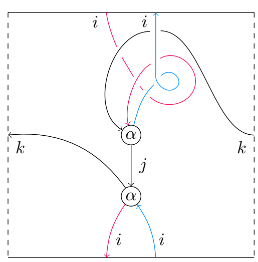
$$Tv^0_{1|1} = \sum_{i,j\in\mathcal{I}(\mathcal{C})} \frac{d_j}{D^2}$$



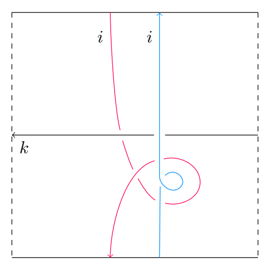
$$= \sum_{i,j,k\in\mathcal{I}(\mathcal{C})} \frac{d_j d_k}{D^2}$$



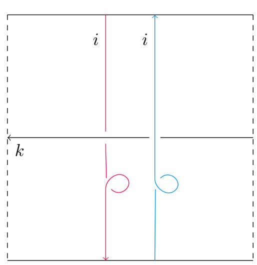
$$= \sum_{i,j,k\in\mathcal{I}(\mathcal{C})} \frac{d_j d_k}{D^2}$$



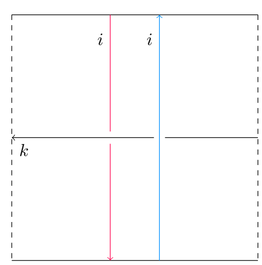
$$= \sum_{i,k\in\mathcal{I}(\mathcal{C})} \frac{d_k}{D^2}$$



$$= \sum_{i,k\in\mathcal{I}(\mathcal{C})} \frac{d_k}{D^2}$$



$$= \sum_{i,k\in\mathcal{I}(\mathcal{C})} \frac{d_k}{D^2}$$



$$= v^0_{1|1}.$$

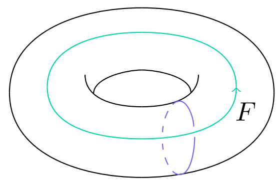
Here we first cut along the the dashed lines, which are identified to produce a cylinder, and insert the completeness relation introduced in proposition 2.3. Then we twist the node on the right counterclockwise so we are able to contract according to the completeness relation along the line that is labeled by j. Twists of the red ribbons and inverse-twists of the blue ribbons are created in this process and they cancel each other in each summand. □

Remark 3.12. If we replace F in the string-net $v^0_{1|1}$ by any other object in $Z(\mathcal{C})$ with trivial twist, by carrying out the same argument we see that the string-net will still be invariant under a Dehn twist.

3.5 Modular invariance

By sewing the cylinder along the boundaries, we get the correlator on a torus:

$$v^1_{0|0} :=$$

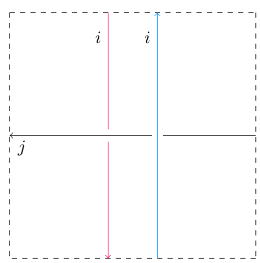


.

Lemma 3.13. *The string-net* $v^1_{0|0}$ *is invariant under the action of the mapping class group* $\mathrm{Map}(\Sigma^1_{0|0})$.

Proof. The mapping class group $\mathrm{Map}(\Sigma^1_{0|0}) \cong \mathrm{SL}(2,\mathbb{Z})$ of the torus is generated by a Dehn twist T and a mapping class S that exchange two generators of the first homology of the torus $H_1(\Sigma^1_0;\mathbb{Z})$. The invariance under the action of T follows immediately from lemma 3.11, so we only need to show the invariance under the action of S. Write

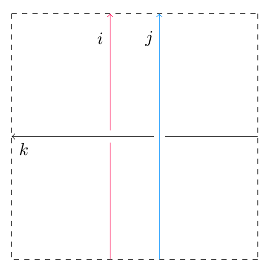
$$v^1_{0|0} = \sum_{i,j\in\mathcal{I}(\mathcal{C})} \frac{d_j}{D^2}$$



$$= \sum_{i\in\mathcal{I}(\mathcal{C})} G_{\bar{i},i},$$

where we define:

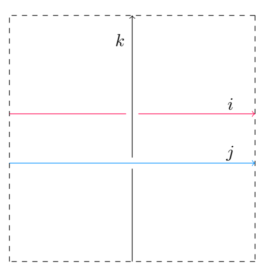
$$G_{i,j} := \sum_{k\in\mathcal{I}(\mathcal{C})} \frac{d_k}{D^2}$$



.

It is sufficient to show that

$$SG_{i,j} = \sum_{k\in\mathcal{I}(\mathcal{C})} \frac{d_k}{D^2}$$



$$= \sum_{a,b\in\mathcal{I}(\mathcal{C})} \frac{s_{\bar{i},a}s_{j,b}}{D^2} G_{a,b},$$

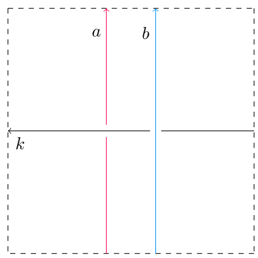
since this implies that

$$Sv^1_{0|0} = \sum_{i\in\mathcal{I}(\mathcal{C})} SG_{\bar{i},i} = \sum_{i,a,b\in\mathcal{I}(\mathcal{C})} \frac{s_{i,a}s_{i,b}}{D^2} G_{a,b} = \sum_{a,b\in\mathcal{I}(\mathcal{C})} \delta_{\bar{a},b} G_{a,b} = \sum_{b\in\mathcal{I}(\mathcal{C})} G_{\bar{b},b} = v^1_{0|0}.$$

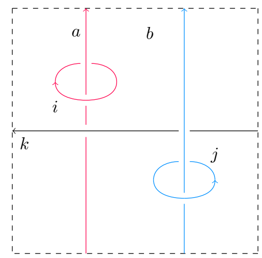
In fact, we have

$$\sum_{a,b\in\mathcal{I}(\mathcal{C})} \frac{s_{\bar{i},a}s_{j,b}}{D^2} G_{a,b}$$

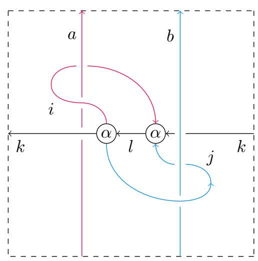
$$= \sum_{a,b,k\in\mathcal{I}(\mathcal{C})} \frac{d_k s_{\bar{i},a} s_{j,b}}{D^4}$$



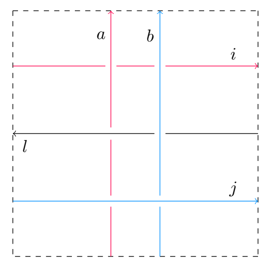
$$= \sum_{a,b,k\in\mathcal{I}(\mathcal{C})} \frac{d_a d_b d_k}{D^4}$$



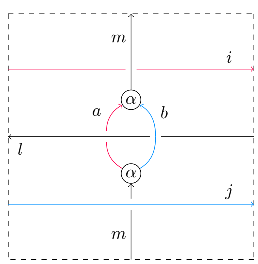
$$= \sum_{a,b,k,l\in\mathcal{I}(\mathcal{C})} \frac{d_a d_b d_k d_l}{D^4}$$



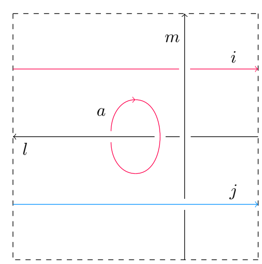
$$= \sum_{a,b,l\in\mathcal{I}(\mathcal{C})} \frac{d_a d_b d_l}{D^4}$$



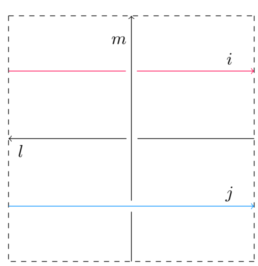
$$= \sum_{a,b,l,m\in\mathcal{I}(\mathcal{C})} \frac{d_a d_b d_l d_m}{D^4}$$



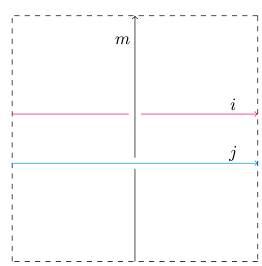
$$= \sum_{a,l,m\in\mathcal{I}(\mathcal{C})} \frac{d_a d_l d_m}{D^4}$$



$$= \sum_{l,m\in\mathcal{I}(\mathcal{C})} \frac{d_l d_m}{D^2} \delta_{0,l}$$



$$= \sum_{m\in\mathcal{I}(\mathcal{C})} \frac{d_m}{D^2}$$

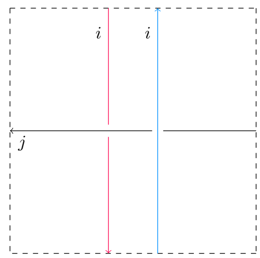


$$= SG_{i,j}.$$

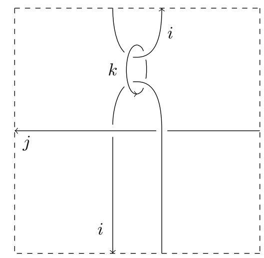
Here we first use lemma 3.2 to insert circles around the red and the blue ribbons, then we cut across the two circles and the line in the middle that is labeled with k and insert a completeness relation. Then we contract along k, cut along the dashed lines one the top and the bottom (they are identified) and insert a completeness relation again. Then we contract along the b line and create a circle around l. Finally we use lemma 3.3 to get rid of the l line. ☐

Remark 3.14. In fact, the modular invariance of the string-net $v^1_{0|0}$ can be made manifest. Namely, the string-net $v^1_{0|0}$ is actually the empty diagram on the torus:

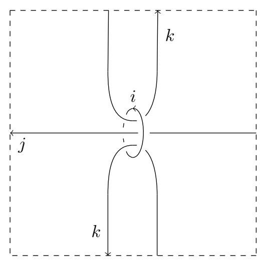
$$v^1_{0|0} = \sum_{i,j\in\mathcal{I}(\mathcal{C})} \frac{d_j}{D^2}$$



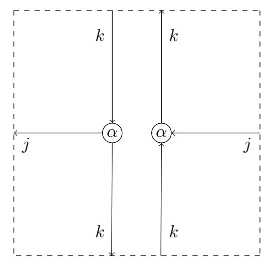
$$= \sum_{i,j,k\in\mathcal{I}(\mathcal{C})} \frac{d_i d_j d_k}{D^4}$$



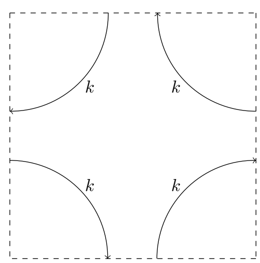
$$= \sum_{i,j,k\in\mathcal{I}(\mathcal{C})} \frac{d_i d_j d_k}{D^4}$$



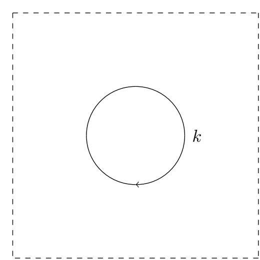
$$= \sum_{j,k\in\mathcal{I}(\mathcal{C})} \frac{d_j d_k}{D^2}$$



$$= \sum_{k\in\mathcal{I}(\mathcal{C})} \frac{d_k}{D^2}$$



$$= \sum_{k\in\mathcal{I}(\mathcal{C})} \frac{d_k}{D^2}$$

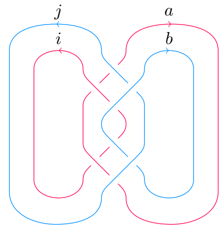
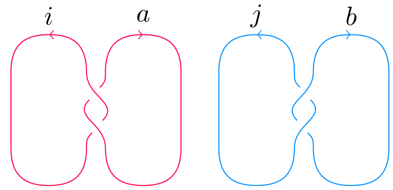


$$=$$

.

Here we have used lemma 3.4. In fact, this is a special case of the more general result we are going to present later in theorem 3.22.

For a modular tensor category $\mathcal{C}$, since $Z(\mathcal{C}) \simeq \mathcal{C}^{rev} \boxtimes \mathcal{C}$, every simple object in $Z(\mathcal{C})$ is isomorphic to $Z_{(i,j)} \coloneqq (X_i \otimes X_j, \gamma_{(i,j)})$ for some $i, j \in \mathcal{I}(\mathcal{C})$ and the half braiding $\gamma_{(i,j)}$ given by the inverse of the braiding of X_i and the braiding of X_j. The s matrix for $Z(\mathcal{C})$ is then given by

$s_{(i,j),(a,b)} =$  $=$  $= s_{\bar{i},a} s_{j,b}.$

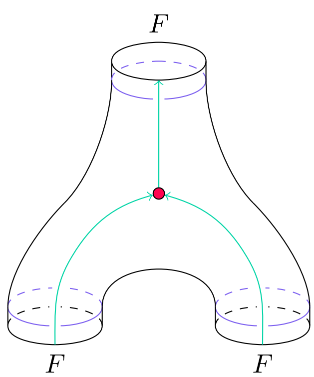
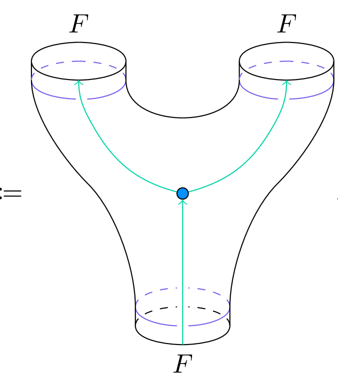
Moreover, the global dimension of $Z(\mathcal{C})$ is given by $D^2_{Z(\mathcal{C})} = \sum_{i,j \in \mathcal{I}(\mathcal{C})} d_i^2 d_j^2 = D^4$, i.e. the square of the global dimension of $\mathcal{C}$. Now, it can be seen from the following representation of the string-net space associated to the torus that follows from factorization

$$Z_{SN,\mathcal{C}}(\Sigma^1_0) \cong \bigoplus_{k \in \mathcal{I}(Z(\mathcal{C}))} Z_{SN,\mathcal{C}}(\Sigma^0_2, Z_k^\vee, Z_k) \cong \bigoplus_{i,j \in \mathcal{I}(\mathcal{C})} Z_{SN,\mathcal{C}}(\Sigma^0_2, Z^\vee_{(i,j)}, Z_{(i,j)})$$

that $\{G_{i,j}\}_{i,j \in \mathcal{I}(\mathcal{C})}$ is a basis for the vector space $Z_{SN,\mathcal{C}}(\Sigma^1_0)$, which transforms under the S-move as expected from the action of the mapping class group. The basis $\{G_{i,j}\}_{i,j \in \mathcal{I}(\mathcal{C})}$ is canonical up to a choice of circle (we think of a torus as the product of two circles), but the coefficient of the $v^1_{0|0}$ is unchanged under a change of the choice of circle since the corresponding change of basis is implemented by the action of the S-move. Hence, the invariant we get is indeed the *charge conjugation* matrix $(\delta_{i,\bar{j}})_{i,j \in \mathcal{I}(\mathcal{C})}$, the entities of which are the coefficients of $v^1_{0|0}$ under the bases $\{G_{i,j}\}_{i,j \in \mathcal{I}(\mathcal{C})}$, even though $v^1_{0|0}$ is in fact the empty diagram on the torus.

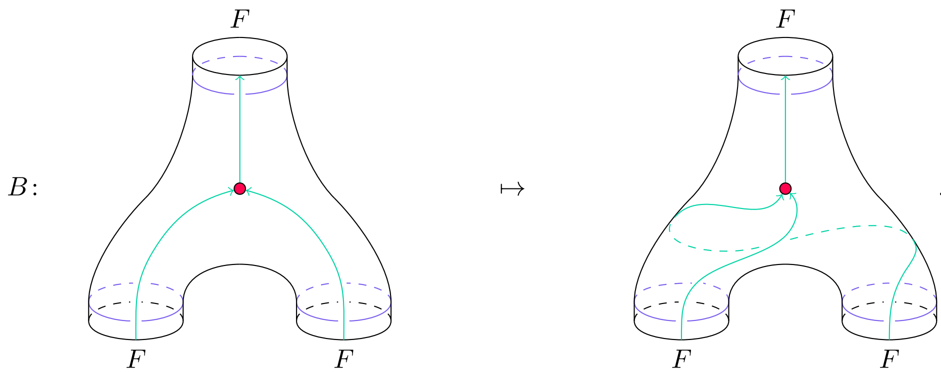
3.6 Invariance under braid moves

We now consider the natural candidates for $v_F(\Sigma^0_{2|1})$ and $v_F(\Sigma^0_{1|2})$:

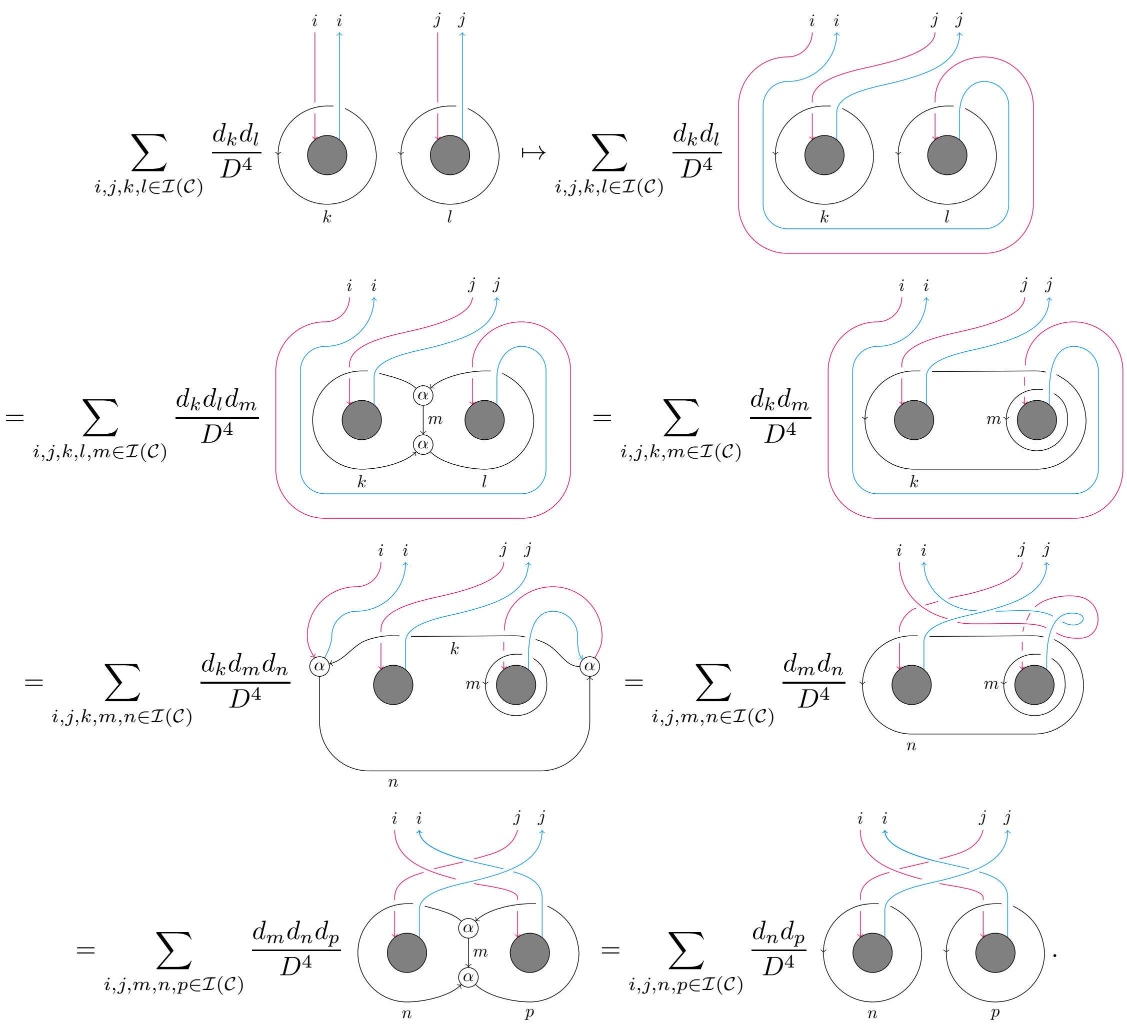
$v^0_{2|1} :=$  and $v^0_{1|2} :=$  .

Lemma 3.15. *The string-net $v^0_{2|1}$ is invariant under the action of the mapping class group* $\mathrm{Map}(\Sigma^0_{2|1})$ *and the string-net $v^0_{1|2}$ is invariant under the action of the mapping class group* $\mathrm{Map}(\Sigma^0_{1|2})$.

Proof. The three projectors guarantee the invariance under the Dehn twists, hence we only have to show the invariance under the braid move, which for instance acts on $v^0_{2|1}$ as:



Since the product and coproduct of the Frobenius algebra F are only commutative with respect to the braiding in the Drinfeld center $Z(\mathcal{C})$, we need to show that the braid move produces the braiding in $Z(\mathcal{C})$ with the help of the projectors. It turns out that this is indeed the case:

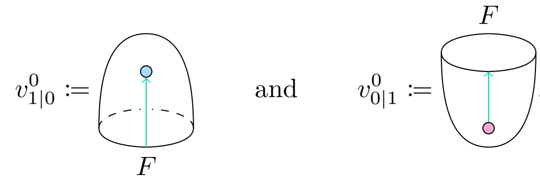


$\square$

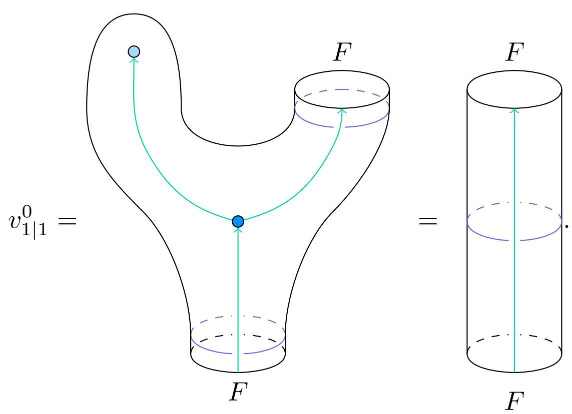
Remark 3.16. As can be seen from the proof above, the proposed string-nets $v^0_{2|1}$ and $v^0_{1|2}$ will be invariant under the braid move if we replace F by any commutative symmetric (hence cocommutative) Frobenius algebra in $Z(\mathcal{C})$.

3.7 Consistency via the Lego-Teichmüller game

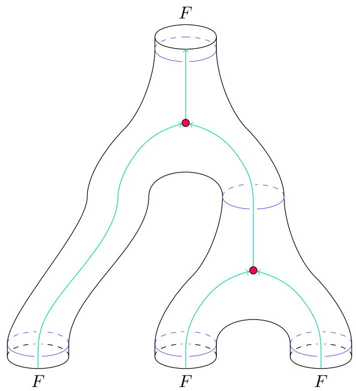
The requirement of consistency with respect to sewing forces us to make the following choices of $v_F(\Sigma^0_{1|0})$ and $v_F(\Sigma^0_{0|1})$, where the counit and the unit of F are used respectively:

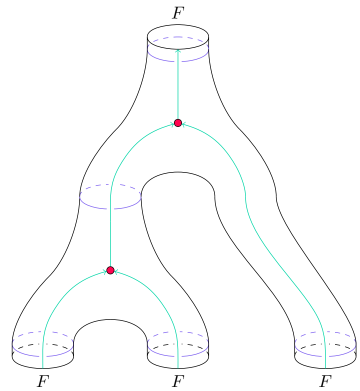


Note that on a disc the projector for the boundary is simply the number $\frac{1}{D^2}\sum_{i\in\mathcal{I}(\mathcal{C})} d_i^2 = 1$. Such a choice of the correlators on the discs leads to the same result on the cylinder. For instance:



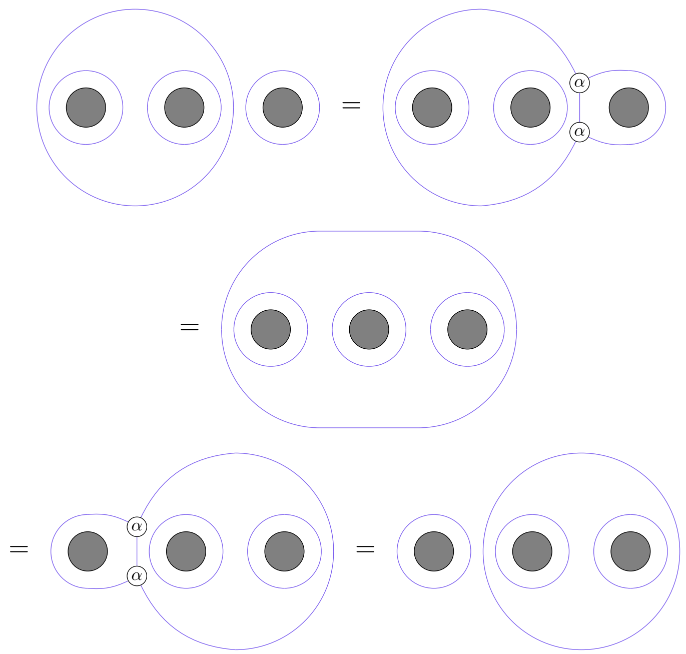
There is a systematic framework of decomposing extended surfaces of any genus and with any number of boundary circles into pairs of pants, cylinders and discs which undergoes the name of *Lego-Teichmüller game*. Combining this with factorization, we can build the correlator for any extended surface $\Sigma^g_{p|q}$ from $v^0_{1|0}$, $v^0_{0|1}$, $v^0_{1|1}$, $v^0_{2|1}$, and $v^0_{1|2}$ (we call them the *elementary string-nets*). The fact that we are using the morphisms of the Frobenius algebra structure on F makes sure that we get the consistent results if we build everything from those elementary correlators and the sewing doesn't involve sewing the boundary components of the same elementary surface. For instance, the associativity of the product gives us:

$v^0_{3|1} :=$ 

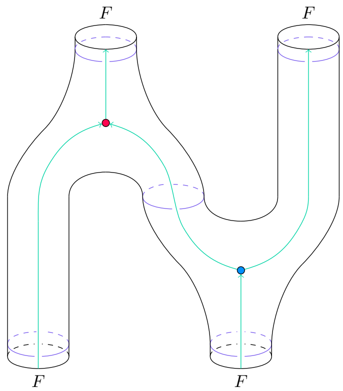
$=$ 

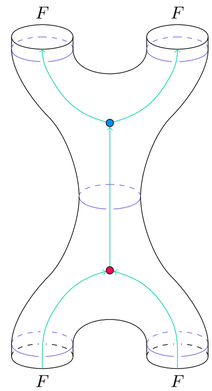
.

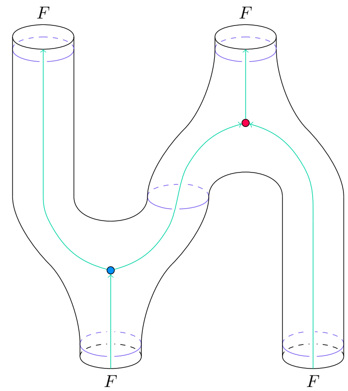
Here we also used the fact that we can pass projectors through each others, as was demonstrated in the proof of lemma 3.15. For instance, we have:



The Frobenius property of F guarantees that:

$v^0_{2|2} :=$ 

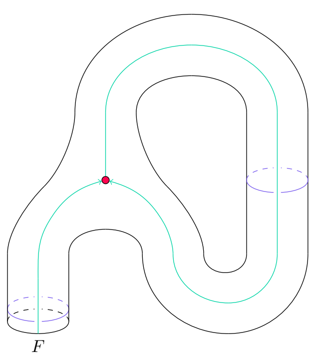
$=$ 

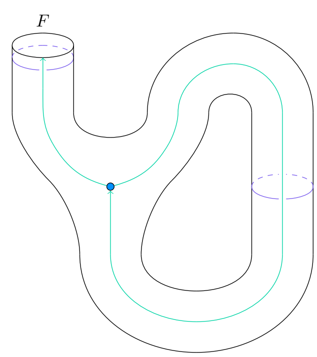
$=$ 

.

In general, given a generic extended surface $\Sigma^g_{p|q}$, there exist different ways of decomposing it into a pairs of pants, cylinders and discs. There exists at least one such decomposition, for which one can label each boundary circle of the components as either ingoing or outgoing, such that only ingoing and outgoing circles are sewed together, and that for each pair of pants there are at most two ingoing and at most two outgoing boundary circles, and that the types of the two boundary circles of each cylinder are different. We call such a decomposition along with a label of boundary circles a *regular decomposition*. Given a regular decomposition Π of an extended surface $\Sigma^g_{p|q}$, we define the string-net $v_F(\Sigma^g_{p|q}, \Pi) \in \mathrm{Bl}^F(\Sigma^g_{p|q})$ to be the one obtained by placing the appropriate elementary string-net on each component. According to the general theory of the Lego-Teichmüller game [BK00], given two regular decompositions Π and Π', there exist a finite sequence of moves, which satisfies finite numbers of relations, that transform Π into Π'. The Lego-Teichmüller game has five types of moves. Among these moves, the only one involving surfaces of genus higher than zero is the S-move on tori with one boundary circle (see lemma 3.17). The invariance under the rest of the moves is taken care of by the (co)associativity, the (co)unity and the Frobenius property of F, along with the special property of the boundary projectors.

We now consider the following string-nets $v^1_{1|0}$ and $v^1_{0|1}$ obtained from regular decompositions of the tori with one boundary circle:

$v^1_{1|0} :=$ 

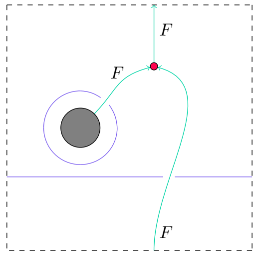
and $v^1_{0|1} :=$ 

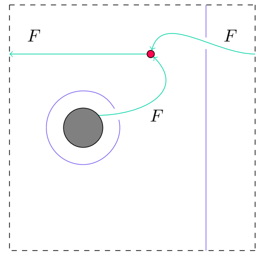
.

It turns out that in order to show that the assignment of the string-net $v_F(\Sigma^g_{p|q}, \Pi) \in \mathrm{Bl}^F(\Sigma^g_{p|q})$ to any extended surface $\Sigma^g_{p|q}$ is independent of the choice of regular decomposition Π and gives rise to a consistent system of bulk field correlators, it remains to show that the string-nets $v^1_{1|0}$ and $v^1_{0|1}$ are invariant under the S-move.

Lemma 3.17. *The string-nets $v^1_{1|0}$ and $v^1_{0|1}$ are invariant under the actions of the corresponding mapping class groups.*

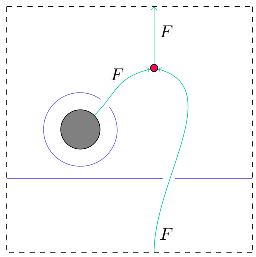
Proof. For $v^1_{1|0}$: we need to show that it is invariant under the S-move:

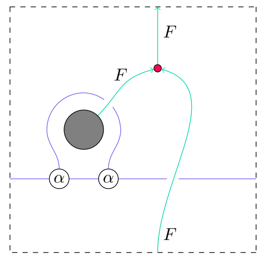
$S:$ 

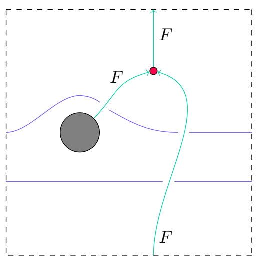
$\mapsto$ 

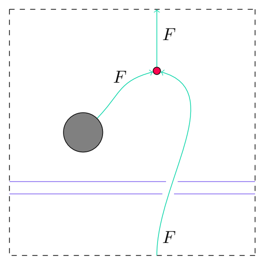
.

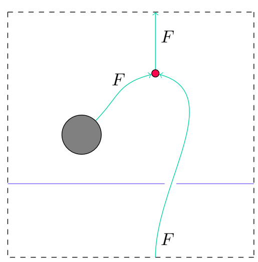
First we notice that the projector around the single boundary component is redundant:

$v^1_{1|0} =$ 

$=$ 

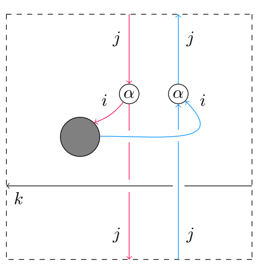
$=$ 

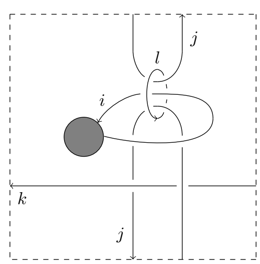
$=$ 

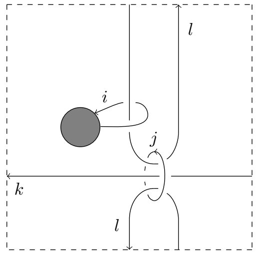
$=$ 

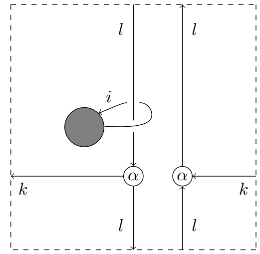
.

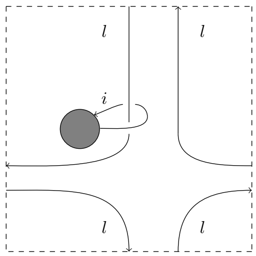
Then by using lemma 3.4, we see:

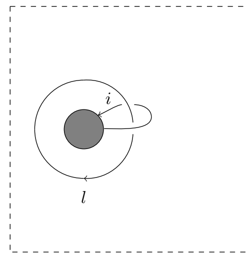
$$v^1_{1|0} = \sum_{i,j,k} \frac{d_j d_k}{D^2}$$


$$= \sum_{i,j,k,l} \frac{d_j d_k d_l}{D^4}$$


$= \sum_{i,j,k,l} \frac{d_j d_k d_l}{D^4}$ 

$= \sum_{i,k,l} \frac{d_k d_l}{D^2}$ 

$= \sum_{i,l} \frac{d_l}{D^2}$ 

$= \sum_{i,l} \frac{d_l}{D^2}$ 

.

The last picture is manifestly invariant under the S-move.

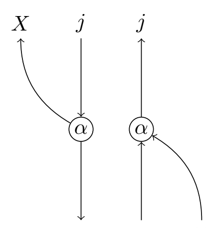
Using both the invariance of $v^1_{1|0}$ and the Frobenius property, it can be decuced that $v^1_{0|1}$ is also invariant under the S-move and hence the action of mapping class group. $\square$

Theorem 3.18. *The assignment of the string-net* $v_F(\Sigma^g_{p|q}, \Pi) \in \mathrm{Bl}^F(\Sigma^g_{p|q})$ *to any extended surface* $\Sigma^g_{p|q}$ *is independent of the choice of regular decomposition* Π *and such assignments give rise to a consistent system of bulk field correlators* $v_F \colon \Delta_{\mathbb{C}} \to \mathrm{Bl}^F$.

The upshot of the argument we are giving here is that given any commutative symmetric Frobenius algebra in $Z(\mathcal{C})$, a consistent system of correlators can be produced via decomposing each extended surface into smaller pieces and sewing together the elementary string-nets according to the decompositions, provided that the string-nets we get on the tori with one boundary circle are invariant under the S-move. However, the consistency for the Cardy case can be seen in a much more straight forward manner, and a closed form of the correlators can be derived: it turns out that the string-nets we get, in their most simplified forms, are as empty as possible.

3.8 Consistency made explicit

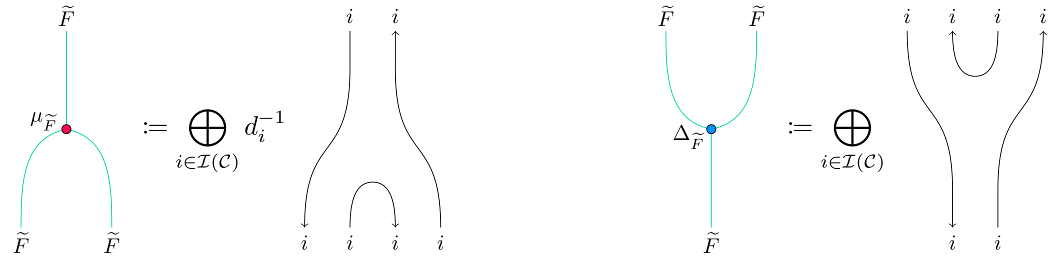
The coend $L = \bigoplus_{i \in \mathcal{I}(\mathcal{C})} X_i^\vee \otimes X_i$ can be also equipped with a different half-braiding that comes from the central monad:

$\gamma^{non}_{L;X} \coloneqq \bigoplus_{i,j \in \mathcal{I}(\mathcal{C})} d_j$ 

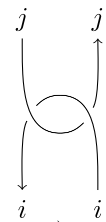
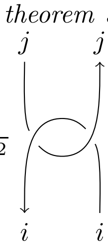
.

We call it the *non-crossing half-braiding* for the obvious reason.

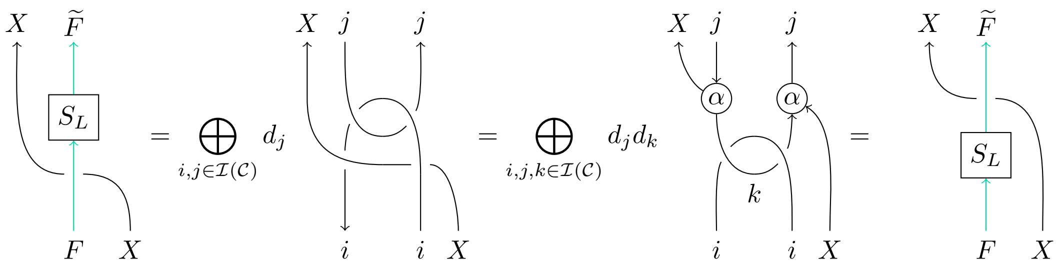
We denote $\widetilde{F} := (L, \gamma_L^{non}) \in Z(\mathcal{C})$. There is also a naturally defined Frobenius algebra structure on this object, with the multiplication and co-multiplication given by:



In this case, it is easy to show that this is a special symmetric Frobenius algebra.

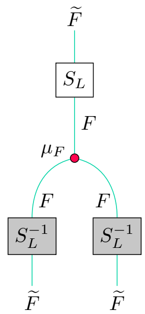
Theorem 3.19. *For a modular tensor category $\mathcal{C}$, the morphism* $S_L := \bigoplus_{i,j \in \mathcal{I}(\mathcal{C})} d_j$  $\in \mathrm{End}_{\mathcal{C}}(L)$ *is an isomorphism of Frobenius algebras in $Z(\mathcal{C})$ from the Cardy bulk algebra $(F, \mu_F, \eta_F, \Delta_F, \varepsilon_F)$ in theorem 3.6 to the Frobenius algebra $(\widetilde{F}, \mu_{\widetilde{F}}, \eta_{\widetilde{F}}, \Delta_{\widetilde{F}}, \varepsilon_{\widetilde{F}})$ defined above, with the inverse given by* $S_L^{-1} := \bigoplus_{i,j \in \mathcal{I}(\mathcal{C})} \frac{d_j}{D^2}$  .

Proof. Using proposition 2.3, it is not hard to see that $S_L \in \mathrm{Hom}_{Z(\mathcal{C})}(F, \widetilde{F})$ and $S_L^{-1} \in \mathrm{Hom}_{Z(\mathcal{C})}(\widetilde{F}, F)$. For instance:

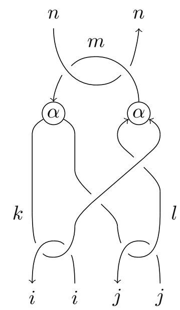


The fact that S_L and S_L^{-1} are inverse to each other is equivalent to lemma 3.4.

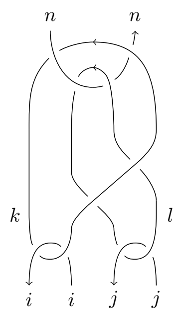
To show that S_L is an isomorphism of algebras, we notice:



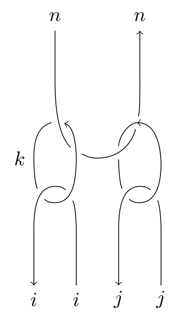
$$= \bigoplus_{i,j,k,l,m,n \in \mathcal{I}(\mathcal{C})} \frac{d_k d_l d_m d_n}{D^4}$$



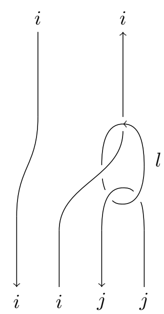
$$= \bigoplus_{i,j,k,l,n \in \mathcal{I}(\mathcal{C})} \frac{d_k d_l d_n}{D^4}$$



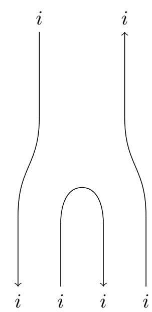
$$= \bigoplus_{i,j,k,l,n \in \mathcal{I}(\mathcal{C})} \frac{d_k d_l d_n}{D^4}$$



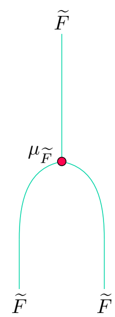
$$= \bigoplus_{i,j,l \in \mathcal{I}(\mathcal{C})} \frac{d_l}{D^2}$$



$$= \bigoplus_{i \in \mathcal{I}(\mathcal{C})} d_i^{-1}$$

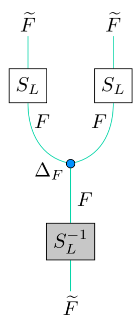


$$=$$

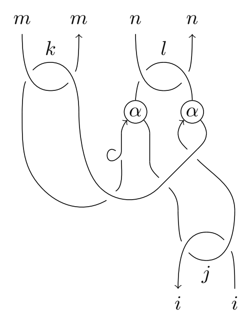


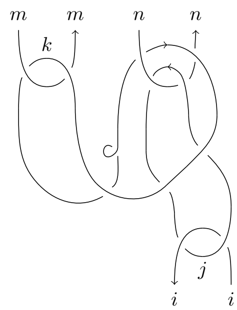
.

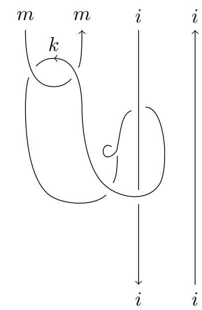
Hence $S_L \circ \mu_F = \mu_{\widetilde{F}} \circ (S_L \otimes S_L)$. Similarly, one shows that S_L is also an isomorphism of coalgebras:

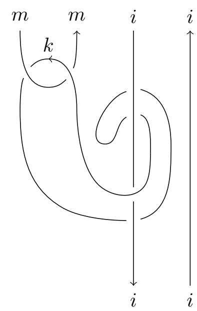


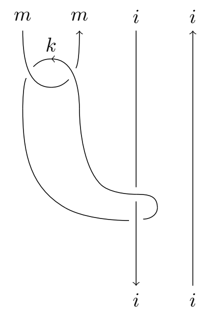
$$= \bigoplus_{i,j,k,l,m,n \in \mathcal{I}(\mathcal{C})} \frac{d_j d_k d_l d_m d_n}{D^4}$$

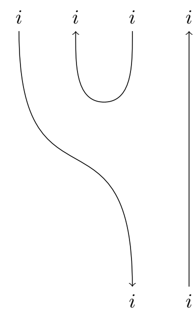


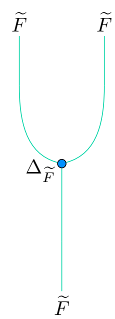
$$= \bigoplus_{i,j,k,m,n \in \mathcal{I}(\mathcal{C})} \frac{d_j d_k d_m d_n}{D^4}$$ 

$$= \bigoplus_{i,k,m \in \mathcal{I}(\mathcal{C})} \frac{d_k d_m}{D^2}$$ 

$$= \bigoplus_{i,k,m \in \mathcal{I}(\mathcal{C})} \frac{d_k d_m}{D^2}$$ 

$$= \bigoplus_{i,k,m \in \mathcal{I}(\mathcal{C})} \frac{d_k d_m}{D^2}$$ 

$$= \bigoplus_{i \in \mathcal{I}(\mathcal{C})}$$ 

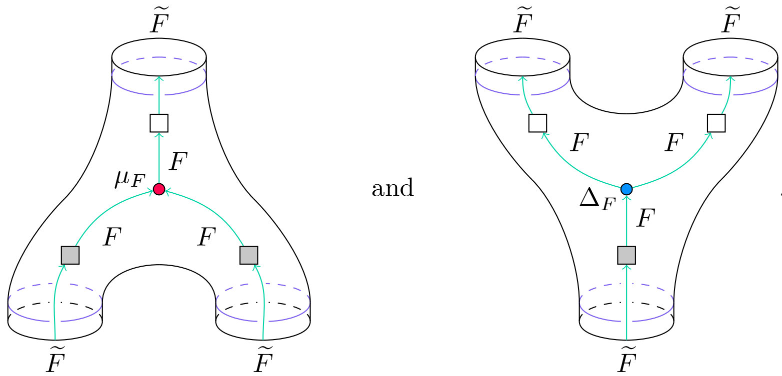
$=$ 

.

□

Corollary 3.20. *For a modular tensor category $\mathcal{C}$, $(\widetilde{F}, \mu_{\widetilde{F}}, \eta_{\widetilde{F}}, \Delta_{\widetilde{F}}, \varepsilon_{\widetilde{F}})$ is a commutative, symmetric Frobenius algebra in $Z(\mathcal{C})$. In particular, $\widetilde{F}$ has trivial twist.*

The isomorphism S_L induces isomorphisms of string-net spaces. This is implemented by composing the string-net with S_L near the outgoing boundary and precomposing the string-net with S_L^{-1} near the ingoing boundary. For instance, applying to the invariants on pairs of pants, we get



Here the white boxes stand for S_L and the gray ones stand for S_L^{-1}. Since both are morphisms in $Z(\mathcal{C})$, it makes no difference which side of the projectors we put the boxes on, as long as we use the correct half-braidings.

On the other hand, if we take $\widetilde{F}$ as our bulk object, we get a new set of conformal blocks

$$\mathrm{Bl}^{\widetilde{F}} \colon \mathcal{S}\mathrm{urf} \to \mathcal{V}\mathrm{ect}_{\mathbb{C}}$$

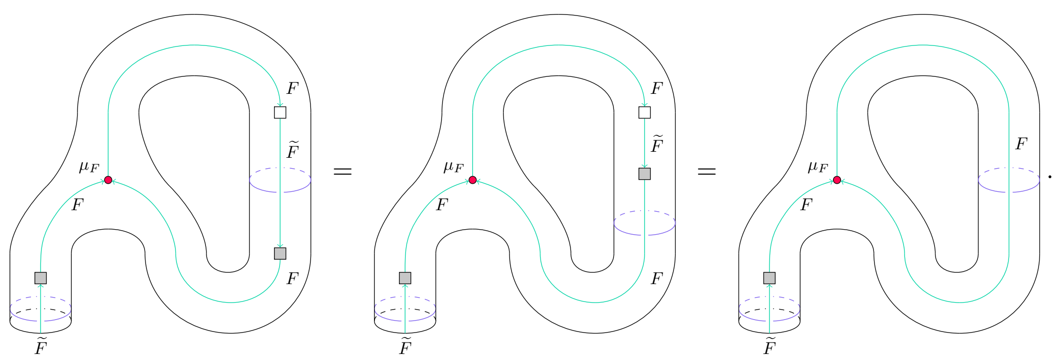
as well as a new set of correlators

$$v_{\widetilde{F}} \colon \Delta_{\mathbb{C}} \to \mathrm{Bl}^{\widetilde{F}}.$$

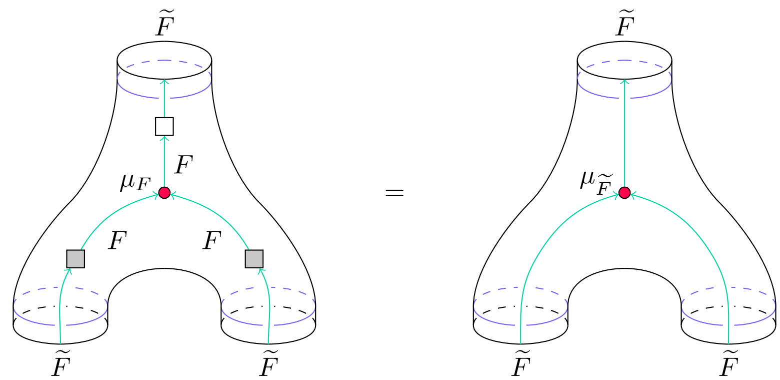
In fact, the induced isomorphisms of string-net spaces give rise to a natural isomorphisms of conformal blocks

$$\mathrm{Bl}^{S_L} \colon \mathrm{Bl}^F \to \mathrm{Bl}^{\widetilde{F}},$$

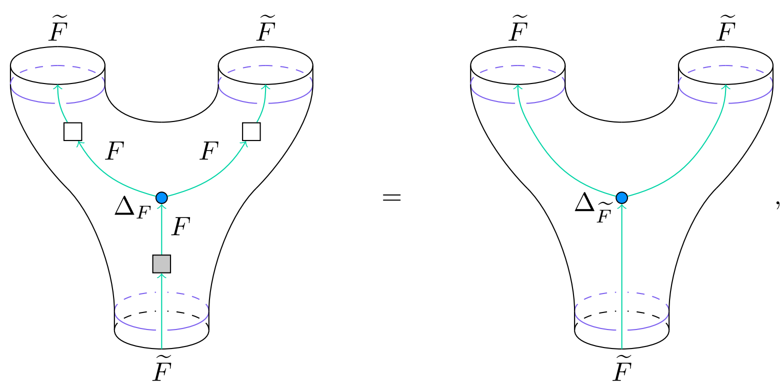
since the isomorphisms intertwine the action of mapping class groups and sewing. For example, by sewing a pair of ingoing and outgoing boundaries of $\Sigma^0_{2|1}$, we get:



Moreover, due to the fact that



and



we get a commutative diagram of natural transformations:

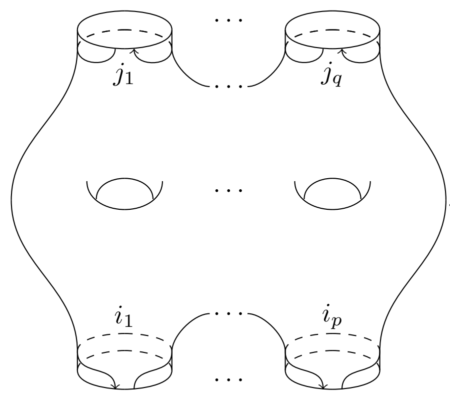
$$\begin{array}{ccc} & \Delta_{\mathbb{C}} & \\ {}^{v_F}\swarrow & & \searrow^{v_{\widetilde{F}}} \\ \mathrm{Bl}^{F} & \xrightarrow[\mathrm{Bl}^{S_L}]{} & \mathrm{Bl}^{\widetilde{F}} \end{array} .$$

Intuitively, the two isomorphic Frobenius algebras produce equivalent sets of correlators. The natural isomorphism Bl^{S_L} gives the precise way to relate them.

It turns out that the correlators given by the Frobenius algebra $(\widetilde{F}, \mu_{\widetilde{F}}, \eta_{\widetilde{F}}, \Delta_{\widetilde{F}}, \varepsilon_{\widetilde{F}})$ are particularly easy to compute:

Theorem 3.21. *For all* $p, q, g \in \mathbb{Z}_{\geq 0}$*, we have*

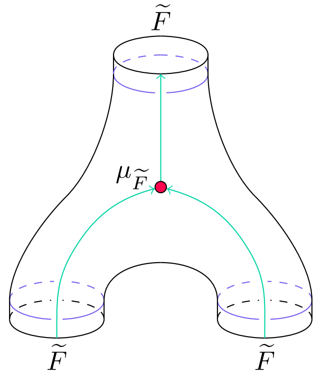
$$v_{\widetilde{F}}(\Sigma^g_{p|q}) = \sum_{i_1,\dots,i_p,j_1,\dots,j_q \in \mathcal{I}(\mathcal{C})} \frac{d_{j_1}\dots d_{j_q}}{D^{2p}}$$



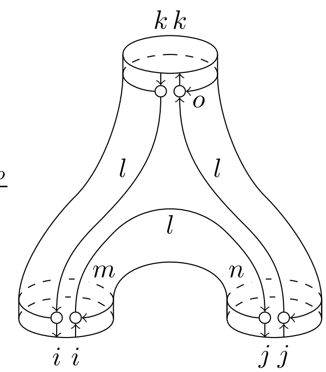
.

Proof. Essentially, we only have to check the cases in which $g = 0$ and $p + q \leq 3$.

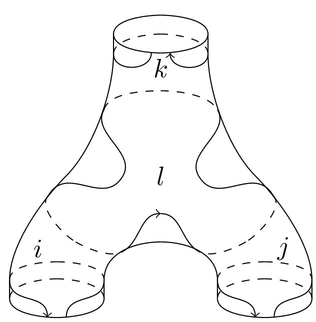
$$v_{\widetilde{F}}(\Sigma^0_{2|1}) =$$



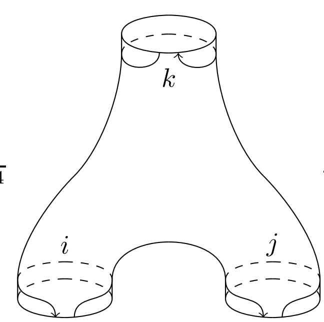
$$= \sum_{i,j,k,l,m,n,o \in \mathcal{I}(\mathcal{C})} \frac{d_k d_l d_m d_n d_o}{D^6}$$



$$= \sum_{i,j,k,l \in \mathcal{I}(\mathcal{C})} \frac{d_k d_l}{D^6}$$



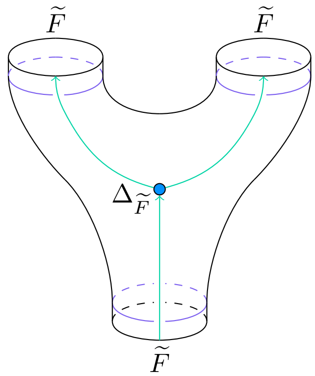
$$= \sum_{i,j,k \in \mathcal{I}(\mathcal{C})} \frac{d_k}{D^4}$$



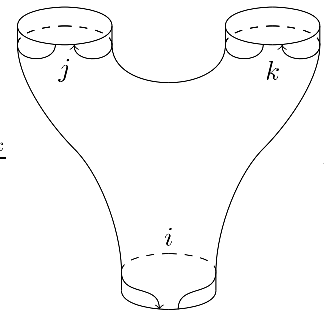
.

Similarly, we have

$v_{\widetilde{F}}(\Sigma^0_{1|2}) =$



$= \sum_{i,j,k \in \mathcal{I}(\mathcal{C})} \frac{d_j d_k}{D^2}$



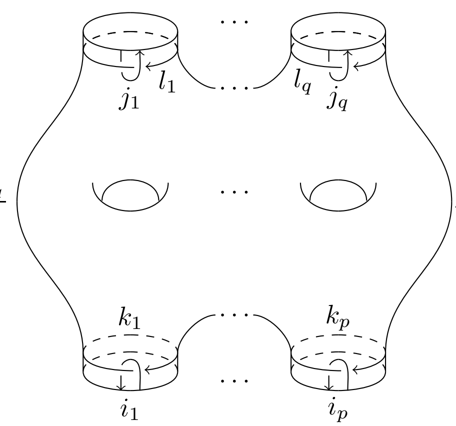
.

The arguments concerning the unit and counit are even more straight forward. Notice that, whenever we sew together a pair of boundaries, we get a contractible circle that cancels out a factor of D^2. □

By applying the inverse of the natural isomorphism Bl^{S_L}, we get the simplified form of the consistent system of correlators v_F given by the Cardy bulk algebra $(F, \mu_F, \eta_F, \Delta_F, \varepsilon_F)$ that manifests both the invariance and consistency:

Theorem 3.22. *For all $p, q, g \in \mathbb{Z}_{\geq 0}$, we have*

$v_F(\Sigma^g_{p|q}) = \sum_{i_1,\dots,i_p,j_1,\dots,j_q,k_1,\dots,k_p,l_1,\dots,l_q \in \mathcal{I}(\mathcal{C})} \frac{d_{j_1} \dots d_{j_q} d_{k_1} \dots d_{k_p} d_{l_1} \dots d_{l_q}}{D^{2(p+q)}}$



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