RELATIONS BETWEEN NOTIONS OF GAPLESSNESS FOR NON-ARCHIMEDEAN FIELDS

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Abstract. In this paper, we are comparing three generalised Bolzano-Weierstraß properties: the Sikorski property (which fails for saturated fields), the Keisler-Schmerl property, and the weak Bolzano-Weierstraß property introduced by Carl, Galeotti, and Löwe. We show that the weak Bolzano-Weierstraß property is "Bolzano-Weierstraß minus Cauchy completeness" and is equivalent the tree property of the base number of the field. We furthermore improve on a number of results by Carl, Galeotti, and Löwe.

1. Introduction

The research area of *generalised Baire spaces* (cf. [8] for a recent survey with a list of open questions) deals with generalisations of real analysis to uncountable cardinals, in particular, analysis in non-Archimedean ordered fields.

One of the important features of the real number field $\mathbb{R}$ for analysis is the fact that it does not have gaps: this is usually expressed in the form of *completeness* (either Dedekind or Cauchy), the *intermediate value theorem*, or the *Bolzano-Weierstraß theorem*. In order to be an analogue of $\mathbb{R}$, a non-Archimedean field should therefore exhibit some forms of gaplessness.

In this paper, we consider Cauchy completeness, saturation, two Bolzano-Weierstraß properties that we shall call the the *Sikorski property* and the *Keisler-Schmerl property*, respectively, and a weak Bolzano-Weierstraß property due to Carl, Galeotti, and Löwe (cf. [10, 7, 1]; definitions are given in § 5) and study their relationships.

Some of these forms of gaplessness are in conflict with each other: e.g., the Sikorski property cannot hold in saturated fields:

Theorem 1.1 (cf., e.g., [1, Corollary 4.8])**.** *If $\mathfrak{K}$ is a κ-saturated totally ordered field with weight and base number κ, then $\mathfrak{K}$ does not have the Sikorski property.*

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In contrast, both the Keisler-Schmerl property and the weak Bolzano-Weierstraß property can hold in saturated fields. Carl, Galeotti, and Löwe characterised the set-theoretic strength of the weak Bolzano-Weierstraß property on saturated fields:

Theorem 1.2 (Carl, Galeotti, & Löwe; [1, Corollary 4.23])**.** *Let $\mathfrak{K}$ be a Cauchy complete and κ-spherically complete totally ordered field with* $\mathrm{bn}(\mathfrak{K}) = \kappa$ *such that κ is uncountable and strongly inaccessible. Then the following are equivalent:*

(1) *κ has the tree property and*
(2) $\mathsf{wBWT}_{\mathfrak{K}}$ *holds.*

We shall show that the weak Bolzano-Weierstraß property is the remainder of the Bolzano-Weierstraß property after removing Cauchy completeness (Theorem 5.4). Furthermore, we shall see that the set-theoretic strength of the Bolzano-Weierstraß phenomenon rests in this remainder of the property and does not require Cauchy completeness (Theorem 7.1).

The structure of the paper is as follows: in § 2, we remind the reader of basic definitions from [1] and characterise weak compactness in terms of the long total order property in § 3. We give definitions and basic properties of our generalised notions of boundedness in § 4 and use these notions to define the three generalisations of Bolzano-Weierstraß in § 5 where we also show the implications and non-implications between the three properties.

The final two sections, §§ 6 & 7 improve on results from [1]: in § 6, we shall improve Theorem 1.1 by showing that even for spherically complete fields with base number κ, the Sikorski property implies that κ is a large cardinal (Theorem 6.3). As a corollary we obtain a result about successor cardinals that was claimed in [1] but whose proof was flawed (Claim 6.1). Finally, in § 7, we shall improve Theorem 1.2 by removing the additional assumptions of Cauchy completeness and strong inaccessibility (Theorem 7.1).

2. Basic definitions

Since this paper is a sequel to [1], we shall freely use the notation introduced in detail in [1, § 2]. In this section, we highlight those definitions that are crucial for this paper and provide a few additional definitions.

If $\mathfrak{K} := (K, +, \cdot, 0, 1, \leq)$ is a totally ordered field, we write $K^+ := \{x \in K\,;\, 0 < x\}$ for the positive part of $\mathfrak{K}$. We let $\mathrm{bn}(\mathfrak{K})$ be the shortest length of a sequence converging to zero in $\mathfrak{K}$, called the *base number* of $\mathfrak{K}$; clearly, the cardinal $\mathrm{bn}(\mathfrak{K})$ is always regular. For several definitions, it is useful to fix a null sequence $\varepsilon_\beta \to 0$

of length κ. If $s, s' : \kappa \to K$ are sequences of length κ, we say that s' is an *approximation* to s if for all $\beta < \kappa$, we have $|s(\beta) - s'(\beta)| < \varepsilon_\beta$.[1] Clearly, if s' is an approximation to s, then s is convergent if s' is, and consequently s has a convergent subsequence if and only if s' does. If $\mathfrak{K}$ is an ordered field, then non-trivial (i.e., not eventually constant) sequences can only be convergent if their length has cofinality $\mathrm{bn}(\mathfrak{K})$; cf., e.g., [1, Corollary 2.6].

Notions of saturation. Two notions of saturation play a major role in this paper: saturation and spherical completeness. If κ is a regular cardinal, then we say that a total order $(X, \leq)$ is *κ-saturated* (or *an η_κ-set*) if for any $L, R \subseteq X$ such that $L < R$ and $|L| + |R| < \kappa$, there is $x \in X$ such that $L < x < R$. We say that it is *κ-spherically complete* if for every $\alpha < \kappa$ and for every nested family $\mathcal{I} = \{I_\gamma\,;\, \gamma < \alpha\}$ of closed intervals, we have that $\bigcap \mathcal{I} \neq \varnothing$. If $\mathfrak{K}$ is an ordered field with base number κ, then one can replace "closed" with "open" in the above definition of spherical completeness (cf. [1, Lemma 2.4]).

If κ is a regular cardinal, every κ-saturated total order is κ-spherically complete, but the converse does not hold (the real line $\mathbb{R}$ is $\aleph_1$-spherically complete, but not $\aleph_1$-saturated).

Some useful facts about totally ordered fields. If $\mathfrak{K}$ and $\mathfrak{K}'$ be totally ordered fields such that $\mathfrak{K}$ is a dense subfield of $\mathfrak{K}'$, then $\mathrm{bn}(\mathfrak{K}) = \mathrm{bn}(\mathfrak{K}')$. Furthermore, if $\mathfrak{K}$ is κ-spherically complete or κ-saturated, then $\mathfrak{K}'$ is κ-spherically complete or κ-saturated.

As usual, we can define a distance function on a totally ordered field and obtain the usual notions of *Cauchy sequence*, *Cauchy completeness*, and *Cauchy completion* (cf. [1, § 2.3] for details); we denote the Cauchy completion of $\mathfrak{K}$ by $\overline{\mathfrak{K}}$. A field is dense in its Cauchy completion; thus, $\mathrm{bn}(\overline{\mathfrak{K}}) = \mathrm{bn}(\mathfrak{K})$. If s and s' are sequences such that s' is an approximation to s, then s is or contains a Cauchy sequence if and only if s' is or contains a Cauchy sequence.

If C and C' are two convex sets, we say that C and C' are *separated by a distance of at least $\varepsilon \in K^+$* if for all $x \in C$ and all $y \in C'$, we have that $|x - y| > \varepsilon$.

We call a sequence in a total order *strictly monotone* if it is either strictly ascending or strictly descending.

Lemma 2.1. *Let $\mathfrak{K}$ be a totally ordered field with* $\mathrm{bn}(\mathfrak{K}) = \kappa$ *and let $s : \kappa \to K$ be a convergent sequence which is not eventually constant. Then s has a strictly monotone subsequence of length κ.*

[1] Formally, being an approximation depends on the choice of the sequence ε_β; for the results in this paper, the choice of null sequence does not matter.

Proof. Let $\ell \in K$ be the limit of s and $S := \mathrm{ran}(s)$. By the pigeon hole principle and the assumption that s is not eventually constant, without loss of generality, we can assume that $s(\alpha) < \ell$ for all $\alpha < \kappa$. Since $\mathrm{bn}(\mathfrak{K}) = \kappa$, we know that no subsequence of s of length $< \kappa$ can converge to ℓ, so for every $A \subseteq S$ with $|A| < \kappa$, there is some α such that $A < s(\alpha) < \ell$. From this, we now recursively construct a strictly increasing subsequence. $\square$

3. The long total order property

A cardinal κ is said to have the *long total order property* if every total order of size κ has a strictly monotone sequence of length κ. It is a folklore result that the long total order property is equivalent to the weak compactness of κ. For completeness, we shall give a proof of this fact here.

Lemma 3.1. *If κ is cardinal, $(X, \leq)$ and $(Y, \leq)$ are total orders, X has size κ, and $(Y, \leq)$ is κ-saturated, then $(X, \leq)$ embeds into $(Y, \leq)$.*

Proof. Order $X = \{x_\alpha\,;\, \alpha < \kappa\}$ in order type κ and construct an embedding $\pi : X \to Y$ by recursion (using AC to pick the images of elements in X). At each stage α, the number of previous elements that lie below x_α and the number of previous elements that lie above x_α have size $|\alpha| < \kappa$, and so the κ-saturation of $(Y, \leq)$ allows us to pick the image of x_α. $\square$

We denote the lexicographic order of ordinal sequences of a fixed ordinal length by $\leq_{\mathrm{lex}}$.

Lemma 3.2. *If κ is a regular cardinal, then every total order of size κ can be embedded in $(2^\kappa, \leq_{\mathrm{lex}})$.*

Proof. We first observe that the map $0 \mapsto 01$, $1 \mapsto 10$, $2 \mapsto 11$ induces an embedding from 3^κ to 2^κ, so it is enough to embed the total order into 3^κ. By Lemma 3.1, we only need to find a κ-saturated suborder of $(3^\kappa, \leq_{\mathrm{lex}})$. If $\lambda < \kappa$ and $s \in 2^\lambda$, we define

$$\widehat{s}(\alpha) := \begin{cases} 0 & \text{if } \alpha < \lambda \text{ and } s(\alpha) = 0, \\ 2 & \text{if } \alpha < \lambda \text{ and } s(\alpha) = 1, \text{ and} \\ 1 & \text{if } \alpha \geq \lambda. \end{cases}$$

It follows immediately from the regularity of κ that the set $H := \{\widehat{s}\,;\, s \in 2^{<\kappa}\} \subseteq 3^\kappa$ is κ-saturated. $\square$

Note that the set H is order isomorphic to the subfield $\mathrm{No}_{<\kappa}$ of the surreal numbers which is well-known to be κ-saturated (cf., e.g., [6, 2] for an introduction to the theory of surreal numbers).

Lemma 3.3. *Let κ be regular and $\lambda < \kappa$. Then there is no strictly monotone κ-sequence in $(2^\lambda, \leq_{\mathrm{lex}})$.*

PROOF. This follows directly from the regularity of κ. Without loss of generality, let s be a monotone sequence in 2^λ of length κ. By definition of $\leq_{\mathrm{lex}}$, for each $\alpha < \lambda$, the value of $s(\alpha)$ can flip back and forth between 0 and 1 at most α many times. By regularity of κ, find β_α such that $s(\alpha)$ is fixed after β_α. Once more, by regularity, find an upper bound $\beta < \kappa$ to all of the β_α. Then the sequence s is constant after β. □

Theorem 3.4 (Folklore)**.** *If κ is an uncountable cardinal. Then the following are equivalent:*

(1) *κ is weakly compact and*
(2) *κ has the long total order property.*

PROOF. For κ inaccessible, this claim can be found in the textbook literature (cf., e.g., [4, Theorem 2.1]). So, it is enough to show that (2) implies that κ is inaccessible.

We first prove that κ is regular. Assume $\lambda = \mathrm{cf}(\kappa) < \kappa$. For each $\alpha < \lambda$ let $A_\alpha = \alpha \times \{\alpha\}$. Moreover, let $A = \bigcup_{\alpha<\lambda} A_\alpha$ and $<_A$ be the following linear order: $(\beta, \gamma) <_A (\beta', \gamma')$ if and only if $\gamma > \gamma'$, or $\gamma = \gamma'$ and $\beta < \beta'$. Note that $|A| = \kappa$. Moreover every strictly monotone λ-sequence s in A must be strictly decreasing and such that for all $\alpha, \beta, \gamma < \lambda$ with $\alpha \neq \beta$ we have that if $s(\alpha) \in A_\gamma$ then $s(\beta) \notin A_\gamma$. So no strictly monotone sequence in A can have size κ contradicting the hypothesis.

We now show that κ is a strong limit: suppose there is a $\lambda < \kappa$ such that $2^\lambda \geq \kappa$. Then there is a suborder of $(2^\lambda, \leq_{\mathrm{lex}})$ of cardinality κ which by the long total order property has a strictly monotone sequence of length κ, contradicting Lemma 3.3. □

4. Generalisations of boundedness

In this section, we introduce strengthenings of the standard notion of boundedness that will be used in § 5 to define the properties studied in this paper. In the entire section, $\mathfrak{K}$ and $\mathfrak{K}'$ are totally ordered fields with $\mathrm{bn}(\mathfrak{K}) = \mathrm{bn}(\mathfrak{K}') = \kappa$. As usual, if $X \subseteq K$, we call $\mathrm{ch}(X) := \{z \in K\,;\, \exists x, y \in X (x \leq z \leq y)\}$ the *convex hull of* X. If $\mathfrak{K}$ is a subfield of $\mathfrak{K}'$ and D is convex in $\mathfrak{K}'$, then $D \cap K$ is convex in $\mathfrak{K}$ and $\mathrm{ch}(D \cap K) = D$.

Definition 4.1. Let $s : \kappa \to K$ be a κ-sequence in $\mathfrak{K}$ and $S := \mathrm{ran}(s)$.

(1) We say that s is *totally bounded* if for all $\varepsilon \in K^+$ there is $\beta_\varepsilon < \kappa$ such that for all $\beta < \kappa$ there is $\gamma < \beta_\varepsilon$ and $|s(\beta) - s(\gamma)| < \varepsilon$.

(2) Let C be a bounded convex subset of $\mathfrak{K}$ such that $|S \cap C| = \kappa$, and let $\varepsilon \in K^+$. We say that $\mathcal{I}$ is a *witnessing family* for s, C, and ε if $|\mathcal{I}| = \mu < \kappa$ and $\mathcal{I} = \{\langle \ell_\alpha, r_\alpha \rangle\,;\ \alpha < \mu\}$ is family of pairs of elements in K such that
   (a) for each $\alpha < \mu$, we have $\ell_\alpha < r_\alpha$, $r_\alpha - \ell_\alpha < \varepsilon$, and $I_\alpha := (\ell_\alpha, r_\alpha) \subseteq C$, and
   (b) $|(S \cap C) \setminus \bigcup_{\alpha<\mu} I_\alpha| < \kappa$.

(3) The sequence s is called *interval witnessed* if for every bounded convex set C in $\mathfrak{K}$ such that $|S \cap C| = \kappa$ and every $\varepsilon \in K^+$ there is a witnessing family for s, C, and ε.

Lemma 4.2. *Let $s : \kappa \to K$ be a κ-sequence in $\mathfrak{K}$. Then the following implications hold:*

$$\begin{tikzcd}[column sep=small] & s \textit{ is convergent} \arrow[dl, Rightarrow, "(1)"'] \arrow[dr, Rightarrow, "(2)"] & \\ s \textit{ is a Cauchy sequence} \arrow[dr, Rightarrow, "(3)"'] & & s \textit{ is interval witnessed} \arrow[dl, Rightarrow, "(4)"] \\ & s \textit{ is totally bounded.} & \end{tikzcd}$$

PROOF. (1) is the standard proof and (2) is straightforward.

(3) Fix $\varepsilon \in K^+$; since s is Cauchy, find some γ_ε such that for all $\beta, \beta' \geq \gamma_\varepsilon$, we have that $|s(\beta) - s(\beta')| < \varepsilon$. Then $\beta_\varepsilon := \gamma_\varepsilon + 1$ is a total bound for s.

(4) If $|S| < \kappa$, then it is clearly totally bounded: let $\beta < \kappa$ be such that $\{s_\gamma\,;\ \gamma < \beta\} = S$; then $\beta_\varepsilon := \beta$ works for all $\varepsilon \in K^+$. Thus assume that $|S| = \kappa$ and fix $\varepsilon \in K^+$; since s is interval witnessed, find a witnessing family $\mathcal{I} = \{I_\alpha\,;\ \alpha < \mu\}$ for s, $\mathrm{ch}(S)$, and ε (if $|S| < \kappa$, then s is clearly totally bounded). Define

$$\eta := \min\{\beta\,;\ \forall \beta' \geq \beta (s(\beta') \in \bigcup \mathcal{I})\}, \tag{*}$$

$$\eta' := \min\{\beta\,;\ \forall \alpha < \mu \exists \beta' < \beta (s(\beta') \in I_\alpha)\}, \text{ and} \tag{**}$$

$$\eta^* := \max\{\eta, \eta'\}.$$

Then η^* is a total bound for s: if $\beta \geq \eta^*$, find α such that $s(\beta) \in I_\alpha$ by (*). Then, (**) implies that there is some $\beta' < \eta' \leq \eta^*$ such that $s(\beta') \in I_\alpha$, so $|s(\beta) - s(\beta')| < \varepsilon$. □

Observation 4.3. If $\mathfrak{K}$ is not Cauchy complete, then there are Cauchy sequences that are not interval witnessed. (In particular, implication (4) in Lemma 4.2 cannot be reversed.)

Proof. Let s be a Cauchy sequence that does not converge and let $\ell \in \overline{K}$ be its limit in the Cauchy completion $\overline{\mathfrak{K}}$ of $\mathfrak{K}$. Without loss of generality, assume that s is strictly increasing and entirely contained within $C := (0, \ell)$. Since s is Cauchy, every subinterval of C with positive distance from ℓ contains only $<\kappa$ many elements of s. Since $\ell \notin K$, all intervals that are part of any sequence of intervals satisfying Definition 4.1 (2a) will have positive distance from ℓ, so for cardinality reasons no such sequence can satisfy Definition 4.1 (2b) for any ε. $\square$

Let $s : \kappa \to K$ be a κ-sequence and C be a convex set. We say that s is *padded within* C if there is an ε such that for every α, we have that $(s(\alpha)-\varepsilon, s(\alpha)+\varepsilon) \subseteq C$.

Lemma 4.4. *Let $s : \kappa \to K$ be a sequence, $S := \mathrm{ran}(s)$, C be any bounded convex set such that $S \subseteq C$, and assume that no element of S is either the least upper bound or greatest lower bound of C. Then either s has a Cauchy subsequence or s is padded within C.*

Proof. We use our fixed null sequence ε_β: if s is not padded within C, then for each β, there is α_β such that $(s(\alpha_\beta) - \varepsilon_\beta, s(\alpha_\beta) + \varepsilon_\beta) \not\subseteq C$. By the pigeon hole principle, without loss of generality, there is a set E of size κ such that $s(\alpha_\beta)+\varepsilon_\beta \notin C$ for each $\beta \in E$. Consider $\{s(\alpha_\beta)\,;\, \beta \in E\}$. If this set has cardinality κ, then it forms a Cauchy subsequence of s. If it does not have cardinality κ, then by regularity of κ, there is some x and a set $B \subseteq E$ of cardinality κ such that $x = s(\alpha_\beta)$ for all $\beta \in B$. Thus, the sequence $\{x + \varepsilon_\beta\,;\, \beta \in B\}$ lies entirely outside of C, but converges to $x \in C$: thus x is the largest element of C, contradicting our assumption. $\square$

Lemma 4.5. *Let $\mathfrak{K}$ be cofinal in $\mathfrak{K}'$. If $s : \kappa \to K$ is interval witnessed in $\mathfrak{K}$, then s is interval witnessed in $\mathfrak{K}'$.*

Proof. Let D be bounded convex in $\mathfrak{K}'$ and $\eta > 0$. Since $\mathfrak{K}$ was cofinal in $\mathfrak{K}'$, $C := D \cap K$ is bounded convex in $\mathfrak{K}$ and there is some $0 < \varepsilon < \eta$ in K^+. Let $\mathcal{I} := \{\langle \ell_\alpha, r_\alpha \rangle\,;\, \alpha < \mu\}$ be a witnessing family for s, C, and ε in $\mathfrak{K}$. Clearly, $\mathcal{I}$ is also a witnessing family for s, D, and η in $\mathfrak{K}'$. $\square$

Lemma 4.6. *Let $s, s' : \kappa \to K$ be sequences such that s is interval witnessed and s' is an approximation to s. Then either s contains a convergent subsequence or s' is interval witnessed.*

Proof. Fix C and ε and find a witnessing family $\mathcal{I} = \{I_\alpha\,;\,\alpha < \mu\}$ for s, C, and ε. If this is a witnessing family for s', C, and ε, we are done, so let us assume that this is not the case. This means that the set $B := \{\beta\,;\, s'(\beta) \notin \bigcup_{\alpha<\mu} I_\alpha\}$ has cardinality κ. By the pigeon hole principle, find $\langle \ell_\alpha, r_\alpha\rangle \in \mathcal{I}$ such that $B' := \{\beta \in B\,;\, \ell_\alpha < s(\beta) < r_\alpha\}$ has cardinality κ. This means that for each $\beta \in B'$, we either have $s'(\beta) < \ell_\alpha < s(\beta)$ or $s(\beta) < r_\alpha < s'(\beta)$. Once more using the pigeon hole principle, we find a set B'' of cardinality κ where one of the two alternatives holds, say, without loss of generality, $s(\beta) < r_\alpha < s'(\beta)$. But since s' was an approximation of s, we have that

$$|s(\beta) - r_\alpha| \leq |s(\beta) - s'(\beta)| < \varepsilon_\beta,$$

and hence the subsequence of the $s(\beta)$ with $\beta \in B''$ converges to r_α, so s has a convergent subsequence. □

Lemma 4.7. *Let $\overline{\mathfrak{K}}$ be the Cauchy completion of $\mathfrak{K}$. If $s : \kappa \to K$ is interval witnessed in $\overline{\mathfrak{K}}$, then either s has a convergent subsequence or s is interval witnessed in $\mathfrak{K}$.*

Proof. If s has a convergent subsequence, we are done, so let us assume that it does not and show that it is interval witnessed in $\mathfrak{K}$. Fix a bounded convex set C and $\varepsilon > 0$, and write $S := \mathrm{ran}(s)$. Since s is interval witnessed in $\overline{\mathfrak{K}}$, let $\mathcal{I}$ be a witnessing family for s, C, and ε in $\overline{K}$. We show that for each pair $\langle \ell_\alpha, r_\alpha\rangle \in \mathcal{I}$ we can find $b, b' \in K$ such that $\ell_\alpha < b < b' < r_\alpha$ such that resulting set of pairs is still a witnessing family for s, C, and ε.

Since $\ell_\alpha \in \overline{K}$, there is a strictly decreasing sequence $t : \kappa \to K$ converging to ℓ_α. If for an unbounded number of β, there is some β' with $\ell_\alpha < s(\beta') < t(\beta)$, then these form a subsequence of s converging to ℓ_α. Contradiction! Thus, there is a bound $b \in K$ such that $\ell_\alpha < b < r_\alpha$ and $(\ell_\alpha, b) \cap S = \varnothing$. We use the same argument for the upper bound r_α and obtain $b' \in K$ with $b < b' < r_\alpha$ such that $(b', r_\alpha) \cap S = \varnothing$. So, we can replace (ℓ_α, r_α) by (b, b') without changing the fact that the set of intervals is a witnessing family. Do this for every pair occurring in $\mathcal{I}$ and obtain a witnessing family for s, C, and ε in K. □

5. Three Bolzano-Weierstrass properties

As before, $\mathfrak{K}$ and $\mathfrak{K}'$ are totally ordered fields with $\mathrm{bn}(\mathfrak{K}) = \mathrm{bn}(\mathfrak{K}') = \kappa$. We introduce the three generalisations of the Bolzano-Weierstraß theorem mentioned in the introduction.

Sikorski: We say that $\mathfrak{K}$ has the *Bolzano-Weierstraß property due to Sikorski* (in short, the *Sikorski property*) if every bounded κ-sequence in $\mathfrak{K}$ has a convergent subsequence and write $\mathsf{BWT}_{\mathfrak{K}}$ for this statement [10] (cf. also [9, 3]).

Keisler & Schmerl: We say that $\mathfrak{K}$ has the *Bolzano-Weierstraß property due to Keisler and Schmerl* (in short, the *Keisler-Schmerl property*) if every totally bounded κ-sequence in $\mathfrak{K}$ has a convergent subsequence and write $\mathsf{BWT}^*_{\mathfrak{K}}$ for this statement [7].

Carl, Galeotti, & Löwe: We say that $\mathfrak{K}$ has the *weak Bolzano-Weierstraß property* if every interval witnessed κ-sequence in $\mathfrak{K}$ has a convergent subsequence and write $\mathsf{wBWT}_{\mathfrak{K}}$ for this statement [1].

Proposition 5.1. *The following implications hold:*

$$\begin{tikzcd}
\mathsf{BWT}_{\mathfrak{K}} \arrow[r, Rightarrow, "(1)"] & \mathsf{BWT}^*_{\mathfrak{K}} \arrow[r, Rightarrow, "(2)"] \arrow[d, Rightarrow, "(3)"] & \mathfrak{K} \textit{ is Cauchy complete.} \\
& \mathsf{wBWT}_{\mathfrak{K}} &
\end{tikzcd}$$

Proof. (1) is obvious; (2) is Lemma 4.2 (3); (3) is Lemma 4.2 (4). $\square$

The following proposition tells us that the weak Bolzano-Weierstraß property is preserved by thinning out a field (this is not the case for the other two properties since they both imply Cauchy completeness by Proposition 5.1):

Proposition 5.2. *Let $\mathfrak{K}$ be cofinal in $\mathfrak{K}'$. Then* $\mathsf{wBWT}_{\mathfrak{K}'}$ *implies* $\mathsf{wBWT}_{\mathfrak{K}}$.

Proof. Let $s : \kappa \to K$ be an interval witnessed sequence in $\mathfrak{K}$. By Lemma 4.5, s is interval witnessed in $\mathfrak{K}'$. Thus, using $\mathsf{wBWT}_{\mathfrak{K}'}$, we get a convergent subsequence s' of s with limit ℓ. We claim that $\ell \in K$, proving the claim.

Case 1: s' has an eventually constant subsequence. Then clearly, $\ell \in K$.

Case 2: s' has no eventually constant subsequence. By the pigeon hole principle, we can assume, without loss of generality, that the range of s' lies within the bounded convex set $C := \{x \in K\,;\, \ell - 1 < x < \ell\}$. Since s was interval witnessed in K, we find an witnessing family $\mathcal{I}$ for s, C, and 1. Using once more the pigeon hole principle, we find $(x, y) \in \mathcal{I}$ that contains κ many elements of $\operatorname{ran}(s')$ (and hence is unbounded below ℓ). Note that $x, y \in K$. Clearly, since (x, y) is unbounded below ℓ, we have $\ell \leq y$. But since $(x, y) \subseteq C = \{x \in K\,;\, \ell - 1 < x < \ell\}$, we know that $y \leq \ell$. So $\ell = y \in K$ which proves the claim. $\square$

Theorem 5.3. *Let $\overline{\mathfrak{K}}$ be the Cauchy completion of $\mathfrak{K}$. Then* $\mathsf{wBWT}_{\mathfrak{K}}$ *if and only if* $\mathsf{wBWT}_{\overline{\mathfrak{K}}}$.

Proof. Since $\mathfrak{K}$ is cofinal in $\overline{\mathfrak{K}}$, the "if" direction is just Proposition 5.2. Now assume $\mathsf{wBWT}_{\mathfrak{K}}$ and let $s : \kappa \to \overline{K}$ be interval witnessed in $\overline{\mathfrak{K}}$. We need to show that s has a convergent subsequence.

Since $\overline{\mathfrak{K}}$ is the Cauchy completion of $\mathfrak{K}$, we find an approximation $s : \kappa \to K$ to s'. By Lemma 4.6, we can assume that s' is interval witnessed in $\overline{\mathfrak{K}}$ (otherwise s has a convergent subsequence and we are done). If s' is not interval witnessed in $\mathfrak{K}$, then it contains a convergent subsequence by Lemma 4.7; if it is interval witnessed in $\mathfrak{K}$, then it contains a convergent subsequence by $\mathsf{wBWT}_{\mathfrak{K}}$. So, in either case, s' has a convergent subsequence, but since s' was an approximation to s, the sequence s must contain a convergent subsequence. □

Theorem 5.4. *A totally ordered field $\mathfrak{K}$ has the Keisler-Schmerl property if and only if it has the weak Bolzano-Weierstraß property and is Cauchy complete.*

Proof. The "only if" direction is Proposition 5.1 (2) & (3), so let us prove the "if" direction.

Let $s : \kappa \to K$ be a totally bounded sequence and $S := \mathrm{ran}(s)$. We need to show that it has a convergent subsequence.

Fix a bounded convex set $C \supseteq S$ and $\varepsilon > 0$. If C has a least upper bound or greatest lower bound in S, then check whether this value occurs cofinally often in s. If so, then s has a constant subsequence of length κ, and hence a convergent subsequence. If not, then we replace s with the sequence cut after the last occurrence of this value and can henceforth assume that C satisfies the conditions of Lemma 4.4.

As a consequence, we know from Lemma 4.4 that s either contains a Cauchy subsequence or is padded in C. Since we assumed that $\mathfrak{K}$ is Cauchy complete, the first case implies that s contains a convergent subsequence and we are done. So, we can assume that s is padded in C, say, by some value ε'. Let $\eta := \min\{\varepsilon, \varepsilon'\}$.

By total boundedness, we find some $\beta_\eta < \kappa$ such that for all $\beta < \kappa$ there is a $\gamma < \beta_\eta$ such that $|s(\beta) - s(\gamma)| < \eta$. Thus, if $J_\alpha := (s(\alpha) - \eta, s(\alpha) + \eta)$ for $\alpha < \beta_\eta$, then $S \subseteq \bigcup_{\alpha<\beta_\eta} J_\alpha \subseteq C$ (since $\eta \leq \varepsilon'$). It is now easy to transform this family of intervals into a witnessing family for s, C, and ε of size $|\beta_\eta| < \kappa$. Hence s is interval witnessed, and thus by $\mathsf{wBWT}_{\mathfrak{K}}$ has a convergent subsequence. □

Corollary 5.5. *None of the converses of the implications of Proposition 5.1 hold in general.*

Proof. If κ is weakly compact, then consider $\mathrm{No}_{<\kappa}$ and its Cauchy completion $\mathbb{R}_\kappa$. By Theorem 1.2, $\mathbb{R}_\kappa$ has the weak Bolzano-Weierstraß property. Since it is Cauchy complete, it has the Keisler-Schmerl property, but since it is κ-saturated, by Theorem 1.1, it cannot have the Sikorski property, so implication (1) cannot be reversed. By Theorem 5.3, $\mathrm{No}_{<\kappa}$ inherits the weak Bolzano-Weierstraß property from $\mathbb{R}_\kappa$, but since it is not Cauchy complete, it cannot have the Keisler-Schmerl property, so implication (3) cannot be reversed. By Theorem 1.2 once more, if κ is inaccessible, but not weakly compact, then $\mathbb{R}_\kappa$ is Cauchy complete, but does not have the weak Bolzano-Weierstraß property (and thus not the Keisler-Schmerl property), so implication (2) cannot be reversed. □

6. Spherical completeness & the Sikorski property

Theorem 1.1 shows that saturation and the Sikorski property are incompatible. This raises the natural question whether that also holds for the weaker notion of spherical completeness. In [1], the authors claimed that this is true for successor ordinals:

Claim 6.1 (Carl, Galeotti, & Löwe; [1, Corollary 4.10])**.** *If $\mathfrak{K}$ is a field with $\mathrm{w}(\mathfrak{K}) = \mathrm{bn}(\mathfrak{K}) = \kappa^+$ which is κ^+-spherically complete, then $\mathsf{BWT}_{\mathfrak{K}}$ does not hold.*

This claim is correct, but the proof relied on [1, Lemma 2.8] whose proof is flawed (cf. [5, pp. 45–46] for a detailed discussion). Galeotti provided a separate proof of the special case $\kappa = \aleph_1$ in [5, Lemma 3.11]. In this section, we fix the issue by proving (a considerable strengthening of) Claim 6.1.

Lemma 6.2. *Let $\mathfrak{K}$ be a totally ordered field with $\mathrm{bn}(\mathfrak{K}) = \kappa$. Assume that $\mathfrak{K}$ is κ-spherically complete. Then there is a map I with $\mathrm{dom}(I) = 2^{<\kappa}$ such that for $s \in 2^\alpha$, we have that $I(s) = (\ell_s, r_s)$ with $r_s - \ell_s \leq \varepsilon_\alpha$ and if $s \subseteq t$, then $I(s) \supseteq I(t)$.*

Proof. We construct the map I by recursion. Let $I(\varnothing) := (0, 1)$. Suppose that $I(s) = (\ell_s, r_s)$ is defined. Then let

$$I(s^\frown 0) := (\ell_s, \min\{\ell_s + \varepsilon_{\mathrm{dom}(s)+1}, \frac{r_s + \ell_s}{2}\}) \text{ and}$$
$$I(s^\frown 1) := (\max\{r_s - \varepsilon_{\mathrm{dom}(s)+1}, \frac{r_s + \ell_s}{2}\}, r_s).$$

If λ is a limit ordinal and s a binary sequence of length λ, consider the sequence $I_\alpha := I_{s\restriction\alpha}$ for $\alpha < \lambda$. This is a sequence of nested intervals of size less than κ, and thus by κ-spherical completeness, we know that $\bigcap_{\alpha<\lambda} I_\alpha \neq \varnothing$. But we also know that the sequences $\{\ell_{s\restriction\alpha}\,;\, \alpha < \lambda\}$ and $\{r_{s\restriction\alpha}\,;\, \alpha < \lambda\}$ cannot be convergent (since they are too short), so the intersection must contain at least two (in fact, many) elements, say, $x < y$. Let $\ell_s := x$ and $r_s := \min\{x + \varepsilon_\lambda, y\}$. □

Theorem 6.3. *Let $\mathfrak{K}$ be a κ-spherically complete totally ordered field with* $\mathrm{bn}(K) = \kappa$. *If $\mathfrak{K}$ has the Sikorski property, then κ is weakly compact.*

Proof. We are going to use Theorem 3.4 and prove that κ has the long total order property. Fix any total order $(X, \leq)$ of size κ.

The field $\mathfrak{K}$ is κ-spherically complete, so we can use Lemma 6.2 obtain a function I assigning intervals $I(s) = (\ell_s, r_s)$ to the sequences in the full binary tree s. If $x \in 2^\kappa$, consider $\ell_\alpha := \ell_{x\restriction\alpha}$ and $r_\alpha := r_{x\restriction\alpha}$. By construction, these sequences are Cauchy sequences and $r_\alpha - \ell_\alpha \leq \varepsilon_\alpha$.

The Sikorski property of $\mathfrak{K}$ implies that $\mathfrak{K}$ is Cauchy complete, so these sequences converge to a unique element of the field which we denote by $I(x)$. By construction, we get that if $x \leq_{\mathrm{lex}} y$, then $I(x) < I(y)$, so I is an order preserving embedding of $(2^\kappa, \leq_{\mathrm{lex}})$ into $(K, \leq)$, or, more precisely, into the unit interval $(0, 1)$ of the field. Thus, by Lemma 3.2 $(X, \leq)$ is embeddable into $((0, 1), \leq)$; we denote the image of X under this embedding as $\widehat{X} \subseteq (0, 1) \subseteq K$. Consequently, $\widehat{X}$ is a bounded set of size κ and thus, by $\mathsf{BWT}_{\mathfrak{K}}$, there is a convergent sequence consisting of elements of $\widehat{X}$ which, by Lemma 2.1, has a strictly monotone subsequence. This establishes the long total order property of κ which by Theorem 3.4 implies that κ is weakly compact. □

Theorem 6.3 immediately implies Claim 6.1 since successor cardinals are not weakly compact.

7. Weak compactness & weak Bolzano-Weierstrass

In this section, we strengthen Theorem 1.2 by removing the assumptions of Cauchy completeness of K and of strong inaccessibility of κ. As before, we fix a totally ordered field $\mathfrak{K}$ with $\mathrm{bn}(\mathfrak{K}) = \kappa$.

Theorem 7.1. *If $\mathfrak{K}$ is κ-spherically complete, then the following are equivalent:*

(1) *κ has the tree property and*
(2) $\mathsf{wBWT}_{\mathfrak{K}}$ *holds.*

Note that Theorem 5.4 allows us to reformulate Theorem 7.1 in terms of the Keisler-Schmerl property:

Corollary 7.2. *If $\mathfrak{K}$ is κ-spherically complete and Cauchy complete, then the following are equivalent:*

(1) *κ has the tree property and*
(2) $\mathsf{BWT}^*_{\mathfrak{K}}$ *holds.*

Proof of Theorem 7.1. By Theorem 5.3, it is enough to show the claim for Cauchy complete fields (in particular, we can use the proofs from [1] that used the assumption of Cauchy completeness). Thus, "(2)⇒(1)" is just [1, Theorem 4.17].

The proof of "(1)⇒(2)" follows the proof of [1, Theorem 4.21]; however, that proof has as additional assumption that κ is strongly inaccessible, so we need to modify those steps in the construction that use this assumption. The proof of [1, Theorem 4.21] takes an interval witnessed sequence $s : \kappa \to K$ with $S := \mathrm{ran}(s)$ and constructs for each $\alpha < \kappa$, a set of pairwise disjoint intervals T_α by recursion; these intervals form the tree that produces the convergent subsequence via the tree property of κ. The conditions that need to be checked in the tree construction are [1, p. 1103]:

(1) for each $\alpha < \kappa$, $|T_\alpha| < \kappa$,
(2) for each $\alpha < \kappa$ and each $I \in T_\alpha$, we have that $|S \cap I| = \kappa$,
(3) for each $\alpha < \beta < \kappa$ and every $I \in T_\beta$, there is a $J \in T_\alpha$ such that I is a subinterval of J, and
(4) for each $\alpha < \kappa$, we have $|M_\alpha| < \kappa$ (where $S_\alpha := S \cap \bigcup\{I\,;\, I \in T_\alpha\}$ and $M_\alpha := S \backslash S_\alpha$).

If E_α is the set of all interval endpoints occurring in T_α, then $|E_\alpha| < \kappa$, and thus (since $\mathrm{bn}(\mathfrak{K}) = \kappa$), we find $\varepsilon_\alpha > 0$ such that for all $x, y \in E_\alpha$, $\varepsilon_\alpha < |x - y|$. Hence, any two intervals $I, I' \in T_\alpha$ are separated by at least ε_α.

The only step in the construction in [1] that uses the assumption that κ is strongly inaccessible is the limit case of the recursion, so this is the only part of the proof that requires a change:

Let α be a limit ordinal and let the previous sets of intervals T_β, for $\beta < \alpha$ be defined, satisfying (1) to (4). Let $T_{<\alpha} := \bigcup_{\beta<\alpha} T_\beta$ be ordered by reverse inclusion. Note that $\{\varepsilon_\beta\,;\, \beta < \alpha\}$ is a set of positive elements of K of size smaller than κ, and thus (since $\mathrm{bn}(\mathfrak{K}) = \kappa$), there is some $\varepsilon > 0$ smaller than all of the ε_β (for $\beta < \alpha$). Any two intervals in $T_{<\alpha}$ are separated by at least ε.

Let $\mathcal{B}$ be the set of branches through this tree; then for any $b \in \mathcal{B}$, the set $C_b := \bigcap\{I\,;\, I \in b\}$ is a convex set and the proof of the following claim remains the same [1, Claim 4.22]:

Claim 7.3. *We have that* $S \backslash \bigcup_{\beta<\alpha} M_\beta = S \cap \bigcup_{b\in\mathcal{B}} C_b$.

When κ is strongly inaccessible, we know that $|\mathcal{B}| < \kappa$, but this need not be the case in our setting. However, we can argue that only fewer than κ many branches matter for the construction:

By our choice of ε, we know that for every two distinct branches $b, b' \in \mathcal{B}$ the sets C_b and $C_{b'}$ are separated by a distance of at least ε. Since s is interval witnessed, we obtain a witnessing family $\mathcal{I}$ with $|\mathcal{I}| < \kappa$ for s, (x^*, y^*), and ε. Since any two of the sets C_b are separated by a distance of at least ε, we get that for any $I \in \mathcal{I}$,

$$|\{b\,;\, C_b \cap I \neq \varnothing\}| \leq 1.$$

Therefore, the set $\mathcal{B}' := \{b \in \mathcal{B}\,;\, \exists I \in \mathcal{I}(C_b \cap I \neq \varnothing)\}$ has cardinality smaller than κ.

By inductive hypothesis (4), we know that $|M_\beta| < \kappa$ for each $\beta < \alpha$, and thus, by regularity of κ, $\bigcup_{\beta<\alpha} M_\beta$ has size less than κ, hence by Claim 7.3, we have that $|S \cap \bigcup_{b\in\mathcal{B}} C_b| = \kappa$. Moreover, since $|S \setminus \bigcup \mathcal{I}| < \kappa$ we have that $|S \setminus \bigcup_{b\in\mathcal{B}'} C_b| < \kappa$.

For each $b \in \mathcal{B}'$ we can apply the fact that s was interval witnessed to such a convex set C_b and find a set $\mathcal{I}_b$ of fewer than κ many subintervals of C_b with diameter $<\delta(\alpha)$ such that $|S \cap (C_b \setminus \bigcup \mathcal{I}_b)| < \kappa$. Now let $T_\alpha := \{I\,;$ there is a $b \in \mathcal{B}'$ such that $|S \cap C_b| = \kappa$ and $I \in \mathcal{I}_b$ and $|S \cap I| = \kappa\}$.

Property (1) follows from the facts that κ is regular, $|\mathcal{B}'| < \kappa$, and for each $b \in \mathcal{B}'$, $|\mathcal{I}_b| < \kappa$. As before, property (4) follows from (1) and fact that $\mathrm{bn}(\mathfrak{K}) = \kappa$. Property (2) and (3) are clear by construction. Let $W_0 := \bigcup\{S \cap C_b\,;\, |S \cap C_b| < \kappa\}$; once more, by regularity of κ and $|\mathcal{B}'| < \kappa$, we get that $|W_0| < \kappa$. Furthermore, let $W_1 := \bigcup\{S \cap (C_b \setminus \bigcup \mathcal{I}_b)\,;\, |S \cap C_b| = \kappa\}$; again, regularity of κ and the choice of $\mathcal{I}_b$ implies that $|W_1| < \kappa$. But $M_\alpha = \bigcup_{\beta<\alpha} M_\beta \cup W_0 \cup W_1$, so it has size less than κ, and thus we checked that property (4) holds as well.

This finishes the construction for the limit case; the rest of the proof is the same as in [1, Theorem 4.21]. $\square$

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