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Algebras of non-local screenings and diagonal Nichols algebras

Ilaria Flandoli, Simon Lentner

Fachbereich Mathematik, Universität Hamburg
Bereich Algebra und Zahlentheorie
Bundesstraße 55, D-20146 Hamburg

Abstract

In a vertex algebra setting, we consider non-local screening operators, which are associated to any non-integral lattice. We have previously shown that under certain restrictions these screening operators satisfy the relations of a quantum shuffle algebra or Nichols algebra, with a diagonal braiding associated to the non-locality and non-integrality.

In the present article, we take all finite-dimensional diagonal Nichols algebras, as classified by Heckenberger, and find all realisations of the braiding by a lattice that are compatible with reflections. Usually the realising lattices are unique or parametrised by one or two parameters. We then study the associated algebra of screenings with improved methods. Typically, for positive definite lattices we obtain the Nichols algebra, such as the positive part of the quantum group, and for negative definite lattices we obtain a certain extension of the Nichols algebra generalising the infinite quantum group with a large centre. Our result in particular covers so-called (p, p')-models and Lie superalgebras, which had been of interest to other research.

Contents

1 Introduction

Given a (non-integral) lattice Λ with basis $v_1, \ldots, v_n$, we associate the Gram matrix $m_{ij} = (v_i, v_j)$ and a braiding matrix $q_{ij} = e^{\pi i (v_i, v_j)}$. For the Heisenberg vertex algebra, we consider the non-local screening operators $\mathfrak{Z}_{v_i}$. Then in [Len17] it was proven that the screening operators obey the relations of the diagonal Nichols algebra $\mathcal{B}(q)$, as long as the so-called *smallness* condition on m_{ij} is fulfilled. The application one has in mind is the construction of vertex algebras with the same representation theory as quantum groups [Wak86][FGST06a][AM08][FT10] and beyond [Sem11].

The goal of this article is to find all lattices Λ, such that the associated braiding q_{ij} gives a finite-dimensional Nichols algebra as classified in [Hec06], and such that the Weyl reflections on q_{ij} lift to reflections on m_{ij} in a suitable sense. In this case we say that Λ, m_{ij} *realise* the braiding matrix q_{ij}.

As a second goal, for each realising lattice Λ we then study the algebra of screening operators by analysing the smallness condition which does usually not hold. Depending on the free parameters in the realisation we find extensions of Nichols algebras.

As a main example, let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra with simple roots $\alpha_1, \ldots, \alpha_n$. Let q be a root of unity and $q_{ij} = q^{(\alpha_i, \alpha_j)}$ be the braiding with associated Nichols algebra $\mathcal{B}(q)$ the positive part of the small quantum group $u_q(\mathfrak{g})^+$.

For every real number r with $q = e^{i\pi r}$ we get a realising lattice, namely the root lattice of $\mathfrak{g}$ rescaled by r. Then the screening algebra is for $r > 0$ the Nichols algebra $u_q(\mathfrak{g})^+$ and for $r < 0$ conjecturally the positive part of the Kac-Procesi-DeConcini quantum group $U_q^{\mathcal{K}}(\mathfrak{g})$ (see 5.4).

These are all solutions, if q has sufficiently large order. But for e.g. $q = -1$, $\mathfrak{g} = \mathfrak{sl}_n$ we get an additional family of realising solutions. They are associated to the Lie superalgebras $\mathfrak{sl}(n'|n'')$, $n' + n'' = n$, which give incidentally the same braiding matrices but different realising lattices. The screening algebra is for

$r > 0$ again the Nichols algebra $u_q(\mathfrak{sl}_n)^+ = u_q(\mathfrak{sl}(n'|n''))^+$, $q = -1$, and for $r < 0$ conjecturally the positive part of the Kac-Procesi-DeConcini quantum super group $U_q^{\mathcal{K}}(\mathfrak{sl}(n'|n''))$, which is different from $U_q^{\mathcal{K}}(\mathfrak{sl}_n)$.

The paper is organized as follows:
In section 2 we present some preliminary notions on Nichols algebras.
In section 3, we briefly present the notion of vertex algebras and their representation theory. In particular we look at the Heisenberg algebra $\mathcal{H}$ and its modules $\mathcal{H}_v$, $v \in \mathbb{C}^n$. Then we introduce screening operators and review in theorem 3.2 the result from [Len17] that if m_{ij} fulfils the smallness condition then the screening operators $\mathfrak{Z}_{v_i}$ generate the Nichols algebra $\mathcal{B}(q)$, $q_{ij} = e^{i\pi m_{ij}}$. Then we prove in theorem 3.5 our first main result, which weakens the smallness conditions on m_{ij} by analytical continuation.
In section 4 we state the classification problem: for a given braiding q_{ij}, we classify lattices Λ we Gram matrix $m_{ij} = (v_i, v_j)$ such that $q_{ij} = e^{i\pi m_{ij}}$ and such that m_{ij} is compatible with Nichols algebra reflections in the sense of 5.
In section 5 we classify realising lattices for braidings of Cartan type: starting from a simple Lie algebra $\mathfrak{g}$ we rescale its root lattice by a parameter r and prove that this lattice Λ always realises the braiding. Except small values of q, we prove this solution to be unique. Then we calculate the screening algebra depending on r.
In section 6 we proceed for Lie superalgebras. A classification of the realisable lattices of this type is presented and explicit examples in rank 2 and arbitrary rank are shown.
In section 7.1 we construct realising lattices for all other finite dimensional diagonal Nichols algebras in rank 2.
In section 7.2 we show that the examples presented in the previous sections exhaust all realising lattices for rank 2 braidings of finite dimensional diagonal Nichols algebras.
In section 8 we present the construction and classification for rank 3.
In section 9 we indicate how this determines the realising lattices for rank ≥ 4.
Final tables show all realising Λ, m_{ij} for rank 2 and 3.

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2 Preliminaries on Nichols algebras

We start by giving the basic definitions and examples regarding Nichols algebras.

2.1 Definition and properties

Let $M = \langle x_1, \ldots, x_{rank} \rangle_{\mathbb{C}}$ be a complex vector space and let $(q_{ij})_{i,j=1,\ldots rank}$ be an arbitrary matrix with $q_{ij} \in \mathbb{C}^{\times}$. This defines a *braiding of diagonal type* on M via:

$$c: \; c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i.$$

Hence we get an action ρ_n of the *braid group* $\mathbb{B}_n$ on $M^{\otimes n}$ via:

$$c_{i,i+1} := \mathrm{id} \otimes \cdots \otimes c \otimes \cdots \otimes \mathrm{id}.$$

Definition 2.1. Let (M, c) be a braided vector space. We consider the canonical projections $\mathbb{B}_n \twoheadrightarrow \mathbb{S}_n$ sending the braiding $c_{i,i+1}$ to the transposition $(i, i+1)$. There exists the *Matsumoto section* of sets $s : \mathbb{S}_n \to \mathbb{B}_n$ given by $(i, i+1) \mapsto c_{i,i+1}$ which has the property $s(xy) = s(x)s(y)$ whenever $\mathrm{length}(xy) = \mathrm{length}(x) + \mathrm{length}(y)$. Then we define the *quantum symmetrizer* by

$$Ш_{q,n} := \sum_{\tau \in \mathbb{S}_n} \rho_n(s(\tau)) \tag{1}$$

where ρ_n is the representation of $\mathbb{B}_n$ on $M^{\otimes n}$ induced by the braiding c. Then the *Nichols algebra* or *quantum shuffle algebra* generated by (M, c) is defined by

$$\mathcal{B}(M) := \bigoplus_n M^{\otimes n} / \ker(Ш_{q,n}).$$

Remark 2.2. This characterization enables one in principle to compute $\mathcal{B}(M)$ in each degree, but it is very difficult to find generators and relations for $\mathcal{B}(M)$ since in general the kernel of the map $Ш_{q,n}$ is hard to calculate in explicit terms. In fact $\mathcal{B}(M)$ is a Hopf algebra in a braided sense and as such it enjoys several equivalent universal properties.

2.2 Examples

We now present some examples.

Example 2.3 (Rank 1)**.** [Nichols78] Let $M = x\mathbb{C}$ be a 1-dimensional vector space with braiding given by $q_{11} = q \in \mathbb{C}^\times$, then

$$\mathbb{C} \ni Ш_{q,n} = \sum_{\tau \in \mathbb{S}_n} q_{11}^{|\tau|} = \prod_{k=1}^{n} \frac{1 - q^k}{1 - q} =: [n]_q!$$

Because this polynomial has zeros all $q \neq 1$ of order $\leq n$ the Nichols algebra is

$$\mathcal{B}(M) = \begin{cases} \mathbb{C}[x]/(x^\ell), & q_{11} \text{ primitive } \ell\text{-th root of unity} \\ \mathbb{C}[x], & \text{else} \end{cases}$$

Example 2.4 (Quantum group)**.** Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank n with root system Φ and simple roots $\alpha_1, \ldots, \alpha_n$ and Killing form (α_i, α_j). Let q be a primitive ℓ-th root of unity. Consider the n-dimensional vector space M with diagonal braiding $q_{ij} := q^{(\alpha_i, \alpha_j)}$ Then the Nichols algebra $\mathcal{B}(M)$ is isomorphic to the positive part $u_q(\mathfrak{g})^+$ of the small quantum group $u_q(\mathfrak{g})$, which is a deformation of the universal enveloping of a Lie algebra $U(\mathfrak{g})$.

2.3 Generalized root system and Weyl groupoid

Every finite-dimensional Nichols algebra comes with a generalized *root system*, a *Weyl groupoid* and a *PBW-type basis* [Kha00], [Hec06], [HS08].

The Weyl groupoid plays a similar role as the Weyl group does for ordinary root systems in Lie algebras, but in the general case not all Weyl chambers look the same: different braiding matrices and even different Dynkin diagrams are attached to different Weyl chambers (i.e. groupoid objects). This behaviour already appears for Lie superalgebras.
The finite Weyl groupoids are classified in [CH09], [CH10]; apart from the finite Weyl groups there are additional series $D_{n,m}$ and 74 sporadic examples.

Remark 2.5. We remark that the generalized root systems do not provide a complete classification as they do in the theory of complex semisimple Lie algebras: there are non-isomorphic Nichols algebras whose corresponding Weyl groupoids are equivalent and there are Weyl groupoids to which no finite dimensional diagonal Nichols algebra corresponds.

Definition 2.6. To every braiding matrix q_{ij} we define the associated Cartan matrix (a_{ij}) for all $i \neq j$ by

$$a_{ii} = 2 \quad \text{and} \quad a_{ij} := -\min\left\{ m \in \mathbb{Z} \mid q_{ii}^{-m} = q_{ij}q_{ji} \quad \text{or} \quad q_{ii}^{(1+m)} = 1 \right\}. \tag{2}$$

Definition 2.7. We call a root α_i q-Cartan, respectively q-truncation, if it satisfies:

$$q_{ii}^{a_{ij}} = q_{ij}q_{ji}, \qquad \text{respectively} \qquad q_{ii}^{1-a_{ij}} = 1. \tag{3}$$

We observe that a root can be both q-Cartan *and* q-truncation. In particular we will call a root *only* q-Cartan, respectively *only* q-truncation, if it is exclusively so.

Definition 2.8. The Weyl groupoid is generated by reflections, defined for every k as:

$$\begin{aligned} \mathcal{R}^k : \mathbb{Z}^n &\longrightarrow \mathbb{Z}^n \\ \alpha_i &\longmapsto \alpha_i - a_{ki}\alpha_k \end{aligned}$$

which transform the bicharacter q_{ij} into the bicharacter $\mathcal{R}^k(q_{ij})$. As we said, this is a new braiding matrix, possibly different from the original one and with possibly different associated Cartan matrix. However, the Nichols algebras have the same dimension and are closely related [HS11, BLS15].

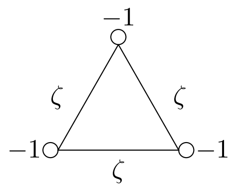
Remark 2.9. With $\mathcal{R}^k$ we mean the reflection around the k-th simple root in the respective Weyl chamber, which can be again expressed in coordinates with respect to the simple roots $\alpha_1, \ldots, \alpha_n$ in some fixed initial Weyl chamber.

Example 2.10. We consider, as an example, the finite dimensional diagonal Nichols algebra of rank 3 with the following braiding in an initial Weyl chamber

$$q_{ii} = -1, \qquad q_{ij}q_{ji} = \zeta,$$

with $i \neq j$ and $\zeta \in \mathcal{R}_3$ a primitive third root of unity.

Following Heckenberger, we write the braiding as a diagram, where nodes correspond to the simple roots α_i and are decorated by the braiding q_{ii} and each edge is decorated by the double braiding $q_{ij}q_{ji}$:



As it turns out, the overall root system has seven positive roots. If $\{\alpha_1, \alpha_2, \alpha_3\}$ are the simple roots in the Weyl chamber shown above, then the positive roots in this basis are:

$$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_{12},\ \alpha_{23},\ \alpha_{13},\ \alpha_{123}\}$$

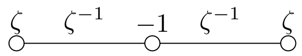
and the Cartan matrix of this Weyl chamber is:

$$a^I_{ij} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

We now reflect around α_2. Then the new simple roots are $\{\alpha_{12}, -\alpha_2, \alpha_{23}\}$ and the new braiding matrix is:

$$q_{12,12} = q_{23,23} = \zeta \qquad q_{22} = -1$$
$$q_{12,2}q_{2,12} = q_{23,2}q_{2,23} = \zeta^{-1} \qquad q_{12,23}q_{23,12} = 1$$

which is in diagram notation:



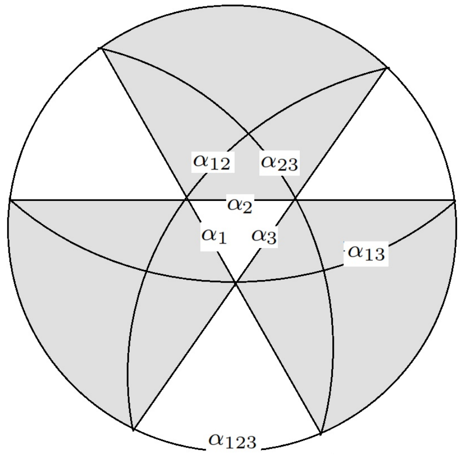
In this new basis the positive roots are:

$$\{\alpha_{12},\ -\alpha_2,\ \alpha_{23},\ \alpha_1,\ \alpha_3,\ \alpha_{123},\ \alpha_{13}\}$$

and the new Cartan matrix is hence

$$a^{II}_{ij} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Even though this Cartan matrix is of standard type A_3, the root system has one additional root. The following figure shows the hyperplane arrangement of the root system in $\mathbb{R}^3$ in a projective picture:



Each of the seven lines corresponds to the hyperplane orthogonal to a root. Each triangle is a Weyl chamber with the three adjacent hyperplanes corresponding to the three simple roots. Equilateral triangles (white) correspond to the Cartan matrix I and right triangles (grey) to the Cartan matrix II.

3 Preliminaries on screening operators

3.1 Vertex algebras and their representation theory

Vertex operator algebras (VOA) are algebras with an extra layer of analysis [FBZ04] [Kac98]. In particular the multiplication map Y, called vertex operator, depends on a formal parameter z, and there is a compatible action of the Virasoro algebra.

There is a notion of vertex algebra modules. Under certain finiteness-assumptions on a vertex operator algebra $\mathcal{V}$, the category of $\mathcal{V}$-modules has a tensor product $\boxtimes$ and a braiding [HLZ10].

Example 3.1. The easiest example of a vertex operator algebra is the n-dimensional Heisenberg algebra $\mathcal{H}$. Then, the irreducible modules $\mathcal{H}_v$ are parametrised by vectors $v \in \mathbb{C}^n$, with $\mathcal{H} = \mathcal{H}_0$, tensor product $\mathcal{H}_v \boxtimes \mathcal{H}_w = \mathcal{H}_{v+w}$, and braiding given by the scalar $e^{\pi i (v,w)}$.

From the perspective of our article, this is already an interesting vertex operator algebra: In the next section we will define screening operators $\mathfrak{Z}_{v_i}$ for $v_i \in \mathbb{C}^n$, and the idea of this article is to analyse the algebra generated by these screening operators, which will be largely determined by the braidings $q_{ij} = e^{\pi i (v_i, v_j)}$.

3.2 Screening operators

We now briefly review the notion of *screening operators*. They go back to [DF84] and appear throughout vertex operator literature. Our main focus are screening operators for elements in a module different than the vacuum module, and those are called *non-local screening operators*.
Given $\mathcal{V}$ a VOA, W module of $\mathcal{V}$ and $w \in W$. The tensor product $W \boxtimes U$ with some other module U is defined in [HLZ06] by the universal property of having an intertwining operator

$$Y : W \otimes_{\mathbb{C}} U \to (W \boxtimes U)[\log z][[z^{\pm 1/N}]].$$

If we evaluate Y on our fixed $w \in W$, we get a map

$$Y(w, z) : U \to (W \boxtimes U)[\log z][[z^{\pm 1/N}]]$$

Taking monodromies around $z = 0$, we get linear maps into the algebraic closure

$$\mathfrak{Z}_w : U \to \overline{W \boxtimes U}.$$

These maps are called *screening operators*. If the singularity of $Y(w, z)$ at $z = 0$ is mild enough, then by the next theorem these screening operators should fulfil the relations of the Nichols algebra $\mathcal{B}(W)$ associated to the module W. The braiding of W in the representation category expresses the non-locality

of $Y(w,z)$. If the singularity at $z=0$ is severe, then the screening operators generate some algebra extension of the Nichols algebra $\mathcal{B}(W)$.
For general VOAs this is work in progress by Huang-Lentner, but for Heisenberg VOAs and lattice VOAs it has been proven in [Len17]:

Theorem 3.2. *Given a non-integral lattice Λ and elements $v_1,\ldots,v_n \in \Lambda$, we consider the elements e^{v_i} in the modules $\mathcal{H}_{v_i}$ of the associated Heisenberg VOA $\mathcal{H}$. The braiding between two elements is*

$$e^{v_i} \otimes e^{v_j} \mapsto q_{ij}\, e^{v_j} \otimes e^{v_i},$$

$$\text{where } q_{ij} := e^{i\pi m_{ij}}, \quad m_{ij} := (v_i, v_j).$$

Consider the diagonal Nichols algebra $\mathcal{B}(q)$ for braiding matrix $q=(q_{ij})_{i,j}$ generated by elements x_{v_i}, then any relation in the Nichols algebra, in degree $(d_1,\ldots,d_n) \in \mathbb{N}^n$, holds for the screening operators $\mathfrak{Z}_{v_i}$, under the additional assumption of smallness*:*

$$\forall J \subseteq I,\ \forall i,j \in J \qquad \sum_{i<j} d_i d_j m_{ij} + \sum_i \binom{d_i}{2} m_{ii} > 1 - \sum_i d_i$$

where $I = \{1,\ldots,n\}$ is the index set.

Example 3.3. In the case $\Lambda = \frac{1}{\sqrt{p}}\Lambda_{\mathfrak{g}}$, with $\Lambda_{\mathfrak{g}}$ the root-lattice of a complex finite-dimensional simple Lie algebra $\mathfrak{g}$, and $\ell = 2p$ even integer, we obtain as $\mathcal{B}(q)$ the positive part of the small quantum group $u_q(\mathfrak{g})^+$ where q is a primitive ℓ-th root of unity and the braiding is

$$q_{ij} = e^{i\pi(\frac{1}{\sqrt{p}}\alpha_i, \frac{1}{\sqrt{p}}\alpha_j)} = e^{\frac{2i\pi}{\ell}(\alpha_i,\alpha_j)} = q^{(\alpha_i,\alpha_j)},$$

where $\alpha_i \in \Lambda_{\mathfrak{g}}$.

Lemma 3.4. *In particular, by theorem 6.1 of [Len17], theorem 3.2 holds if Λ is positive definite and $m_{ii} = (v_i, v_i) \leq 1$ for v_i in a fixed basis.*

Theorem 3.2 is a general result. We will now present a refined version, which will appear in our examples. Roughly, it shows that for the definition of smallness the assumption not-too-negative can be replaced by not-a-negative-integer, by analytic continuation. We prove this only in two special cases:

Theorem 3.5 (Continued Smallness)**.** *As in the previous theorem, we consider the action of linear combinations of monomials $\mathfrak{Z}_{v_1}\cdots\mathfrak{Z}_{v_n}$ of n screening operators on the module V_λ, we will denote $m_i := (v_i, \lambda)$ and $m_{ij} := (v_i, v_j)$ for $1 \leq i,j \leq n$.*

a) If all m_i are equal $\forall i \in I$ and all m_{ij} are equal $\forall i,j \in I$, then a relation in the Nichols algebra $\mathcal{B}(q)$ holds for the screening operators $\mathfrak{Z}_{v_i}$, under the weaker assumptions which we call continued smallness

$$m_{ij} \notin -\mathbb{N}\frac{2}{k} \qquad k = 1,\ldots,n.$$

b) If there is a distinguished element $1 \in I$ such that all m_i are equal $\forall i \in I, i \neq 1$ and all m_{ij} are equal $\forall i,j \in I, i \neq 1$, then a relation in the Nichols algebra $\mathcal{B}(q)$ holds for the screening operators $\mathfrak{Z}_{v_i}$, under the weaker assumption of continued smallness

$$m_{ij} \notin -\mathbb{N}\frac{2}{k} \qquad k = 1, \ldots, n-1$$
$$m_{1j} + k\frac{m_{ij}}{2} \notin -\mathbb{N} \qquad k = 0, \ldots, n-2.$$

Proof. Retaking the steps in the proof of theorem 3.2 in [Len17] we consider the following function (which play roughly the role of structure constants for multiplying screenings)

$$F((m_i, m_{ij})_{ij}) = \int \cdots \int_{[e^0, e^{2\pi}]^n} dz_1 \ldots dz_n \prod_i z_i^{m_i} \prod_{i<j} (z_i - z_j)^{m_{ij}}$$

We express this function as quantum symmetrizer of another function:

$$F(m_i, m_{ij}) = \text{Ш}\ \tilde{F}(m_i, m_{ij})$$
$$\tilde{F}((m_i; m_{ij})_{ij}) := \frac{1}{(2\pi i)^n} \sum_{k=0}^{n} (-1)^k \left(\prod_{i=k+1}^{n} e^{2\pi i\ m_i} \right) \sum_{\eta \in \mathbb{S}_{k,\overline{n-k}}} \left(\prod_{i<j,\ \eta(i)>\eta(j)} e^{\pi i\ m_{ij}} \right)$$
$$\cdot \operatorname{Sel}((m_{\eta^{-1}(i)}; 0; m_{\eta^{-1}(i)\eta^{-1}(j)})_{ij})$$

Where Sel indicates the Selberg integral

$$\operatorname{Sel}(m_i, \bar{m}_i, m_{ij}) = \operatorname{Sel}((m_i; \bar{m}_i; m_{ij})_{i<j})$$
$$:= \int \cdots \int_{1>z_1>\ldots>z_n>0} dz_1 \cdots dz_n \prod_i z_i^{m_i} \prod_i (1-z_i)^{\bar{m}_i} \prod_{i<j} (z_i - z_j)^{m_{ij}}.$$

By this result, the Nichols algebra relations are thereby proven to hold if $\tilde{F}$ is analytic at the parameters m_i, m_{ij} under consideration.

a) In our special situation with equal $m_{ij} =: m_{vv}$ and $m_i =: m_{v\lambda}$ we find from the factorization in [Len17] resp. from the Selberg integral formula:

$$\tilde{F}(m_{v\lambda}; m_{vv}) := \prod_{s=0}^{n-1} \left((e^{\pi i m_{vv}})^s e^{2\pi i m_{v\lambda}} - 1 \right) \cdot \operatorname{Sel}(m_{v\lambda}; 0; m_{vv})$$
$$\operatorname{Sel}(a-1, b-1, 2c) = \prod_{k=0}^{n-1} \frac{\Gamma(a+kc)\Gamma(b+kc)\Gamma(1+(k+1)c)}{\Gamma(a+b+(n+k-1)c)\Gamma(1+c)}$$

Our goal is to prove that under the assumptions on $m_{vv}, m_{v\lambda}$ the function $\tilde{F}$ is analytic.

The Gamma function does not have zeros, and it has poles for negative integer values of z. Thus the only possible poles are for:

- Poles:
$$a + kc \in -\mathbb{N}_0, \qquad k = 0, \ldots, n-1.$$
These simple poles cancel with the factors $((e^{\pi i m_{vv}})^s e^{2\pi i m_{v\lambda}} - 1)$. So at these values $\tilde{F}$ is analytic and thus F vanishes according to the quantum symmetrizer formula and Nichols algebra relations hold. We remark however, that these exceptionally non-zero values of $\tilde{F}$ give rise to reflection operators [Len17].
- Poles:
$$\begin{aligned} 1 + kc &\in -\mathbb{N}_0, & k &= 0, \ldots, n-1 \\ \iff kc &\in -\mathbb{N}, & k &= 0, \ldots, n-1 \\ \iff kc &\in -\mathbb{N}, & k &= 1, \ldots, n-1 \end{aligned}$$
- Poles:
$$\begin{aligned} 1 + (k+1)c &\in -\mathbb{N}_0, & k &= 0, \ldots, n-1 \\ \iff (k+1)c &\in -\mathbb{N}, & k &= 0, \ldots, n-1 \\ \iff \qquad kc &\in -\mathbb{N}, & k &= 1, \ldots, n \end{aligned}$$
where in the last step we substituted $k+1$ with k.

We thus found that to avoid poles we need to ask the condition
$$k\frac{m_{ij}}{2} \notin -\mathbb{N}, \qquad k = 1, \ldots, n.$$

b) To prove the second point we proceed in the same way, this time isolating the distinguish element with index equals to 1.

$$\begin{aligned} &\mathrm{Sel}(m_i, m_{ij}, m_1, m_{1j}) \\ &= \int_0^1 \cdots \int_0^1 \prod_{i=2}^n z_1^{m_1} z_i^{m_i} \prod_{j=2}^n (z_1 - z_j)^{m_{1j}} \prod_{2 \le i < j \le n} (z_i - z_j)^{m_{ij}} dz_1 \cdot dz_2 \ldots dz_n \\ &= \int_0^1 dz_1 \; z_1^{m_1 + (n-1) + \sum m_{1j} + \sum m_{ij} + \sum m_i} \\ &\qquad \cdot \int_0^1 \cdots \int_0^1 \prod_{i=2}^n \tilde{z}_i^{m_i} \prod_{j=2}^n (1 - \tilde{z}_j)^{m_{1j}} \prod_{2 \le i < j \le n} (\tilde{z}_i - \tilde{z}_j)^{m_{ij}} d\tilde{z}_2 \ldots d\tilde{z}_n. \end{aligned}$$

Calling m the power of z_1, we have:

$$\begin{aligned} &\mathrm{Sel}(m_i, m_{ij}, m_1, m_{1j}) \\ &= \frac{(e^{2\pi i m} - 1)/2\pi i}{1 + m} \int_0^1 \cdots \int_0^1 \prod_{i=2}^n \tilde{z}_i^{m_i} \prod_{j=2}^n (1 - \tilde{z}_j)^{m_{1j}} \\ &\qquad \cdot \prod_{2 \le i < j \le n} (\tilde{z}_i - \tilde{z}_j)^{m_{ij}} d\tilde{z}_2 \ldots d\tilde{z}_n \\ &= \frac{(e^{2\pi i m} - 1)/2\pi i}{1 + m} \prod_{k=0}^{n-2} \frac{\Gamma(a + kc)\Gamma(b + kc)\Gamma(1 + (k+1)c)}{\Gamma(a + b + (n + k - 1)c)\Gamma(1 + c)} \end{aligned}$$

A similar calculation as in the previous case disregard the pole at $1+m=0$ and $a+kc \in -\mathbb{N}_0$.
Then we have poles just for:

- Poles:

$$\begin{aligned} & b+kc \in -\mathbb{N}_0, && k=0,\dots,n-2 \\ \iff & m_{1j}+k\frac{m_{ij}}{2} \notin -\mathbb{N} && k=0,\dots,n-2. \end{aligned}$$

- Poles:

$$\begin{aligned} & 1+(k+1)c \in -\mathbb{N}_0, && k=0,\dots,n-2 \\ \iff & kc \in -\mathbb{N}, && k=1,\dots,n-1 \\ \iff & k\frac{m_{ij}}{2} \in -\mathbb{N} && k=1,\dots,n-1 \end{aligned}$$

which are the asserted conditions.

□

3.3 Central charge

The output of our article are elements $v_1,\dots v_n \in \mathbb{C}^n$, the respective screening operators of the Heisenberg algebra $\mathcal{H}$, and their algebra relations. We also wish to fix an action of the Virasoro algebra at a certain central charge on $\mathcal{H}$. As discussed in [FL17] it is usually desirable to choose the Virasoro structure in such a way, that it is compatible with the screening operators associated to the $v_1,\dots,v_n$. This gives a unique solution of Virasoro structure and a characteristic central charge for the situation at hand, which we now compute:

Proposition 3.6. *For the Heisenberg algebra, there is a family of Virasoro structures parametrised by the choice of an element $Q \in \mathbb{C}^n$, called background charge [FF06]. The compatibility condition means that*

$$\frac{1}{2}(v_i,v_i)-(v_i,Q)=1 \qquad i=1,\dots,n.$$

The central charge of the system will be:

$$c = rank - 12(Q,Q).$$

In particular for $rank=2$, we have as in [Sem11] the explicit formula:

$$c = 2-3\frac{|v_1(m_{22}-2)-v_2(m_{11}-2)|^2}{m_{11}m_{22}-m_{12}^2} \tag{4}$$

4 Formulation of the classification problem

Definition 4.1. Let Λ be a lattice of rank n, basis $\{v_1,\dots,v_n\}$, bilinear form $(\ ,\ )$ and Cartan matrix a_{ij} and let $m_{ij}:=(v_i,v_j)$. Given a braiding matrix q_{ij}, we say that the lattice Λ and the matrix m_{ij} *realise* q_{ij} iff

- we have: $e^{i\pi m_{ij}} = q_{ij}$
- m_{ij} satisfies:

$$\text{A: } 2m_{ij} = a_{ij} m_{ii} \qquad \text{or} \qquad \text{B: } (1 - a_{ij}) m_{ii} = 2 \tag{5}$$

- all the reflected matrices $\mathcal{R}^k(m_{ij})$ fulfil again (5).

We will say with respect to a realisation m_{ij} that a root v_i is *m-Cartan* if m_{ii} satisfies (5)A, and *m-truncation* if it satisfies (5)B.

[Sem11] asks this condition only for one specific Weyl chamber.

Remark 4.2. We observe that condition (5) is the logarithmic version of (3).

Remark 4.3. Clearly m-Cartan implies q-Cartan and m-truncation implies q-truncation. The converse is not always true. If a root is both q-Cartan and q-truncation, then there are two possible solutions in terms of the m_{ij} matrix. An example is the $\mathfrak{sl}(2|1)$ superalgebra, presented below in example (4.5).

Proposition 4.4. *Clearly, if v_k is m-Cartan, then $\mathcal{R}^k(m_{ij}) = m_{ij}$.*

Our goals are as follows:

- Given a braiding q_{ij} from Heckenberger lists in [Hec05], [Hec06], construct all the realising m_{ij}. In sections 5, 6 and 7.1 we construct the m_{ij} while in section 7.2 we prove that the constructed m_{ij} exhaust all cases of Heckenberger list in rank 2. In section 8 we do the same for rank 3.
- We compute the central charges for each solution.
- We analyse which Nichols algebras relations hold and which don't, for the associated screening operators. This may depend on a free parameter in the family of solutions.

Example 4.5. We now show an example of this procedure. We consider row 3 of table 1 in [Hec05], described by the braiding matrices:

$$q^{\mathrm{I}}_{ij} = \begin{bmatrix} q^2 & q^{-1} \\ q^{-1} & -1 \end{bmatrix} \qquad q^{\mathrm{II}}_{ij} = \begin{bmatrix} -1 & q \\ q & -1 \end{bmatrix}$$

and corresponding diagrams:

I II

with $q \in \mathbb{C}^\times$, $q^2 \neq \pm 1$, simple roots $\{\alpha_1, \alpha_2\}$ and $\{\alpha_{12}, \alpha_2\}$ respectively, and a unique associated Cartan matrix

$$a^{\mathrm{I}}_{ij} = a^{\mathrm{II}}_{ij} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

This describes the Lie superalgebra $\mathfrak{sl}(2|1)$. The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}\}$ where α_1 is only q-Cartan and α_2, α_{12} are only q-truncation (for $q^2 = -1$ this is not true: all roots are both q-Cartan and q-truncation, which gives more solution, see remark 4.7).

Proposition 4.6. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} 2r & -r \\ -r & 1 \end{bmatrix}, \qquad m^{II}_{ij} = \begin{bmatrix} 1 & -1+r \\ -1+r & 1 \end{bmatrix}$$

for all $r = \frac{p'}{p} \in \mathbb{Q}$ with $(p', p) = 1$ such that $e^{i\pi r} = q$.

Proof. We check that condition (5)B is satisfied for α_2, α_{12}:

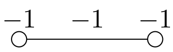
$$m_{22} = \frac{2}{1 - a_{21}} = 1$$
$$m_{12,12} = \frac{2}{1 - a_{12,2}} = 1$$

while condition (5)A is satisfied for the root α_1:

$$m_{11} = \frac{2m_{12}}{a_{1,2}} = 2r.$$

The reflection on α_1 preserves q_{ij} as well as m_{ij}, because α_1 is m-Cartan. We check that reflections on α_2 and α_{12}, which interchange q^{I}_{ij} and q^{II}_{ij}, also interchange our choices of m^{I}_{ij} and m^{II}_{ij}. □

Remark 4.7. For $q^2 \neq \pm 1$ this family gives all solutions.
For $q^2 = -1$, we have more choices for the m_{ij}-matrices because the roots become both q-truncation and q-Cartan. Thus we may have solutions fulfilling either (5A) or (5B). The new (unique) diagram in this case is:



to whom correspond several solutions of m_{ij}-matrices; for simple roots α_1, α_2: let $p' \in \mathbb{Z}$ with $(p', 2) = 1$,

- if we assume α_1 and α_2 m-truncation, the unique family of solutions is given by

$$m_{ij} = \begin{bmatrix} 1 & -\frac{p''}{2} \\ -\frac{p''}{2} & 1 \end{bmatrix} \qquad m_{ij} = \begin{bmatrix} 1 & -\frac{p'}{2} \\ -\frac{p'}{2} & p' \end{bmatrix} \qquad m_{ij} = \begin{bmatrix} p' & -\frac{p'}{2} \\ -\frac{p'}{2} & 1 \end{bmatrix}.$$

  These are reflections one of the other by Proposition 4.6 with $p'' = 2 - p'$. Other combinations bring to the same solution in different Weyl chambers.

- if we assume α_1 and α_2 m-Cartan, the unique family of solutions is given by

$$m_{ij} = \begin{bmatrix} p' & -\frac{p'}{2} \\ -\frac{p'}{2} & p' \end{bmatrix}$$

  which can be interpreted as coming from $\mathfrak{sl}_3$ for $p = 2$.

5 Cartan type

5.1 q diagram

Let $\mathfrak{g}$ be a simple Lie algebra with simple roots $\alpha_1, \dots, \alpha_n$ and Killing form $(\alpha_i, \alpha_j)_{\mathfrak{g}} \in \{-3, -2, -1, 0, 2, 4, 6\}$. Let $q \in \mathbb{C}^\times$ be a primitive ℓ-th root of unity with $\ell \in \mathbb{Z}$ and let $\mathrm{ord}(q^2) > d$ with d half length of the long roots. Define a braiding matrix by

$$q_{ij} = q^{(\alpha_i, \alpha_j)_{\mathfrak{g}}}.$$

Definition 5.1. The finite dimensional Nichols algebra $\mathcal{B}(q)$ is called of *Cartan type.*

We have that:

- q_{ij} is invariant under reflections $\mathcal{R}^k$,
- the Weyl groupoid is the Weyl group associated to $\mathfrak{g}$,
- the set of positive roots is the set of roots associated to $\mathfrak{g}$,
- the Cartan matrix a_{ij} is exactly the Cartan matrix for $\mathfrak{g}$.

5.2 Construction of m_{ij}

Definition 5.2. Given $r \in \mathbb{Q}$, such that $\frac{r}{2} = \frac{k}{\ell}$, with $k \in \mathbb{Z}$, $(k, \ell) = 1$ we define

$$m_{ij} := (\alpha_i, \alpha_j) r.$$

Differently spoken, the lattice Λ of definition 4.1 is, in this case, exactly the root lattice of $\mathfrak{g}$ rescaled by r.

Remark 5.3. Usually in literature $r = \frac{p'}{p}$, e.g. $r = \frac{1}{p}$ and $\ell = 2p$, $q = e^{\frac{\pi i}{p}}$.

Lemma 5.4. *The matrix m_{ij} realises the braiding q_{ij} for all reflections, and every simple root is m-Cartan.*

Proof. Condition (5) asks

$$2m_{ij} = a_{ij} m_{ii} \qquad \text{or} \qquad (1 - a_{ij}) m_{ii} = 2 \tag{6}$$
$$2m_{ji} = a_{ji} m_{jj} \qquad \text{or} \qquad (1 - a_{ji}) m_{jj} = 2. \tag{7}$$

But from the last point of enumeration 5.1 we have $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. Hence

$$m_{ii} = (\alpha_i, \alpha_i) r = 2(\alpha_i, \alpha_j) r \frac{(\alpha_i, \alpha_i)}{2(\alpha_i, \alpha_j)} = 2 \frac{m_{ij}}{a_{ij}}$$

which is (5)A, saying that the roots are m-Cartan.
Since any reflection leaves the m_{ij} invariant (not just the q_{ij}) because is a m-Cartan reflection, condition (5) holds also after reflections. □

Lemma 5.5. *If $\ell_i > 1 - a_{ij}$ for $i = 1, \dots, n$, with $\ell_i := \frac{\ell}{gdc(\ell, 2d_i)}$ as in [Lus90], then none of the roots are m-truncation.*

Proof. Assume the root α_i is m-truncation, i.e. $(1-a_{ij})m_{ii} = 2$, this implies: $q_{ii}^{(1-a_{ij})} = e^{i\pi m_{ii}(1-a_{ij})} = e^{i\pi\cdot 2} = 1$. But $ord(q_{ii}) = ord(q^{2d_i}) = \ell_i > 1 - a_{ij}$ and we find a contradiction. ☐

Lemma 5.6. *If all roots are m-Cartan, then the unique solution for the matrix m_{ij} is the one of definition 5.2. In particular this is the case if $\ell_i > 1 - a_{ij}$ for $i = 1, \ldots, n$.*

Proof. If all the roots are m-Cartan then if we fix m_{ii} for some root α_i, the mixed term m_{ij} is fixed by condition 5(A) and so is m_{jj} by the same condition with reversed indices. Moreover the reflections around m-Cartan roots leave the system invariant, so the m_{ij} are fixed $\forall i, j$. But then, up to a rescaling there is a unique solution for m_{ij} and this is the one defined in 5.2. ☐

Example 5.7. As a counterexample of the condition of lemma 5.5, we consider $\mathfrak{sl}_3$. In this case, $a_{ij} = -1\ \forall i, j$ and for $2\ell_i = \ell = 2p = 4$, i.e. $q_{ii} = -1$, the roots can be considered as m-truncation as well. We thus obtain an additional solution of m_{ij}, which will be understood from reinterpreting $\mathfrak{sl}_3$, $\ell = 2p = 4$, as the Lie superalgebra $\mathfrak{sl}(2|1)$, $\ell = 2p = 4$, treated in the remark of example 4.5.

5.3 Central charge

Recall $\{v_1, \ldots, v_n\}$ as basis of Λ with $m_{ij} = (v_i, v_j)$.

Proposition 5.8. *The central charge of the system is*

$$c = \text{rank}\mathfrak{g} - 12(\frac{1}{r} \mid \rho^\vee \mid^2 - 2(\rho, \rho^\vee) + r \mid \rho \mid^2) \tag{8}$$

where ρ is the sum of all positive roots.

Proof. The central charge is:

$$c = rank - 12(Q, Q)$$

where $Q = \sum_j a_j v_j$ is the unique combination such that for every i

$$\frac{1}{2}(v_i, v_i)_\Lambda - (v_i, Q) = 1$$
$$\frac{1}{2}(v_i, v_i)_\Lambda - \sum_j a_j (v_i, v_j)_\Lambda = 1$$

Rewriting $v_i = -\sqrt{r}\alpha_i$, with α_i root of $\mathfrak{g}$, this set of equations bring us to

$$Q = \sqrt{\frac{1}{r}}\rho^\vee - \sqrt{r}\rho$$

that on turn gives the central charge as in the statement. ☐

Remark 5.9. The central charge matches with the one of the affine Lie algebra $\hat{\mathfrak{g}}_k$ at level $k + h^\vee = \frac{1}{r}$ as in [Ara07].

Remark 5.10. For rank 2 the central charge is

$$c = 1 - 3\frac{(2p' - 2p)^2}{2pp'} = 13 - 6\frac{p}{p'} - 6\frac{p'}{p}$$

which is the central charge of the p, p' model.

5.4 Algebra relations

We now want to determine when the algebra of screenings satisfies Nichols algebra relations. We will again denote d the half length of the long roots.

With the definition of smallness and the results in [Len17], see theorem 3.4, we get for a rescaled root lattice $m_{ij} = (\alpha_i, \alpha_j)r$:

Corollary 5.11. *If $\frac{1}{2d} \geq r > 0$, then all Nichols algebra relations hold.*

Proof. Since we are rescaling by $\sqrt{r}$ a positive definite lattice Λ, the only condition for the new lattice to be positive definite is $r > 0$. We ask moreover $m_{ii} = 2dr \leq 1$ for all i. This implies $r \leq \frac{1}{2d}$. □

Now we want to analyse the algebra relations in the screening algebra for arbitrary values of r. To do so we study relation by relation using theorem 3.5.

Definition 5.12. A generator x_i is said to satisfy the *truncation* relation if

$$x_i^{\ell_i} = 0, \qquad \ell_i = ord(q_{ii}).$$

A pair of generators x_i, x_j are said to satisfy the *Serre* relation if

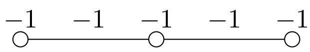
$$(ad_c x_i)^{1-a_{ij}} x_j = 0, \qquad a_{ij} = -\min\left\{ m \in \mathbb{Z} \mid q_{ii}^{-m} = q_{ij}q_{ji} \quad \text{or} \quad q_{ii}^{(1+m)} = 1. \right\}$$

We denoted the braided commutator by $(ad_c x_i)x_j := [x_i, x_j]_c = [x_i, x_j]_q$.

Theorem 5.25 of [Ang08] states a set of defining relations for each finite dimensional Nichols algebra of Cartan type:

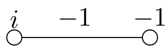
Theorem 5.13. *For finite dimensional Nichols algebra of Cartan type $u_q(\mathfrak{g})^+$, i.e. with diagonal braiding $q_{ij} = q^{(\alpha_i, \alpha_j)}$ associated to the root system of a Lie algebra $\mathfrak{g}$, the defining relations are as follows*

1. *For each root α the truncation relation and for each pair of simple roots α_i, α_j with $q_{ii}^{1-a_{ij}} \neq 1$ the Serre relation.*
2. *For the following subdiagrams the following additional relations:*
   - *For type A_3 with $q = -1$*

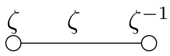


$$[(\mathrm{ad}x_2)x_1, (\mathrm{ad}x_2)x_3]_c = 0$$

   - *For type B_2 or C_2 with $q = i$*

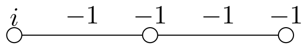


or with $q = \zeta \in \mathcal{R}_3$

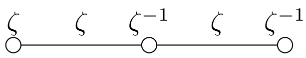


$$[(\mathrm{ad}x_1)^2 x_2, (\mathrm{ad}x_1)x_2]_c = 0$$

- *For type* B_3 *with* $q = i$

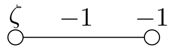


or with $q = \zeta \in \mathcal{R}_3$

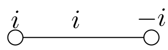


$$[(\mathrm{ad}x_1)^2(\mathrm{ad}x_2)x_3, (\mathrm{ad}x_1)x_2]_c = 0$$

- *For type* G_2 *with* $q = \zeta \in \mathcal{R}_6$



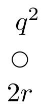
or with $q = i$



$$\begin{aligned} [(\mathrm{ad}x_1)^3 x_2, (\mathrm{ad}x_1)^2 x_2]_c &= 0 \\ [x_1, [x_1^2 x_2 x_1 x_2]_c]_c &= 0 \\ [[x_1^2 x_2 x_1 x_2]_c, [x_1, x_2]_c]_c &= 0 \\ [[x_1^2 x_2]_c, [x_1^2 x_2 x_1 x_2]_c]_c &= 0. \end{aligned}$$

We now apply our refined smallness criteria of theorem 3.5 to these explicit set of relations to determine the algebra of screening operators in comparison to the Nichols algebra.

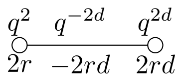
Example 5.14. Let us consider a rank 1 Cartan q-diagram and corresponding m-solution:



The truncation relation $(\mathfrak{Z}_1)^n = 0$, $n = ord(q^2)$ holds, according to 3.5, iff $r > 0$.

For $r < 0$ it is further calculated in [Len17] that $(\mathfrak{Z}_{\sqrt{r}\alpha_1})^n = \mathfrak{Z}_{n\sqrt{r}\alpha_1}$ which is a *local* screening. The algebra of screenings is therefore an extension of the Nichols algebra by a long screening.

Example 5.15. Let us consider a rank 2 Cartan q-diagram and corresponding m-solution:



– By the previous example the simple truncation relations hold for $r \geq 0$.

  We conjecture that in this case the non-simple truncation ($[\mathfrak{z}_1, \mathfrak{z}_2]^n$ etc.) also hold for $r > 0$. But this would either require a reflection theory for algebra of screenings or a generalization of theorem 3.5.

– The long Serre relation $[\mathfrak{z}_2, [\mathfrak{z}_2, \mathfrak{z}_1]] = 0$ holds if $2dr \notin -\mathbb{N}$. Does the long Serre relation may fail if $q_{22} = -1$ and $r < 0$, which is when the long root α_2 is both q-Cartan and q-truncation and when the truncation relation fails. But for these cases the Serre relation was in theorem 5.13 not required as an independent relation.

– The short Serre relation $[\mathfrak{z}_1, \ldots [\mathfrak{z}_1, \mathfrak{z}_2] \ldots] = 0$ which involves $d+1$ times the first screening, holds if

$$2r, 3r, \ldots, (d+1)r \notin -\mathbb{N}$$
$$dr, (d-1)r, (d-2)r, \ldots, 2r \notin \mathbb{N}.$$

  In particular:

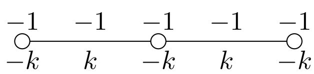
  * for $d = 1$ see the long Serre relations.
  * for $d = 2$ holds iff $3r \notin -\mathbb{N}$.
  * for $d = 3$ holds iff $2r, 4r \notin -\mathbb{N}$ and $2r \notin \mathbb{N}$. But $2r \in \mathbb{Z}$ is not admissible because $q \neq -1$.

  So again the short Serre relation may fail if the short root α_1 is both q-Cartan and q-truncation and the truncation relation fails.

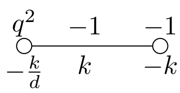
– Extra relations as listed in point (2) of theorem 5.13 apply exactly in the exceptional cases for the Serre relations above.

Summarizing we have the following possible exceptions:

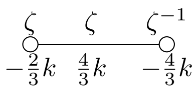
- for $q^2 = -1$, $k \in \mathbb{N}$, k odd, ($r < 0$, $d = 1$):



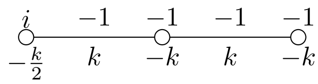
- for $q^2 \in \mathcal{R}_{2d}$, $k \in \mathbb{N}$, k odd, $\forall d$ ($r < 0$):



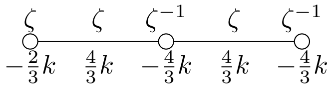
- for $q^2 = \zeta \in \mathcal{R}_3$, $k \in \mathbb{N}$, k odd, ($r < 0$, $d = 2$):



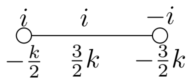
- for $q^2 = i$, $k \in \mathbb{N}$, k odd, ($r < 0$, $d = 2$):



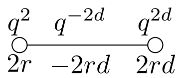
- for $q^2 = \zeta \in \mathcal{R}_3$, $k \in \mathbb{N}$, k odd, ($r < 0$, $d = 2$):



- for $k \in \mathbb{N}$, k odd, ($r < 0$, $d = 3$):



Proposition 5.16. *We consider again a rank 2 Cartan q-diagram*



1. *If $q^{2d} = -1$, $r < 0$ the long Serre relation holds.*
2. *If ord(q^2)$= d+1$, $r < 0$ the short Serre relation holds.*

Proof. 1. The long Serre relation reads

$$[\mathfrak{z}_2, [\mathfrak{z}_2, \mathfrak{z}_1]_{-1}]_{+1} = (\mathfrak{z}_2)^2\mathfrak{z}_1 + \mathfrak{z}_2\mathfrak{z}_1\mathfrak{z}_2 - \mathfrak{z}_2\mathfrak{z}_1\mathfrak{z}_2 - \mathfrak{z}_1(\mathfrak{z}_2)^2 = [(\mathfrak{z}_2)^2, \mathfrak{z}_1].$$

Since $r < 0$ this is not automatically zero because $(\mathfrak{z}_2)^2 \neq 0$. Despite this it was studied in [Len17] that $(\mathfrak{z}_2)^2 \sim \mathfrak{z}_{22}$. Then standard OPE calculations give: $[(\mathfrak{z}_2)^2, \mathfrak{z}_1] = [\mathfrak{z}_{22}, \mathfrak{z}_1] = 0$.

2. This point is a generalization of the previous. We have:

$$\begin{aligned}
[\mathfrak{z}_1, \ldots, [\mathfrak{z}_1, \mathfrak{z}_2]\ldots] &= (\mathfrak{z}_1)^{d+1}\mathfrak{z}_2 - (q^{-d} + q^{-d+2} + \ldots + q^d)(\mathfrak{z}_1)^d\mathfrak{z}_2\mathfrak{z}_1 + \ldots \\
&= \sum_{i+j=d+1} (-1)^i(\mathfrak{z}_1)^i\mathfrak{z}_2(\mathfrak{z}_1)^j \frac{[i+j]!}{[i]![j]!} = [(\mathfrak{z}_1)^{d+1}, \mathfrak{z}_2] \\
&= [\mathfrak{z}_{(d+1)\alpha_1}, \mathfrak{z}_2] = 0
\end{aligned}$$

where for the penultimate equality we used again results from [Len17] and for the last one theorem 3.2.

□

Remark 5.17. Alternatively this follows conceptually from the fact that this holds for generic q. One can argue similarly for the other relations or proceed as in the previous proposition.

In conclusion:

Corollary 5.18. *The screening operators algebra is as follows:*

- *For $r \geq 0$ all Nichols algebra relations hold (conjecturally also the non-simple truncation relations).*
- *For $r < 0$ all Nichols algebra relations hold except the truncation relations.*

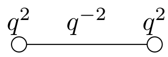
  Conjecturally, the non-zero result of the truncation relation are, as above, themselves local screenings and in the centre of the algebra of screenings. Hence in these cases we get the positive part of the infinite-dimensional Kac-Procesi-DeConcini quantum group, also called non-restricted specialization *[CP94].*

Remark 5.19. We remark that for $r < 0$ products of screenings can be not well defined.

5.5 Examples in rank 2

Heckenberger row 2

This case of the list is described by the braiding diagram:



with $q \in \mathbb{C}$ $q^2 \neq 1$ and simple roots $\{\alpha_1, \alpha_2\}$. The realising lattice is a rescaled A_2 root lattice i.e. $\mathfrak{sl}_3$.

The set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_{12}\}$ with unique associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

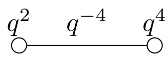
Proposition 5.20. *Defining r as in 5.2 we find that the following m_{ij}-matrix is a realising solution:*

$$m_{ij} = \begin{bmatrix} 2r & -r \\ -r & 2r \end{bmatrix}.$$

Remark 5.21. For $q^2 = -1$, these are all solutions and the roots are both q-Cartan *and* q-truncation.
This case is shown in detail in remark 4.7 of example 4.5.

Heckenberger row 4

This case of the list is described by the braiding diagram:



with $q \in \mathbb{C}$ $q^2 \neq \pm 1$ and simple roots $\{\alpha_1, \alpha_2\}$. The realising lattice is a rescaled B_2 root lattice.

The set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}\}$ with unique associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}.$$

Proposition 5.22. *If $q^2 \neq \pm 1$, then for every possible r defined as in 5.2 the following m_{ij}-matrix*

$$m_{ij} = \begin{bmatrix} 2r & -2r \\ -2r & 4r \end{bmatrix}$$

is a realising solution for the braiding.

Remark 5.23. 1. When $q^2 \in \mathcal{R}_4$, the root α_2 is q-Cartan *and* q-truncation. There is an additional family of solutions when it is m-truncation:

$$m_{ij}^{\mathrm{I}} = \begin{bmatrix} 2r & -2r \\ -2r & 1 \end{bmatrix} \qquad m_{ij}^{\mathrm{II}} = \begin{bmatrix} -2r+1 & 2r-1 \\ 2r-1 & 1 \end{bmatrix} \qquad \text{for } r = \frac{p'}{4}, \ p' \text{ odd},$$

with simple roots I: $\{\alpha_1, \alpha_2\}$ and II: $\{\alpha_{12}, -\alpha_2\}$.

This lattice can be interpreted as lattice realising the Lie superalgebra $B(1,1)$ described in case Heckenberger row 5, which for this choice of q^2 has the same q-diagram.

2. When $q^2 \in \mathcal{R}_3$, the root α_1 is q-Cartan *and* q-truncation. There is an additional family of solutions when it is m-truncation:

$$m_{ij}^{\mathrm{I}} = \begin{bmatrix} \frac{2}{3} & -2r \\ -2r & 4r \end{bmatrix} \qquad m_{ij}^{\mathrm{II}} = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3}+2r \\ -\frac{4}{3}+2r & \frac{8}{3}-4r \end{bmatrix}$$

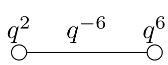
for $r = \frac{2+3p'}{6}$, $p' \in \mathbb{Z}$, with simple roots I: $\{\alpha_1, \alpha_2\}$ and II: $\{-\alpha_1, \alpha_{112}\}$.

This lattice can be interpreted as lattice realising the case Heckenberger row 6 (a colour Lie algebra), which for this choice of q^2 has the same q-diagram.

Remark 5.24. Note that $q^2 = -1$ is excluded. Indeed for that value, the system degenerates and the short truncation roots form a lattice of type A_1^n as described in [FL17]. Physically it corresponds to n pair of symplectic fermions.

Heckenberger row 11

This case of the list is described by the braiding diagram:



with $q^2 \neq \pm 1, q^2 \notin \mathcal{R}_3$ and simple roots $\{\alpha_1, \alpha_2\}$.
The realising lattice is of type G_2.

The set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{1112}, \alpha_{11122}\}$ with unique associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

Proposition 5.25. *If $q^2 \neq \pm 1, q^2 \notin \mathcal{R}_3$, then for every possible r defined as in 5.2 the following m_{ij}-matrix*

$$m_{ij} = \begin{bmatrix} 2r & -3r \\ -3r & 6r \end{bmatrix}$$

is a realising solution for the braiding.

Remark 5.26. When $q^2 \in \mathcal{R}_4$, the root α_1 is q-Cartan *and* q-truncation.
When it is m-truncation we get:

$$m^{\mathrm{I}}_{ij} = \begin{bmatrix} \frac{1}{2} & -3r \\ -3r & 6r \end{bmatrix} \qquad m^{\mathrm{II}}_{ij} = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} + 3r \\ -\frac{3}{2} + 3r & \frac{9}{2} - 12r \end{bmatrix}$$

with simple roots I: $\{\alpha_1, \alpha_2\}$ and II: $\{-\alpha_1, \alpha_{1112}\}$.

The root α_{1112} is never m-truncation and it is m-Cartan iff $r = \frac{1}{4}$. But for this value of r, α_1 is also m-Cartan and thus this is *not* a new solution.

Remark 5.27. When $q^2 \in \mathcal{R}_6$, the root α_2 is m-Cartan *and* m-truncation.
When it is m-truncation we get:

$$m^{\mathrm{I}}_{ij} = \begin{bmatrix} 2r & -3r \\ -3r & 1 \end{bmatrix} \qquad m^{\mathrm{II}}_{ij} = \begin{bmatrix} 1-4r & -1+3r \\ -\frac{3}{2} + 3r & 1 \end{bmatrix}$$

with simple roots I: $\{\alpha_1, \alpha_2\}$ and II: $\{\alpha_{12}, -\alpha_2\}$.

The root α_{12} is never m-truncation and it is m-Cartan iff $r = \frac{1}{6}$. But for this value of r, α_2 is also m-Cartan and thus this is *not* a new solution.

6 Super Lie type

6.1 q diagram

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a simple Lie superalgebra of *classical*, *basic* type [FSS96], i.e. of type $A(m,n), B(m,n), C(n+1), D(m,n), F(4), G(3), D(2,1;\alpha)$. For these Lie superalgebras a (non degenerate or zero) Killing form $(\ ,\ )_{\mathfrak{g}}$ is defined.

We now choose a Weyl chamber $\alpha_1, \ldots, \alpha_{f-1},\ \alpha_f,\ \alpha_{f+1}, \ldots, \alpha_n$ with just one simple fermionic root α_f. We call it the *standard chamber* according to [Kac77]. Given α positive root in the standard chamber, we define $f(\alpha)$ the multiplicity of α_f in α.

We can then split $\mathfrak{g}$ as the direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}'' \oplus \mathfrak{m},$$

where $\mathfrak{g}'$ and $\mathfrak{g}''$ are two bosonic connected component generated by the simple roots $\alpha_1, \ldots, \alpha_{f-1}$ and $\alpha_{f+1}, \ldots, \alpha_n$ respectively, while $\mathfrak{m}$ is the $\mathfrak{g}' \oplus \mathfrak{g}''$-module spanned by all other roots.

We have that $\mathfrak{m}$ contains $\mathfrak{g}_{\bar{1}}$ and thus in particular contains the $\mathfrak{g}' \oplus \mathfrak{g}''$-submodule generated by the fermion α_f, i.e. the vector space of fermions γ, with $f(\gamma) = 1$. Moreover $\mathfrak{m}$ may contain bosonic roots δ, with $f(\delta)$ positive even.

Definition 6.1. We can write the inner product $(\ ,\ )_{\mathfrak{g}}$ of two arbitrary simple roots as

$$(\alpha_i, \alpha_j)_{\mathfrak{g}} = (\alpha_i, \alpha_j)_{\mathfrak{g}'} + (\alpha_i, \alpha_j)_{\mathfrak{g}''} = \begin{cases} (\alpha_i, \alpha_j)_{\mathfrak{g}'} & \text{if } i \leq f,\ j < f \\ 0 & \text{if } i \leq f \leq j \\ (\alpha_i, \alpha_j)_{\mathfrak{g}''} & \text{if } i \geq f,\ j > f. \end{cases}$$

In particular we assume $(\alpha_f, \alpha_f)_{\mathfrak{g}} = (\alpha_f, \alpha_f)_{\mathfrak{g}'} = (\alpha_f, \alpha_f)_{\mathfrak{g}''} = 0$.

Definition 6.2. Let q', q'' be primitive roots of unity of the same order. Then to the above data in the standard chamber we associate the braiding matrix q_{ij}:

$$q_{ij} = \begin{cases} (q')^{(\alpha_i, \alpha_j)_{\mathfrak{g}'}} & \text{if } i \leq f,\ j < f \\ (q'')^{(\alpha_i, \alpha_j)_{\mathfrak{g}''}} & \text{if } i \geq f,\ j > f \\ 1 & \text{if } i > f > j \\ -1 & \text{if } i = f = j. \end{cases}$$

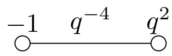
Under certain conditions on the q_{ij}, this braiding gives a finite dimensional Nichols algebra $\mathcal{B}(q)$, which we call of *Super Lie type.*

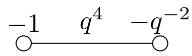
The reflections will act on the braiding as follow:

- Reflections $\mathcal{R}^k$ around bosonic roots α_k leave q_{ij} invariant.
- Reflections $\mathcal{R}^k$ around fermionic roots α_k interchange fermionic and bosonic roots and may produce a braiding containing $-q$.

Remark 6.3. In the classification of Nichols algebras in [Hec05] and [Hec06] we find that the fermion (as in the Lie superalgebra sense of the term) in the standard chamber α_f has $q_{ff} = -1$, i.e. it is q-truncation. This is not true in general for every fermion as we can see in the following example.

Example 6.4. The case Heckenberger row 5 of table 1 in [Hec05] is described by two diagrams:





I II

corresponding to the simple roots:

$$\text{I}: \{\alpha_1, \alpha_2\} \qquad \text{II}: \{-\alpha_1, \alpha_{12}\}.$$

This is the Lie superalgebra **B(1,1)** and α_{12} is a fermion with $q_{12,12} \neq -1$. We will describe this example in detail later on in this section.

6.2 Construction of m_{ij}

Definition 6.5. Given $p', p'' \in \mathbb{Z}$ such that $(p', p) = (p'', p) = 1$, we define $r' := \frac{p'}{p}$, $r'' := \frac{p''}{p}$ and in the standard chamber:

$$m_{ij}^S = \begin{cases} (\alpha_i, \alpha_j)_{\mathfrak{g}'} r' & \text{if } i \leq f,\ j < f \\ (\alpha_i, \alpha_j)_{\mathfrak{g}''} r'' & \text{if } i \geq f,\ j > f \\ 0 & \text{if } i > f > j \\ 1 & \text{if } i = f = j. \end{cases}$$

We notice that if we restrict to $\mathfrak{g}'$ or $\mathfrak{g}''$, we get exactly the same result as in the Cartan type section for p', p respectively p'', p.

Lemma 6.6. *If we call $q' = e^{i\pi r'}$ and $q'' = e^{i\pi r''}$, then $q_{ij} = e^{i\pi m_{ij}}$ is the braiding defined in definition 6.2.*

Proof. We have $m_{ij} = 0$ if α_i and α_j are disconnected, so that $1 = e^{i\pi \cdot 0}$ and $m_{ff} = 1$ for the fermionic root which gives $-1 = e^{i\pi \cdot 1}$ as demanded. □

Lemma 6.7. *In an arbitrary chamber $C_{\gamma_1, \ldots, \gamma_{rank}}$ we have*

$${m_{ij}}^C = (\gamma_i, \gamma_j)_{\mathfrak{g}'} r' + (\gamma_i, \gamma_j)_{\mathfrak{g}''} r'' + f(\gamma_i) f(\gamma_j).$$

Proof. We write $\gamma_i = \sum_k x_{ik}\alpha_k$ and $\gamma_j = \sum_l x_{jl}\alpha_l$ and we extend for linearity:

$$\begin{aligned} {m_{ij}}^C &= \sum_{k,l} x_{ik} x_{jl} {m_{kl}}^S \\ &= \sum_{k,l \in \mathfrak{g}' \cup \{f\}} x_{ik} x_{jl} (\alpha_k, \alpha_l)_{\mathfrak{g}'} r' + \sum_{k,l \in \mathfrak{g}'' \cup \{f\}} x_{ik} x_{jl} (\alpha_k, \alpha_l)_{\mathfrak{g}''} r'' + x_{if} x_{jf} = \\ &= (\gamma_i, \gamma_j)_{\mathfrak{g}'} r' + (\gamma_i, \gamma_j)_{\mathfrak{g}''} r'' + f(\gamma_i) f(\gamma_j) \end{aligned}$$

where the last equality follows from the definition of $f(\gamma)$ as the multiplicity of α_f in γ and the fact that on each component $\mathfrak{g}'$ and $\mathfrak{g}''$ the roots are spanned as in a Lie algebra. □

Corollary 6.8. *A root γ in an arbitrary chamber is*

- *m-truncation if* $\quad (\gamma, \gamma)_{\mathfrak{g}'} r' + (\gamma, \gamma)_{\mathfrak{g}''} r'' + f(\gamma) f(\gamma) = 1$
- *m-Cartan if, for every simple root β_i in the standard chamber,*

$$\begin{aligned} &(\gamma, \beta_i)_{\mathfrak{g}'} r' + 2(\gamma, \beta_i)_{\mathfrak{g}''} r'' + 2f(\gamma) f(\beta_i) \\ &= a_{\gamma, \beta_i}((\gamma, \gamma)_{\mathfrak{g}'} r' + (\gamma, \gamma)_{\mathfrak{g}''} r'' + f(\gamma) f(\gamma)). \end{aligned}$$

Example 6.9. We consider as an example the Lie superalgebra $\mathbf{A(1,1)}$ of rank 3. The simple roots in the standard chamber are $\{\alpha_1,\ \alpha_2 = \alpha_f,\ \alpha_3\}$ with inner product:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Hence:

$$m^S_{ij} = \begin{bmatrix} 2r' & -r' & 0 \\ -r' & 1 & -r'' \\ 0 & -r'' & 2r'' \end{bmatrix}, \qquad q_{ij} = \begin{bmatrix} (q')^2 & (q')^{-1} & 1 \\ (q')^{-1} & -1 & (q'')^{-1} \\ 1 & (q'')^{-1} & (q'')^2 \end{bmatrix}.$$

Remark 6.10. According to [Kac77] we can write the simple roots as

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \alpha_f = \epsilon_2 - \delta_1, \quad \alpha_3 = \delta_1 - \delta_2,$$

with vectors ϵ_i generating $\mathfrak{g}'$ and δ_i generating $\mathfrak{g}''$.

What remains to do is to see under which conditions the defined m_{ij} are realising solutions of the given braidings.

Lemma 6.11. *If $\gamma \in \mathfrak{g}'$ i.e. $\gamma = \sum_{i=1}^{f-1} a_i\alpha_i$ or $\gamma \in \mathfrak{g}''$ i.e. $\gamma = \sum_{i=f+1}^{n} a_i\alpha_i$, then γ is m-Cartan.*

Proof. Suppose $\gamma \in \mathfrak{g}'$; then

$$(\gamma,\gamma) = (\gamma,\gamma)_{\mathfrak{g}'} \qquad (\gamma,\gamma)_{\mathfrak{g}''} = 0$$
$$(\gamma,\alpha_i) = (\gamma,\alpha_i)_{\mathfrak{g}'} \qquad (\gamma,\alpha_i)_{\mathfrak{g}''} = 0$$

for every arbitrary simple root α_i. Moreover $f(\gamma) = 0$. So we have that (5)A:

$$2(\gamma,\alpha_i)_{\mathfrak{g}'}r' + 2(\gamma,\alpha_i)_{\mathfrak{g}''}r'' + 2f(\gamma)f(\alpha_i) = a_{\gamma,i}((\gamma,\gamma)_{\mathfrak{g}'}r' + (\gamma,\gamma)_{\mathfrak{g}''}r'' + f(\gamma)f(\gamma))$$

becomes:

$$2(\gamma,\alpha_i)r' = a_{\gamma,i}((\gamma,\gamma)r').$$

Since $\gamma \in \mathfrak{g}'$, we are restricting to one bosonic sector and thus the latter is true because of definition of $a_{\gamma,i}$ in the Lie algebra setting.
By linearity in the simple roots α_i it is possible to extend this result to every arbitrary root $\alpha = \sum b_i\alpha_i$. ☐

Lemma 6.12. *If $\gamma \neq \alpha_f$ is isotropic, i.e. $(\gamma,\gamma) = (\gamma,\gamma)_{\mathfrak{g}'} = (\gamma,\gamma)_{\mathfrak{g}''} = 0$, and $f(\gamma) = \pm 1$ then γ is m-truncation.*

Proof. Condition (5)B for a root to be m-truncation reads:

$$(\gamma,\gamma)_{\mathfrak{g}'} + (\gamma,\gamma)_{\mathfrak{g}''} + f(\gamma)f(\gamma) = 1$$

which is clearly true under these hypothesis. ☐

We summarize these results in the following:

Corollary 6.13. *The matrix m_{ij} defined in 6.5 realises the braiding q_{ij} for every root α, with the following possible exceptions:*

1. *α is a boson in $\mathfrak{g}' \cup \mathfrak{g}''$, i.e. $f(\alpha)$ is a strictly positive even integer.*

2. *α is an isotropic fermion with $f \neq \pm 1$.*

3. *α is a non-isotropic fermion.*

4. *α is a fermion strong orthogonal to another fermion γ, i.e. in their real span $\langle \alpha, \gamma \rangle_{\mathbb{R}}$ there aren't roots.*

Proof.

- If a boson α belongs to only one bosonic side $\mathfrak{g}'$ or $\mathfrak{g}''$, then lemma 6.11 tells us it must be m-Cartan. Otherwise, α is like in (1) and must be spanned by the standard fermion as well, thus $f(\alpha) > 0$ even. In this case lemma 6.11 fails since no Lie algebra Killing form is a priori holding. We then have to check explicitly for which r' and r'' one of condition (5) holds using Corollary 6.8.

- Let now α be a fermion which is never strong orthogonal to other fermions. If it is isotropic and $f(\alpha) = \pm 1$, thanks to lemma 6.12, it satisfies the M-condition truncation. If $f \neq \pm 1$ or it is non-isotropic, we are back to the points (2) and (3) of the lemma and we have to check explicitly for which r' and r'' one of condition (5) holds using Corollary 6.8.

- If α and γ are two strong orthogonal fermions, then $a_{\alpha\beta} = 0$. In this case we have to check for which r' and r''

$$m_{\alpha,\beta} = (\alpha, \beta)_{\mathfrak{g}'} r' + (\alpha, \beta)_{\mathfrak{g}''} r'' + f(\alpha) f(\beta) = 0$$

□

Remark 6.14. In the examples we didn't find any boson with $f > 2$ and any fermion with $f > 1$. Thus, point (1) concerns then just bosons with $f = 2$ and point (2) never happens.

In conclusion we will now have to look, in every example, if one or more of the situations described by lemma 6.13 is happening.

Now as last result we state a classification Lemma:

Lemma 6.15. *If all the bosonic roots are m-Cartan, then the unique possible realising solution for the given braiding is the matrix m_{ij} of definition 6.5. In particular this is the case if $\ell_i > 1 - a_{ij}$ for $\forall i \neq f$.*

Proof. Condition (5) gives a unique solution for the m_{ij} in the standard chamber: the fermionic root is m-truncation and thus fixed to $m_{ff} = 1$, while, since all the other roots are m-Cartan, restricting our study to the two bosonic sectors separately we end up in the same situation of lemma 5.6. Moreover the compatibility with the reflections fixes the m_{ij} in all the chambers. □

Example 6.16. We apply lemma 6.13 to example 6.9: after reflecting the standard chamber set of roots around the fermion α_2, we find for new simple roots: $\{\alpha_{12}, -\alpha_2, \alpha_{23}\}$ the matrix:

$$m_{ij}^C = \begin{bmatrix} 1 & -1 + r' & -1 + r' + r'' \\ -1 + r' & 1 & -1 + r'' \\ -1 + r' + r'' & -1 + r'' & 1 \end{bmatrix}.$$

Exception (4) of lemma 6.13 appears. We then have to ask $m_{23} = 0$, i.e. $r' + r'' = 1$. In that case m_{ij} is a realising solution.
This construction realises the Nichols algebra $\mathcal{B}(q)$ described by case row 8 of table 2 in [Hec05] when $q \neq \pm 1$.

6.3 Central charge

We will compute the central charge of systems associated to Lie superalgebras $\mathfrak{g}$, with non degenerate Killing form $(\ , \ )$.

Proposition 6.17. *The central charge of the system is $c = rank - 12(Q, Q)$ with*

$$Q = \frac{\rho^\vee_{\mathfrak{g}'}}{\sqrt{r'}} - \rho_{\mathfrak{g}'}\sqrt{r'} + \frac{\rho^\vee_{\mathfrak{g}''}}{\sqrt{r''}} - \rho_{\mathfrak{g}''}\sqrt{r''} - \rho^\vee_{rest}$$

where we denoted by $\rho_{\mathfrak{g}'}$ the sum of positive roots in $\mathfrak{g}'$, $\rho_{\mathfrak{g}''}$ the sum of positive roots in $\mathfrak{g}''$ and ρ_{rest} the sum of the remaining positive roots of $\mathfrak{g}$.

Proof. The central charge is $c = rank - 12(Q, Q)$ if Q is such that $\forall \alpha_i$ simple root of $\mathfrak{g}$

$$\frac{1}{2}(-\sqrt{r_i}\alpha_i, -\sqrt{r_i}\alpha_i) - (-\sqrt{r_i}\alpha_i, Q) = 1 \qquad \text{where} \quad r_i = \begin{cases} \frac{p'}{p} \text{ if } i < f \\ 1 \text{ if } i = f \\ \frac{p''}{p} \text{ if } i > f. \end{cases}$$

Let $\lambda^\vee_j = \frac{\lambda_j}{d_j}$ be such that $(\alpha_i, \lambda^\vee_j) = \delta_{ij}$. Since $\rho_{\mathfrak{g}} = \sum_{i=1}^{n} \lambda_i$, we have that $\rho_{\mathfrak{g}'} = \sum_{i<f} \lambda_i$, $\rho_{\mathfrak{g}''} = \sum_{i>f} \lambda_i$ and then $\rho_{\text{rest}} = \lambda_f$. We can thus rewrite Q as:

$$Q = \frac{\rho^\vee_{\mathfrak{g}'}}{\sqrt{r'}} - \rho_{\mathfrak{g}'}\sqrt{r'} + \frac{\rho^\vee_{\mathfrak{g}''}}{\sqrt{r''}} - \rho_{\mathfrak{g}''}\sqrt{r''} - \rho^\vee_{\text{rest}} = \sum_i \left(\frac{1}{\sqrt{r_i}} - \sqrt{r_i} d_i \right) \lambda^\vee_i.$$

Hence the previous equation becomes:

$$\begin{aligned} &\frac{1}{2}(-\alpha_i\sqrt{r_i}, -\alpha_i\sqrt{r_i}) - (-\alpha_i\sqrt{r_i}, Q) \\ &= \frac{1}{2} 2 d_i r_i + \sum_j \sqrt{r_i}(\frac{1}{\sqrt{r_j}} - \sqrt{r_j} d_j)(\alpha_i, \lambda^\vee_j) = 1 \end{aligned}$$

□

6.4 Algebra relations

We now want to determine when the algebra of screenings satisfies Nichols algebras relations for braiding q_{ij}.
We will denote again d', d'' the half length of the long bosonic root in $\mathfrak{g}', \mathfrak{g}''$.

Lemma 6.18. *For m_{ij} as above, smallness holds under the condition*

$$\frac{1}{2d'} \geq r' > 0, \qquad \frac{1}{2d''} \geq r'' > 0, \qquad \det(m_{ij}) > 0.$$

Proof. Smallness 3.2 for all monomials holds under the assumptions $|\alpha_i| \leq 1$, which means $2d'r' \leq 1$, $2d''r'' \leq 1$, and m_{ij} positive definite. By Sylvester's criterion, this is equivalent to $\det(m_{ij}) > 0$ and to the principal minor being positive definite. The principal minor is a rescaling of the root lattices $\mathfrak{g}', \mathfrak{g}''$, so it is positive definite for $r', r'' > 0$.

□

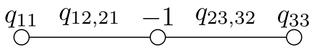
Example 6.19. For type $A(n,m)$ these conditions read

$$\frac{1}{2} \geq r' > 0, \qquad \frac{1}{2} \geq r'' > 0, \qquad \frac{n}{n+1}r' + \frac{m}{m+1}r'' < 1.$$

In [Ang15], theorem 3.1, we find a set of defining relations for each finite dimensional Nichols algebra of super Lie type. We report them in the following theorem.

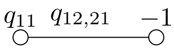
Theorem 6.20. *For finite dimensional Nichols algebra of super Lie type with diagonal braiding $q_{ij} = q^{(\alpha_i,\alpha_j)_{\mathfrak{g}',\mathfrak{g}''}}$ for bosonic roots and $q_{ii} = -1$ for the fermionic root in the standard chamber, associated to the root system of a Lie superalgebra $\mathfrak{g}$, the defining relations are as follows*

1. *For each root α the truncation relation and for each pair of simple roots α_i, α_j with $q_{ii}^{1-a_{ij}} \neq 1$ the Serre relation.*
2. *For the following subdiagrams the following additional relations:*
   - *For type $A(2,0)$, $A(1,1)$, $D(2,1;\alpha)$:*



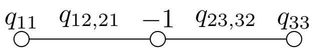
$$[(\mathrm{ad}x_2)x_1, (\mathrm{ad}x_2)x_3]_c = 0$$

   - *For type $B(1,1)$:*



$$[(\mathrm{ad}x_1)^2x_2, (\mathrm{ad}x_1)x_2]_c = 0$$

   - *For type $B(2,1)$*

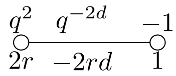


$$[(\mathrm{ad}x_1)^2(\mathrm{ad}x_2)x_3, (\mathrm{ad}x_1)x_2]_c = 0$$

Example 6.21. Let us consider a rank 1 q-diagram and corresponding m-solution, for a bosonic and fermionic root respectively:

The bosonic truncation relation $(\mathfrak{z}_b)^n = 0$, $n = ord(q^2)$ holds, according to 3.5, iff $r > 0$.
The fermionic truncation relation $(\mathfrak{z}_f)^2 = 0$ always holds according to 3.5.

Example 6.22. Let us consider a rank 2 super Lie q-diagram and corresponding m-solution:



In the examples we will found such a diagram just if $d = 1, 2$.

– By the previous example the simple truncation relations hold for $r \geq 0$.

  We conjecture that in this case the non-simple truncation ($[\mathfrak{z}_1, \mathfrak{z}_2]^n$ etc.) also hold for $r > 0$. But this would either require a reflection theory for algebra of screenings or a generalization of theorem 3.5.

– The bosonic Serre relation $[\mathfrak{z}_1, \ldots [\mathfrak{z}_1, \mathfrak{z}_2] \ldots] = 0$ (already studied in the Cartan section), involves $d + 1$ times the first screening and holds

  * for $d = 1$ iff $2r \notin -\mathbb{N}$.
  * for $d = 2$ iff $3r \notin -\mathbb{N}$.

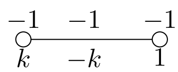
  So it may fail if the bosonic root α_1 is both q-Cartan and q-truncation and the truncation relation fails.

– The fermionic Serre relation $[\mathfrak{z}_2, [\mathfrak{z}_2, \mathfrak{z}_1]] = 0$ holds

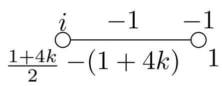
  * for $d = 1$ iff $-r + \frac{1}{2} \notin -\mathbb{N}$.
  * for $d = 2$ iff $2r, -2r + \frac{1}{2} \notin -\mathbb{N}$.
    But $2r \in \mathbb{Z}$ is not admissible because $q^{-2d} \neq 1$.

Summarizing we have the following possible exceptions:

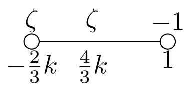
- for $k \in \mathbb{Z}$, k odd, $d = 1$:



- for $q^2 = i$, $k \in \mathbb{N}$, $d = 2$:



- for $q^2 = \zeta \in \mathcal{R}_3$, $k \in \mathbb{N}$, k odd, $d = 2$:



In conclusion:

Corollary 6.23. *Apart from the possible exceptions above, the screening operators algebra is as follows:*

- *For $r', r'' \geq 0$ all Nichols algebra relations hold (conjecturally also the non-simple truncation relations).*
- *For $r', r'' < 0$ all Nichols algebra relations hold except the bosonic truncation relations.*

  Conjecturally the algebra of screenings is again the positive part of an infinite-dimensional Kac-Procesi-DeConcini quantum super group.
- *For $r' > 0, r'' < 0$ or $r' < 0, r'' > 0$ the truncation relations on one side of the Dynkin diagram of the standard chamber fail, and we conjecturally get the positive part of an corresponding version of an infinite-dimensional Kac-Procesi-DeConcini quantum super group.*

Regarding the Kac-Procesi-DeConcini version of an arbitrary Nichols algebra, see the concept of a pre-Nichols algebra in [Ang14].

6.5 Examples in rank 2

We now present the cases of table 1 in [Hec05] rising from Lie superalgebras of rank 2. We will check in every case whether the exceptions of corollary 6.13 appear.
In rank 2, there is obviously always just one bosonic sector $\mathfrak{g}'$.
In the respective remarks we will express the simple roots in the standard chamber using as in [Kac77] the standard basis ϵ_i and δ_i.

Heckenberger row 3

The case row 3 of table 1 in [Hec05], studied in example 4.5, is realised by the Lie superalgebra lattice $\mathbf{A(1,0)}$. This case is described by the diagrams:

I II

with $q^2 \neq \pm 1$ and simple roots $\mathrm{I} : \{\alpha_1, \alpha_2\}$, $\mathrm{II} : \{\alpha_{12}, -\alpha_2\}$. The set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_{12}\}$ with unique associate Cartan matrix and inner products

$$a_{ij} = \begin{bmatrix} 2 & -1 \\ \\ -1 & 2 \end{bmatrix}, \qquad (\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}.$$

Therefore the m_{ij} matrix in the standard basis and after reflecting around α_2 are given by:

$$m_{ij}^{\mathrm{I}} = \begin{bmatrix} 2r & -r \\ -r & 1 \end{bmatrix}, \qquad m_{ij}^{\mathrm{II}} = \begin{bmatrix} 1 & -1+r \\ -1+r & 1 \end{bmatrix}.$$

None of the exceptions of lemma 6.13 appears; therefore m_{ij} is a realising solution $\forall r$. This result matches with example 4.5.

Remark 6.24. As observed in example 4.5, if we allow the value $q^2 = -1$ we obtain row 2 of table 1 in [Hec05].

Remark 6.25. The simple roots in the standard chamber of $A(1,0)$ can be expressed by

$$\alpha_1 = \epsilon_1 - \epsilon_2, \qquad \alpha_2 = \alpha_f = \epsilon_2 - \delta_1.$$

Heckenberger row 5

Row 5 of table 1 in [Hec05] is realised by the Lie superalgebra lattice **B(1,1)**. This case is described by the diagrams:

I II

with $q^2 \neq \pm 1, q^2 \notin \mathcal{R}_4$ and simple roots I : $\{\alpha_1, \alpha_2\}$, II : $\{\alpha_{12}, -\alpha_2\}$.

The set of positive roots is given by $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}\}$ with unique associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$$

and inner product:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -2 \\ -2 & 0 \end{bmatrix}.$$

Therefore the m_{ij} matrix in the standard basis and after reflecting around α_1 are given by:

$$m_{ij}^{\mathrm{I}} = \begin{bmatrix} 2r & -2r \\ -2r & 1 \end{bmatrix} \qquad m_{ij}^{\mathrm{II}} = \begin{bmatrix} -2r+1 & 2r-1 \\ 2r-1 & 1 \end{bmatrix}.$$

None of the exceptions of lemma 6.13 appears; therefore m_{ij} is a realising solution $\forall r$.

Remark 6.26. When $q^2 \in \mathcal{R}_3$, the root α_1 is q-Cartan *and* q-truncation. When it is m-truncation we get:

$$m_{ij}^{\mathrm{I}} = \begin{bmatrix} \frac{2}{3} & -2r \\ -2r & 1 \end{bmatrix} \qquad m_{ij}^{\mathrm{II}} = \begin{bmatrix} \frac{5}{3} - 4r & 2r-1 \\ 2r-1 & 1 \end{bmatrix} \qquad m_{ij}^{\mathrm{III}} = \begin{bmatrix} \frac{2}{3} & 2r - \frac{4}{3} \\ 2r - \frac{4}{3} & \frac{11}{3} - 8r \end{bmatrix}$$

where III: $\{-\alpha_1, \alpha_{112}\}$ comes after reflecting around α_1. The root α_{112} is never m-Cartan and it is m-truncation iff $r = \frac{1}{3}$. But for this value of r, α_1 is also m-Cartan and thus this is *not* a new solution.

Remark 6.27. If we allow $q = i$ the system is the one described in row 4 in section 5. Also in this case it corresponds to the Lie superalgebra $B(1,1)$.

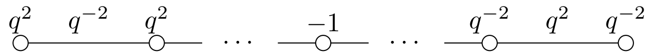
Remark 6.28. The roots can be expressed by

$$\alpha_1 = \epsilon_1, \qquad \alpha_2 = \alpha_f = \delta_1 - \epsilon_1.$$

6.6 Arbitrary rank

We generalize our study to arbitrary rank cases. In every case we will see under which assumptions the constructed m_{ij} matrices are realising solutions.

A(m,n)



The simple roots in the standard chamber are:

$$\alpha_1, \ldots, \alpha_f = \alpha_{m+1}, \ldots, \alpha_{m+n+1}$$

with inner product matrix

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -1 & & & & \\ -1 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 0 & \ddots & \\ & & & \ddots & \ddots & -1 \\ & & & & -1 & 2 \end{bmatrix}$$

We list all the positive roots. We denote by Δ_0 the set of bosons and by Δ_1 the set of fermions according to the literature [Kac77].

$$\Delta_0 = \{\alpha_l + \ldots + \alpha_k, \text{ with } l, k < f \text{ or } l, k > f\}$$
$$\Delta_1 = \{\alpha_l + \ldots + \alpha_k, \text{ with } l \leq f \leq k, \ l \neq k\}$$

We now apply the lemmas of the previous section to determine possible conditions on r' and r'' such that the m_{ij} matrix defined as in 6.5 is a realising solution.

- All the bosons are either in $\mathfrak{g}'$ or $\mathfrak{g}''$. Then, thanks to lemma 6.11, we know they are always m-Cartan.
- All the fermions are isotropic and have $f = \pm 1$. Thanks to lemma 6.12 we know that if they are not strong orthogonal to any other root they are m-truncation.
- We now focus on the case of strong orthogonal fermions. Let us consider two fermions:

  $$\gamma_1 = \alpha_{l_1} + \ldots + \alpha_{k_1} \qquad \text{with } \ l_1 \leq f \leq k_1,$$
  $$\gamma_2 = \alpha_{l_2} + \ldots + \alpha_{k_2} \qquad \text{with } \ l_2 \leq f \leq k_2.$$

  They are strong orthogonal if $l_1 \neq l_2, \ k_1 \neq k_2$. In this case we have to check that $m_{12} = (\gamma_1, \gamma_2)_{\mathfrak{g}'} r' + (\gamma_1, \gamma_2)_{\mathfrak{g}''} r'' + f(\gamma_1) f(\gamma_2) = 0$.
  We thus compute the inner products in the two bosonic sides. We assume $l_1 < l_2$ and $k_1 < k_2$, because every other combination works analogously

and gives the same result.
Wlog we can assume $l_2 = l_1 + 1$ and $k_2 = k_1 + 1$ and thus

$$\begin{aligned}(\gamma_1, \gamma_2) =& (\alpha_{l_1}, \gamma_2) + (\alpha_{l_1+1}, \gamma_2) + \ldots + (\alpha_f, \gamma_2) + \ldots + (\gamma_{k_1}, \gamma_2) \\ =& (\alpha_{l_1}, \alpha_{l_1+1})_{\mathfrak{g}'} \\ &+ (\alpha_{l_1+1}, \alpha_{l_1+1})_{\mathfrak{g}'} + (\alpha_{l_1+1}, \alpha_{l_1+2})_{\mathfrak{g}'} \\ &+ \ldots \\ &+ (\alpha_f, \alpha_{f-1})_{\mathfrak{g}'} + (\alpha_f, \alpha_f) + (\alpha_f, \alpha_{f+1})_{\mathfrak{g}''} \\ &+ \ldots \\ &+ (\alpha_{k_1}, \alpha_{k_1} - 1)_{\mathfrak{g}''} + (\alpha_{k_1}, \alpha_{k_1})_{\mathfrak{g}''} + (\alpha_{k_1}, \alpha_{k_1+1})_{\mathfrak{g}''}\end{aligned}$$

The only term that contributes is $(\alpha_f, \alpha_{f-1})_{\mathfrak{g}'} + (\alpha_f, \alpha_f) + (\alpha_f, \alpha_{f+1})_{\mathfrak{g}''}$ since the previous terms sum up to zero in $\mathfrak{g}'$, and the following terms sum up to zero in $\mathfrak{g}''$. Hence we have $(\gamma_1, \gamma_2) = -1_{\mathfrak{g}'} - 1_{\mathfrak{g}''}$. Asking m_{12} to be zero, means to ask

$$-1 \cdot r' - 1 \cdot r'' + 1 = 0 \qquad \Rightarrow \qquad r' + r'' = 1$$

To conclude, the only condition needed for the m_{ij} matrix to be a realising solution is $r' + r'' = 1$.

Remark 6.29. This condition matches with the formulation of $A(m, n)$ in terms of Nichols algebra diagram ([Hec06], Table C, row 2), where $q_{\mathfrak{g}'} = q$ and $q_{\mathfrak{g}''} = q^{-1}$. Indeed, if $r' + r'' = 1$ then

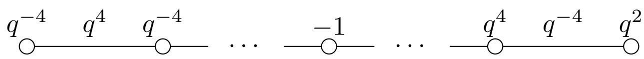
$$q_{\mathfrak{g}'} q_{\mathfrak{g}''} = e^{i\pi(\alpha_i,\alpha_i)r'} e^{i\pi(\alpha_j,\alpha_j)r''} = e^{i\pi 2r'} e^{i\pi 2r''} = e^{i\pi 2(r'+r'')} = 1,$$

calling α_i a root in $\mathfrak{g}'$ and α_j a root in $\mathfrak{g}''$.

Remark 6.30. We can write the simple roots in the standard chamber using as in [Kac77] the standard basis $\epsilon_1, \ldots, \epsilon_{m+1}, \delta_1, \ldots, \delta_{n+1}$:

$$\begin{aligned}\{\alpha_1 = \epsilon_1 - \epsilon_2,\ \ \alpha_2 = \epsilon_2 - \epsilon_3, \ldots,\ \ \alpha_{m+1} = \epsilon_{m+1} - \delta_1, \\ \alpha_{m+2} = \delta_1 - \delta_2, \ldots,\ \ \alpha_{m+n+1} = \delta_n - \delta_{n+1}\}\end{aligned}$$

B(m,n)



The simple roots in the standard chamber are:

$$\alpha_1, \ldots, \alpha_f = \alpha_n, \ldots, \alpha_{m+n}$$

with inner product matrix

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 4 & -2 & & & & \\ -2 & \ddots & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & 0 & \ddots & \\ & & & \ddots & \ddots & -2 \\ & & & & -2 & 2 \end{bmatrix}$$

All the positive roots are:

$$\begin{aligned}
\Delta_0 = \{&\alpha_l + \ldots + \alpha_k, && \text{with } l, k < f \\
&\alpha_l + \ldots + \alpha_k, && \text{with } l, k > f,\ k \neq m+n \\
&\alpha_l + \ldots + \alpha_{m+n}, && \text{with } l > f \\
&\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n}, && \text{with } l < f,\ k \leq f \\
&\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n}, && \text{with } l, k > f\}
\end{aligned}$$

$$\begin{aligned}
\Delta_1 = \{&\alpha_l + \ldots + \alpha_{m+n}, && \text{with } l \leq f \\
&\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n}, && \text{with } l < f < k \\
&\alpha_l + \ldots + \alpha_k, && \text{with } l < f < k,\ k \neq m+n\}
\end{aligned}$$

We now apply the lemmas of the previous section to determine possible conditions on r' and r'' such that the m_{ij} matrix defined as in 6.5 is a realising solution.

- All the bosons which are not of the type $\gamma_{lk} := \alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n}$, with $l < f,\ k \leq f$, are either in $\mathfrak{g}'$ or $\mathfrak{g}''$. Then, thanks to lemma 6.11, we know they are always m-Cartan.
- For γ_{lk}, we need to explicitly ask condition (5).
  The inner product is $(\gamma_{lk}, \gamma_{lk}) = -2_{\mathfrak{g}'} - 4_{\mathfrak{g}''}$.
  - γ_{lk} is m-truncation if $2r' + 4r'' = 3$.
  - γ_{lk} is m-Cartan if $r' + r'' = 1$.
- All the fermions which are not of the type $\gamma_l := \alpha_l + \ldots + \alpha_{m+n}$, are isotropic and have $f = \pm 1$. Thanks to lemma 6.12 we then know that if they are not strong orthogonal to any other root they are m-truncation.
- For γ_l, we need to explicitly ask condition (5).
  The inner product is $(\gamma_l, \gamma_l) = -1_{\mathfrak{g}''}$.
  - γ_l is m-truncation holds if $r'' = 0$.
  - γ_l is m-Cartan holds if $r' + r'' = 1$.
- We now focus on the case of strong orthogonal fermions. Let us consider the fermions:

$$\begin{aligned}
\{&\gamma_1 := \alpha_{l_1} + \ldots + \alpha_{m+n} \\
&\gamma_2 := \alpha_{l_2} + \ldots + 2\alpha_{k_2} + \ldots + 2\alpha_{m+n} \\
&\gamma_3 := \alpha_{l_3} + \ldots + \alpha_{k_3}\}
\end{aligned}$$

  The fermions γ_1 and γ_2 are strong orthogonal iff $l_1 \neq l_2$;
  The fermions γ_2 and γ_3 are strong orthogonal iff $l_2 \neq l_3$ or $k_2 \neq k_3 + 1$;
  The fermions γ_1 and γ_3 are strong orthogonal iff $l_1 \neq l_3$;
  Two fermions of type γ_2 are strong orthogonal iff have different l_2 and k_2;
  Two fermions of type γ_3 are strong orthogonal iff have different l_3 and k_3;
  Asking the condition $m_{ij} = 0$ for those cases, we find again the condition $r' + r'' = 1$.

In conclusion, the only condition needed for the m_{ij} matrix to be a realising solution is $r' + r'' = 1$. If this condition is satisfied the bosons with $f = 2$ as well as the non isotropic fermions are m-Cartan. If moreover $r' = r'' = \frac{1}{2}$ then the bosons with $f = 2$ are also m-truncation.

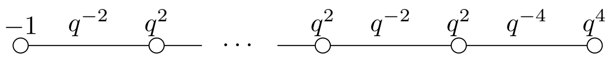
Remark 6.31. As in the case of the Lie superalgebras of type $A(m, n)$, the condition $r' + r'' = 1$ matches with the formulation of $B(m, n)$ in terms of Nichols algebra diagram ([Hec06], Table C, row 4), where $q_{\mathfrak{g}'} = q$ and $q_{\mathfrak{g}''} = q^{-1}$.

Remark 6.32. We can write the simple roots in the standard chamber using as in [Kac77] the standard basis $\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n$:

$$\{\alpha_1 = \delta_1 - \delta_2,\ \ \alpha_2 = \delta_2 - \delta_3, \dots,\ \ \alpha_n = \delta_n - \epsilon_1,$$
$$\alpha_{n+1} = \epsilon_1 - \epsilon_2, \dots,\ \ \alpha_{m+n} = \epsilon_m.\}$$

The bosons with $f = 2$ will be of the form $\delta_i + \delta_j$, while the non isotropic fermions will be δ_i.

C(n)



The simple roots in the standard chamber are:

$$\alpha_f = \alpha_1, \dots, \alpha_n$$

with inner product matrix

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 0 & -1 & & & & \\ -1 & 2 & \ddots & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -2 & 4 \end{bmatrix}.$$

All the positive roots are:

$$\begin{aligned} \Delta_0 = \{&\alpha_l + \dots + \alpha_k, && \text{with } l \neq 1\ k \neq n \\ &\alpha_l + \dots + 2\alpha_k + \dots + 2\alpha_{n-1} + \alpha_n, && \text{with } l \neq 1\ k \neq n \\ &\alpha_l + \dots + \alpha_n, && \text{with } l \neq 1 \\ &2\alpha_l + \dots + 2\alpha_{n-1} + \alpha_n, && \text{with } l \neq 1\} \end{aligned}$$

$$\begin{aligned} \Delta_1 = \{&\alpha_1 + \dots + \alpha_n \\ &\alpha_1 + \dots + \alpha_k, && \text{with } k \neq 1 \\ &\alpha_1 + \dots + 2\alpha_k + \dots + 2\alpha_{n-1} + \alpha_n, && \text{with } k \neq n\} \end{aligned}$$

We now apply the lemmas of the previous section to determine possible conditions on r' such that the m_{ij} matrix defined as in 6.5 is a realising solution.

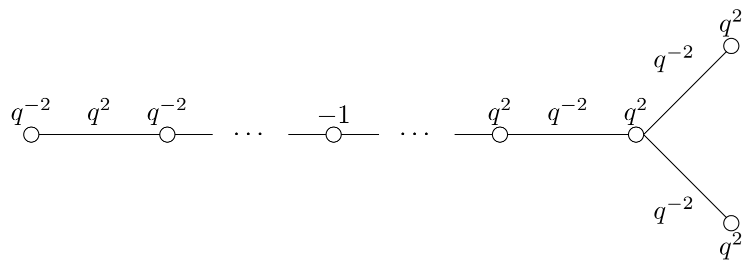
- Since there is just one bosonic side it is obvious that all the bosons are m-Cartan.
- All the fermions are isotropic, non strong orthogonal to each other, and have $f = \pm 1$. Thanks to lemma 6.12 we then know that they are m-truncation.

To conclude, the m_{ij} matrix is always a realising solution.

Remark 6.33. We can write the simple roots in the standard chamber using as in [Kac77] the standard basis $\epsilon_1, \delta_1 \ldots, \delta_{n-1}$:

$$\{\alpha_1 = \epsilon_1 - \delta_1,\ \ \alpha_2 = \delta_1 - \delta_2, \ldots,\ \ \alpha_{n-1} = \delta_{n-2} - \delta_{n-1},\ \ \alpha_n = 2\delta_{n-1}\}$$

D(m,n)



The simple roots in the standard chamber are:

$$\alpha_1, \ldots, \alpha_n = \alpha_f, \ldots, \alpha_{n+m}$$

with inner product matrix

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -1 & & & & \\ -1 & \ddots & \ddots & & & \\ & \ddots & 0 & \ddots & & \\ & & \ddots & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{bmatrix}$$

All the positive roots are:

$$
\begin{aligned}
\Delta_0 = \{&\alpha_l + \ldots + \alpha_k, && \text{with } l, k < f\\
&\alpha_l + \ldots + \alpha_k, && \text{with } l, k > f\\
&\alpha_l + \ldots + \alpha_{m+n-2} + \alpha_{m+n}, && \text{with } l > f\\
&\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n-2} + \alpha_{m+n-1} + \alpha_{m+n}, && \text{with } l < f,\ k \leq f\\
&\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n-2} + \alpha_{m+n-1} + \alpha_{m+n}, && \text{with } l, k > f\\
&2\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n-2} + \alpha_{m+n-1} + \alpha_{m+n}, && \text{with } l < f,\ k \leq f\}
\end{aligned}
$$

$$
\begin{aligned}
\Delta_1 = \{&\alpha_l + \ldots + \alpha_k, && \text{with } l \leq f \leq k\\
&\alpha_l + \ldots + \alpha_{n+m-2} + \alpha_{n+m}, && \text{with } l \leq f\\
&\alpha_l + \ldots + 2\alpha_k + \ldots + 2\alpha_{m+n-2} + \alpha_{n+m-1} + \alpha_{n+m}, && \text{with } l < f < k\}
\end{aligned}
$$

We now apply the lemmas of the previous section to determine possible conditions on r' and r'' such that the m_{ij} matrix defined as in 6.5 is a realising solution.

- All bosons except the IV or VI type in the list, are either in $\mathfrak{g}'$ or $\mathfrak{g}''$. Then, thanks to lemma 6.11, we know they are always m-Cartan.
- The bosons of type IV have inner product $-2_{\mathfrak{g}'} - 4_{\mathfrak{g}''}$.
  - it is m-truncation if $2r' + 4r'' = 3$.
  - it is m-Cartan if $r' + r'' = 1$.

  The bosons of type VI have inner product $-4_{\mathfrak{g}''}$.
  - it is m-truncation if $4r'' = 3$.
  - it is m-Cartan if $r' + r'' = 1$.
- All fermions are isotropic and have $f = \pm 1$. Thanks to lemma 6.12 we then know that if they are not strong orthogonal to any other root they are m-truncation.
- There are many possibility for two fermions to be strong orthogonal. Asking the condition $m_{ij} = 0$ for those cases, we find again the condition $r' + r'' = 1$.

In conclusion, the only condition needed for the m_{ij} matrix to be a realising solution is $r' + r'' = 1$. If this condition is satisfied the bosons with $f = 2$ are m-Cartan. If moreover $r' = r'' = \frac{1}{2}$ then the boson of type IV are also m-truncation. Instead if $r' = \frac{1}{4}$, $r'' = \frac{3}{4}$ then the boson of type VI are also m-truncation.

Remark 6.34. As in the previous cases the condition $r' + r'' = 1$ matches with the formulation of $D(m, n)$ in terms of Nichols algebra diagram ([Hec06], Table C, row 10), where $q_{\mathfrak{g}'} = q$ and $q_{\mathfrak{g}''} = q^{-1}$.

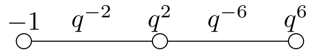
Remark 6.35. We can write the simple roots in the standard chamber using as in [Kac77] the standard basis $\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n$:

$$\{\alpha_1 = \delta_1 - \delta_2, \ldots, \quad \alpha_n = \delta_n - \epsilon_1, \quad \alpha_{n+1} = \epsilon_1 - \epsilon_2, \; \ldots$$
$$\ldots \; \alpha_{m+n-1} = \epsilon_{m-1} - \epsilon_m, \quad \alpha_{m+n} = \epsilon_{m-1} + \epsilon_m\}.$$

The bosons of type IV will be of the form $\delta_i + \delta_j$, while the one of type VI will be of the form $2\delta_i$.

Sporadic cases

G(3)



The simple roots in the standard chamber are $\{\alpha_1 = \alpha_f, \alpha_2, \alpha_3\}$ with inner product

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -3 \\ 0 & -3 & 6 \end{bmatrix}.$$

There is only one bosonic part $\mathfrak{g}'$ and the positive roots are:

$$\{\alpha_1, \; \alpha_2, \; \alpha_3, \; \alpha_{12}, \; \alpha_{23}, \; \alpha_{223}, \; \alpha_{123}, \; \alpha_{1223},$$
$$\alpha_{12223}, \; \alpha_{2223}, \; \alpha_{22233}, \; \alpha_{1222233}, \; \alpha_{122233}.\}$$

The m_{ij} matrix is given by

$$m^{\mathrm{I}}_{ij} = \begin{bmatrix} 1 & -r & 0 \\ -r & 2r & -3r \\ 0 & -3r & 6r \end{bmatrix}.$$

- Since there is just one bosonic side it is obvious that all the bosons satisfy are m-Cartan.
- All the fermions, except for α_{1223}, are isotropic and have $f = \pm 1$. Thanks to lemma 6.12 we then know that they are m-truncation.
- The fermion α_{1223} is m-Cartan without further assumptions.
- There are no couples of strong orthogonal fermions.

To conclude the m_{ij} matrix is a realising solution $\forall r$.
This construction realise the Nichols algebra $\mathcal{B}(q)$ described row 7 of table 2 in [Hec05] when $q \neq \pm 1, \; q \notin \mathcal{R}_3$.

For this lower rank case we can also show explicitly all the reflections of the m_{ij} matrix: reflecting m^{I}_{ij} around α_1 we find the following

$$m^{\mathrm{II}}_{ij} = \begin{bmatrix} 1 & -1+r & 0 \\ -1+r & 1 & -3r \\ 0 & -3r & 6r \end{bmatrix}.$$

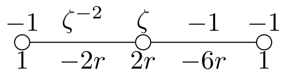
Reflecting it around α_{12} we find the following

$$m_{ij}^{\text{III}} = \begin{bmatrix} 2r & -r & -2r \\ -r & 1 & -1+3r \\ -2r & -1+3r & 1 \end{bmatrix}.$$

Reflecting it around α_{123} we find the following

$$m_{ij}^{\text{IV}} = \begin{bmatrix} 6r & -3r & 0 \\ -3r & 1 & -1+2r \\ 0 & -1+2r & 1-2r. \end{bmatrix}$$

Remark 6.36. If $q^2 \in \mathcal{R}_6$, α_3 is both q-Cartan and q-truncation. When it is m-truncation we find

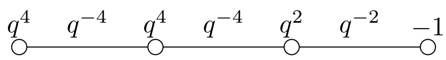


with $\zeta \in \mathcal{R}_6$. This is a solution iff $r = \frac{1}{6}$. But for this value of r, α_3 is also m-Cartan and thus this is not a new solution.

Remark 6.37. The roots can be expressed by

$$\alpha_1 = \alpha_f = \delta + \epsilon_1, \qquad \alpha_2 = \epsilon_2 \qquad \alpha_3 = \epsilon_3 - \epsilon_2$$

F(4)



The simple roots in the standard chamber are $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4 = \alpha_f\}$ with inner product

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 4 & -2 & & \\ -2 & 4 & -2 & \\ & -2 & 2 & -1 \\ & & -1 & 0 \end{bmatrix}.$$

There is only one bosonic part $\mathfrak{g}'$ and the rest of the positive roots are:

$$\{\alpha_{12},\ \alpha_{23},\ \alpha_{34},\ \alpha_{233},\ \alpha_{123},\ \alpha_{234},\ \alpha_{1233},\ \alpha_{2334},$$
$$\alpha_{1234},\ \alpha_{12233},\ \alpha_{12334},\ \alpha_{1223334},\ \alpha_{122334},\ \alpha_{12233344}.\}$$

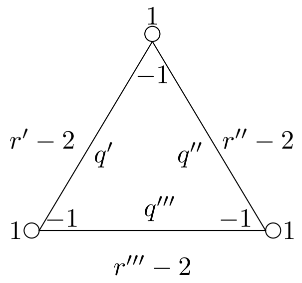
- All bosons except $\alpha_{12233344}$ are completely in the bosonic sector and thus are m-Cartan.
- The boson $\alpha_{12233344}$ is m-Cartan without further assumptions.
- All fermions are isotropic and have $f = \pm 1$. Thanks to lemma 6.12 we then know they are m-truncation.
- We have two couples of strong orthogonal fermions:

  $$\{\alpha_{34}, \alpha_{122334}\} \qquad \{\alpha_{234}, \alpha_{12334}\}$$

  which give the condition $r = \frac{1}{3}$.

To conclude, the condition for the m_{ij} matrix to be a realising solution is $r = \frac{1}{3}$.

D(2,1; α)



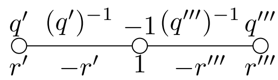
The simple roots in the standard chamber are $\{\alpha_1, \alpha_2 = \alpha_f, \alpha_3\}$ with inner product:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

The positive roots are:

$$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_{12},\ \alpha_{23},\ \alpha_{123},\ \alpha_{1223}.\}$$

Reflecting the diagram around one of the root (the system is completely symmetric in the three roots), we obtain:



Exception (4) of lemma 6.13 appears. Imposing that the first and the third roots are not connected we find the condition $r' + r'' + r''' = 2$. In this case these m_{ij} matrices are realising solution.
This corresponds to the condition $q' \cdot q'' \cdot q''' = 1$ of case 9 (as well as 10 and 11), rank 3, in table 2 of [Hec05].

7 Rank 2

7.1 Other cases in rank 2: construction

In this section we are going to present the examples of $rank = 2$ Nichols algebra which don't belong to the Cartan and super Lie study of the previous two sections.

Heckenberger row 6

This case of table 1 in [Hec05] is described by two diagrams:

where $\zeta \in \mathcal{R}_3$ and $q^2 \neq 1, \zeta, \zeta^2$ and with respectively simple roots:

$$\mathrm{I} : \{\alpha_1, \alpha_2\} \qquad \mathrm{II} : \{-\alpha_1, \alpha_{112}\}.$$

There is just one associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}\}$ where α_2 and α_{112} are only q-Cartan while the others are only q-truncation.

Proposition 7.1. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{2}{3} & -r \\ -r & 2r \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3}+r \\ -\frac{4}{3}+r & \frac{8}{3}-2r \end{bmatrix}.$$

Proof. First we check that condition (5)B is satisfied for α_1:

$$m_{11} = \frac{2}{1-a_{12}} = \frac{2}{3}$$

and condition (5)A is satisfied for α_{22} and α_{112}:

$$m_{22,22} = \frac{2m_{12}}{a_{21}} = 2r$$
$$m_{112,112} = \frac{2m_{112,-1}}{a_{112,1}} = \frac{8}{3} - 2r.$$

We then check that the reflection around α_1 send one m_{ij}-matrix to the other as follows:

$$m^{\mathrm{II}}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3}+r \\ -\frac{4}{3}+r & \frac{8}{3}-2r \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} \frac{2}{3} & -r \\ -r & 2r \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} = \mathcal{R}^1(m^{\mathrm{I}}_{ij})$$

□

Remark 7.2. When $q^2 \in \mathcal{R}_2$, the root α_2 is q-Cartan *and* q-truncation.
When it is m-truncation we get:

$$m^{\mathrm{I}}_{ij} = \begin{bmatrix} \frac{2}{3} & -r \\ -r & 1 \end{bmatrix} \qquad m^{\mathrm{II}}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{4}{3}+r \\ -\frac{4}{3}+r & \frac{11}{3}-4r \end{bmatrix} \qquad m^{\mathrm{III}}_{ij} = \begin{bmatrix} \frac{5}{3}-2r & r-1 \\ r-1 & 1 \end{bmatrix}.$$

with III: $\{\alpha_{12}, -\alpha_2\}$.
The root α_{112} is never m-truncation and it is m-Cartan iff $r = \frac{1}{2}$. But for this value of r, α_2 is also m-Cartan and thus this is *not* a new solution.

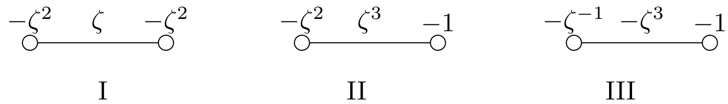
As we can see in [Hel10] truncation and Serre relations are the only defining relations. We have the following:

Proposition 7.3. *The truncation relations hold for every* $r \geq 0$*, while the Serre relations hold for* $2r \notin -\mathbb{N}$ *and* $r \neq \frac{1+3k}{3}, \frac{2+3k}{3}$.

Remark 7.4. We could call this case of *colour* type. It indeed behaves as a super Lie case except for the fact that $m_{ff} = \frac{2}{3}$, and not 1. In particular lemma 6.15 trivially extends to this case as a classification lemma, with the appropriate changes.

Heckenberger row 9

This case of table 1 in [Hec05] is described by three diagrams:



where $\zeta \in \mathcal{R}_{12}$ and with respectively simple roots:

$$\text{I} : \{\alpha_1, \alpha_2\} \qquad \text{II} : \{-\alpha_1, \alpha_{112}\} \qquad \text{III} : \{\alpha_{12}, -\alpha_{122}\}.$$

The associate Cartan matrices are:

$$a^{\text{I}}_{ij} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \qquad a^{\text{II}}_{ij} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \qquad a^{\text{III}}_{ij} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{122}\}$ where α_{12} is only q-Cartan while the others are only q-truncation.

Proposition 7.5. *The following* m_{ij} *matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{7}{12} \\ -\frac{7}{12} & \frac{2}{3} \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{3}{4} \\ -\frac{3}{4} & 1 \end{bmatrix} \qquad m^{III}_{ij} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{4} \\ -\frac{1}{4} & 1 \end{bmatrix}$$

Proof. First we check that condition (5)B is satisfied for all the roots:

$$\begin{aligned} m_{11} &= \frac{2}{1 - a_{12}} = \frac{2}{3} \\ m_{22} &= \frac{2}{1 - a_{21}} = \frac{2}{3} \\ m_{112,112} &= \frac{2}{1 - a_{112,1}} = 1 \\ m_{122,122} &= \frac{2}{1 - a_{122,12}} = 1 \end{aligned}$$

and condition (5)A is satisfied for the root α_{12}:

$$m_{12,12} = \frac{2m_{-122,12}}{a_{12,112}} = \frac{1}{6}.$$

We then check that the reflections send one m_{ij}-matrix to the other as follows:

$$m_{ij}^{\mathrm{II}} = \begin{bmatrix} \frac{2}{3} & -\frac{3}{4} \\ \\ -\frac{3}{4} & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ \\ 0 & 1 \end{bmatrix}^T \begin{bmatrix} \frac{2}{3} & -\frac{7}{12} \\ \\ -\frac{7}{12} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ \\ 0 & 1 \end{bmatrix} = \mathcal{R}^1(m_{ij}^{\mathrm{I}})$$

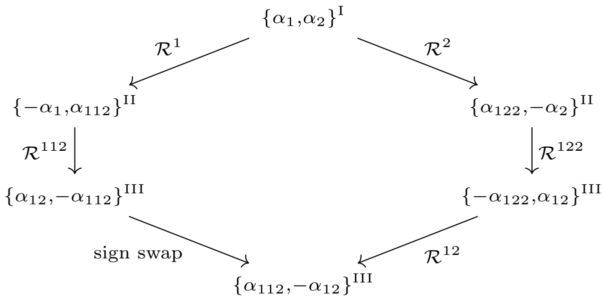
$$m_{ij}^{\mathrm{III}} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{4} \\ \\ -\frac{1}{4} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} \frac{2}{3} & -\frac{3}{4} \\ \\ -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \\ 1 & -1 \end{bmatrix} = \mathcal{R}^{122} \circ \mathcal{R}^2(m_{ij}^{\mathrm{I}})$$

□

Corollary 7.6. *By formula (4) for rank 2, we have that the central charge of the system is $c = -126$.*

Proposition 7.7. *Truncation and Serre relations always hold, by lemma 3.4.*

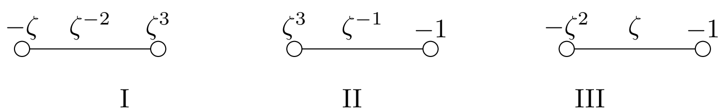
We conclude this case with a picture illustrating how the set of simple roots behave under reflections. We write I, II, III, to indicate to which diagram do the simple roots in each case belong.



(9)

Heckenberger row 10

This case of table 1 in [Hec05] is described by three diagrams:



where $\zeta \in \mathcal{R}_9$ and with respectively simple roots:

$$\mathrm{I} : \{\alpha_1, \alpha_2\} \qquad \mathrm{II} : \{-\alpha_2, \alpha_{122}\} \qquad \mathrm{III} : \{\alpha_{12}, -\alpha_{122}\}.$$

The associate Cartan matrices are:

$$a_{ij}^{\mathrm{I}} = \begin{bmatrix} 2 & -2 \\ \\ -2 & 2 \end{bmatrix} \qquad a_{ij}^{\mathrm{II}} = \begin{bmatrix} 2 & -2 \\ \\ -1 & 2 \end{bmatrix} \qquad a_{ij}^{\mathrm{III}} = \begin{bmatrix} 2 & -4 \\ \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{122}, \alpha_{11122}\}$ where α_1 and α_{12} are only q-Cartan while the others are only q-truncation.

Proposition 7.8. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{5}{9} & -\frac{5}{9} \\ -\frac{5}{9} & \frac{2}{3} \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{2}{3} & -\frac{7}{9} \\ -\frac{7}{9} & 1 \end{bmatrix} \qquad m^{III}_{ij} = \begin{bmatrix} \frac{1}{9} & -\frac{2}{9} \\ -\frac{2}{9} & 1 \end{bmatrix}$$

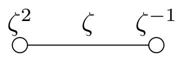
Proof. We check that the roots $\{\alpha_2, \alpha_{112}, \alpha_{122}, \alpha_{11122}\}$ satisfy condition (5)B, while the root α_1 and α_{12} satisfy condition (5)A.
We check that the reflections send one m_{ij}-matrix to the other. □

Corollary 7.9. *By formula (4) for rank 2, we have that the central charge of the system is $-\frac{1088}{5}$.*

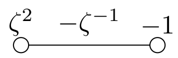
Proposition 7.10. *Truncation and Serre relations always hold, by lemma 3.4.*

Heckenberger row 12

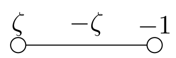
This case of table 1 in [Hec05] is described by three diagrams:



I



II



III

where $\zeta \in \mathcal{R}_8$ and with respectively simple roots:

$$\mathrm{I}: \{\alpha_1, \alpha_2\} \qquad \mathrm{II}: \{-\alpha_1, \alpha_{1112}\} \qquad \mathrm{III}: \{\alpha_{112}, -\alpha_{1112}\}.$$

There is just one associate Cartan matrix:

$$a_{ij} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{1112}, \alpha_{11122}\}$ where α_2 and α_{112} are only q-Cartan while the others are only q-truncation.

Proposition 7.11. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{1}{2} & -\frac{7}{8} \\ -\frac{7}{8} & \frac{7}{4} \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{1}{2} & -\frac{5}{8} \\ -\frac{5}{8} & 1 \end{bmatrix} \qquad m^{III}_{ij} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{8} \\ -\frac{3}{8} & 1 \end{bmatrix}$$

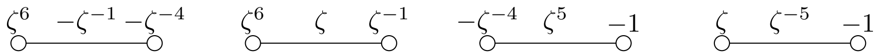
Proof. We check that the roots $\{\alpha_1, \alpha_{12}, \alpha_{1112}, \alpha_{11122}\}$ satisfy condition (5)B, while the root α_2 and α_{112} satisfy condition (5)A.
We check that the reflections send one m_{ij}-matrix to the other. □

Corollary 7.12. *By formula (4) for rank 2, we have that the central charge of the system is $-\frac{874}{7}$.*

Proposition 7.13. *Truncation and Serre relations always hold, by lemma 3.4.*

Heckenberger row 13

This case of table 1 in [Hec05] is described by four diagrams:



I II III IV

where $\zeta \in \mathcal{R}_{24}$ and with respectively simple roots:

$$\mathrm{I} : \{\alpha_1, \alpha_2\} \qquad \mathrm{II} : \{-\alpha_1, \alpha_{1112}\} \qquad \mathrm{III} : \{-\alpha_2, \alpha_{122}\} \qquad \mathrm{IV} : \{\alpha_{12}, -\alpha_{122}\}..$$

The associate Cartan matrices are:

$$a^{\mathrm{I}}_{ij} = \begin{bmatrix} 2 & -3 \\ -2 & 2 \end{bmatrix} \quad a^{\mathrm{II}}_{ij} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \quad a^{\mathrm{III}}_{ij} = \begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \quad a^{\mathrm{IV}}_{ij} = \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{122}, \alpha_{1112}, \alpha_{11122}, \alpha_{1111222}\}$ where α_{12} and α_{1112} are the only q-Cartan roots while the others are only q-truncation.

Proposition 7.14. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^{I}_{ij} = \begin{bmatrix} \frac{1}{2} & -\frac{13}{24} \\ -\frac{13}{24} & \frac{2}{3} \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{1}{2} & -\frac{23}{24} \\ -\frac{23}{24} & \frac{23}{12} \end{bmatrix}$$

$$m^{III}_{ij} = \begin{bmatrix} 1 & -\frac{19}{24} \\ -\frac{19}{24} & \frac{2}{3} \end{bmatrix} \qquad m^{IV}_{ij} = \begin{bmatrix} 1 & -\frac{5}{24} \\ -\frac{5}{24} & \frac{1}{12} \end{bmatrix}$$

Proof. We check that the roots α_{12} and α_{1112} satisfy condition (5)A, while the rest condition (5)B.
We check that the reflections send one m_{ij}-matrix to the other. □

Corollary 7.15. *By formula (4) for rank 2, we have that the central charge of the system is $-\frac{7826}{23}$.*

Proposition 7.16. *Truncation and Serre relations always hold, by lemma 3.4.*

Heckenberger row 14

This case of table 1 in [Hec05] is described by two diagrams:

I II

where $\zeta \in \mathcal{R}_5$ and with respectively simple roots:

$$\mathrm{I} : \{\alpha_1, \alpha_2\} \qquad \mathrm{II} : \{\alpha_{12}, -\alpha_2\}.$$

The associate Cartan matrices are:

$$a_{ij}^{\mathrm{I}} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \qquad a_{ij}^{\mathrm{II}} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is $\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{1112}, \alpha_{1111222}, \alpha_{11122}, \alpha_{11111222}\}$ where $\alpha_1, \alpha_{12}, \alpha_{112}$ and α_{11122} are only q-Cartan while the others are only q-truncation.

Proposition 7.17. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m_{ij}^{I} = \begin{bmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{3}{5} & 1 \end{bmatrix} \qquad m_{ij}^{II} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & 1 \end{bmatrix}.$$

Proof. We check that the roots $\alpha_1, \alpha_{12}, \alpha_{112}$ and α_{11122} satisfy condition (5)A, while the others satisfy condition (5)B.
We check that the reflections send one m_{ij}-matrix to the other. □

Corollary 7.18. *By formula (4) for rank 2, we have that the central charge of the system is* -364.

Proposition 7.19. *Truncation and Serre relations always hold, by lemma 3.4.*

Heckenberger row 17

This case of table 1 in [Hec05] is described by two diagrams:

I II

where $\zeta \in \mathcal{R}_7$ and with respectively simple roots:

$$\mathrm{I} : \{\alpha_1, \alpha_2\} \qquad \mathrm{II} : \{\alpha_{12}, -\alpha_2\}.$$

The associate Cartan matrices are:

$$a_{ij}^{\mathrm{I}} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \qquad a_{ij}^{\mathrm{II}} = \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}.$$

The set of positive roots is

$$\{\alpha_1, \alpha_2, \alpha_{12}, \alpha_{112}, \alpha_{1112}, \alpha_{11122}, \alpha_{1111222}, \alpha_{111112222}, \alpha_{111111122222}, \alpha_{11111222}, \alpha_{1111111122222}, \alpha_{11111112222}\}$$

where $\{\alpha_1, \alpha_{12}, \alpha_{112}, \alpha_{11122}, \alpha_{1111222}, \alpha_{11111222}\}$ are only q-Cartan while the others are only q-truncation.

Proposition 7.20. *The following m_{ij} matrices are realising solutions of the given braiding and its reflections:*

$$m^I_{ij} = \begin{bmatrix} \frac{6}{14} & -\frac{9}{14} \\ & \\ -\frac{9}{14} & 1 \end{bmatrix} \qquad m^{II}_{ij} = \begin{bmatrix} \frac{2}{14} & -\frac{5}{14} \\ & \\ -\frac{5}{14} & 1 \end{bmatrix}.$$

Proof. We check that the roots $\{\alpha_1, \alpha_{12}, \alpha_{112}, \alpha_{11122}, \alpha_{1111222}, \alpha_{11111222}\}$ satisfy condition (5)A, while the others satisfy condition (5)B.
We check that the reflections send one m_{ij}-matrix to the other. □

Corollary 7.21. *By formula (4) for rank 2, we have that the central charge of the system is -962.*

Proposition 7.22. *Truncation and Serre relations always hold, by lemma 3.4.*

7.2 Classification: rank 2

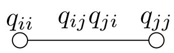
In this section we are going to prove the following

Theorem 7.23. *For all finite dimensional diagonal Nichols algebras of rank = 2, all m_{ij} matrices which are realising solutions of the given braiding are the ones constructed in sections 5, 6 or 7.1.*

In order to prove it, we are going to go through table 1 in [Hec05], see which roots are q-truncation, q-Cartan and compute for every diagram the corresponding m_{ij}. We will see that for every case, the m_{ij} match with one of the constructed in the previous sections, and that there are no other possible solutions.

To prove this result we will need the following tools:

Proposition 7.24. *We consider a diagram*



where we assume that both $\{\alpha_i, \alpha_j\}$ are q-truncation, and apply a reflection $\mathcal{R}^i$ around the root α_i

$$\mathcal{R}^i: \quad \alpha_i \longmapsto -\alpha_i$$
$$\alpha_j \longmapsto \alpha$$

arriving to a new diagram with simple roots $\{-\alpha_i,\ \alpha := \alpha_j - a_{ij}\alpha_i\}$.
We have:

1. *if β is m-truncation then*

$$m_{ij} = \frac{a_{ij}}{1-a_{ij}} - \frac{1}{a_{ij}(1-a_{\beta,-i})} + \frac{1}{a_{ij}(1-a_{ji})} \tag{10}$$

2. *if β is m-Cartan then*

$$m_{ij} = \frac{a_{ij}}{1-a_{ij}} + \frac{\left(\frac{1}{1-a_{ji}} - \frac{a_{ij}}{(1-a_{ij})a_{\beta i}}\right)}{\left(-\frac{1}{a_{\beta i}} + a_{ij}\right)} \tag{11}$$

Proof. Since $\{\alpha_i, \alpha_j\}$ are only q-truncation, thus m-truncation, we have the relations

$$m_{ii} = \frac{2}{1 - a_{ij}} \qquad m_{jj} = \frac{2}{1 - a_{ji}}.$$

1. If β is m-truncation then $m_{\beta\beta} = \frac{2}{1-a_{\beta,-i}}$. But for definition of β we have:

$$m_{\beta\beta} = m_{jj} - 2a_{ij}m_{ij} + a_{ij}^2 m_{ii}.$$

   Gathering all the information together we get:

$$\frac{2}{1 - a_{\beta,-i}} = \frac{2}{1 - a_{ji}} - 2a_{ij}m_{ij} + a_{ij}^2 \frac{2}{1 - a_{ij}}$$

   and from this the final result.

2. This case is completely analogous, with the only difference that β is m-Cartan and thus $m_{\beta\beta} = \frac{2m_{\beta,-i}}{a_{\beta i}}$ we will then have:

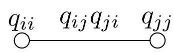
$$m_{\beta\beta} = \frac{2m_{\beta,-i}}{a_{\beta i}} = -2\frac{m_{ij}}{a_{\beta i}} + \frac{2a_{ij}(\frac{2}{1-a_{ij}})}{a_{\beta i}}$$
$$m_{\beta\beta} = \frac{2}{1 - a_{ji}} - 2a_{ij}m_{ij} + a_{ij}^2 \frac{2}{1 - a_{ij}}.$$

   The two equations together give the thesis.

□

Analogously:

Proposition 7.25. *We consider a diagram*



where we assume that $\{\alpha_i, \alpha_j\}$ are the first q-Cartan and the latter q-truncation. We apply a reflection around the q-truncation root α_j,

$$\mathcal{R}^j : \quad \alpha_j \longmapsto -\alpha_j$$
$$\alpha_i \longmapsto \beta$$

arriving to a new diagram associated to the roots: $\{\beta := \alpha_i - a_{ji}\alpha_j,\ -\alpha_j\}$. We have:

1. *if β is m-truncation then*

$$m_{ij} = \frac{a_{ij}}{1 - a_{ij}a_{ji}} \left(\frac{1}{1 - a_{\beta,-j}} - \frac{a_{ji}^2}{1 - a_{ji}} \right) \tag{12}$$

2. *if β is m-Cartan then*

$$m_{ij} = \frac{a_{ij}a_{ji}}{1 - a_{ji}} \cdot \frac{a_{ji}a_{\beta,-j} - 2}{a_{ji}a_{ij}a_{\beta,-j} - a_{\beta,-j} - a_{ij}} \tag{13}$$

Heckenberger row 2

We have $d = 1$ and then $\ell_1 = \ell_2 = \frac{\ell}{gdc(\ell,2)}$. Therefore $\ell \neq 2$ and since $a_{ij} = -1$ we have the following:
If $\ell > 4$ or $\ell = 3$ then by classification lemma 5.6 we get a unique solution, presented in section 5 Heckenberger row 2.
If $\ell = 4$ then $q_{ii} = q^2 = -1$ and the roots are both q-Cartan and q-truncation:

- If both are m-Cartan, we find a unique solution, by lemma 5.6 presented in section 5 Heckenberger row 2, in the limit case $q^2 = -1$.
- If one of the two is m-truncation, we find a unique solution, presented in section 6, Heckenberger row 3, in the limit case $q^2 = -1$. This result is a consequence of lemma 6.15.
- If both are only m-truncation we recognize the matrix $\begin{bmatrix} 1 & -\frac{p'}{2} \\ -\frac{p'}{2} & 1 \end{bmatrix}$ which is the other Weyl chamber in example 4.5.

Heckenberger row 3

We have $d = 1$ and then $\ell_1 = \ell_2 = \frac{\ell}{gdc(\ell,2)}$. Therefore $\ell \neq 2$ and since $a_{12} = -1$ we have the following:
If $\ell > 4$ or $\ell = 3$ then by classification lemma 6.15 we get a unique solution, presented in section 6 case Heckenberger row 3.
If $\ell = 4$, α_1 is both q-Cartan and q-truncation.

- If it is m-Cartan, we find again the unique solution presented in section 6 Heckenberger row 3, in the limit case $q^2 = -1$. This result is a consequence of lemma 6.15.
- If it is m-truncation we recognize again the matrix $\begin{bmatrix} 1 & -\frac{p'}{2} \\ -\frac{p'}{2} & 1 \end{bmatrix}$ which is the other Weyl chamber in example 4.5.

Heckenberger row 4

We have $d = d_2 = 2$ and then $\ell_1 = \frac{\ell}{gdc(\ell,2)}$, $\ell_2 = \frac{\ell}{gdc(\ell,4)}$. Moreover $\ell \neq 2, 4$, because $q^2 \neq \pm 1$, and since $a_{12} = -2, a_{21} = -1$ we have the following:
If $\ell > 8$ or $\ell = 5, 7$ then by classification lemma 5.6 we get a unique solution, presented in section 5 Heckenberger row 4.
If $\ell = 8$ then the long root α_2 is both q-Cartan and q-truncation, while α_1 is only q-Cartan.

- If α_2 is m-Cartan, we find again the unique solution presented in section 5, Heckenberger row 4, by lemma 5.6.
- If α_2 is m-truncation, we find the unique solution presented in section 6, Heckenberger row 5, in the limit case $q^2 = i$, by lemma 6.15.

If $\ell = 3, 6$ then the short root α_1 is both q-Cartan and q-truncation, while α_2 is only q-Cartan.

- If α_1 is m-Cartan, we find a unique solution, presented in section 5 Heckenberger row 4, again thanks to lemma 5.6.
- If α_1 is m-truncation, we find a family of solution, presented in section 7.1, Heckenberger row 6, up to rescaling. The uniqueness follows from lemma 6.15, as observed in remark 7.4.

Heckenberger row 5

We have $d = 1$ and then $\ell_1 = \frac{\ell}{gdc(\ell,2)}$. Moreover $\ell \neq 2, 4$, because $q^2 \neq \pm 1$, and since $a_{12} = -2$ we have the following:
If $\ell > 6$ or $\ell = 5$ then by classification lemma 6.15 we get a unique solution, presented in section 6 Heckenberger row 5.
If $\ell = 3, 6$ then the bosonic root α_1 is both q-Cartan and q-truncation.

- If α_1 is m-Cartan, we find again the unique solution presented in section 6 Heckenberger row 5, by lemma 6.15.
- If α_1 is m-truncation, we recognize the matrix $\begin{bmatrix} \frac{2}{3} & -2r \\ -2r & 1 \end{bmatrix}$ of remark 6.26 which is a solution only for $r = \frac{1}{3}$.

Heckenberger row 6

We have $d = 1$ and then $\ell_2 = \frac{\ell}{gdc(\ell,2)}$. Moreover $\ell \neq 2, 3, 6$, because $q^2 \neq 1, \zeta, \zeta^2$, with $\zeta \in \mathcal{R}_3$. Since $a_{12} = -1$ we have the following:
If $\ell > 6$ or $\ell = 5$ then by classification lemma 6.15 we get a unique solution, presented in section 7.1 Heckenberger row 6 (see remark 7.4).
If $\ell = 4$ then the root α_2 is both q-Cartan and q-truncation.

- If α_2 is m-Cartan, we find again the unique solution presented in section 7.1 Heckenberger row 6, by lemma 6.15.
- If α_2 is m-truncation, we recognize the matrix $\begin{bmatrix} \frac{2}{3} & -r \\ -r & 1 \end{bmatrix}$ of remark 7.2 which is a solution only for $r = \frac{1}{2}$.

Heckenberger row 7

We apply formula (10) to the reflection $\mathcal{R}^1$ and $\mathcal{R}^2$, since the simple roots α_1 and α_2 as well as the ones after reflections are only q-truncation and thus m-truncation. From the first reflection we obtain $m_{12} = -\frac{2}{3}$, while from the latter $m_{12} = -\frac{1}{2}$. Since these results don't match, it means that there is no possible formulation of the Nichols Algebra braiding in terms of the m_{ij} matrix.

Remark 7.26. We have q-truncation roots α_i, α_j, with $q_{ii} = \zeta$, $q_{jj} = \zeta^{-1}$, both third roots of unity and it is not possible to realise both of them with $m_{ii} = m_{jj} = \frac{2}{3}$. This is another way to see that this case is not realisable.

Heckenberger row 8

We apply formula (10) to the reflections $\mathcal{R}^1$ and $\mathcal{R}^2$, since the simple roots α_1 and α_2 as well as the ones after reflections are only q-truncation and thus m-truncation. From the first reflection we obtain $m_{12} = -\frac{3}{4}$, while from the latter $m_{12} = -\frac{7}{12}$. Since these results don't match, it means that there is no possible formulation of the Nichols Algebra braiding in terms of the m_{ij} matrix.

Heckenberger row 9

We apply formula (10) to the reflection $\mathcal{R}^1$ or $\mathcal{R}^2$, since the simple roots α_1 and α_2 as well as the ones after reflections are only q-truncation and thus m-truncation. The resulting m_{12} shows that this is the m_{ij} appearing in section 7.1. This is thus the only possible solution.

Heckenberger row 10

We apply formula (12) to the reflection $\mathcal{R}^2$, since the simple root α_1 is only q-Cartan and thus m-Cartan, while α_2 as well as the ones after reflections are only q-truncation and thus m-truncation. The resulting m_{12} shows that this is the m_{ij} appearing in section 7.1. This is thus the only possible solution.

Heckenberger row 11

We have $d = d_2 = 3$ and then $\ell_1 = \frac{\ell}{gdc(\ell,2)}$, $\ell_2 = \frac{\ell}{gdc(\ell,6)}$. Moreover $\ell \neq 2, 3, 4, 6$ because $q^2 \neq \pm 1$, $q^2 \notin \mathcal{R}_3$. Since $a_{12} = -3$ and $a_{21} = -1$ we have the following: If $\ell > 12$ or $\ell = 5, 7, 9, 10, 11$ then by classification lemma 5.6 we get a unique solution, presented in section 5 Heckenberger row 11.
If $\ell = 12$ then the root α_2 is both q-Cartan and q-truncation, while the root α_1 is only q-Cartan.

- If α_2 is m-Cartan, we find again the unique solution presented in section 5 Heckenberger row 11, by lemma 5.6.
- If α_2 is m-truncation, we recognize the matrix $\begin{bmatrix} 2r & -3r \\ -3r & 1 \end{bmatrix}$ of remark 5.26 which is a solution only for $r = \frac{1}{6}$.

If $\ell = 8$ then the root α_1 is both q-Cartan and q-truncation, while the root α_2 is only q-Cartan.

- If α_1 is m-Cartan, we find again the unique solution presented in section 5 Heckenberger row 11, by lemma 5.6.
- If α_1 is m-truncation, we recognize the matrix $\begin{bmatrix} \frac{1}{2} & -3r \\ -3r & 6r \end{bmatrix}$ of remark 5.27 which is a solution only for $r = \frac{1}{4}$.

Heckenberger row 12

We apply formula (12) to the reflections $\mathcal{R}^1$, since the simple roots α_1 as well as the ones after reflections are only q-truncation and thus m-truncation, while α_2 is only q-Cartan, and thus m-Cartan. The result is $m_{12} = -\frac{7}{8}$, which matches with the one of section 7.1.

Heckenberger row 13

We apply formula (10) to the reflection $\mathcal{R}^1$ or $\mathcal{R}^2$, since the simple roots α_1 and α_2 as well as the ones after reflections are only q-truncation and thus m-truncation. The resulting m_{12} shows that this is the m_{ij} appearing in section 7.1. This is thus the only possible solution.

Heckenberger row 14

We apply formula (13) to the reflections $\mathcal{R}^2$, since the simple roots α_1 as well as the ones after reflections are only q-Cartan and thus m-Cartan, while α_2 is only q-truncation, and thus m-truncation. The result is $m_{12} = -\frac{3}{5}$, which matches with the one of section 7.1.

Heckenberger row 15

We apply formula (10) to the reflections $\mathcal{R}^1$ and (11) to $\mathcal{R}^2$ since the simple roots α_1 and α_2 as well as the ones after $\mathcal{R}^1$ are only q-truncation and thus m-truncation, while the ones after $\mathcal{R}^2$ are only q-Cartan, and thus m-Cartan. From the first reflection we obtain $m_{12} = -\frac{4}{5}$, while from the latter $m_{12} = -\frac{11}{20}$. Since these results don't match, it means that there is no possible formulation of the Nichols Algebra braiding in terms of the m_{ij} matrix.

Heckenberger row 16

The root α_1 is q-Cartan so we can't start with the system of simple roots α_1, α_2 if we want to compare the results of the reflections around them. We then start with the simple roots α_{122} and $-\alpha_2$ which are only q-truncation and thus m-truncation. After reflection $\mathcal{R}^{122}$ we obtain a only q-Cartan, and thus m-Cartan, simple root. While after reflection $\mathcal{R}^2$ we obtain a only q-truncation, and thus m-truncation, simple root. We then apply (11) to $\mathcal{R}^{122}$ and (10) to $\mathcal{R}^2$ obtaining to different results. Hence there is no possible formulation of the Nichols Algebra braiding in terms of the m_{ij} matrix.

Heckenberger row 17

We apply formula (12) to the reflections $\mathcal{R}^2$, since the simple roots α_2 as well as the ones after reflections are only q-truncation and thus m-truncation, while α_1 is only q-Cartan, and thus m-Cartan. The result is $m_{12} = -\frac{5}{14}$, which matches with the one of section 7.1.

8 Rank 3

We now rise the rank by one and construct all m_{ij}-matrices which realise finite dimensional diagonal Nichols algebras of rank 3, listed in table 2 of [Hec05].

For Cartan type we will refer to the study of section 5. For super Lie type we will explicitly compute the realising solutions.
For the other cases, we will see that the m_{ij} matrices are completely fixed by the lower rank: this will imply uniqueness of the solution and make it not just a construction result but also a classification one.

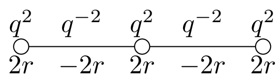
In particular for these latter cases we will proceed as follows:

- Given a q-diagram in rank 3, we will consider it as two rank 2 q-diagrams joined in the middle node. We will then associate to both sides the m_{ij}-matrices realising them, found in the rank 2 study. For these m_{ij}-matrices to be compatible, some restriction on the parameter of which they depend will possibly appear.
- We will then reflect the q-diagram on its q-truncation roots and proceed again as in the first point for the new diagram.
  We reflect until we arrive not just to an already found q-diagram, but also when the m_{ij} realisation is repeated (the m_{ij} matrix can be different also if associated to the same q-diagram).
- We will then have to make sure that all the conditions found on the parameters are compatible and acceptable, in order for the rank 3 m_{ij}-matrices to be realising solutions.

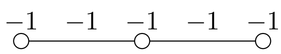
The q-diagrams and the associated realising solutions are listed in table 2 of the Appendix.

Heckenberger row 1

This case belongs to the Cartan section. In particular it corresponds to the Lie algebras A_3 and it is described by the following q-diagram with corresponding m_{ij} solution:

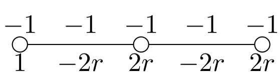


Remark 8.1. When $q^2 \in \mathcal{R}_2$ the roots are both q-Cartan and q-truncation and the q-diagram reads



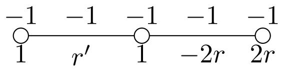
We have the following extra solutions:

– When α_1 is m-truncation and α_2, α_3 are m-Cartan we find



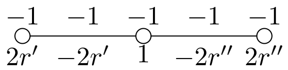
which is one chamber of the Lie superalgebra $A(2,0)$ described in Heckenberger row 4.

- When α_1, α_2 are m-truncation and α_3 is m-Cartan we find



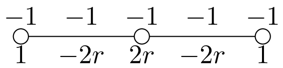
  which is a m-solution just for $r = \frac{1}{2}$ and $r' = -1$. But for these values of r, r' the roots α_1, α_2 are also m-Cartan and thus this is not a new solution.

- When α_2 is m-truncation and α_1, α_3 are m-Cartan we find



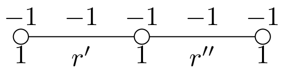
  This is a solution either for $r' = \frac{1}{2}$ for which we end up again in the previous point, or for $r' = 1 - r''$, which gives us one chamber of the Lie superalgebra $A(1,1)$ described in Heckenberger row 8.

- When α_1, α_3 are m-truncation and α_2 is m-Cartan we find



  which is another chamber of the Lie superalgebra $A(1,1)$ described in Heckenberger row 8.

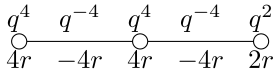
- When the roots are all m-truncation we find



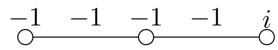
  This is a solution either for $r' = -r'' - 2$ which is again a chamber of the Lie superalgebra $A(1,1)$, or for $r' = r'' = -1$ for which the roots are also m-Cartan and thus does not give a new solution.

Heckenberger row 2

This case belongs to the Cartan section. In particular it corresponds to the Lie algebras B_3 and it is described by the following q-diagram with corresponding m_{ij} solution:

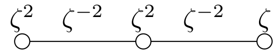


Remark 8.2. When $q^2 \in \mathcal{R}_4$ the roots α_1, α_2 are both q-Cartan and q-truncation and the q-diagram reads



For all the possible combinations of m-truncation and m-Cartan roots, no new solution is found. In some cases we find the Lie superalgebra $B(2,1)$ described in Heckenberger row 5.

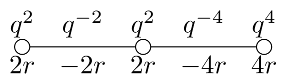
Remark 8.3. When $q^2 \in \mathcal{R}_3$ the root α_3 is both q-Cartan and q-truncation and the q-diagram reads



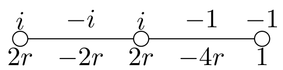
with $\zeta \in \mathcal{R}_3$. The case when it is m-truncation is a solution only for $r = \frac{1}{3}$ for which the root is also m-Cartan and thus does not give a new solution.

Heckenberger row 3

This case belongs to the Cartan section. In particular it corresponds to the Lie algebras C_3 and it is described by the following q-diagram with corresponding m_{ij} solution:



Remark 8.4. If $q^2 \in \mathcal{R}_4$, α_3 is both q-Cartan and q-truncation and the q-diagram reads



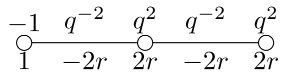
The case when it is m-truncation is a solution iff $r = \frac{1}{4}$ for which it is actually also m-Cartan. So this is not a new solution.

Heckenberger row 4

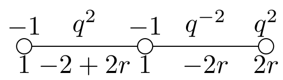
Row 4 of table 2 in [Hec05] corresponds to the Lie superalgebra $A(2,0)$.
The simple roots in the standard chamber are $\{\alpha_1 = \alpha_f, \alpha_2, \alpha_3\}$. We then have just a bosonic part $\mathfrak{g}'$. The inner products is given by:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

and therefore:



Reflecting around α_1 we find the following



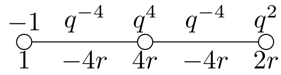
Reflecting around the second root we find a symmetric result.
The roots satisfy condition (5) $\forall r$ and therefore this m_{ij} is a realising solution.

Heckenberger row 5

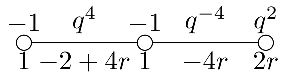
Row 5 of table 2 in [Hec05] corresponds to the Lie superalgebra $B(2,1)$.
The simple roots in the standard chamber are $\{\alpha_1 = \alpha_f, \alpha_2, \alpha_3\}$. We then have just a bosonic part $\mathfrak{g}'$. The inner products is given by:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

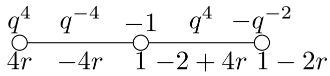
and therefore:



Reflecting around α_1 we find the following



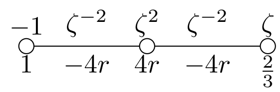
and after another reflection around the second root we find the following



The roots satisfy condition (5) $\forall r$ and therefore this m_{ij} is a realising solution.

Remark 8.5. If $q^2 \in \mathcal{R}_4$ then the root α_2 is both q-Cartan and q-truncation. This case has been already studied in details in Heckenberger row 2 remark 8.2.

Remark 8.6. If $q^2 \in \mathcal{R}_3$ then the root α_3 is both q-Cartan and q-truncation. When it is m-truncation we get:



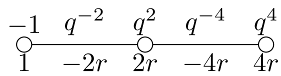
This is a solution iff $r = \frac{1}{3}$. But for this value of r, α_3 is also m-Cartan and thus this is not a new solution.

Heckenberger row 6

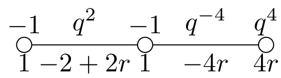
Row 6 of table 2 in [Hec05] corresponds to the Lie superalgebra $C(3)$.
The simple roots in the standard chamber are $\{\alpha_1 = \alpha_f, \alpha_2, \alpha_3\}$. We then have just a bosonic part $\mathfrak{g}'$. The inner products is given by:

$$(\alpha_i, \alpha_j) = - \begin{bmatrix} 0 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

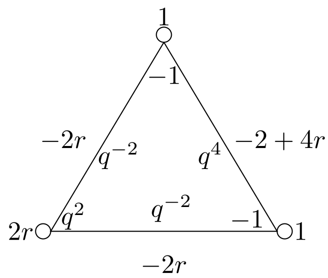
and therefore:



Reflecting around α_1 we find the following

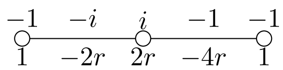


Reflecting around α_{12} we find the following



The roots satisfy condition (5) $\forall r$ and therefore this m_{ij} is a realising solution.

Remark 8.7. If $q^2 \in \mathcal{R}_4$, α_3 is both q-Cartan and q-truncation. When it is m-truncation we find



This is a solution iff $r = \frac{1}{4}$. But for this value of r, α_3 is also m-Cartan and thus this is not a new solution.

Remark 8.8. The simple roots in the standard chamber can be expressed according to [Kac77] by

$$\alpha_1 = \alpha_f = \epsilon_1 - \delta_1, \qquad \alpha_2 = \delta_1 - \delta_2 \qquad \alpha_3 = 2\delta_2.$$

Heckenberger row 7

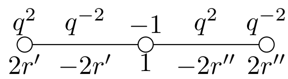
Row 7 of table 2 in [Hec05] corresponds to the Lie superalgebra $G(3)$ and it has been already explicitly treated as sporadic case of super Lie type in section 6.6.

Heckenberger row 8

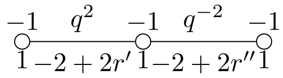
Row 8 of table 2 in [Hec05] corresponds to the Lie superalgebra $A(1,1)$.
The simple roots in the standard chamber are $\{\alpha_1,\ \alpha_2 = \alpha_f,\ \alpha_3\}$. We then have two bosonic parts $\mathfrak{g}'$ and $\mathfrak{g}''$. The inner products is given by:

$$(\alpha_i, \alpha_j) = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

and therefore:



Reflecting around α_2 we find the following



Other reflections give different m_{ij} matrices as shown in table 2. However, exception (4) of lemma 6.13, already appears. Indeed to the latter diagram is associated the following:

$$m_{ij}^C = \begin{bmatrix} 1 & -1+r' & -1+r'+r'' \\ -1+r' & 1 & -1+r'' \\ -1+r'+r'' & -1+r'' & 1 \end{bmatrix}.$$

We then have to ask $m_{13}^C = 0$, i.e. $r' + r'' = 1$. In this case these m_{ij} matrices are realising solution.

Remark 8.9. The simple roots in the standard chamber can be expressed according to [Kac77] by

$$\alpha_1 = \epsilon_1 - \epsilon_2, \quad \alpha_2 = \alpha_f = \epsilon_2 - \delta_1, \quad \alpha_3 = \delta_1 - \delta_2,$$

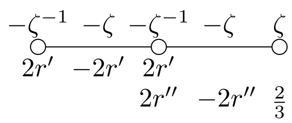
with vectors ϵ_i generating $\mathfrak{g}'$ and δ_i generating $\mathfrak{g}''$.

Heckenberger row 9-10-11

Rows 9,10,11 of table 2 in [Hec05] correspond to the Lie superalgebra $D(2,1;\alpha)$ and it has been already explicitly treated as sporadic case of super Lie type in section 6.6.

Heckenberger row 12

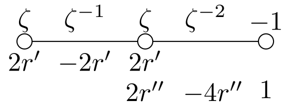
The first diagram is a composition of the diagrams of rank 2: #2 with $q = -\zeta^{-1}$ and #6 with $q = -\zeta^{-1}$, with $\zeta \in \mathcal{R}_3$.



For them to be joint in the middle circle we find $r' = r'' =: r$.
The only q-truncation root is the third. Reflecting on it we find the same diagram and as matching condition $2r = \frac{8}{3} - 2r$, i.e. $r = \frac{2}{3}$. But $q = e^{i\pi r} \in \mathcal{R}_6$. So this case is not realisable.

Heckenberger row 13

This case has two sub cases: $\zeta \in \mathcal{R}_3$ and $\zeta \in \mathcal{R}_6$ and diagram:



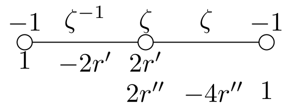
1. Suppose $\zeta \in \mathcal{R}_3$. The first diagram is a composition of the diagrams of rank 2: #2 with $q = \zeta$ and #5 with $q = \zeta$. For them to be joint in the middle circle we find $r' = r'' =: r$.
   The only q-truncation root is the third. Reflecting on it we find a diagram composition of #4 with $q = -\zeta^{-1}$ and #5 with $q = \zeta$. As matching condition we find $r = -2r + 1$, i.e. $r = \frac{1}{3}$ which is an acceptable condition.
   This case is thus realisable by the unique solution with parameter $r = \frac{1}{3}$.
2. Suppose $\zeta \in \mathcal{R}_6$. We proceed analogously, but after reflecting around the third root we find a diagram which is composition of #6 with $q = \zeta$ and #5 with $q = \zeta$. The condition now is $r = \frac{1}{6}$ which is an acceptable condition.
   This case is thus realisable by the unique solution with parameter $r = \frac{1}{6}$.

Heckenberger row 14

This case is not realisable, since one of the diagrams contains diagram #7 of rank 2 which is on turn not realisable.

Heckenberger row 15

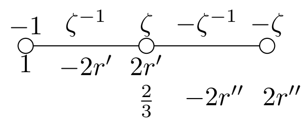
The first diagram is a composition of the diagrams of rank 2: #3 with $q = \zeta$ and #5 with $q = \zeta$, where $\zeta \in \mathcal{R}_3$.



For them to be joint in the middle circle we find $r' = r'' =: r$.
After the reflections around $\mathcal{R}^{12} \circ \mathcal{R}^1$ we find the condition $r = \frac{1}{3}$ which is acceptable and gives a unique realisable solution.

Heckenberger row 16

The first diagram is a composition of the diagrams of rank 2: #3 with $q = \zeta$ and #6 with $q = -\zeta$, where $\zeta \in \mathcal{R}_3$.



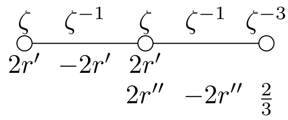
For them to be joint in the middle circle we find $r' = \frac{1}{3}$.
After reflecting on the second root we find the condition $r'' = \frac{5}{6}$.
This case is thus realisable by the unique solution with parameters $r' = \frac{1}{3}$ and $r'' = \frac{5}{6}$.

Heckenberger row 17

This case is not realisable, since one of the diagrams contains diagram #7 of rank 2 which is on turn not realisable.

Heckenberger row 18

The first diagram is a composition of the diagrams of rank 2: #2 with $q = \zeta$ and #6 with $q = \zeta$, with $\zeta \in \mathcal{R}_9$.



For them to be joint in the middle circle we find $r' = r'' =: r$.
The only q-truncation root is the third. Reflecting on it we find the same diagram and as matching condition $r = -\frac{8}{3} + 2r$, i.e. $r = \frac{8}{9}$.
This case is thus realisable by the unique solution with parameter $r = \frac{8}{9}$.

9 Rank ≥ 4

The construction of all m_{ij}-matrices, which realise finite dimensional diagonal Nichols algebras of rank ≥ 4 can be obtained directly from rank 3. Namely, for a given q-diagram one has to combine in a coherent way the m_{ij} for some overlapping subdiagrams. It is indeed enough to know rank 3 because the effect of a reflection $\mathcal{R}^k$ on a pair of roots α_i, α_j and q_{ij}, m_{ij} only depends on the rank 3 subdiagram α_i, α_j, α_k.

10 Tables: realising lattices of Nichols algebras in rank 2 and 3

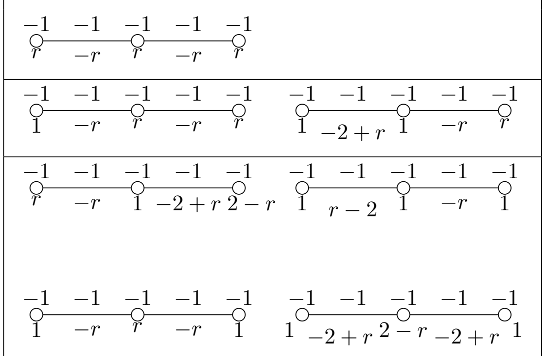
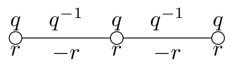
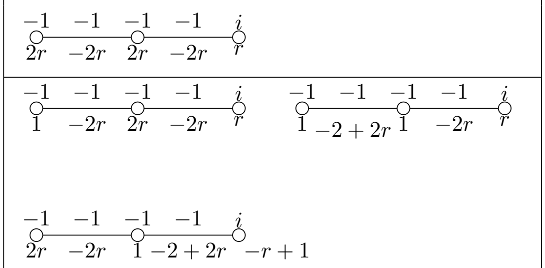
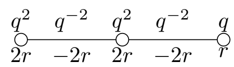
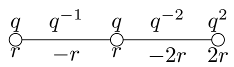
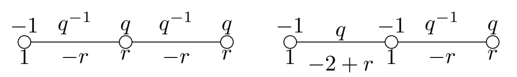
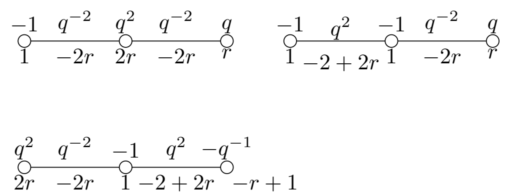
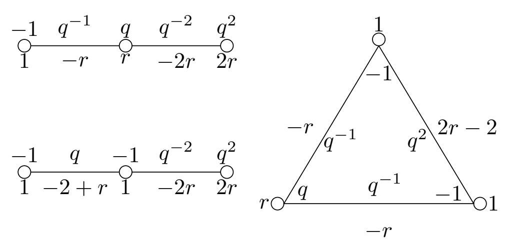
We now list from [Hec05] all finite-dimensional diagonal Nichols algebras in rank 2 and 3 in terms of their q-diagrams, and below each of them we display the corresponding realising lattice in terms of m_{ij}-diagrams, such that $q_{ij} = e^{i\pi m_{ij}}$ and the reflection compatibility 5 holds.
The numbers of the rows are Heckenberger's numbering, but sometimes we subdivide the cases, e.g. $2'$, $2''$. Note that we display the Nichols algebras associated to quantum groups as Heckenberger, in contrast to the notation used for quantum groups and used in section 5, 6, which means that there is an additional 2 factor in the q-exponent missing.

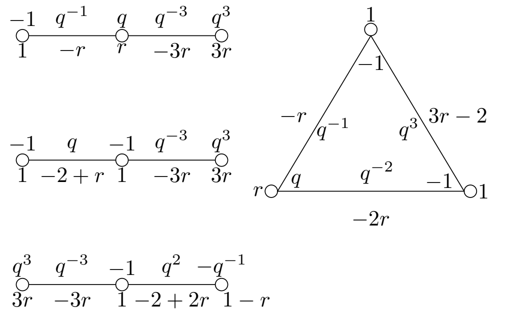
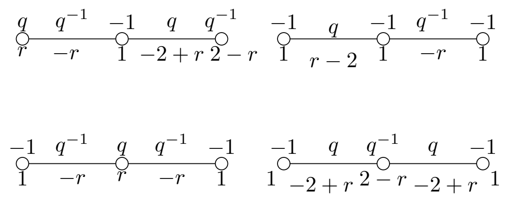
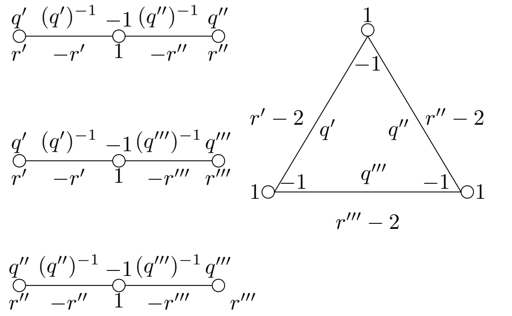
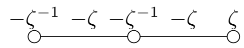
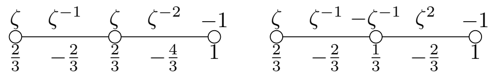
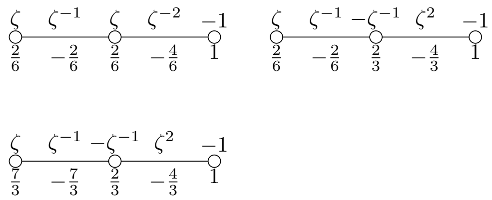
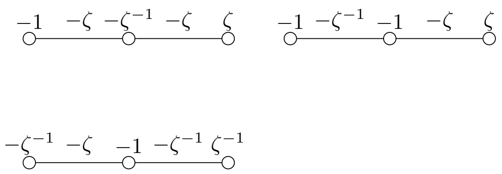
Table 1: Realisation of finite dimensional diagonal Nichols algebras of rank 2.

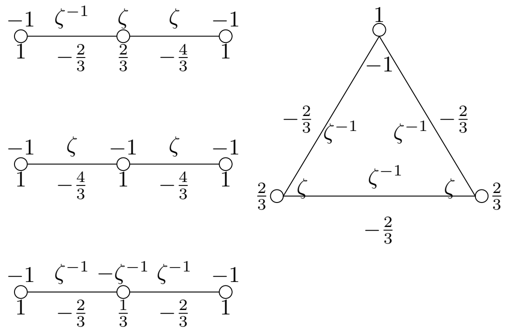
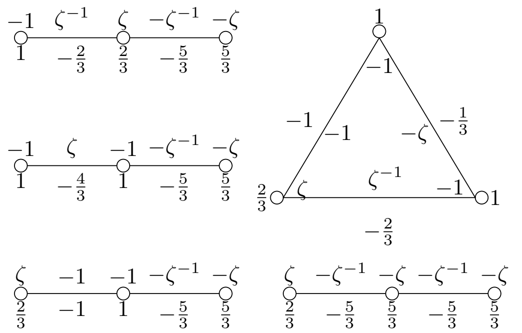
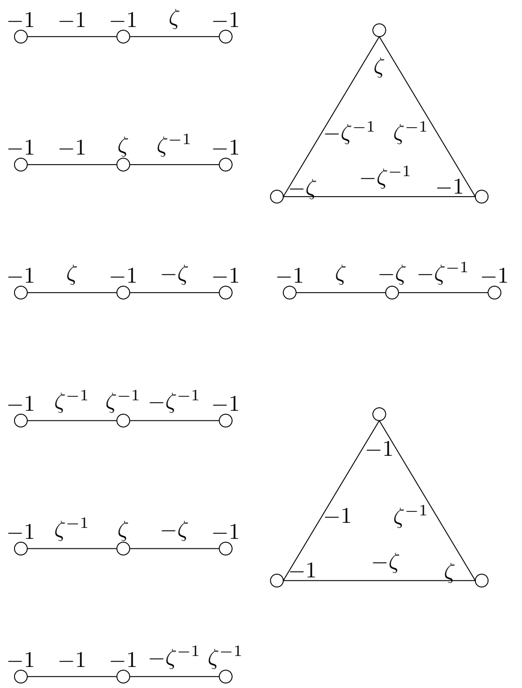
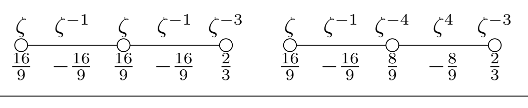
row	**Braiding**	**Conditions**	
$2'$	$\overset{-1}{\underset{r}{\circ}}\overset{-1}{\underset{-r}{\text{———}}}\overset{-1}{\underset{r}{\circ}}$ $\overset{-1}{\underset{r}{\circ}}\overset{-1}{\underset{-r}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$ $\quad$ $\overset{-1}{\underset{1}{\circ}}\overset{-1}{\underset{-2+r}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$		One solution according to A_2 (see $2''$). One solution according to $A(1,0)$ (see 3).
$2''$	$\overset{q}{\underset{r}{\circ}}\overset{q^{-1}}{\underset{-r}{\text{———}}}\overset{q}{\underset{r}{\circ}}$	$q \neq \pm 1$	Cartan, A_2
3	$\overset{q}{\underset{r}{\circ}}\overset{q^{-1}}{\underset{-r}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$ $\quad$ $\overset{-1}{\underset{1}{\circ}}\overset{q}{\underset{-2+r}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$	$q \neq \pm 1$	Super Lie, $A(1,0)$
$4'$	$\overset{i}{\underset{r}{\circ}}\overset{-1}{\underset{-2r}{\text{———}}}\overset{-1}{\underset{2r}{\circ}}$ $\overset{i}{\underset{r}{\circ}}\overset{-1}{\underset{-2r}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$ $\quad$ $\overset{i}{\underset{-r+1}{\circ}}\overset{-1}{\underset{-2+2r}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$	$i \in \mathcal{R}_4$	One solution according to B_2 (see $4'''$). One solution according to $B(1,1)$ (see 5).
$4''$	$\overset{\zeta}{\underset{r}{\circ}}\overset{\zeta}{\underset{-2r}{\text{———}}}\overset{\zeta^{-1}}{\underset{2r}{\circ}}$ $\overset{\zeta}{\underset{\frac{2}{3}}{\circ}}\overset{\zeta}{\underset{-2r}{\text{———}}}\overset{\zeta^{-1}}{\underset{2r}{\circ}}$ $\quad$ $\overset{\zeta}{\underset{\frac{2}{3}}{\circ}}\overset{\zeta}{\underset{-\frac{8}{3}+2r}{\text{———}}}\overset{\zeta^{-1}}{\underset{\frac{8}{3}-2r}{\circ}}$	$\zeta \in \mathcal{R}_3$	One solution according to B_2 (see $4'''$). One solution according to 6.
$4'''$	$\overset{q}{\underset{r}{\circ}}\overset{q^{-2}}{\underset{-2r}{\text{———}}}\overset{q^{2}}{\underset{2r}{\circ}}$	$q \neq \pm 1,\ q \notin \mathcal{R}_3, \mathcal{R}_4$	Cartan, B_2
5	$\overset{q}{\underset{r}{\circ}}\overset{q^{-2}}{\underset{-2r}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$ $\quad$ $\overset{-q^{-1}}{\underset{1-r}{\circ}}\overset{q^{2}}{\underset{-2+2r}{\text{———}}}\overset{-1}{\underset{1}{\circ}}$	$q \neq \pm 1,\ q \notin \mathcal{R}_4$	Super Lie, $B(1,1)$
6	$\overset{\zeta}{\underset{\frac{2}{3}}{\circ}}\overset{q^{-1}}{\underset{-r}{\text{———}}}\overset{q}{\underset{r}{\circ}}$ $\quad$ $\overset{\zeta}{\underset{\frac{2}{3}}{\circ}}\overset{\zeta^{-1}q}{\underset{-\frac{8}{3}+r}{\text{———}}}\overset{\zeta q^{-1}}{\underset{\frac{8}{3}-r}{\circ}}$	$\zeta \notin \mathcal{R}_3,\ q \neq 1, \zeta, \zeta^2$	
7	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\quad$ $\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in \mathcal{R}_3$	No solution
8	$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{———}}\overset{-\zeta^{2}}{\circ}$ $\quad$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$ $\quad$ $\overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ $\overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{———}}\overset{-1}{\circ}$ $\quad$ $\overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$	$\zeta \in \mathcal{R}_{12}$	No solution

9	$\underset{\frac{2}{3}}{-\zeta^2}\ \overset{\zeta}{\underset{-\frac{7}{6}}{\text{———}}}\ \underset{\frac{2}{3}}{-\zeta^2}$; $\underset{\frac{2}{3}}{-\zeta^2}\ \overset{\zeta^3}{\underset{-\frac{3}{2}}{\text{———}}}\ \underset{1}{-1}$; $\underset{\frac{1}{6}}{-\zeta^{-1}}\ \overset{-\zeta^3}{\underset{-\frac{1}{2}}{\text{———}}}\ \underset{1}{-1}$	$\zeta \in \mathcal{R}_{12}$	
10	$\underset{\frac{5}{9}}{-\zeta}\ \overset{\zeta^{-2}}{\underset{-\frac{10}{9}}{\text{———}}}\ \underset{\frac{2}{3}}{\zeta^3}$; $\underset{\frac{2}{3}}{\zeta^3}\ \overset{\zeta^{-1}}{\underset{-\frac{14}{9}}{\text{———}}}\ \underset{1}{-1}$; $\underset{\frac{1}{9}}{-\zeta^2}\ \overset{\zeta}{\underset{-\frac{4}{9}}{\text{———}}}\ \underset{1}{-1}$	$\zeta \in \mathcal{R}_9$	
11	$\underset{r}{q}\ \overset{q^{-3}}{\underset{-3r}{\text{———}}}\ \underset{3r}{q^3}$	$q \notin \mathcal{R}_3,\ q \neq \pm 1$	Cartan, G_2
12	$\underset{\frac{1}{2}}{\zeta^2}\ \overset{\zeta}{\underset{-\frac{7}{4}}{\text{———}}}\ \underset{\frac{7}{4}}{\zeta^{-1}}$; $\underset{\frac{1}{2}}{\zeta^2}\ \overset{-\zeta^{-1}}{\underset{-\frac{5}{4}}{\text{———}}}\ \underset{1}{-1}$; $\underset{\frac{1}{4}}{\zeta}\ \overset{-\zeta}{\underset{-\frac{3}{4}}{\text{———}}}\ \underset{1}{-1}$	$\zeta \in \mathcal{R}_8$	
13	$\underset{\frac{1}{2}}{\zeta^6}\ \overset{-\zeta^{-1}}{\underset{-\frac{13}{12}}{\text{———}}}\ \underset{\frac{2}{3}}{-\zeta^{-4}}$; $\underset{\frac{1}{2}}{\zeta^6}\ \overset{\zeta}{\underset{-\frac{23}{12}}{\text{———}}}\ \underset{\frac{23}{12}}{\zeta^{-1}}$; $\underset{\frac{2}{3}}{-\zeta^{-4}}\ \overset{\zeta^5}{\underset{-\frac{19}{12}}{\text{———}}}\ \underset{1}{-1}$; $\underset{\frac{1}{12}}{\zeta}\ \overset{\zeta^{-5}}{\underset{-\frac{5}{12}}{\text{———}}}\ \underset{1}{-1}$	$\zeta \in \mathcal{R}_{24}$	
14	$\underset{\frac{2}{5}}{\zeta}\ \overset{\zeta^2}{\underset{-\frac{6}{5}}{\text{———}}}\ \underset{1}{-1}$; $\underset{\frac{1}{5}}{-\zeta^{-2}}\ \overset{\zeta^{-2}}{\underset{-\frac{4}{5}}{\text{———}}}\ \underset{1}{-1}$	$\zeta \in \mathcal{R}_5$	
15	$\zeta\ \overset{\zeta^{-3}}{\text{———}}\ -1$; $-\zeta\ \overset{-\zeta^{-3}}{\text{———}}\ -1$; $-\zeta^{-2}\ \overset{\zeta^3}{\text{———}}\ -1$; $-\zeta^{-2}\ \overset{-\zeta^{-3}}{\text{———}}\ -1$	$\zeta \in \mathcal{R}_{20}$	No solution
16	$-\zeta\ \overset{-\zeta^{-3}}{\text{———}}\ \zeta^5$; $\zeta^3\ \overset{-\zeta^4}{\text{———}}\ -\zeta^{-4}$; $\zeta^5\ \overset{-\zeta^{-2}}{\text{———}}\ -1$; $\zeta^3\ \overset{-\zeta^2}{\text{———}}\ -1$	$\zeta \in \mathcal{R}_{15}$	No solution
17	$\underset{\frac{6}{14}}{-\zeta}\ \overset{-\zeta^{-3}}{\underset{-\frac{9}{7}}{\text{———}}}\ \underset{1}{-1}$; $\underset{\frac{2}{14}}{-\zeta^{-2}}\ \overset{-\zeta^3}{\underset{-\frac{5}{7}}{\text{———}}}\ \underset{1}{-1}$	$\zeta \in \mathcal{R}_7$	

Table 2: Realisation of finite dimensional diagonal Nichols algebras of rank 3.

<table>
<tr><th>row</th><th>Braiding</th><th>Conditions</th><th></th></tr>
<tr><td>1′</td><td>
</td><td></td><td>One solution according to A_3 (see 1″).
One solution according to $A(2,0)$ (see 4).
One solution according to $A(1,1)$ (see 8).</td></tr>
<tr><td>1″</td><td>
</td><td>$q \neq \pm 1$</td><td>Cartan, A_3</td></tr>
<tr><td>2′</td><td>
</td><td>$i \in \mathcal{R}_4$</td><td>One solution according to B_3 (see 2″).
One solution according to $B(2,1)$ (see 5).</td></tr>
<tr><td>2″</td><td>
</td><td>$q \neq \pm 1$, $q \notin \mathcal{R}_4$</td><td>Cartan, B_3</td></tr>
<tr><td>3</td><td>
</td><td>$q \neq \pm 1$</td><td>Cartan, C_3</td></tr>
<tr><td>4</td><td>
</td><td>$q \neq \pm 1$</td><td>Super Lie, A(2,0)</td></tr>
<tr><td>5</td><td>
</td><td>$q \neq \pm 1$, $q \notin \mathcal{R}_4$</td><td>Super Lie, B(2,1)</td></tr>
<tr><td>6</td><td>
</td><td>$q \neq \pm 1$</td><td>Super Lie, C(3)</td></tr>
</table>

7		$q \neq \pm 1$, $q \notin \mathcal{R}_3$	Super Lie, $G(3)$
8		$q \neq \pm 1$	Super Lie, $A(1,1)$
9, 10, 11		$q', q'', q''' \neq 1$, $q' \cdot q'' \cdot q''' = 1$	Super Lie, D(2, 1;α), $r' + r'' + r''' = 2$
12		$\zeta \in \mathcal{R}_3$	No solution.
$13'$		$\zeta \in \mathcal{R}_3$	$r = \frac{1}{3}$
$13''$		$\zeta \in \mathcal{R}_6$	$r = \frac{1}{6}$
14		$\zeta \in \mathcal{R}_3$	No solution.

15		$\zeta \in \mathcal{R}_3$	$r = \frac{1}{3}$
16		$\zeta \in \mathcal{R}_3$	$r = \frac{5}{6}$
17		$\zeta \in \mathcal{R}_3$	No solution.
18		$\zeta \in \mathcal{R}_9$	$r = \frac{8}{9}$

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