

A NEW OBSTRUCTION FOR NORMAL SPANNING TREES

MAX PITZ

ABSTRACT. In a paper from 2001 (Journal of the LMS), Diestel and Leader offered a proof that a connected graph has a normal spanning tree if and only if it does not contain a minor from two specific forbidden classes of graphs, all of cardinality \aleph_1 .

Unfortunately, their proof contains a gap, and their result is incorrect. In this paper, we construct a third type of obstruction: an \aleph_1 -sized graph without a normal spanning tree that contains neither of the two types described by Diestel and Leader as a minor. Further, we show that any list of forbidden minors characterising the graphs with normal spanning trees must contain graphs of arbitrarily large cardinality.

§1. INTRODUCTION

A rooted spanning tree T of a graph G is called *normal* if the ends of any edge of G are comparable in the natural tree order of T . Intuitively, the edges of G run ‘parallel’ to branches of T , but never ‘across’. All countable connected graphs have normal spanning trees, but uncountable graphs might not, as demonstrated by complete graphs on uncountably many vertices.

Halin observed in [6] that the property of having a normal spanning tree is minor-closed, i.e. preserved under taking connected minors. Recall that a graph H is a *minor* of another graph G , written $H \preceq G$, if to every vertex $x \in H$ we can assign a (possibly infinite) connected set $V_x \subseteq V(G)$, called the *branch set* of x , so that these sets V_x are disjoint for different x and G contains a $V_x - V_y$ edge whenever xy is an edge of H .

In [6, Problem 7.3] Halin asked for a forbidden minor characterisation for the property of having a normal spanning tree. In the universe of finite graphs, the famous Seymour-Robertson Theorem asserts that any minor-closed property of finite graphs can be characterised by *finitely* many forbidden minors, see e.g. [4, §12.7]. Whilst for infinite graphs, we generally need an infinite list of forbidden minors, Diestel and Leader [5] published a proof claiming that for the property of having a normal spanning tree, the forbidden minors come in two structural types:

First, the class of (\aleph_0, \aleph_1) -graphs, bipartite graphs (A, B) such that $|A| = \aleph_0$, $|B| = \aleph_1$, and every vertex in B has infinite degree. Structural results on this graph class can be found in [1].

2010 *Mathematics Subject Classification.* 05C83, 05C05, 05C63.

Key words and phrases. normal spanning trees, minor, colouring number, stationary sets.

And second, the class of *Aronszajn-tree graphs*, graphs whose vertex set is an order theoretic Aronszajn tree \mathcal{T} (an order tree (\mathcal{T}, \leq) of size \aleph_1 in which all levels and branches are countable) such that the down-neighbourhood of any node $t \in \mathcal{T}$ is cofinal below t .

However, there is a gap in Diestel and Leader’s proof, and it turns out that their list of forbidden minors is incomplete: In Section 3, we exhibit a third obstruction for normal spanning trees – a graph without normal spanning tree containing neither an (\aleph_0, \aleph_1) -graph nor an Aronszajn-tree graph as a minor. More significantly, we will see in Section 5 why any list of forbidden minors that works just under the usual axioms of set theory ZFC must contain graphs of arbitrary large cardinality. In between, in Section 4, we discuss how these new obstructions occur naturally when trying to build a normal spanning tree.

Where does this leave us? Fortunately, all is not lost. Indeed, this new third obstruction only demonstrates that the 3-way interaction between normal spanning trees, graphs on order trees and the colouring number of infinite graphs is deeper and more intriguing than initially thought. Recall that a graph G has *countable colouring number* if there is a well-order \leq^* on $V(G)$ such that every vertex of G has only finitely many neighbours preceding it in \leq^* . Every graph with a normal spanning tree, and hence every minor of it, has countable colouring number, as witnessed by well-ordering the graph level by level.

The most important consequence of Diestel and Leader’s proposed forbidden minor characterisation was that it would have implied Halin’s conjecture [6, Conjecture 7.6], that a connected graph has a normal spanning tree if and only if every minor of it has countable colouring number. In a paper in preparation [9], I will give a direct proof of Halin’s conjecture. From this, a revised forbidden minor characterisation of graphs with normal spanning trees can be deduced.

§2. PRELIMINARIES

We follow the notation in [4]. Given a subgraph $H \subseteq G$, write $N(H)$ for the set of vertices in $G - H$ with a neighbour in H .

2.1. Normal spanning trees. If T is a (graph-theoretic) tree with root r , we write $x \leq y$ for vertices $x, y \in T$ if x lies on the unique r - y path in T . A rooted spanning tree $T \subseteq G$ is *normal* if the ends of any edge of G are comparable in this tree order on T .

A set of vertices $U \subseteq V(G)$ is *dispersed* (in G) if every ray in G can be separated from U by a finite set of vertices. The following theorem of Jung from [7], from which we will use the implication (1) \Rightarrow (2) further below, characterises graphs with normal spanning trees, see also [10] for a short proof.

Theorem 2.1 (Jung). *The following are equivalent for a connected graph G :*

- (1) G has a normal spanning tree,
- (2) G has a normal spanning tree for every choice of $r \in V(G)$ as the root, and
- (3) $V(G)$ is a countable union of dispersed sets.

2.2. Normal tree orders and T -graphs. A partially ordered set (T, \leq) is called an *order tree* if it has a unique minimal element (called the *root*) and all subsets of the form $[t] = [t]_T := \{t' \in T : t' \leq t\}$ are well-ordered. Our earlier partial ordering on the vertex set of a rooted graph-theoretic tree is an order tree in this sense.

Let T be an order tree. A maximal chain in T is called a *branch* of T ; note that every branch inherits a well-ordering from T . The *height* of T is the supremum of the order types of its branches. The *height* of a point $t \in T$ is the order type of $\overset{\circ}{[t]} := [t] \setminus \{t\}$. The set T^i of all points at height i is the i th *level* of T , and we write $T^{<i} := \bigcup \{T^j : j < i\}$.

The intuitive interpretation of a tree order as expressing height will also be used informally. For example, we may say that t is *above* t' if $t > t'$, call $\lceil X \rceil = \lceil X \rceil_T := \bigcup \{[x] : x \in X\}$ the *down-closure* of $X \subseteq T$. And we say that X is *down-closed*, or X is a *rooted subtree*, if $X = \lceil X \rceil$.

An order tree T is *normal* in a graph G , if $V(G) = T$ and the two ends of any edge of G are comparable in T . We call G a *T -graph* if T is normal in G and the set of lower neighbours of any point t is cofinal in $\overset{\circ}{[t]}$. For later use recall down the following standard results about T -graphs, and refer the reader to [2, §2] for details.

Lemma 2.2. *Let (T, \leq) be an order tree and G a T -graph.*

- (1) For incomparable vertices t, t' in T , the set $[t] \cap [t']$ separates t from t' in G .
- (2) Every connected subgraph of G has a unique T -minimal element.
- (3) Every subgraph of G induced by an up-set $[t]$ is connected.
- (4) If $T' \subseteq T$ is down-closed, the components of $G - T'$ are spanned by the sets $[t]$ for t minimal in $T - T'$.

2.3. Stationary sets and Fodor's lemma. We denote ordinals by i, j, k, ℓ , and identify $i = \{j : j < i\}$. Let ℓ be any limit ordinal. A subset $A \subseteq \ell$ is *unbounded* if $\sup A = \ell$, and *closed* if $\sup(A \cap m) = m$ implies $m \in A$ for all limits $m < \ell$. The set A is a *club-set* in ℓ if it is both closed and unbounded. A subset $S \subseteq \ell$ is *stationary* (in ℓ) if S meets every club-set of ℓ . For the following standard results about stationary sets see e.g. [8, §III.6].

Lemma 2.3. (1) *If κ is a regular uncountable cardinal, $S \subseteq \kappa$ is stationary and $S = \bigcup \{S_n : n \in \mathbb{N}\}$, then some S_n is stationary.*

(2) [Fodor's lemma] *If κ is a regular uncountable cardinal, $S \subseteq \kappa$ stationary and $f : S \rightarrow \kappa$ is such that $f(s) < s$ for all $s \in S$, then there is $i < \kappa$ such that $f^{-1}(i)$ is stationary.*

§3. A NEW OBSTRUCTION FOR NORMAL SPANNING TREES OF SIZE \aleph_1

In this section we encounter a third obstruction for normal spanning trees – a graph without normal spanning tree, but also without (\aleph_0, \aleph_1) -graph and Aronszajn-tree graph as a minor.

Consider the ω_1 -regular tree with tops, i.e. the order tree (T, \leq) where the nodes of T are all sequences of elements of ω_1 of length $\leq \omega$, including the empty sequence. Let $t \leq t'$ if t is a proper initial segment of t' , and let G_{ω_1} be any T -graph.

Given a set $S \subseteq \omega_1$ of limit ordinals, choose for each $s \in S$ a cofinal (not necessarily increasing) sequence $f_s: \mathbb{N} \rightarrow s$, and let $F = F(S) = \{f_s: s \in S\}$ be the corresponding collection of sequences in ω_1 . Let $T(S)$ denote the subtree of T given by all finite sequences in T together with $F(S)$, and let $G_{\omega_1}(S)$ denote the corresponding induced subgraph of G_{ω_1} .

To our knowledge, such a collection $F(S) = \{f_s: s \in S\}$ of tree branches was first considered by Stone in [11, §5] where it is shown that F is not Borel in the end space of $G_{\omega_1}[T^{<\omega}]$.

Theorem 3.1. *Let $S \subseteq \omega_1$ be stationary. Then $G_{\omega_1}(S)$ does not have a normal spanning tree, despite the fact that it contains neither an (\aleph_0, \aleph_1) -graph nor an Aronszajn-tree graph as a minor.*

The three assertions of Theorem 3.1 are divided into Lemmas 3.2, 3.3 and 3.4 below.

Lemma 3.2. *Let $S \subseteq \omega_1$ be stationary. Then $G_{\omega_1}(S)$ does not have a normal spanning tree.*

Proof. Suppose for a contradiction that $G_{\omega_1}(S)$ has a normal spanning tree R . By Lemma 2.3(1), for some level $n \in \mathbb{N}$ the set $S' = \{s \in S: f_s \in R^n\}$ is stationary. By Lemma 2.2(1), any two vertices $f_s \neq f_{s'}$ in $F(S')$ can be separated by $[f_s]_R \cap [f_{s'}]_R$, a set of at most n vertices.

However, by Lemma 2.3(1), there is a stationary subset $S'' \subseteq S'$ and $m \in \mathbb{N}$ such that the first $n + 1$ neighbours of f_s for $s \in S''$ are contained on exactly the same levels of $T^{\leq m}$. After applying Fodor's Lemma 2.3(2) iteratively m times, we get a stationary subset $S''' \subseteq S''$ such that $f_s(i) = f_{s'}(i)$ for all $s, s' \in S'''$ and $i \leq m$. So the vertices in $F(S''')$ have at least $n + 1$ common neighbours, a contradiction. \square

Lemma 3.3. *If an (\aleph_0, \aleph_1) -graph H is a minor of some T -graph G with $\text{height}(T) = \omega + 1$, then some countable subtree of $T^{<\omega}$ has uncountably many tops.*

Proof. Let $H = (A, B)$ be an (\aleph_0, \aleph_1) -graph with $H \preceq G$. For $v \in V(H)$ write t_v for the unique minimal node of the branch set of v in \mathcal{T} , which exists by Lemma 2.2(2). Observe that if $ab \in E(H)$, then t_a and t_b are comparable in T . Since B is uncountable, there is a first level $\alpha \leq \omega$ such that $B' = \{b \in B: t_b \in T^\alpha\}$ is uncountable.

Case 1: If $\alpha = n < \omega$ is finite. Since every $b \in B'$ has infinite degree, there must be infinitely many t_a comparable to such an $t_b \in T^n$. As $[t_b]$ is finite, it follows that every such t_b for $b \in B'$ has some t_a above it. But then all these t_a are distinct, contradicting that A is countable.

Case 2: If $\alpha = \omega$. Then the branch sets of $b \in B'$ are $\{t_b\} \subseteq T^\omega$. And $\{t_a : a \in N(b)\}$ forms an infinite, and hence cofinal chain below t_b for any $b \in B'$. Since A is countable, $[A \cap T^{<\omega}]$ is a countable subtree of $T^{<\omega}$ that picks up uncountable many tops t_b for $b \in B'$, a contradiction. \square

To see that Lemma 3.3 applies to $G_{\omega_1}(S)$, note that any countably subtree of T contains only sequences with values in i for some $i < \omega_1$. But then no f_s for $s > i$ is a top of that subtree: since the sequence f_s is cofinal in s , we have $f_s(n) \geq i$ for at least one $n \in \mathbb{N}$.

Lemma 3.4. *If an Aronszajn tree graph H is a minor of some T -graph G , then $\text{height}(T) \geq \omega_1$.*

Proof. Let \mathcal{T} be an Aronszajn tree and suppose that a \mathcal{T} -graph H embeds into some T -graph G as a minor. Using the notation of the previous proof, if T has countable height, there is a level T^α of T such that $Y = \{v \in V(H) : t_v \in T^\alpha\}$ is uncountable. If we choose α minimal, then deleting the countable set $X = \{v \in V(H) : t_v \in T^{<\alpha}\}$ in H separates all vertices of Y from each other.

However, we have $X \subseteq \mathcal{T}^{<\beta}$ for some $\beta < \omega_1$. By the Aronszajn property, both $\mathcal{T}^{<\beta}$ and \mathcal{T}^β are countable. Since H is a \mathcal{T} -graph, all but countably many vertices of $H - X$ are contained in a connected subgraph (Lemma 2.2(3)) of the form $[t]$ for some $t \in \mathcal{T}^\beta$. This contradicts that Y is uncountable. \square

§4. DISCUSSION OF DIESTEL & LEADER'S PROOF

In this section we discuss the gap in Diestel and Leader's proof, and see how clubs and stationary sets of ω_1 appear naturally when taking Diestel and Leader's proof strategy to its logical conclusion.

Very briefly, given the task of constructing a normal spanning tree for some connected graph G , Diestel and Leader aim to partition G into countable subgraphs H_1, H_2, H_3, \dots such that each component of H_n has only finitely many neighbours in $\overline{H}_n = \bigcup_{i < n} H_i$, and these finitely many neighbours span a clique. Given this setup, using Jung's Theorem 2.1(2), it is then not hard to extend a normal spanning tree of \overline{H}_n to a normal spanning tree of $\overline{H}_n \cup H_n$. However, since G is uncountable and the subgraphs are just countable, one needs a transfinite sequence $H_1, H_2, H_3, \dots, H_\omega, H_{\omega+1}, \dots$ of such subgraphs. This is where the gap in Diestel and Leader's proof occurs, when they advise in [5, §5] to “repeat[...] this step transfinitely until T_F is exhausted”, for this strategy may fail at limit steps. Indeed, even if one carefully constructs, as Diestel and Leader do, the first ω many graphs H_1, H_2, H_3, \dots such that each graph has only finitely many neighbours in the union of the earlier ones, there might be trouble finding a suitable H_ω that has

only finitely many neighbours in $\overline{H}_\omega = \bigcup_{n < \omega} H_n$; indeed, the current attempt is doomed at this point if there exists just one vertex $v \in G - \overline{H}_\omega$ with infinitely many neighbours in \overline{H}_ω : to which H_i for $i \geq \omega$ shall it belong?

Hence, implementing this strategy successfully requires a certain amount of “looking ahead” in order to avoid problems at limit steps. As case in point, consider again the ω_1 -regular tree with all tops (T, \leq) , and let G_{ω_1} be any T -graph. Select an arbitrary collection $F \subseteq T^\omega$ of tops, let $T(F)$ denote the subtree of T induced by all finite sequences in T together with F , and $G_{\omega_1}(F)$ be the corresponding subgraph of G_{ω_1} .

For $i < \omega_1$ write G_i for the subgraph of $G_{\omega_1}(F)$ induced by all sequences in $T(F)$ with values strictly less than i . Implementing the strategy following Diestel and Leader, one could select for example $H_{n+1} = G_{n+1} \setminus G_n$, as is readily verified using Lemma 2.2(4). However, any top $f \in F$ with $f(n) < \omega$ for all $n \in \mathbb{N}$ but $\sup \{f(n) : n \in \mathbb{N}\} = \omega$ is then precisely such a vertex in $G - \overline{H}_\omega$ with infinitely many neighbours in \overline{H}_ω that we are trying to avoid.

Formalizing this observation, for $f \in F$ define $f^* := \sup \{f(n) : n \in \mathbb{N}\} < \omega_1$, $F^* = \{f^* : f \in F\}$, and $F_{>}^* = \{f^* \in F^* : f^* > f(n) \text{ for all } n \in \mathbb{N}\} \subseteq \omega_1$, the subset of F^* where the supremum is proper. Using this notation, selecting $H_{n+1} = G_{c_{n+1}} \setminus G_{c_n}$ for some increasing sequence $c_1 < c_2 < \dots$ avoids this problem at the first limit step precisely if $c_\omega := \sup \{c_n : n \in \mathbb{N}\}$ does not belong to $F_{>}^*$; and it avoids the problem altogether if $C = \{c_i : i < \omega_1\}$ is a club set of ω_1 with $C \cap F_{>}^* = \emptyset$. Such a club set C exists if and only if $F_{>}^*$ fails to be stationary. In other words, the strategy suggested by Diestel and Leader can be carried out precisely when there is a suitable club set C along which we can decompose the graph:

Theorem 4.1. *A graph of the form $G_{\omega_1}(F)$ has a normal spanning tree if and only if it contains no (\aleph_0, \aleph_1) -subgraph and some club set $C \subseteq \omega_1$ avoids $F_{>}^*$.*

Proof. We first prove the forwards implication. If $G_{\omega_1}(F)$ has a normal spanning tree, then it clearly cannot contain an (\aleph_0, \aleph_1) -subgraph. Now assume for a contradiction that $F_{>}^*$ meets every club set of ω_1 . Then $S = F_{>}^*$ is stationary. For every $s \in S$ choose some $f_s \in F$ with $f_s^* = s$. Then $F(S) = \{f_s : s \in S\}$ gives rise to a subgraph of the form $G_{\omega_1}(S) \subseteq G_{\omega_1}(F)$ that fails to have a normal spanning tree by Lemma 3.2, a contradiction.

Conversely, assume that there is a club-set $C \subseteq \omega_1$ avoiding $F_{>}^*$. We show that $G_{\omega_1}(F)$ has a normal spanning tree unless it contains an (\aleph_0, \aleph_1) -subgraph. Without loss of generality, $G_{\omega_1}(F)$ is the $T(F)$ graph where all comparable nodes are connected by an edge (if this graph has a normal spanning tree, then also all its connected subgraphs have normal spanning trees). This ensures that $\lceil t \rceil$ spans a clique for all $t \in T(F)$.

Write G_i for the subgraph of $G_{\omega_1}(F)$ induced by all sequences with values strictly less than i . If some G_i is uncountable, there must be uncountably many tops from F above the countable subtree $G_i \cap T^{<\omega}$, giving rise to an (\aleph_0, \aleph_1) -subgraph. Hence, all G_i are countable.

Let $C = \{c_i : i < \omega_1\}$ be an increasing enumeration of the club set C . Then G is the increasing union over $\bigcup \{G_{c_i} : i < \omega_1\}$. Moreover, this union is continuous precisely because $F_{>}^* \cap C = \emptyset$: an element $f \in G_{c_\ell} \setminus \bigcup_{i < \ell} G_{c_i}$ for a limit $\ell < \omega_1$ would satisfy $f^* = c_\ell$ and $f(n) < c_\ell$ for all $n \in \mathbb{N}$, and hence $f^* \in F_{>}^* \cap C$.

This allows us to construct – by a transfinite recursion on $i < \omega_1$ – an increasing chain of normal spanning trees R_i of G_{c_i} all with the same root extending each other. Assume that the normal spanning tree R_i of G_{c_i} is already defined. By Lemma 2.2(4), the components of $G_{c_{i+1}} \setminus G_{c_i}$ are spanned by the upsets $[t]$ for t the T -minimal elements of $G_{c_{i+1}} \setminus G_{c_i}$. By definition of G_{c_i} , the down-closure $\overline{[t]}$ for any such t forms a finite clique in G . Hence, $\overline{[t]}$ forms a chain in the normal spanning tree R_i . Let r_t denote the maximal element of $\overline{[t]}$ in R_i . Since $G_{c_{i+1}}$ is countable, there is by Theorem 2.1(2) for every component $[t]$ of $G_{c_{i+1}} \setminus G_{c_i}$ a normal spanning tree R_t with root t . Attaching the R_t 's to R_i with t a successor of r_t gives a normal spanning tree R_{i+1} of $G_{c_{i+1}}$.

By the continuity of our sequence, for any limit $\ell < \omega_1$, the union $R_\ell = \bigcup_{i < \ell} R_i$ is a normal spanning tree for G_{c_ℓ} . Then $R = \bigcup_{i < \omega_1} R_i$ is the desired normal spanning tree for $G_{\omega_1}(F)$. \square

Corollary 4.2. *A graph of the form $G_{\omega_1}(F)$ has a normal spanning tree if and only if it does not contain an (\aleph_0, \aleph_1) -subgraph or a subgraph isomorphic to $G_{\omega_1}(S)$ for $S \subseteq \omega_1$ stationary.*

§5. IRREDUCIBLE OBSTRUCTIONS OF SIZE κ

One particular consequence of Diestel and Leader's proposed forbidden minor characterisation would have been that 'not having a normal spanning tree' is a property reflecting to at least one minor of size \aleph_1 . However, it turns out that for all uncountable regular cardinals κ there are κ -sized graphs without a normal spanning tree that consistently have the property that all their minors of size $< \kappa$ do have normal spanning trees.

These examples are natural generalisations of our earlier examples from Theorem 3.1 to arbitrary regular uncountable cardinals κ . They are defined as follows: Consider the κ -regular tree with all tops, represented by all sequences of elements of κ of length $\leq \omega$, and let G_κ be any T -graph. Given a set $S \subseteq \kappa$ of limit ordinals of countable cofinality, choose for each $s \in S$ a cofinal sequence $f_s : \mathbb{N} \rightarrow s$, and let $F = F(S) = \{f_s : s \in S\}$ be the corresponding collection of sequences in κ . Let $T(S)$ denote the subtree of T given by all finite sequences in T together with $F(S)$, and let $G_\kappa(S)$ denote the corresponding induced subgraph of G_κ .

Theorem 5.1. *Let κ be a regular uncountable cardinal. Whenever $S \subseteq \kappa$ is stationary consisting just of cofinality ω ordinals, then $G_\kappa(S)$ does not have a normal spanning tree.*

Furthermore, it is consistent with the axioms of set theory ZFC that for any regular uncountable cardinal κ there exists such a stationary set $S \subseteq \kappa$ such that all minors of $G_\kappa(S)$ of cardinality strictly less than κ have normal spanning trees.

Proof. Indeed, that $G_\kappa(S)$ does not have a normal spanning tree follows as in Lemma 3.2.

To see the furthermore part of the theorem, recall that assuming Jensen's square principle \square_κ , which holds for example in the constructible universe, there exists a *non-reflecting* stationary set in κ , i.e. a stationary set $S \subseteq \kappa$ consisting just of cofinality ω ordinals such that for any limit ordinal $\ell < \kappa$, the restriction $S \cap \ell$ is not stationary in ℓ , see e.g. [3, §4].

It remains to show that for any non-reflecting stationary set $S \subseteq \kappa$ of cofinality ω ordinals, the graph $G_\kappa(S)$ is a graph without normal spanning tree such that all minors of size $< \kappa$ do have normal spanning trees. As in the proof of Theorem 4.1, we may assume that $G_\kappa(S)$ is the T -graph where all comparable vertices of $T(S)$ are adjacent.

For $i < \kappa$, denote by G_i once again the subgraph of $G_\kappa(S)$ induced by all sequences in $T(S)$ with values strictly less than i . Every minor of size $< \kappa$ of $G_\kappa(S)$ is a contraction of some $< \kappa$ -sized subgraph of $G_\kappa(S)$, and hence by regularity of κ , a minor of G_i for some $i < \kappa$. Hence, it suffices to show that every G_i for $i < \kappa$ has a normal spanning tree. This will be done by induction on i , following the proof idea in Theorem 4.1.

The base case for the induction is trivial. In the successor step, fix a normal spanning tree R for G_i . By definition of $F(S)$, no component D of $G_{i+1} \setminus G_i$ can contain an element from F . Hence, by Lemma 2.2(4), the tree order of (T, \leq) restricted to such a component prescribes a normal spanning tree R_D for D . As $N(D)$ is finite and complete in G_i , its elements form a chain in R . Hence, attaching R_D to R as a further uptree behind the highest element of $N(D)$ in R for every such component D gives rise to a normal spanning tree for G_{i+1} .

At a limit step $\ell < \kappa$, by choice of S there is a club set $C \subseteq \ell$ which misses S . Let $C = \{c_i : i < \text{cf}(\ell)\}$ be an increasing enumeration of C . From $S \cap C = \emptyset$ it follows that $\{G_{c_i} : i < \text{cf}(\ell)\}$ is an increasing, continuous chain in G_ℓ such that every component of $G_{c_{i+1}} \setminus G_{c_i}$ has finite neighbourhood in G_{c_i} . As in Theorem 4.1, this allows us to construct an increasing chain of normal spanning trees R_i of G_{c_i} extending each other all with the root of T as their root. Then $R = \bigcup_{i < \text{cf}(\ell)} R_i$ is a normal tree in G_ℓ .

If $\text{cf}(\ell) > \omega$, then $G_\ell = \bigcup \{G_{c_i} : i < \text{cf}(\ell)\}$ and we are done. Otherwise, if $\text{cf}(\ell) = \omega$, we could have $f_\ell \in G_\ell \setminus \bigcup \{G_{c_i} : i < \text{cf}(\ell)\}$ if $\ell \in S$. In this case, we attach f_ℓ below the root of R to form a normal spanning tree of G_ℓ rooted in f_ℓ . \square

It might come as a surprise to hear that no such examples as in the furthermore-part of Theorem 5.1 are possible at singular uncountable cardinals κ . Indeed, the property of having a normal spanning tree exhibits the following singular compactness-type behaviour: If a connected graph G of singular uncountable size κ has the property that all its subgraphs of size $< \kappa$ have normal spanning trees, then so does G itself. For details, we refer the reader to [9].

§6. FURTHER PROBLEMS ON NORMAL SPANNING TREES AND FORBIDDEN MINORS

Problem 1. Is there an Aronszajn tree graph that contains neither an (\aleph_0, \aleph_1) -graph nor a graph $G_{\omega_1}(S)$ as in Theorem 3.1 as a minor?

Problem 2. Does every \aleph_1 -sized graph without normal spanning tree contain either an (\aleph_0, \aleph_1) -graph, an Aronszajn tree graph or a graph $G_{\omega_1}(S)$ as in Theorem 3.1 as a minor?

Problem 3. Is it consistent with the axioms of set theory ZFC that every graph without normal spanning tree contains an \aleph_1 -sized subgraph or minor without normal spanning tree?

REFERENCES

- [1] N. Bowler, G. Geschke, and M. Pitz. Minimal obstructions for normal spanning trees. *Fund. Math.*, 241:245–263, 2018.
- [2] J.-M. Brochet and R. Diestel. Normal tree orders for infinite graphs. *Transactions of the American Mathematical Society*, 345(2):871–895, 1994.
- [3] J. Cummings. Notes on singular cardinal combinatorics. *Notre Dame Journal of Formal Logic*, 46(3):251–282, 2005.
- [4] R. Diestel. *Graph Theory*. Springer, 5th edition, 2015.
- [5] R. Diestel and I. Leader. Normal spanning trees, Aronszajn trees and excluded minors. *Journal of the London Mathematical Society*, 63:16–32, 2001.
- [6] R. Halin. Miscellaneous problems on infinite graphs. *Journal of Graph Theory*, 35(2):128–151, 2000.
- [7] H.A. Jung. Wurzelbäume und unendliche Wege in Graphen. *Mathematische Nachrichten*, 41(1-3):1–22, 1969.
- [8] K. Kunen. *Set theory*. Volume 34 of Studies in Logic, College Publications, London, 2011.
- [9] M. Pitz. Proof of Halin’s normal spanning tree conjecture. In preparation, 2020.
- [10] M. Pitz. A unified existence theorem for normal spanning trees. <https://arxiv.org/abs/2003.11575>, 2020.
- [11] A.H. Stone. On σ -discreteness and Borel isomorphism. *American Journal of Mathematics*, 85(4):655–666, 1963.

HAMBURG UNIVERSITY, DEPARTMENT OF MATHEMATICS, BUNDESSTRASSE 55 (GEOMATIKUM), 20146 HAMBURG, GERMANY

E-mail address: max.pitz@uni-hamburg.de