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Sheaves of Higher Categories and Presentations of Smooth Field Theories

Severin Bunk

Abstract

We study the extension of higher presheaves on a category $\mathcal{C}$ to its free cocompletion $\widehat{\mathcal{C}}$. Here, higher presheaves take values in ∞-categories of (∞, n)-categories, for any $n \in \mathbb{N}_0$. We first observe that any pretopology on $\mathcal{C}$ induces a pretopology of generalised coverings on $\widehat{\mathcal{C}}$. Our main result is that the ∞-categorical right Kan extension along the Yoneda embedding $\mathcal{Y} \colon \mathcal{C} \to \widehat{\mathcal{C}}$ preserves the descent property: given a pretopology on $\mathcal{C}$, any higher sheaf on $\mathcal{C}$ extends to a higher sheaf on $\widehat{\mathcal{C}}$ with the induced pretopology. Our proofs are performed using model categorical presentations of the relevant ∞-categories. We then present two applications of the main result in geometry and topology: first, we prove a descent property for smooth vector bundles on diffeological spaces. Diffeological spaces form a categorically well-behaved generalisation of manifolds, which includes many infinite-dimensional spaces. Our second application is to smooth bordism-type field theories on a manifold. Here, in contrast to TQFTs, diffeomorphism groups do not generally act through mapping class groups, and the smooth structure of diffeomorphism groups becomes relevant. We show how incorporating these data allows to construct smooth field theories on a manifold from generators and relations.

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1 Introduction and main results

1.1 Descent properties and extension of higher sheaves

Local-to-global properties are ubiquitous in topology, geometry, and quantum field theory. The prototypical example of a local-to-global, or *descent*, property is the gluing of local sections of a sheaf: given a manifold M with an open covering $\{U_i\}_{i\in I}$, then global sections of a sheaf F on M are in bijection with families $\{f_i \in F(U_i)\}_{i\in I}$ such that $f_{i|U_{ij}} = f_{j|U_{ij}}$ for all $i, j \in I$, with $U_{ij} \coloneqq U_i \cap U_j$.

However, sheaves are not sufficient to describe all geometric structures on manifolds. For example, principal G-bundles, for any Lie group G, do not glue in this sense: they form a sheaf valued in groupoids rather than a sheaf valued in sets. While the theory of sheaves of groupoids is much richer than that of sheaves of sets, it is still not sufficient in order to describe generic geometric structures. For instance, in the works of Schreiber [Sch13], n-gerbes are described as sections of certain sheaves of n-groupoids, for any $n \in \mathbb{N}$. In order to obtain a unified framework for sheaves of n-groupoids for all $n \in \mathbb{N}$, one passes to sheaves of ∞-groupoids. These form an ∞-topos $\mathbb{H}$ [Lur09] which is presented by various model categories of simplicial presheaves [Lur09, Sch13]. Here we work with the following presentation: for a small category $\mathcal{C}$, let $\mathcal{H}_\infty \coloneqq (\mathcal{S}\mathrm{et}_\Delta)^{\mathcal{C}^{\mathrm{op}}}$ be the category of simplicial presheaves on $\mathcal{C}$, endowed with the projective model structure. Then, $\mathbb{H}$ is the underlying ∞-category of $\mathcal{H}_\infty$.

Let $\mathcal{M}\mathrm{fd}$ denote the category of smooth manifolds and smooth maps. Our prime example in this paper is the case $\mathcal{C} = \mathcal{C}\mathrm{art}$, where $\mathcal{C}\mathrm{art}$ is the full subcategory of $\mathcal{M}\mathrm{fd}$ on the *cartesian spaces* [Sch13], i.e. those manifolds diffeomorphic to $\mathbb{R}^n$, for any $n \in \mathbb{N}_0$. It is an important (well-known) observation that sheaves of ∞-groupoids on $\mathcal{C}\mathrm{art}$ allow to describe geometric structures not just on objects of $\mathcal{C}\mathrm{art}$, but on *all* manifolds [Sch13]. This is achieved as follows: given a presheaf F of ∞-groupoids on $\mathcal{C}\mathrm{art}$ and a manifold M, one defines the ∞-groupoid of (derived) sections of F on M as the mapping space $\underline{\mathcal{H}_\infty}(Q\underline{M}, F)$. Here, $\underline{M}$ is the presheaf on $\mathcal{C}\mathrm{art}$ associated to M, the functor Q is a cofibrant replacement in $\mathcal{H}_\infty$, and $\underline{\mathcal{H}_\infty}(-,-)$ denotes the $\mathcal{S}\mathrm{et}_\Delta$-valued hom functor in $\mathcal{H}_\infty$. If F is the higher presheaf on cartesian spaces which describes G-bundles or n-gerbes, for instance, then the Kan complex $\underline{\mathcal{H}_\infty}(Q\underline{M}, F)$ is the ∞-groupoid of G-bundles or n-gerbes on M, respectively.

In the present paper, we study the above observation systematically. We first note that we could have used any presheaf on $\mathcal{C}\mathrm{art}$ in place of $\underline{M}$. More generally, let $\mathcal{C}$ be a generic small category and let $\widehat{\mathcal{C}}$ denote its category of presheaves. For $F \in \mathcal{H}_\infty$, mapping each $X \in \widehat{\mathcal{C}}$ to the simplicial set $\underline{\mathcal{H}_\infty}(QX, F)$ defines a simplicial presheaf $S^Q_\infty F$ on $\widehat{\mathcal{C}}$; we thus obtain a functor

$$S^Q_\infty \colon \mathcal{H}_\infty \longrightarrow \widehat{\mathcal{H}}_\infty\,, \qquad F \longmapsto \underline{\mathcal{H}_\infty}\big(Q(-), F\big) \tag{1.1}$$

from the projective model category $\mathcal{H}_\infty$ of simplicial presheaves on $\mathcal{C}$ to the projective model category $\widehat{\mathcal{H}}_\infty$ of simplicial presheaves on $\widehat{\mathcal{C}}$. (One needs to take some care in order to avoid set-theoretical issues here – in the main text we achieve this by working with a nested pair of Grothendieck universes – but in this introduction we swipe such issues under the rug.)

Throughout the paper, we use a particularly nice choice for the cofibrant replacement functor Q in $\mathcal{H}_\infty$ (as introduced in [Dug01]) and construct homotopy (co)limits in terms of (co)bar constructions [Rie14]. For $F \in \mathcal{H}_\infty$ fibrant, this allows us to identify $S^Q_\infty F \colon \widehat{\mathcal{C}}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$ as the homotopy right Kan extension of F along the Yoneda embedding $\mathcal{Y}^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \widehat{\mathcal{C}}^{\mathrm{op}}$ (Proposition 3.9). Consequently, on the underlying ∞-categories of $\mathcal{H}_\infty$ and $\widehat{\mathcal{H}}_\infty$, the functor S^Q_∞ is nothing but the ∞-categorical right Kan extension along the Yoneda embedding.

Our main motivation for studying sheaves of higher categories arises from field theory, and in particular from the study of so-called *smooth* functorial field theories (FFTs) on a manifold M [BW19, ST11] (see also Section 1.3). In such an FFT, objects and morphisms in the bordism category carry smooth maps to M, and one keeps track of smooth variations of these maps; this is achieved by constructing the bordism category as a presheaf of (higher) categories on $\mathcal{C}\mathrm{art}$. Correspondingly, also the targets of smooth FFTs should be presheaves of higher categories on $\mathcal{C}\mathrm{art}$. In order to relate smooth FFTs on M to geometric structures on M, however, one needs to consider targets which satisfy descent.

Therefore, we set up a theory of sheaves of (∞, n)-categories for any $n \in \mathbb{N}_0$, similarly to [Bar05, Sec. 3.2]. We start by considering presheaves valued in an ∞-category of (∞, n)-categories. The ∞-category of such presheaves is modelled by the projective model category $\mathcal{H}_{\infty,n}$ of functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{CSS}_n$, where $\mathcal{CSS}_n$ is the (injective) model category of complete Segal spaces. Analogously, we let $\widehat{\mathcal{H}}_{\infty,n}$ denote the projective model category of $\mathcal{CSS}_n$-valued presheaves on $\widehat{\mathcal{C}}$.

We observe that for any Grothendieck pretopology τ on $\mathcal{C}$ there is an induced Grothendieck pretopology $\widehat{\tau}$ on $\widehat{\mathcal{C}}$: its coverings are the *τ-local epimorphisms*, also called *generalised coverings* [DHI04], which are defined as follows. Let $\mathcal{Y}\colon \mathcal{C} \to \widehat{\mathcal{C}}$ denote the Yoneda embedding of $\mathcal{C}$. A morphism $\pi\colon Y \to X$ in $\widehat{\mathcal{C}}$ is a τ-local epimorphism if, for every $c \in \mathcal{C}$ and every map $\mathcal{Y}_c \to X$, there exists a covering $\{c_i \to c\}_{i \in I}$ in the site $(\mathcal{C}, \tau)$ such that each composition $\mathcal{Y}_{c_i} \to \mathcal{Y}_c \to X$ factors through π; this is an abstract way of saying that the morphism π has local sections. For instance, the category $\mathcal{C}\mathrm{art}$ admits a pretopology τ_{dgop} of differentiably good open coverings (see Example 2.3 for details). A smooth map $\pi\colon M \to N$ between manifolds induces a τ_{dgop}-local epimorphism $\underline{\pi}\colon \underline{M} \to \underline{N}$ on the associated presheaves precisely if π is a surjective submersion.

Given a Grothendieck pretopology τ on $\mathcal{C}$, the model category of sheaves of (∞, n)-categories on the site $(\mathcal{C}, \tau)$ is the $\mathcal{CSS}_n$-enriched left Bousfield localisation $\mathcal{H}^{loc}_{\infty,n}$ of $\mathcal{H}_{\infty,n}$ at the Čech nerves of coverings in $(\mathcal{C}, \tau)$. Similarly, we let $\widehat{\mathcal{H}}^{loc}_{\infty,n}$ denote the $\mathcal{CSS}_n$-enriched left Bousfield localisation of $\widehat{\mathcal{H}}_{\infty,n}$ at the Čech nerves of the τ-local epimorphisms. We extend the functor S^Q_∞ from (1.1) to a functor $S^Q_{\infty,n}\colon \mathcal{H}_{\infty,n} \to \widehat{\mathcal{H}}_{\infty,n}$ and show:

Theorem 1.2 *Let $n \in \mathbb{N}_0$, and let $\mathcal{C}$ be a U-small category.*

(1) For any fibrant objects $\mathcal{F} \in \mathcal{H}_{\infty,n}$, the presheaf $S^Q_{\infty,n}\mathcal{F}$ is naturally equivalent to the homotopy right Kan extension $\mathrm{hoRan}_{\mathcal{Y}^{\mathrm{op}}}\mathcal{F}$ *of $\mathcal{F}$ along the Yoneda embedding $\mathcal{Y}^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \widehat{\mathcal{C}}^{\mathrm{op}}$. In particular, $S^Q_{\infty,n}$ presents the ∞-categorical right Kan extension of presheaves of (∞, n)-categories along the Yoneda embedding on the level of the underlying ∞-categories.*

(2) If $(\mathcal{C}, \tau)$ is a Grothendieck site, there is a Quillen adjunction

$$Re^Q_{\infty,n} : \widehat{\mathcal{H}}^{loc}_\infty \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}^{loc}_\infty : S^Q_{\infty,n}\,.$$

Thus, if a presheaf $\mathcal{F}$ of higher categories on $\mathcal{C}$ satisfies τ-descent, then $S^Q_{\infty,n}\mathcal{F}$ satisfies $\widehat{\tau}$-descent.

This subsumes a result of Nikolaus and Schweigert [NS11], who proved that presheaves of 2-groupoids satisfy descent along open coverings of manifolds if and only if they satisfy descent with respect to surjective submersions. Further, there are results similar in spirit to Theorem 1.2 in [Lur18, Prop. 1.1.4.4, Cor. 1.1.4.5]. There, it is shown that the ∞-categorical right Kan extension establishes even an equivalence between the ∞-categories of sheaves on a topological space X and sheaves defined only on elements of a basis for the topology of X. We expect that on the level of ∞-categories it should be possible to find a mutual generalisation of Theorem 1.2 and these results of Lurie's; this will be the subject of future work. Here, we work with model categorical presentations throughout because we aim for our constructions to be directly applicable to problems in field theory, where explicit models for bordism categories have been constructed as higher Segal spaces in model categorical language [CS19, SP17], and where fields are usually written explicitly in terms of simplicial presheaves [FSS12]. Since all our model categories are simplicial, we readily obtain the corresponding results on their underlying ∞-categories.

1.2 Diffeological vector bundles

After proving Theorem 1.2, we present two of its consequences. First, consider the sequence of fully faithful inclusions

$$\mathcal{M}\mathrm{fd} \subset \mathcal{D}\mathrm{fg} \subset \widehat{\mathcal{C}\mathrm{art}}$$

of the category of smooth manifolds into the category of *diffeological spaces* into the category of presheaves on $\mathcal{C}\mathrm{art}$. Diffeological spaces are a generalisation of manifolds that share many of their geometric features. In particular, diffeological spaces have an underlying set X and a smooth structure, defined by specifying which maps $c \to X$ from cartesian spaces to X should be called smooth. Diffeological spaces are useful in describing geometric problems outside the scope of manifolds, such as many infinite-dimensional geometries, but they also exhibit a better categorical and homotopy-theoretic behaviour than manifolds (see, for instance, [BH11, BW18, CW14, Kih19]). The category $\mathcal{D}\mathrm{fg}$ carries a canonical Grothendieck pretopology whose coverings are usually called *subductions* [IZ13]. A morphism of diffeological spaces is a subduction precisely if its associated morphism of presheaves is a τ_{dgop}-local epimorphism. Thus, both the sites of manifolds with surjective submersions and the site of diffeological spaces with subductions form subsites of $(\widehat{\mathcal{C}\mathrm{art}}, \widehat{\tau}_{dgop})$.

There exists a notion of vector bundles internal to the category $\mathcal{D}\mathrm{fg}$ (see Definition 4.1), which reproduces smooth vector bundles on manifolds when restricted to the subcategory $\mathcal{M}\mathrm{fd} \subset \mathcal{D}\mathrm{fg}$. However, a proof that diffeological vector bundles satisfy descent along subductions has so far been missing in the literature; here we provide such a proof. Diffeological vector bundles form a pseudo-functor $\mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}} \colon \mathcal{D}\mathrm{fg}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$, valued in the 2-category of categories. We first provide a strictification of this pseudo-functor to a functor $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}} \colon \mathcal{D}\mathrm{fg}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$. The latter gives rise to a presheaf $N_{rel} \circ \mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}$ of ∞-categories on $\mathcal{D}\mathrm{fg}$ via Rezk's classification diagram functor [Rez01]. Finally, we show that $N_{rel} \circ \mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}$ is the restriction to $\mathcal{D}\mathrm{fg}$ of an object $S^Q_{\infty,1}\mathcal{F}$ in the image of $S^Q_{\infty,1} \colon \mathcal{H}^{loc}_{\infty,1} \to \widehat{\mathcal{H}}^{loc}_{\infty,1}$. The desired descent property then follows from the realisation that $\mathcal{F}$ is fibrant in $\mathcal{H}^{loc}_{\infty,1}$, together with Theorem 1.2.

Theorem 1.3 *Diffeological vector bundles satisfy descent along subductions.*

1.3 Smooth functorial field theories

Our second application is to smooth FFTs on a background manifold M. We are not going to be precise about what an FFT is, since the main point of the present paper is the development of formalism which is applicable in much wider contexts; for background on FFTs we refer to [Ati88, Saf]. In its simplest form, a d-dimensional FFT is a symmetric monoidal functor

$$\mathcal{Z} \colon \mathcal{B}\mathrm{ord}_d \to \mathcal{V}\mathrm{ect}$$

from a category of closed, oriented $(d-1)$-dimensional manifolds and d-dimensional oriented bordisms to the category of vector spaces. The symmetric monoidal structures are disjoint union and tensor product, respectively. FFTs of this form are also called topological quantum field theories (TQFTs).

In low dimensions, TQFTs can be studied, and even classified, using presentations of the bordism category $\mathcal{B}\mathrm{ord}_d$ in terms of generators and relations [Koc04, SP09]. For $d = 2$, for instance, $\mathcal{B}\mathrm{ord}_d$ is generated by the circle $\mathbb{S}^1$ on the level of objects and by cylinder and pair-of-pants bordisms on the level of morphisms. Since bordisms are taken up to diffeomorphism it follows that (i) there is an action of $\mathrm{Diff}_+(\mathbb{S}^1)$ (orientation-preserving diffeomorphisms) on $\mathcal{Z}(\mathbb{S}^1)$ and that (ii) isotopic diffeomorphisms act by the same automorphism. Thus, the action of $\mathrm{Diff}_+(\mathbb{S}^1)$ on $\mathcal{Z}(\mathbb{S}^1)$ is trivial, and one can consistently set $\mathcal{Z}(Y) = \mathcal{Z}(\mathbb{S}^1)$ for any oriented 1-manifold Y that is diffeomorphic to $\mathbb{S}^1$, without specifying such a diffeomorphism.

Smooth FFTs on a manifold M differ from TQFTs in two key aspects: manifolds and bordisms additionally carry smooth maps to M, and one keeps track of smooth variations of these maps by considering smooth families of manifolds and bordisms, parameterised over a category of test spaces. We refer the reader to [BW19] for a more extensive discussion of smooth FFTs on a manifold.

For concreteness, consider 2-dimensional smooth FFTs test spaces given by objects $c \in \mathcal{C}\mathrm{art}$. A c-parameterised family of closed 1-manifolds with smooth maps to M is a pair (Y, γ) consisting of a closed 1-manifold Y and a smooth map $\gamma\colon c \times Y \to M$. If we restrict ourselves to $Y = \mathbb{S}^1$, these pairs are in bijection to the smooth maps from c to the free loop space $LM = M^{\mathbb{S}^1}$ of M (seen as diffeological spaces). A smooth FFT on M should assign to each loop a vector space, and for every c-parameterised family $(\mathbb{S}^1, \gamma)$ as above the vector space should vary smoothly over c. In other words, $(\mathbb{S}^1, \gamma)$ should be sent to a vector bundle on c. Varying c and γ, one expects the bundles over c to describe a smooth (i.e. diffeological) vector bundle E on LM.

A significant complication compared to the TQFTs case arises from the fact that the action of $\mathrm{Diff}_+(\mathbb{S}^1)$ on LM is non-trivial: for $f \in \mathrm{Diff}_+(\mathbb{S}^1)$, the objects $(\mathbb{S}^1, \gamma)$ and $(\mathbb{S}^1, \gamma \circ f)$ are different, in general. Hence, knowing the bundle $E \to LM$ is not sufficient in order to define the value of a smooth 1-dimensional FFT on M on all pairs (Y, γ) with $Y \cong \mathbb{S}^1$ – here, the choice of diffeomorphism $Y \cong \mathbb{S}^1$ matters. An isotopy $h\colon f_0 \to f_1$ in $\mathrm{Diff}_+(\mathbb{S}^1)$ gives rise to a bordism $\Sigma_h\colon (\mathbb{S}^1, \gamma \circ f_0) \to (\mathbb{S}^1, \gamma \circ f_1)$. If our FFT $\mathcal{Z}$ is *superficial* (see [BW19, Def. 4.2.3] for details), its value $\mathcal{Z}(\Sigma_h)$ depends only on f_0 and f_1, and $\mathcal{Z}$ induces a $\mathrm{Diff}_+(\mathbb{S}^1)$-equivariant structure on the bundle E. Using descent theory for vector bundles, one can employ this equivariant structure to coherently extend $E \to LM$ to a bundle over all mapping spaces M^Y for $Y \cong \mathbb{S}^1$. We expect all smooth FFTs that stem from geometric structures on M to be superficial, so that this procedure should be broadly applicable.

To treat this coherence problem generally, given an oriented manifold Y we consider the groupoid $\mathcal{M}_Y$ of all oriented manifolds diffeomorphic to Y. This is a $\mathcal{D}\mathrm{fg}$-enriched category, and we let $P\colon \mathcal{M}_Y^{\mathrm{op}} \to \mathcal{D}\mathrm{fg}$ be a $\mathcal{D}\mathrm{fg}$-enriched functor. In the above discussion of 2-dimensional FFTs, for instance, we have $PY = M^Y$. We write $\mathrm{D}(Y)$ for the diffeological group of orientation-preserving diffeomorphisms of Y; this acts on $P(Y)$. For $\mathcal{F} \in \mathcal{H}_{\infty,n}$, we define an (∞, n)-category $\mathcal{F}(P)^{\mathrm{D}(Y)}$ of $\mathrm{D}(Y)$-*equivariant sections* of $\mathcal{F}$ on $P(Y)$. On the other hand, we can define an (∞, n)-category $\mathcal{F}(P)^{coh}$ of *coherent sections* of $\mathcal{F}$ over P. Roughly, these consist of a section E_{Y_0} of $S^Q_{\infty,n}\mathcal{F}$ over $P(Y_0)$ for each $Y_0 \in \mathcal{M}_Y$, together with a 1-isomorphism $d_0^* E_{Y_1} \to d_1^* E_{Y_1}$ over $P(Y_1) \times \mathrm{D}(Y_0, Y_1)$, where d_0 is the projection onto $P(Y_1)$, and where d_1 is the action of $\mathrm{D}(Y_0, Y_1)$ on P. There is further coherence data for compositions of diffeomorphisms.

For $\mathcal{F} \in \mathcal{H}_{\infty,n}$ projectively fibrant, one can construct coherent sections of $\mathcal{F}$ on P from equivariant sections of $\mathcal{F}$ on $P(Y)$ by choosing a diffeomorphism $Y_0 \to Y$ for every $Y_0 \in \mathcal{M}_Y$ (Theorem 4.28). However, in practise choosing infinitely many such diffeomorphisms is not feasible, and the following result is more useful:

Theorem 1.4 *If $\mathcal{F} \in \mathcal{H}^{loc}_{\infty,n}$ is fibrant, there exists a span of weak equivalences in $\mathcal{CSS}_n$:*

$$\mathcal{F}(P)^{\mathrm{D}(Y)} \xrightarrow{\ \sim\ } Z(\mathcal{F}, P, Y) \xleftarrow{\ \sim\ } \mathcal{F}(P)^{coh}\,.$$

We construct $Z(\mathcal{F}, P, Y)$ explicitly and show that here exists a homotopy inverse to the second morphism. On the level of complete Segal objects in the ∞-category of spaces, this is essentially unique. In particular, if $S^Q_{\infty,n}\mathcal{F}$ has functorial descent, the descent functor provides an inverse as desired. Any such homotopy inverse induces an equivalence $\mathcal{F}(P)^{\mathrm{D}(Y)} \xrightarrow{\sim} \mathcal{F}(P)^{coh}$. In particular, this makes precise the coherent transfer of an equivariant bundle over LM to $M^{Y'}$, for any $Y' \cong \mathbb{S}^1$, from our 2-dimensional example above. In this way, the value of a smooth FFT $\mathcal{Z}$ on any object can be obtained coherently and in an essentially unique way from the value of $\mathcal{Z}$ on generating objects. In [BW19] this has been utilised to construct 2-dimensional open-closed smooth FFTs on a manifold M from equivariant diffeological bundles on spaces of loops and paths in M. There, the bundles were obtained via transgression of higher geometric structures on M [Wal16, BW18].

1.4 Conventions

Enriched categories If $\mathcal{V}$ is a monoidal category and $\mathcal{C}$ is a $\mathcal{V}$-enriched category (or $\mathcal{V}$-category), we denote the $\mathcal{V}$-enriched hom-objects of $\mathcal{C}$ by $\underline{\mathcal{C}}^{\mathcal{V}}(-,-)$. If $\mathcal{V} = \mathcal{S}\mathrm{et}_{\Delta}$ is the category of simplicial sets, and only then, we will omit the superscript and write $\underline{\mathcal{C}}(-,-) := \underline{\mathcal{C}}^{\mathcal{S}\mathrm{et}_{\Delta}}(-,-)$. If $\mathcal{V}$ is a symmetric monoidal model category and $\mathcal{C}$ is a $\mathcal{V}$-enriched model category (in the sense of [Hov99]), we will equivalently say that $\mathcal{C}$ is a model $\mathcal{V}$-category.

Sizes and universes Throughout, we choose and fix a nested pair of Grothendieck universes $U \in V$. We assume that U contains the natural numbers. We write $\mathcal{S}\mathrm{et}_U$ and $\mathcal{S}\mathrm{et}_{\Delta U}$ for the categories of U-small sets and U-small simplicial sets, respectively. All indexing sets will be assumed to be U-small. Whenever we write $\mathcal{S}\mathrm{et}$ or $\mathcal{S}\mathrm{et}_{\Delta}$, we shall mean $\mathcal{S}\mathrm{et}_V$ or $\mathcal{S}\mathrm{et}_{\Delta V}$, respectively.

Let $\mathcal{C}$ be a U-small category. Consider the category $\widehat{\mathcal{C}} := \mathcal{S}\mathrm{et}_U^{\mathcal{C}^{\mathrm{op}}}$ of $\mathcal{S}\mathrm{et}_U$-valued presheaves on $\mathcal{C}$. Observe that $\widehat{\mathcal{C}}$ is no longer U-small, since $\mathcal{S}\mathrm{et}_U$ is not U-small. However, since $\mathcal{S}\mathrm{et}_U \subset U \in V$, it follows that $\mathcal{S}\mathrm{et}_U \in V$ is V-small. Thus, $\widehat{\mathcal{C}}$ is a V-small category.

The Yoneda embedding $\mathcal{Y}\colon \mathcal{C} \to \mathcal{S}\mathrm{et}_U^{\mathcal{C}^{\mathrm{op}}} = \widehat{\mathcal{C}}$ is fully faithful. Likewise, the Yoneda embedding $\widehat{\mathcal{Y}}\colon \widehat{\mathcal{C}} \to \mathcal{S}\mathrm{et}_V^{\widehat{\mathcal{C}}^{\mathrm{op}}}$ is fully faithful; observing that an object of $\mathcal{S}\mathrm{et}_U$ is also an object of $\mathcal{S}\mathrm{et}_V$, the standard proof applies. Further, the Yoneda Lemma holds true for both $\mathcal{Y}$ and $\widehat{\mathcal{Y}}$ (again by the usual method of proof). Finally, observe for any $c \in \mathcal{C}$ and for any $X \in \widehat{\mathcal{C}}$, by the Yoneda Lemma, there are canonical isomorphisms

$$\widehat{\mathcal{Y}}_X(\mathcal{Y}_c) = \widehat{\mathcal{C}}(\mathcal{Y}_c, X) \cong X(c) \in \mathcal{S}\mathrm{et}_U \subset \mathcal{S}\mathrm{et}_V\,.$$

Diagrams For $\mathcal{J}$ a V-small category and $\mathcal{C}$ a V-tractable model category (cf. [Bar05, Bar10]), the projective and the injective model structures on $\mathcal{C}^{\mathcal{J}}$ exist; we denote them by $(\mathcal{C}^{\mathcal{J}})_{proj}$ and $(\mathcal{C}^{\mathcal{J}})_{inj}$, respectively. If $\mathcal{J}$ is a Reedy category, we denote the Reedy model structure on $\mathcal{C}^{\mathcal{J}}$ by $(\mathcal{C}^{\mathcal{J}})_{Reedy}$.

Enriched left Bousfield localisation Throughout this article we follow the conventions of [Bar10] to describe enriched left Bousfield localisations. We refer the reader there for details and background on this formalism. Let us briefly recall that if $\mathcal{V}$ is a symmetric monoidal model V-category with a cofibrant replacement functor Q, and if $\mathcal{M}$ is a model $\mathcal{V}$-category with a chosen collection of morphisms A, then

- $z \in \mathcal{M}$ is *$A/\mathcal{V}$-local* if it is fibrant in $\mathcal{M}$ and for every morphism $f: x \to y$ in A the induced morphism

$$\underline{\mathcal{M}}^{\mathcal{V}}(Qy, z) \xrightarrow{(Qf)^*} \underline{\mathcal{M}}^{\mathcal{V}}(Qx, z)$$

  is a weak equivalence in $\mathcal{V}$, and
- $f \in \mathcal{M}(x, y)$ is an *$A/\mathcal{V}$-local weak equivalence* if for every $A/\mathcal{V}$-local object $z \in \mathcal{M}$ the morphism

$$\underline{\mathcal{M}}^{\mathcal{V}}(Qy, z) \xrightarrow{(Qf)^*} \underline{\mathcal{M}}^{\mathcal{V}}(Qx, z)$$

  is a weak equivalence in $\mathcal{V}$.

In that case, the enriched left Bousfield localisation $L_{A/\mathcal{V}}\mathcal{M}$, provided it exists, is a model $\mathcal{V}$-category with a left Quillen $\mathcal{V}$-functor $\mathcal{M} \to L_{A/\mathcal{V}}\mathcal{M}$ which is universal among left Quillen $\mathcal{V}$-functors out of $\mathcal{M}$ that send $A/\mathcal{V}$-local weak equivalences to weak equivalences (cf. [Bar10, Def.4.42, Def. 4.45]). Note that for simplicial model categories, simplicial Bousfield localisation was already described in [Hir03]. In the case of $\mathcal{V} = \mathcal{S}\mathrm{et}_{\Delta}$ we will speak of local objects rather than $\mathcal{S}\mathrm{et}_{\Delta}$-local objects, and analogously for local weak equivalences.

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2 Grothendieck sites and diffeological spaces

In this section we review the notions of a Grothendieck site, of concrete (pre)sheaves, and of diffeological spaces. We mostly follow [BH11] in our presentation, but we make a connection between the approaches to diffeological spaces in [IZ13] and [BH11]. As slight variation to the standard conventions, we define diffeological spaces as concrete sheaves on a site which differs from the usual choice, but this is just for technical convenience (cf. Remark 2.20).

2.1 Sites and local epimorphisms

We start with the definition of a site, using the notion of Grothendieck pretopologies.

Definition 2.1 ([BH11, Def. 11, 12]) *Let $\mathcal{C}$ be a U-small category.*

(1) A coverage, *or* Grothendieck pretopology, *on $\mathcal{C}$ is given by assigning to every object $c \in \mathcal{C}$ a U-small set $\tau(c)$ of families of morphisms $\{f_i: c_i \to c\}_{i \in I}$ (with $I \in \mathcal{S}\mathrm{et}_U$) satisfying the following properties: for each $c \in \mathcal{C}$, the identity 1_c is a covering family, and for every morphism $g: c' \to c$ in $\mathcal{C}$ there exists a family $\{f'_j: c'_j \to c'\}_{j \in J} \in \tau(c')$ such that for every $j \in J$ we find some $i \in I$ and a commutative diagram*

$$\begin{tikzcd} c'_j \arrow[r] \arrow[d, "f'_j"'] & c_i \arrow[d, "f_i"] \\ c' \arrow[r, "g"'] & c \end{tikzcd}$$

(2) The families in $\tau(c)$ are called covering families for *c.*

(3) A (Grothendieck) site *is a category* $\mathcal{C}$ *equipped with a coverage* τ.

Later we will use the following technical condition:

Definition 2.2 *We call a site* $(\mathcal{C}, \tau)$ closed *if it satisfies the following condition: let* $\{c_i \to c\}_{i \in I}$ *be any covering family in* $(\mathcal{C}, \tau)$. *Further, for each* $i \in I$, *let* $\{c_{i,j} \to c_i\}_{j \in J_i}$ *be a covering family in* $(\mathcal{C}, \tau)$. *Then, there exists a covering family* $\{d_k \to c\}_{k \in K}$ *such that every morphism* $d_k \to c$ *factors through one of the composites* $c_{i,j} \to c_i \to c$.

Example 2.3 Let $\mathcal{C}\mathrm{art}$ be the category of cartesian spaces, i.e. of sub-manifolds of $\mathbb{R}^\infty$ that are diffeomorphic to some $\mathbb{R}^n$, with smooth maps between these manifolds as morphisms. This category is small and has finite products. A coverage on $\mathcal{C}\mathrm{art}$ is defined by calling a family $\{\iota_i : c_i \to c\}_{i \in I}$ a covering family if it satisfies

(1) all ι_i are open embeddings (in particular $\dim(c_i) = \dim(c)$ for all $i \in I$),

(2) the ι_i cover c, i.e. $c = \bigcup_{i \in I} \iota_i(c_i)$, and

(3) every finite intersection $\iota_{i_0}(c_{i_0}) \cap \cdots \cap \iota_{i_m}(c_{i_m})$, with $i_0, \ldots, i_m \in I$, is a cartesian space.

Coverings $\{\iota_i : c_i \to c\}_{i \in I}$ with these properties are called *differentiably good open coverings* of c. They induce a coverage τ_{dgop} on $\mathcal{C}\mathrm{art}$, which turns $(\mathcal{C}\mathrm{art}, \tau_{dgop})$ into a site. This site is even closed because every open cover of $c \in \mathcal{C}\mathrm{art}$ admits a differentiably good refinement [FSS12, Cor. A.1]. Via the embedding $\mathcal{C}\mathrm{art} \hookrightarrow \mathcal{M}\mathrm{fd}$ of $\mathcal{C}\mathrm{art}$ into the category of all smooth manifolds, any manifold M defines a presheaf $\underline{M}$ on $\mathcal{C}\mathrm{art}$, defined by $\underline{M}(c) = \mathcal{M}\mathrm{fd}(c, M)$. ◁

Example 2.4 Let $\mathcal{O}\mathrm{p}$ be the U-small category whose objects are open subsets of $\mathbb{R}^n$ for any (varying) $n \in \mathbb{N}_0$ and whose morphisms are all smooth maps between such open subsets. The category $\mathcal{O}\mathrm{p}$ has finite products and carries a coverage whose covering families are the open coverings. The resulting site is denoted $(\mathcal{O}\mathrm{p}, \tau_{op})$; observe that it is closed. ◁

Suppose that $\mathcal{C}$ comes endowed with a Grothendieck coverage τ. Let $\mathcal{Y} : \mathcal{C} \to \widehat{\mathcal{C}}$, $c \mapsto \mathcal{Y}_c$, denote the Yoneda embedding of $\mathcal{C}$.

Definition 2.5 *A morphism* $\pi : Y \to X$ *in* $\widehat{\mathcal{C}}$ *is a* τ-local epimorphism *if for every morphism* $\varphi : \mathcal{Y}_c \to X$ *there exists a covering* $\{f_j : c_j \to c\}_{j \in J} \in \tau(c)$ *and morphisms* $\{\varphi_j : \mathcal{Y}_{c_j} \to Y\}_{j \in J}$ *with* $\pi \circ \varphi_j = \varphi \circ f_j$ *for all* $j \in J$.

Proposition 2.6 *For any site* $(\mathcal{C}, \tau)$, *the class of* τ*-local epimorphisms is stable under pullback. In particular, the collection of* τ*-local epimorphisms defines a coverage* $\widehat{\tau}$ *on* $\widehat{\mathcal{C}}$.

Proof. Consider a pullback diagram

$$\begin{tikzcd} A \times_X Y \arrow[r] \arrow[d, "\pi'"'] & Y \arrow[d, "\pi"] \\ A \arrow[r, "g"'] & X \end{tikzcd}$$

in $\widehat{\mathcal{C}}$, where π is a τ-local epimorphism. We will show that π' is a τ-local epimorphism. Let $\varphi : \mathcal{Y}_c \to A$ be an arbitrary morphism. Since π is a τ-local epimorphism, we find a covering $\{f_j : c_j \to c\}_{j \in J} \in \tau(c)$ and morphisms $\{\psi_j : \mathcal{Y}_{c_j} \to Y\}_{j \in J}$ that are local lifts of $g \circ \varphi$. Then we have

$$g \circ \varphi \circ f_j = \pi \circ \psi_j \qquad \forall\, j \in J\,.$$

The universal property of the pullback thus yields uniquely determined morphisms $\varphi_j : \mathcal{Y}_{c_j} \to A \times_A Y$, which satisfy $\pi' \circ \varphi_j = \varphi$ for all $j \in J$. Hence, π' is a τ-local epimorphism.

A covering family in $\widehat{\tau}$ is of the form $\{\pi\colon Y \to X\}$, where π is a τ-local epimorphism. The coverage property (Definition 2.1(1)) is then precisely the statement that τ-local epimorphisms are stable under pullback. $\square$

Note that in the Grothendieck site $(\widehat{\mathcal{C}}, \widehat{\tau})$ every covering family consists of a single morphism. We list some general properties of τ-local epimorphisms:

Lemma 2.7 *Let $(\mathcal{C}, \tau)$ be a site.*

(1) Consider morphisms $p \in \widehat{\mathcal{C}}(Y, X)$, $q \in \widehat{\mathcal{C}}(Z, Y)$. If $p \circ q$ is a τ-local epimorphism, then so is p.

(2) For any covering family $\{c_i \to c\}_{i\in I}$, the induced morphism $\coprod_{i\in I} \mathcal{Y}_{c_i} \to \mathcal{Y}_c$ is a τ-local epimorphism.

(3) τ-local epimorphisms are stable under colimits: let $\mathcal{J}$ be a U-small category, let $D', D\colon \mathcal{J} \to \widehat{\mathcal{C}}$ be diagrams in $\widehat{\mathcal{C}}$, and let $\pi\colon D' \to D$ be a morphism of diagrams such that each component $\pi_j\colon D'_j \to D_j$ is a τ-local epimorphism, for any $j \in \mathcal{J}$. Then, the induced morphism $\operatorname{colim}\pi\colon \operatorname{colim} D' \to \operatorname{colim} D$ is a τ-local epimorphism.

(4) The site $(\mathcal{C}, \tau)$ is closed if and only if τ-local epimorphisms are stable under composition. In that case, $(\widehat{\mathcal{C}}, \widehat{\tau})$ is closed.

Proof. Claims (1) and (2) follow straightforwardly from the definition of τ-local epimorphisms.

To see (3), consider an arbitrary morphism $\varphi\colon \mathcal{Y}_c \to \operatorname{colim} D$. Since the functor $\widehat{\mathcal{C}}(\mathcal{Y}_c, -)\colon \widehat{\mathcal{C}} \to \mathcal{S}\mathrm{et}_U$ preserves colimits, φ must factor through D_j for some $j \in \mathcal{J}$. Since $\pi_j\colon D'_j \to D_j$ is a τ-local epimorphism by assumption, there exists a covering family $\{c_i \to c\}_{i\in I} \in \tau(c)$ and lifts $\varphi_i\colon \mathcal{Y}_{c_i} \to D'_j$ of φ for each $i \in I$. Composing each φ_i with the cocone morphism $D'_j \to \operatorname{colim} D'_j$ yields morphisms $\mathcal{Y}_{c_i} \to \operatorname{colim} D'$ as desired.

For (4), it readily follows from the closedness of $(\mathcal{C}, \tau)$ that τ-local epimorphisms are stable under composition. On the other hand, assume that τ-local epimorphisms are stable under composition. Consider an object $c \in \mathcal{C}$, a covering family $\{f_i\colon c_i \to c\}_{i\in I}$, and for each $i \in I$ a covering family $\{c_{i,k} \to c_i\}_{k\in K_i}$. By claims (2) and (3), we thus obtain τ-local epimorphisms

$$\coprod_{i\in I} \coprod_{k\in K_i} \mathcal{Y}_{c_{i,k}} \longrightarrow \coprod_{i\in I} \mathcal{Y}_{c_i} \longrightarrow \mathcal{Y}_c\,.$$

By assumption, their composition is a τ-local epimorphism again, and hence it follows (again using that $\widehat{\mathcal{C}}(\mathcal{Y}_c, -)$ preserves colimits) that $(\mathcal{C}, \tau)$ is closed. $\square$

Let $(\mathcal{C}, \tau)$ be a site, and consider a covering $\mathcal{U} = \{c_i \to c\}_{i\in I}$. We can form its *Čech nerve*, which is the simplicial object in $\widehat{\mathcal{C}}$ whose level-n object reads as

$$\check{C}\mathcal{U}_n \coloneqq \coprod_{i_0,\ldots,i_n\in I} C_{i_0\ldots i_n}\,, \quad \text{with} \quad C_{i_0\ldots i_n} \coloneqq \mathcal{Y}_{c_{i_0}} \underset{\mathcal{Y}_c}{\times} \cdots \underset{\mathcal{Y}_c}{\times} \mathcal{Y}_{c_{i_n}} \quad \in \widehat{\mathcal{C}}\,. \tag{2.8}$$

Note that $C_{i_0\ldots i_n}$ is an element in $\widehat{\mathcal{C}}$ which is not necessarily representable as soon as $n \neq 0$. The simplicial structure morphisms are given by projecting out or doubling the i-th factor, respectively. Observe that we have $C_i = \mathcal{Y}_{c_i}$ for each $i \in I$. Depending on the context we will view the Čech nerve $\check{C}\mathcal{U}$ either as a simplicial object $\check{C}\mathcal{U}_\bullet$ in $\widehat{\mathcal{C}}$ or as an augmented simplicial object $\check{C}\mathcal{U}_\bullet \to \mathcal{Y}_c$ in $\widehat{\mathcal{C}}$.

Definition 2.9 *Let $(\mathcal{C}, \tau)$ be a site, and let $X \in \widehat{\mathcal{C}}$ be a presheaf on $\mathcal{C}$. Then, X is called a* sheaf *on $(\mathcal{C}, \tau)$ if for every object $c \in \mathcal{C}$ and for every covering family $\{f_i\colon c_i \to c\}_{i\in I} \in \tau(c)$ the diagram*

$$X(c) = \widehat{\mathcal{C}}(\mathcal{Y}_c, X) \xrightarrow{X(f_i)} \prod_{i_0\in I} \widehat{\mathcal{C}}(C_{i_0}, X) \rightrightarrows \prod_{i_0,i_1\in I} \widehat{\mathcal{C}}\big(C_{i_0 i_1}, X\big)$$

is an equaliser diagram in $\mathcal{S}\mathrm{et}$. *The* category $\mathcal{S}\mathrm{h}(\mathcal{C},\tau)$ of sheaves on $(\mathcal{C},\tau)$ *is the full subcategory of* $\widehat{\mathcal{C}}$ *on the sheaves.*

Remark 2.10 One can check that this definition of a sheaf is equivalent to [BH11, Def. 14]; the 'compatible collections of plots of X' of [BH11] are in canonical bijection with the elements of the equaliser in Definition 2.9. $\triangleleft$

2.2 Concrete sites and diffeological spaces

In this section we focus on a special class of sites and on a special class of sheaves thereon, which frequently occur in infinite-dimensional geometry, for instance. The material in this section is relevant only to Section 4 and can otherwise be skipped. Let $\mathcal{C}$ be a U-small category with a Grothendieck coverage τ.

Definition 2.11 ([BH11, Def. 17]) *A site* $(\mathcal{C},\tau)$ *is called* subcanonical *if every representable presheaf on* $\mathcal{C}$ *is a sheaf.*

Let $c \in \mathcal{C}$ be an arbitrary object. We write $\mathrm{Ev}_c = \widehat{\mathcal{C}}(\mathcal{Y}_c, -) \colon \widehat{\mathcal{C}} \to \mathcal{S}\mathrm{et}_U$ for the functor that evaluates a presheaf on $\mathcal{C}$ at c. On morphisms $\varphi \in \widehat{\mathcal{C}}(X,Y)$ the functor Ev_c acts as $\varphi \mapsto \varphi_{|c}$, i.e. it sends a natural transformation to its component at the object $c \in \mathcal{C}$.

Definition 2.12 ([BH11, Def. 18]) *A site* $(\mathcal{C},\tau)$ *is* concrete *if it satisfies the following conditions:*

(1) $\mathcal{C}$ *has a terminal object* $*$.

(2) The functor $\mathrm{Ev}_* = \mathcal{C}(*,-) \colon \mathcal{C} \to \mathcal{S}\mathrm{et}_U$ *is faithful.*

(3) $(\mathcal{C},\tau)$ *is subcanonical.*

(4) Every covering family $\{f_i \colon c_i \to c\}_{i\in I} \in \tau(c)$ *in* $\mathcal{C}$ *is* jointly surjective*: we have*

$$\bigcup_{i\in I} (f_i)_*\big(\mathcal{C}(*,c_i)\big) = \mathcal{C}(*,c)\,.$$

Example 2.13 The site $(\mathcal{C}\mathrm{art},\tau_{dgop})$ from Example 2.3 is concrete: first, note that $\mathcal{C}\mathrm{art}$ has a terminal object $* = \mathbb{R}^0$. Further, the functor $\mathcal{C}\mathrm{art}(\mathbb{R}^0,-) \colon \mathcal{C}\mathrm{art} \to \mathcal{S}\mathrm{et}_U$ just forgets the smooth structure on $c \in \mathcal{C}\mathrm{art}$; thus, it is faithful. Since smooth maps to a target manifold form a sheaf with respect to open coverings, $(\mathcal{C}\mathrm{art},\tau_{dgop})$ is subcanonical. Finally, the fact that covering families in $(\mathcal{C}\mathrm{art},\tau_{dgop})$ are jointly surjective is immediate from the definition of τ_{dgop}. Analogously, one checks that the site $(\mathcal{O}\mathrm{p},\tau_{op})$ from Example 2.4 is concrete. $\triangleleft$

Definition 2.14 ([BH11, Def. 19, Def. 46]) *A presheaf* X *on a concrete site* $(\mathcal{C},\tau)$ *is called* concrete *if for every object* $c \in \mathcal{C}$ *the map*

$$\mathrm{Ev}_* \colon X(c) \cong \widehat{\mathcal{C}}(\mathcal{Y}_c, X) \longrightarrow \mathcal{S}\mathrm{et}_U\big(\mathcal{Y}_c(*), X(*)\big)\,, \qquad \varphi \longmapsto \varphi_{|*}$$

is injective. In this case, we define the subset $\mathrm{Plot}_X(c) \coloneqq \mathrm{Ev}_*(X(c)) \subset \mathcal{S}\mathrm{et}_U\big(\mathcal{Y}_c(*), X(*)\big)$ *of* plots of X over $c \in \mathcal{C}$.

Thus, for $X \in \widehat{\mathcal{C}}$ concrete there is a canonical bijection $X(c) \cong \mathrm{Plot}_X(c)$ for any $c \in \mathcal{C}$. This is the most crucial feature of concrete presheaves $X \in \widehat{\mathcal{C}}$: the set of sections of X over any object $c \in \mathcal{C}$ can be identified with a subset of the maps of sets $\mathcal{Y}_c(*) \to X(*)$. Therefore, for a concrete presheaf X on $\mathcal{C}$ it makes sense to call $\mathrm{Ev}_* X = X(*) \in \mathcal{S}\mathrm{et}_U$ the *underlying set of* X.

Lemma 2.15 *Let* $(\mathcal{C},\tau)$ *be a concrete site.*

(1) Given any concrete presheaf $X \in \widehat{\mathcal{C}}$, the assignment $c \mapsto \mathrm{Plot}_X(c)$ is a concrete presheaf on $\mathcal{C}$, and there is a canonical isomorphism $X \cong \mathrm{Plot}_X$. in $\widehat{\mathcal{C}}$

(2) Let $\widehat{\mathcal{C}}_c \subset \widehat{\mathcal{C}}$ denote the full subcategory on the concrete presheaves. We obtain an auto-equivalence $\mathrm{Plot}\colon \widehat{\mathcal{C}}_c \to \widehat{\mathcal{C}}_c$*. It is isomorphic to the identity functor and preserves sheaves. Explicitly,* Plot *acts on objects as $X \mapsto \mathrm{Plot}_X$, and it sends a morphism $\psi \in \widehat{\mathcal{C}}(X, Y)$ to the morphism* $\mathrm{Plot}_\psi\colon \mathrm{Plot}_X \to \mathrm{Plot}_Y$ *whose component on $c \in \mathcal{C}$ is given by the composition*

$$X(c) \xrightarrow{\mathrm{Ev}_*} \mathrm{Plot}_X(c) \xrightarrow{\psi} \mathrm{Plot}_Y(c) \xrightarrow{\mathrm{Ev}_*^{-1}} Y(c)\,.$$

(3) For any $X \in \widehat{\mathcal{C}}_c$ and any $c \in \mathcal{C}$, the set $\mathrm{Plot}_X(c)$ contains all constant maps $\mathcal{Y}_c(*) \to X(*)$.

Proof. Ad (1): Let $X \in \widehat{\mathcal{C}}$ be concrete, and let $f\colon c' \to c$ be a morphism in $\mathcal{C}$. This defines a map

$$\mathrm{Plot}_X(f)\colon \mathrm{Plot}_X(c) \longrightarrow \mathrm{Plot}_X(c')\,, \qquad \varphi_{|*} \longmapsto \varphi_{|*} \circ f_{|*} = (\varphi \circ f)_{|*}\,,$$

where $\varphi \in X(c) \cong \widehat{\mathcal{C}}(\mathcal{Y}_c, X)$. The functoriality of $c \mapsto \mathrm{Plot}_X(c)$ is evident from this prescription; it stems from the composition law for natural transformations.

Ad (2): The functoriality of $X \mapsto \mathrm{Plot}_X$ is again evident from the definition of Plot. The canonical isomorphism $X \cong \mathrm{Plot}_X$ is induced by Ev_*, which is natural again by the composition law for natural transformations. Since Plot_X is isomorphic to a concrete presheaf, it is concrete itself. Thus, Plot is an endofunctor on $\widehat{\mathcal{C}}_c$ which is isomorphic to the identity functor; hence, Plot is an auto-equivalence. If X is even a sheaf, then so is Plot_X, since it is isomorphic to a sheaf.

Ad (3): Observe that $\mathcal{Y}_*(*) \cong * \in \mathcal{S}\mathrm{et}_U$. The canonical morphism $c \to *$ in $\mathcal{C}$ induces the inclusion $X(*) \hookrightarrow \mathrm{Plot}_X(c)$ as the constant maps $\mathcal{Y}_c(*) \to X(*)$. □

Definition 2.16 ([BH11, Def. 20]) *Let $(\mathcal{C}, \tau)$ be a concrete site. The* category $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$ of $(\mathcal{C}, \tau)$-spaces *is the full subcategory of $\widehat{\mathcal{C}}$ on the concrete sheaves. We refer to $(\mathcal{C}\mathrm{art}, \tau_{dgop})$-spaces as* diffeological spaces*, and we abbreviate $\mathcal{D}\mathrm{fg} := \mathcal{D}\mathrm{fg}(\mathcal{C}\mathrm{art}, \tau_{dgop})$.*

Corollary 2.17 *Let $(\mathcal{C}, \tau)$ be a concrete site, and let $X, Y \in \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$. There is a canonical bijection*

$$\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)(X, Y) \cong \big\{\psi\colon X(*) \to Y(*) \,\big|\, \psi \circ \mathrm{Ev}_* f \in \mathrm{Plot}_Y(c)\ \forall\, f \in \mathrm{Plot}_X(c)\big\}\,.$$

We thus generalise the definition of diffeological spaces from [IZ13] to arbitrary concrete sites:

Definition 2.18 *Let $(\mathcal{C}, \tau)$ be a concrete site. We define a category $\mathcal{D}\mathrm{fg}^{IZ}(\mathcal{C}, \tau)$ as follows. Its objects are pairs (S, Plot_S) of a set $S \in \mathcal{S}\mathrm{et}_U$ and a functor $\mathrm{Plot}_S\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_U$ such that*

(1) $\mathrm{Plot}_S(c) \subset \mathcal{S}\mathrm{et}_U(\mathrm{Ev}_*\mathcal{Y}_c, S)$ for all $c \in \mathcal{C}$; elements of $\mathrm{Plot}_S(c)$ for $c \in \mathcal{C}$ are called plots of *(S, Plot_S),*

(2) $\mathrm{Plot}_S(c)$ contains all constant maps $\mathrm{Ev}_*\mathcal{Y}_c \to S$, and

(3) Plot_S is a sheaf on $(\mathcal{C}, \tau)$, where for $f \in \mathcal{C}(c, d)$ the map $\mathrm{Plot}_S(f)$ is precomposition by $\mathrm{Ev}_*\mathcal{Y}_f$.

A morphism $(S, \mathrm{Plot}_S) \to (T, \mathrm{Plot}_T)$ is a map $\psi\colon S \to T$ such that, for every $c \in \mathcal{C}$ and every $\varphi \in \mathrm{Plot}_S(c)$, we have $\psi \circ f \in \mathrm{Plot}_T(c)$.

Observe that there is a pair of functors

$$F : \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau) \rightleftarrows \mathcal{D}\mathrm{fg}^{IZ}(\mathcal{C}, \tau) : G$$

where $F(X) = (X(*), \mathrm{Plot}_X)$ and $G(S, \mathrm{Plot}_S) = \mathrm{Plot}_S$. These satisfy $F \circ G = 1$ and $G \circ F \cong 1$ by Lemma 2.15(2). This proves

Proposition 2.19 *For any concrete site $(\mathcal{C}, \tau)$, there is an equivalence $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau) \cong \mathcal{D}\mathrm{fg}^{IZ}(\mathcal{C}, \tau)$.*

From now on we drop the distinction between these two categories and just talk about $(\mathcal{C}, \tau)$-spaces. We will pass freely between the description of $(\mathcal{C}, \tau)$-spaces via concrete sheaves X and via the pair $(X(*), \mathrm{Plot}_X)$.

Remark 2.20 In [IZ13], diffeological spaces are defined as concrete presheaves on $(\mathcal{O}\mathrm{p}, \tau_{op})$, whereas here we define them on $(\mathcal{C}\mathrm{art}, \tau_{dgop})$. However, the two categories are equivalent because the canonical inclusion of $(\mathcal{C}\mathrm{art}, \tau_{dgop})$ into $(\mathcal{O}\mathrm{p}, \tau_{op})$ is an inclusion of a dense subsite. For any manifold M, the presheaf $\underline{M}$ from Example 2.3 is a diffeological space. $\triangleleft$

The relation between Sections 3 and Section 4 relies on the following observation.

Remark 2.21 The category $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$, together with the collection of those τ-local epimorphisms whose source and target are $(\mathcal{C}, \tau)$-spaces, form a site which is contained in the site $(\widehat{\mathcal{C}}, \widehat{\tau})$ as a full subcategory. The τ-local epimorphisms between $(\mathcal{C}, \tau)$-spaces are also called *subductions* [IZ13]. $\triangleleft$

We now turn to categorical constructions in categories of $(\mathcal{C}, \tau)$-spaces, where the description via plots will be very helpful.

Proposition 2.22 ([BH11, Section 5.3]) *Let $(\mathcal{C}, \tau)$ be a concrete site. There exists an adjunction*

$$\mathsf{Dfg} : \widehat{\mathcal{C}} \rightleftarrows \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau) : \iota \qquad (\perp)$$

whose right adjoint is a fully faithful inclusion, i.e. $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau) \subset \widehat{\mathcal{C}}$ is a reflective localisation.

Corollary 2.23 ([BH11]) *For any concrete site $(\mathcal{C}, \tau)$, the category $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$ of $\mathcal{C}$-spaces is cartesian closed and has all U-small limits and colimits. Limits of diagrams $D \colon \mathcal{I} \to \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$ can be computed in $\widehat{\mathcal{C}}$, and colimits of diagrams $D \colon \mathcal{I} \to \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$ can be computed via*

$$\mathrm{colim}^{\mathcal{D}\mathrm{fg}(\mathcal{C},\tau)}(D) \cong \mathsf{Dfg}\big(\mathrm{colim}^{\widehat{\mathcal{C}}}(\iota \circ D)\big)\,.$$

Here the superscripts indicate in which category the respective colimit is formed. For later use, we give an explicit prescription for computing colimits in $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$:

Lemma 2.24 *Let $(\mathcal{C}, \tau)$ be a site. The underlying-set functor $\mathrm{Ev}_* \colon \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau) \to \mathcal{S}\mathrm{et}_U$, $X \mapsto X(*)$, is part of a triple adjunction $Disc \dashv \mathrm{Ev}_* \dashv Indisc$. Consequently, Ev_* preserves both limits and colimits.*

Proof. Let $S \in \mathcal{S}\mathrm{et}_U$ be a set. We define objects $Disc(S), Indisc(S) \in \widehat{\mathcal{C}}$ by setting

$$\begin{aligned} Disc(S)(c) &\coloneqq \{\varphi \colon \mathcal{Y}_c(*) \to S \,|\, \varphi \text{ is constant}\}\,, \\ Indisc(S)(c) &\coloneqq \mathcal{S}\mathrm{et}_U\big(\mathcal{Y}_c(*), S\big)\,. \end{aligned}$$

It is straightforward to check that these define $(\mathcal{C}, \tau)$-spaces.

Given $X \in \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$, recall the canonical bijection from Corollary 2.17. We find that $\mathrm{Plot}_{Disc(S)}(c)$ is the set of constant maps $\mathcal{Y}_c(*) \to S$. Thus, any map $\psi \in \mathcal{S}\mathrm{et}_U(S, X(*))$ satisfies that $\psi \circ \varphi \colon \mathcal{Y}_c(*) \to X(*)$ is constant, for every plot $\varphi \in \mathrm{Plot}_{Disc(S)}(c)$. Hence, $\psi \circ \varphi$ is a plot of X by Lemma 2.15(3). The adjunction $Disc \dashv \mathrm{Ev}_*$ thus follows from Corollary 2.17. Analogously one shows that $\mathrm{Ev}_* \dashv Indisc$. $\square$

Proposition 2.25 *Let $(\mathcal{C}, \tau)$ be a* closed *site (cf. Definition 2.2), and let $D\colon \mathcal{J} \to \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$ be a U-small diagram. The colimit of D can be described as follows: its underlying set reads as*

$$\mathrm{Ev}_*\big(\mathrm{colim}_{\mathcal{J}}^{\mathcal{D}\mathrm{fg}(\mathcal{C},\tau)} D\big) = \mathrm{colim}_{\mathcal{J}}^{\mathcal{S}\mathrm{et}_U}(\mathrm{Ev}_* \circ D)\,.$$

A plot of $\mathrm{colim}_{\mathcal{J}}^{\mathcal{D}\mathrm{fg}(\mathcal{C},\tau)} D$ is a map of sets $\varphi\colon \mathcal{Y}_c(*) \to \mathrm{colim}_{\mathcal{J}}^{\mathcal{S}\mathrm{et}_U}(\mathrm{Ev}_* \circ D)$ such that there exists a covering $\{f_i\colon c_i \to c\}_{i\in I}$ in $(\mathcal{C}, \tau)$, a map $i \mapsto j_i$ from I to the objects of $\mathcal{J}$, and a family of maps $\{\varphi_i\colon \mathcal{Y}_{c_i}(*) \to D(j_i)(*)\}_{i\in I}$ such that the diagram

$$\begin{tikzcd}
\mathcal{Y}_{c_i}(*) \arrow[r, "\varphi_i"] \arrow[d, "\mathcal{Y}_{f_i}(*)"'] & D(j_i)(*) \arrow[d] \\
\mathcal{Y}_c(*) \arrow[r] & \mathrm{colim}_{\mathcal{J}}^{\mathcal{S}\mathrm{et}_U}(\mathrm{Ev}_* \circ D)
\end{tikzcd} \tag{2.26}$$

in $\mathcal{S}\mathrm{et}_U$ commutes for every $i \in I$, and such that φ_i is a plot of $D(j_i)$ for every $i \in I$.

Proof. Let $Z(*) := \mathrm{colim}_{\mathcal{J}}^{\mathcal{S}\mathrm{et}_U}(\mathrm{Ev}_* \circ D)$. For $c \in \mathcal{C}$, let $Z(c)$ denote the set of maps $\varphi\colon \mathcal{Y}_c(*) \to Z(*)$ with the above lifting property. We first show that $c \mapsto Z(c)$ defines a presheaf on $\mathcal{C}$. Thus, we consider a morphism $f \in \mathcal{C}(c', c)$; it induces a map $f_{|*}\colon \mathcal{Y}_{c'}(*) \to \mathcal{Y}_c(*)$. The morphism $Z(f)\colon Z(c) \to Z(c')$ is given by $\varphi \mapsto \varphi \circ f_{|*}$. Hence, we need to show that for any $\varphi \in Z(c)$ and $f \in \mathcal{C}(c', c)$, the composition $\varphi \circ f_{|*}$ again has the lifting property. This follows readily from the factorisation property of covering families; see Definition 2.1(1).

The presheaf Z is concrete since we have constructed the value $Z(c)$ as a subset of $\mathcal{S}\mathrm{et}_U(\mathcal{Y}_c(*), Z(*))$ and since constant maps to $Z(*)$ trivially have the local lifting property.

Next, we show that Z is a sheaf. To that end, suppose that $\{c_i \to c\}_{i\in I}$ is a covering family for c and that we are given morphisms $\{\varphi_i\colon \mathcal{Y}_{c_i} \to Z\}_{i\in I}$ such that the diagram

$$\begin{tikzcd}
\mathcal{Y}_{c_i} \underset{\mathcal{Y}_c}{\times} \mathcal{Y}_{c_j} \arrow[r] \arrow[d] & \mathcal{Y}_{c_i} \arrow[d, "\varphi_i"] \\
\mathcal{Y}_{c_j} \arrow[r, "\varphi_j"'] & Z
\end{tikzcd}$$

commutes for every $i, j \in I$. Since evaluation at any object of $\mathcal{C}$ is a limit-preserving functor $\widehat{\mathcal{C}} \to \mathcal{S}\mathrm{et}_U$, these data induce afamily of maps $\{\mathcal{Y}_{c_i}(*) \to \mathcal{Y}_c(*)\}_{i\in I}$ and maps $\{\varphi_{i|*}\colon \mathcal{Y}_{c_i}(*) \to Z(*)\}_{i\in I}$ such that the diagram

$$\begin{tikzcd}
\mathcal{Y}_{c_i}(*) \underset{\mathcal{Y}_c(*)}{\times} \mathcal{Y}_{c_j}(*) \arrow[r] \arrow[d] & \mathcal{Y}_{c_i}(*) \arrow[d, "\varphi_{i|*}"] \\
\mathcal{Y}_{c_j}(*) \arrow[r, "\varphi_{j|*}"'] & Z(*)
\end{tikzcd}$$

in $\mathcal{S}\mathrm{et}_U$ commutes for every $i, j \in I$. Since $(\mathcal{C}, \tau)$ is a concrete site, the family $\{\mathcal{Y}_{c_i}(*) \to \mathcal{Y}_c(*)\}_{i\in I}$ is jointly surjective, so that these data determine a unique map $\varphi\colon \mathcal{Y}_c(*) \to Z(*)$ such that all diagrams

$$\begin{tikzcd}
\mathcal{Y}_{c_i}(*) \arrow[r] \arrow[dr, "\varphi_{i|*}"'] & \mathcal{Y}_c(*) \arrow[d, "\varphi"] \\
& Z(*)
\end{tikzcd}$$

commute. We claim that $\varphi \in Z(c)$. First, for each $i \in I$, let $\{c_{i,k} \to c_i\}_{k \in K_i}$ be a covering family for c_i such that there exist lifts $\varphi_{i,k} \colon \mathcal{Y}_{c_i} \to D(j_{i,k})$ of the morphisms $\varphi_i \colon \mathcal{Y}_{c_i} \to Z$ as in (2.26). By the assumption that $(\mathcal{C}, \tau)$ is closed we can choose a covering family $\{c_l \to c\}_{l \in L}$ of c such that each morphism $c_l \to c$ factors through some morphism $c_{i,k} \to c_i \to c$. The compositions $c_l \to c_{i,k} \to D(j_{i,k})$ then provide the desired lifts of φ.

Our next step is to make Z into the vertex of a cocone under the diagram D. Consider the map $\iota_j \colon D(j)(*) \to Z(*)$ induced by the definition of $Z(*)$ as the colimit in $\mathcal{S}\mathrm{et}_U$ of the diagram $\mathrm{Ev}_* \circ D$. This map induces a morphism in $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$ by Corollary 2.17. Explicitly, given any plot $\mathcal{Y}_c(*) \to D(j)(*)$, the composition $\mathcal{Y}_c \to Z(*)$ trivially admits a lifting along a covering family of c to maps to $D(j)(*)$. Thus, composition by ι_j sends plots of $D(j)$ to plots of Z.

Finally, we need to show that Z is a colimit of the diagram $D \colon \mathcal{J} \to \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$. To see this, let $\{\psi_j \colon D(j) \to A\}_{j \in \mathcal{J}}$ be a cocone under D in the category $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$. Evaluating at $* \in \mathcal{C}$, we obtain a cocone $\psi_{j|*} \colon D(j)(*) \to A(*)$ under the digram $\mathrm{Ev}_* \circ D$ in $\mathcal{S}\mathrm{et}_U$. By construction, the set $Z(*)$ presents a colimit of that diagram; hence these data induce a unique map of sets $\psi \colon Z(*) \to A(*)$. Recall that for any pair of objects $X, Y \in \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$ the map $\mathrm{Ev}_* \colon \mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)(X, Y) \to \mathcal{S}\mathrm{et}_U(X(*), Y(*))$, $\phi \mapsto \phi_{|*}$ is injective (cf. Corollary 2.17). It follows that if the map ψ gives rise to a morphism of $(\mathcal{C}, \tau)$-spaces, then that morphism is the unique morphism in $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$ inducing a morphism of cocones under D.

Therefore, we are left to show that $\psi \colon Z(*) \to A(*)$ gives rise to a morphism in $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$. By Corollary 2.17, that is equivalent to showing that composition by ψ sends plots of Z to plots of A. Thus, let $\varphi \colon \mathcal{Y}_c \to Z$ be an arbitrary morphism. As before, by definition of Z, we find a covering family $\{f_i \colon c_i \to c\}_{i \in I}$ and lifts $\varphi_i \colon \mathcal{Y}_{c_i} \to D(j_i)$ of φ along the morphisms f_i as in (2.26). We claim that $\{\psi_{j_i} \circ \varphi_i\}_{i \in I}$ is a compatible family of morphisms $\mathcal{Y}_{c_i} \to A$ in $\widehat{\mathcal{C}}$. For $i, k \in I$, consider the diagram

$$
\begin{tikzcd}
 & \mathcal{Y}_{c_i} \arrow[r, "\varphi_i"] \arrow[d, "f_i"] & D(j_i) \arrow[d, "\iota_{j_i}"] \arrow[dr, bend left, "\psi_{j_i}"] & \\
\mathcal{Y}_{c_i} \underset{\mathcal{Y}_c}{\times} \mathcal{Y}_{c_k} \arrow[ur] \arrow[dr] & \mathcal{Y}_c \arrow[r, "\varphi"] & Z \arrow[r, dashed, "\psi"] & A \\
 & \mathcal{Y}_{c_k} \arrow[r, "\varphi_k"'] \arrow[u, "f_k"] & D(j_k) \arrow[u, "\iota_{j_k}"] \arrow[ur, bend right, "\psi_{j_k}"'] &
\end{tikzcd}
\tag{2.27}
$$

in $\widehat{\mathcal{C}}$. The left-hand triangle commutes by definition of the pullback. The two central squares commute by definition of φ_i. We do not yet know whether ψ is actually a morphism in $\widehat{\mathcal{C}}$. However, evaluating the whole diagram at $* \in \mathcal{C}$, we obtain a diagram in $\mathcal{S}\mathrm{et}_U$ in which the two right-hand triangles also commute, since ψ is a morphism of cocones under the diagram $\mathrm{Ev}_* \circ D$ in $\mathcal{S}\mathrm{et}_U$. Thus, we infer that the outer hexagon in (2.27) commutes as maps on the underlying sets. Recalling that the map which sends morphisms in $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$ to maps of underlying sets is injective, it thus follows that the outer hexagon is commutative already as a diagram in $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$ (i.e. before evaluating at the terminal object).

Consequently, $\{\psi_{j_i} \circ \varphi_i\}_{i \in I}$ is indeed a compatible family of morphisms $\mathcal{Y}_{c_i} \to A$. Thus, as A is a sheaf, it defines a unique morphism $\varrho \colon \mathcal{Y}_c \to A$ in $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$. It follows from the construction of ϱ that $\mathrm{Ev}_* \varrho = \varrho_{|*} = \psi \circ \varphi_{|*}$; hence, composition with ψ sends plots of Z to plots of A, so that ψ gives rise to a morphism in $\mathcal{D}\mathrm{fg}(\mathcal{C}, \tau)$. $\square$

We now specialise to the case $(\mathcal{C}, \tau) = (\mathcal{C}\mathrm{art}, \tau_{dgop})$. Recall that we refer to $(\mathcal{C}\mathrm{art}, \tau_{dgop})$-spaces as *diffeological spaces* and that we write $\mathcal{D}\mathrm{fg} = \mathcal{D}\mathrm{fg}(\mathcal{C}\mathrm{art}, \tau_{dgop})$. We will denote the underlying set $\mathcal{Y}_c(*)$ of a cartesian space $c \in \mathcal{C}\mathrm{art}$ again by c.

Lemma 2.28 *The embedding* $\iota\colon \mathcal{D}\mathrm{fg} \to \widehat{\mathcal{C}\mathrm{art}}$ *preserves coproducts.*

Proof. Let $\{X_k\}_{k\in K}$ be a family of diffeological spaces. For $c \in \mathcal{C}\mathrm{art}$, a map $\varphi\colon c \to X := \coprod_{k\in K} X_k$ is a plot if there exists an open covering $\{c_i \to c\}_{i\in I}$ and plots $\{\varphi_i\colon c_i \to X_{k(i)}\}_{i\in I}$ such that the diagram

$$\begin{tikzcd} c_i \arrow[r, "\varphi_i"] \arrow[d, hook] & X_{k(i)} \arrow[d, hook] \\ c \arrow[r, "\varphi"'] & \coprod_{k\in K} X_i k = X \end{tikzcd}$$

commutes for every $i \in I$. Let $i, j \in I$ such that $c_i \cap c_j \neq \emptyset$ (where we identify c_i with its image in c). Consider a point $z \in c_i \cap c_j$. We have $\varphi_i(z) = \varphi(z) = \varphi_j(z)$ in $X(*)$. Since $X(*) = \coprod_{k\in K} X_k(*)$, we infer that $k(i) = k(j)$.

Now consider $x, y \in c$ arbitrary. Let $x \in c_i$ and $y \in c_j$, for some appropriate $i, j \in I$. Since c is a connected manifold, we find a smooth path $\gamma\colon [0,1] \to c$ from x to y, which can even be chosen to be an embedding. (For instance, one chooses a diffeomorphism $f\colon \mathbb{R}^n \to c$ and then sets $\gamma(t) = f(tf(y) + (1-t)f(x))$.) Since the image of γ is a compact subset of c, there exists a finite subset $\{i_0, \ldots, i_n\} \subset I$ satisfying that $i_0 = i$, $i_n = j$, that $\gamma([0,1]) \subset \bigcup_{i=0}^n c_i$, and that $c_{i_l} \cap c_{i_{l+1}} \neq \emptyset$ for all $l = 0, \ldots, n-1$. Using the above argument, it now follows by induction that $k(i) = k(j)$. Hence, all plots $\varphi_i\colon c_i \to X_{k(i)}$ take their values in a single diffeological space X_k for some $k \in K$. Finally, since the plots of any diffeological space form a sheaf, it now follows that φ itself factors through X_k. $\square$

Remark 2.29 Note that the Lemma 2.28 does not hold if one works over the site $(\mathcal{O}\mathrm{p}, \tau_{op})$, as in [IZ13]. The connectedness of the parameter spaces $c \in \mathcal{C}\mathrm{art}$ is crucial in the proof. $\triangleleft$

3 Sheaves of higher categories

In this section we prove Theorem 1.2 in three steps: we first consider (pre-)sheaves of ∞-groupoids, then of ∞-categories, and finally of (∞, n)-categories for $n \geq 2$. In each case, we first set up the theory of sheaves of (∞, n)-categories which we use to prove our results. To a large extent, the idea for how to define model categories of sheaves of higher categories is already contained in [Bar05, TV05]. We then show how, given a U-small site $(\mathcal{C}, \tau)$, sheaves of higher categories on $(\mathcal{C}, \tau)$ induce sheaves of higher categories on the site $(\widehat{\mathcal{C}}, \widehat{\tau})$, where $\widehat{\mathcal{C}}$ is the category of $\mathcal{S}\mathrm{et}_U$-valued presheaves on $\mathcal{C}$, and where $\widehat{\tau}$ is the Grothendieck pretopology of τ-local epimorphisms.

Remark 3.1 It might be interesting to consider the simplicial model category $\mathcal{S}\mathrm{et}_{\Delta U}^{\mathcal{C}^{\mathrm{op}}}$ in place of $\widehat{\mathcal{C}}$ and view it as a model site in the sense of [TV05], using the τ-local epimorphisms. However, for our present applications, and in order to unravel the relevant concepts, it is sufficient to work with $\widehat{\mathcal{C}}$. $\triangleleft$

3.1 Sheaves of ∞-groupoids

Let $\mathcal{C}$ be a U-small category. Let $\mathcal{H} := \mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}^{\mathrm{op}}}$ denote the category of V-small simplicial presheaves on $\mathcal{C}$ (recall that $\mathcal{S}\mathrm{et}$ without a subscript shall always mean $\mathcal{S}\mathrm{et}_V$). It is enriched, tensored and cotensored over $\mathcal{S}\mathrm{et}_{\Delta}$. Further, since $\mathcal{C}$ is V-small (it is even U-small), $\mathcal{H}$ can be endowed with the projective model structure, turning $\mathcal{H}$ into a model $\mathcal{S}\mathrm{et}_{\Delta}$-category. We denote this model category by $\mathcal{H}_\infty$.

Definition 3.2 *The model structure on $\mathcal{H}_\infty$ is called the* projective model category for presheaves of ∞-groupoids on $\mathcal{C}$. *A* presheaf of ∞-groupoids on $\mathcal{C}$ *is a fibrant object in $\mathcal{H}_\infty$.*

Proposition 3.3 *$\mathcal{H}_\infty$ is a left proper, tractable, model $\mathcal{S}\mathrm{et}_{\Delta}$-category. If $\mathcal{C}$ has finite products, then $\mathcal{H}_\infty$ is additionally a symmetric monoidal model category.*

Proof. The existence and tractability of the projective model structure follow from [Bar10, Thm. 2.14]. $\mathcal{H}_\infty$ is left proper since pushouts and weak equivalences are defined objectwise and cofibrations are (in particular) objectwise cofibrations. Its simplicial enrichment is a consequence of [Bar10, Prop. 4.50], and the fact that $\mathcal{H}_\infty$ is symmetric monoidal follows from [Bar10, Prop. 4.52], using that the Yoneda embedding preserves limits and that $\mathcal{H}_\infty$ is a simplicial model category. □

Note that [Bar10, Prop. 4.52] gives a more general criterion for when a projective model structure inherits a symmetric monoidal structure; however, for us the main case of interest will be where $\mathcal{C}$ admits finite products. Composing the Yoneda embedding $\mathcal{Y}$ on the category $\mathcal{C}$ with the functor $\mathsf{c}_\bullet \colon \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}_{\Delta}$, we obtain a fully faithful functor

$$\mathcal{Y} \colon \mathcal{C} \to \mathcal{H}_\infty\,.$$

Projective model categories of simplicial presheaves have an explicit cofibrant replacement functor Q, found by Dugger [Dug01]. Throughout this paper, we will use this functor for cofibrant replacements in all model categories of simplicial presheaves. The action of Q on $F \in \mathcal{H}_\infty$, reads as

$$(QF)_n \cong \coprod_{c_0,\ldots,c_n \in \mathcal{C}} \mathcal{Y}_{c_0} \times \mathcal{C}(c_0,c_1) \times \cdots \times \mathcal{C}(c_{n-1},c_n) \times F_n(c_n) \tag{3.4}$$

This functor has several good properties, some of which we present in Appendix A.

Now, let $\mathcal{C}$ be endowed with a Grothendieck pretopology τ. Recall that to any covering $\mathcal{U} = \{c_i \to c\}_{i\in I}$ in $(\mathcal{C},\tau)$ gives rise to Čech nerve $\check{C}\mathcal{U}_\bullet \to \mathcal{Y}_c$, as defined in (2.8), which we now view as a morphism in $\mathcal{H}_\infty$.

Definition 3.5 *Let $(\mathcal{C},\tau)$ be a Grothendieck site, and let $\check{\tau}$ denote the class of morphisms in $\mathcal{H}_\infty$ consisting of Čech nerves of coverings in $(\mathcal{C},\tau)$. The* projective model category $\mathcal{H}_\infty^{loc}$ for sheaves of ∞-groupoids on $\mathcal{C}$ *is the $\mathcal{S}\mathrm{et}_{\Delta}$-enriched left Bousfield localisation of $\mathcal{H}_\infty$ at $\check{\tau}$,*

$$\mathcal{H}_\infty^{loc} \coloneqq L_{\check{\tau}/\mathcal{S}\mathrm{et}_{\Delta}} \mathcal{H}_\infty\,.$$

A sheaf of ∞-groupoids *is a fibrant object in $\mathcal{H}_\infty^{loc}$.*

The model structure on $\mathcal{H}_\infty^{loc}$ is also called the *local projective model structure*. In particular, $\mathcal{H}_\infty^{loc}$ is a model $\mathcal{S}\mathrm{et}_{\Delta}$-category. The $\mathcal{S}\mathrm{et}_{\Delta}$-enriched internal hom in both $\mathcal{H}_\infty$ and $\mathcal{H}_\infty^{loc}$ is given by

$$\underline{\mathcal{H}}_\infty(F,G) = \int_{c \in \mathcal{C}^{\mathrm{op}}} (G(c))^{F(c)} \qquad \in \mathcal{S}\mathrm{et}_{\Delta}\,,$$

where the argument of the end is given by the internal hom in $\mathcal{S}\mathrm{et}_{\Delta}$. In particular, there is a natural isomorphism $\underline{\mathcal{H}}_\infty(\mathcal{Y}_c,F) \cong F(c)$ for any $c \in \mathcal{C}$ and any $F \in \mathcal{H}_\infty$.

An object F in $\mathcal{H}_\infty^{loc}$ is fibrant if

(1) $F(c)$ is a Kan complex for every $c \in \mathcal{C}$, and

(2) for every covering family $\mathcal{U} = \{c_i \hookrightarrow c\}_{i \in I}$ in $(\mathcal{C}, \tau)$, the morphism

$$\mathcal{F}(c) \longrightarrow \underset{n \in \mathbb{\Delta}}{\mathrm{holim}}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}} \Big(\cdots \underline{\mathcal{H}}_\infty\big(Q(\check{C}\mathcal{U}_n),\, F\big)\ \cdots \Big) = \underset{n \in \mathbb{\Delta}}{\mathrm{holim}}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}} \Big(\cdots \prod_{i_0, \ldots, i_n \in I} \underline{\mathcal{H}}_\infty(QC_{i_0 \ldots i_n},\, F)\ \cdots \Big)$$

is a weak equivalence in $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$. Here we have used the notation from (2.8) and that Q preserves colimits (it is a left adjoint).

Proposition 3.6 *$\mathcal{H}_\infty^{loc}$ is a proper, tractable, $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$-model category. If $\mathcal{H}_\infty$ is symmetric monoidal (as a model category), then so is $\mathcal{H}_\infty^{loc}$.*

Proof. The first claim follows form [Bar10, Thm. 2.14, Thm 4.56] and the fact that $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} = \mathcal{S}\mathrm{et}_{\mathbb{\Delta} V}$ is a proper, tractable model V-category. The second claim is an application of [Bar10, Thm. 4.58]. ☐

Remark 3.7 If $\mathcal{C}$ has finite products, then the conditions of [Bar10, Prop. 4.52] are satisfied, so that in this case $\mathcal{H}_\infty$, and hence also $\mathcal{H}_\infty^{loc}$, are symmetric monoidal model categories. ◁

Since both $\mathcal{C}$ and $\mathcal{S}\mathrm{et}_U$ are V-small categories, the category $\widehat{\mathcal{C}} = \mathcal{S}\mathrm{et}_U^{\mathcal{C}^{\mathrm{op}}}$ is a V-small category. Thus, we may define the category

$$\widehat{\mathcal{H}} \coloneqq \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}{}^{\widehat{\mathcal{C}}^{\mathrm{op}}}\,.$$

As $\widehat{\mathcal{C}}$ is V-small, $\widehat{\mathcal{H}}$ carries a projective model structure [Bar10]. We let $\widehat{\mathcal{H}}_\infty$ denote the category $\widehat{\mathcal{H}}$ endowed with the projective model structure. Recall from Proposition 2.6 that if $(\mathcal{C}, \tau)$ is a site, then also $(\widehat{\mathcal{C}}, \widehat{\tau})$ is a site. Let $(\widehat{\tau})\check{}$ denote the collection of Čech nerves of coverings in $(\widehat{\mathcal{C}}, \widehat{\tau})$, and define the $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$-enriched left Bousfield localisation

$$\widehat{\mathcal{H}}_\infty^{loc} \coloneqq L_{(\widehat{\tau})\check{}/\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}} \widehat{\mathcal{H}}_\infty\,. \tag{3.8}$$

Both $\widehat{\mathcal{H}}_\infty$ and $\widehat{\mathcal{H}}_\infty^{loc}$ are symmetric monoidal by Proposition 3.3 and Proposition 3.6, since $\widehat{\mathcal{C}}$ has all finite products.

The Yoneda embedding $\mathcal{Y} \colon \mathcal{C} \to \widehat{\mathcal{C}}$ induces a pullback functor

$$\mathcal{Y}^* \colon \widehat{\mathcal{H}} \to \mathcal{H}\,.$$

Since both $\mathcal{C}$ and $\widehat{\mathcal{C}}$ are V-small, and since both $\widehat{\mathcal{H}}$ and $\mathcal{H}$ have all V-small limits and colimits, the functor $\mathcal{Y}^*$ has both a left adjoint $\mathcal{Y}_!$ and a right adjoint $\mathcal{Y}_*$; there is a triple of adjunctions

$$\begin{tikzcd} \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}{}^{\widehat{\mathcal{C}}^{\mathrm{op}}} \arrow[r, "\mathcal{Y}^*" description] & \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}{}^{\mathcal{C}^{\mathrm{op}}} \arrow[l, bend right, "\mathcal{Y}_!"'] \arrow[l, bend left, "\mathcal{Y}_*"] \end{tikzcd}$$

The adjoints can be computed using the usual (co)end formulas for pointwise Kan extensions. In particular, the expression

$$(\mathcal{Y}_* F)(X) = \int_{c \in \mathcal{C}^{\mathrm{op}}} F(c)^{(\widehat{\mathcal{C}})^{\mathrm{op}}(X, \mathcal{Y}_c)} \cong \int_{c \in \mathcal{C}^{\mathrm{op}}} F(c)^{X(c)} \cong \underline{\mathcal{H}}_\infty(X, F)$$

gives an explicit formula for the right adjoint $\mathcal{Y}_*$.

The question we are interested in is whether one of the functors $\mathcal{Y}_!$ or $\mathcal{Y}_*$ maps sheaves of ∞-groupoids on $(\mathcal{C}, \tau)$ to sheaves of ∞-groupoids on $(\widehat{\mathcal{C}}, \widehat{\tau})$. That is, we would like to know whether one of these functors preserves fibrant objects as a functor $\mathcal{H}_\infty^{loc} \to \widehat{\mathcal{H}}_\infty^{loc}$. Since this indicates that we are looking for a right Quillen functor between these model categories, we focus our attention on the right adjoint $\mathcal{Y}_*$. In general there is no reason that $\mathcal{Y}_*$ should even preserve projectively fibrant objects as a functor $\mathcal{H}_\infty \to \widehat{\mathcal{H}}_\infty$. However, if $Q \colon \mathcal{H}_\infty \to \mathcal{H}_\infty$ is a cofibrant replacement functor, then the functor

$$S_\infty^Q \colon \mathcal{H}_\infty \longrightarrow \widehat{\mathcal{H}}_\infty \,, \qquad \big(S_\infty^Q(F)\big)(X) := \underline{\mathcal{H}}(QX, F)$$

does preserve fibrant objects because $\mathcal{H}_\infty$ is a model $\mathcal{S}\mathrm{et}_\Delta$-category. Note that here we view $X \in \widehat{\mathcal{C}}$ as an object in $\mathcal{H}_\infty$ via the canonical inclusion of sets into simplicial sets.

We investigate the functor S_∞^Q further: let $F \in \mathcal{H}_\infty$ and $X \in \widehat{\mathcal{C}}$. Using the observation that QX is a bar construction (Appendix A) we compute

$$\begin{aligned}
(S_\infty^Q F)(X) &= \underline{\mathcal{H}}_\infty(QX, F) \\
&= \underline{\mathcal{H}}_\infty\big(B^{\mathcal{H}_\infty}(*, \mathcal{C}_{/X}, \mathcal{Y}),\, F\big) \\
&\cong C_{\mathcal{S}\mathrm{et}_\Delta}\big(*, (\mathcal{C}_{/X})^{\mathrm{op}}, \underline{\mathcal{H}}_\infty(\mathcal{Y}, F)\big) \\
&\cong C_{\mathcal{S}\mathrm{et}_\Delta}\big(*, (\mathcal{C}_{/X})^{\mathrm{op}}, F\big)\,,
\end{aligned}$$

where $C_{\mathcal{S}\mathrm{et}_\Delta}$ and $B^{\mathcal{H}_\infty}$ are the cobar and bar constructions in $\mathcal{S}\mathrm{et}_\Delta$ and in $\mathcal{H}_\infty$, respectively. If F is projectively fibrant, then the last expression is a model for the homotopy limit [Rie14],

$$(S_\infty^Q F)(X) \cong \mathrm{holim}^{\mathcal{S}\mathrm{et}_\Delta}\big((\mathcal{C}_{/X})^{\mathrm{op}} \xrightarrow{\mathrm{pr}_{\mathcal{C}^{\mathrm{op}}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathcal{S}\mathrm{et}_\Delta\big)\,.$$

Now, using the Yoneda Lemma and [Rie14, Ex. 9.2.11] we conclude

Proposition 3.9 *If $F \in \mathcal{H}_\infty$ is fibrant, $S_\infty^Q F$ agrees with the homotopy right Kan extension of F along the (opposite) Yoneda embedding $\mathcal{Y}^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \hookrightarrow \widehat{\mathcal{C}}^{\mathrm{op}}$:*

$$S_\infty^Q F \cong \mathrm{hoRan}_{\mathcal{Y}^{\mathrm{op}}}(F)\,.$$

In particular, S_∞^Q presents the ∞-categorical right Kan extension along the Yoneda embedding on the underlying ∞-categories of $\mathcal{H}_\infty$ and $\widehat{\mathcal{H}}_\infty$.

Since $\widehat{\mathcal{C}}$ is V-small, $\widehat{\mathcal{C}}$-indexed (co)ends exist in $\mathcal{S}\mathrm{et}_\Delta$. Therefore, a left adjoint to S_∞^Q is given by

$$\widetilde{Re}_\infty^Q \colon \widehat{\mathcal{H}} \to \mathcal{H}\,, \qquad \widetilde{Re}_\infty^Q(\widehat{G}) := \int^{X \in \widehat{\mathcal{C}}^{\mathrm{op}}} \widehat{G}(X) \otimes QX\,.$$

Lemma 3.10 *There are canonical natural isomorphisms*

$$\widetilde{Re}_\infty^Q(\widehat{G}) \cong \int^{c \in \mathcal{C}^{\mathrm{op}}} (\mathcal{Y}^*\widehat{G})(c) \otimes Q(\mathcal{Y}_c) \cong Q \circ \mathcal{Y}^*(\widehat{G}) =: Re_\infty^Q(G)\,.$$

Proof. We have the following natural isomorphisms:

$$\begin{aligned}
\big(\widetilde{Re}_\infty^Q(\widehat{G})\big)_n &= \int^{X \in \widehat{\mathcal{C}}^{\mathrm{op}}} \widehat{G}_n(X) \otimes (QX)_n \\
&= \int^{X \in \widehat{\mathcal{C}}^{\mathrm{op}}} \widehat{G}_n(X) \otimes \coprod_{c_0, \ldots, c_n \in \mathcal{C}} \mathcal{Y}_{c_0} \times \mathcal{C}(c_0, c_1) \times \cdots \times \mathcal{C}(c_{n-1}, c_n) \times X(c_n)
\end{aligned}$$

$$
\begin{aligned}
&\cong \int^{X \in \widehat{\mathcal{C}}^{\mathrm{op}}} \coprod_{c_0,\dots,c_n \in \mathcal{C}} \widehat{G}_n(X) \times \mathcal{Y}_{c_0} \times \mathcal{C}(c_0, c_1) \times \cdots \times \mathcal{C}(c_{n-1}, c_n) \times \widehat{\mathcal{C}}(\mathcal{Y}_{c_n}, X) \\
&\cong \coprod_{c_0,\dots,c_n \in \mathcal{C}} \mathcal{Y}_{c_0} \times \mathcal{C}(c_0, c_1) \times \cdots \times \mathcal{C}(c_{n-1}, c_n) \times \widehat{G}_n(\mathcal{Y}_{c_n}) \\
&\cong \int^{c \in \mathcal{C}^{\mathrm{op}}} \widehat{G}_n(\mathcal{Y}_c) \otimes (Q\mathcal{Y}_c)_n \\
&\cong \big((Q \circ \mathcal{Y}^*)(\widehat{G})\big)_n \\
&= Re^Q_\infty(\widehat{G})_n \,.
\end{aligned}
$$

The first and second isomorphisms result from the Yoneda Lemma for the category $\widehat{\mathcal{C}}$. The second-to-last isomorphism is an application of the Yoneda Lemma for $\mathcal{C}$. ☐

Proposition 3.11 *The functors Re^Q_∞ and S^Q_∞ give rise to a simplicial Quillen adjunction sitting inside the following non-commutative diagram*

$$
\begin{tikzcd}
\widehat{\mathcal{C}} \arrow[r, "\widehat{\mathcal{Y}}"] \arrow[d, "Q"'] & \widehat{\mathcal{H}}_\infty \arrow[dl, shift right, "Re^Q_\infty"'] \\
\mathcal{H}_\infty \arrow[ur, shift right, "S^Q_\infty"'] &
\end{tikzcd}
$$

Further, there exists a natural isomorphism $\eta \colon Re^Q_\infty \circ \widehat{\mathcal{Y}} \xrightarrow{\cong} Q$.

Proof. This is a direct application of [Dug01, Prop. 2.3] (see also [Dug, Prop. 3.2.8]); the fact that S^Q_∞ is a right Quillen adjoint follows immediately from the fact that $\mathcal{H}_\infty$ is a simplicial model category. In the present case, η as written down by Dugger turns out to be an isomorphism as a consequence of the Yoneda Lemma. ☐

Proposition 3.12 *The functors Re^Q_∞ and S^Q_∞ have the following properties:*

(1) The functor Re^Q_∞ is homotopical.

(2) Let $\widehat{Q} \colon \widehat{\mathcal{H}}_\infty \to \widehat{\mathcal{H}}_\infty$ denote Dugger's cofibrant replacement functor on $\widehat{\mathcal{H}}_\infty$ (defined in analogy with (3.4)*). For every fibrant object $F \in \mathcal{H}_\infty$, there is a natural zig-zag of weak equivalences*

$$
F \xleftarrow{\sim} QF \xrightarrow{\sim} Re^Q_\infty \circ S^Q_\infty(F) \xleftarrow{\sim} Re^Q_\infty \circ \widehat{Q} \circ S^Q_\infty(F) \,.
$$

Proof. By Lemma 3.10, Re^Q_∞ is homotopical if and only if $Q \circ \mathcal{Y}^*$ is so. Since weak equivalences in both $\mathcal{H}_\infty$ and in $\widehat{\mathcal{H}}_\infty$ are defined objectwise, the functor $\mathcal{Y}^* \colon \widehat{\mathcal{H}}_\infty \to \mathcal{H}_\infty$ is homotopical. Now claim (1) follows since Q is homotopical. This implies that the morphism $Re^Q_\infty \circ \widehat{Q} \circ S^Q_\infty(F) \longrightarrow Re^Q_\infty \circ S^Q_\infty(F)$ is a weak equivalence. For $F \in \mathcal{H}_\infty$, we find that

$$
\big(\mathcal{Y}^* \circ S^Q_\infty(F)\big)(c) = S^Q_\infty(F)(\mathcal{Y}_c) = \underline{\mathcal{H}}_\infty(Q\mathcal{Y}_c, F) \,.
$$

If F is fibrant, the weak equivalence $Q\mathcal{Y}_c \xrightarrow{\sim} \mathcal{Y}_c$ induces a weak equivalence

$$
F(c) \cong \underline{\mathcal{H}}_\infty(\mathcal{Y}_c, F) \xrightarrow{\sim} \underline{\mathcal{H}}_\infty(Q\mathcal{Y}_c, F) \,,
$$

since $\mathcal{Y}_c$ is projectively cofibrant. Hence, there is a natural weak equivalence $F \xrightarrow{\sim} \mathcal{Y}^* \circ S^Q_\infty(F)$. To complete the proof, we apply the homotopical functor Q to this morphism. ☐

The question now is whether the Quillen adjunction $Re_{\infty}^{Q} \dashv S_{\infty}^{Q}$ between the projective model structures also induces a Quillen adjunction between the *local* projective model structures. This will be a consequence of the following two statements.

Proposition 3.13 *Let $\mathcal{V}$ be a symmetric monoidal model V-category. Let $\mathcal{M}$ and $\mathcal{N}$ be two model $\mathcal{V}$-categories, and let $F \dashv G$ be a Quillen $\mathcal{V}$-adjunction from $\mathcal{M}$ to $\mathcal{N}$. Suppose that A, B are V-sets of morphisms in $\mathcal{M}$ and in $\mathcal{N}$, respectively, such that the $\mathcal{V}$-enriched left Bousfield localisations $L_{A/\mathcal{V}}\mathcal{M}$ and $L_{B/\mathcal{V}}\mathcal{N}$ exist. If G maps B-local objects to A-local objects, then $F \dashv G$ descends to a Quillen adjunction between the localised model categories.*

Proof. The proof is a straightforward adaptation of [Hir03, Prop. 3.1.6, Prop. 3.3.18] to Barwick's formalism of enriched left Bousfield localisation. First, the Quillen $\mathcal{V}$-adjunction $\mathcal{N} \rightleftarrows L_{B/\mathcal{V}}\mathcal{N}$ implies that

$$F : \mathcal{M} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} L_{B/\mathcal{V}}\mathcal{N} : G$$

is a Quillen $\mathcal{V}$-adjunction. By the universal property of enriched left Bousfield localisations [Bar10, Def. 4.42] we are thus left to check that the functor $F\colon \mathcal{M} \to L_{B/\mathcal{V}}\mathcal{N}$ maps every cofibrant approximation to an element of A to a weak equivalence in $L_{B/\mathcal{V}}\mathcal{N}$, i.e. to a $B/\mathcal{V}$-local weak equivalence in $\mathcal{N}$. By the enriched adjointness $F \dashv G$, however, this is equivalent to checking that the functor $G\colon L_{B/\mathcal{V}}\mathcal{N} \to \mathcal{M}$ maps $B/\mathcal{V}$-local objects to $A/\mathcal{V}$-local objects. □

Theorem 3.14 ([DHI04, Cor. A.3]) *Let $\mathcal{C}$ be a U-small category endowed with a Grothendieck coverage τ, and let $\pi\colon Y \to X$ be a τ-local epimorphism in $\widehat{\mathcal{C}}$. We denote its Čech nerve by $\check{C}\pi$. The augmentation map*

$$\pi^{[\bullet]}\colon \check{C}\pi \to X$$

from the Čech nerve of π to X is a weak equivalence in $\mathcal{H}_{\infty}^{loc}$.

The proof of Theorem 3.14 is rather technical; it requires a "wrestling match with the small object argument" ([DHI04] at the beginning of Subsection A.12). We refer the reader to that reference for details.

Remark 3.15 In [DHI04], Theorem 3.14 is proven for the Čech localisation of the *injective* model structure $\mathcal{H}^{i}$ on $\mathcal{H}$. However, it also holds true in $\mathcal{H}_{\infty}$: observe that the Quillen equivalence $L : \mathcal{H}_{\infty} \leftrightarrows \mathcal{H}^{i} : R$ induces a Quillen equivalence $L : \mathcal{H}_{\infty}^{loc} \leftrightarrows \mathcal{H}^{i,loc} : R$ between the Čech localisations. Given a Čech nerve $\pi^{[\bullet]}\colon \check{C}\pi \to X$, it follows that also $Q(\pi^{[\bullet]})\colon Q(\pi) \to QX$ is a weak equivalence in $\mathcal{H}^{i}$, since there exists a natural weak equivalence $Q \xrightarrow{\sim} 1$ and since $\mathcal{H}^{i}$ and $\mathcal{H}_{\infty}$ have the same weak equivalences. Thus, $Q(\pi^{[\bullet]}) = L(Q(\pi^{[\bullet]}))$ is a weak equivalence, and since L reflects weak equivalences between cofibrant objects (it is a left Quillen equivalence) and since Q is weakly equivalent to the identity functor, it follows that $\pi^{[\bullet]}$ is also a weak equivalence in $\mathcal{H}_{\infty}^{loc}$. ◁

Proposition 3.16 *The functor $S_{\infty}^{Q}\colon \mathcal{H}_{\infty}^{loc} \to \widehat{\mathcal{H}}_{\infty}^{loc}$ preserves local objects.*

Proof. Let $\pi\colon Y \to X$ be a τ-local epimorphism in $\widehat{\mathcal{C}}$, and let $F \in \mathcal{H}_{\infty}^{loc}$ be fibrant. We have a commutative diagram

$$\begin{tikzcd} S^Q_\infty(F)(X) = \underline{\mathcal{H}}_\infty(QX, F) \arrow[r, "(Q\pi^{[\bullet]})^*", "\sim"'] \arrow[d] & \underline{\mathcal{H}}_\infty\big(Q(\check{C}\pi), F\big) \arrow[d, "\sim"] \\ \underset{n\in\mathbb{\Delta}^{\mathrm{op}}}{\mathrm{holim}}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}} \big((S^Q_\infty F)(\check{C}\pi_n)\big) & \underline{\mathcal{H}}_\infty\Big(\underset{n\in\mathbb{\Delta}^{\mathrm{op}}}{\mathrm{hocolim}}^{\mathcal{H}^{loc}_\infty} Q(\check{C}\pi_n),\, F\Big) \arrow[l, "\cong"] \end{tikzcd} \tag{3.17}$$

Since $\mathcal{H}^{loc}_\infty$ is a Bousfield localisation of $\mathcal{H}_\infty$, the natural weak equivalence $q\colon Q \xrightarrow{\sim} 1$ in $\mathcal{H}_\infty$ is a natural weak equivalence in $\mathcal{H}^{loc}_\infty$ as well. Thus, the morphism $Q\pi^{[\bullet]}\colon Q(\check{C}\pi) \to QX$ is a weak equivalence in $\mathcal{H}^{loc}_\infty$ by Theorem 3.14. Since F is fibrant in $\mathcal{H}^{loc}_\infty$ and since $\mathcal{H}^{loc}_\infty$ is a simplicial model category, it follows that the top morphism in diagram (3.17) is a weak equivalence in $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$.

Consider the diagram $\mathbb{\Delta}^{\mathrm{op}} \to \mathcal{H}_\infty$, $n \mapsto (\check{C}\pi)_n$, where $(\check{C}\pi)_n$ is seen as a simplicially constant presheaf. By Proposition A.3, there exists a canonical weak equivalence f from the homotopy colimit of this diagram to $Q(\check{C}\pi)$. The right vertical morphism is obtained by applying the right Quillen functor $\underline{\mathcal{H}}_\infty(-, F)$ to f. Since both source and target of f are cofibrant (we model the homotopy colimit using the bar construction [Rie14]), the resulting morphism is a weak equivalence.

The bottom horizontal isomorphism merely stems from the fact that the functor $\underline{\mathcal{H}}_\infty(-, F)$ translates the bar construction in $\mathcal{H}_\infty$ into a cobar construction in $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ and the fact that that both diagrams under the homotopy (co)limit are pointwise (co)fibrant, respectively. □

Combining Proposition 3.13 and Proposition 3.16, we obtain

Theorem 3.18 *The Quillen adjunction $Re^Q_\infty : \widehat{\mathcal{H}}_\infty \rightleftarrows \mathcal{H}_\infty : S^Q_\infty$ induces a Quillen adjunction*

$$Re^Q_\infty : \widehat{\mathcal{H}}^{loc}_\infty \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}^{loc}_\infty : S^Q_\infty\,.$$

3.2 Sheaves of ∞-categories

In this section we extend our results from Section 3.1 to sheaves of $(\infty, 1)$-categories.

Model structures and the sheaf condition

Let $s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} = \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\mathbb{\Delta}})$ denote the category V-small bisimplicial sets. We will also refer to it as the category of simplicial spaces. There is a canonical embedding

$$\mathsf{c}_\bullet\colon \mathcal{S}\mathrm{et}_{\mathbb{\Delta}} \hookrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\mathbb{\Delta}})\,, \qquad (\mathsf{c}_\bullet K)_n = K \quad \forall\, [n] \in \mathbb{\Delta}\,.$$

We will sometimes use this functor implicitly when there is no danger of confusion. The inclusion $\mathsf{c}_\bullet$ is left adjoint to the evaluation functor

$$\mathrm{Ev}_{[0]}\colon \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}) \longrightarrow \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}\,, \qquad X_\bullet \longmapsto X_0\,.$$

We let

$$\mathcal{Y}^{\mathbb{\Delta}}\colon \mathbb{\Delta} \longrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \longrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}_{\mathbb{\Delta}})$$

denote the composition of the Yoneda embedding of $\mathbb{\Delta}$ with the inclusion $\mathsf{c}_\bullet \colon \mathcal{S}\mathrm{et} \hookrightarrow \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$. The category $s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ is cartesian closed [Rez01], with products taken objectwise, and mapping objects

$$(X_\bullet, Y_\bullet) \mapsto Y_\bullet^{X_\bullet}\,, \qquad \big(Y_\bullet^{X_\bullet}\big)_n = \underline{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(X_\bullet \times \mathcal{Y}^{\mathbb{\Delta}}_{[n]}, Y_\bullet)\,. \tag{3.19}$$

The adjunction $\mathsf{c}_\bullet \dashv \mathrm{Ev}_{[0]}$ induces a simplicial enrichment (i.e. a two-variable adjunction [Hov99, Rie14]) on $s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ from its cartesian closed structure. For any $k, n \in \mathbb{N}_0$ and $Y_\bullet \in s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ there are canonical natural isomorphisms

$$s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}\big(\mathcal{Y}^{\mathbb{\Delta}}_{[n]} \times \mathsf{c}_\bullet \Delta^k, Y_\bullet\big) \cong \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}(\Delta^k, Y_n) \cong Y_{n,k}\,.$$

For a simplicial set $K \in \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ and $n \in \mathbb{N}_0$ this yields a natural isomorphism

$$\underline{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(\mathcal{Y}^{\mathbb{\Delta}}_{[n]} \times \mathsf{c}_\bullet K, Y) \cong \underline{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(K, Y_n)\,. \tag{3.20}$$

Remark 3.21 In the following we will work with the injective model structure on bisimplicial sets; that is, as a model category, we set $s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} = (\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}{}^{\mathbb{\Delta}^{\mathrm{op}}})_{inj}$. Note that this model structure coincides with the Reedy model structure [Hir03]. The reason for our choice of the injective model structure over the projective model structure is that we need a cartesian closed model category for ∞-categories. This is desirable for purely conceptual reasons already – for instance, in order to form functor ∞-categories simply by taking an enriched internal hom-object – but we will actually *need* cartesian closure in order for Theorem 3.18 to extend to (pre)sheaves of ∞-categories. So far it appears to be unknown whether the projective model structure on $s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ localises to a cartesian closed model category of complete Segal spaces. For the injective case, cartesian closure has been shown already in [Rez01]. $\triangleleft$

We let $\mathcal{CSS}$ denote the category of (V-small) bisimplicial sets endowed with the model structure for complete Segal spaces. It is constructed as a left Bousfield localisation of the injective model structure $s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ on bisimplicial sets, i.e. $\mathcal{CSS} = L_S s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ for some class S of morphisms of bisimplicial sets. We do not need the explicit definition of S here; we refer the reader to [Rez01] for details. The fibrant objects in $\mathcal{CSS}$ are called *complete Segal spaces*, or *∞-categories*. They are those diagrams $X_\bullet \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ satisfying that

(1) the simplicial space $X_\bullet \in s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ is Reedy fibrant,

(2) the Segal maps

$$X_n \xrightarrow{\sim} X_1 \overset{h}{\underset{X_0}{\times}} \cdots \overset{h}{\underset{X_0}{\times}} X_1$$

are weak equivalences in $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ (in the Kan-Quillen model structure), and

(3) the degeneracy s_0 induces a weak equivalence

$$s_0 \colon X_0 \xrightarrow{\sim} X_{weq}\,,$$

where $X_{weq} \subset X_1$ is the subspace generated by the weakly invertible morphisms [Rez01].

Proposition 3.22 [Rez01, Bar05, Bar10] *The model category $\mathcal{CSS}$ is left proper, tractable, simplicial, and cartesian closed.*

Definition 3.23 *Let $\mathcal{C}$ be a small category. The* projective model category for presheaves of ∞-categories on $\mathcal{C}$ *is*

$$\mathcal{H}_{\infty,1} \coloneqq \mathcal{F}\mathrm{un}\big(\mathcal{C}^{\mathrm{op}}, \mathcal{CSS}\big)_{proj}\,.$$

A presheaf of ∞-categories *on $\mathcal{C}$ is a fibrant object in $\mathcal{H}_{\infty,1}$.*

Proposition 3.24 *The category $\mathcal{H}_{\infty,1}$ is a left proper, tractable $\mathcal{CSS}$-model category. If $\mathcal{C}$ has finite products, then $\mathcal{H}_{\infty,1}$ is symmetric monoidal.*

Proof. Tractability follows from [Bar10, Thm. 2.14]. The enrichment over $\mathcal{CSS}$ is an application of [Bar10, Prop. 4.50]. Left properness follows since pushouts are computed objectwise, weak equivalences are defined objectwise, and cofibrations are (in particular) objectwise cofibrations. Hence, the condition for left properness of $\mathcal{H}_{\infty,1}$ reduces to the condition of left properness of $\mathcal{CSS}$. Finally, if $\mathcal{C}$ has finite products, then $\mathcal{H}_{\infty,1}$ is symmetric monoidal by [Bar10, Prop. 4.52]. □

Recall that for any V-small category $\mathcal{I}$ there are Quillen adjunctions

$$\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} \underset{\mathsf{c}_*}{\overset{\mathsf{c}^*}{\rightleftarrows}} \big(\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}{}^{\mathcal{I}^{op}}\big)_{proj} \rightleftarrows \big(\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}{}^{\mathcal{I}^{op}}\big)_{inj}$$

where $\mathsf{c}\colon \mathcal{I} \to *$ is the collapse functor to the terminal category. If $\mathcal{I}$ has a terminal object, c_* is given by evaluation at that object. Thus, in the case of $\mathcal{I} = \mathbb{\Delta}$, we obtain a Quillen adjunction

$$\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} \underset{\mathrm{Ev}_{[0]}}{\overset{\mathsf{c}_\bullet}{\rightleftarrows}} s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} = (\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}{}^{\mathbb{\Delta}^{op}})_{Reedy}$$

This induces a simplicial Quillen adjunction

$$\mathsf{c}_\bullet : \mathcal{H}_\infty \underset{\perp}{\rightleftarrows} \mathcal{H}_{\infty,1} : \mathrm{Ev}_{[0]}$$

on presheaf categories. We obtain similar Quillen adjunctions by applying the same argument to the diagram categories $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}{}^{(\mathcal{C}^{op})}$ and $\mathcal{CSS}^{(\mathcal{C}^{op})}$. This leads to a commutative diagram

$$\begin{tikzcd}
\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} \arrow[r, shift left, "\tilde{\mathsf{c}}"] \arrow[d, shift right, "\mathsf{c}_\bullet"'] & \mathcal{H}_\infty \arrow[l, shift left, "\tilde{\mathsf{c}}_*"] \arrow[d, shift right, "\mathsf{c}_\bullet"'] \\
\mathcal{CSS} \arrow[r, shift left, "\tilde{\mathsf{c}}"] \arrow[u, shift right, "\mathrm{Ev}_{[0]}"'] & \mathcal{H}_{\infty,1} \arrow[l, shift left, "\tilde{\mathsf{c}}_*"] \arrow[u, shift right, "\mathrm{Ev}_{[0]}"']
\end{tikzcd}$$

of Quillen adjunctions, in which the left-facing and the downwards-facing arrows are the left adjoints. The functors $\tilde{\mathsf{c}}$ are given by $(\tilde{\mathsf{c}}X)(c) = X$ for all $c \in \mathcal{C}$; if $\mathcal{C}$ has a terminal object $* \in \mathcal{C}$, then their right adjoints are given by $\tilde{\mathsf{c}}_* = \mathrm{Ev}_*$, the evaluation functor at $* \in \mathcal{C}$. We readily deduce:

Lemma 3.25 *The following statements hold true:*

(1) Every projectively cofibrant presheaf of ∞-groupoids F gives rise to a projectively cofibrant presheaf of ∞-categories $\mathsf{c}_\bullet F$ on $\mathcal{C}$.

(2) In particular, $\mathsf{c}_\bullet \mathcal{Y}_c$ and $\tilde{\mathsf{c}}K$ are cofibrant in $\mathcal{H}_{\infty,1}$, for every $c \in \mathcal{C}$ and every $K \in \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$.

(3) The functor $\mathsf{c}_\bullet \colon \mathcal{H}_\infty \to \mathcal{H}_{\infty,1}$ is homotopical, i.e. it preserves weak equivalences.

To avoid heavy notation, we denote the composition of the Yoneda embedding of $\mathcal{C}$ with any of the canonical embeddings $\mathcal{S}\mathrm{et} \hookrightarrow \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ or additionally $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} \hookrightarrow s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ again by $\mathcal{Y}$; this yields a functor

$$\mathcal{Y}\colon \mathcal{C} \to \mathcal{H}_{\infty,1}\,.$$

Further, we will often leave the functor $\mathsf{c}_\bullet$ implicit. In particular, if $(\mathcal{C},\tau)$ is a site, we can use $\mathsf{c}_\bullet$ to promote the Čech nerve $\check{C}\mathcal{U}_\bullet \to \mathcal{Y}_c$ of any covering family $\mathcal{U} = \{c_i \to c\}_{i\in I}$ in $(\mathcal{C},\tau)$ to a morphism in $\mathcal{H}_{\infty,1}$. We still refer to this morphism as the *Čech nerve* of the covering $\mathcal{U}$.

Definition 3.26 *Let $(\mathcal{C}, \tau)$ be a Grothendieck site, and let $\check{\tau}$ denote the class of morphisms in $\mathcal{H}_{\infty,1}$ consisting of Čech nerves of coverings in $(\mathcal{C}, \tau)$. The* projective model category for sheaves of ∞-categories on $\mathcal{C}$ *is the $\mathcal{CSS}$-enriched left Bousfield localisation of $\mathcal{H}_{\infty,1}$ at $\check{\tau}$,*

$$\mathcal{H}_{\infty,1}^{loc} \coloneqq L_{\check{\tau}/\mathcal{CSS}} \mathcal{H}_{\infty,1} \,.$$

A sheaf of ∞-categories *on $(\mathcal{C}, \tau)$ is a fibrant object in $\mathcal{H}_{\infty,1}^{loc}$.*

An object $\mathcal{F} \in \mathcal{H}_{\infty,1}^{loc}$ is fibrant precisely if

(1) it is objectwise a complete Segal space and
(2) for every covering family $\mathcal{U} = \{c_i \to c\}_{i \in I}$ in $(\mathcal{C}, \tau)$, the morphism

$$\mathcal{F}(c) \longrightarrow \operatorname{holim}^{\mathcal{CSS}}_{\mathbb{\Delta}} \Big(\cdots \prod_{i_0, \ldots, i_n \in I} \underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}(QC_{i_0 \ldots i_n}, \mathcal{F}) \ \cdots \Big)$$

is an equivalence in $\mathcal{CSS}$. Here, Q is the cofibrant replacement functor in $\mathcal{H}_\infty$ as used in Section 3.1. It produces a simplicial presheaf, which is then viewed as a presheaf of bisimplicial sets via the functor $\mathsf{c}_\bullet$.

Proposition 3.27 *$\mathcal{H}_{\infty,1}^{loc}$ is a proper, tractable $\mathcal{CSS}$-model category. If $\mathcal{H}_{\infty,1}$ is symmetric monoidal, then so is $\mathcal{H}_{\infty,1}^{loc}$.*

Proof. This follows form Proposition 3.24, together with [Bar10, Thm 4.46, Prop. 4.58]. □

The $\mathcal{CSS}$-enriched internal hom in both $\mathcal{H}_{\infty,1}$ and $\mathcal{H}_{\infty,1}^{loc}$ is given by

$$\underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}(\mathcal{F}, \mathcal{G}) = \int_{c \in \mathcal{C}^{\mathrm{op}}} \big(\mathcal{G}(c)\big)^{\mathcal{F}(c)} \qquad \in \mathcal{CSS} \,,$$

where the exponential is taken in $\mathcal{CSS}$. In particular, there is a natural isomorphism $\underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}(\mathcal{Y}_c, \mathcal{F}) \cong \mathcal{F}(c)$ for any $c \in \mathcal{C}$ and any $\mathcal{F} \in \mathcal{H}_{\infty,1}$.

Lemma 3.28 *Let $\mathcal{I}$ be a V-small category, and let $D \colon \mathcal{I} \to \mathcal{CSS}$ be a projectively fibrant diagram. Then, the homotopy limit of D can be computed levelwise. That is, there exist canonical natural isomorphisms*

$$\big(\operatorname{holim}^{\mathcal{CSS}}_{\mathcal{J}}(D)\big)_n \cong \operatorname{holim}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}_{\mathcal{J}}(D_n) \,,$$

where for $i \in \mathcal{J}$ the space $D_n i = (Di)_n \in \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ is the n-th simplicial level of $Di \in s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$.

Proof. Since D is objectwise fibrant, a model for the homotopy limit of D is given by the two-sided cobar construction $C(*, \mathcal{I}, D) \in \mathcal{CSS}$ [Rie14, Cor. 5.1.3]. Let $G \colon \mathcal{J} \to \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ and $F \colon \mathcal{J} \to \mathcal{CSS}$ be functors. The cosimplicial cobar construction is the cosimplicial object in $s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ whose n-th level reads as

$$C^n(G, \mathcal{I}, F) = \prod_{i \in (N\mathcal{J})_n} (Fi_n)^{\mathsf{c}_\bullet(Gi_0)} \cong \prod_{i_0, \ldots, i_n \in \mathcal{J}} (Fi_n)^{\mathsf{c}_\bullet(Gi_0) \otimes (\mathcal{J}(i_0,i_1) \times \cdots \times \mathcal{J}(i_{n-1}, i_n))} \qquad \in s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} \,,$$

where $N\mathcal{J} \in \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ denotes the nerve of $\mathcal{J}$. The cobar construction $C(G, \mathcal{J}, F)$ is the totalisation

$$C(G, \mathcal{J}, F) = \int_{n \in \mathbb{\Delta}} \big(C^n(G, \mathcal{J}, F)\big)^{\mathsf{c}_\bullet \Delta^n} \,.$$

For a simplicial category $\mathcal{D}$, we will sometimes write $C_{\mathcal{D}}$ for the cobar construction in $\mathcal{D}$ if the ambient category is not clear from context.

For any $n \in \mathbb{N}_0$, there are canonical natural isomorphisms of simplicial sets

$$\begin{aligned}
\big(\operatorname*{holim}_{\mathcal{J}}{}^{\mathcal{CSS}}(D)\big)_n &= \big(C_{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D)\big)_n \\
&= \underline{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\big(\mathcal{Y}^{\mathbb{\Delta}}_{[n]}, C_{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D)\big) \\
&\cong \int_{k\in\mathbb{\Delta}} \underline{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\Big(\mathcal{Y}^{\mathbb{\Delta}}_{[n]}, \big(C^k_{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D)\big)^{\mathsf{c}_\bullet\Delta^k}\Big) \\
&\cong \int_{k\in\mathbb{\Delta}} \underline{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\big(\mathcal{Y}^{\mathbb{\Delta}}_{[n]} \otimes \mathsf{c}_\bullet\Delta^k, C^k_{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D)\big) \\
&\cong \int_{k\in\mathbb{\Delta}} \underline{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\big(\Delta^k, C^k_{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D)_n\big) \\
&\cong \int_{k\in\mathbb{\Delta}} \underline{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\Big(\Delta^k, \prod_{i\in(\mathrm{N}\mathcal{J})_k} (Di_k)_n\Big) \\
&\cong \int_{k\in\mathbb{\Delta}} \big(C^k_{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D_n)\big)^{\Delta^k} \\
&= C_{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D_n) \\
&= \operatorname*{holim}_{\mathcal{J}}{}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(D_n)\,.
\end{aligned}$$

In the fifth setp we have made use of the formula (3.20). The first identity is, as said above, an application of [Rie14, Cor. 5.1.3]. The last identity also relies on this result, together with the fact that D being objectwise fibrant in $\mathcal{CSS}$ implies that the functor

$$D_n = \underline{\mathcal{CSS}}\big(\mathcal{Y}^{\mathbb{\Delta}}_{[n]}, D\big) \colon \mathcal{J} \longrightarrow \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$$

is objectwise fibrant as well. □

Lemma 3.29 *Let $\mathcal{M}$ be a model $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$-category and $\mathcal{J}$ a V-small category. For any objectwise fibrant diagram $D\colon \mathcal{J} \to \mathcal{M}$, the object $C(*,\mathcal{J},D)$ is fibrant in $\mathcal{M}$. In particular, for any diagram $D\colon \mathcal{J} \to \mathcal{M}$, the object $C(*,\mathcal{J},R^{\mathcal{M}} \circ D)$ is fibrant in $\mathcal{M}$, where $R^{\mathcal{M}}$ is a functorial fibrant replacement in $\mathcal{M}$.*

Proof. This follows from combining (the duals of) [Rie14, Lem. 5.2.1] and [Rie14, Thm. 5.2.3]. □

Consequently, we have that for every projectively fibrant diagram $D\colon \mathcal{J} \to \mathcal{CSS}$ the homotopy limit $\operatorname{holim}^{\mathcal{CSS}}_{\mathcal{J}}(D) = C(*,\mathcal{J},D)$ is a complete Segal space itself.

Now consider a presheaf $\mathcal{F}$ of ∞-categories in the sense of Definition 3.26. Observe that composing $\mathcal{F}$ with the evaluation of its first simplicial direction on $[n] \in \mathbb{\Delta}$ gives to a simplicial presheaf $\mathcal{F}_n \in \mathcal{H}_\infty$, with $\mathcal{F}_n(c) \coloneqq (\mathcal{F}(c))_n$ for all $c \in \mathcal{C}$ and $n \in \mathbb{N}_0$.

Theorem 3.30 *Let $(\mathcal{C},\tau)$ be a site, and let $\mathcal{F}$ be a presheaf of ∞-categories on $\mathcal{C}$; that is, $\mathcal{F}$ is a fibrant object in $\mathcal{H}_{\infty,1}$. Then, the following are equivalent:*

(1) $\mathcal{F}$ is a sheaf of ∞-categories.

(2) $\mathcal{F}_n$ is a sheaf of ∞-groupoids for every $n \in \mathbb{N}_0$.

(3) $\mathcal{F}_0$ and $\mathcal{F}_1$ are sheaves of ∞-groupoids.

Proof. (1) ⇒ (2): If $\mathcal{F}$ is a sheaf of ∞-categories on $\mathcal{C}$, then for every $c \in \mathcal{C}$ the bisimplicial set $\mathcal{F}(c)$ is a complete Segal space, and for every covering $\{c_i \to c\}_{i\in I}$ in $(\mathcal{C},\tau)$ the morphism

$$\eta\colon \mathcal{F}(c) \longrightarrow \operatorname{holim}^{\mathcal{CSS}}_{\mathbb{\Delta}}\Big(\cdots \prod_{i_0,\ldots,i_n\in I} \underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}(QC_{i_0\ldots i_n},\mathcal{F})\ \cdots\Big)$$

is an equivalence of complete Segal spaces (cf. Lemma 3.29). Since both sides are fibrant objects in the left Bousfield localisation $\mathcal{CSS} = L_S(s\mathcal{S}\mathrm{et}_\Delta)$, this is equivalent to η being a weak equivalence in $s\mathcal{S}\mathrm{et}_\Delta$, i.e. it is a levelwise weak equivalence. Lemma 3.28 together with Equations (3.19) and (3.20) imply that, if $\mathcal{F} \in \mathcal{H}_{\infty,1}$ is fibrant, there are canonical isomorphisms, natural in $\mathcal{F}$ and $k \in \mathbb{N}_0$,

$$\begin{aligned}
&\Big(\mathrm{holim}^{\mathcal{CSS}}_{\Delta}\Big(\cdots \prod_{i_0,\dots,i_n \in I} \underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}(QC_{i_0\dots i_n}, \mathcal{F}) \ \cdots\Big)\Big)_k \\
&\cong \mathrm{holim}^{\mathcal{S}\mathrm{et}_\Delta}_{\Delta}\Big(\cdots \prod_{i_0,\dots,i_n \in I} \big(\underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}(QC_{i_0\dots i_n}, \mathcal{F})\big)_k \ \cdots\Big) \\
&\cong \mathrm{holim}^{\mathcal{S}\mathrm{et}_\Delta}_{\Delta}\Big(\cdots \prod_{i_0,\dots,i_n \in I} \underline{\mathcal{H}}_{\infty}(QC_{i_0\dots i_n}, \mathcal{F}_k) \ \cdots\Big)\,.
\end{aligned}$$

Combining this with Lemma 3.28 we obtain that, if $\mathcal{F} \in \mathcal{H}^{loc}_{\infty,1}$ is fibrant, the map

$$\begin{aligned}
\eta_k \colon \mathcal{F}_k(c) \longrightarrow &\Big(\mathrm{holim}^{\mathcal{CSS}}_{\Delta}\Big(\cdots \prod_{i_0,\dots,i_n \in I} \underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}(QC_{i_0\dots i_n}, \mathcal{F}) \ \cdots\Big)\Big)_k \\
&\cong \mathrm{holim}^{\mathcal{S}\mathrm{et}_\Delta}_{\Delta}\Big(\cdots \prod_{i_0,\dots,i_n \in I} \underline{\mathcal{H}}_{\infty}(QC_{i_0\dots i_n}, \mathcal{F}_k) \ \cdots\Big)
\end{aligned}$$

is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$ for every $k \in \mathbb{N}_0$. Hence, each level $\mathcal{F}_k$ is a sheaf of ∞-groupoids.

The implication (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1): If $\mathcal{F}$ is projectively fibrant, then η is a morphism between complete Segal spaces. Assuming further that both η_0 and η_1 are weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$, it follows from the Segal condition that η_k is an equivalence for every $k \in \mathbb{N}_0$. Again using the fact that weak equivalences in $\mathcal{CSS}$ between fibrant objects are precisely the levelwise weak equivalences of simplicial spaces, this implies that η is a weak equivalence in $\mathcal{CSS}$, and hence that $\mathcal{F}$ is a sheaf of ∞-categories. □

Transfer to the large site of $\mathcal{C}$

We now extend Theorem 3.18 to $(\infty,1)$-categories. To that end, observe that for each U-small site $(\mathcal{C},\tau)$ the site $(\widehat{\mathcal{C}},\widehat{\tau})$ is V-small, so that the projective model structure $\widehat{\mathcal{H}}_{\infty,1} = (\mathcal{CSS}^{\widehat{\mathcal{C}}^{\mathrm{op}}})_{proj}$ exists. Consider the functor

$$S^Q_{\infty,1} \colon \mathcal{H}_{\infty,1} \longrightarrow \widehat{\mathcal{H}}_{\infty,1}\,, \qquad (S^Q_{\infty,1}\mathcal{F})(X) = \underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}(\mathsf{c}_\bullet QX, \mathcal{F})\,.$$

Proposition 3.31 *There exists a Quillen adjunction*

$$Re^Q_{\infty,1} : \widehat{\mathcal{H}}_{\infty,1} \rightleftarrows_{\perp} \mathcal{H}_{\infty,1} : S^Q_{\infty,1}\,.$$

Proof. The left adjoint $Re^Q_{\infty,1}$ is given by

$$Re^Q_{\infty,1}(\widehat{\mathcal{G}}) = \int^{c \in \mathcal{C}^{\mathrm{op}}} \widehat{\mathcal{G}}(\mathcal{Y}_c) \otimes \mathsf{c}_\bullet Q\mathcal{Y}_c = \int^{c \in \mathcal{C}^{\mathrm{op}}} (\mathcal{Y}^*\widehat{\mathcal{G}})(c) \otimes \mathsf{c}_\bullet Q\mathcal{Y}_c\,.$$

The adjointness is proven entirely in parallel to the proof of Proposition 3.11. The fact that $S^Q_{\infty,1}$ is right Quillen follows from the properties of the functor $\mathsf{c}_\bullet$ (it is left Quillen) and the fact that $\mathcal{H}_{\infty,1}$ is a $\mathcal{CSS}$-model category (Lemma 3.24). □

In parallel to (3.8), we let $(\widehat{\tau})\check{}$ denote the set of Čech nerves in $\widehat{\mathcal{H}}_{\infty,1}$ of coverings in $(\widehat{\mathcal{C}}, \widehat{\tau})$ and set

$$\widehat{\mathcal{H}}^{loc}_{\infty,1} := L_{(\widehat{\tau})\check{}/\mathcal{CSS}} \widehat{\mathcal{H}}_{\infty,1} \,.$$

Theorem 3.32 *The Quillen adjunction $Re^Q_{\infty,1} \dashv S^Q_{\infty,1}$ induces a Quillen adjunction*

$$Re^Q_{\infty,1} : \widehat{\mathcal{H}}^{loc}_{\infty,1} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}^{loc}_{\infty,1} : S^Q_{\infty,1} \,.$$

Proof. We aim to apply Proposition 3.13 to $S^Q_{\infty,1}$. Hence, we need to check that $S^Q_{\infty,1}$ sends $\check{\tau}/\mathcal{CSS}$-local objects to $(\widehat{\tau})\check{}/\mathcal{CSS}$-local objects. Let $\mathcal{F} \in \mathcal{H}^{loc}_{\infty,1}$ be a fibrant object, and let $\pi \colon Y \to X$ be a τ-local epimorphism in $\widehat{\mathcal{C}}$. We denote its Čech nerve by $\check{C}\pi_\bullet \to X$. We have to show that the morphism

$$\pi^* \colon (S^Q_{\infty,1}\mathcal{F})(X) \longrightarrow \operatorname{holim}^{\mathcal{CSS}}_{\mathbb{\Delta}} \big((S^Q_{\infty,1}\mathcal{F})(\check{C}\pi_\bullet) \big)$$

is a weak equivalence in $\mathcal{CSS}$. Since $\mathcal{F}$ is a fibrant object in $\mathcal{H}_{\infty,1}$, and since $S^Q_{\infty,1} \colon \mathcal{H}_{\infty,1} \to \widehat{\mathcal{H}}_{\infty,1}$ is right Quillen, it follows that π^* is a morphism between complete Segal spaces. Consequently, π^* is a weak equivalence in $\mathcal{CSS}$ if and only if it is a levelwise weak equivalence of simplicial objects in $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$.

F $X \in \widehat{\mathcal{C}}$ and $n \in \mathbb{N}_0$ we compute

$$\begin{aligned}
\big((S^Q_{\infty,1}\mathcal{F})(X) \big)_n &= \big(\underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}(\mathsf{c}_\bullet QX, \mathcal{F}) \big)_n \\
&\cong \int_c \underline{\mathcal{CSS}}^{\mathcal{CSS}} \big(\mathsf{c}_\bullet(QX(c)), \mathcal{F}(c) \big)_n \\
&\cong \int_c \underline{s\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}} \big(\mathcal{Y}^{\mathbb{\Delta}}_{[n]} \times \mathsf{c}_\bullet(QX(c)), \mathcal{F}(c) \big) \\
&\cong \int_c \underline{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}} \big(QX(c), \mathcal{F}_n(c) \big) \\
&\cong \underline{\mathcal{H}_\infty}(QX, \mathcal{F}_n) \\
&= S^Q_\infty \mathcal{F}_n(X) \,.
\end{aligned}$$

Therefore, using Lemma 3.28, the n-th level of the morphism π^* fits into a commutative square

$$\begin{tikzcd}
\big((S^Q_{\infty,1}\mathcal{F})(X) \big)_n \arrow[r, "(\pi^*)_n"] \arrow[d, "\cong"'] & \Big(\operatorname{holim}^{\mathcal{CSS}}_{\mathbb{\Delta}} \big((S^Q_{\infty,1}\mathcal{F})(\check{C}\pi_\bullet) \big) \Big)_n \arrow[d, "\cong"] \\
(S^Q_\infty \mathcal{F}_n)(X) \arrow[r, "(\pi_n)^*"'] & \operatorname{holim}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}_{\mathbb{\Delta}} \big((S^Q_\infty \mathcal{F}_n)(\check{C}\pi_\bullet) \big)
\end{tikzcd}$$

If $\mathcal{F}$ is fibrant in $\mathcal{H}^{loc}_{\infty,1}$, then by Theorem 3.30 the simplicial presheaves $\mathcal{F}_n$ are sheaves of ∞-groupoids for each $n \in \mathbb{N}_0$, i.e. they are fibrant objects in $\mathcal{H}^{loc}_\infty$. Thus, a direct application of Theorem 3.18 implies that $(\pi_n)^*$, and hence also $(\pi^*)_n$, is a weak equivalence in $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ for every $n \in \mathbb{N}_0$. □

3.3 Sheaves of (∞, n)-categories

We proceed to consider n-fold simplicial spaces, i.e. objects of

$$s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} = (\mathcal{S}\mathrm{et}_{\mathbb{\Delta}})^{\mathbb{\Delta}^{\mathrm{op}\times n}} \,.$$

For fixed $n \in \mathbb{N}$, we use the notation $([k_0], \dots, [k_{n-1}]) =: \vec{k} \in \mathbb{\Delta}^n$. The category $s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ is simplicial, with simplicial mapping space

$$\underline{s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(X,Y) = \int_{\vec{k}\in\mathbb{\Delta}^n} \underline{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(X_{\vec{k}}, Y_{\vec{k}})\,.$$

Further, it is cartesian closed, with internal hom given by

$$(Y^X)_{\vec{k}} = \underline{s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(\mathcal{Y}^{\mathbb{\Delta}^n}_{\vec{k}} \times X, Y) \cong \int_{\vec{l}\in\mathbb{\Delta}^n} \underline{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\big(\mathcal{Y}^{\mathbb{\Delta}^n}_{\vec{k}}(\vec{l}) \times X_{\vec{l}}, Y_{\vec{l}}\big)\,.$$

Here, $\mathcal{Y}^{\mathbb{\Delta}^n} : \mathbb{\Delta}^n \to \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ is the Yoneda embedding of $\mathbb{\Delta}^n$, composed with the embedding $\mathcal{S}\mathrm{et} \hookrightarrow \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$. Via the monoidal adjunctions

$$\mathsf{c}_\bullet : s^{n-1}\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} : \mathrm{Ev}_{[0]}\,,$$

where $(\mathsf{c}_\bullet X)_{k_0,\dots,k_{n-1}} = X_{k_1,\dots,k_{n-1}}$, the category $s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ is enriched, tensored, and cotensored over $s^k\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ for any $0 \leq k \leq n$.

Theorem 3.33 [Bar05, Scholium 2.3.18] *We set $\mathcal{CSS}_0 \coloneqq \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ and $\mathcal{CSS}_1 \coloneqq \mathcal{CSS}$. For $n \geq 2$ there exist cartesian closed, left proper, tractable model categories that are defined iteratively via*

$$\mathcal{CSS}_n \coloneqq L_{S_n}\big((\mathcal{CSS}_{n-1}^{\mathbb{\Delta}^{\mathrm{op}}})_{inj}\big)\,.$$

Fibrant objects in $\mathcal{CSS}_n$ are called *complete n-uple Segal spaces*, or *(∞, n)-categories*. The set of morphisms S_n is the image of S_{n-1} under $\mathsf{c}_\bullet$, for $n \geq 2$; we refer to [Bar05] for details.

Remark 3.34 We divert slightly from Barwick's construction of $\mathcal{CSS}_n$ here, in that we use the injective model structure on $\mathcal{CSS}_{n-1}^{\mathbb{\Delta}^{\mathrm{op}}}$ rather than the Reedy model structure. The resulting model structure is Quillen equivalent to Barwick's, since in both models the Quillen equivalent Reedy and injective model structures are localised at the same morphisms of n-fold simplicial sets. However, for us it will be convenient to work with a model category for n-uple complete Segal spaces in which every object is cofibrant, which is built into our definition. ◁

We record the following straightforward fact:

Lemma 3.35 *For any $n \in \mathbb{N}_0$ there exist symmetric monoidal Quillen adjunctions*

$$\begin{tikzcd} \mathcal{CSS}_n \arrow[r, shift left, "\mathsf{c}_\bullet"] \arrow[r, phantom, "\perp"] & (\mathcal{CSS}_n)^{\mathbb{\Delta}^{\mathrm{op}}}_{inj} \arrow[l, shift left, "\mathrm{Ev}_{[0]}"] \arrow[r, shift left, "1"] \arrow[r, phantom, "\perp"] & \mathcal{CSS}_{n+1}\,. \arrow[l, shift left, "1"] \end{tikzcd}$$

This implies that we can consider $\mathcal{CSS}_n$ as a model $\mathcal{CSS}_k$-category for any $0 \leq k \leq n$.

Lemma 3.36 *Let $n \in \mathbb{N}$, let $\mathcal{I}$ be a V-small category, and let $D \colon \mathcal{I} \to \mathcal{CSS}_n$ be a projectively fibrant diagram, i.e. Di is an n-uple complete Segal space for every $i \in \mathcal{J}$.*

(1) The homotopy limit of D can be computed levelwise: for each $k \in \mathbb{N}_0$, there exist canonical natural isomorphisms

$$\big(\mathrm{holim}^{\mathcal{CSS}_n}_{\mathcal{J}}(D)\big)_k \cong \mathrm{holim}^{\mathcal{CSS}_{n-1}}_{\mathcal{J}}(D_k)\,,$$

where for $i \in \mathcal{J}$ the object $D_k i = (Di)_k$ is the k-th simplicial level of $Di \in s(s^{n-1}\mathcal{S}\mathrm{et}_{\mathbb{\Delta}})$.

(2) The homotopy limit of D can be computed objectwise when viewing the diagram as a functor $D \colon \mathcal{J} \times \mathbb{\Delta}^n \to \mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$. That is, for any $\vec{k} \in \mathbb{\Delta}^n$ there is an isomorphism

$$\big(\mathrm{holim}^{\mathcal{CSS}_n}_{\mathcal{J}}(D)\big)_{\vec{k}} \cong \mathrm{holim}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}_{\mathcal{J}}(D_{\vec{k}})\,,$$

which is natural in $\vec{k} \in \mathbb{\Delta}^n$.

Proof. In order to prove part (1), we proceed as in the proof of Lemma 3.28: let $\mathcal{Y}^n \colon \mathbb{\Delta} \to s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$ denote the functor defined via

$$(\mathcal{Y}^n_{[k]})_{l_0,\ldots,l_n} = \mathbb{\Delta}([l_0],[k])\,.$$

Since D is objectwise fibrant, the homotopy limit is modelled by the cobar construction $C_{s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D)$ in the simplicial category $s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$. Using the identification $s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}} \cong (s^{n-1}\mathcal{S}\mathrm{et}_{\mathbb{\Delta}})^{\mathbb{\Delta}^{\mathrm{op}}}$, we compute its value on the object $[m] \in \mathbb{\Delta}$: there are canonical natural isomorphisms

$$\begin{aligned}
\big(\mathrm{holim}^{\mathcal{CSS}_n}_{\mathcal{J}}(D)\big)_m &= \big(C_{s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D)\big)_m \\
&= \underline{\mathcal{CSS}_n}^{\mathcal{CSS}_{n-1}}\big(\mathcal{Y}^n_{[m]}, C_{s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D)\big) \\
&\cong \int_{k\in\mathbb{\Delta}} \underline{s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}^{s^{n-1}\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\Big(\mathcal{Y}^n_{[m]}, \big(C^k_{s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D)\big)^{\Delta^k}\Big) \\
&\cong \int_{k\in\mathbb{\Delta}} \underline{s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}^{s^{n-1}\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\big(\mathcal{Y}^n_{[m]}\otimes\Delta^k, C^k_{s^n\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D)\big) \\
&\cong \int_{k\in\mathbb{\Delta}} \underline{s^{n-1}\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}^{s^{n-1}\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\big(\Delta^k, C^k(*,\mathcal{J},D)_m\big) \\
&\cong \int_{k\in\mathbb{\Delta}} \underline{s^{n-1}\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}^{s^{n-1}\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\Big(\Delta^k, \prod_{i\in(N\mathcal{J})_k}(Di_k)_m\Big) \\
&\cong \int_{k\in\mathbb{\Delta}} \Big(\prod_{i\in(N\mathcal{J})_k}(Di_k)_m\Big)^{\Delta^k} \\
&= C_{s^{n-1}\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(*,\mathcal{J},D_m) \\
&= \mathrm{holim}^{\mathcal{CSS}_{n-1}}_{\mathcal{J}}(D_m)\,.
\end{aligned}$$

In the last step we have again used that injective fibrant diagrams are in particular object-wise fibrant. In particular, since $D \colon \mathcal{J} \to \mathcal{CSS}_n$ is objectwise fibrant, it follows that, for every $[m] \in \mathbb{\Delta}$ and each $i \in \mathcal{J}$, the object $(Di)_m \in \mathcal{CSS}_{n-1}$ is again fibrant. That is, for every $[m] \in \mathbb{\Delta}$, the diagram $D_m \colon \mathcal{J} \to \mathcal{CSS}_{n-1}$ is again objectwise fibrant, so that the cobar construction in the second-to-last line indeed models the homotopy limit.

Part (2) follows by iterating part (1) to reduce to the case $n = 1$ and then applying Lemma 3.28. □

Definition 3.37 *Let $\mathcal{C}$ be a small category.*

(1) The projective model category for presheaves of (∞,n)-categories on $\mathcal{C}$ *is*

$$\mathcal{H}_{\infty,n} \coloneqq \mathcal{F}\mathrm{un}(\mathcal{C}^{\mathrm{op}},\mathcal{CSS}_n)_{proj}\,.$$

A fibrant object in $\mathcal{H}_{\infty,n}$ is called a presheaf of (∞,n)-categories on $\mathcal{C}$.

(2) Let $(\mathcal{C},\tau)$ be a Grothendieck site, and let $\check{\tau}$ denote the class of morphisms in $\mathcal{H}_\infty$ consisting of Čech nerves of coverings in $(\mathcal{C},\tau)$, promoted to morphisms in $\mathcal{H}_{\infty,n}$ via iterated applications of $\mathsf{c}_\bullet \colon \mathcal{CSS}_{n-1} \to \mathcal{CSS}_n$. The projective model category for sheaves of (∞,n)-categories on $(\mathcal{C},\tau)$ *is the enriched left Bousfield localisation*

$$\mathcal{H}^{loc}_{\infty,n} \coloneqq L_{\check{\tau}/\mathcal{CSS}_n}\mathcal{H}_{\infty,n}\,.$$

A fibrant object in $\mathcal{H}^{loc}_{\infty,n}$ is called a sheaf of (∞,n)-categories on $(\mathcal{C},\tau)$.

Proposition 3.38 *Let $\mathcal{C}$ be a U-small category. Then $\mathcal{H}_{\infty,n}$ is a left proper, tractable $\mathcal{CSS}_n$-model category. Further, if $\mathcal{C}$ is endowed with a Grothendieck pretopology τ, then $\mathcal{H}^{loc}_{\infty,n}$ is a $\mathcal{CSS}_n$-model category, and it is symmetric monoidal if $\mathcal{H}_{\infty,n}$ is so.*

Proof. The statement about $\mathcal{H}_{\infty,n}$ is a consequence of [Bar10, Prop. 4.50]. For $\mathcal{H}^{loc}_{\infty,n}$, the claim follows from [Bar10, Prop. 4.47]. The statement about symmetric monoidal structures is an application of [Bar10, Thm. 4.58]. □

Analogously to Sections 3.1 and 3.2, $\mathcal{H}_{\infty,n}$ is symmetric monoidal whenever $\mathcal{C}$ has finite products.

Lemma 3.39 *The following statements hold true:*

(1) Every projectively cofibrant presheaf of $(\infty, n-1)$-categories $\mathcal{F}$ gives rise to a projectively cofibrant presheaf of (∞, n)-categories $\mathsf{c}_\bullet\mathcal{F}$ on $\mathcal{C}$.

(2) In particular, each $\mathcal{Y}_c$ and each $\tilde{\mathsf{c}}K$ is cofibrant in $\mathcal{H}_{\infty,n}$, for every $c \in \mathcal{C}$ and every $K \in \mathcal{CSS}_k$, for any $k \in \mathbb{N}_0$, where $\tilde{\mathsf{c}}K$ is the constant sheaf with value K.

Proof. This is an immediate consequence of Lemma 3.35 and the definition of projective model structures. □

Proposition 3.40 *The functor $\mathsf{c}_\bullet$ induces a left Quillen functor $\mathsf{c}_\bullet\colon \mathcal{H}^{loc}_{\infty,l} \longrightarrow \mathcal{H}^{loc}_{\infty,l+1}$.*

Proof. We know that there is a Quillen adjunction $\mathsf{c}_\bullet : \mathcal{H}_{\infty,l} \rightleftarrows \mathcal{H}_{\infty,l+1} : \mathrm{Ev}_{[0]}$, where the evaluation takes place in the first simplicial argument. Since weak equivalences between fibrant objects in $\mathcal{CSS}_n$ are nothing but levelwise weak equivalences, and since homotopy limits in $\mathcal{CSS}_n$ can be computed levelwise (Lemma 3.36), it follows that $\mathrm{Ev}_{[0]}$ sends local objects to local objects. Then, the claim follows from Proposition 3.13. □

We now extend Theorem 3.32 to (∞, n)-categories. To that end, observe that $\widehat{\mathcal{C}}$ is V-small category, so that the projective model structure $\widehat{\mathcal{H}}_{\infty,1} = (\mathcal{CSS}^{\widehat{\mathcal{C}}^{\mathrm{op}}})_{proj}$ exists. Consider the functor

$$S^Q_{\infty,n}\colon \mathcal{H}_{\infty,n} \longrightarrow \widehat{\mathcal{H}}_{\infty,n}\,, \qquad (S^Q_{\infty,n}\mathcal{F})(X) = \underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}(\mathsf{c}_\bullet QX, \mathcal{F})\,.$$

Proposition 3.41 *On $\mathcal{F} \in \mathcal{H}_{\infty,n}$ fibrant, the functor $S^Q_{\infty,n}$ agrees with the homotopy right Kan extension of $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{CSS}_n$ along the Yoneda embedding,*

$$S^Q_{\infty,n}\mathcal{F} \cong \mathrm{hoRan}_{\mathcal{Y}^{\mathrm{op}}}(\mathcal{F})\,.$$

Proof. Analogous to Proposition 3.9. □

Proposition 3.42 *There exists a Quillen adjunction*

$$Re^Q_{\infty,n} : \widehat{\mathcal{H}}_{\infty,n} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}_{\infty,n} : S^Q_{\infty,n}\,.$$

Proof. Analogous to the proof of Proposition 3.31. □

We let $(\widehat{\tau})\check{}$ denote the image in $\widehat{\mathcal{H}}_{\infty,n}$ of the Čech nerves of τ-local epimorphisms in $\widehat{\mathcal{H}}_\infty$ under the functor $\mathsf{c}_\bullet$. Then, we define the model category for sheaves of (∞, n)-categories on $\widehat{\mathcal{C}}$ as

$$\widehat{\mathcal{H}}^{loc}_{\infty,n} \coloneqq L_{(\widehat{\tau})\check{}/\mathcal{CSS}_n}\widehat{\mathcal{H}}_{\infty,n}\,.$$

Theorem 3.43 *Let $(\mathcal{C}, \tau)$ be a site. For any $n \in \mathbb{N}$, the Quillen adjunction $Re^Q_{\infty,n} \dashv S^Q_{\infty,n}$ induces a Quillen adjunction*

$$Re^Q_{\infty,n} : \widehat{\mathcal{H}}^{loc}_{\infty,n} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}^{loc}_{\infty,n} : S^Q_{\infty,n}\,.$$

Proof. Our goal is to apply Proposition 3.13 to $S^Q_{\infty,n}$. Hence, we need to check that $S^Q_{\infty,n}$ sends $\check{\tau}/\mathcal{CSS}_n$-local objects to $(\widehat{\tau})\check{}/\mathcal{CSS}_n$-local objects. Let $\mathcal{F} \in \mathcal{H}^{loc}_{\infty,n}$ be a fibrant object, and let $\pi \colon Y \to X$ be a $\widehat{\tau}$-local epimorphism in $\widehat{\mathcal{C}}$. We have to show that the morphism

$$\pi^* \colon (S^Q_{\infty,n}\mathcal{F})(X) \longrightarrow \mathrm{holim}^{\mathcal{CSS}_n}_{\mathbb{\Delta}} \big((S^Q_{\infty,n}\mathcal{F})(\check{C}\pi_\bullet)\big)$$

is a weak equivalence in $\mathcal{CSS}_n$. Since $\mathcal{F}$ is fibrant, π^* is a morphism between fibrant objects in $\mathcal{CSS}_n$ (cf. Lemma 3.29). Consequently, by the construction of $\mathcal{CSS}_n$ as a left Bousfield localisation of $(\mathcal{CSS}_{n-1})^{\mathbb{\Delta}^{\mathrm{op}}}_{inj}$, it follows that π^* is a weak equivalence if and only if it is a levelwise weak equivalence of simplicial objects in $\mathcal{CSS}_{n-1}$. However, since injectively fibrant objects are projectively fibrant, the levels π^*_k of π^* as a morphism of simplicial objects are again maps between fibrant objects (this time in $\mathcal{CSS}_{n-1}$). Iterating this argument, we see that π^* is a weak equivalence in $\mathcal{CSS}_n$ if and only if each of its n-fold simplicial components $\pi^*_{k_0,\ldots,k_{n-1}}$ are weak equivalences in $\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}$. Now the proof is completed entirely in parallel to that of Theorem 3.32. □

4 Two applications

We provide two applications of the theory developed in Section 3. First, we prove that diffeological vector bundles satisfy descent along subductions, i.e. along τ_{dgop}-local epimorphisms between diffeological spaces. Then we show a result that allows to construct smooth FFTs on a manifold M from their values on generators of the bordism category. For motivation and background, we refer the reader to Sections 1.2 and 1.3. Throughout this section we work over the site $(\mathcal{C}\mathrm{art}, \tau_{dgop})$ from Example 2.3.

4.1 Diffeological vector bundles descend along subductions

Recall the notion of a diffeological space (cf. Definition 2.16); in our conventions, these are built from U-small sets.

Definition 4.1 *Let $X \in \mathcal{D}\mathrm{fg}$ be a diffeological space.*

(1) A (complex rank-k) vector bundle *on X is a pair (E, π) of a diffeological space E and a morphism $\pi \colon E \to X$ in $\mathcal{D}\mathrm{fg}$ with the structure of a $\mathbb{C}$-vector space on each fibre $E_{|x} := \pi^{-1}(\{x\})$, satisfying the following condition: for each plot $\varphi \colon c \to X$ there exists an isomorphism of diffeological spaces*

$$\Phi \colon c \times \mathbb{C}^k \longrightarrow c \underset{X}{\times} E$$

such that $\mathrm{pr}_c \circ \Phi = \mathrm{pr}_c$ and such that for every $x \in c$ the restriction $\Phi_{|x} \colon \mathbb{C}^k \to E_{|\varphi(x)}$ is linear.

(2) Let (E, π) and (E', π') be vector bundles on X. A morphism *$(E, \pi) \to (E', \pi')$ is a morphism $\Psi \in \mathcal{D}\mathrm{fg}(E, E')$ with $\pi' \circ \Psi = \pi$ and such that $\Psi_{|x} \colon E_{|x} \to E'_{|x}$ is a linear map for every $x \in X$.*

This defines a category $\mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}(X)$ of diffeological vector bundles on X.

Example 4.2 If M is a smooth manifold, which we view as a diffeological space, then the category $\mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}(X)$ is canonically equivalent to the ordinary category of vector bundles on M. ◁

Any morphism $f \in \mathcal{D}\mathrm{fg}(X', X)$ gives rise to a pullback functor

$$f^* \colon \mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}(X) \longrightarrow \mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}(X'), \qquad (E, \pi) \longmapsto \big(X' \underset{X}{\times} E, \pi'\big),$$

where π' is the pullback of π along f. As for vector bundles on manifolds, this defines a pseudo-functor $\mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}} \colon \mathcal{D}\mathrm{fg}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ (equivalently, there is a cartesian fibration $\mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}} \to \mathcal{D}\mathrm{fg}$). Here, $\mathcal{C}\mathrm{at}$ is the 2-category of V-small categories.

We define a strict presheaf $\mathcal{VB}\mathrm{un}^{\mathrm{triv}}_{\mathcal{C}\mathrm{at}}$ of categories on $\mathcal{C}\mathrm{art}$ as follows: for $c \in \mathcal{C}\mathrm{art}$, the category $\mathcal{VB}\mathrm{un}^{\mathrm{triv}}_{\mathcal{C}\mathrm{at}}(c)$ has objects (c,n), where $n \in \mathbb{N}_0$, and a morphism $(c,n) \to (c,m)$ is a smooth function $\psi \colon c \to \mathsf{Mat}(m{\times}n, \mathbb{C})$ from c to the vector space of complex m-by-n matrices; to make clear that ψ is defined over c we also write (c,ψ) instead of just ψ. A smooth map $f \colon c' \to c$ acts via

$$f^* \colon \mathcal{VB}\mathrm{un}^{\mathrm{triv}}_{\mathcal{C}\mathrm{at}}(c) \longrightarrow \mathcal{VB}\mathrm{un}^{\mathrm{triv}}_{\mathcal{C}\mathrm{at}}(c') \,, \qquad (c,n) \mapsto (c',n) \,, \quad (c,\psi) \mapsto (c', \psi \circ f) \,.$$

Let $\mathcal{H}_{\infty,1}$ denote the model category for presheaves of ∞-categories on the site $(\mathcal{C}\mathrm{art}, \tau_{dgop})$ (as in Definition 3.23). Further, let $N_{rel} \colon \mathcal{C}\mathrm{at} \to s\mathcal{S}\mathrm{et}_{\Delta}$ denote Rezk's classifying diagram functor [Rez01]. Explicitly, if $\mathcal{D}^\sim$ denotes the maximal subgroupoid of a category $\mathcal{D}$, we have

$$(N_{rel}\mathcal{D})_n = N\big((\mathcal{D}^{[n]})^\sim\big) \,.$$

That is, the elements of the set $N_{rel}(\mathcal{D})_{n,m}$ are commutative diagrams in $\mathcal{D}$ of the form

$$\begin{tikzcd}
d_{0,m} \arrow[r] & d_{1,m} \arrow[r] & \cdots \arrow[r] & d_{n,m} \\
\vdots \arrow[u, "\cong"] & \vdots \arrow[u, "\cong"] & \ddots & \vdots \arrow[u, "\cong"] \\
d_{0,1} \arrow[r] \arrow[u, "\cong"] & d_{1,1} \arrow[r] \arrow[u, "\cong"] & \cdots \arrow[r] & d_{n,1} \arrow[u, "\cong"] \\
d_{0,0} \arrow[r] \arrow[u, "\cong"] & d_{1,0} \arrow[r] \arrow[u, "\cong"] & \cdots \arrow[r] & d_{n,0} \arrow[u, "\cong"]
\end{tikzcd}$$

Definition 4.3 *We define the following two presheaves of ∞-categories:*

$$\mathcal{VB}\mathrm{un}^{\mathrm{triv}} := N_{rel} \circ \mathcal{VB}\mathrm{un}^{\mathrm{triv}}_{\mathcal{C}\mathrm{at}} \quad \in \mathcal{H}_{\infty,1} \qquad \textit{and} \qquad \mathcal{VB}\mathrm{un} := S^Q_{\infty,1}\big(\mathcal{VB}\mathrm{un}^{\mathrm{triv}}\big) \quad \in \widehat{\mathcal{H}}_{\infty,1} \,.$$

Proposition 4.4 *Both $\mathcal{VB}\mathrm{un}^{\mathrm{triv}}$ and $\mathcal{VB}\mathrm{un}$ are fibrant objects in $\mathcal{H}^{loc}_{\infty,1}$ and in $\widehat{\mathcal{H}}^{loc}_{\infty,1}$, respectively.*

Proof. We only need to show that $\mathcal{VB}\mathrm{un}^{\mathrm{triv}} \in \mathcal{H}^{loc}_{\infty,1}$ is fibrant. The statement that $\mathcal{VB}\mathrm{un}$ is a fibrant object in $\widehat{\mathcal{H}}^{loc}_{\infty,1}$ will then follow from Theorem 3.32.

Note that $\mathcal{VB}\mathrm{un}^{\mathrm{triv}}$ is fibrant in $\mathcal{H}_{\infty,1}$, since for each $c \in \mathcal{C}\mathrm{art}$, the bisimplicial set $\mathcal{VB}\mathrm{un}^{\mathrm{triv}}(c) = N_{rel}(\mathcal{VB}\mathrm{un}^{\mathrm{triv}}_{\mathcal{C}\mathrm{at}}(c))$ is the classification diagram of an ordinary category; hence, it is a complete Segal space by [Rez01, Prop. 6.1]. We are thus left to show that, in the notation introduced in (2.8), the canonical morphism

$$u \colon \mathcal{VB}\mathrm{un}^{\mathrm{triv}}(c) \longrightarrow \mathrm{holim}^{\mathcal{CSS}}_{\Delta}\Bigg(\cdots \prod_{i_0,\ldots,i_n \in I} \mathcal{VB}\mathrm{un}^{\mathrm{triv}}(C_{i_0,\ldots i_n}) \cdots\Bigg) =: \mathrm{holim}^{\mathcal{CSS}}_{\Delta}\big(\mathcal{VB}\mathrm{un}^{\mathrm{triv}}(\check{C}\mathcal{U}_\bullet)\big)$$

is a weak equivalence in $\mathcal{CSS}$ for each differentiably good open covering $\mathcal{U} = \{c_i \hookrightarrow c\}_{i\in I}$. We recall from Lemma 3.28 that the homotopy limit may be computed level-wise. Furthermore, by Theorem 3.30 it suffices to check that the morphisms u_0 and u_1 are weak equivalences in $\mathcal{S}\mathrm{et}_{\Delta}$. That is, we need to check whether the simplicial presheaves $\mathcal{VB}\mathrm{un}^{\mathrm{triv}}_0$ and $\mathcal{VB}\mathrm{un}^{\mathrm{triv}}_1$ are fibrant in $\mathcal{H}^{loc}_{\infty}$.

The simplicial presheaf $\mathcal{VB}\mathrm{un}_0^{\mathrm{triv}}$ assigns to $c \in \mathcal{C}\mathrm{art}$ the nerve $\mathcal{VB}\mathrm{un}_0^{\mathrm{triv}}(c) = N(\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}^{\mathrm{triv}\sim}(c))$ of the maximal subgroupoid of $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}^{\mathrm{triv}}(c)$. Concretely, $\mathcal{VB}\mathrm{un}_0^{\mathrm{triv}}(c)$ has objects (c, n) with $n \in \mathbb{N}_0$ and morphisms

$$\mathcal{VB}\mathrm{un}_0^{\mathrm{triv}}(c)\big((c,n),(c,m)\big) = \begin{cases} \mathcal{M}\mathrm{fd}\big(c, \mathsf{GL}(n,\mathbb{C})\big)\,, & n = m\,, \\ \emptyset\,, & n \neq m\,. \end{cases}$$

The homotopy limit $\mathrm{holim}_{\mathbb{\Delta}}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(\mathcal{VB}\mathrm{un}_0^{\mathrm{triv}}(\check{C}\mathcal{U}_\bullet))$ is the nerve of the groupoid whose objects are the $\mathsf{GL}(n,\mathbb{C})$-valued Čech 1-cocycles on c, subordinate to the covering $\mathcal{U}$, for any $n \in \mathbb{N}_0$, and whose morphisms are the $\mathsf{GL}(n,\mathbb{C})$-valued Čech 1-coboundaries subordinate to $\mathcal{U}$. The morphism

$$u_0 \colon \mathcal{VB}\mathrm{un}_0^{\mathrm{triv}}(c) \longrightarrow \mathrm{holim}_{\mathbb{\Delta}}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\big(\mathcal{VB}\mathrm{un}_0^{\mathrm{triv}}(\check{C}\mathcal{U}_\bullet)\big)$$

is induced via the nerve from the functor $\widetilde{u_0}$ that sends the object (c, n) to the cocycle given by $g_{ij} = \mathbb{1}_n \in \mathsf{GL}(n,\mathbb{C})$, for $i, j \in I$ and $n \in \mathbb{N}_0$, and that sends the morphism $\psi \colon (c,n) \to (c,n)$ to the coboundary $h_i = \psi_{|c_i}$, for $i \in I$. The functor $\widetilde{u_0}$ is an equivalence: it is fully faithful since $\underline{\mathsf{GL}(n,\mathbb{C})}$ is a sheaf on $(\mathcal{C}\mathrm{art}, \tau_{dgop})$ (see Example 2.3 and Remark 2.20). It is essentially surjective since the Čech cohomology group $\check{\mathrm{H}}^1(c, \mathsf{GL}(n,\mathbb{C}))$ is trivial for all $n \in \mathbb{N}_0$ (since $c \cong \mathbb{R}^k$ for some $k \in \mathbb{N}_0$). Consequently, the morphism $u_0 = N\widetilde{u_0}$ is a weak equivalence of simplicial sets.

We turn to u_1: the simplicial set $\mathcal{VB}\mathrm{un}_1^{\mathrm{triv}}(c)$ is the nerve of a groupoid whose objects are morphisms $(c,\psi) \colon (c,n) \to (c,m)$ in $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}^{\mathrm{triv}}(c)$, where $\psi \colon c \to \mathsf{Mat}(m{\times}n,\mathbb{C})$ is a smooth map. Let $(c,\psi_0), (c,\psi_1)$ be objects with $(c,\psi_a) \colon (c, n_{0,a}) \to (c, n_{1,a})$ for $a = 0,1$. Morphisms $(c,\psi_0) \to (c,\psi_1)$ are given by triples (c, g_0, g_1) such that

$$\begin{tikzcd}
(c, n_{0,1}) \arrow[r, "{(c,\psi_1)}"] & (c, n_{1,1}) \\
(c, n_{0,0}) \arrow[r, "{(c,\psi_0)}"'] \arrow[u, "{(c,g_0)}"', "\cong"] & (c, n_{1,0}) \arrow[u, "{(c,g_1)}", "\cong"']
\end{tikzcd}$$

is a commutative diagram in $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}^{\mathrm{triv}}(c)$ whose vertical morphisms are isomorphisms. In particular, we have that $n_{a,0} = n_{a,1}$ for $a = 0,1$.

The homotopy limit $\mathrm{holim}_{\mathbb{\Delta}}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(\mathcal{VB}\mathrm{un}_1^{\mathrm{triv}}(\check{C}\mathcal{U}_\bullet))$ is the nerve of the following groupoid: its objects are triples (g, g', ψ), where $g = (g_{ij})$ (resp. $g' = (g'_{ij})$) is a $\mathsf{GL}(n,\mathbb{C})$-valued (resp. $\mathsf{GL}(n',\mathbb{C})$-valued) 1-cocycle on c, subordinate to $\mathcal{U}$. Further, $\psi = (\psi_i)$ is a collection of smooth maps $\psi_i \colon c_i \to \mathsf{Mat}(n'{\times}n,\mathbb{C})$ such that

$$g'_{ij}\,\psi_i = \psi_j\,g_{ij} \qquad \forall\, i,j \in I\,. \tag{4.5}$$

In particular, (c, g) and (c, g') are vertices in $\mathrm{holim}_{\mathbb{\Delta}}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(\mathcal{VB}\mathrm{un}_0^{\mathrm{triv}}(\check{C}\mathcal{U}_\bullet))$. A morphism $(g, g', \psi) \to (\tilde{g}, \tilde{g}', \tilde{\psi})$ consists of a pair (h, h') of morphisms $(c, h) \colon (c, g) \to (c, \tilde{g})$ and $(c, h') \colon (c, g') \to (c, \tilde{g}')$ in $\mathrm{holim}_{\mathbb{\Delta}}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(\mathcal{VB}\mathrm{un}_0^{\mathrm{triv}}(\check{C}\mathcal{U}_\bullet))$, satisfying the additional condition that

$$h'_i\psi_i = \tilde{\psi}_i h_i \qquad \forall\, i \in I\,.$$

The morphism u_1 is the nerve of the functor $\widetilde{u_1}$ which sends an object (c,ψ) to the object $(\mathbb{1}_n, \mathbb{1}_{n'}, \psi)$ and a morphism (c, g_0, g_1) to (g_0, g_1). The functor $\widetilde{u_1}$ is fully faithful, again because $\underline{\mathsf{GL}(n,\mathbb{C})}$ is a sheaf on $(\mathcal{C}\mathrm{art}, \tau_{dgop})$. Moreover, $\widetilde{u_1}$ is essentially surjective: let (g, g', ψ) be an object in the groupoid whose nerve is $\mathrm{holim}_{\mathbb{\Delta}}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}(\mathcal{VB}\mathrm{un}_1^{\mathrm{triv}}(\check{C}\mathcal{U}_\bullet))$. Since $\check{\mathrm{H}}^1(c, \mathsf{GL}(n,\mathbb{C})) \cong 0$ for all $n \in \mathbb{N}_0$, we find coboundaries $h = (h_i)$ and $h' = (h'_i)$ that establish equivalences $g \sim \mathbb{1}_n$ and $g' \sim \mathbb{1}_{n'}$. These provide an isomorphism $(h, h') \colon (g, g', \psi) \to (\mathbb{1}_n, \mathbb{1}_{n'}, \tilde{\psi})$, where $\tilde{\psi}_i = h'_i\psi_i h_i^{-1}$ for all $i \in I$. The condition (4.5) on the object

$(\mathbb{1}_n, \mathbb{1}_{n'}, \tilde{\psi})$ now reads as $\tilde{\psi}_{i|c_{ij}} = \tilde{\psi}_{j|c_{ij}}$ for all $i, j \in I$. The essential surjectivity of $\widetilde{u_1}$ thus follows from the fact that $\underline{\mathsf{Mat}(n' \times n, \mathbb{C})}$ is a sheaf on $(\mathcal{C}\mathrm{art}, \tau_{dgop})$. Consequently, $u_1 = N\widetilde{u_1}$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$. This completes the proof that $\mathcal{VB}\mathrm{un}^{\mathrm{triv}}$ is fibrant in $\mathcal{H}^{loc}_{\infty,n}$. ☐

Proposition 4.6 *There is a strict presheaf of categories $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}} \colon \mathcal{D}\mathrm{fg}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ on the category of diffeological spaces such that*

$$\mathcal{VB}\mathrm{un} = N_{rel} \circ \mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}\,.$$

Proof. We construct $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}$ explicitly: recall the explicit formula

$$\mathcal{VB}\mathrm{un}(X) = \big(S^Q_{\infty,1}\,\mathcal{VB}\mathrm{un}\big)(X) = \underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}\big(QX, \mathcal{VB}\mathrm{un}^{\mathrm{triv}}\big)\,.$$

At level zero we thus have

$$\mathcal{VB}\mathrm{un}_0(X) = \underline{\mathcal{H}}^{\mathcal{CSS}}_{\infty,1}\big(QX, \mathcal{VB}\mathrm{un}^{\mathrm{triv}}\big)_0 \cong \underline{\mathcal{H}}_{\infty}\big(QX, \mathcal{VB}\mathrm{un}^{\mathrm{triv}}_0\big)\,.$$

A vertex of the simplicial set $\mathcal{VB}\mathrm{un}_0(X)$ is therefore a morphism $QX \to \mathcal{VB}\mathrm{un}^{\mathrm{triv}}_0 = N(\mathcal{VB}\mathrm{un}^{\mathrm{triv}\sim}_{\mathcal{C}\mathrm{at}})$ of simplicial presheaves on $\mathcal{C}\mathrm{art}$. In explicit terms, such a morphism consists of the following data: for every plot $\varphi \colon c \to X$ we have an object of $\mathcal{VB}\mathrm{un}^{\mathrm{triv}}_{\mathcal{C}\mathrm{at}}(c)$, which we denote by $(\varphi, n(\varphi))$. For every morphism $f \colon c_0 \to c_1$ in $\mathcal{C}\mathrm{art}$, and every plot $\varphi \in X(c_1)$ with assigned object $(\varphi, n_1(\varphi))$, we have an isomorphism

$$h_f \colon \big(\varphi \circ f, n_0(\varphi \circ f)\big) \longrightarrow f^*\big(\varphi, n_1(\varphi)\big)$$

in $\mathcal{VB}\mathrm{un}^{\mathrm{triv}}_{\mathcal{C}\mathrm{at}}(c_0)$. Observe that this is just a smooth map $h_f \colon c_0 \to \mathsf{GL}(n_0, \mathbb{C})$. Finally, for every composable sequence $(f_0 \colon c_0 \to c_1,\ f_1 \colon c_1 \to c_2)$ in $\mathcal{C}\mathrm{art}$ and for every $\varphi \in X(c_2)$ with assigned objects $(\varphi, n_2(\varphi))$, $(\varphi \circ f_1, n_1(\varphi \circ f))$, and $(\varphi \circ f_1 \circ f_0, n_0(\varphi \circ f_1 \circ f_0))$, as well as assigned morphisms h_{f_0}, h_{f_1}, and $h_{f_1 \circ f_0}$, we have the relation

$$f_0^* h_{f_1} \cdot h_{f_0} = h_{f_1 \circ f_0}\,.$$

Furthermore, if $f = 1_c$, then $h_f = \mathbb{1}_{n(\varphi)}$. We denote the data of a vertex of $\mathcal{VB}\mathrm{un}_0(X)$ by (n, h), where n assigns to each plot $\varphi \colon c \to X$ the number $n(\varphi)$, and where h assigns to each morphism $f \colon \varphi' \to \varphi$ of plots of X the map h_f.

A 1-simplex in $\mathcal{VB}\mathrm{un}_0(X)$ with initial vertex denoted (n_0, h_0) and final vertex denoted (n_1, h_1) is given by specifying, for each plot $\varphi \colon c \to X$, a smooth map $g_\varphi \colon c \to \mathsf{GL}(n_0, \mathbb{C})$ such that, for every morphism $f \colon \varphi_0 \to \varphi_1$ of plots of X, we have that

$$f^* g_{\varphi_1} \cdot h_f = h_f \cdot g_{\varphi_0}\,. \tag{4.7}$$

All higher simplices in $\mathcal{VB}\mathrm{un}_0(X)$ are degenerate. In fact, $\mathcal{VB}\mathrm{un}_0(X)$ is the nerve of a groupoid (which is uniquely determined by the 0- and 1-simplices described above); we write

$$\mathcal{VB}\mathrm{un}_0(X) = N\big(\mathcal{VB}\mathrm{un}^{\sim}_{\mathcal{C}\mathrm{at}}(X)\big)\,.$$

The notation $\mathcal{VB}\mathrm{un}^{\sim}_{\mathcal{C}\mathrm{at}}(X)$ suggests that this groupoid is in fact the maximal subgroupoid of a category. This is indeed the case: a vertex in $\mathcal{VB}\mathrm{un}_1(X)$ is given by a pair (n_i, h_i), $i = 0, 1$, of vertices of $\mathcal{VB}\mathrm{un}_0(X)$ and an assignment of a smooth map $g_\varphi \colon c \to \mathsf{Mat}(n_1 \times n_0, \mathbb{C})$ to each plot $\varphi \colon c \to X$ of X, such that g satisfies the condition (4.7). A 1-simplex in $\mathcal{VB}\mathrm{un}_1(X)$ from g_0 to g_1 is an assignment of a pair of smooth maps $(k_{\varphi_0}, k'_{\varphi_1}) \colon c \to \mathsf{GL}(n_0(\varphi), \mathbb{C}) \times \mathsf{GL}(n_1(\varphi), \mathbb{C})$ to each plot $\varphi \colon c \to X$ such that

$$k'_\varphi \circ g_{0,\varphi} = g_{1,\varphi} \circ k_\varphi$$

holds true for every plot φ of X. Further, k and k' each satisfy an analogue of condition (4.7) with respect to g_0 and g_1, respectively. All higher simplices of $\mathcal{VB}\mathrm{un}_1(X)$ are degenerate. In fact, $\mathcal{VB}\mathrm{un}(X)$ is the classification diagram of a category,

$$\mathcal{VB}\mathrm{un}(X) = N_{rel}\big(\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}(X)\big)\,.$$

Then, $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}$ is turned into a (strict) functor $\mathcal{D}\mathrm{fg}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ by letting a morphism $F \in \mathcal{D}\mathrm{fg}(Y,X)$ act on objects as $(n,h) \mapsto (F^*n, F^*h)$, where $(F^*n)(\varphi) = n(\varphi \circ F)$ and $(F^*h)_f = h_f$, and on morphisms as $g \mapsto F^*g$, with $(F^*g)_\varphi = g_{F\circ\varphi}$. □

Theorem 4.8 *There is an objectwise equivalence*

$$\mathcal{A}\colon \mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}} \xrightarrow{\sim} \mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}$$

of (non-strict) presheaves of categories on $\mathcal{D}\mathrm{fg}$.

The proof of Theorem 4.8 is somewhat lengthy; we provide it in Appendix B.

Corollary 4.9 *For any diffeological space X, there is a weak equivalence of complete Segal spaces*

$$N_{rel}\mathcal{A}\colon \mathcal{VB}\mathrm{un}(X) \xrightarrow{\sim} N_{rel}\mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}(X)\,.$$

Recall that $\mathcal{D}\mathrm{fg}$ with the τ_{dgop}-local epimorphisms between diffeological spaces forms a site (Remark 2.21). Its covering families each consist of a single τ_{dgop}-local epimorphism.

Corollary 4.10 *The presheaf of categories $\mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}$ on $\mathcal{D}\mathrm{fg}$ satisfies descent with respect to τ_{dgop}-local epimorphisms.*

Proof. We have already shown in Proposition 4.4 that $\mathcal{VB}\mathrm{un}$ is fibrant in $\widehat{\mathcal{H}}^{loc}_{\infty,1}$. Thus, for any τ_{dgop}-local epimorphism $\pi\colon Y \to X$ in $\mathcal{D}\mathrm{fg}$ the canonical morphism

$$\mathcal{VB}\mathrm{un}(X) \longrightarrow \mathrm{holim}^{\mathcal{CSS}}_{\mathbb{\Delta}}\big(\mathcal{VB}\mathrm{un}(\check{C}\pi_\bullet)\big)$$

is a weak equivalence, where $\check{C}\pi_\bullet$ denotes the Čech nerve of π. The homotopy limit on the right-hand side is precisely the result of applying the classification diagram functor N_{rel} to the descent category of $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}$ with respect to π. The fact that N_{rel} reflects weak equivalences [Rez01, Thm. 3.7] now implies that the presheaf of categories $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}$ satisfies descent with respect to π. We finally observe that a weak equivalence of presheaves of categories also induces an equivalence of descent categories. Then, the two-out-of-three property of weak equivalences implies that $\mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}$ satisfies descent with respect to π. □

4.2 Descent and coherence for smooth functorial field theories

Let Y be an oriented manifold, let $\mathcal{M}\mathrm{fd}^{\mathsf{or}}$ denote the groupoid of oriented manifolds and orientation-preserving diffeomorphisms, and let $\mathcal{M}_Y$ be the connected component of Y in $\mathcal{M}\mathrm{fd}^{\mathsf{or}}$. For $X, X' \in \mathcal{M}\mathrm{fd}^{\mathsf{or}}$, we let $\mathrm{D}(X,X')$ denote the diffeological space of diffeomorphisms from X to X'. Concretely, a map $\varphi\colon c \to \mathrm{D}(X,X')$ is a plot if and only if its adjoint map $\varphi^\dashv\colon c \times X \to X'$ is a smooth map of manifolds. This establishes both $\mathcal{M}\mathrm{fd}^{\mathsf{or}}$ and $\mathcal{M}_Y$ as categories enriched in diffeological spaces; the $\mathcal{D}\mathrm{fg}$-enriched mapping spaces are $\underline{\mathcal{M}}_Y^{\mathcal{D}\mathrm{fg}}(Y_0,Y_1) = \mathrm{D}(Y_0,Y_1)$. We also use the shorthand notation $\mathrm{D}(Y) := \mathrm{D}(Y,Y)$.

Remark 4.11 One could replace the category $\mathcal{D}\mathrm{fg}$ by the category $\widehat{\mathcal{C}\mathrm{art}}$ throughout this subsection without changing any of the results. The reason that we work with the more special choice of $\mathcal{D}\mathrm{fg}$ rather than $\widehat{\mathcal{C}\mathrm{art}}$ is that it allows to use diffeological vector bundles as discussed in Section 4.1 in this context. This is desirable in field theory constructions – see for instance [BW19]. $\triangleleft$

Recall that $\mathcal{D}\mathrm{fg}$ is cartesian closed [BH11], so that it is in particular enriched, tensored, and cotensored over $\mathcal{D}\mathrm{fg}$ itself. Consider $\mathcal{D}\mathrm{fg}$-enriched functors

$$P\colon \mathcal{M}_Y^{\mathrm{op}} \longrightarrow \mathcal{D}\mathrm{fg}\,, \qquad G\colon \mathcal{M}_Y \longrightarrow \mathcal{D}\mathrm{fg}\,,$$

as well as the functors G^n, with $G^n(X) \coloneqq (G(X))^n$. Setting $G^0(X) \coloneqq *$ gives rise to an augmented simplicial object

$$G^{\bullet+1} \in \big(\mathcal{C}\mathrm{at}(\mathcal{M}_Y, \mathcal{D}\mathrm{fg})\big)^{\mathbb{\Delta}_+^{\mathrm{op}}}\,,$$

where $\mathbb{\Delta}_+$ is the simplex category $\mathbb{\Delta}$ with an initial object $[-1]$ adjoined to it.

Example 4.12 In the context of smooth field theories on a manifold M, the relevant choice of P is $M^{(-)}$, which assigns to $Y' \in \mathcal{M}_Y$ the diffeological mapping space $M^{Y'}$. $\triangleleft$

This setup allows us to form, for each $n \in \mathbb{N}_0$, the enriched two-sided simplicial bar construction [Rie14, Section 9.1], which produces a simplicial object

$$B_\bullet\big(G^n, \mathcal{M}_Y^{\mathrm{op}}, P\big) \ \in \mathcal{D}\mathrm{fg}^{(\mathbb{\Delta}^{\mathrm{op}})}\,.$$

Explicitly, we have

$$B_k\big(G^n, \mathcal{M}_Y^{\mathrm{op}}, P\big) = \coprod_{Y_0,\ldots,Y_k \in \mathcal{M}_Y} P(Y_0) \times \mathrm{D}(Y_1, Y_0) \times \cdots \times \mathrm{D}(Y_k, Y_{k-1}) \times G^n(Y_k)\,.$$

We now consider the explicit case of $G = \mathrm{D}(Y, -)\colon \mathcal{M}_Y \to \mathcal{D}\mathrm{fg}$, so that we have

$$B_0\big(\mathrm{D}(Y,-)^n, \mathcal{M}_Y^{\mathrm{op}}, P\big) = \coprod_{Y_0 \in \mathcal{M}_Y} P(Y_0) \times \mathrm{D}(Y, Y_0)^n\,.$$

Define morphisms Φ_n as the composition

$$\begin{tikzcd}
P(Y_0) \times \mathrm{D}(Y,Y_0)^{n+1} \arrow[rr] \arrow[d, dashed, "\Phi_n"'] & & P(Y_0) \times \mathrm{D}(Y,Y_0) \times \mathrm{D}(Y,Y_0)^n \arrow[d, "1\times\Delta\times 1"] \\
P(Y) \times \mathrm{D}(Y,Y)^n & & P(Y_0) \times \mathrm{D}(Y,Y_0)^2 \times \mathrm{D}(Y,Y_0)^n \arrow[ll, "\mathrm{Ev}\times\delta^n"]
\end{tikzcd}$$

where Δ denotes the diagonal morphism, and where $\mathrm{Ev}\colon P(Y_0) \times \mathrm{D}(Y,Y_0) \to P(Y)$ is defined via the tensor adjunction: it is the adjoint of the morphism $P_{Y_0,Y}\colon \mathrm{D}(Y,Y_0) = \underline{\mathcal{M}}_Y^{\mathrm{op},\mathcal{D}\mathrm{fg}}(Y_0, Y) \longrightarrow \underline{\mathcal{D}\mathrm{fg}}^{\mathcal{D}\mathrm{fg}}(PY_0, PY)$. The morphism δ acts as

$$(f, f_1, \ldots, f_n) \longmapsto (f^{-1}f_1, f_1^{-1}f_2, \ldots, f_{n-1}^{-1}f_n)\,.$$

Let $\mathsf{BD}(Y) \subset \mathcal{M}_Y$ denote the full $\mathcal{D}\mathrm{fg}$-enriched subcategory on the object Y. We call the simplicial object

$$\big(P(Y) /\!\!/ \mathrm{D}(Y)\big)_\bullet \coloneqq B_\bullet\big(*, \mathsf{BD}(Y)^{\mathrm{op}}, P\big) \in \mathcal{D}\mathrm{fg}^{(\mathbb{\Delta}^{\mathrm{op}})}$$

the *action groupoid* of the $\mathrm{D}(Y)$-action via Ev on $P(Y) \in \mathcal{D}\mathrm{fg}$.

Lemma 4.13 *For every $n \in \mathbb{N}_0$, the morphism*

$$\Phi_n \colon B_0\big(\mathrm{D}(Y,-)^{n+1}, \mathcal{M}_Y^{\mathrm{op}}, P\big) \longrightarrow \big(P(Y) /\!/ \mathrm{D}(Y)\big)_n$$

is an augmentation of the simplicial object $B_\bullet\big(\mathrm{D}(Y,-)^{n+1}, \mathcal{M}_Y^{\mathrm{op}}, P\big)$ in $\mathcal{D}\mathrm{fg}$.

Proof. This follows directly from the compatibility of Ev with compositions. $\square$

We further define morphisms

$$\Psi_k \colon B_k\big(\mathrm{D}(Y,-), \mathcal{M}_Y^{\mathrm{op}}, P\big) \longrightarrow B_k\big(*, \mathcal{M}_Y^{\mathrm{op}}, P\big)\,,$$

which arise from the augmentation $\mathrm{D}(Y,-)^\bullet \to \mathrm{D}(Y,-)^0 = *$.

Proposition 4.14 *Let $k \in \mathbb{N}_0$.*

(1) The morphism

$$\Psi_k \colon B_k\big(\mathrm{D}(Y,-), \mathcal{M}_Y^{\mathrm{op}}, P\big) \longrightarrow B_k\big(*, \mathcal{M}_Y^{\mathrm{op}}, P\big)$$

is an augmentation of the simplicial object $B_k\big(\mathrm{D}(Y,-)^{\bullet+1}, \mathcal{M}_Y^{\mathrm{op}}, P\big)$ in $\mathcal{D}\mathrm{fg}$.

(2) The morphism Ψ_k is a subduction, and the simplicial object $B_k\big(\mathrm{D}(Y,-)^{\bullet+1}, \mathcal{M}_Y^{\mathrm{op}}, P\big)$ is isomorphic the Čech nerve of Ψ_k.

Proof. Part (1) is immediate from the construction of Ψ_k. For part (2), we only need to observe that Ψ_k is a coproduct of projections onto a factor in a product and use Lemmas 2.7 and 2.28. $\square$

Remark 4.15 Recall from Proposition 2.22 that there is a fully faithful right adjoint $\iota \colon \mathcal{D}\mathrm{fg} \to \widehat{\mathcal{C}\mathrm{art}}$, which we will be using implicitly. Under the functor ι, the Čech nerve of Ψ_k in $\mathcal{D}\mathrm{fg}$ agrees with the Čech nerve of $\iota(\Psi_k)$ in $\widehat{\mathcal{C}\mathrm{art}}$. ◁

For $k, l \in \mathbb{N}_0$, we introduce the short-hand notation

$$\mathrm{Coh}_{k,l}(Y,P) \coloneqq B_k\big(\mathrm{D}(Y,-)^{l+1}, \mathcal{M}_Y^{\mathrm{op}}, P\big) \quad \in \widehat{\mathcal{C}\mathrm{art}}\,.$$

Corollary 4.16 *We obtain a bisimplicial object in $\mathcal{D}\mathrm{fg}$ (and hence in $\widehat{\mathcal{C}\mathrm{art}}$) which is augmented in each direction,*

$$\begin{tikzcd} \mathrm{Coh}(Y,P) \arrow[r, "\Phi_\bullet"] \arrow[d, "\Psi_\bullet"] & \big(P(Y) /\!/ \mathrm{D}(Y)\big)_\bullet \\ B_\bullet\big(*, \mathcal{M}_Y^{\mathrm{op}}, P\big) & \end{tikzcd} \tag{4.17}$$

and where the vertical simplicial objects are the Čech nerves of the subductions Ψ_k.

Explicitly, diagram (4.17) reads as

$$
\begin{tikzcd}
 & \vdots & & \vdots & & \vdots \\
\cdots \coprod\limits_{Y_0, Y_1 \in \mathcal{M}_Y} & P(Y_0) \times \mathrm{D}(Y_1, Y_0) \times \mathrm{D}(Y, Y_1)^2 \arrow[rr, shift left, "d_i^h"] \arrow[rr, shift right] \arrow[d, shift left] \arrow[d, shift right, "d_i^v"'] & & \coprod\limits_{Y_0 \in \mathcal{M}_Y} P(Y_0) \times \mathrm{D}(Y, Y_0)^2 \arrow[rr, "\Phi_1"] \arrow[d, shift left] \arrow[d, shift right, "d_i^v"'] & & P(Y) \times \mathrm{D}(Y) \arrow[d, shift left] \arrow[d, shift right, "d_i"'] \\
\cdots \coprod\limits_{Y_0, Y_1 \in \mathcal{M}_Y} & P(Y_0) \times \mathrm{D}(Y_1, Y_0) \times \mathrm{D}(Y, Y_1) \arrow[rr, shift left, "d_i^h"] \arrow[rr, shift right] \arrow[d, "\Psi_1"] \arrow[u] & & \coprod\limits_{Y_0 \in \mathcal{M}_Y} P(Y_0) \times \mathrm{D}(Y, Y_0) \arrow[rr, "\Phi_0"] \arrow[d, "\Psi_0"] \arrow[u] \arrow[ll] & & P(Y) \arrow[u] \\
\cdots \coprod\limits_{Y_0, Y_1 \in \mathcal{M}_Y} & P(Y_0) \times \mathrm{D}(Y_1, Y_0) \arrow[rr, shift left, "d_i"] \arrow[rr, shift right] & & \coprod\limits_{Y_0 \in \mathcal{M}_Y} P(Y_0) \arrow[ll] & &
\end{tikzcd}
$$

Applying ι to diagram (4.17), we obtain a bisimplicial object in $\widehat{\mathcal{C}\mathrm{art}}$ with augmentations in each direction, which we will denote by the same symbols as in (4.17). Observe that by Lemma 2.28 ι commutes with coproducts.

Proposition 4.18 *The morphisms $\Phi_\bullet$ and $\Psi_\bullet$ have the following properties:*

(1) Each Φ_n induces a weak equivalence in $\mathcal{H}_\infty$:

$$
\Phi_n \colon \mathrm{Coh}_{\bullet,n}(Y,P) \xrightarrow{\sim} \mathsf{c}_\bullet\big(P(Y)/\!\!/\mathrm{D}(Y)\big)_n\,.
$$

(2) Each Ψ_k is a $\widehat{\tau}_{dgop}$-local epimorphism. Consequently, it induces a weak equivalence in $\mathcal{H}_\infty^{loc}$:

$$
\Psi_k \colon \mathrm{Coh}_{k,\bullet}(Y,P) \xrightarrow{\sim} \mathsf{c}_\bullet B_k\big(*, \mathcal{M}_Y^{\mathrm{op}}, P\big)\,.
$$

Proof. Ad (1): We claim that the augmented simplicial object

$$
\mathrm{Coh}_{\bullet,n}(Y,P) = B_\bullet\big(\mathrm{D}(Y,-)^{n+1}, \mathcal{M}_Y^{\mathrm{op}}, P\big) \xrightarrow{\Phi_n} P(Y) \times \mathrm{D}(Y)^n
$$

in $\mathcal{H}_\infty$ admits extra degeneracies. To see this, we define maps of diffeological spaces

$$
\begin{aligned}
s_{-1|-1} \colon P(Y) \times \mathrm{D}(Y)^n &\longrightarrow \coprod_{Y_0 \in \mathcal{M}_Y} P(Y_0) \times \mathrm{D}(Y, Y_0)^{n+1}\,,\\
(\gamma, f_1, \ldots, f_n) &\longmapsto (\gamma, 1_Y, f_1, \ldots, f_n)\,,
\end{aligned}
$$

where the image lies in the summand labelled by $Y \in \mathcal{M}_Y$. For $k \in \mathbb{N}_0$, we set

$$
\begin{aligned}
s_{-1|k} \colon &\coprod_{Y_0, \ldots, Y_k \in \mathcal{M}_Y} P(Y_0) \times \mathrm{D}(Y_1, Y_0) \times \ldots \times \mathrm{D}(Y_k, Y_{k-1}) \times \mathrm{D}(Y, Y_k)^{n+1}\\
&\longrightarrow \coprod_{Y_0, \ldots, Y_{k+1} \in \mathcal{M}_Y} P(Y_0) \times \mathrm{D}(Y_1, Y_0) \times \ldots \times \mathrm{D}(Y_{k+1}, Y_k) \times \mathrm{D}(Y, Y_{k+1})^{n+1}\,,\\
&(\gamma, g_0, \ldots, g_{k-1}, f_0, \ldots, f_n) \longmapsto (\gamma, g_0, \ldots, g_{k-1}, 1_Y, f_0, \ldots, f_n)\,,
\end{aligned}
$$

where the image lies in the component of the coproduct labelled by $(Y_0, \ldots, Y_k, Y)$. These are indeed morphisms of diffeological spaces, and their images under the forgetful functor $\iota \colon \mathcal{D}\mathrm{fg} \to \widehat{\mathcal{C}\mathrm{art}}$ yield the desired extra degeneracies.

Ad (2): This follows from Proposition 4.14 together with Theorem 3.14. $\square$

Proposition 4.19 *With the above notation, the following statements hold true:*

(1) We have a weak equivalence in $\mathcal{H}_\infty$*:*

$$Q\,\mathrm{diag}(\Phi)\colon Q \circ \mathrm{diag}\big(\mathrm{Coh}(Y,P)\big) \xrightarrow{\sim} Q\big(P(Y)/\!\!/\mathrm{D}(Y)\big)\,.$$

(2) We have a weak equivalence in $\mathcal{H}_\infty^{loc}$*:*

$$Q\,\mathrm{diag}(\Psi)\colon Q \circ \mathrm{diag}\big(\mathrm{Coh}(Y,P)\big) \xrightarrow{\sim} Q\big(B(*,\mathcal{M}_Y^{\mathrm{op}},P)\big)\,.$$

Proof. By Proposition 4.18, the morphism $\Phi\colon \mathrm{Coh}(Y,P) \to \mathsf{c}_\bullet(P(Y)/\!\!/\mathrm{D}(Y))$ is a levelwise weak equivalence of simplicial objects in $\mathcal{H}_\infty$. Since the functor $\mathrm{diag} \cong |-|\colon s\mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ is homotopical (it is left Quillen and all objects in $s\mathcal{S}\mathrm{et}_\Delta$ are cofibrant), $\mathrm{diag}(\Phi)$ is a weak equivalence in $\mathcal{H}_\infty$. This implies (1), since Q is homotopical.

For claim (2), we observe that we have a commutative diagram

$$\begin{tikzcd}
\operatorname*{hocolim}^{\mathcal{H}_\infty}_{k\in\Delta^{\mathrm{op}}} Q\big(\mathrm{Coh}(Y,P)_k\big) \arrow[r, "\mathrm{hocolim}\,Q\Psi_k", "\sim"'] \arrow[d, "\sim"] & \operatorname*{hocolim}^{\mathcal{H}_\infty}_{k\in\Delta^{\mathrm{op}}} Q\big(B(*,\mathcal{M}_Y^{\mathrm{op}},P)_k\big) \arrow[d, "\sim"] \\
Q\circ\mathrm{diag}\big(\mathrm{Coh}(Y,P)\big) \arrow[r, "Q\circ\mathrm{diag}(\Psi)"'] & Q\big(B(*,\mathcal{M}_Y^{\mathrm{op}},P)\big)
\end{tikzcd}$$

The top morphism is a local weak equivalence by Proposition 4.18, and the vertical morphisms are projective weak equivalence by Proposition A.3. It follows that the bottom morphism is a local weak equivalence. □

Definition 4.20 *Let* $n \in \mathbb{N}_0$*, and let* $\mathcal{F} \in \mathcal{H}_{\infty,n}$*.*

(1) We define (∞,n)*-categories*

$$\mathcal{F}(P)^{\mathrm{D}(Y)} \coloneqq C_{\mathcal{CSS}_n}\big(*,\Delta,(S^Q_{\infty,n}\mathcal{F})\big(P(Y)/\!\!/\mathrm{D}(Y)\big)\big)\,,$$
$$\mathcal{F}(P)^{\mathrm{D}(Y)}_{red} \coloneqq \underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}\big(Q(P(Y)/\!\!/\mathrm{D}(Y)),\mathcal{F}\big)\,,$$

and we call $\mathcal{F}(P)^{\mathrm{D}(Y)}$ *the* (∞,n)*-category of* equivariant sections of $\mathcal{F}$ over $P(Y)$*.*

(2) We define (∞,n)*-categories*

$$\mathcal{F}(P)^{coh} \coloneqq C_{\mathcal{CSS}_n}\big(*,\Delta,(S^Q_{\infty,n}\mathcal{F})\big(B_\bullet(*,\mathcal{M}_Y^{\mathrm{op}},P)\big)\big)$$
$$\mathcal{F}(P)^{coh}_{red} \coloneqq \underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}\big(Q\big(B_\bullet(*,\mathcal{M}_Y^{\mathrm{op}},P)\big),\mathcal{F}\big)\,,$$

and we call $\mathcal{F}(P)^{coh}$ *the* (∞,n)*-category of* coherent sections of $\mathcal{F}$ over P*.*

Note that $\mathcal{F}(P)^{\mathrm{D}(Y)}$ and $\mathcal{F}(P)^{coh}$ are models for the homotopy limits

$$\mathcal{F}(P)^{\mathrm{D}(Y)} = \operatorname*{holim}^{\mathcal{CSS}_n}_{k\in\Delta}\ (S^Q_{\infty,n}\mathcal{F})\big((P(Y)/\!\!/\mathrm{D}(Y))_k\big)\,,$$
$$\mathcal{F}(P)^{coh} = \operatorname*{holim}^{\mathcal{CSS}_n}_{k\in\Delta}\ (S^Q_{\infty,n}\mathcal{F})\big(B_k(*,\mathcal{M}_Y^{\mathrm{op}},P)\big)\,.$$

We chose to use these specific models for their good computational properties.

Theorem 4.21 *For any $n \in \mathbb{N}_0$ and for any fibrant $\mathcal{F} \in \mathcal{H}^{loc}_{\infty,n}$ there is a canonical zig-zag of weak equivalences between fibrant objects in $\mathcal{CSS}_n$,*

$$\begin{tikzcd}[column sep=small] & \mathcal{F}(P)^{\mathrm{D}(Y)}_{red} \arrow[dl, "\sim"'] \arrow[dr, "\sim", "(Q\,\mathrm{diag}(\Phi))^*"'] & & \mathcal{F}(P)^{coh}_{red} \arrow[dl, "\sim"', "(Q\,\mathrm{diag}(\Psi))^*"] \arrow[dr, "\sim"] & \\ \mathcal{F}(P)^{\mathrm{D}(Y)} & & \underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}\big(Q \circ \mathrm{diag}\big(\mathrm{Coh}(Y,P)\big), \mathcal{F}\big) & & \mathcal{F}(P)^{coh} \end{tikzcd} \tag{4.22}$$

Proof. Let $G \in \mathcal{H}_\infty$ be any object. Then, since $\mathcal{H}_{\infty,n}$ is a $\mathcal{CSS}_n$-model category, the functor

$$\mathbb{\Delta} \to \mathcal{CSS}_n\,, \qquad [k] \mapsto \underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}(Q(G_k), \mathcal{F})$$

is objectwise fibrant. Hence, its homotopy limit is modelled by the cobar construction in $\mathcal{CSS}_n$. Since $\mathcal{F}$ is fibrant in $\mathcal{H}_{\infty,n}$, the functor $\underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}(-,\mathcal{F})\colon \mathcal{H}_{\infty,n} \to \mathcal{CSS}_n$ is left Quillen. Further, recall from Lemma 3.35 that the functor $\mathsf{c}_\bullet\colon \mathcal{H}^{loc}_{\infty,l} \to \mathcal{H}^{loc}_{\infty,l+1}$ is left Quillen for every $l \in \mathbb{N}_0$. Hence, using the cobar and bar constructions to model homotopy (co)limits, there exists a canonical isomorphism

$$\underset{k\in\mathbb{\Delta}}{\mathrm{holim}}^{\mathcal{CSS}_n}\ \underline{\mathcal{H}}_{\infty,n}\big(Q(G_k), \mathcal{F}\big) \cong \underline{\mathcal{H}}_{\infty,n}\big(\underset{k\in\mathbb{\Delta}}{\mathrm{hocolim}}^{\mathcal{H}_\infty} Q(G_k), \mathcal{F}\big)\,.$$

By Proposition A.3 there exists a canonical weak equivalence $\mathrm{hocolim}^{\mathcal{H}_\infty}_{k\in\mathbb{\Delta}} Q(G_k) \xrightarrow{\sim} Q\,\mathrm{diag}(G)$. Since homotopy colimits are cofibrant (when computed using (co)bar constructions, following [Rie14]), this is a weak equivalence between cofibrant objects in $\mathcal{H}_\infty$. It hence induces a weak equivalence

$$\underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}\big(Q \circ \mathrm{diag}(G), \mathcal{F}\big) \xrightarrow{\sim} \underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}\big(\underset{k\in\mathbb{\Delta}}{\mathrm{hocolim}}^{\mathcal{H}_\infty} Q(G_k), \mathcal{F}\big)\,.$$

The cases $G = P(Y)/\!\!/\mathrm{D}(Y)$ and $G = B_\bullet(*, \mathcal{M}^{\mathrm{op}}_Y, P)$ yield the first and the last weak equivalence in diagram (4.22).

The second and the third weak equivalence are direct consequences of Proposition 4.19. □

We can improve on this result by establishing a more direct relation between equivariant and coherent sections:

Theorem 4.23 *For any $n \in \mathbb{N}_0$ and for any fibrant $\mathcal{F} \in \mathcal{H}^{loc}_{\infty,n}$ there is a canonical zig-zag of weak equivalences between fibrant objects in $\mathcal{CSS}_n$,*

$$\mathcal{F}(P)^{\mathrm{D}(Y)} \xrightarrow[\sim]{\Psi^*} Z(\mathcal{F}, P, Y) \xleftarrow[\sim]{\Phi^*} \mathcal{F}(P)^{coh} \tag{4.24}$$

Proof. For the sake of legibility, we write $\mathrm{Coh} \coloneqq \mathrm{Coh}(Y,P)$ in this proof. We set

$$Z(\mathcal{F}, P, Y) = C_{\mathcal{CSS}_n}\Big(*, \mathbb{\Delta}, C_{\mathcal{CSS}_n}\big(*, \mathbb{\Delta}, (S^Q_{\infty,n}\mathcal{F})(\mathrm{Coh}_{\bullet,\bullet})\big)\Big)\,.$$

First, we define Ψ^* and show that it is an equivalence. Since for each $k \in \mathbb{N}_0$ the morphism Ψ_k is the Čech nerve of a τ_{dgop}-local epimorphism, we can apply Theorem 3.43 (respectively Theorems 3.32 or 3.18) to it. Thus, for each $k \in \mathbb{N}_0$ we obtain a weak equivalence

$$(S^Q_{\infty,n}\mathcal{F})\big(B_k(*, \mathcal{M}^{\mathrm{op}}_Y, P)\big) \xrightarrow{\sim} C_{\mathcal{CSS}_n}\big(*, \mathbb{\Delta}, (S^Q_{\infty,n}\mathcal{F})(\mathrm{Coh}_{k,\bullet})\big)\,.$$

Since the cobar construction preserves weak equivalences between projectively fibrant diagrams [Rie14], this yields the left-hand weak equivalence in (4.24).

We move on to the right-hand weak equivalence: we can commute the two cobar constructions in $Z(\mathcal{F}, P, Y)$ and use the definition of $S^Q_{\infty,n}$ to obtain an isomorphism

$$\begin{aligned} Z(\mathcal{F}, P, Y) &\cong C_{\mathcal{CSS}_n}\Big(*, \mathbb{\Delta}, C_{\mathcal{CSS}_n}\big(*, \mathbb{\Delta}, \underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}(Q(\mathrm{Coh}_{\bullet,\bullet}), \mathcal{F})\big)\Big) \\ &\cong C_{\mathcal{CSS}_n}\Big(*, \mathbb{\Delta}, \underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}\big(B^{\mathcal{H}_\infty}(*, \mathbb{\Delta}, Q(\mathrm{Coh}_{\bullet,\bullet})), \mathcal{F}\big)\Big)\,, \end{aligned}$$

where the inner cobar construction in the first line and the bar construction in the second line refer to the first (horizontal) simplicial degree of Coh. Observe that the functor Q is always applied to the levels $\mathrm{Coh}_{k,l}$, seen as simplicially constant objects in $\mathcal{H}_\infty$. Fix some $l \in \mathbb{N}_0$ and consider the simplicial object $\mathrm{Coh}_{\bullet,l} \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{H}_\infty$. The bar construction $B^{\mathcal{H}_\infty}(*, \mathbb{\Delta}, Q\mathrm{Coh}_{\bullet,l})$ is (a model for) the homotopy colimit of this simplicial diagram. Using Lemma A.2 and the fact that $\mathrm{Coh}_{\bullet,l}$ has an augmentation and extra degeneracies (Proposition 4.18, [Rie14, Cor. 4.5.2]), we obtain projective weak equivalences

$$\begin{aligned} B^{\mathcal{H}_\infty}(*, \mathbb{\Delta}, Q\mathrm{Coh}_{\bullet,l}) &= \underset{k\in\mathbb{\Delta}^{\mathrm{op}}}{\mathrm{hocolim}}^{\mathcal{H}_\infty}\, \mathrm{Coh}_{k,l} \\ &\overset{\sim}{\longrightarrow} Q\big(c \longmapsto \underset{k\in\mathbb{\Delta}^{\mathrm{op}}}{\mathrm{hocolim}}^{\mathcal{S}\mathrm{et}_{\mathbb{\Delta}}}\, \mathrm{Coh}_{k,l}(c)\big) \\ &\overset{\sim}{\longrightarrow} Q\big(c \longmapsto \mathrm{Coh}_{-1,l}(c)\big) \\ &= Q\big((P(Y) /\!\!/ \mathrm{D}(Y))_l\big)\,. \end{aligned}$$

This induces a weak equivalence

$$\begin{aligned} Z(\mathcal{F}, P, Y) &= C_{\mathcal{CSS}_n}\Big(*, \mathbb{\Delta}, \underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}\big(B^{\mathcal{H}_\infty}(*, \mathbb{\Delta}, Q(\mathrm{Coh}_{\bullet,\bullet})), \mathcal{F}\big)\Big) \\ &\overset{\sim}{\longleftarrow} C_{\mathcal{CSS}_n}\Big(*, \mathbb{\Delta}, \underline{\mathcal{H}}^{\mathcal{CSS}_n}_{\infty,n}\big(Q\big((P(Y) /\!\!/ \mathrm{D}(Y))_\bullet\big), \mathcal{F}\big)\Big) \\ &= C_{\mathcal{CSS}_n}\Big(*, \mathbb{\Delta}, (S^Q_{\infty,n}\mathcal{F})\big((P(Y) /\!\!/ \mathrm{D}(Y))_\bullet\big)\Big) \\ &= \mathcal{F}(P)^{\mathrm{D}(Y)}\,, \end{aligned}$$

which proves the claim. □

Remark 4.25 The weak equivalence Φ^* in (4.24) arises from weak equivalences in $\mathcal{H}_\infty$, i.e. from *projective* weak equivalences of simplicial presheaves. The fact that Ψ^* is a weak equivalence relies on the fact that $\mathcal{F}$ is Čech local and that $\Psi_\bullet$ is levelwise a τ_{dgop}-local weak equivalence. ◁

Remark 4.26 The morphisms in diagram (4.24) are weak equivalences between cofibrant-fibrant objects (all objects in $\mathcal{CSS}_n$ are cofibrant, Remark 3.34). Hence, as $\mathcal{CSS}_n$ is in particular a simplicial model category, each of these weak equivalences admits a contractible space of homotopy inverses. Choosing a homotopy inverse Ψ_* for the morphism Ψ^* hence specifies a weak equivalence $\mathcal{F}(P)^{\mathrm{D}(Y)} \overset{\sim}{\longrightarrow} \mathcal{F}(P)^{coh}$, unique up to contractible choice. This establishes an equivalence (unique up to contractible choices) of the (∞, n)-categories of $\mathrm{D}(Y)$-equivariant sections of $\mathcal{F}$ over $P(Y)$ (cf. Definition 4.20) and coherent sections of $\mathcal{F}$ on P. ◁

Remark 4.27 If $\mathcal{F}$ is the sheaf of diffeological vector bundles from Section 4.1 the insights from Theorem 4.23 and Remark 4.26 were used in [BW19] in order to obtain a coherent vector bundle on P, where $Y = \mathbb{S}^1$ and $P = M^{(-)}$ for some manifold M, from the equivariant structure of the transgression line bundle of a bundle gerbe with connection. Moreover, this procedure was also applied to bundles over certain spaces of paths in M, and the coherent vector bundles thus obtained were subsequently assembled into a smooth open-closed FFT on M. In this sense, that smooth FFT was built from its values on generating objects by means of the equivalence of equivariant and coherent sections of a sheaf of higher categories as explored here. ◁

We point out that there is another way of establishing an equivalence between the (∞, n)-categories $\mathcal{F}(P)^{\mathrm{D}(Y)}$ and $\mathcal{F}(P)^{coh}$: we can view the truncations to simplicial levels zero and one of $P(Y)/\!\!/\mathrm{D}(Y)$ and of $B(*, \mathcal{M}_Y^{\mathrm{op}}, P)$ as strict presheaves of groupoids on $\mathcal{C}\mathrm{art}$. Let us denote these by $Gr(P(Y)/\!\!/\mathrm{D}(Y))$ and $Gr(B(*, \mathcal{M}_Y^{\mathrm{op}}, P))$, respectively. Observe that

$$P(Y)/\!\!/\mathrm{D}(Y) = N \circ Gr\big(P(Y)/\!\!/\mathrm{D}(Y)\big) \quad \text{and} \quad B(*, \mathcal{M}_Y^{\mathrm{op}}, P) = N \circ Gr\big(B(*, \mathcal{M}_Y^{\mathrm{op}}, P)\big)\,,$$

and that there is a canonical inclusion $\iota\colon Gr(P(Y)/\!\!/\mathrm{D}(Y)) \hookrightarrow Gr(B(*, \mathcal{M}_Y^{\mathrm{op}}, P))$. Any choice of a family of diffeomorphisms $\{f_{Y_0}\colon Y_0 \to Y\}_{Y_0 \in \mathcal{M}_Y}$ determines a morphism $p\colon Gr(B(*, \mathcal{M}_Y^{\mathrm{op}}, P)) \hookrightarrow Gr\big(P(Y)/\!\!/\mathrm{D}(Y)\big)$, for which there exist 2-isomorphisms (i.e. natural isomorphisms, coherent over $\mathcal{C}\mathrm{art}$) $p \circ \iota = 1$ and $\eta\colon \iota \circ p \to 1$ of presheaves of groupoids over $\mathcal{C}\mathrm{art}$. In particular, $N\eta$ determines a simplicial homotopy $N\eta\colon B(*, \mathcal{M}_Y^{\mathrm{op}}, P) \otimes \Delta^1 \longrightarrow B(*, \mathcal{M}_Y^{\mathrm{op}}, P)$ between the identity and $N(p \circ \iota)$ in $\mathcal{H}_\infty$. This, in turn, shows that both ι and p are homotopy equivalences, weakly inverse to each other. Consequently, we obtain a pair of projective weak equivalences

$$Q\big(P(Y)/\!\!/\mathrm{D}(Y)\big) \underset{Qp}{\overset{Q\iota}{\underset{\sim}{\rightleftarrows}}} Q\big(B(*, \mathcal{M}_Y^{\mathrm{op}}, P)\big)\,.$$

This proves

Theorem 4.28 *Let $\mathcal{F} \in \mathcal{H}_{\infty,n}$* projectively *fibrant. With ι and p defines as above, we obtain weakly inverse weak equivalences*

$$\mathcal{F}(P)^{\mathrm{D}(Y)}_{red} \underset{(Qp)^*}{\overset{(Q\iota)^*}{\underset{\sim}{\leftrightarrows}}} \mathcal{F}(P)^{coh}_{red}\,.$$

Remark 4.29 Observe that Theorem 4.28 has greater scope than Theorem 4.23, because it does not require Čech fibrancy of $\mathcal{F}$. However, if there is a good choice of inverse Ψ_* as described in Remark 4.26 then Theorem 4.23 is more useful in applications, since it builds the coherence data on the output of $\Psi_* \circ \Phi^*$ very explicitly. Furthermore, $\Psi_* \circ \Phi^*$ is a morphism between the (∞, n)-categories of equivariant and coherent sections of $\mathcal{F}$ as one would like to describe them in practise, whereas in Theorem 4.28 one needs to first choose inverses for the weak equivalence $\mathcal{F}(P)^{\mathrm{D}(Y)}_{red} \stackrel{\sim}{\longrightarrow} \mathcal{F}(P)^{\mathrm{D}(Y)}$, before one arrives at a weak equivalence $\mathcal{F}(P)^{\mathrm{D}(Y)} \stackrel{\sim}{\longrightarrow} \mathcal{F}(P)^{coh}$. $\triangleleft$

A Cofibrant replacement and homotopy colimits in $\mathcal{H}_\infty$

Here we collect some technical results concerning Dugger's cofibrant replacement functor Q from (3.4). First, we observe that we can write Q by means of the two-sided simplicial bar construction as follows:

Lemma A.1 *Let $\mathcal{C}$ be any V-small category, and let $\mathcal{Y}$ denote its Yoneda embedding.*

(1) There is a canonical natural isomorphism

$$QF \cong B(F, \mathcal{C}, \mathcal{Y})\,.$$

(2) For any presheaf $X \in \widehat{\mathcal{C}}$ there is a canonical isomorphism

$$Q(\mathsf{c}_\bullet X) \cong B(*, \mathcal{C}/X, \mathcal{Y}) = \mathrm{hocolim}^{\mathcal{H}_\infty}_{\mathcal{C}/X} \mathcal{Y}\,.$$

Proof. We compute

$$\big(B(F, \mathcal{C}, \mathcal{Y})\big)_n = \bigg(\int^{k \in \mathbb{\Delta}^{\mathrm{op}}} B_k(F, \mathcal{C}, \mathcal{Y}) \otimes \Delta^k\bigg)_n \cong \int^k \big(B_k(F, \mathcal{C}, \mathcal{Y})\big)_n \otimes \Delta^k_n \cong \big(B_n(F, \mathcal{C}, \mathcal{Y})\big)_n\,,$$

and we have

$$\begin{aligned}
\big(B_n(F,\mathcal{C},\mathcal{Y})\big)_n &= \Big(\coprod_{c_0,\dots,c_n\in\mathcal{C}} \mathcal{Y}_{c_0}\times\mathcal{C}(c_0,c_1)\times\cdots\times\mathcal{C}(c_{n-1},c_n)\times F(c_n)\Big)_n \\
&= \coprod_{c_0,\dots,c_n\in\mathcal{C}} \mathcal{Y}_{c_0}\times\mathcal{C}(c_0,c_1)\times\cdots\times\mathcal{C}(c_{n-1},c_n)\times F_n(c_n) \\
&= (QF)_n\,.
\end{aligned}$$

The second claim follows readily from claim (1), the fact that the functor $\mathcal{Y}\colon \mathcal{C}\to\mathcal{H}_\infty$ takes values in cofibrant objects of $\mathcal{H}_\infty$, and the expression of the homotopy colimit in terms of the two-sided simplicial bar construction [Rie14, Cor. 5.1.3]. □

Lemma A.2 *Let $\mathcal{J}$ be a V-small category, and let $D\colon \mathcal{J}\to\mathcal{H}_\infty$ be a functor.*

(1) There is a canonical isomorphism, natural in D,

$$\mathrm{hocolim}^{\mathcal{H}_\infty}_{\mathcal{J}}(D) \cong Q\big(c\mapsto \mathrm{hocolim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{J}}\mathrm{Ev}_c D\big)\,,$$

where $\mathrm{Ev}_c\colon \mathcal{H}_\infty\to\mathcal{S}\mathrm{et}_\Delta$, $X\mapsto X(c)$ is the evaluation functor at $c\in\mathcal{C}$.

(2) There is a canonical natural weak equivalence

$$\mathrm{hocolim}^{\mathcal{H}_\infty}_{\mathcal{J}}(Q\circ D) \xrightarrow{\sim} Q\big(c\mapsto \mathrm{hocolim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{J}}\mathrm{Ev}_c D\big)\,.$$

Proof. Following [Rie14, Cor. 5.1.3], we write the homotopy colimit using the bar construction and the cofibrant replacement functor Q; then the statement is a consequence of the following commutation of colimits in $\mathcal{H}_\infty$:

$$\begin{aligned}
&\big(\mathrm{hocolim}^{\mathcal{H}_\infty}_{\mathcal{J}}(D)\big)_n \\
&= \big(B^{\mathcal{H}_\infty}_n(*,\mathcal{J},Q\circ D)\big)_n \\
&= \coprod_{j_0,\dots,j_n\in\mathcal{J}} \mathcal{J}(j_0,j_1)\times\cdots\times\mathcal{J}(j_{n-1},j_n)\times\big(Q\circ D(j_0)\big)_n \\
&= \coprod_{j_0,\dots,j_n} \mathcal{J}(j_0,j_1)\times\cdots\times\mathcal{J}(j_{n-1},j_n)\times\Big(\coprod_{c_0,\dots,c_n\in\mathcal{C}} \mathcal{Y}_{c_0}\times\mathcal{C}(c_0,c_1)\times\cdots\times\mathcal{C}(c_{n-1},c_n)\times D(j_0)_n(c_n)\Big) \\
&\cong \coprod_{c_0,\dots,c_n\in\mathcal{C}} \mathcal{Y}_{c_0}\times\mathcal{C}(c_0,c_1)\times\cdots\times\mathcal{C}(c_{n-1},c_n) \\
&\qquad\times\Big(\coprod_{j_0,\dots,j_n\in\mathcal{J}} D(j_0)_n(c_n)\times\mathcal{J}(j_0,j_1)\times\cdots\times\mathcal{J}(j_{n-1},j_n)\Big) \\
&= \coprod_{c_0,\dots,c_n\in\mathcal{C}} \mathcal{Y}_{c_0}\times\mathcal{C}(c_0,c_1)\times\cdots\times\mathcal{C}(c_{n-1},c_n)\otimes B^{\mathcal{S}\mathrm{et}_\Delta}_n\big(*,\mathcal{J},\mathrm{Ev}_{c_n}D\big) \\
&= Q\big(c\mapsto \mathrm{hocolim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{J}}(\mathrm{Ev}_c D)\big)\,.
\end{aligned}$$

In the last step we have used that every object in $\mathcal{S}\mathrm{et}_\Delta$ is cofibrant, so that the bar construction correctly models the homotopy colimit. The second claim follows from the fact that hocolim is homotopical and that there exists a natural weak equivalence $q\colon Q\xrightarrow{\sim}1$. □

Proposition A.3 *Given a simplicial object $F_\bullet\colon \Delta^{\mathrm{op}}\to\mathcal{H}_\infty$, there is a natural weak equivalence*

$$\underset{k\in\Delta^{\mathrm{op}}}{\mathrm{hocolim}}^{\mathcal{H}_\infty}\; Q(F_k) \xrightarrow{\sim} Q\,\mathrm{diag}(F)\,.$$

Proof. This follows by a combination of Proposition A.2(2) and the Bousfield-Kan map for simplicial spaces [Hir03, Section 18.7]. ☐

B Proof of Theorem 4.8

Definition of the functor $\mathcal{A}$: We start by describing the functor $\mathcal{A}$ on objects: consider the category $\mathcal{C}\mathrm{art}/X$, whose objects are plots $\varphi \in X(c)$ for any $c \in \mathcal{C}\mathrm{art}$ and whose morphisms $(\varphi \in X(c')) \to (\varphi' \in X(c))$ are smooth maps $f\colon c \to c'$ such that $f^*\varphi' = \varphi$. Given an object $(n,h) \in \mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}(X)$ we define a functor $D_{(n,h)}\colon \mathcal{C}\mathrm{art}/X \to \mathcal{D}\mathrm{fg}$, which acts as

$$\begin{aligned} (\varphi\colon c \to X) &\longmapsto c \times \mathbb{C}^{n(\varphi)}\,, \\ (f\colon \varphi \to \varphi') &\longmapsto \big((f,h_f)\colon c \times \mathbb{C}^n \longrightarrow c' \times \mathbb{C}^n\big)\,. \end{aligned}$$

We then set

$$E \coloneqq \underset{\mathcal{C}\mathrm{art}/X}{\mathrm{colim}}{}^{\mathcal{D}\mathrm{fg}}\, D_{(n,h)}$$

and denote the canonical morphism $D_{(n,h)}(\varphi) \to E$ by ι_φ. Observe that $\mathcal{C}\mathrm{art}/X$ is a U-small category, since $\mathcal{C}\mathrm{art}$ is U-small. The object $X \in \mathcal{D}\mathrm{fg}$ is a cocone under $D_{(n,h)}$, which is established by the morphism of diagrams

$$\widehat{\pi}\colon D_{(n,h)} \to X\,, \qquad \widehat{\pi}_{|\varphi}\colon D_{(n,h)}(\varphi) = c \times \mathbb{C}^{n(\varphi)} \xrightarrow{\mathrm{pr}} c \xrightarrow{\varphi} X\,.$$

We let $\pi\colon E \to X$ denote the unique morphism induced on the colimit, and we set

$$\mathcal{A}(n,h) \coloneqq (E,\pi)\,.$$

Lemma 2.24 implies that on the level of underlying sets we have

$$\mathrm{Ev}_*\Big(\underset{\mathcal{C}\mathrm{art}/X}{\mathrm{colim}}{}^{\mathcal{D}\mathrm{fg}}\, D_{(n,h)}\Big) \cong \underset{\mathcal{C}\mathrm{art}/X}{\mathrm{colim}}{}^{\mathcal{D}\mathrm{fg}}\, (\mathrm{Ev}_* \circ D_{(n,h)}) = \Big(\coprod_{\varphi \in X(c)} c \times \mathbb{C}^{n(\varphi)}\Big)\Big/\!\sim\,,$$

where $\sim$ is the equivalence relation generated by setting $(\varphi, y, v) \sim (\varphi', y', v')$ if there exists a morphism $f\colon \varphi \to \varphi'$ in $\mathcal{C}\mathrm{art}/X$ such that $h_f(\varphi, y, v) = (\varphi', y', v')$. Here, we use the convention $(\varphi, y, v) \in X(c) \times c \times \mathbb{C}^{n(\varphi)}$. The morphism π sends an equivalence class $[\varphi, y, v]$ to $\varphi(y)$. Since the morphisms h act via linear isomorphisms of vector spaces, the fibre of π carries a canonical $\mathbb{C}$-vector space structure.

$\mathcal{A}(n,h) = (E,\pi)$ is a diffeological vector bundle on X: To see this, let $\psi \in X(d)$ be a plot of X over $d \in \mathcal{C}\mathrm{art}$. Consider the morphism of diffeological spaces

$$\Phi_\psi\colon d \times \mathbb{C}^n \xrightarrow{\;1\;} D_{(n,h)}(\psi) \xrightarrow{\mathrm{pr}_d \times \iota_\psi} d \times_X E\,. \tag{B.1}$$

Note that Φ_ψ is linear on the fibres. We need to show that it is an isomorphism. For $x \in X(*)$ and $c \in \mathcal{C}\mathrm{art}$, let $\mathsf{c}_x\colon \mathrm{pt} \to X(*)$ denote the constant plot of X with value x. Every element $(y, [\varphi, z, v]) \in (d \times_X E)(*)$ has a unique representative of the form

$$\big(y, [\varphi, z, v]\big) = \big(y, [\psi, y, w]\big)\,.$$

This is established by the pairs (z, h_z) and (y, h_y) (from the data of (n, h)) associated to the morphism $z\colon \mathsf{c}_{\psi(y)} \to \varphi$ and $y\colon \mathsf{c}_{\psi(y)} \to \psi$ in $\mathcal{C}\mathrm{art}/X$, i.e. to the commutative diagram

$$
\begin{tikzcd}
 & d \arrow[dr, "\psi"] & \\
\mathrm{pt} \arrow[ur, "y"] \arrow[rr, "\mathsf{c}_{\psi(y)} = \mathsf{c}_{\varphi(z)}"] \arrow[dr, "z"'] & & X \\
 & c \arrow[ur, "\varphi"'] &
\end{tikzcd}
$$

in $\mathcal{D}\mathrm{fg}$. The representative is well-defined by the composition properties of the pairs (f, h) in the data of E. We now define a map of sets

$$
\Psi_\psi\colon (d \times_X E)(*) \longrightarrow d \times \mathbb{C}^n\,, \qquad \big(y, [\psi, y, w]\big) \longmapsto (y, w)\,.
$$

We readily see that the map Ψ_ψ thus defined is an inverse for Φ_ψ as maps of sets. It remains to prove that Ψ_ψ is a morphism of diffeological spaces; that is, we need to show that for any plot $\varrho\colon c \to d \times_X E$, the composition $\Psi_\psi \circ \varrho\colon c \to c \times \mathbb{C}^n$ is a plot. Now, a plot $\varrho\colon c \to d \times_X E$ is equivalently a pair of a smooth map $\varrho_d\colon c \to d$ and a plot $\varrho_E\colon c \to E$ such that $\psi \circ \varrho_d = \pi \circ \varrho_E$. By Proposition 2.25 there exists a covering $\{f_i\colon c_i \hookrightarrow c\}_{i \in I}$ of c together with morphisms $\{\varrho_i\colon c_i \to D_{(n,h)}(\psi_i)\}_{i \in I}$ in $\mathcal{D}\mathrm{fg}$ such that $\varrho_E \circ f_i = \iota_{\psi_i} \circ \varrho_i$ for all $i \in I$. Note that $\psi_i\colon d_i \to X$ are plots of X. Using that $D_{(n,h)}(\psi_i) = d_i \times \mathbb{C}^n$, we can further decompose ϱ_i into pairs of smooth maps $\varrho_{i,d_i}\colon c_i \to d_i$ and $\varrho_{i,\mathbb{C}^n}\colon c_i \to \mathbb{C}^n$. By construction, we thus obtain a commutative diagram

$$
\begin{tikzcd}
c_i \arrow[r, "\varrho_i"] \arrow[d, "f_i"'] & d \times_X D_{(n,h)}(\psi_i) \arrow[d, "1 \times \iota_{\psi_i}"] \\
c \arrow[r, "\varrho"'] & d \times_X E
\end{tikzcd}
$$

in $\mathcal{D}\mathrm{fg}$ for every $i \in I$. Observe that this implies that

$$
\begin{tikzcd}
 & d \arrow[dr, "\psi"] & \\
c_i \arrow[ur, "\varrho_d \circ f_i"] \arrow[r, "f_i"] \arrow[dr, "\varrho_{i,d_i}"'] & c \arrow[u, "\varrho_d"'] & X \\
 & d_i \arrow[ur, "\psi_i"'] &
\end{tikzcd}
$$

commutes as well, for each $i \in I$. Using the notation h_f for morphisms as in the proof of Proposition 4.6, we compute the action of the composition $\Psi_\psi \circ \varrho \circ f_i = \Psi_\psi \circ (1 \times \iota_{\psi_i}) \circ \varrho_i\colon c_i \to d \times \mathbb{C}^n$ on $y_i \in c_i$ as

$$
\begin{aligned}
y_i \longmapsto & \Psi_\psi\big(\varrho_d \circ f_i(y_i), [\psi_i,\ \varrho_{i,d_i}(y_i),\ \varrho_{i,\mathbb{C}^n}(y_i)]\big) \\
&= \Psi_\psi\big(\varrho_i(y_i), [\psi_i \circ \varrho_{i,d_i},\ y_i,\ h^{-1}_{\varrho_{i,d_i}} \circ \varrho_{i,\mathbb{C}^n}(y_i)]\big) \\
&= \Psi_\psi\big(\varrho_i(y_i), [\psi \circ \varrho_d \circ f_i,\ y_i,\ h^{-1}_{\varrho_{i,d_i}} \circ \varrho_{i,\mathbb{C}^n}(y_i)]\big) \\
&= \Psi_\psi\big(\varrho_i(y_i), [\psi,\ \varrho_d \circ f_i(y_i),\ h_{\varrho_d \circ f_i} \circ h^{-1}_{\varrho_{i,d_i}} \circ \varrho_{i,\mathbb{C}^n}(y_i)]\big) \\
&= \big(\varrho_i(y_i),\ h_{\varrho_d \circ f_i} \circ h^{-1}_{\varrho_{i,d_i}} \circ \varrho_{i,\mathbb{C}^n}(y_i)\big)\,.
\end{aligned}
$$

Both components of this map are smooth. Thus, each composition $(\Psi_\psi \circ \varrho) \circ f_i$ is smooth. Since, by construction, these maps agree on intersections c_{ij}, it follows that $\Psi_\psi \circ \varrho\colon c \to d \times \mathbb{C}^n$ is smooth. Therefore, Ψ_ψ is a morphism of diffeological spaces, and (E, π) is a diffeological vector bundle on X.

$\mathcal{A}$ on morphisms: Let $g\colon (n_0, h_0) \to (n_1, h_1)$ be a morphism in $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}(X)$. By the relation (4.7), g gives rise to a morphism of diagrams $D_g\colon D_{(n_0,h_0)} \to D_{(n_1,h_1)}$, and hence induces a morphism

$$D_g\colon \underset{\mathcal{C}\mathrm{art}/X}{\mathrm{colim}}^{\mathcal{D}\mathrm{fg}} D_{(n_0,h_0)} \longrightarrow \underset{\mathcal{C}\mathrm{art}/X}{\mathrm{colim}}^{\mathcal{D}\mathrm{fg}} D_{(n_1,h_1)}$$

of diffeological spaces. By construction, this map is a map of diffeological spaces over X. Since for any plot $\varphi\colon c \to X$ the map g_φ induces a morphism of vector bundles $c \times \mathbb{C}^{n_0} \to c \times \mathbb{C}^{n_1}$, the map D_g is linear on each fibre. Therefore, D_g is indeed a morphism of diffeological vector bundles on X. This completes the construction of the functor $\mathcal{A}$.

$\mathcal{A}$ is fully faithful: For (n_0, h_0), $(n_1, h_1) \in \mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}(X)$, we write $E_i \coloneqq \mathrm{colim}^{\mathcal{D}\mathrm{fg}}_{\mathcal{C}\mathrm{art}/X} D_{(n_i,h_i)}$, where $i = 0, 1$. Recall from (B.1) that for each plot $\varphi\colon c \to X$ we have constructed *canonical* trivialisations

$$\Phi^{E_i}_\varphi\colon c \times \mathbb{C}^{n_i(\varphi)} \xrightarrow{\cong} c \times_X E_i\,.$$

Thus, given a morphism $\xi\colon E_0 \to E_1$ in $\mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}(X)$ and a plot $\varphi\colon c \to X$, we obtain a morphism

$$\begin{tikzcd}
c \times \mathbb{C}^{n_0(\varphi)} \arrow[r, "\Phi^{E_0}_\varphi"] \arrow[d, dashed, "\widetilde{\xi}_\varphi"'] & c \times_X E_0 \arrow[d, "1 \times_X \xi"] \\
c \times \mathbb{C}^{n_1(\varphi)} & c \times_X E_1 \arrow[l, "(\Phi^{E_1}_\varphi)^{-1}"]
\end{tikzcd}$$

which is linear on fibres; hence, $\widetilde{\xi}_\varphi$ corresponds to a smooth map $\widetilde{\xi}_\varphi\colon c \to \mathsf{Mat}(n_1 \times n_0, \mathbb{C})$. Observe that for any morphism $f\colon \varphi_0 \to \varphi_1$ in $\mathcal{C}\mathrm{art}/X$ the diagram

$$\begin{tikzcd}
c_0 \times \mathbb{C}^{n_i(\varphi_0)} \arrow[r, "\Phi^{E_i}_{\varphi_0}"] \arrow[d, "f \times 1"'] & c_0 \times_X E_i \arrow[d, "f \times_X 1"] \\
c_1 \times \mathbb{C}^{n_i(\varphi_1)} \arrow[r, "\Phi^{E_i}_{\varphi_1}"'] & c_1 \times_X E_i
\end{tikzcd}$$

commutes (in this case, $n_0(\varphi_1) = n_0(\varphi_0)$). This implies that $\widetilde{\xi}$ satisfies condition (4.7), and hence that it is a morphism $\widetilde{\xi}\colon (n_0, h_0) \to (n_1, h_1)$ in $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}(X)$.

The map

$$\widetilde{(-)}\colon \mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}(X)(E_0, E_0) \longrightarrow \mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}(X)\big((n_0, h_0), (n_1, h_1)\big)\,, \qquad \xi \mapsto \widetilde{\xi}$$

is injective: consider all constant plots $x\colon \mathrm{pt} \to X$, for $x \in X(*)$. The set $\{\widetilde{\xi}_x\}_{x \in X(*)}$ uniquely determines the value of ξ at every point $x \in X(*)$; hence it fully determines the morphism ξ.

Furthermore, if $g\colon (n_0, h_0) \to (n_1, h_1)$ is a morphism in $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}(X)$, then, again by construction of Φ_φ from the cocone data, we have that

$$\big(\widetilde{(-)} \circ \mathcal{A}(g)\big)_\varphi = g_\varphi$$

for all plots $\varphi\colon c \to X$. Hence, the map $g \mapsto \mathcal{A}(g)$ is a right inverse for $\widetilde{(-)}$. It follows that $\widetilde{(-)}$ is a bijection, and hence that $\mathcal{A}$ is fully faithful.

$\mathcal{A}$ is essentially surjective: Let $\pi_F\colon F \to X$ be a diffeological vector bundle on X. For each plot $\varphi\colon c \to X$ choose a trivialisation

$$\Xi_\varphi\colon c \times \mathbb{C}^{n(\varphi)} \xrightarrow{\cong} c \times_X F\,. \tag{B.2}$$

Given a morphism $f\colon \varphi_0 \to \varphi_1$ in $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}(X)$, the universal property and the pasting law for pullbacks determine a canonical isomorphism $c_0 \times_X F \cong c_0 \times_{c_1} (c_1 \times_X F)$. This allows us to define an isomorphism

$$\begin{tikzcd}
c_0 \times \mathbb{C}^{n(\varphi_0)} \arrow[r, "\Xi_{\varphi_0}"] \arrow[d, dashed, "h_f"'] & c_0 \times_X F \arrow[r, "\cong"] & c_0 \times_{c_1} (c_1 \times_X F) \arrow[d, "f^*\Xi_{\varphi_1}^{-1}"] \\
c_0 \times \mathbb{C}^{n(\varphi_1)} & & c_0 \times_{c_1} (c_1 \times \mathbb{C}^{n(\varphi_1)}) \arrow[ll, "\cong"]
\end{tikzcd}$$

The map h_f is linear on fibres and hence determines a unique smooth map $h_f\colon c_0 \to \mathsf{GL}(n(\varphi_0), \mathbb{C})$. Since the morphisms labelled '$\cong$' in the above diagram are chosen as canonical isomorphisms between different representatives for the same limits, the collection of morphisms h_f assembles into an object $(n, h) \in \mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}(X)$.

We claim that there is an isomorphism $\mathcal{A}(n,h) \xrightarrow{\cong} F$ in $\mathcal{VB}\mathrm{un}_{\mathcal{D}\mathrm{fg}}(X)$. By a convenient abuse of notation, we denote this isomorphism by Ξ. We set

$$\Xi[\varphi, y, v] \coloneqq \mathrm{pr}_F \circ \Xi_\varphi(y, v)\,,$$

where Ξ_φ was chosen in (B.2), and where $\mathrm{pr}_F\colon c \times_X F \to F$ is the projection to F. This defines a map $\Xi\colon \mathcal{A}(n,h) \to F$, which is linear on fibres. We need to show that it is smooth. Consider a plot $\varrho\colon c \to \mathrm{colim}^{\mathcal{D}\mathrm{fg}} D_{(n,h)} = \mathcal{A}(n,h)$. By Proposition 2.25, there exist a covering $\{f_i\colon c_i \hookrightarrow c\}_{i\in I}$ of c and lifts $\varrho_i\colon c_i \to D_{(n,h)}(\psi_i)$ for some plots $\psi_i\colon d_i \to X$. Consider the diagram

$$\begin{tikzcd}
c_i \arrow[r, "\varrho_i"] \arrow[d, "f_i"'] & D_{(n,h)}(\psi_i) \arrow[d, "\iota_{\psi_i}"'] \arrow[dr, bend left, "\mathrm{pr}_F \circ \Xi_{\psi_i}"] & \\
c \arrow[r, "\varrho"'] & \mathrm{colim}^{\mathcal{D}\mathrm{fg}} D_{(n,h)} \arrow[r, "\Xi"'] & F
\end{tikzcd}$$

The left-hand square commutes by definition of ϱ_i, and the right-hand triangle commutes by construction of Ξ and of (n,h). Thus, the map $\Xi \circ \varrho$ is locally given by plots $\mathrm{pr}_F \circ \Xi_{\psi_i} \circ \varrho_i$. By the sheaf property of diffeological spaces, $\Xi \circ \varrho$ is a plot of F itself.

Finally, we need to prove that Ξ is an isomorphism. As a map, it has an inverse, which is given by

$$\Xi'\colon F \longrightarrow \mathrm{colim}^{\mathcal{D}\mathrm{fg}} D_{(n,h)}\,, \qquad \zeta \longmapsto \big[\psi, y, \Xi_\psi^{-1}(y, \zeta)\big]\,,$$

where $\psi\colon d \to X$ is some plot and $y \in d$ is any point such that $\psi(y) = \pi_F(\zeta)$. It remains to show that Ξ' is a morphism of diffeological spaces. To that end, let $\widehat{\zeta}\colon d \to F$ be a plot. This induces a plot $\psi \coloneqq \pi_F \circ \widehat{\zeta}\colon d \to X$. Now, given any $y \in d$, we can write

$$\Xi' \circ \widehat{\zeta}(y) = \big[\psi, y, \Xi_\psi^{-1}(y, \widehat{\zeta}(y))\big] = \iota_\psi \circ \Xi_\psi^{-1}(y, \widehat{\zeta}(y)) = \iota_\psi \circ \Xi_\psi^{-1} \circ (1_d \times \widehat{\zeta})(y)\,.$$

Thus, the composition $\Xi' \circ \widehat{\zeta}$ factors through a plot of $D_{(n,h)}(\psi)$, so that Ξ' is a plot by Proposition 2.25.

<u>$\mathcal{A}$ is a morphism of presheaves of categories:</u> Finally, we need to show that $\mathcal{A}$ is compatible with pullbacks of vector bundles along morphisms $F\colon X \to Y$ in $\mathcal{D}\mathrm{fg}$. Let $(n,h) \in \mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}(Y)$. As a diffeological space, we have

$$F^*\big(\mathcal{A}(n,h)\big) = F^*\big(\operatorname*{colim}_{\mathcal{C}\mathrm{art}/Y}{}^{\mathcal{D}\mathrm{fg}} D_{(n,h)}\big) = X \times_Y \big(\operatorname*{colim}_{\mathcal{C}\mathrm{art}/Y}{}^{\mathcal{D}\mathrm{fg}} D_{(n,h)}\big) = X \times_Y \mathcal{A}(n,h)\,.$$

We have to compare this to

$$\mathcal{A}\big(F^*(n,h)\big) = \operatorname*{colim}_{\mathcal{C}\mathrm{art}/X}{}^{\mathcal{D}\mathrm{fg}}\big(D_{F^*(n,h)}\big)\,.$$

To that end, we consider the map

$$\Xi_F\colon \mathcal{A}\big(F^*(n,h)\big) \longrightarrow F^*\big(\mathcal{A}(n,h)\big)\,, \qquad [\varphi, z, v] \longmapsto \big(\varphi(z), [F\circ\varphi, z, v]\big)\,,$$

where $\varphi\colon c \to X$ is a plot, $z \in c$ is some point, and where $v \in \mathbb{C}^{n(\varphi)}$ is a vector. We also define a map

$$\Xi'_F\colon F^*\big(\mathcal{A}(n,h)\big) \longrightarrow \mathcal{A}\big(F^*(n,h)\big)\,, \qquad \big(x, [F(x), \mathrm{pt}, w]\big) \longmapsto [x, \mathrm{pt}, w]\,,$$

where $x \in X(*)$ is any point in the underlying set of X, $w \in \mathbb{C}^{n(F(x))}$ is a vector, and where we denote a constant plot $\mathrm{pt} \to X$ by its value in $X(*)$. We readily see that Ξ_F and Ξ'_F are mutually inverse maps, fibrewise linear, compatible with morphisms in $\mathcal{VB}\mathrm{un}_{\mathcal{C}\mathrm{at}}$, and that the diagram

$$\begin{tikzcd}
G^*F^*\big(\mathcal{A}(n,h)\big) \arrow[r, "\cong"] \arrow[d, "G^*\Xi_F"'] & (FG)^*\big(\mathcal{A}(n,h)\big) \arrow[d, "\Xi_{FG}"] \\
G^*\big(\mathcal{A}_{F^*(n,h)}\big) \arrow[r, "\Xi_G"'] & \mathcal{A}_{G^*F^*(n,h)} = \mathcal{A}_{(FG)^*(n,h)}
\end{tikzcd}$$

commutes for every morphism $G \in \mathcal{D}\mathrm{fg}(W,X)$. It thus remains to show that both Ξ_F and Ξ'_F are morphisms of diffeological spaces.

We start with Ξ_F: let $\varrho\colon c \to \mathcal{A}(F^*(n,h))$ be a plot. Let $\{f_i\colon c_i \hookrightarrow c\}_{i\in I}$, $\psi_i\colon d_i \to X$, and $\varrho_i\colon c_i \to D_{F^*(n,h)}(\psi_i)$ be lifting data for ϱ as before. Then, we have a commutative diagram

$$\begin{tikzcd}
c_i \arrow[r, "\varrho_i"] \arrow[d, "f_i"'] & D_{(n,h)}(\psi_i) \arrow[r, "(\psi_i\circ \mathrm{pr}_{d_i})\times_Y(1_{d_i\times\mathbb{C}^n})"] \arrow[d, "\iota_{\psi_i}"'] & X \times_Y \big(D_{(n,h)}(F\circ\psi_i)\big) \arrow[d, "1_X\times_Y(\iota_{F\circ\psi_i})"] \\
c \arrow[r, "\varrho"'] & \mathcal{A}_{F^*(n,h)} \arrow[r, "\Xi_F"'] & X \times_Y \mathcal{A}(n,h)
\end{tikzcd}$$

which shows that the map $\Xi_F \circ \varrho$ is smooth (the $\mathcal{A}(n,h)$-valued component factors through $D_{(n,h)}$ locally).

For Ξ'_F, consider a plot $\varrho\colon c \to X \times_Y \mathcal{A}(n,h)$. It decomposes into a plot $\varrho_X\colon c \to X$ and a plot $\varrho_{\mathcal{A}}\colon c \to \mathcal{A}_{(n,h)}$ such that $\pi\circ\varrho_{\mathcal{A}} = F\circ\varrho_X$, where $\pi\colon \mathcal{A}_{(n,h)} \to Y$ is the vector bundle projection. As before, for the plot $\varrho_{\mathcal{A}}$ there exists a covering $\{f_i\colon c_i \hookrightarrow c\}_{i\in I}$, plots $\psi\colon d_i \to Y$, and lifts $\varrho_{\mathcal{A},i}\colon c_i \to D_{(n,h)}(\psi_i)$ such that $\iota_{\psi_i}\circ\varrho_{\mathcal{A},i} = \varrho_{\mathcal{A}}\circ f_i$ for each $i \in I$. Using the morphisms $h_{\varrho_{\mathcal{A},i}}$, it is in fact always possible to choose $d_i = c_i$ and $\varrho_{\mathcal{A},i}\colon c_i \to c_i \times \mathbb{C}^{n(\psi_i)}$ to be a section, i.e. to satisfy $\mathrm{pr}_{c_i}\circ\varrho_{i,\mathcal{A}} = 1_{c_i}$. Observe that then $\psi_i = F\circ\varrho_X\circ f_i$ factors through F. We obtain a commutative diagram

$$\begin{tikzcd}
c_i \arrow[r, "(\varrho_X\circ f_i)\times\varrho_{\mathcal{A},i}"] \arrow[d, "f_i"'] & X \times_Y \big(D_{(n,h)}(\psi_i)\big) \arrow[r, "\mathrm{pr}_{c_i\times\mathbb{C}^n}"] \arrow[d, "1\times\iota_{\psi_i}"'] & D_{(n,h)}(\varrho_X\circ f_i) \arrow[d, "\iota_{(\varrho_X\circ f_i)}"] \\
c \arrow[r, "\varrho_X\times\varrho_{\mathcal{A}}"'] & X \times_Y \mathcal{A}(n,h) \arrow[r, "\Xi'_F"'] & \mathcal{A}\big(F^*(n,h)\big)
\end{tikzcd}$$

This shows that $\Xi'_F\circ\varrho$ is a plot of $\mathcal{A}(F^*(n,h))$ by Proposition 2.25, which completes the proof.

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Universität Hamburg, Fachbereich Mathematik, Bereich Algebra und Zahlentheorie,
Bundesstraße 55, 20146 Hamburg, Germany
severin.bunk@uni-hamburg.de