

Symmetries of quaternionic Kähler manifolds with S^1 -symmetry

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Abstract

We study symmetry properties of quaternionic Kähler manifolds obtained by the HK/QK correspondence. To any Lie algebra \mathfrak{g} of infinitesimal automorphisms of the initial hyper-Kähler data we associate a central extension of \mathfrak{g} , acting by infinitesimal automorphisms of the resulting quaternionic Kähler manifold. More specifically, we study the metrics obtained by the one-loop deformation of the c -map construction, proving that the Lie algebra of infinitesimal automorphisms of the initial projective special Kähler manifold gives rise to a Lie algebra of Killing fields of the corresponding one-loop deformed c -map space. As an application, we show that this construction increases the cohomogeneity of the automorphism groups by at most one. In particular, if the initial manifold is homogeneous then the one-loop deformed metric is of cohomogeneity at most one. As an example, we consider the one-loop deformation of the symmetric quaternionic Kähler metric on $SU(n, 2)/S(U(n) \times U(2))$, which we prove is of cohomogeneity exactly one. This family generalizes the so-called universal hypermultiplet ($n = 1$), for which we determine the full isometry group.

Keywords: quaternionic Kähler manifolds, HK/QK correspondence, c -map, one-loop deformation, isometry groups, cohomogeneity one

MSC classification: 53C26.

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1 Introduction

In the late 1980's, physicists working on string theory discovered a new method of constructing quaternionic Kähler manifolds of negative scalar curvature, which was dubbed the (supergravity) c -map. A local description of the resulting metrics was first given by Ferrara and Sabharwal [FS90], and they were described in differential-geometric terms by Hitchin [Hit09].

The input for the c -map (recalled in more detail in Section 2) is a projective special Kähler (PSK) manifold, which we will denote by \bar{M} . It arises as the Kähler quotient by a distinguished circle action on an associated manifold M , which carries a conical affine special Kähler (CASK) structure. As shown in [ACM13, ACDM15], the supergravity c -map can be described as a two-step process. The first, called the rigid c -map, consists of passing from the corresponding CASK manifold M to its cotangent bundle $N := T^*M$, and defining a natural (pseudo-)hyper-Kähler structure on the latter. The circle action on M lifts to a so-called rotating (circle) symmetry of the hyper-Kähler structure on N , and plays a crucial role in the second step, the HK/QK correspondence. This correspondence produces a quaternionic Kähler manifold \bar{N} of the same dimension out of any hyper-Kähler manifold with rotating symmetry, completing the following diagram:

$$\begin{array}{ccc}
 M & \xrightarrow{\text{rigid } c} & N \\
 \mathbb{C}^* \downarrow & & \downarrow \text{HK/QK} \\
 \bar{M} & \xrightarrow{\text{supergravity } c} & \bar{N}
 \end{array} \tag{1}$$

A one-parameter deformation of the c -map was discovered by Robles-Llana, Saueressig and Vandoren [RSV06]. It is known as the one-loop deformed c -map because of its physical origin as a perturbative quantum correction to the Ferrara-Sabharwal metric. It is obtained by modifying the Ferrara-Sabharwal metric by introducing a real parameter c . When $c = 0$, one recovers the Ferrara-Sabharwal metric. Curiously, c has a very simple meaning on the hyper-Kähler side: It parametrizes the additive freedom in choosing a Hamiltonian for the lifted circle action. This key observation means that, by working on the hyper-Kähler side of the correspondence, we can treat the c -map and its one-loop correction simultaneously.

The HK/QK correspondence has been a subject of interest for both physicists and mathematicians (see, for instance, [APP11, ACM13, Hit13, Hit14, ACDM15, MS15]). It was first discovered by Haydys in the Riemannian setting [Hay08] and subsequently generalized to allow for indefinite metrics [ACM13, ACDM15]. It is this latter form which can be used for the c -map construction. Nevertheless, the output of the c -map and its one-loop deformation is a positive-definite quaternionic Kähler manifold, which is moreover complete if the initial PSK manifold was complete and $c \geq 0$ [CDS17]. Summarizing and extending the previous remarks, the HK/QK correspondence assigns a quaternionic Kähler manifold with a nowhere-vanishing Killing field to any (possibly indefinite) hyper-Kähler manifold with rotating circle symmetry. Conversely, any quaternionic Kähler manifold with a Killing field locally arises in this fashion.

Another perspective on the HK/QK correspondence was advanced by Macia and Swann [MS15]: They showed that it can be recovered as a variation on the twist construction,

a general duality between manifolds endowed with torus action, introduced by Swann in earlier work [Swa10]. One advantage of the twist construction, which we review in Section 2 together with some facts about the relevant aspects of special Kähler geometry, is that it provides a precise prescription for carrying tensor fields on N which are invariant under the natural circle action over to \bar{N} . Invariant vector fields, in particular, arise naturally in the context of the c -map. Infinitesimal automorphisms of the CASK structure on M respect its distinguished circle action, and we show that they can be lifted to vector fields on N which respect the rigid c -map structure on N .

With this in mind, we investigate the HK/QK correspondence in the presence of additional symmetries preserving the rigid c -map structure in Section 3. We prove that, under the HK/QK correspondence as well as under its one-parameter deformation, these give rise to isometries of the resulting quaternionic Kähler manifold. Applying this to the c -map, we show that the (identity component of the) automorphism group of the PSK manifold \bar{M} injects into the isometry group of its (one-loop deformed) c -map image \bar{N} . As a corollary, we find that if $\text{Aut } \bar{M}$ acts with cohomogeneity k , then the isometry group of \bar{N} acts with cohomogeneity at most $k + 1$ for every $c \neq 0$, and at most k for $c = 0$ (the latter was previously known).

In Section 4, we apply our results to the study of a series of examples, namely the quaternionic Kähler manifolds obtained by applying the c -map to complex hyperbolic space $\mathbb{C}H^n = SU(n, 1)/U(n)$. The undeformed c -map yields the non-compact Wolf spaces $SU(n+1, 2)/S(U(n+1) \times U(2))$, which is a homogeneous PSK manifold. These can be characterized as the only complete, simply connected quaternionic Kähler manifolds with negative scalar curvature which are also Kähler. In the physics literature, the case $n = 0$ is known as the “universal hypermultiplet”. The one-loop deformed c -map yields a one-parameter deformation of the symmetric metric, and we investigate its isometry group. It follows from our results that it acts with cohomogeneity at most one and by using results from [CST20], where curvature formulas for the HK/QK correspondence are derived, we prove that the isometry group acts by cohomogeneity exactly one in all dimensions. This shows that our results are sharp in this family of examples and proves, in particular, that the one-loop deformed Ferrara–Sabharwal metric is not homogeneous, even locally. In the case of the universal hypermultiplet, we determine the full isometry group.

Acknowledgements

D.T. thanks Markus Röser for useful discussions. This work was supported by the German Science Foundation (DFG) under the Research Training Group 1670, and under Germany’s Excellence Strategy – EXC 2121 “Quantum Universe” – 390833306.

2 Preliminaries

In this section, we set the stage for the rest of this paper, briefly recalling the most important background material. In particular, the terminology and notation introduced here will be used throughout Section 3, where we prove our main results.

2.1 Twist construction and the HK/QK correspondence

As mentioned in the introduction, Macia and Swann [MS15] studied the HK/QK correspondence from the point of view of Swann’s twist construction. This is a general procedure which, given a manifold with torus action, produces a new (“twist”) manifold with twisted torus action, uniquely determined by some choices of *twist data* on the original manifold. Moreover, geometric structures which are invariant under the torus action on the original manifold unambiguously correspond to invariant (twisted) structures on the twist manifold. The twisting construction is a duality in the sense that the original manifold can be recovered by appropriately twisting the twist manifold.

For example, given a (pseudo-)hyper-Kähler manifold with circle action—as produced, for instance, by the rigid c -map—the twist construction can be applied. Macia and Swann investigated the geometry of the resulting twist manifold and showed that, after deforming the original metric and choosing appropriate twist data, the twist of the deformed metric defines a quaternionic Kähler metric on the resulting manifold. Their formulas reproduce the explicit expressions for the HK/QK correspondence found in [ACDM15], casting it as a special case of the twist construction. In this section, we introduce the basic language of the twist construction and, following Macia and Swann, outline its application to the HK/QK correspondence.

2.1.1 The twist construction

For our purposes, it suffices to describe the twist construction for *circle* actions; the interested reader is referred to [Swa10] for twists of higher-dimensional torus actions.

Consider a manifold N equipped with an S^1 -action generated by a vector field $Z \in \mathfrak{X}(N)$. To apply the twist construction, we consider an auxiliary manifold, namely the total space P of an S^1 -principal bundle $\pi_N : P \rightarrow N$. We furthermore equip this principal bundle with a principal S^1 -connection $\eta \in \Omega^1(P)$ with curvature ω . We wish to lift the vector field Z to P such that it preserves the connection η and commutes with the principal circle action. We will then obtain the *twist* of M by dividing out the action generated by this lifted vector field. Regarding the existence of such a lift, we have:

Proposition 2.1 ([Swa10]). *A lift as above exists if and only if $\iota_Z \omega$ is an exact one-form, i.e. $\iota_Z \omega = -df_Z$ for some Hamiltonian function $f_Z \in C^\infty(N)$. \square*

Concretely, the lift is given by $Z_P = \tilde{Z} + \pi_N^* f_Z X_P$, where \tilde{Z} denotes the horizontal lift with respect to η , and X_P generates the principal circle action. Note in particular that, in order to uniquely specify Z_P , one needs to fix a choice of Hamiltonian function. Thus, in order to lift the circle action we must specify the following *twist data*:

- (i) A vector field $Z \in \mathfrak{X}(N)$ generating a circle action on N .
- (ii) A closed, integral two-form ω (i.e. with integral periods) with respect to which Z is Hamiltonian. Up to isomorphism, this form determines a unique principal circle bundle $\pi_N : P \rightarrow N$, which admits a connection η whose curvature is ω .
- (iii) A choice of Hamiltonian function f_Z for Z with respect to ω , which we assume is nowhere-vanishing (equivalently, the lifted vector field Z_P is transverse to $\mathcal{H} := \ker \eta$).

For given twist data, we can define the twist of N as follows:

Definition 2.2. For (Z, ω, f_Z) as above, we call the quotient space $\bar{N} := P/\langle Z_P \rangle$ the *twist* of N with respect to the twist data (Z, ω, f_Z) .

If both Z and Z_P generate free and proper actions, the twist is a smooth manifold.

We now have the double fibration

$$N \xleftarrow{\pi_N} P \xrightarrow{\pi_{\bar{N}}} \bar{N}$$

The connection η on P determines a horizontal distribution \mathcal{H} , which is pointwise identified with the tangent spaces to N and to \bar{N} , by means of the respective projections. This induces identifications between tensor fields in the following manner:

Definition 2.3. We say that vector fields $X \in \mathfrak{X}(N)$ and $X' \in \mathfrak{X}(\bar{N})$ are \mathcal{H} -related, and write $X' \sim_{\mathcal{H}} X$, if their horizontal lifts to P (with respect to η) are equal. We say that differential forms α on N and α' on \bar{N} are \mathcal{H} -related if $(\pi_N^* \alpha) \circ P_{\mathcal{H}} = (\pi_{\bar{N}}^* \alpha') \circ P_{\mathcal{H}}$, where $P_{\mathcal{H}} : TP \rightarrow \mathcal{H}$, $U \mapsto U - \eta(U)X_P$, is the projection onto \mathcal{H} . The definition of \mathcal{H} -relatedness is now extended to arbitrary (p, q) -tensors by demanding compatibility with tensor products.

Remark 2.4.

- (i) Twisting is automatically compatible with contractions of tensor fields.
- (ii) Given some tensor field on N , the existence of a (well-defined) \mathcal{H} -related tensor field on \bar{N} implies that its lift is invariant under both X_P and Z_P . The expression $\tilde{Z} = Z_P - (\pi_N^* f_Z)X_P$ (where \tilde{Z} denotes the horizontal lift) then forces the original tensor field on N to be Z -invariant. Thus, the twist construction only allows one to carry Z -invariant tensors on N over to \bar{N} .
- (iii) The vector field $Z_{\bar{N}} := (\pi_{\bar{N}})_*(X_P) \in \mathfrak{X}(\bar{N})$ plays a role dual to Z in the following sense. If one performs a twist on \bar{N} with respect to the data $(Z_{\bar{N}}, (f_Z^{-1}\omega)', (f_Z^{-1})')$, where primes denote \mathcal{H} -relatedness, one recovers the original manifold N .

Z -invariant tensor fields on N and, dually, $Z_{\bar{N}}$ -invariant tensor fields on \bar{N} , are determined by their respective lifts to P . Therefore, one can uniquely determine \mathcal{H} -related tensor fields in terms of the twist data. For instance, the twist construction uniquely assigns to $X \in \mathfrak{X}(N)$ the vector field $X' = (\pi_{\bar{N}})_*(\tilde{X})$, where \tilde{X} denotes the horizontal lift of X . One can also check how natural operations, such as the Lie bracket, behave under twisting:

Lemma 2.5 ([Swa10]).

- (i) If h is a Z -invariant symmetric $(0, 2)$ -tensor on N , then the \mathcal{H} -related symmetric $(0, 2)$ -tensor h' on \bar{N} is determined by

$$\pi_N^* h' = \pi_N^* h - 2\eta \vee \pi_N^*(f_Z^{-1} \iota_Z h) + \eta^2 \pi_N^*(f_Z^{-2} h(Z, Z))$$

- (ii) If $\mathfrak{X}(N) \ni X, Y \sim_{\mathcal{H}} X', Y'$, i.e. $\tilde{X} = \widehat{X}'$ and similarly for Y (where the tilde and hat denote the respective horizontal lifts), we have

$$[X', Y'] \sim_{\mathcal{H}} [X, Y] + f_Z^{-1} \omega(X, Y) Z \quad \square$$

Since we will be interested in producing Killing fields on twist manifolds, we show explicitly how the Lie derivative of a symmetric $(0, 2)$ -tensor behaves under the twist:

Lemma 2.6. *Let $V \sim_{\mathcal{H}} V'$ be vector fields and $h \sim_{\mathcal{H}} h'$ be symmetric $(0, 2)$ -tensors. Then $L_{V'} h' \sim_{\mathcal{H}} L_V h - 2f_Z^{-1}(\iota_V \omega) \vee (\iota_Z h)$.*

Proof. The Leibniz rule tells us that for two vector fields X, Y on M , we have

$$\pi_N^*((L_V h)(X, Y)) = L_{\tilde{V}}(\pi_N^* h(\tilde{X}, \tilde{Y})) - \pi_N^* h(\widetilde{[V, X]}, \tilde{Y}) - \pi_N^* h(\tilde{X}, \widetilde{[V, Y]})$$

From lemma 2.5, we have

$$\widetilde{[V, Y]} = \widetilde{[V', Y']} - \pi_N^*(f_Z^{-1} F(V, Y)) \cdot \tilde{Z}$$

and therefore we find

$$\begin{aligned} \pi_N^*((L_V h)(X, Y)) &= (L_{\tilde{V}}(\pi_N^* h'))(\widehat{X'}, \widehat{Y'}) \\ &\quad + \pi_N^*(f_Z^{-1} \omega(V, X)) \pi_N^* h'(\tilde{Z}, \tilde{Y}) + \pi_N^*(f_Z^{-1} \omega(V, Y)) \pi_N^* h'(\tilde{X}, \tilde{Z}) \\ &= \pi_N^*((L_{V'} h')(X', Y')) + \pi_N^*(2f_Z^{-1}(\iota_V \omega) \vee (\iota_Z h))(\tilde{X}, \tilde{Y}) \end{aligned}$$

Bringing the second term to the left-hand side yields the claimed result. \square

2.1.2 Elementary deformations

Given a hyper-Kähler manifold with circle action, it is natural to expect that—after choosing suitable twist data—its twist manifold also admits interesting (quaternionic) structures. In this setting, Macia and Swann [MS15] investigated when the twist construction yields a quaternionic Kähler manifold. They discovered that, in order to obtain a quaternionic Kähler metric on the twist manifold, the original hyper-Kähler structure must first be modified. To study these modifications, they introduced the notion of *elementary deformations* of hyper-Kähler metrics, which we now discuss.

We start from a hyper-Kähler manifold (N, g, I_i) with Kähler forms ω_i ($i = 1, 2, 3$). For notational convenience, we will also define $\omega_0 = g$ and $I_0 = \text{id}$. To apply the twist construction, we need a circle action generated by a vector field Z . We will require that the action generated by Z is compatible with the quaternionic Hermitian structure in the sense that Z preserves g , as well as the subbundle of $\text{End}(TM)$ spanned by I_1, I_2 and I_3 . We define one-forms $\alpha_i(\cdot) = \omega_i(Z, \cdot)$, $i = 0, 1, 2, 3$, as well as the symmetric $(0, 2)$ -tensor $g_\alpha = \sum_{i=0}^3 \alpha_i^2$. When Z is nowhere-vanishing, this is proportional to the restriction of the metric g to the quaternionic span $\mathbb{H}Z = \langle Z, I_1 Z, I_2 Z, I_3 Z \rangle$ of Z .

Definition 2.7. Given a pseudo-hyper-Kähler manifold (M, g, I_i) , an *elementary deformation* of g is a (pseudo-)Riemannian metric on (M, I_i) of the form $g_{\mathbb{H}} = a \cdot g + b \cdot g_{\alpha}$ for nowhere-vanishing $a, b \in C^{\infty}(M)$.

The action of L_Z on $\langle I_1, I_2, I_3 \rangle \subset \text{End } TM$ is linear, because $\nabla I_i = 0$. Furthermore, the Leibniz rule implies that $L_Z \in \mathfrak{so}(3)$. Therefore, the action of Z is either trivial, or conjugate to

$$L_Z I_1 = 0 \qquad L_Z I_2 = I_3 \qquad L_Z I_3 = -I_2$$

after suitable normalization of Z . In the latter case, we say that Z generates a *rotating circle symmetry* of (M, g, I_i) . In the following, we will consider only such rotating symmetries. In fact, we will make one further assumption, namely that the action generated by Z is ω_1 -Hamiltonian: $\iota_Z \omega_1 = -df_Z$ for some moment map $f_Z \in C^{\infty}(M)$. Such a hyper-Kähler manifold, equipped with ω_1 -Hamiltonian rotating circle symmetry, is precisely the input of the HK/QK correspondence.

2.1.3 The HK/QK correspondence

It was shown by Haydys [Hay08] that, given a rotating circle symmetry on a hyper-Kähler manifold, one can construct a bundle over it which admits the structure of a hyper-Kähler cone. The hyper-Kähler cone then gives rise to a quaternionic Kähler manifold by the inverse of the Swann bundle construction [Swa91]. His construction is known as the HK/QK correspondence.

These results (which Haydys obtained under the assumption that all metrics involved are positive definite), were extended to metrics of arbitrary signature in [ACM13], with control over the signature of the resulting quaternionic Kähler metrics. In particular, necessary and sufficient conditions for the resulting metric to be positive definite were given, and shown to include not only definite but also indefinite initial data. In this way, the authors of [ACM13, ACDM15] were able to obtain the supergravity c -map as a special case of the HK/QK correspondence with indefinite initial data. As part of this work, they gave a simple expression (cf. [ACDM15, Theorem 2]) for the quaternionic Kähler metric constructed in the HK/QK correspondence, which is precisely the metric given in the theorem below.

Macia and Swann [MS15] then showed that the above-mentioned metric can be obtained from a combination of an elementary deformation and a twist, applied to the initial hyper-Kähler manifold. Furthermore, they characterized those elementary deformations and twist data that lead to a quaternionic Kähler metric. In summary, one obtains the following result:

Theorem 2.8 ([Hay08]; [ACM13]; [ACDM15]; [MS15]). *Let (M, g, I_i) be a (possibly indefinite) hyper-Kähler manifold equipped with rotating circle symmetry Z , such that ω_1 is integral and the lift Z_P generates a free and proper action. Denoting the Hamiltonian function of Z with respect to ω_1 by f_Z , one can associate a (possibly indefinite) quaternionic Kähler manifold $(\bar{N}, g_{\mathbb{Q}})$ as follows. Let $g_{\mathbb{H}}$ be an elementary deformation of g with respect to Z , and consider the twist manifold $(\bar{N}, g_{\mathbb{Q}})$, where $g_{\mathbb{Q}} \sim_{\mathcal{H}} g_{\mathbb{H}}$ with respect to the twist data*

$(Z, \omega_{\mathbb{H}}, f_{\mathbb{H}})$. Then $(\bar{N}, g_{\mathbb{Q}})$ is quaternionic Kähler if and only if

$$\begin{aligned}\omega_{\mathbb{H}} &= k(\omega_1 + d\alpha_0) & f_{\mathbb{H}} &= k(f_Z + g(Z, Z)) \\ g_{\mathbb{H}} &= \frac{B}{f_Z}g + \frac{B}{f_Z^2}g_{\alpha}\end{aligned}$$

where $B, k \in \mathbb{R} \setminus \{0\}$ are constants.

Remark 2.9.

- (i) B and k simply scale the metric and curvature of P , respectively. In the following, we will set them to 1. Since the Hamiltonian function f_Z is unique only up to an additive constant, there is an implicit dependence of the construction on this additional parameter. In the case of the c -map, this parameter governs the so-called one-loop deformation of the quaternionic Kähler metric.
- (ii) For the convenience of the reader, we provide a dictionary to translate between the notation used for the various objects living on the hyper-Kähler manifold in this article, in the work by Macia and Swann [MS15], and in the paper [ACDM15] by Alekseevsky, Cortés, Dyckmanns and Mohaupt:

This article	Macia & Swann	ACDM
Z	X	$-\frac{1}{2}Z$
f_Z	$-(\mu - c)$	$-\frac{1}{2}f$
η	θ	η
α_0	α_0	$-\frac{1}{2}\beta$
$\omega_{\mathbb{H}}$	F	$\omega_1 - \frac{1}{2}d\beta$
$f_{\mathbb{H}}$	$-a$	$\frac{1}{2}f_1$
$g_{\mathbb{H}}$	$-g_{\mathbb{H}}$	$2g'$

2.2 Special Kähler geometry and the c -map

2.2.1 Special Kähler structures

In this section, we review the c -map and the various manifolds involved in this construction.

Definition 2.10. An *affine special Kähler manifold* (M, g, ω, ∇) is a pseudo-Kähler manifold (M, g, ω) endowed with a flat, torsion-free connection ∇ such that $\nabla\omega = 0$ and $d_{\nabla}J = 0$, where J is the complex structure on M .

For the c -map, the relevant class of affine special Kähler manifolds is the following:

Definition 2.11. An affine special Kähler manifold (M, g, ω, ∇) is called *conical* (or a *CASK manifold*) if it admits a vector field ξ , called the *Euler field*, which satisfies $\nabla\xi = \nabla^{\text{LC}}\xi = \text{id}_{TM}$, where ∇^{LC} denotes the Levi-Civita connection of g . We will furthermore assume throughout this paper that $\{\xi, J\xi\}$ generates a principal \mathbb{C}^* -action, and that $\mathcal{D} := \text{span}\{\xi, J\xi\}$ is negative definite with respect to g , while the orthogonal complement \mathcal{D}^{\perp} is positive definite.

The Euler field is homothetic and holomorphic, while $J\xi$ is holomorphic and Killing (cf. Proposition 3 of [CM09]).

Lemma 2.12. *The Euler field is always ∇ -affine, while $J\xi$ is ∇ -affine if and only if $\nabla = \nabla^{LC}$.*

Proof. Since ∇ is flat we may use local, parallel vector fields X, Y and compute

$$(L_\xi \nabla)_X Y = L_\xi(\nabla_X Y) - \nabla_{[\xi, X]} Y - \nabla_X([\xi, Y]) = \nabla_X Y = 0$$

where we used $[\xi, Y] = -\nabla_Y \xi = -Y$. Analogously, we obtain

$$\begin{aligned} (L_{J\xi} \nabla)_X Y &= -\nabla_X([J\xi, Y]) = \nabla_X(\nabla_Y(J\xi)) = \nabla_X((\nabla_Y J)\xi + J\nabla_Y \xi) \\ &= \nabla_X((\nabla_\xi J)Y + J\nabla_Y \xi) = \nabla_X(JY) = (\nabla_X J)Y \end{aligned}$$

where we used $\nabla_\xi J = (L_\xi + \nabla \xi)J = 0$ and $\nabla \xi = \text{id}_{TM}$ in the penultimate step. This expression vanishes if and only if ∇ is the Levi-Civita connection. \square

The following fact is well-known (see, for instance, [CHM12, MS15]).

Lemma 2.13. *The function $f = \frac{1}{2}g(\xi, \xi)$ defines both a Hamiltonian function for $J\xi$ and a Kähler potential for g_M .*

Proof. $\nabla \xi = \text{id}_{TM}$ implies $df = g(\xi, \cdot) = -\omega(J\xi, \cdot)$. Furthermore $dd^c f = -dJ^*df = d(\omega(J\xi, J\cdot)) = d(\omega(\xi, \cdot)) = \omega$, where we used $\nabla \omega = 0$ and $\nabla \xi = \text{id}_{TM}$. \square

Since ξ generates a homothetic $\mathbb{R}_{>0}$ -action, dividing out the \mathbb{C}^* -action on a CASK manifold amounts to fixing a level set for the moment map $\frac{1}{2}g(\xi, \xi)$ —we will take the level where $g(\xi, \xi) = -1$ for notational convenience—and subsequently taking the quotient by the circle action generated by $-J\xi$.

Definition 2.14. A *projective special Kähler* (PSK) manifold \bar{M} is the Kähler quotient $M // S^1$ of a CASK manifold M by the S^1 -action generated by the Hamiltonian Killing field $-J\xi$.

The natural notion of symmetry in this setting is the following:

Definition 2.15.

- (i) An automorphism of a CASK manifold $(M, g_M, \omega_M, \nabla, \xi)$ is a diffeomorphism of M which preserves g_M, ω_M, ∇ and ξ .
- (ii) Let $\bar{M} = M // S^1$ be a PSK manifold. An automorphism of \bar{M} is a diffeomorphism of \bar{M} induced by an automorphism of the CASK manifold M .

Lemma 2.16. *If $X \in \mathfrak{X}(M)$ generates a one-parameter group φ_t of automorphisms of a CASK manifold $(M, g_M, \omega_M, \nabla, \xi)$, then X is ω_M -Hamiltonian.*

Proof. Defining $\alpha := \frac{1}{2}d^c(g(\xi, \xi)) \in \Omega^1(M)$, Lemma 2.13 asserts $\omega_M = d\alpha$. Then $\rho := \alpha(X)$ is a candidate Hamiltonian function. We compute its differential:

$$d\rho = d\iota_X \alpha = (L_X - \iota_X d)\alpha = L_X \alpha - \iota_X \omega_M$$

Thus, it suffices to check that $L_X \alpha = 0$. Since $\alpha = -\frac{1}{2}J^*(dg(\xi, \xi))$, this follows from the fact that X preserves J, g and ξ . \square

Definition 2.17.

- (i) An infinitesimal automorphism of a CASK manifold $(M, g_M, \omega_M, \nabla, \xi)$ is a vector field $X \in \mathfrak{aut}(M)$ such that its local flow preserves the CASK data on M . The Lie algebra of such vector fields is denoted by $\mathfrak{aut}(M)$.
- (ii) An infinitesimal automorphism of a PSK manifold is a vector field \bar{X} induced by an infinitesimal automorphism of the corresponding CASK manifold (note that the latter always project, since they commute with ξ and $J\xi$). The corresponding Lie algebra is denoted by $\mathfrak{aut}(\bar{M})$.

Proposition 2.18. *Consider the natural map $\varphi : \mathfrak{aut}(M) \rightarrow \mathfrak{aut}(\bar{M})$ induced by the projection $\pi : M \rightarrow \bar{M}$.*

- (i) *If $\nabla \neq \nabla^{LC}$, then φ is a isomorphism, with inverse $\bar{X} \mapsto X^{\mathcal{H}} - \pi^* h_{\bar{X}} J\xi$, where $X^{\mathcal{H}}$ denotes the horizontal (that is, perpendicular to \mathcal{D}) lift of \bar{X} and $h_{\bar{X}}$ is a uniquely determined Hamiltonian for \bar{X} with respect to the Kähler form $\omega_{\bar{M}}$ on \bar{M} .*
- (ii) *If $\nabla = \nabla^{LC}$, then φ induces an isomorphism between $\mathfrak{aut}(M)/(\mathbb{R} \cdot J\xi)$ and $\mathfrak{aut}(\bar{M})$, with inverse $\bar{X} \mapsto X^{\mathcal{H}} - \pi^* h_{\bar{X}} J\xi$, where $h_{\bar{X}}$ is now any Hamiltonian for \bar{X} with respect to $\omega_{\bar{M}}$*

Proof. Since the \mathbb{C}^* -action generated by $\{\xi, J\xi\}$ is homothetic, the distribution \mathcal{D}^\perp of vectors orthogonal to the vertical distribution $\mathcal{D} = \text{span}\{\xi, J\xi\}$ defines a principal connection in the \mathbb{C}^* -bundle $M \rightarrow \bar{M}$.

Now consider an infinitesimal automorphism $X \in \mathfrak{aut}(M)$, which projects to $\bar{X} \in \mathfrak{aut}(\bar{M})$. We can decompose $X = X^{\mathcal{H}} + X^{\mathcal{V}}$ into its horizontal and vertical components, where $X^{\mathcal{H}}$ is the horizontal lift of \bar{X} . Since X preserves both g_M and ξ , we have $0 = L_X(g_M(\xi, \xi)) = 2g_M(\nabla_X^{LC}\xi, \xi) = 2g_M(X, \xi)$, i.e. X is orthogonal to ξ .

Thus, we may write $X^{\mathcal{V}} = -f_X J\xi$ for some function $f_X \in C^\infty(M)$. Since $X^{\mathcal{H}}$ is \mathbb{C}^* -invariant, we obtain that $0 = [X, \xi] = [X^{\mathcal{V}}, \xi] = \xi(f_X)J\xi$ and similarly for $J\xi$. This shows that f_X is constant on the fibers of the projection, hence of the form $\pi^* h_{\bar{X}}$ for some $h_{\bar{X}} \in C^\infty(\bar{M})$. We now show that $h_{\bar{X}}$ is uniquely determined if $\nabla \neq \nabla^{LC}$, and up to additive constant in case $\nabla = \nabla^{LC}$. Indeed, let X and X' be two infinitesimal CASK automorphisms projecting to \bar{X} . Then $X' - X = \pi^*(h_{\bar{X}} - h'_{\bar{X}})J\xi$ must be both Killing and ∇ -affine. The first condition implies that $h_{\bar{X}} - h'_{\bar{X}}$ is constant. In case $\nabla \neq \nabla^{LC}$, the latter then implies that it vanishes (by Lemma 2.12).

We claim that $h_{\bar{X}}$ is a Hamiltonian of \bar{X} with respect to the Kähler form $\omega_{\bar{M}}$ of \bar{M} . The principal connection \mathcal{D}^\perp induces a principal S^1 -connection on the bundle $S = \{g(\xi, \xi) = -1\} \subset M$ over \bar{M} , which has $-J\xi$ as its fundamental vector field. We denote its connection one-form by $\theta = g(J\xi, \cdot)$. Differentiating this equation, we obtain $d\theta = \omega_M|_S = \pi_S^* \omega_{\bar{M}}$, where $\pi_S : S \rightarrow \bar{M}$ is the projection map. Since X is an infinitesimal CASK automorphism, $L_X \theta = 0$, which means that

$$0 = d\iota_X \theta + \iota_X d\theta = \pi^* dh_{\bar{X}} + (\pi^* \omega_M)(X, \cdot) = \pi^*(dh_{\bar{X}} + \omega_{\bar{M}}(\bar{X}, \cdot))$$

This finishes the proof. \square

An extrinsic perspective on special Kähler geometry is developed in [ACD02]. It is proven that any simply connected affine special Kähler manifold M of complex dimension $n + 1$ admits a (holomorphic, non-degenerate, Lagrangian) immersion into $T^*\mathbb{C}^{n+1}$ such that the standard special Kähler structure on $T^*\mathbb{C}^{n+1}$ induces the given affine special Kähler structure on M . In fact, such an immersion is given by the graph of an exact one-form. This implies that the special Kähler structure is determined by a holomorphic function on an open subset of \mathbb{C}^{n+1} satisfying a certain non-degeneracy condition. Thus, any simply connected special Kähler manifold may be regarded as a domain $U \subset \mathbb{C}^{n+1}$ endowed with a holomorphic function F which determines the special Kähler structure completely. F is known as the holomorphic prepotential of U . An analogous theorem holds for CASK manifolds, requiring that the domain and the holomorphic function are conical in the appropriate sense, and by projectivizing one obtains an identification of the corresponding PSK manifold \bar{M} with an open set in \mathbb{CP}^n , inducing the PSK structure.

More generally, any affine special Kähler manifold M can be covered by open sets U_α , called special Kähler domains, on which the affine special Kähler structure arises in this fashion. By picking suitable global coordinates $\{z_I\}_{I=0,1,\dots,n}$ on \mathbb{C}^{n+1} and possibly shrinking the open sets U_α , we may assume that (the image of) each U_α is contained in $\{z_0 \neq 0\} \subset \mathbb{C}^{n+1}$. From now on, we will only consider special Kähler domains of this type.

In case M is a CASK manifold, projectivizing CASK domains yields local identifications of the corresponding PSK manifold \bar{M} with subsets of $\{z_0 \neq 0\} \subset \mathbb{CP}^n$. To each such PSK domain \bar{U}_α we may canonically associate a Kähler potential, defined analogously to the standard potential for the Fubini–Study metric on \mathbb{CP}^n . Recall from lemma 2.13 that $\frac{1}{2}g(\xi, \xi)$ defines a global Kähler potential on M . Identifying the CASK and PSK domains with their images in \mathbb{C}^{n+1} and \mathbb{CP}^n , the function

$$\mathcal{K}_\alpha := \log \left(\frac{\frac{1}{2}g(\xi, \xi)}{|z_0|^2} \right)$$

now defines a Kähler potential on \bar{U}_α .

Remark 2.19. We revisit the case $\nabla = \nabla^{\text{LC}}$ from the previous lemma. Then, the short exact sequence

$$0 \longrightarrow \mathbb{R} \cdot J\xi \longrightarrow \mathfrak{aut}(M) \longrightarrow \mathfrak{aut}(\bar{M}) \longrightarrow 0$$

of Lie algebras splits. To see this, let us first assume that M is simply connected. Then, in light of the above discussion, we can (globally) realize it as a complex Lagrangian submanifold of $T^*\mathbb{C}^{n+1}$ (where $n = \dim_{\mathbb{C}} \bar{M}$). From the condition $\nabla = \nabla^{\text{LC}}$, it follows that it is contained in a linear subspace, and that there exists a global, holomorphic prepotential of the form $F = \sum a_{ij} z^i z^j$, where the real matrix $(N_{ij}) := 2\text{Im}(a_{ij})$ is of signature $(n, 1)$. In fact, the condition $\nabla = \nabla^{\text{LC}}$ is equivalent to the vanishing of the third derivatives of F [Fre99]. This implies that $\mathfrak{aut}(M) = \mathfrak{u}(n, 1)$ and $\mathfrak{aut}(\bar{M}) = \mathfrak{su}(n, 1)$, with $\mathbb{R} \cdot J\xi = \mathfrak{u}(1)$ the center of $\mathfrak{u}(n, 1)$, so the sequence splits. In case $\pi_1(M) = \Gamma$ is non-trivial, we can write $M = \tilde{M}/\Gamma$, where \tilde{M} is the universal covering, which is also a CASK manifold. Then we have $\mathbb{R} \cdot J\xi = \mathfrak{u}(1) \subset \mathfrak{aut}(M) = \mathfrak{aut}(\tilde{M})^\Gamma \subset \mathfrak{aut}(\tilde{M}) = \mathfrak{u}(n, 1) = \mathfrak{u}(1) \oplus \mathfrak{su}(n, 1)$, which implies that $\mathfrak{aut}(M) = \mathfrak{u}(1) \oplus \mathfrak{h}$, where $\mathfrak{h} \subset \mathfrak{su}(n, 1)$.

2.2.2 The rigid and supergravity c -maps

As mentioned in the introduction, the (supergravity) c -map and its one-loop deformation construct a complete quaternionic Kähler manifold \bar{N} of negative scalar curvature out of any complete PSK manifold \bar{M} . We first describe it under the assumption that \bar{M} is a PSK domain [CDS17]. Then, we may regard \bar{M} as a subset of $\mathbb{C}P^n$ endowed with a canonical Kähler potential \mathcal{K} , and the corresponding CASK manifold M as a domain in \mathbb{C}^{n+1} equipped with a holomorphic function F .

As a smooth manifold, \bar{N} is simply the direct product $\bar{M} \times \mathbb{R}^{2n+4}$, but its Riemannian structure given by the one-loop deformed Ferrara–Sabharwal metric g_{FS}^c is not that of a product. To write this metric down explicitly, we use global coordinates. On $\bar{M} \subset \mathbb{C}P^n$ we use homogeneous coordinates $(z^0 : z^1 : \dots : z^n)$, which under the standard identification $\{z^0 \neq 0\} \cong \mathbb{C}^n$ correspond to coordinates $(X^0 \equiv 1, X^1, \dots, X^n)$ on \mathbb{C}^n . Though $X^0 \equiv 1$, it will nevertheless appear in the following expressions to simplify notation. We further have coordinates $(\rho, \tilde{\phi}, \zeta^I, \tilde{\zeta}_I)$ on $\mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \cong \mathbb{R}^{2n+4}$. With respect to these, and denoting the Kähler metric on \bar{M} by $g_{\bar{M}}$, g_{FS}^c , $c \geq 0$, is given by

$$g_{\text{FS}}^c = \frac{\rho + c}{\rho} g_{\bar{M}} + g_G^c \quad (2)$$

where

$$\begin{aligned} g_G^c := & \frac{1}{4\rho^2} \frac{\rho + 2c}{\rho + c} d\rho^2 + \frac{1}{4\rho^2} \frac{\rho + c}{\rho + 2c} \left(d\tilde{\phi} + \sum_{I=0}^n (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + cd^c\mathcal{K} \right)^2 \\ & + \frac{1}{2\rho} \sum_{I,J=0}^n \left(\mathcal{J}_{IJ} d\zeta^I d\zeta^J + \mathcal{J}^{IJ} (d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K) (d\tilde{\zeta}_J + \mathcal{R}_{JL} d\zeta^L) \right) \\ & + \frac{2c}{\rho^2} e^{\mathcal{K}} \left| \sum_{I=0}^n (X^I d\tilde{\zeta}_I + F_I d\zeta^I) \right|^2 \end{aligned} \quad (3)$$

In this expression, (F_I) , (\mathcal{J}_{IJ}) and (\mathcal{R}_{IJ}) are determined by the derivatives of the holomorphic prepotential F , and constant on each copy of \mathbb{R}^{2n+4} . Their precise definitions can be found in [CDS17], but will not be relevant in the following. We note that, for any two $c_1, c_2 > 0$, the metrics $g_{\text{FS}}^{c_1}$ and $g_{\text{FS}}^{c_2}$ are isometric [CDS17].

It was explained in [CDS17] how to define the c -map for arbitrary PSK manifolds. In the case $c = 0$ the construction is locally identical to the c -map for special Kähler domains, with the only global difference being that \bar{N} may fail to be globally trivial as a bundle over \bar{M} . This corresponds to the fact that, after covering \bar{M} by PSK domains, transition functions must be used to patch the different domains together.

For $c > 0$, however, this patching does not immediately preserve the metric g_{FS}^c : One must first divide out the isometric \mathbb{Z} -action which sends $\tilde{\phi} \mapsto \tilde{\phi} + 2\pi c$, thereby making the coordinate $\tilde{\phi}$ periodic. Thus, in order to define the one-loop deformed c -map on arbitrary PSK manifolds, one locally works with bundles with fiber $\mathbb{R}_{>0} \times S^1 \times \mathbb{R}^{2n+2}$ instead. Then, the metric g_{FS}^c is invariant under the transition functions, and therefore patches together to a global quaternionic Kähler metric. Note that its coordinate expression does not change.

The resulting manifold \bar{N} is best thought of as a bundle of Lie groups over the PSK manifold \bar{M} . If \bar{M} is a PSK domain, we may endow each fiber with a group structure by casting

$\mathbb{R}^{2n+4} \cong \mathbb{R}_{>0} \times \mathbb{R}^{2n+3}$ as the Iwasawa subgroup of $SU(n+2, 1)$, which we will denote by $G(n+2)$. This is a one-dimensional, solvable extension of the Heisenberg group of dimension $2n+3$, which is parametrized by the coordinates $(\tilde{\phi}, \tilde{\zeta}_I, \zeta^I)$ in the above notation. With respect to these coordinates, the group multiplication takes the following form:

$$(e^\lambda, \alpha, \tilde{v}_I, v^I) \cdot (\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I) = \left(e^\lambda \rho, e^\lambda \tilde{\phi} + \alpha + e^{\lambda/2} \left(\sum_I \tilde{v}_I \zeta^I - v^I \tilde{\zeta}_I \right), \tilde{v}_I + e^{\lambda/2} \tilde{\zeta}_I, v^I + e^{\lambda/2} \zeta^I \right) \quad (4)$$

In the case of PSK domains (or if $c = 0$), this action endows \bar{N} with the structure of a principal $G(n+2)$ -bundle. In the general case (for $c > 0$), when the $\tilde{\phi}$ -coordinate is periodic, one no longer has an action of all of $G(n+2)$ on the fibers, since the one-dimensional extension fails to respect the quotient. However, shifts in $\tilde{\phi}$ are central in Heis_{2n+3} , hence its action descends to \bar{N} , and we may take the appropriate cyclic quotient to obtain a free action. Regarding the compatibility of the Ferrara–Sabharwal metric and its one-loop deformation with these actions, we have the following basic result.

Lemma 2.20. *Consider a c -map space \bar{N} equipped with the (one-loop deformed) Ferrara–Sabharwal metric g_{FS}^c . Then Heis_{2n+3} acts by isometries for every $c \geq 0$, and all of $G(n+2)$ acts by isometries in case $c = 0$.*

Proof. Let $U \subset \bar{M}$ be a PSK domain, so that \bar{N} restricts to a trivial bundle over U . Using the notation introduced above, the metric g_{FS}^c then takes the form given by equations (2) and (3), and the action of $G(n+2)$ is given by (4). In case \bar{M} was not globally a PSK domain and $c > 0$, only the subgroup Heis_{2n+3} has a well-defined action on \bar{N} . Since the group acts purely in the fiber directions, it is enough to check that its elements leave g_G^c invariant and preserve the function $\frac{\rho+c}{\rho}$ to verify that they act by isometries. That this is the case for elements of the form $(0, \alpha, \tilde{v}_I, v^I) \in \text{Heis}_{2n+3} \subset G(n+2)$ follows easily from an explicit computation.

The remaining elements, which are of the form $(e^\lambda, 0, 0, 0) \in G(n+2)$, only act on \bar{N} when either $c = 0$ or \bar{M} is globally a PSK domain (and $c > 0$). In the former case, they act by isometries. In the latter, we observe that for $\lambda \neq 0$, these elements preserve neither the function $\frac{\rho+c}{\rho}$ nor g_G^c , and therefore do not induce isometries. \square

Corollary 2.21. *Let \bar{M} be a complete, simply connected PSK manifold of (real) dimension $2n$, and $\Gamma \subset \text{Heis}_{2n+3}$ a cocompact lattice. Then the (universal covering of the) one-loop deformed c -map space $(\bar{N}, g_{\text{FS}}^c)$, $c \geq 0$, associated to \bar{M} admits a smooth quotient \bar{N}/Γ , which is a complete quaternionic Kähler manifold with fundamental group Γ .*

Proof. If either $c = 0$ or \bar{M} is a PSK domain, \bar{N} is itself simply connected, and carries a free and isometric action of Heis_{2n+3} . In case \bar{M} is not a PSK domain and $c > 0$, this is still true for its universal covering (which we also denote by \bar{N} for notational convenience). The quotient by Γ is smooth, complete, and has fundamental group Γ by construction. \square

This direct description gives a satisfactory (global) definition of the c -map, but makes it difficult to study the properties of the c -map. For instance, there is no obvious way to compute the isometry group of $(\bar{N}, g_{\text{FS}}^c)$ other than by direct inspection of the coordinate

expressions. An alternative, locally equivalent, formulation is summarized by (1), which we now explain in more detail.

The cotangent bundle of an affine special Kähler manifold carries a (pseudo-)hyper-Kähler structure which admits a rather simple description. Let $(M, g_M, \omega_M, \nabla)$ be a special Kähler manifold. The special Kähler connection induces a splitting

$$T(T^*M) \cong T^{\mathcal{H}}N \oplus T^{\mathcal{V}}N \cong \pi^*(TM) \oplus \pi^*(T^*M) \quad (5)$$

With respect to this splitting, we define

$$g = \begin{pmatrix} g_M & 0 \\ 0 & g_M^{-1} \end{pmatrix} \quad I_1 = \begin{pmatrix} J_M & 0 \\ 0 & J_M^* \end{pmatrix} \quad I_2 = \begin{pmatrix} 0 & -\omega_M^{-1} \\ \omega_M & 0 \end{pmatrix} \quad I_3 = I_1 I_2 \quad (6)$$

where we omitted pullbacks to simplify the notation. A priori, these tensor fields (only) endow $N := T^*M$ with an almost hyper-Hermitian structure. However, (N, g, ω_a) is in fact hyper-Kähler if (and only if) $(M, g_M, \omega_M, \nabla)$ is special Kähler (see, for instance, [ACD02]). In case M is a CASK manifold, the map which sends M to N , equipped with the above hyper-Kähler structure, is called the *rigid c-map*. It is also well-known that, in this case, N carries a natural rotating circle symmetry (cf. [ACM13]):

Lemma 2.22. *Let $(M, g_M, \omega_M, \nabla, \xi)$ be a CASK manifold, and (N, g, I_i) , $i = 1, 2, 3$, its cotangent bundle, equipped with the rigid c-map hyper-Kähler structure. Then the vector field $Z = -\tilde{J}\xi$ on N , where the tilde denotes the ∇ -horizontal lift, generates a rotating circle symmetry. In other words, $L_Z g = L_Z \omega_1 = 0$, $L_Z \omega_2 = \omega_3$ and $L_Z \omega_3 = -\omega_2$. \square*

Since Z is horizontal, $\omega_1(Z, \cdot) = \pi^* \omega_M(Z, \cdot)$. By Lemma 2.13, the rotating circle symmetry is Hamiltonian with Hamiltonian function $-\frac{1}{2}\pi^*(g_M(\xi, \xi)) = -\frac{1}{2}g(Z, Z)$. As explained in the previous section, one may now apply the HK/QK correspondence to obtain the Ferrara–Sabharwal metric [ACDM15, MS15].¹ Though $-\frac{1}{2}g(Z, Z)$ is a natural choice of Hamiltonian function for Z with respect to ω_1 , it is not unique. We may choose to work with the function $f_Z = -\frac{1}{2}g(Z, Z) - \frac{1}{2}c$ for any $c \in \mathbb{R}$ instead (the normalization of c is conventional). This induces a change in the twist data used in the HK/QK correspondence, since $f_H = f_Z + g(Z, Z)$ is shifted as well. Consequently, there is a one-parameter freedom in the HK/QK correspondence (cf. Remark 2.9), which precisely reproduces the one-loop deformation of the Ferrara–Sabharwal metric [MS15].

There are two main advantages to the HK/QK correspondence over the direct description of the c -map. Firstly, since the one-loop deformation of the c -map corresponds to a simple shift in the Hamiltonian function f_Z on the hyper-Kähler side of the correspondence, it is much simpler to treat the undeformed c -map simultaneously with its one-loop deformation. Secondly, the twist construction makes it possible to compute tensor fields on the c -map image \tilde{N} in terms of their counterparts on the hyper-Kähler manifold N . We will exploit both of these in the following.

¹Due to global issues when dividing out the lifted circle action on the S^1 -bundle over N , the HK/QK correspondence may produce a manifold which is not globally diffeomorphic to the manifold \tilde{N} obtained by the direct c -map construction given above. Nevertheless, the approaches are locally equivalent (as can be seen by picking a local transversal slice for the action generated by Z_P), and therefore we may use the HK/QK correspondence to study tensor fields on \tilde{N} .

3 Symmetry properties of the HK/QK correspondence and the c -map

We now investigate the behavior of the HK/QK correspondence and the c -map in the presence of symmetries. We will show that under the supergravity c -map as well as under its one-loop deformation, the identity (connected) component of the automorphism group of the initial projective special Kähler manifold is a subgroup of the isometry group of the resulting quaternionic Kähler manifold. In the undeformed case, this result was already known (see e.g. [CDJL17, App. A]), but our usage of the HK/QK correspondence ensures that our results remain valid also for the one-loop corrected supergravity c -map.

In addition to isometries of the quaternionic Kähler manifold coming from the automorphisms of the PSK manifold, there is a solvable group $G(n+2)$ of dimension $2n+4$ acting on the quaternionic Kähler manifold, with orbits transverse to the orbits of the automorphism group of the PSK manifold (see, for example, [CHM12]). In the undeformed case, it acts isometrically. In the deformed case, however, only its nilradical, a codimension one Heisenberg subgroup, acts by isometries (see Theorem 3.16).

3.1 Automorphisms under the HK/QK correspondence

As explained in Section 2, the input for the HK/QK correspondence is a (connected) hyper-Kähler manifold (N, g, I_i) ($i = 1, 2, 3$) equipped with a so-called rotating symmetry, i.e. a circle action which is isometric and ω_1 -Hamiltonian—with Hamiltonian function f_Z —and whose generating vector field Z satisfies $L_Z\omega_2 = \omega_3$, $L_Z\omega_3 = -\omega_2$.

We now want to consider this set-up in the presence of additional symmetries. Since we are interested in the connected component of the identity of the appropriate symmetry groups, we may work infinitesimally.

Definition 3.1. By an (*infinitesimal*) *symmetry* of a hyper-Kähler manifold with rotating symmetry, we understand a vector field $Y \in \mathfrak{X}(N)$ which is Killing and triholomorphic as well as ω_1 -Hamiltonian, and preserves the Hamiltonian function f_Z . The space of such vector fields will be denoted by $\mathbf{aut}(N, f_Z)$.

We note that this implies that $[Y, Z] = 0$, since $Z = -\omega_1^{-1}(df_Z)$, where we interpret ω_1 as an isomorphism $TM \rightarrow T^*M$.

Our aim is to produce a Killing field Y_Q , associated with Y , on the quaternionic Kähler manifold \bar{N} , which results from applying the HK/QK correspondence to N with the above data. The first step is to apply the elementary deformation, chosen according to Theorem 2.8, such that the deformed metric $g_H = \frac{1}{f_Z}g + \frac{1}{f_Z^2}g_\alpha$ is \mathcal{H} -related to a quaternionic Kähler metric g_Q on the twist manifold \bar{N} . The relevant twist data are (Z, ω_H, f_H) , defined as in Theorem 2.8 (we set $k = 1$, cf. Remark 2.9).

Lemma 3.2. *The vector field Y is ω_H -Hamiltonian, and preserves the elementary deformation g_H of the metric. The corresponding Hamiltonian function f_Y (unique up to an additive constant) is Z -invariant.*

Proof. We assumed that Y is ω_1 -Hamiltonian; call its Hamiltonian function φ . Then we have

$$\iota_Y \omega_H = -d\varphi + \iota_Y d\alpha_0 = -d(\varphi + g(Z, Y))$$

where we used Cartan's formula and the fact that Y preserves both g and Z .

It follows directly from our assumptions $L_Y g = 0$, $L_Y Z = 0$, $L_Y f_Z = 0$ and $L_Y \omega_i = 0$ ($i = 1, 2, 3$) that Y also preserves $g_H = \frac{1}{f_Z} g + \frac{1}{f_Z^2} g_\alpha$.

To check Z -invariance of $f_Y := \varphi + g(Z, Y)$, we note $L_Z(g(Z, Y)) = 0$ because Z is Killing and commutes with Y . Thus, it suffices to calculate $L_Z \varphi = \iota_Z d\varphi = -\omega_1(Y, Z) = -df_Z(Y) = 0$. \square

Since Y is Z -invariant, we can now use the twist construction to define an \mathcal{H} -related vector field Y' on the (quaternionic Kähler) twist manifold \bar{N} . Since we want to produce an infinitesimal isometry of the quaternionic Kähler metric g_Q , we use Lemma 2.6 to see that Y' will be Killing with respect to g_Q precisely if we have

$$L_Y g_H - 2f_H^{-1}(\iota_Y \omega_H) \vee (\iota_Z g_H) = 0 \quad (7)$$

Since $L_Y g_H = 0$ and the second term does not generally vanish, we will modify Y (in a Z -invariant way!) to ensure its twist will be a Killing field. A natural choice is to add a term proportional to the distinguished vector field Z . The following shows that this is indeed the right approach:

Lemma 3.3. *There exists a Z -invariant function $\psi \in C^\infty(N)$ such that the twist of $Y_H := Y + \psi Z$ is Killing.*

Proof. We write $Y_H = Y + \psi Z$ for some arbitrary $\psi \in C^\infty(N)$, and attempt to solve Equation (7). Since both Y and Z preserve g_H , we find $L_{Y_H} g_H = L_{\psi Z} g_H = 2(d\psi)\iota_Z g_H$. Since Y and Z are both ω_H -Hamiltonian, we have:

$$L_{Y_H} g_H - 2f_H^{-1}(\iota_{Y_H} \omega_H) \vee (\iota_Z g_H) = \left(2(d\psi) - 2f_H^{-1}(-df_Y - \psi df_H)\right) \vee \iota_Z g_H$$

and thus it suffices, in order for this expression to vanish, to solve the differential equation

$$d\psi = -\frac{1}{f_H} df_Y - \frac{\psi}{f_H} df_H$$

This equation is solved by $\psi = -\frac{f_Y}{f_H}$, which therefore satisfies our requirements. The function ψ is Z -invariant because both f_Y and f_H are. \square

Thus, the twist of the vector field $Y_H = Y - \frac{f_Y}{f_H} Z$ yields a Killing field $Y'_H =: Y_Q$ of g_Q . Recall that $Z_{\bar{N}}$ is the Killing field on \bar{N} which plays a role analogous to Z under the HK/QK correspondence.

Lemma 3.4. *The above-constructed Killing field Y_Q on \bar{N} commutes with $Z_{\bar{N}}$.*

Proof. Since $Y_Q \sim_{\mathcal{H}} Y_H$, its horizontal lift to P is X_P -invariant—otherwise Y_H would not be well-defined. The claim now follows from compatibility of the Lie bracket with the pushforward. \square

Denoting the space of Killing fields on \bar{N} that commute with $Z_{\bar{N}}$ by $\mathfrak{aut}(\bar{N}, Z_{\bar{N}})$, we have now proven:

Proposition 3.5. *Given any $Y \in \mathfrak{aut}(N, f_Z)$, the vector field $Y_Q = (Y - \frac{f_Y}{f_H}Z)'$ is an element of $\mathfrak{aut}(\bar{N}, Z_{\bar{N}})$. \square*

Remark 3.6. Our choice of a Hamiltonian function f_Y introduces some non-uniqueness in the definition of Y_Q . A different choice $\hat{f}_Y := f_Y + C$ for some $C \in \mathbb{R}$ leads to a new vector field $\hat{Y}_Q = Y_Q - \frac{C}{f_Z}Z' = Y_Q + CZ_{\bar{N}}$, where we used that $\tilde{Z} = Z_P - f_Z X_P$, hence $(\pi_{\bar{N}})_*\tilde{Z} = -f_Z Z_{\bar{N}}$. In summary, Y_Q is unique up to a constant multiple of $Z_{\bar{N}}$.

Running through our argument for linearly independent vector fields, we find:

Proposition 3.7. *Given a basis $\{Y_j\}$, $j = 1, \dots, \dim \mathfrak{aut}(N, f_Z)$, of $\mathfrak{aut}(N, f_Z)$, any set of choices of Hamiltonian functions f_{Y_j} with respect to ω_H gives rise to an injective, linear map*

$$\begin{aligned} \varphi : \mathfrak{aut}(N, f_Z) &\longrightarrow \mathfrak{aut}(\bar{N}, Z_{\bar{N}}) \\ Y = \sum_j \alpha_j Y_j &\longmapsto Y_Q = \sum_j \alpha_j Y_j^Q \end{aligned}$$

where $Y_j^Q := Y_{j,Q} = \left(Y_j - \frac{f_{Y_j}}{f_H}Z\right)'$.

Proof. Linearity of the map is clear. To prove injectivity, note that the process of twisting a vector field consists of the composition of two point-wise isomorphisms, and therefore introduces no kernel. Thus, $\varphi(Y) = 0$ if and only if $Y_H = 0$, i.e. $Y = \frac{\sum_j \alpha_j f_{Y_j}}{f_H}Z$. Any vector field of the form ψZ for $\psi \in C^\infty(N)$ satisfies $L_{\psi Z}g = 2(d\psi)\iota_Z g$, hence is Killing if and only if ψ is constant. But since Z generates a rotating circle symmetry, no non-zero multiple of it preserves ω_2 or ω_3 . Because Y is triholomorphic, $\psi \equiv 0$, so $\ker \varphi = \{0\}$. \square

Both $\mathfrak{aut}(N, f_Z)$ and $\mathfrak{aut}(\bar{N}, Z_{\bar{N}})$ naturally come equipped with the structure of a Lie algebra. It is natural to ask if (any of) the maps from Proposition 3.7 are Lie algebra homomorphisms. It turns out that this is generally not quite the case:

Theorem 3.8. *Let $\{Y_j\}$, $j = 1, \dots, k$ be a basis of a Lie subalgebra $\mathfrak{g} \subset \mathfrak{aut}(N, f_Z)$ satisfying $[Y_j, Y_k] = \sum_l c_{jk}^l Y_l$. Then*

$$[Y_j^Q, Y_k^Q] = \sum_l c_{jk}^l Y_l^Q + A_{jk} Z_{\bar{N}}$$

for constants $A_{jk} = \omega'_H(Y'_j, Y'_k) - \sum_l c_{jk}^l f'_{Y_l}$. Thus, \mathfrak{g} induces a $(k+1)$ -dimensional Lie subalgebra $\mathfrak{g}_Q = \text{span}\{Y_j^Q, Z_{\bar{N}} \mid j = 1, \dots, k\} \subset \mathfrak{aut}(\bar{N}, Z_{\bar{N}})$, which is a (possibly trivial) central extension of \mathfrak{g} .

Proof. Expanding in terms of the Y_j , we have:

$$[Y_j^H, Y_k^H] = \sum_l c_{jk}^l Y_l - \left(\left[\frac{f_{Y_j}}{f_H} Z, Y_k \right] + \left[Y_j, \frac{f_{Y_k}}{f_H} Z \right] \right) + \left[\frac{f_{Y_j}}{f_H} Z, \frac{f_{Y_k}}{f_H} Z \right]$$

As in the proof of Lemma 3.4, the last term vanishes, and $df_{\mathbb{H}}(Y_j) = 0$ for every j . Thus, the expression simplifies to

$$[Y_j^{\mathbb{H}}, Y_k^{\mathbb{H}}] = \sum_l c_{jk}^l Y_l + \frac{1}{f_{\mathbb{H}}}(Y_k(f_{Y_j}) - Y_j(f_{Y_k}))Z = \sum_l c_{jk}^l Y_l - \frac{2}{f_{\mathbb{H}}}\omega_{\mathbb{H}}(Y_j, Y_k)Z$$

Now we use Lemma 2.5:

$$\begin{aligned} [Y_j^{\mathbb{Q}}, Y_k^{\mathbb{Q}}] &\sim_{\mathcal{H}} [Y_j^{\mathbb{H}}, Y_k^{\mathbb{H}}] + \frac{1}{f_{\mathbb{H}}}\omega_{\mathbb{H}}(Y_j^{\mathbb{H}}, Y_k^{\mathbb{H}})Z = [Y_j^{\mathbb{H}}, Y_k^{\mathbb{H}}] + \frac{1}{f_{\mathbb{H}}}\omega_{\mathbb{H}}(Y_j, Y_k)Z \\ &= \sum_l c_{jk}^l Y_l - \frac{1}{f_{\mathbb{H}}}\omega_{\mathbb{H}}(Y_j, Y_k)Z \end{aligned}$$

Since $-(\frac{1}{f_{\mathbb{H}}}Z)' = Z_{\bar{N}}$, we have $[Y_j^{\mathbb{Q}}, Y_k^{\mathbb{Q}}] = \sum_l c_{jk}^l Y_l' + \omega'_{\mathbb{H}}(Y_j', Y_k')Z_{\bar{N}}$. Using the relation $Y_j^{\mathbb{Q}} = (Y_j^{\mathbb{H}})' = Y_j' - (f_{Y_j}f_{\mathbb{H}}^{-1}Z)' = Y_j' + f'_{Y_j}Z_{\bar{N}}$ can also write this as

$$[Y_j^{\mathbb{Q}}, Y_k^{\mathbb{Q}}] = \sum_l c_{jk}^l Y_l^{\mathbb{Q}} + (\omega'_{\mathbb{H}}(Y_j', Y_k') - \sum_l c_{jk}^l f'_{Y_l})Z_{\bar{N}}$$

which was to be shown.

To check that the combination $\omega'_{\mathbb{H}}(Y_j', Y_k') - \sum_l c_{jk}^l f'_{Y_l}$ is indeed constant, it suffices to show that its twist is. Thus, we compute the differential of $\omega_{\mathbb{H}}(Y_j, Y_k) - \sum_l c_{jk}^l f_{Y_l}$. Applying Cartan's formula multiple times and using $d\omega_{\mathbb{H}} = 0$ we write the first term as

$$L_{Y_k}\iota_{Y_j}\omega_{\mathbb{H}} - \iota_{Y_k}L_{Y_j}\omega_{\mathbb{H}} = \iota_{Y_j}L_{Y_k}\omega_{\mathbb{H}} + \iota_{[Y_k, Y_j]}\omega_{\mathbb{H}} - \iota_{Y_k}L_{Y_j}\omega_{\mathbb{H}}$$

Both the first and last term vanish, since the vector fields Y_j all preserve $\omega_{\mathbb{H}}$. Thus, we find

$$d\left(\omega_{\mathbb{H}}(Y_j, Y_k) - \sum_l c_{jk}^l f_{Y_l}\right) = \omega_{\mathbb{H}}([Y_k, Y_j], \cdot) + \sum_l c_{jk}^l \omega_{\mathbb{H}}(Y_l, \cdot) = 0$$

since $[Y_k, Y_j] = \sum_l c_{kj}^l Y_l$. \square

The constants A_{jk} define a two-cocycle $\alpha := \frac{1}{2}A_{jk}Y_j^* \wedge Y_k^* \in Z^2(\mathfrak{g})$, where $\{Y_j^*\}$ denotes the dual basis of \mathfrak{g}^* . It is well-known that the corresponding cohomology class $[\alpha]$ vanishes if and only if the extension is trivial. In other words, we have:

Corollary 3.9. *Under the assumptions of Theorem 3.8, if the class $[\alpha] \in H^2(\mathfrak{g})$ is trivial, there exists a redefinition $\bar{Y}_j^{\mathbb{Q}} := Y_j^{\mathbb{Q}} + \beta_j Z_{\bar{N}}$ such that $\bar{\mathfrak{g}} := \text{span}\{\bar{Y}_j^{\mathbb{Q}} \mid j = 1, \dots, k\} \subset \mathfrak{g}_{\mathbb{Q}}$ is a subalgebra isomorphic to \mathfrak{g} , and $\mathfrak{g}_{\mathbb{Q}} = \bar{\mathfrak{g}} \oplus \mathbb{R}Z_{\bar{N}}$. \square*

Corollary 3.10. *If $\mathfrak{g} \subset \text{aut}(N, f_Z)$ is semi-simple, then it gives rise to an isomorphic subalgebra $\bar{\mathfrak{g}} \subset \text{aut}(\bar{N}, Z_{\bar{N}})$.*

Proof. This follows from Whitehead's lemma, which asserts that $H^2(\mathfrak{g}) = 0$ for every semisimple Lie algebra \mathfrak{g} . \square

3.2 Application to the c -map

Now we apply the above results to the supergravity c -map. We consider a PSK manifold \bar{M} , which, by definition, is the Kähler quotient $M // S^1$ of a CASK manifold $(M, g_M, \omega_M, \nabla, \xi)$ by the circle action generated by $-J\xi$, where ξ is the Euler field on M , and J its complex structure.

Recall from Section 2 that applying the supergravity c -map to \bar{M} yields the same result as applying the rigid c -map to M , and composing it with the HK/QK correspondence. This fact allows us to study the supergravity c -map in the presence of symmetries through the HK/QK correspondence, which we discussed in the previous section. We will now verify that (infinitesimal) automorphisms of the PSK manifold \bar{M} naturally lead, through the rigid c -map, to the set-up considered in Section 3.1.

An automorphism of the PSK manifold \bar{M} is, by definition, induced by an automorphism φ of the CASK manifold M . Any diffeomorphism of M admits a canonical lift to a diffeomorphism Φ of its cotangent bundle, using the pullback on one-forms: If α_p is a covector at $p \in M$, we define $\Phi((p, \alpha_p)) = (\varphi(p), (\varphi^{-1})^*\alpha_p)$ (or in other words, $\Phi = (\varphi^{-1})^*$).

Lemma 3.11. *Let M be a CASK manifold and $N = T^*M$ its cotangent bundle, endowed with the rigid c -map hyper-Kähler structure. If φ is an automorphism of the CASK structure on M and Φ its canonical lift to N , then Φ preserves the full hyper-Kähler structure of N , i.e. $\Phi^*g = g$ and $\Phi^*\omega_i = \omega_i$, $i = 1, 2, 3$.*

Proof. Since φ preserves the special Kähler connection, Φ preserves the splitting (5). Thus, we may use the expressions in Equation (6). But then, g , I_1 , and I_2 are explicitly defined in terms of tensor fields preserved by φ , hence they are preserved by Φ . \square

Now we consider the infinitesimal case. Assume that the initial CASK manifold M (or equivalently the PSK manifold \bar{M}) admits a one-parameter group of automorphisms φ_t , generated by some vector field X . The one-parameter group Φ_t of canonical lifts generates a vector field $Y \in \mathfrak{X}(N)$, which can be expressed as follows:

Lemma 3.12. *Let $(\nabla X)^* : T^*M \rightarrow T^*M$ be the adjoint of the endomorphism ∇X of TM , and $\eta \in \Gamma(T^\vee N)$ the (fiberwise) Euler field on $T^*M = N$. Under the isomorphism $T^\vee N \cong \pi^*(T^*M)$, we may view $(\nabla X)^*$ as an endomorphism of the vertical tangent bundle, and hence apply it to η . Then $Y = \tilde{X} - (\nabla X)^*(\eta)$, where \tilde{X} is the ∇ -horizontal lift of X .*

Proof. Since the proposed formula for Y certainly specifies a lift of X , we need only check that it is the *canonical* one. It is a basic fact from symplectic geometry that a diffeomorphism of T^*M is the canonical lift of a diffeomorphism of M if and only if it preserves the tautological one-form λ . Therefore, it suffices to check $L_{\tilde{X} - (\nabla X)^*(\eta)}\lambda = 0$.

Since ∇ is flat, we may choose local ∇ -affine coordinates (q^i) on M . With respect to local canonical coordinates (q^i, p_j) on T^*M , we have $\eta = \sum p_i \frac{\partial}{\partial p_i}$ and consequently

$$\tilde{X} - (\nabla X)^*(\eta) = \sum \left(X^i \frac{\partial}{\partial q^i} - \frac{\partial X^i}{\partial q^j} p_i \frac{\partial}{\partial p_j} \right)$$

With respect to any set of canonical coordinates, $\lambda = \sum p_i dq^i$, so we compute

$$L_{\tilde{X} - (\nabla X)^*(\eta)} \lambda = \sum \left(p_i dX^i - \frac{\partial X^i}{\partial q^j} p_i dq^j \right) = 0$$

proving our claim. \square

Lemma 3.11 shows that the canonical lift Y of an infinitesimal CASK automorphism X preserves the full hyper-Kähler structure of $N = T^*M$. Hence, to prove that our discussion of automorphisms under the HK/QK correspondence applies, we need only verify that the lifted action is ω_1 -Hamiltonian and preserves f_Z , the Hamiltonian function of $Z = -\widetilde{J}\xi$ with respect to ω_1 .

Lemma 3.13. *Let φ_t, Φ_t , and $Y \in \mathfrak{X}(N)$ be as above. Then*

- (i) *The action generated by Y is ω_1 -Hamiltonian as well as ω_3 -Hamiltonian.*
- (ii) *Y preserves f_Z .*

Proof.

- (i) Since ω_3 is the canonical symplectic structure on T^*M , it is given by $\omega_3 = -d\lambda$. Since Y is the canonical lift of X , it preserves λ , and is ω_3 -Hamiltonian with Hamiltonian function $-\lambda(Y)$.

Since the special Kähler connection on M is symplectic, ω_M has constant coefficients $(\omega_M)_{ij}$ with respect to any choice of ∇ -affine coordinates (q^i) . With respect to the corresponding canonical coordinates (q^i, p_j) on N , we therefore have

$$\omega_1 = \sum \left((\omega_M)_{ij} dq^i \wedge dq^j + (\omega_M)^{ij} dp_i \wedge dp_j \right)$$

where $(\omega_M)^{ij}$ are the coefficients of the inverse matrix. Using Lemma 3.12, we find

$$\iota_Y \omega_1 = \sum \left(2(\omega_M)_{ij} X^i dq^j - 2(\omega_M)^{ij} \frac{\partial X^k}{\partial q^i} p_k dp_j \right)$$

By Lemma 2.16, the first term is given by $-\pi^* df_X$, where f_X is a Hamiltonian function for X with respect to ω_M . Since X preserves the special Kähler connection, its coefficients with respect to the ∇ -affine coordinates are affine functions, i.e. $\frac{\partial X^i}{\partial q^j}$ is constant for every i, j . Therefore, the second term is given by $-d\left(\sum S^{jk} p_k dp_j\right)$, where we defined a new tensor field $S \in \text{Sym}^2((T^\vee N)^*)$ by setting $S^{jk} := \sum (\omega_M)^{ij} \frac{\partial X^k}{\partial q^i}$. In invariant terms, $S = \omega_1((\nabla X)^* -, \cdot)|_{T^\vee N}$, where we regard $(\nabla X)^*$ as an endomorphism of $T^\vee N$; its symmetry follows from the fact that $0 = L_X \omega_M = (\nabla_X - \nabla X)\omega_M$ and that ω_M is parallel with respect to ∇ . Then $\sum S^{ij} p_i p_j = S(\eta, \eta)$, where $\eta = \sum p_i \frac{\partial}{\partial p_i} \in \Gamma(T^\vee N)$ is the fiberwise Euler field. Thus, we have found

$$\iota_Y \omega_1 = -d(\pi^* f_X + S(\eta, \eta))$$

In particular, Y is ω_1 -Hamiltonian.

- (ii) Since $f_Z = -\frac{1}{2}\pi^* g_M(\xi, \xi)$ and X preserves both g_M and ξ , its canonical lift Y automatically preserves f_Z .

□

Proposition 3.14. *Given a PSK manifold \bar{M} and underlying CASK manifold M , there is an injective linear map $\mathfrak{aut}(\bar{M}) \rightarrow \mathfrak{aut}(N, f_Z)$, where N is the image of M under the rigid c -map.*

Proof. By Proposition 2.18, there exists an injective linear map $\mathfrak{aut}(\bar{M}) \rightarrow \mathfrak{aut}(M)$. In case the special Kähler connection ∇ on M does not equal the Levi-Civita connection, this is even a canonical isomorphism. By Lemmata 3.11 and 3.13, the canonical lift of any element of $\mathfrak{aut}(M)$ to $N = T^*M$ lies in $\mathfrak{aut}(N, f_Z)$. By functoriality of the pullback this defines a homomorphism $\mathfrak{aut}(M) \rightarrow \mathfrak{aut}(N, f_Z)$, which is obviously injective as well. □

Combining this with our results from Section 3.1, we have several interesting consequences:

Theorem 3.15. *Let \bar{M} be any PSK manifold, and \bar{N} its image under the supergravity c -map. Then there exists an injective, linear map from $\mathfrak{aut}(\bar{M})$ into $\mathfrak{aut}(\bar{N}, Z_{\bar{N}})$, the algebra of Killing fields on \bar{N} which commute with $Z_{\bar{N}}$. This is true for the quaternionic Kähler metrics on \bar{N} induced by the (undeformed) supergravity c -map, as well as their one-loop deformations.*

Proof. This follows immediately by composing the maps constructed in Proposition 3.14 and Proposition 3.7 (note that the latter introduces a certain freedom of choice, cf. Remark 3.6). □

Theorem 3.16. *Let \bar{M} be a PSK manifold of real dimension $2n$ and set $m = \dim \mathfrak{aut}(\bar{M})$. Let (\bar{N}, g_{FS}^c) , $c \geq 0$, be its image under the (one-loop corrected) supergravity c -map. Then the isometry group of (\bar{N}, g_{FS}^c) has dimension at least $m + 2n + 3$.*

Proof. Thinking of \bar{N} as a bundle over \bar{M} , we already know from Lemma 2.20 that, for every $c \geq 0$, we have an isometric action of Heis_{2n+3} (possibly with discrete stabilizer), which restricts trivially to \bar{M} . By contrast, all isometries generated by the Killing fields constructed by means of Theorem 3.15 certainly act non-trivially on the base PSK manifold, hence they are independent of such fiberwise isometries. □

In case $c = 0$ the group of fiberwise isometries is even of dimension $2n + 4$ (again, see Lemma 2.20), so we recover Proposition 23 of [CDJL17]:

Corollary 3.17. *Let \bar{M}^{2n} be a PSK manifold with $m = \dim \mathfrak{aut}(\bar{M})$. Then the isometry group of (\bar{N}, g_{FS}^0) , its image under the (undeformed) supergravity c -map, has dimension at least $m + 2n + 4$.* □

The next corollary generalizes Corollary 24 of [CDJL17], where only the case $c = 0$ was considered:

Corollary 3.18. *Let \bar{M}^{2n} be a PSK manifold such that its automorphism group acts with cohomogeneity k . Then the isometry group of (\bar{N}, g_{FS}^c) acts with cohomogeneity at most $k + 1$. If $c = 0$, the isometry group acts with cohomogeneity at most k .* □

Recall that all known homogeneous quaternionic Kähler manifolds of negative scalar curvature are *Alekseevsky spaces*, that is, quaternionic Kähler manifolds which admit a simply transitive, completely solvable group of isometries [Ale75, dWvP92, Cor96]. It is known that, with exception of the quaternionic hyperbolic spaces, they can all be obtained from the c -map [dWvP92].

Corollary 3.19. *For all Alekseevsky spaces, with exception of the quaternionic hyperbolic spaces, the one-loop deformation of the Ferrara-Sabharwal metric defines a one-parameter deformation by complete quaternionic Kähler manifolds (\bar{N}, g_{FS}^c) , $c \geq 0$, which admit a group of isometries acting with cohomogeneity one. \square*

Since there are many examples of complete, simply connected PSK manifolds [CHM12], including homogeneous spaces [dWvP92, AC00] and manifolds of cohomogeneity one [CDJL17], this result yields many examples of quaternionic Kähler manifolds with non-trivial fundamental group.

In the next section, we study the one-loop deformations of the non-compact Wolf spaces $SU(2, n)/S(U(2) \times U(n))$ in more detail. For this series of examples, we show that the above results are sharp, i.e. that the one-loop deformed Ferrara–Sabharwal metric is of cohomogeneity precisely one.

4 Examples

Certainly the most basic example of a PSK manifold is the space consisting of a single point. Applying the c -map to it, we obtain the simplest and (certainly in the physics literature) most prominent example of the c -map. The resulting four-dimensional quaternionic Kähler manifold² is known as the *universal hypermultiplet*. The universal hypermultiplet, which we will denote by \bar{N}_0 , is diffeomorphic to \mathbb{R}^4 , but the (undeformed) Ferrara–Sabharwal metric casts it as a non-compact Wolf space, namely the complex hyperbolic plane $\mathbb{C}H^2 = SU(1, 2)/U(2)$. The one-loop deformed universal hypermultiplet is not symmetric, and in fact not even locally homogeneous. This follows from the results of [CS18], which we will review below. Thus, even the trivial PSK manifold yields an interesting quaternionic Kähler manifold.

The (undeformed) universal hypermultiplet $\bar{N}_0 = SU(1, 2)/S(U(1) \times U(2))$ is the first in an infinite series of non-compact Wolf spaces. Indeed, for any $k \geq 0$ the $(4k + 4)$ -dimensional symmetric space

$$\bar{N}_k := \frac{SU(k + 1, 2)}{S(U(k + 1) \times U(2))}$$

is quaternionic Kähler. These spaces can be characterized as the only Riemannian manifolds which are simultaneously quaternionic Kähler of negative scalar curvature, and Kähler.

Each \bar{N}_k can be constructed by applying the c -map. Consider

$$M_k = \left\{ (z_0, \dots, z_k) \in \mathbb{C}^{k+1} \mid |z_0|^2 > \sum_{j=1}^k |z_j|^2 \right\}$$

We can view this domain in \mathbb{C}^{k+1} as an affine special Kähler manifold by equipping it with the trivial special Kähler connection $\nabla = d$ and the standard flat, indefinite Kähler structure of signature $(k, 1)$:

$$\begin{aligned} g_{M_k} &= \sum_{j=1}^k |dz_j|^2 - |dz_0|^2 \\ \omega_{M_k} &= \frac{i}{2} \left(\sum_{j=1}^k dz_j \wedge d\bar{z}_j - dz_0 \wedge d\bar{z}_0 \right) \end{aligned} \tag{8}$$

With the position vector field $\xi = \sum_{\mu=0}^k \left(z_\mu \frac{\partial}{\partial z_\mu} + \bar{z}_\mu \frac{\partial}{\partial \bar{z}_\mu} \right)$ as Euler field, M_k is a CASK domain. The vector field $-J\xi$ is (up to sign) just the generator of the standard $U(1)$ -action of the unit complex numbers on \mathbb{C}^{k+1} which leaves the full CASK structure invariant, and by taking the Kähler quotient we obtain (the projective model of) complex hyperbolic space $\mathbb{C}H^k$, which is therefore a PSK domain. Applying the supergravity c -map to $\mathbb{C}H^k$, one obtains \bar{N}_k (see, for instance, [CDS17]). In the following, our main interest is in the one-loop deformations of these symmetric metrics.

The coordinate z_0 on M_k plays a distinguished role in the c -map construction. Accordingly, the case $k = 0$ is qualitatively different and fundamentally simpler from its higher-dimensional generalizations. In this case, which we discuss first, we are able to compute the

²As is usual, we call a four-manifold quaternionic Kähler if it is (anti-)self-dual and Einstein.

full isometry group of the one-loop deformed Ferrara–Sabharwal metric. Its action on \bar{N}_0 is of cohomogeneity one, and it follows from Corollary 3.18 that the isometry group acts with cohomogeneity at most one on the higher-dimensional generalizations as well. In fact, our results in [CST20] imply that it acts with cohomogeneity precisely one. We summarize these results below.

4.1 The universal hypermultiplet

Since the universal hypermultiplet is diffeomorphic to $\mathbb{R}_{>0} \times \mathbb{R}^3$, we may use global coordinates $(\rho, \tilde{\phi}, \zeta, \tilde{\zeta})$, with respect to which the (one-loop deformed) Ferrara–Sabharwal metric takes the following form [CS18]:

$$g_{\bar{N}_0}^c = \frac{1}{2\rho^2} \left[\frac{\rho + 2c}{\rho + c} d\rho^2 + \frac{\rho + c}{\rho + 2c} (d\tilde{\phi} + \zeta d\tilde{\zeta} - \tilde{\zeta} d\zeta)^2 + 2(\rho + 2c)(d\tilde{\zeta}^2 + d\zeta^2) \right]$$

For $c = 0$, this is the complex hyperbolic metric on $\mathbb{C}H^2$, normalized such that the reduced scalar curvature $\nu = \frac{1}{4n(n+2)} \text{scal}$ (where n is the quaternionic dimension) equals -1 . On the other hand, as $c \rightarrow \infty$, $g_{\bar{N}_0}^c$ tends to a metric of constant curvature. While both of these limiting cases are well understood, less is known for finite $c > 0$. In this section, we compute the full isometry group of this metric. First, let us note that, by Lemma 2.20, we have a free and isometric action of the three-dimensional Heisenberg group, which is transitive on the level sets of the coordinate ρ .

Lemma 4.1. *The norm of the curvature endomorphism $\mathcal{R} : \Lambda^2 T^*M \rightarrow \Lambda^2 T^*M$ of $(\bar{N}_0, g_{\bar{N}_0}^c)$ is given by*

$$\|\mathcal{R}\|^2 = 6 \left(1 + \frac{\rho^6}{(\rho + 2c)^6} \right)$$

In particular, for any $c > 0$, $\|\mathcal{R}\|^2$ is an injective function of ρ .

Proof. An explicit eigenbasis for the (self-adjoint) curvature operator was constructed in [CS18]. With our normalization, the corresponding eigenvalues are

$$\lambda_1 = - \left(1 + 2 \frac{\rho^3}{(\rho + 2c)^3} \right), \quad \lambda_2 = -1, \quad \lambda_3 = - \left(1 - \frac{\rho^3}{(\rho + 2c)^3} \right)$$

with multiplicities $\mu_1 = 1$, $\mu_2 = 3$ and $\mu_3 = 2$, respectively. This means that

$$\|\mathcal{R}\|^2 = \sum_i \mu_i \lambda_i^2 = 6 \left(1 + \frac{\rho^6}{(\rho + 2c)^6} \right)$$

as claimed. \square

Since all scalar curvature invariants are invariant under isometries, we deduce:

Corollary 4.2. *There are no isometries of the deformed Ferrara–Sabharwal metric which change the value of the coordinate ρ . In particular, $(\bar{N}_0, g_{\bar{N}_0}^c)$ is not a homogeneous space.*

\square

Lemma 4.3. *Any isometry φ of the one-loop deformed Ferrara–Sabharwal metric $g_{\bar{N}_0}^{c>0}$ is of the form $\varphi(\rho, \tilde{\phi}, \tilde{\zeta}, \zeta) = (\rho, \psi(\tilde{\phi}, \tilde{\zeta}, \zeta))$, where $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ preserves the symmetric bilinear form $\alpha_\lambda := \lambda(d\tilde{\phi} + \zeta d\tilde{\zeta} - \tilde{\zeta}d\zeta)^2 + d\tilde{\zeta}^2 + d\zeta^2$ for every $\lambda \in (0, c/8)$.*

Proof. We rewrite the metric as

$$g_{\bar{N}_0}^c = \frac{1}{2\rho^2} \left(\frac{\rho + 2c}{\rho + c} d\rho^2 + 2(\rho + 2c)\alpha_{\lambda(\rho)} \right)$$

where $\lambda(\rho) = \frac{1}{2} \frac{\rho+c}{(\rho+2c)^2} \in (0, c/8)$ for all $\rho > 0$. This makes it clear that any diffeomorphism of the given form is an isometry.

Conversely, any isometry φ must preserve the level sets of ρ . Since it is an isometry, it furthermore preserves the unit normal bundle of each such hypersurface, or equivalently leaves the normal vector field $\frac{\partial}{\partial \rho}$ invariant up to sign. The first observation implies that $\varphi(\rho, \tilde{\phi}, \tilde{\zeta}, \zeta) = (\rho, \psi(\rho, \tilde{\phi}, \tilde{\zeta}, \zeta))$, where ψ preserves α_λ for every $\lambda \in (0, c/8)$. The second fact implies that ψ is independent of ρ , proving that φ is of the claimed form. \square

Lemma 4.4. *The Killing algebra of $(\bar{N}_0, g_{\bar{N}_0}^{c>0})$ is spanned by the Killing vectors*

$$X_1 := \frac{\partial}{\partial \tilde{\phi}}, \quad X_2 = \frac{\partial}{\partial \zeta} - \tilde{\zeta} \frac{\partial}{\partial \tilde{\phi}}, \quad X_3 = \frac{\partial}{\partial \tilde{\zeta}} + \zeta \frac{\partial}{\partial \tilde{\phi}}, \quad X_4 = \tilde{\zeta} \frac{\partial}{\partial \zeta} - \zeta \frac{\partial}{\partial \tilde{\zeta}}$$

Proof. We know that any Killing vector must be tangent to the level sets \mathcal{H}_ρ of ρ . For any such hypersurface, $\{X_1, X_2, X_3\}$ provides a global trivialization of the tangent bundle. Therefore, any Killing vector is of the form $X = f_1 X_1 + f_2 X_2 + f_3 X_3$ for ρ -independent functions $f_i \in C^\infty(\mathcal{H}_\rho)$, which we assume are not all constant. By the previous Lemma, $L_X \alpha_\lambda = 0$. Combining this with the fact that each X_i , $i = 1, 2, 3$, preserves α_λ , we obtain

$$2\lambda(df_1 - 2\tilde{\zeta}df_2 + 2\zeta df_3)(d\tilde{\phi} + \zeta d\tilde{\zeta} - \tilde{\zeta}d\zeta) + 2(d\tilde{\zeta} \vee df_3 + d\zeta \vee df_2) = 0$$

Since this equation must hold for every $\lambda \in (0, c/8)$, the two terms actually vanish independently:

$$\begin{aligned} (df_1 - 2\tilde{\zeta}df_2 + 2\zeta df_3)(d\tilde{\phi} + \zeta d\tilde{\zeta} - \tilde{\zeta}d\zeta) &= 0 \\ (d\tilde{\zeta} \vee df_3 + d\zeta \vee df_2) &= 0 \end{aligned}$$

The latter implies that f_2 depends only on $\tilde{\zeta}$ while f_3 only depends on ζ , with the constraint that $\frac{\partial f_2}{\partial \zeta} = -\frac{\partial f_3}{\partial \tilde{\zeta}}$. This forces these derivatives to be constant, and we conclude that f_2 and f_3 are affine functions. By adding constant multiples of X_2 and X_3 , we may in fact arrange that these functions are linear, i.e. $f_2 = k\tilde{\zeta}$ and $f_3 = -k\zeta$ for some $k \in \mathbb{R}$. Plugging this into the first equation, we find

$$(df_1 - kd(\tilde{\zeta}^2 + \zeta^2))(d\tilde{\phi} + \zeta d\tilde{\zeta} - \tilde{\zeta}d\zeta) = 0$$

and deduce that $f_1 = k(\tilde{\zeta}^2 + \zeta^2)$, up to a constant which we may remove by adding a multiple of X_1 . In conclusion, we have

$$X = k((\tilde{\zeta}^2 + \zeta^2)X_1 + \tilde{\zeta}X_2 - \zeta X_3) = kX_4$$

This proves the claim. \square

Theorem 4.5. *The isometry group of the one-loop deformed Ferrara–Sabharwal metric on the universal hypermultiplet is $\text{Heis}_3 \rtimes O(2)$.*

Proof. The isometry group acts on the Killing algebra by isomorphisms, and in particular preserves its center, which is spanned by $\frac{\partial}{\partial \phi}$. Thus, any isometry φ of $g_{N_0}^{c>0}$ must be of the form $\varphi(\rho, \tilde{\phi}, \tilde{\zeta}, \zeta) = (\rho, \chi(\tilde{\phi}, \tilde{\zeta}, \zeta), \psi(\tilde{\zeta}, \zeta))$, where $\chi : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are smooth maps. By the same argument as in the proof of the preceding Lemma, φ must also preserve the symmetric bilinear forms $(d\tilde{\phi} + \zeta d\tilde{\zeta} - \tilde{\zeta} d\zeta)^2$ and $d\tilde{\zeta}^2 + d\zeta^2$. Recognizing the latter as the Euclidean metric on \mathbb{R}^2 , we see that ψ must be a Euclidean motion. Thinking of \mathbb{R}^2 as \mathbb{C} , with complex coordinate $\xi = \tilde{\zeta} + i\zeta$, we have $\psi(\xi) = e^{i\theta}(\xi + u)$ or $\psi(\xi) = e^{-i\theta}(\bar{\xi} + \bar{u})$, where $u \in \mathbb{C}$.

Imposing that φ also preserves $(d\tilde{\phi} + \text{Im}(\bar{\xi}d\xi))^2$, we find

$$d(\tilde{\phi} \pm \chi(\tilde{\phi}, \xi)) = \text{Im}(\xi d\bar{\xi}) \pm \text{Im}(\psi(\xi) d\overline{\psi(\xi)})$$

We look for solutions of this equation, given our general form for $\psi(\xi)$. Since the left-hand side is exact, so must the right-hand side. For $\psi(\xi) = e^{i\theta}(\xi + u)$, this only happens if the negative sign is chosen, while for $\psi(\xi) = e^{-i\theta}(\bar{\xi} + \bar{u})$ the positive sign must be chosen. This leads to the differential equation

$$d(\tilde{\phi} + \text{Im}(u\bar{\xi}) \pm \chi(\tilde{\phi}, \xi)) = 0$$

whose general solutions are easily read off:

$$\chi(\tilde{\phi}, \xi) = \mp(\tilde{\phi} + \text{Im}(u\bar{\xi}) + k)$$

where $k \in \mathbb{R}$ is an arbitrary constant. The translations in $\tilde{\phi}$ and ξ determine a normal subgroup isomorphic to Heis_3 , while rotations and reflections in the $(\tilde{\zeta}, \zeta)$ -plane give rise to the $O(2)$ -subgroup. \square

4.2 Higher-dimensional hypermultiplet manifolds

To obtain the higher-dimensional symmetric spaces $\tilde{N}_k = \frac{SU(k+1,2)}{S(U(k+1) \times U(2))}$ and their one-loop deformations, we start with complex hyperbolic space $\mathbb{C}H^k$. Introducing complex coordinates $\{X_j\}$ on the unit ball $B^{2n} \subset \mathbb{C}^n$, the complex hyperbolic metric can be expressed as:

$$g_{\mathbb{C}H^k} = \frac{1}{1 - |X|^2} \left(\sum_j |dX_j|^2 + \frac{1}{1 - |X|^2} |\bar{X}_j dX_j|^2 \right)$$

where $|X| = \sum_j |X_j|^2 < 1$. On the corresponding CASK manifold M_k we have the standard linear action of $SU(k, 1)$. It descends to $\mathbb{C}H^k$, casting it as the Hermitian symmetric space $SU(k, 1)/U(k)$. In fact, $SU(k, 1)$ acts by automorphisms of the PSK structure because it preserves the full CASK structure on M_k (here we use that, in this case, the special Kähler and Levi-Civita connection coincide), and so we see that $\mathbb{C}H^k$ is homogeneous as a PSK manifold.

The one-loop deformed Ferrara–Sabharwal metric on $\bar{N}_k \cong \mathbb{C}H^k \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}^{k+1} \times \mathbb{R}^{k+1}$ is given in global coordinates $(X_j, \rho, \tilde{\phi}, \zeta^I, \tilde{\zeta}_I)$ by the following bulky expression:

$$\begin{aligned} g_{\bar{N}_k}^c &= \frac{\rho+c}{\rho} g_{\mathbb{C}H^k} + \frac{1}{4\rho^2} \frac{\rho+2c}{\rho+c} d\rho^2 \\ &+ \frac{1}{4\rho^2} \frac{\rho+c}{\rho+2c} \left(d\tilde{\phi} + \sum_{I=0}^k (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + \frac{2c}{1-|X|^2} \operatorname{Im} \left[\sum_{j=1}^k \bar{X}_j dX_j \right] \right)^2 \\ &+ \frac{1}{2\rho} \left(\sum_{j=1}^k ((d\tilde{\zeta}_j)^2 + (d\zeta^j)^2) - (d\tilde{\zeta}_0)^2 - (d\zeta^0)^2 \right) \\ &+ \frac{\rho+c}{\rho} \frac{1}{1-|X|^2} \left| (d\tilde{\zeta}_0)^2 + (d\zeta^0)^2 + \sum_{j=1}^k X^k ((d\tilde{\zeta}_j)^2 + (d\zeta^j)^2) \right|^2 \end{aligned}$$

Theorem 3.15 implies that the full group $SU(k, 1)$ of PSK symmetries of $\mathbb{C}H^k$ lifts to a subgroup of the isometry group of this metric; its orbits are of course $2k$ -dimensional. Adding to this the fiberwise isometries (cf. Lemma 2.20), we have found a group of isometries whose orbits are of dimension $4k + 3$, i.e. hypersurfaces, in agreement with the general result Corollary 3.18.

As in the case of the universal hypermultiplet, we have constructed a group acting by isometries, we have found that the isometry group acts with cohomogeneity at most one. A natural question is whether this result can be improved, or in other words, whether it is possible to find additional isometries which act non-trivially on the coordinate ρ . The following Theorem, the proof of which is given in [CST20], provides a negative answer in all dimensions:

Theorem 4.6. *The norm of the curvature operator $\mathcal{R} : \Lambda^2 T^*M \rightarrow \Lambda^2 T^*M$ of $(\bar{N}_k, g_{\bar{N}_k}^c)$ is given by*

$$\|\mathcal{R}\|^2 = n(5n+1) + 3 \left((n-1) \frac{\rho}{\rho+2c} + \frac{\rho^3}{(\rho+2c)^3} \right)^2 + 3 \left((n-1) \frac{\rho^2}{(\rho+2c)^2} + \frac{\rho^6}{(\rho+2c)^6} \right)$$

□

Corollary 4.7. *For any $c > 0$, the norm of the curvature operator of $(\bar{N}_k, g_{\bar{N}_k}^c)$ is an injective function of ρ . As a consequence, any isometry of $g_{\bar{N}_k}^{c>0}$ must preserve ρ .*

Proof. Up to an additive constant, the function $|\mathcal{R}|^2 : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$ is the composition of two functions $\varphi, \psi : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$, given by

$$\begin{aligned} \varphi(x) &= \frac{x}{x+2c} \\ \psi(x) &= 3((x^3 + (n-1)x)^2 + x^6 + (n-1)x^2) \end{aligned}$$

The first function is easily checked to be injective. For ψ , assume that $\psi(x_1) - \psi(x_2) = 0$ for some $x_1, x_2 \in \mathbb{R}_{>0}$. Then

$$(x_1^3 + (n-1)x_1)^2 + x_1^6 + (n-1)x_1^2 - (x_2^3 + (n-1)x_2)^2 - x_2^6 - (n-1)x_2^2 = 0$$

Clearly, this expression vanishes if $x_1 = x_2$, and since ψ is even, it also vanishes if $x_1 = -x_2$. Factorizing, we find:

$$(x_1 - x_2)(x_1 + x_2) \left(2(x_1^4 + x_2^4) + 2x_1^2x_2^2 + 2(x_1^2 + x_2^2)(n - 1) + n(n - 1) \right) = 0$$

The second and third factor are each manifestly positive for $x_1, x_2 \in \mathbb{R}_{>0}$, so the only possibility is $x_1 = x_2$. \square

As an immediate consequence, we have:

Theorem 4.8. *The one-loop deformation $g_{\bar{N}_k}^{c>0}$ of the Ferrara–Sabharwal metric on \bar{N}_k is of cohomogeneity one. In particular, $(\bar{N}_k, g_{\bar{N}_k}^{c>0})$ is not a homogeneous space.*

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