ANDRÁSFAI AND VEGA GRAPHS IN RAMSEY-TURÁN THEORY

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ABSTRACT. Given nonnegative integers $n \geqslant s$, we let $\mathrm{ex}(n,s)$ denote the maximum number of edges in a triangle-free graph G on n vertices with $\alpha(G) \leqslant s$. In the early sixties Andrásfai conjectured that for $n/3 < s < n/2$ the function $\mathrm{ex}(n,s)$ is piecewise quadratic with critical values at $s/n = k/(3k-1)$ for $k \in \mathbb{N}$. We confirm that this is indeed the case whenever s/n is slightly larger than a critical value, thus determining $\mathrm{ex}(n,s)$ for all n and s such that $s/n \in [k/(3k-1), k/(3k-1) + \gamma_k]$, where $\gamma_k = \Theta(k^{-6})$.

§1. INTRODUCTION

The structure of dense triangle-free graphs has been the subject of extensive studies for a long time. The first result in this direction is Mantel's celebrated theorem [21] from 1907, which states that balanced bipartite graphs are the densest triangle-free graphs. It is natural to ask for the densest triangle-free graphs when we impose some additional restrictions on them; for instance, we may bound their chromatic or independence number.

Let us first discuss the case when we require the chromatic number of a dense triangle-free graph to be large. It is easy to see that in this case the appropriate measure of the density of a graph is not the number of its edges (as one can always add a small graph of large chromatic number to a complete bipartite graph), but rather its minimum degree. This avenue of research was started by Andrásfai, Erdős, and Sós [3] who showed that among triangle-free graphs with chromatic number three those with the largest minimum degree are 'balanced blow-ups' of the pentagon. Erdős and Simonovits [11] noticed that a construction due to Hajnal shows that for every $k \geqslant 2$, $\varepsilon > 0$, and sufficiently large n, there exists a triangle-free graph on n vertices whose minimum degree is larger than $(1/3-\varepsilon)n$, and whose chromatic number is k. On the other hand, they conjectured that every triangle-free graph on n vertices whose minimum degree is larger than $n/3$ is 3-colourable. This was refuted by Häggkvist [14], who found a 10-regular triangle-free graph on 29 vertices whose chromatic number is four. Jin [15] showed that this example is insofar optimal that every triangle-free graph whose minimum degree is strictly larger than $10n/29$ has chromatic number three. Moreover, Thomassen [24] proved that for every $\varepsilon > 0$ there exists a constant c_ε such that

every triangle-free n-vertex graph with minimum degree at least $(1/3 + \varepsilon)n$ has chromatic number at most c_ε, and Łuczak [16] supplemented this result by proving, roughly speaking, that for some constant C_ε there are at most C_ε 'types' of such graphs. Finally, Brandt and Thomassé [8] characterised all triangle-free graphs on n vertices whose minimum degree is larger than $n/3$; their theorem, stated in Section 5 below, plays a decisive rôle in the proof of our main result.

In this article, however, we mainly study triangle-free graphs G with bounded independence number $\alpha(G)$. More specifically, we are interested in the behaviour of the function $\mathrm{ex}(n, s)$, which for $n \geqslant s \geqslant 0$ gives the largest number of edges in a triangle-free graph on n vertices whose independent sets have at most the size s, i.e.,

$$\mathrm{ex}(n, s) = \max\{e(G) \colon v(G) = n, \ G \not\supseteq K_3, \ \text{and} \ \alpha(G) \leqslant s\}\,.$$

Notice that Mantel's theorem yields

$$\mathrm{ex}(n, s) = \lfloor n^2/4 \rfloor \quad \text{for every } s > \lfloor n/2 \rfloor\,.$$

Next we observe that in a triangle-free graph the neighbourhood of every vertex forms an independent set, which implies the so-called *trivial bound*

$$\mathrm{ex}(n, s) \leqslant ns/2$$

for all n and s. Brandt [6] provided several explicit constructions showing that this upper bound is asymptotically optimal for $s \leqslant n/3$, i.e., that we have

$$\mathrm{ex}(n, s) = ns/2 + o(n^2)$$

in this range. Thus, it remains to study the behaviour of $\mathrm{ex}(n, s)$ for $s/n \in (1/3, 1/2)$.

This line of research was started over 50 years ago by Andrásfai [2], who proved

$$\mathrm{ex}(n, s) = n^2 - 4ns + 5s^2 \quad \text{for } s/n \in [2/5, 1/2]\,.$$

He also speculated that $\mathrm{ex}(n, s)$ might be a piecewise quadratic function with cusps at points of the form $s = kn/(3k - 1)$. We slightly revised his conjecture in [17] and resolved the next case by showing

$$\mathrm{ex}(n, s) = 3n^2 - 15ns + 20s^2 \quad \text{for } s/n \in [3/8, 2/5]\,.$$

The new version of the conjecture reads as follows.

Conjecture 1.1. *If $n/3 < s \leqslant n/2$, then*

$$\mathrm{ex}(n, s) = \min_k g_k(n, s)\,, \tag{1.1}$$

where

$$g_k(n, s) = k(k - 1)n^2/2 - k(3k - 4)ns + (3k - 4)(3k - 1)s^2/2 \tag{1.2}$$

for every $k \geqslant 1$.

Let us remark that we also have a conjecture on the extremal graphs for which equality holds in (1.1). Since the definition of these graphs requires some preparation, we state this stronger conjecture only at the end the article.

As for the function $g(n,s) = \min_k g_k(n,s)$, which stands on the right side of (1.1), an elementary calculation (see [17, Cor. 2.7]) shows that for $k \geqslant 2$ and $\frac{k}{3k-1}n \leqslant s < \frac{k-1}{3k-4}n$ we have $g(n,s) = g_k(n,s)$. Thus, for fixed n the function $g(n,s)$ is piecewise quadratic in $s \in (n/3, n/2)$ with cusps at the points $s = kn/(3k-1)$ for $k \geqslant 2$.

The main goal of this work is to add further plausibility to Conjecture 1.1 by proving it whenever s/n is slightly larger than one of the 'critical points' $k/(3k-1)$.

Theorem 1.2. *For every $k \geqslant 2$ there exists $\gamma = \gamma(k) > 0$ such that*

$$\mathrm{ex}(n,s) = g_k(n,s) = \min_\ell g_\ell(n,s)$$

whenever

$$\frac{k}{3k-1}n \leqslant s \leqslant \left(\frac{k}{3k-1} + \gamma\right)n\,.$$

For instance, this holds for $\gamma(k) = (600k^6)^{-1}$.

Along the way, we establish the following minimum degree version of Conjecture 1.1.

Theorem 1.3. *Let $k \geqslant 2$ and $n \geqslant s \geqslant 0$. If H denotes a triangle-free graph on n vertices with $\alpha(H) \leqslant s$ and*

$$\delta(H) > \frac{k+1}{3k+2}n\,,$$

then $e(H) \leqslant g_k(n,s)$.

Let us mention that similar problems could be and, in many cases, have been, considered for K_r-free graphs and, more generally, for H-free graphs for any given graph H. It hardly seems necessary to recall that Turán's problem to determine the maximum number of edges in an H-free graph on n vertices is fairly well understood thanks to the work of Turán himself [25], Erdős, Stone, and Simonovits [10, 13]. The studies of the *chromatic threshold* (equal to 1/3 for triangle-free graphs by the aforementioned result of Thomassen [24]) were begun by Łuczak and Thomassé [19] and culminated in the work of Allen *et al.* [1] who determined this parameter for H-free graphs when an arbitrary graph H is given (for the precise definition of the 'chromatic threshold' we refer to either of those two articles).

The question on the behaviour of $\mathrm{ex}(n,s)$ considered in this work belongs to an area called *Ramsey-Turán theory*, which has been initiated by Vera T. Sós and extensively investigated during the last fifty years. There is a comprehensive survey on this subject by Sós and Simonovits [22]. Important milestones in the Ramsey-Turán theory of general K_r-free graphs were obtained by Bollobás, Erdős, Hajnal, Sós, Szemerédi [5, 9, 12, 23], and, more

recently, by Lüders and Reiher [20]. Due to their work, we asymptotically know the value of

$$\mathrm{ex}_r(n, s) = \max\{e(G) \colon v(G) = n, \ G \not\supseteq K_r, \ \text{and} \ \alpha(G) \leqslant s\}$$

for all $r \geqslant 3$ provided that $s/n \ll r^{-1}$ is sufficiently small. It would, of course, be interesting to study this function for larger values of s/n as well, but, as the present article demonstrates, even the case $r = 3$ of triangles seems to be fairly difficult.

The structure of the article is the following. In Section 2 we start with the definition and some basic properties of the blow-up operation. The two subsequent sections define and study Andrásfai and Vega graphs, which are the main protagonists in the story of dense triangle-free graphs (see Theorem 5.1 below). In particular, in this part of the article we prove some special cases of Theorem 1.2 addressing blow-ups of these two types of graphs (see Lemma 3.3 and Lemma 4.1). In Section 5 these results will be employed in the proofs of the Theorems 1.2 and 1.3. Moreover, we shall state there a precise version of our conjecture on extremal cases in Conjecture 1.1. These are the same as the extremal graphs for the two aforementioned lemmata, which we characterise in Lemma 3.5 and Lemma 4.4, respectively.

§2. Blow-ups of graphs

Given a graph F with vertex set $V(F) = \{v_1, \ldots, v_r\}$, a *blow-up of F* is a graph H obtained from F upon replacing its vertices by independent sets $V_1, V_2, \ldots, V_r$, and each of its edges $v_i v_j \in E(F)$, $1 \leqslant i < j \leqslant r$, by the complete bipartite graph $K(V_i, V_j)$ between V_i and V_j. The sets $V_1, \ldots, V_r$ are called the *vertex classes* of H. As above, we shall always denote the vertices of the original graph F by lower case letters and the vertex classes of H by capitalised versions of the same letters. A blow-up is *proper* if all vertex classes are non-empty and *balanced* if all of them are of the same size. As the isomorphism type of H depends only on the sizes of its vertex classes it will be convenient to write $H = F(h)$, where the function $h \colon V(F) \longrightarrow \mathbb{Z}_{\geqslant 0}$ is defined by $h(v_i) = |V_i|$ for every $v_i \in V(F)$. In the special case where h is the constant function attaining always the value t it will be convenient to write $H = F(t)$. For later use we remark that a blow-up of a blow-up of F is again a blow-up of F.

Resuming the discussion of the blow-up H of F with vertex classes $V_1, \ldots, V_r$ we set

$$N(V_i) = \bigcup_{v_j \in N(v_i)} V_j$$

for each of these vertex classes, where $N(v_i)$ denotes the neighbourhood of v_i in F. Clearly, all vertices in V_i have the neighbourhood $N(V_i)$ in H and, consequently, every non-empty vertex class V_i satisfies

$$\delta(H) \leqslant |N(V_i)| \leqslant \Delta(H). \tag{2.1}$$

Now simple averaging leads to the following observation.

Fact 2.1. *Let H be an n-vertex blow-up of a k-regular graph F on the r-element vertex set $V(F) = \{v_1, \ldots, v_r\}$.*

(*a*) *We have*

$$\sum_{i=1}^{r} |N(V_i)| = kn \,. \tag{2.2}$$

(*b*) *If the blow-up H is proper, then*

$$\delta(H) \leqslant \frac{kn}{r} \leqslant \Delta(H) \,.$$

Proof. The double counting argument

$$\sum_{i=1}^{r} |N(V_i)| = \sum_{i=1}^{r} \sum_{v_j \in N(v_i)} |V_j| = \sum_{j=1}^{r} |N(v_j)| \, |V_j| = k \sum_{j=1}^{r} |V_j| = kn$$

establishes part (*a*). If H is a proper blow-up, then (2.1) yields

$$r\delta(H) \leqslant \sum_{i=1}^{r} |N(V_i)| \leqslant r\Delta(H)$$

and part (*b*) follows. $\qquad\square$

The second part of the foregoing fact has the following useful consequence.

Lemma 2.2. *Let $d, k \geqslant 2$ be two integers and let J be a graph. Suppose that J has a k-regular proper blow-up F on $3k - 1$ vertices and that H denotes a further proper blow-up of J having n vertices. If*

$$\frac{d+1}{3d+2}n < \delta(H) \quad and \quad \Delta(H) < \frac{d-1}{3d-4}n \,, \tag{2.3}$$

then $k = d$.

Proof. Notice that every balanced blow-up $H(t)$ of H satisfies the assumption on H as well. By applying this observation to a sufficiently large integer t we learn that we may assume, without loss of generality, that H is a proper blow-up of F. Now Fact 2.1(*b*) reveals

$$\delta(H) \leqslant \frac{kn}{3k-1} \leqslant \Delta(H) \,,$$

which together with (2.3) implies

$$\frac{d+1}{3d+2} < \frac{k}{3k-1} < \frac{d-1}{3d-4} \,.$$

Taking reciprocals we obtain

$$3 - \frac{1}{d-1} < 3 - \frac{1}{k} < 3 - \frac{1}{d+1} \,,$$

i.e., $d - 1 < k < d + 1$. $\qquad\square$

The main result of this section is the following upper bound on the number of edges of a blow-up.

Lemma 2.3. *Suppose that F is a k-regular graph on r vertices and that H is an n-vertex blow-up of F all of whose vertex classes have at least the size x. If we have $|N(Z)| \leqslant s$ for every vertex class Z of H, then*

$$e(H) \leqslant \frac{ns}{2} - \frac{x(rs - kn)}{2} \,. \tag{2.4}$$

Moreover, if $rs \neq kn$ and (2.4) holds with equality, then

- *at least one vertex class of H has size x;*
- *every vertex class V_i of H with $|V_i| > x$ satisfies $|N(V_i)| = s$.*

Proof. As usual, we write the vertex set of F in the form $V(F) = \{v_1, \ldots, v_r\}$. Setting $x_i = |V_i|$ for $i \in [r]$ we have

$$
\begin{aligned}
2e(H) &= \sum_{v \in V(H)} \deg_H(v) = \sum_{i=1}^{r} |V_i| \cdot |N(V_i)| \\
&= \sum_{i=1}^{r} x \cdot |N(V_i)| + \sum_{i=1}^{r} (x_i - x) \cdot |N(V_i)| \\
&\leqslant x \sum_{i=1}^{r} |N(V_i)| + \sum_{i=1}^{r} (x_i - x)s \stackrel{(2.2)}{=} xkn + s(n - xr) = ns - x(rs - kn) \,,
\end{aligned}
$$

which proves the desired upper bound on $e(H)$.

Suppose from now on that this estimate holds with equality and that $rs \neq kn$. This means that $(x_i - x)|N(V_i)| = (x_i - x)s$ holds for every $i \in [r]$, or in other words that $x_i > x$ implies $|N(V_i)| = s$. This proves the second bullet. Now, if the first bullet fails, we have $|N(V_i)| = s$ for every $i \in [r]$ and (2.2) yields the contradiction $rs = kn$. $\square$

Let us notice the following consequence of the above result.

Corollary 2.4. *Suppose that $k \geqslant 2$ is a natural number, F is a k-regular graph on $3k - 1$ vertices and H is an n-vertex blow-up of F. If $|N(Z)| \leqslant s$ and $|Z| \geqslant (k-1)n - (3k-4)s$ hold for every vertex class Z of H, then*

$$e(H) \leqslant g_k(n, s) \,.$$

Moreover, if $e(H) = g_k(n, s)$, then

- *(i) $kn/(3k - 1) \leqslant s \leqslant (k - 1)n/(3k - 4)$;*
- *(ii) H contains a vertex class of size $(k - 1)n - (3k - 4)s$;*
- *(iii) the neighbourhood of each vertex class of H containing more than $(k-1)n - (3k-4)s$ vertices has size s.*

Proof. In order to prove the desired upper bound on $e(H)$ we remark that Lemma 2.3 applied to $r = 3k - 1$ and $x = (k-1)n - (3k-4)s$ yields

$$e(H) \leqslant \tfrac{1}{2}\big[ns - \big((k-1)n - (3k-4)s\big)\big((3k-1)s - kn\big)\big] = g_k(n, s)\,.$$

Let us now study the case that equality holds in this estimate. If clause (i) failed, then the trivial upper bound $e(H) \leqslant ns/2$ would contradict $e(H) = g_k(n, s)$.

The remaining two clauses follow from the moreover-part in Lemma 2.3 provided that its assumption $(3k-1)s \neq kn$ holds. So it remains to deal with the case $s = kn/(3k-1)$. Now $(k-1)n - (3k-4)s = n/(3k-1)$ is at the same time the average size of the vertex classes of H and a lower bound on the sizes of these vertex classes. In other words, H is the balanced blow-up $F\big(n/(3k-1)\big)$ and (ii), (iii) hold trivially. $\square$

§3. Andrásfai graphs and their blow-ups

The characterisation of triangle-free graphs on n vertices whose minimum degree is larger than $n/3$ due to Brandt and Thomassé [8] involves two explicit families of such graphs, called *Andrásfai graphs* and *Vega graphs* (see Theorem 5.1 below). In this section we study the first of these graph sequences, which has been introduced by Andrásfai in [2] and has been rediscovered several times throughout the years.

One way to construct a k-regular triangle-free graph is to take an Abelian group $\mathbb{G}$, a symmetric sum-free subset S of size k, and to form the Cayley graph $\mathrm{Cayley}(\mathbb{G}; S)$. A natural (and, as we shall soon argue, generic) example occurs when we take the cyclic group $\mathbb{Z}_{3k-1}$ and its sum-free subset $S_k = \{k, k+1, \ldots, 2k-1\}$. The *Andrásfai graph* Γ_k is defined to be the corresponding Cayley graph $\mathrm{Cayley}(\mathbb{Z}_{3k-1}; S_k)$. Describing the same graph in more concrete terms, we set

$$V(\Gamma_k) = \{v_0, v_1, \ldots, v_{3k-2}\}$$

and declare the adjacencies in Γ_k by

$$v_i v_j \in E(\Gamma_k) \iff k \leqslant |i - j| \leqslant 2k - 1 \tag{3.1}$$

for all vertices $v_i, v_j \in V(\Gamma_k)$. For instance, $\Gamma_1 = K_2$, $\Gamma_2 = C_5$, and Figure 3.1 shows some further Andrásfai graphs.

Let us remark that given an Abelian group $\mathbb{G}$ containing a sum-free set S with $|S| > |\mathbb{G}|/3$ one can show by means of Kneser's theorem (see [18]) that for some positive integer k there exists a homomorphism $\varphi \colon \mathbb{G} \longrightarrow \mathbb{Z}_{3k-1}$ satisfying $\varphi[S] \subseteq S_k$. Such a group homomorphism φ induces a graph homomorphism $\varphi_\star \colon \mathrm{Cayley}(\mathbb{G}; S) \longrightarrow \Gamma_k$, or, in other words, it indicates that $\mathrm{Cayley}(\mathbb{G}; S)$ is contained in a sufficiently large blow-up of Γ_k. These considerations reveal that balanced blow-ups of Andrásfai graphs are universal in the class of triangle-free Cayley graphs whose density is larger than $1/3$. Somewhat relatedly, a
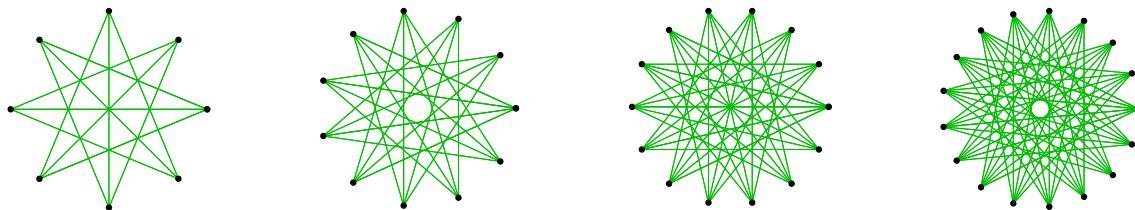
FIGURE 3.1. Andrásfai graphs Γ_3, Γ_4, Γ_5, and Γ_6.

finite graph is a subgraph of a blow-up of an Andrásfai graph if and only if it is isomorphic to a subgraph of the infinite triangle-free Cayley graph $\mathrm{Cayley}\big(\mathbb{R}/\mathbb{Z}; (1/3, 2/3) + \mathbb{Z}\big)$, see [4, Lemma 2.1]. We proceed with three well known, useful properties of Andrásfai graphs.

Fact 3.1. *Let $k \geqslant 2$ be an integer.*

(i) *We have $\Gamma_{k-1} \subseteq \Gamma_k$. Conversely, if we remove a vertex from Γ_k, then the resulting graph is a subgraph of a proper blow-up of Γ_{k-1}. Consequently, every subgraph of Γ_k is a subgraph of a proper blow-up of Γ_ℓ for some $\ell \leqslant k$.*

(ii) *The chromatic number of Γ_k is 3.*

(iii) *Every independent set in Γ_k is contained in the neighbourhood of some vertex of Γ_k. In particular, $\alpha(\Gamma_k) = k$.*

Proof. The first part of (i) follows from $\Gamma_{k-1} \cong \Gamma_k - \{v_0, v_k, v_{2k}\}$. Furthermore, the graph $\Gamma_k^- = \Gamma_k - \{v_k\}$ satisfies $N(v_0, \Gamma_k^-) \subseteq N(v_1, \Gamma_k^-)$ and $N(v_{2k}, \Gamma_k^-) \subseteq N(v_{2k-1}, \Gamma_k^-)$, which proves Γ_k^- to be a subgraph of the blow-up of $\Gamma_k - \{v_0, v_k, v_{2k}\}$ obtained by doubling the vertices v_1 and v_{2k-1}. This establishes the second part of (i) and the last part follows inductively.

Proceeding with (ii) we observe that, due to (i), Γ_k contains a pentagon Γ_2 as a subgraph, whence $\chi(\Gamma_k) \geqslant \chi(\Gamma_2) = 3$. To verify the reverse inequality, we partition $V(\Gamma_k)$ into the three independent sets $\{v_0, v_1, \ldots, v_{k-1}\}$, $\{v_k, v_{k+1}, \ldots, v_{2k-1}\}$, and $\{v_{2k}, v_{2k+1}, \ldots, v_{3k-2}\}$.

Finally, let $S \subseteq V(\Gamma_k)$ be a nonempty independent set we want to cover by the neighbourhood of an appropriate vertex. By symmetry we may suppose that $v_k \in S$, which due to (3.1) entails $S \subseteq \{v_1, v_2, \ldots, v_{2k-1}\}$. Let $i, j \in [2k-1]$ be the smallest and largest index with $v_i, v_j \in S$. Now $S \subseteq \{v_i, \ldots, v_j\}$, a further application of (3.1) shows $j - i \leqslant k - 1$, and altogether we have $S \subseteq N(v_{j+k})$. $\qquad\square$

Let us consider for $k \geqslant 2$ and $n \geqslant s \geqslant 0$ an n-vertex blow-up H of the Andrásfai graph Γ_k with $\alpha(H) \leqslant s$. According to the notation of Section 2, this blow-up comes together with a fixed partition

$$V(H) = V_0 \cup V_1 \cup \ldots \cup V_{3k-2}$$

of its vertex set. Clearly, since Γ_k is triangle-free, so is H. Therefore the neighbourhood of each vertex class of H is an independent set, which proves

$$|N(V_i)| = |V_{i+k}| + \ldots + |V_{i+2k-1}| \leqslant s \qquad (3.2)$$

for every $i \in \mathbb{Z}_{3k-1}$. We use this inequality to bound the sizes of the vertex classes of H.

Fact 3.2. *For every $i \in \mathbb{Z}_{3k-1}$ we have*

$$(k-1)n - (3k-4)s \leqslant |V_i| \leqslant 3s - n \, .$$

Proof. The upper bound holds because of

$$|V_i| = |V_0 \cup V_1 \cup \cdots \cup V_{3k-2}| + |V_i| - n$$

$$= |N(V_i)| + |N(V_{i+k})| + |N(V_{i-k})| - n \overset{(3.2)}{\leqslant} 3s - n \, .$$

By applying this estimate to $V_{i+k+1}, \ldots, V_{i+2k-2}$ instead of V_i we infer

$$|V_i| = n - \big(|N(V_{i+(k-1)})| + |N(V_{i-(k-1)})| + |V_{i+k+1}| + \cdots + |V_{i+2k-2}| \big)$$

$$\overset{(3.2)}{\geqslant} n - \big(2s + (k-2)(3s-n) \big) = (k-1)n - (3k-4)s \, . \qquad \square$$

By combining Corollary 2.4 and Fact 3.2 we arrive at the main result of this section.

Lemma 3.3. *Let k be a natural number. If for $n \geqslant s \geqslant 0$ the graph H is an n-vertex blow-up of the Andrásfai graph Γ_k satisfying $\alpha(H) \leqslant s$, then*

$$e(H) \leqslant g_k(n, s) \, . \qquad \square$$

As remarked in [17, Fact 1.5], this estimate can hold with equality. We would now like to complement this observation by an explicit description of all extremal cases. As it turns out, every n-vertex blow-up H of Γ_k with $\alpha(H) \leqslant s$ and $e(H) = g_k(n, s)$ belongs to the following family of graphs.

Definition 3.4. *Given natural numbers $k \geqslant 2$, n, and $s \in [kn/(3k-1), (k-1)n/(3k-4)]$ the family $\mathscr{G}_k^n(s)$ consists of all graphs obtained from Γ_k by blowing up its vertices*

- *v_0 and v_k by $(k-1)n - (3k-4)s$,*
- *v_{2k-1} by a, v_{2k} by b, where $a, b \in [(k-1)n - (3k-4)s, 3s-n]$ are two integers summing up to $(k-2)n - (3k-7)s$,*
- *and all remaining vertices by $3s - n$.*

Observe that if $s = kn/(3k-1)$, then $(k-1)n - (3k-4)s = 3s - n$ and the only graph in $\mathscr{G}_k^n(s)$ is the balanced blow-up $\Gamma_k(3s-n)$. At the other end of the spectrum we have the case $s = (k-1)n/(3k-4)$, where $(k-1)n - (3k-4)s = 0$ and the graphs in $\mathscr{G}_k^n(s)$ are actually blow-ups of $\Gamma_k \setminus \{v_0, v_k\}$. In this graph, v_{2k} is a twin sister of v_{2k-1} and again the class $\mathscr{G}_k^n(s)$ consists of a single graph. In view of $\Gamma_{k-1} \cong \Gamma_k - \{v_0, v_k, v_{2k}\}$ this graph is

the balanced blow-up $\Gamma_{k-1}(3s - n)$. However, if s lies strictly between $kn/(3k - 1)$ and $(k-1)n/(3k-4)$, then $\mathscr{G}_k^n(s)$ consists of $\lfloor ((3k-1)s - kn)/2 \rfloor + 1$ mutually non-isomorphic graphs.

Lemma 3.5. *If H denotes an n-vertex blow-up of Γ_k satisfying*

$$\alpha(H) \leqslant s \quad and \quad e(H) = g_k(n, s)$$

for some $k \geqslant 2$, then $kn/(3k - 1) \leqslant s \leqslant (k-1)n/(3k-4)$ and H is isomorphic to some graph in $\mathscr{G}_k^n(s)$.

Proof. As usual we denote the vertex classes of H corresponding to the vertices of Γ_k by $V_0, \ldots, V_{3k-2}$. Recall that by Fact 3.2 we have

$$(k-1)n - (3k-4)s \leqslant |V_i| \leqslant 3s - n \quad \text{for every } i \in \mathbb{Z}_{3k-1}. \tag{3.3}$$

In particular, H has the properties enumerated in the moreover-part of Corollary 2.4 and clause (i) corresponds to the estimates on s stated in the lemma. Next, (ii) allows us to assume, without loss of generality, that

$$|V_0| = (k-1)n - (3k-4)s. \tag{3.4}$$

In the special case

$$|V_k| = |V_{2k-1}| = (k-1)n - (3k-4)s$$

we have

$$n = |V_0| + \ldots + |V_{3k-2}| \overset{(3.3)}{\leqslant} 3\big((k-1)n - (3k-4)s\big) + (3k-4)(3s-n) = n,$$

which yields $|V_i| = 3s - n$ for every $i \neq 0, k, 2k - 1$, meaning that $a = (k-1)n - (3k-4)s$ exemplifies $H \in \mathscr{G}_k^n(s)$. By symmetry we may therefore suppose $|V_{2k-1}| > (k-1)n - (3k-4)s$ from now on, which in view of Corollary 2.4(iii) entails

$$s = |N(V_{2k-1})| = |V_0| + \ldots + |V_{k-1}| \overset{(3.3)}{\leqslant} \big((k-1)n - (3k-4)s\big) + (k-1)(3s-n) = s,$$

i.e.,

$$|V_1| = \ldots = |V_{k-1}| = 3s - n. \tag{3.5}$$

Because of $|V_1| + \ldots + |V_k| = |N(V_{2k})| \leqslant s$ and (3.3) this yields

$$|V_k| = (k-1)n - (3k-4)s, \tag{3.6}$$

which together with (3.4) establishes the first bullet in Definition 3.4.

Next we observe that, since by (3.4) the lower bound on $|V_0|$ provided by Fact 3.2 holds with equality, an easy inspection of the proof of Fact 3.2 discloses

$$|V_{k+1}| = \ldots = |V_{2k-2}| = 3s - n$$

and for the same reason (3.6) leads to $|V_{2k+1}| = \ldots = |V_{3k-2}| = 3s-n$. Combined with (3.5) these equations confirm the third bullet in Definition 3.4 and, finally, the second bullet follows easily in the light of $v(H) = n$ and (3.3). $\qquad\square$

§4. VEGA GRAPHS AND THEIR BLOW-UPS

In this section we investigate another important class of dense triangle-free graphs which, unlike Andrásfai graphs, have chromatic number four. Let us recall that the first 4-chromatic triangle-free graph on n vertices whose minimum degree is larger than $n/3$ was a blow-up of the Grötzsch graph discovered in 1981 by Häggkvist [14]. In 1998, Brandt and Pisanski [7] worked with a computer program named Vega and found an infinite sequence of 4-chromatic triangle-free graphs admitting such blow-ups (see Fact 4.2). Due to their origin, these graphs are called *Vega graphs*.
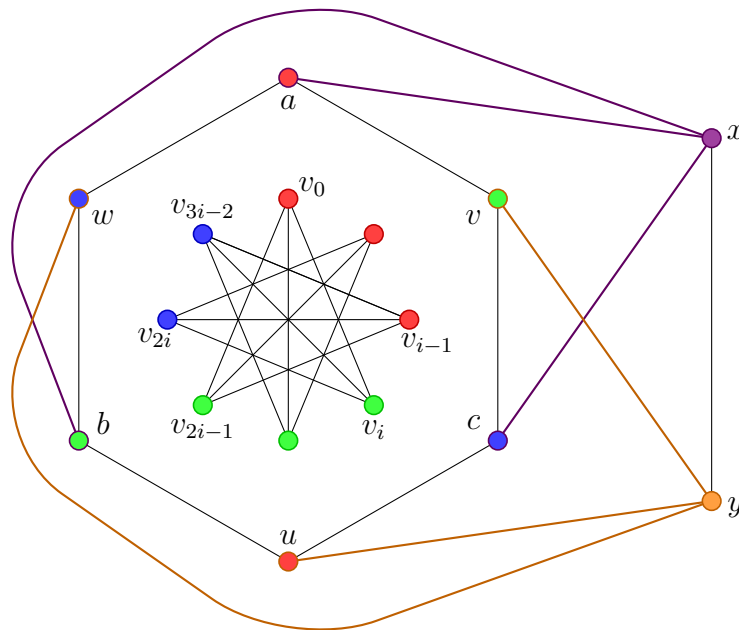

FIGURE 4.1. The Vega graph Υ_i^{00}. The vertices of the external 6-cycle $\mathcal{C}_6$ are connected with the vertices of the same colour of the Andrásfai graph Γ_i in the middle.

4.1. **Definitions and main results.** We commence by presenting a construction of Vega graphs following the work of Brandt and Thomassé [8]. Let an integer $i \geqslant 2$ be given. Start with an Andrásfai graph Γ_i on the vertex set $\{v_0, \ldots, v_{3i-2}\}$ and add an edge xy together with an induced 6-cycle $avcubw$ such that x is joined to a, b, c and y is joined to u, v, w. Moreover, connect

- a, u to $\{v_0, \ldots, v_{i-1}\}$,
- b, v to $\{v_i, \ldots, v_{2i-1}\}$,
- and c, w to $\{v_{2i}, \ldots, v_{3i-2}\}$.

This completes the definition of the sole Vega graph on $3i + 7$ vertices, which we denote by Υ_i^{00} and sometimes just by Υ_i (see Figure 4.1).

There are two Vega graphs on $3i + 6$ vertices obtainable from Υ_i^{00} by simple vertex deletions, namely $\Upsilon_i^{10} = \Upsilon_i^{00} - \{y\}$ and $\Upsilon_i^{01} = \Upsilon_i^{00} - \{v_{2i-1}\}$. Finally, the last Vega graph, $\Upsilon_i^{11} = \Upsilon_i^{00} - \{y, v_{2i-1}\}$, has $3i + 5$ vertices. Observe that the vertex y is present in $\Upsilon_i^{\mu\nu}$ if and only if $\mu = 0$. Similarly, $\nu = 1$ in $\Upsilon_i^{\mu\nu}$ indicates the absence of the vertex v_{2i-1}. For instance, Υ_2^{11} is the well known Grötzsch graph. For later use we would like to remark that $\Upsilon_2^{11} \subseteq \Upsilon_i^{\mu\nu}$ gives a quick proof of the aforementioned estimate $\chi(\Upsilon_i^{\mu\nu}) \geqslant 4$. The main result of this section reads as follows.

Lemma 4.1. *Let integers $i \geqslant 2$ and $\mu, \nu \in \{0, 1\}$ be given and set $k = 9i - (6 + \mu + \nu)$. If for $n \geqslant s \geqslant 0$ the graph H is an n-vertex blow-up of the Vega graph $\Upsilon_i^{\mu\nu}$ satisfying $\alpha(H) \leqslant s$, then*

$$e(H) \leqslant g_k(n, s) \,.$$

The proof of the above lemma is based on Corollary 2.4, which will become applicable once we have exhibited a k-regular blow-up of $\Upsilon_i^{\mu\nu}$ on $3k - 1$ vertices. To this end we shall use an appropriate weight function

$$\omega_{\mu\nu} \colon V(\Upsilon_i) \longrightarrow \mathbb{Z}_{\geqslant 0}$$

from [8]. In the special case $\mu = \nu = 0$ this function is defined by the following table.

vertex z	x, y	a, b, u, v	c, w	v_0, v_{2i-1}	v_j (where $j \neq 0, 2i - 1$)
weight $\omega_{00}(z)$	1	$3i - 2$	$3i - 3$	1	3

In general, one uses the function

$$\omega_{\mu\nu} = \omega_{00} - \mu f - \nu g \,, \tag{4.1}$$

where $f, g \colon V(\Upsilon_i) \longrightarrow \mathbb{Z}$ are defined by

$$f(z) = \begin{cases} 1 & \text{if } z = u, v, w, y \\ -1 & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$

and

$$g(z) = \begin{cases} 1 & \text{if } z = b, v, v_{i-1}, v_{2i-1} \\ -1 & \text{if } z = v_0 \\ 0 & \text{otherwise.} \end{cases}$$

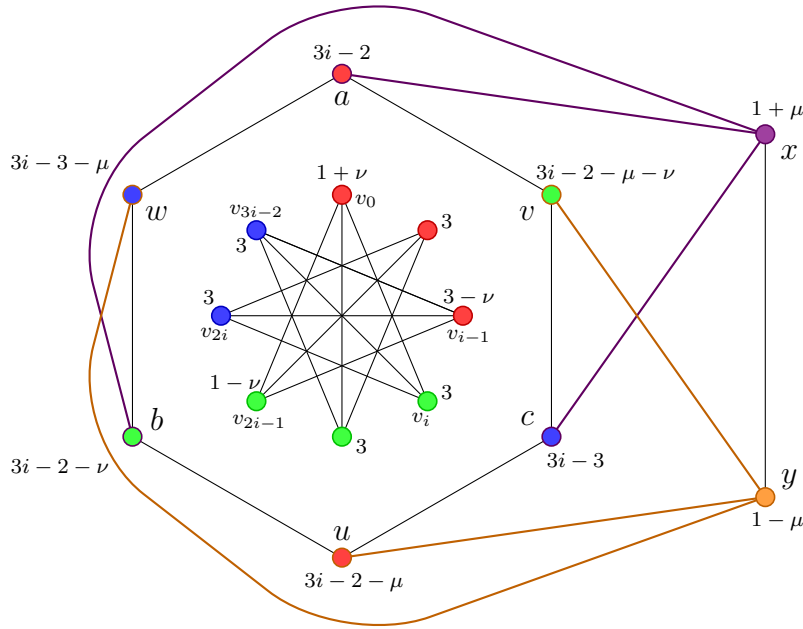The weight function $\omega_{\mu\nu}$ is visualised in Figure 4.2.

FIGURE 4.2. Positive integers assigned to the vertices of the original graph $\Upsilon_i^{\mu\nu}$.

One checks easily that the support of $\omega_{\mu\nu}$ always contains $V(\Upsilon_i^{11})$, that y is in this support if and only if $\mu = 0$, and that v_{2i-1} is in this support if and only if $\nu = 0$. Consequently,

$$\text{the support of } \omega_{\mu\nu} \text{ is } V(\Upsilon_i^{\mu\nu}). \tag{4.2}$$

Given a signed weight function $h\colon V(\Upsilon_i) \longrightarrow \mathbb{Z}$ and a subset $A \subseteq V(\Upsilon_i)$ of the vertex set we define

$$h(A) = \sum_{z \in A} h(z).$$

For instance, some quick calculations disclose the formulae

$$\omega_{00}\big(V(\Upsilon_i)\big) = 27i - 19$$

$$\text{and} \quad f\big(V(\Upsilon_i)\big) = g\big(V(\Upsilon_i)\big) = 3,$$

which by linearity, (4.1), and (4.2) imply

$$\omega_{\mu\nu}\big(V(\Upsilon_i^{\mu\nu})\big) = 3\big(9i - (6 + \mu + \nu)\big) - 1. \tag{4.3}$$

Similarly, for every vertex z we have

$$\omega_{00}\big(N(z)\big) = 9i - 6 \,,$$

$$f\big(N(z)\big) = \begin{cases} 1 & \text{if } z \neq y \\ 2 & \text{if } z = y, \end{cases}$$

$$\text{and} \quad g\big(N(z)\big) = \begin{cases} 1 & \text{if } z \neq v_{2i-1} \\ 2 & \text{if } z = v_{2i-1}, \end{cases}$$

whence

$$\omega_{\mu\nu}\big(N(z)\big) = 9i - (6 + \mu + \nu) \quad \text{for all } z \in V(\Upsilon_i^{\mu\nu}) \,. \tag{4.4}$$

Let $G_i^{\mu\nu} = \Upsilon_i^{\mu\nu}(\omega_{\mu\nu})$ be the blow-up of $\Upsilon_i^{\mu\nu}$ obtained by replacing every vertex z by an independent set Z of size $\omega_{\mu\nu}(z)$, and let

$$k = 9i - (6 + \mu + \nu) \tag{4.5}$$

be the number occurring in Lemma 4.1. We summarise (4.3) and (4.4) in the following observation due to [7].

Fact 4.2. *If $i \geqslant 2$ and $\mu, \nu \in \{0, 1\}$, then $G_i^{\mu\nu}$ is a k-regular blow-up of $\Upsilon_i^{\mu\nu}$ on $3k - 1$ vertices.* $\qquad\square$

Finally, as in the case of Andrásfai graphs, we characterize all extremal Vega graphs.

Definition 4.3. *Given natural numbers $k \geqslant 10$ and $n \geqslant s \geqslant 0$ the family $\mathscr{H}_k^n(s)$ is the smallest collection of blow-ups of Vega graphs with the following properties.*

    (a) If $s = kn/(3k - 1)$, then $G_i^{\mu\nu}(3s - n) \in \mathscr{H}_k^n(s)$ whenever $k = 9i - (6 + \mu + \nu)$.

    (b) If $kn/(3k - 1) < s \leqslant (k - 1)n/(3k - 4)$, then $\Upsilon_i^{0\nu}\big((3s - n)\omega_{\mu\nu} - \lambda f\big) \in \mathscr{H}_k^n(s)$ whenever $k = 9i - (6 + \nu)$.

    (c) If $kn/(3k - 1) < s \leqslant (k - 1)n/(3k - 4)$, then $\Upsilon_i^{\mu 0}\big((3s - n)\omega_{\mu\nu} - \lambda g\big) \in \mathscr{H}_k^n(s)$ whenever $k = 9i - (6 + \mu)$.

Observe that for

- $s \notin [kn/(3k - 1), (k - 1)n/(3k - 4)]$ the family $\mathscr{H}_k^n(s)$ is empty;
- $k \not\equiv 1, 2, 3 \pmod 9$ the family $\mathscr{H}_k^n(s)$ is empty;
- $k \equiv 1 \pmod 9$ we have $\mu + \nu = 2$ and the family $\mathscr{H}_k^n(s)$ is nonempty if and only if $s = nk/(3k - 1)$, in which case it only consists of the graph $G_{(k+8)/9}^{11}(3s - n)$;
- $k \equiv 2, 3 \pmod 9$ and $s = nk/(3k - 1)$ there are one or two graphs in $\mathscr{H}_k^n(s)$ as described in (a).
- $k \equiv 2, 3 \pmod 9$ and $kn/(3k - 1) < s \leqslant (k - 1)n/(3k - 4)$ the family $\mathscr{H}_k^n(s)$ consists of two graphs, one of which is as described in (b) while the other one is as described in (c).

Lemma 4.4. *Given integers $i \geqslant 2$, $\mu, \nu \in \{0, 1\}$, and $n \geqslant s \geqslant 0$, let H be a blow-up of $\Upsilon_i^{\mu\nu}$ on n vertices. If*

$$\alpha(H) \leqslant s \quad and \quad e(H) = g_k(n, s)$$

hold for $k = 9i - (6 + \mu + \nu)$, then H is isomorphic to a graph in the family $\mathscr{H}_k^n(s)$.

4.2. **Proof of Lemma 4.1.** Throughout this subsection, we fix some integers $i \geqslant 2$, $\mu, \nu \in \{0, 1\}$, and $n \geqslant s \geqslant 0$. We keep using the weight function $\omega_{\mu\nu} \colon V(\Upsilon_i) \longrightarrow \mathbb{Z}_{\geqslant 0}$ and the natural number k defined in (4.1) and (4.5), respectively. Let H be an n-vertex blow-up of $\Upsilon_i^{\mu\nu}$ such that $\alpha(H) \leqslant s$. It will be convenient to view H as a blow-up of Υ_i as well by adding an empty vertex class Y in case $\mu = 1$ and an empty V_{2i-1} in case $\nu = 1$. The independence of the neighbourhoods of the vertices in Υ_i entails

$$|N(Z)| \leqslant s \tag{4.6}$$

for every vertex class $Z \subseteq V(H)$ corresponding to a vertex $z \in \Upsilon_i$ (even if this vertex z fails to belong to $\Upsilon_i^{\mu\nu}$.)

We first bound the size of the vertex classes of H from above and below. The ideas in the proofs of the two following facts are similar to those in the proof of Fact 3.2.

Fact 4.5. *Every vertex $z \in V(\Upsilon_i)$ with $\omega_{\mu\nu}(z) \geqslant 2$ satisfies*

$$|Z| \leqslant \omega_{\mu\nu}(z)(3s - n) \,.$$

Moreover, we have

$$|X| + |Y| \leqslant 2(3s - n) = \omega_{\mu\nu}(\{x, y\})(3s - n)$$

$$and \ |V_0| + |V_{2i-1}| \leqslant 2(3s - n) = \omega_{\mu\nu}(\{v_0, v_{2i-1}\})(3s - n) \,.$$

Proof. The upper bound on $|X| + |Y|$ follows from

$$|X| + |Y| + 2n = |N(A)| + |N(V)| + |N(C)| + |N(U)| + |N(B)| + |N(W)| \overset{(4.6)}{\leqslant} 6s \,.$$

Similarly,

$$|V_0| + |V_{2i-1}| + 2n = |N(A)| + |N(B)| + |N(U)| + |N(V)| + |N(V_{i-1})| + |N(V_i)| \overset{(4.6)}{\leqslant} 6s$$

yields the desired upper bound on $|V_0| + |V_{2i-1}|$. It remains to prove $|Z| \leqslant \omega_{\mu\nu}(z)(3s - n)$ for every $z \in V(\Upsilon_i) \smallsetminus \{v_0, v_{2i-1}, x, y\}$. Following the same strategy we have just used, this task reduces to exhibiting a list of $3\omega_{\mu\nu}(z)$ vertices of Υ_i whose neighbourhoods cover the entire vertex set $\omega_{\mu\nu}(z)$ many times and the set Z itself even once more. As there are several cases and plenty of vertices to consider, it seems useful to devise the following notation. For a set $Q \subseteq V(\Upsilon_i)$ we denote its characteristic function by $\mathbb{1}(Q)$. If $Q = \{q_1, \ldots, q_r\}$ comes with an explicit enumeration of its members, it will be convenient to omit a pair

of curly braces, thus writing $\mathbb{1}(q_1, \ldots, q_r)$ for this function. For instance, the functions f and g considered earlier can now be represented as

$$f = \mathbb{1}(u, v, w, y) - \mathbb{1}(x) \quad \text{and} \quad g = \mathbb{1}(b, v, v_{i-1}, v_{2i-1}) - \mathbb{1}(v_0).$$

Instead of $\mathbb{1}\big(V(\Upsilon_i)\big)$ we shall just write $\mathbb{1}$. Next, given a function $h\colon V(\Upsilon_i) \longrightarrow \mathbb{Z}$ we let $\Sigma(h) = h(V(\Upsilon_i)) = \sum_{t \in V(\Upsilon_i)} h(t)$ be the sum of the values it attains and by $\mathscr{N}(h)\colon V(\Upsilon_i) \longrightarrow \mathbb{Z}$ we mean the function $\sum_{t \in V(\Upsilon_i)} h(t)\, \mathbb{1}\big(N(t)\big)$. In other words, this function satisfies

$$\mathscr{N}(h)(t) = \sum_{t' \in N(t)} h(t') = h\big(N(t)\big) \quad \text{for every } t \in V(\Upsilon_i). \tag{4.7}$$

So we used earlier that

$$\mathscr{N}(f) = \mathbb{1} + \mathbb{1}(y), \qquad\qquad\qquad \Sigma(f) = 3,$$
$$\text{and} \quad \mathscr{N}(g) = \mathbb{1} + \mathbb{1}(v_{2i-1}), \qquad\qquad \Sigma(g) = 3, \tag{4.8}$$

the upper bound on $|X| + |Y|$ relies on the fact that the hexagon $\mathscr{C}_6 = \{a, v, c, u, b, w\}$ has the properties

$$\mathscr{N}\big(\mathbb{1}(\mathscr{C}_6)\big) = 2 \cdot \mathbb{1} + \mathbb{1}(x, y), \quad \Sigma\big(\mathbb{1}(\mathscr{C}_6)\big) = |\mathscr{C}_6| = 6 \tag{4.9}$$

and soon we are going to need that the inner Andrásfai graph $\Gamma = \{v_0, v_1, \ldots, v_{3i-2}\}$ satisfies

$$\mathscr{N}\big(\mathbb{1}(\Gamma)\big) = i \cdot \mathbb{1}(\Gamma) + (i-1) \cdot \mathbb{1}(\mathscr{C}_6) + \mathbb{1}(a, b, u, v), \qquad \Sigma\big(\mathbb{1}(\Gamma)\big) = |\Gamma| = 3i - 1. \tag{4.10}$$

Now it suffices to exhibit for every vertex $z \in V(\Upsilon_i) \smallsetminus \{v_0, v_{2i-1}, x, y\}$ a function

$$h^z\colon V(\Upsilon_i) \longrightarrow \mathbb{Z}_{\geqslant 0}$$

such that

(a) $\mathscr{N}(h^z)$ agrees on $V(\Upsilon_i^{\mu\nu})$ with $\omega_{\mu\nu}(z) \cdot \mathbb{1} + \mathbb{1}(z)$

(b) and $\Sigma(h^z) = 3\omega_{\mu\nu}(z)$.

Indeed, once we have such a function h^z at our disposal, we can imitate the above reasoning and argue that

$$\omega_{\mu\nu}(z)n + |Z| \overset{(a)}{=} \sum_{t \in V(\Upsilon_i)} \mathscr{N}(h^z)(t)|T| \overset{(4.7)}{=} \sum_{t \in V(\Upsilon_i)} \sum_{t' \in N(t)} h^z(t')\,|T|$$

$$= \sum_{t' \in V(\Upsilon_i)} h^z(t')\,|N(T')| \overset{(4.6)}{\leqslant} \Sigma(h^z)s \overset{(b)}{=} 3\omega_{\mu\nu}(z)s,$$

which proves the desired inequality $|Z| \leqslant \omega_{\mu\nu}(z)(3s - n)$.

Starting with the vertices in Γ we observe that for every $k \in \mathbb{Z}_{3i-1} \smallsetminus \{0, 2i-1\}$ we have

$$\mathscr{N}\big(\mathbb{1}(v_{k-i}, v_k, v_{k+i})\big) = \mathbb{1}(\mathscr{C}_6) + \mathbb{1}(\Gamma) + \mathbb{1}(v_k).$$

By adding (4.9) we infer

$$\mathscr{N}\left(\mathbb{1}(\mathscr{C}_6) + \mathbb{1}(v_{k-i}, v_k, v_{k+i})\right) = 3 \cdot \mathbb{1} + \mathbb{1}(v_k), \tag{4.11}$$

which shows that for $k \neq 0, i-1, 2i-1$ the function $h^{v_k} = \mathbb{1}(\mathscr{C}_6) + \mathbb{1}(v_{k-i}, v_k, v_{k+i})$ has the required properties. Moreover, for $k = i - 1$ we deduce from (4.11) with the help of (4.8) that

$$\mathscr{N}\left(\mathbb{1}(\mathscr{C}_6) + \mathbb{1}(v_{i-1}, v_{2i-1}, v_{3i-2}) - \nu g\right) = (3 - \nu)\mathbb{1} + \mathbb{1}(v_{i-1}) - \nu\mathbb{1}(v_{2i-1}),$$

meaning that we can take $h^{v_{i-1}} = \mathbb{1}(\mathscr{C}_6) + \mathbb{1}(v_{i-1}, v_{2i-1}, v_{3i-2}) - \nu g$. It remains to deal with the vertices on the hexagon. Starting with a, we observe

$$\mathscr{N}\left(\mathbb{1}(a, x) - \mathbb{1}(v_{2i-1})\right) = \mathbb{1}(a, c, w, x, y), \tag{4.12}$$

which in view of (4.9) and (4.10) entails that the function

$$h^a = (i-1)\mathbb{1}(\mathscr{C}_6) + \mathbb{1}(\Gamma) + \mathbb{1}(a, x) - \mathbb{1}(v_{2i-1})$$

satisfies

$$\mathscr{N}(h^a) = \left[(2i-2) \cdot \mathbb{1} + (i-1) \cdot \mathbb{1}(x, y)\right] + \left[i \cdot \mathbb{1}(\Gamma) + (i-1) \cdot \mathbb{1}(\mathscr{C}_6) + \mathbb{1}(a, b, u, v)\right]$$
$$+ \mathbb{1}(a, c, w, x, y) = (3i - 2) \cdot \mathbb{1} + \mathbb{1}(a).$$

Together with $\Sigma(h^a) = 6(i - 1) + (3i - 1) + 2 - 1 = 3(3i - 2)$ this proves that h^a has all required properties. By symmetry, for $\widetilde{h}^b = (i-1) \cdot \mathbb{1}(\mathscr{C}_6) + \mathbb{1}(\Gamma) + \mathbb{1}(b, x) - \mathbb{1}(v_0)$ we obtain $\mathscr{N}(\widetilde{h}^b) = (3i - 2)\mathbb{1} + \mathbb{1}(b)$ and $\Sigma(\widetilde{h}^b) = 3(3i - 2)$, so by (4.8) we may set $h^b = \widetilde{h}^b - \nu g$. Next,

$$\mathscr{N}\left(\mathbb{1}(x) - \mathbb{1}(v_0, v_{2i-1}, w)\right) = \mathbb{1}(c) - \mathbb{1}(a, b, u, v) - \mathbb{1}(\Gamma) \tag{4.13}$$

reveals that the function $h^c = (i-1) \cdot \mathbb{1}(\mathscr{C}_6) + \mathbb{1}(\Gamma) + \mathbb{1}(x) - \mathbb{1}(v_0, v_{2i-1}, w)$ has the property

$$\mathscr{N}(h^c) = \left[(2i-2) \cdot \mathbb{1} + (i-1) \cdot \mathbb{1}(x, y)\right] + \left[i \cdot \mathbb{1}(\Gamma) + (i-1) \cdot \mathbb{1}(\mathscr{C}_6) + \mathbb{1}(a, b, u, v)\right]$$
$$+ \left[\mathbb{1}(c) - \mathbb{1}(a, b, u, v) - \mathbb{1}(\Gamma)\right] = (3i - 3)\mathbb{1} + \mathbb{1}(c)$$

and because of $\Sigma(h^c) = 6(i - 1) + (3i - 1) + 1 - 3 = 3(3i - 3)$ this establishes the desired bound $|C| \leqslant (3i - 3)(3s - n)$. Utilising that similar to (4.12), (4.13) we have

$$\mathscr{N}\left(\mathbb{1}(u, y) - \mathbb{1}(v_{2i-1})\right) = \mathbb{1}(c, u, w, x, y)$$
$$\mathscr{N}\left(\mathbb{1}(v, y) - \mathbb{1}(v_0)\right) = \mathbb{1}(c, v, w, x, y)$$
$$\text{and} \quad \mathscr{N}\left(\mathbb{1}(y) - \mathbb{1}(c, v_0, v_{2i-1})\right) = \mathbb{1}(w) - \mathbb{1}(a, b, u, v) - \mathbb{1}(\Gamma)$$

one confirms easily that the functions

$$h^u = (i-1)\mathbb{1}(\mathscr{C}_6) + \mathbb{1}(\Gamma) + \mathbb{1}(u,y) - \mathbb{1}(v_{2i-1}) - \mu f$$
$$h^v = (i-1)\mathbb{1}(\mathscr{C}_6) + \mathbb{1}(\Gamma) + \mathbb{1}(v,y) - \mathbb{1}(v_0) - \mu f - \nu g$$
$$\text{and} \quad h^w = (i-1)\mathbb{1}(\mathscr{C}_6) + \mathbb{1}(\Gamma) + \mathbb{1}(y) - \mathbb{1}(c,v_0,v_{2i-1}) - \mu f$$

take care of the three remaining vertex classes. $\square$

Adding all inequalities provided by Fact 4.5 we obtain $n \leqslant (3k-1)(3s-n)$ (recall (4.3) and (4.5)), whence

$$\frac{kn}{(3k-1)} \leqslant s$$

and the number

$$\lambda = (3k-1)s - kn = k(3s-n) - s \tag{4.14}$$

is nonnegative. For later use we rewrite this in the form

$$n = (3k-1)(3s-n) - 3\lambda \overset{(4.3)}{=} \omega_{\mu\nu}\big(V(\Upsilon_i)\big)(3s-n) - 3\lambda. \tag{4.15}$$

Similarly, (4.6) and (4.4) yield

$$|N(Z)| \leqslant s = k(3s-n) - \lambda = \omega_{\mu\nu}\big(N(z)\big)(3s-n) - \lambda \tag{4.16}$$

for every vertex $z \in V(\Upsilon_i^{\mu\nu})$. Now we are ready for a lower bound on the sizes of the vertex classes of H.

Fact 4.6. *If $z \in V(\Upsilon_i)$, then*

$$|Z| \geqslant \omega_{\mu\nu}(z)(3s-n) - \lambda. \tag{4.17}$$

Proof. As we shall prove by means of a complete case analysis, there exist two adjacent non-neighbours z', z'' of z in $\Upsilon_i^{\mu\nu}$ such that each of the sets $\{x,y\}$ and $\{v_0, v_{2i-1}\}$ is either contained in $N(z') \cup N(z'') \cup \{z\}$ or in its complement.

Once we have two such vertices z' and z'', the argument proceeds as follows. Due to $z'z'' \in E(\Upsilon_i)$ and $zz', zz'' \notin E(\Upsilon_i)$ the union $R = \{z\} \cup N(z') \cup N(z'')$ is disjoint. Let $S = V(\Upsilon_i) \smallsetminus R$ be the complement of R and write

$$R_H = Z \cup N(Z') \cup N(Z'') \quad \text{as well as} \quad S_H = V(H) \smallsetminus R_H$$

for the sets corresponding to R and S in H. Fact 4.5 implies

$$|S_H| \leqslant \omega_{\mu\nu}(S)(3s-n) \tag{4.18}$$

and, consequently, we have

$$|Z| + |N(Z')| + |N(Z'')| = |R_H| = n - |S_H|$$

$$\overset{(4.15)}{\geqslant} \omega_{\mu\nu}\big(V(\Upsilon_i)\big)(3s-n) - 3\lambda - \omega_{\mu\nu}(S)(3s-n) = \omega_{\mu\nu}(R)(3s-n) - 3\lambda$$

$$\overset{(4.16)}{\geqslant} \omega_{\mu\nu}(z)(3s-n) - \lambda + |N(Z')| + |N(Z'')|\,,$$

i.e., $|Z| \geqslant \omega_{\mu\nu}(z)(3s-n) - \lambda$. So it remains to check that the auxiliary vertices z' and z'' do indeed exist and we list some possible choices in the table that follows.

z	v_0	$v_1, \ldots, v_{2i-2}$	v_{2i-1}	$v_{2i}, \ldots, v_{3i-2}, a, v$	b, u	c, w	x	y
z', z''	c, v	c, x	c, u	b, u	a, v	v_0, v_i	u, v_0	a, v_0

This concludes the proof of Fact 4.6. □

We are left with the task of proving Lemma 4.1 itself. To this end we remark that H can be regarded as a blow-up of the graph $G_i^{\mu\nu}$ considered in Fact 4.2. In fact, every vertex z of $\Upsilon_i^{\mu\nu}$ corresponds to some vertex class Z of H and due to $\omega_{\mu\nu}(z) \geqslant 1$ we can partition each such vertex class into $\omega_{\mu\nu}(z)$ "particles" of size $\lfloor |Z|/\omega_{\mu\nu}(z)\rfloor$ or $\lceil |Z|/\omega_{\mu\nu}(z)\rceil$ each. Owing to Fact 4.6 the sizes of these particles are at least

$$\left\lfloor \frac{|Z|}{\omega_{\mu\nu}(z)} \right\rfloor \overset{(4.17)}{\geqslant} \left\lfloor \frac{\omega_{\mu\nu}(z)(3s-n) - \lambda}{\omega_{\mu\nu}(z)} \right\rfloor = 3s - n - \left\lceil \frac{\lambda}{\omega_{\mu\nu}(z)} \right\rceil \qquad (4.19)$$

$$\geqslant 3s - n - \lambda \overset{(4.14)}{=} (k-1)n - (3k-4)s\,.$$

As the particles endow H with the structure of a blow-up of $G_i^{\mu\nu}$, Corollary 2.4 shows that we have indeed

$$e(H) \leqslant g_k(n,s)\,.$$

This concludes the proof of Lemma 4.1. □

4.3. **Proof of Lemma 4.4.** Corollary 2.4(*i*) yields

$$\frac{kn}{3k-1} \leqslant s \leqslant \frac{(k-1)n}{3k-4}\,,$$

whence $0 \leqslant \lambda \leqslant 3s - n$. In the special case $\lambda = 0$ the Facts 4.5 and 4.6 imply that every vertex $z \in V(\Upsilon_i^{\mu\nu})$ satisfies $|Z| = \omega_{\mu\nu}(z)(3s-n)$, meaning that statement (*a*) holds. From now on we suppose $\lambda \geqslant 1$, which entails the estimate on s occurring in (*b*) and (*c*).

Claim 4.7. *There exists a vertex* $z \in V(\Upsilon_i^{\mu\nu}) \cap \{x, y, v_0, v_{2i-1}\}$ *such that* $\omega_{\mu\nu}(z) = 1$ *and* $|Z| = 3s - n - \lambda$.

Proof. Recall that in the proof of Lemma 4.1 we viewed H as a blow-up of the k-regular graph $G_i^{\mu\nu}$. By Corollary 2.4 at least one of the particles occurring in this construction has the size $(k-1)n - (3k-4)s = (3s-n) - \lambda$. Let $z \in V(\Upsilon_i^{\mu\nu})$ be a vertex one of whose $\omega_{\mu\nu}(z)$

particles has this size. Now (4.19) needs to hold with equality and we have $\lceil \lambda/\omega_{\mu\nu}(z)\rceil = \lambda$. For this reason at least one of the equations $\omega_{\mu\nu}(z) = 1$ or $\lambda = 1$ is true. In the former case, $z \in \{x, y, v_0, v_{2i-1}\}$ is immediate and we are done.

Now suppose for the sake of contradiction that $\lambda = 1$ and that $|Z| \neq 3s - n - \lambda$ holds for every $z \in V(\Upsilon_i^{\mu\nu}) \cap \{x, y, v_0, v_{2i-1}\}$ with $\omega_{\mu\nu}(z) = 1$. Together with the Facts 4.5 and 4.6 this yields $|Z| \leqslant \omega_{\mu\nu}(z)(3s - n)$ for all vertices $z \in V(\Upsilon_i^{\mu\nu})$.

Concerning the set $M = \{z \in V(\Upsilon_i^{\mu\nu}) : |Z| < \omega_{\mu\nu}(z)(3s - n)\}$ of vertices for which this estimate fails to be sharp we can deduce from

$$n = \sum_{z \in V(\Upsilon_i^{\mu\nu})} |Z| \leqslant \omega_{\mu\nu}\big(V(\Upsilon_i^{\mu\nu})\big)(3s - n) - |M|$$

and (4.15) that $|M| \leqslant 3$. Owing to $\chi(\Upsilon_i^{\mu\nu}) = 4$ this shows that the neigbourhouds of the vertices in M cannot cover the entire vertex sets of $\Upsilon_i^{\mu\nu}$ or, in other words, that there is a vertex $z_\star \in V(\Upsilon_i^{\mu\nu})$ whose neighbourhood is disjoint to M. But now

$$s \geqslant |N(Z_\star)| = \sum_{z \in N(z_\star)} |Z| = \sum_{z \in N(z_\star)} \omega_{\mu\nu}(z)(3s - n) = k(3s - n)$$

contradicts (4.16). This completes the proof of Claim 4.7.   □

Let us observe that if the vertex z delivered by Claim 4.7 is either x or y, then $\mu = 0$, while if it is one of v_0, v_{2i-1}, then $\nu = 0$.

First Case. We have $\mu = 0$ and one of X, Y has size $3s - n - \lambda$.

The product of the four transpositions $x \longleftrightarrow y$, $a \longleftrightarrow u$, $b \longleftrightarrow v$, and $c \longleftrightarrow w$ is an automorphism of $\Upsilon_i^{\mu\nu}$ and, since we only aim at determining H up to isomorphism this fact shows that without loss of generality we may suppose $|Y| = 3s - n - \lambda$.

Now for y in place of z the proof of Fact 4.6 goes through with equality. In particular, if we work with the pair $\{z', z''\} = \{a, v_0\}$ indicated in the table, we need to have equality in (4.18), which in turn implies in view of Fact 4.5 that

$$|Z| = \omega_{\mu\nu}(z)(3s - n) \tag{4.20}$$

holds for all $z \in \{b, c, v_{2i}, \ldots, v_{3i-2}\}$. Working with the pair $\{a, v_{i-1}\}$ or $\{b, v_i\}$ instead we learn that (4.20) is also valid for all $z \in \{v_i, \ldots, v_{2i-2}\}$ and all $z \in \{a, v_1, \ldots, v_{i-1}\}$. Altogether, this proves (4.20) for all $z \neq v_0, v_{2i-1}, u, v, w, x, y$.

Now $|N(V_{2i})| \leqslant s = \omega_{\mu\nu}\big(N(v_{2i})\big)(3s - n) - \lambda$ and Fact 4.6 yield

$$|W| = \omega_{\mu\nu}(w)(3s - n) - \lambda \,,$$

meaning for $z = w$ the estimates entering the proof of Fact 4.6 hold with equality as well. Applying this observation to $\{v_0, v_i\}$ playing the rôle of $\{z', z''\}$ and to (4.18) we conclude

$|X| + |Y| = 2(3s - n)$, which in turn entails $|X| = (3s - n) + \lambda$. Thus $|N(C)| \leqslant s$ and Fact 4.6 are only compatible if $|Z| = \omega_{\mu\nu}(z)(3s - n) - \lambda$ holds for $z = u, v$ as well.

Finally, $|V_i| > \omega_{\mu\nu}(v_i)(3s - n) - \lambda$ and Corollary 2.4(*iii*) yield $|N(V_i)| = s$, whence

$$|V_0| = \omega_{\mu\nu}(v_0)(3s - n).$$

The same argument applied to V_{i-1} discloses $|V_{2i-1}| = \omega_{\mu\nu}(v_{2i-1})(3s - n)$ and altogether this concludes the proof that $|Z| = \omega_{\mu\nu}(z)(3s - n) - \lambda f(z)$ holds for every $z \in V(\Upsilon_i^{\mu\nu})$, i.e., that H is as described in (*b*).

Second Case. We have $\nu = 0$ and one of V_0, V_{2i-1} has size $3s - n - \lambda$.

We will show that outcome (*c*) of our lemma occurs. The argument will be very similar to the one we saw in previous case. First, we note that the reflection $v_j \longmapsto v_{2i-1-j}$ of Γ composed with the transpositions $a \longleftrightarrow b$, $u \longleftrightarrow v$ constitutes an automorphism of $\Upsilon_i^{\mu\nu}$ that exchanges v_0 and v_{2i-1}. Thus we may suppose without loss of generality that $|V_{2i-1}| = 3s - n - \lambda$.

As before, we need to have equality in (4.18) when running the proof of Fact 4.6 for $z = v_{2i-1}$ and $\{z', z''\}$ being one of the pairs $\{a, x\}$, $\{a, w\}$, or $\{c, u\}$. Consequently, (4.20) holds whenever $z \in \{v_i, \ldots, v_{2i-2}, v_{2i}, \ldots, v_{3i-2}, a, c, u, w\}$. Now $|N(V_{3i-2})| \leqslant s$ and Fact 4.6 yield

$$|V_{i-1}| = \omega_{\mu\nu}(v_{i-1})(3s - n) - \lambda.$$

This equality case of Fact 4.6 can be analysed by using the pair $\{c, x\}$ in place of $\{z', z''\}$. In this manner we infer that (4.20) holds for $z \in \{v_1, \ldots, v_{i-2}\}$ as well and, moreover, that $|V_0| = (3s - n) + \lambda$. Together with $|N(V_i)| \leqslant s$ and Fact 4.6 this establishes $|Z| = \omega_{\mu\nu}(z)(3s - n) - \lambda$ for $z = b, v$ and it remains to check (4.20) for $z = x, y$. To this end, we observe that $|A| > \omega_{\mu\nu}(a)(3s - n) - \lambda$ and Corollary 2.4(*iii*) yield $|N(A)| = s$, whence $|X| = \omega_{\mu\nu}(x)(3s - n)$. The argument for Y is similar but considers U instead of A. $\qquad \square$

§5. PROOFS OF THE MAIN RESULTS

The main ingredient of our argument is a result of Brandt and Thomassé [8] which states that all maximal triangle-free graphs of large minimum degree are blow-ups of Andrásfai and Vega graphs. A graph G is *maximal triangle-free* if adding any missing edge to G creates a triangle.

Theorem 5.1. *Every maximal triangle-free graph on n vertices with minimum degree greater than $n/3$ is a proper blow-up of some Andrásfai graph or Vega graph.*

Next we recall [17, Fact 2.6], which will allows us to restrict to the 'correct' range of s when proving Theorem 1.3.

Fact 5.2. *If $n \geqslant s \geqslant 0$ and $k \geqslant 2$ are such that $s \notin \left(\frac{k}{3k-1}n, \frac{k-1}{3k-4}n \right)$, then*

$$\mathrm{ex}(n, s) \leqslant g_k(n, s)$$

We are now ready for the proof of our second main result.

Proof of Theorem 1.3. Observe that adding any edges to H can neither increase its independence number nor decrease its minimum degree. Thus, we may and shall assume that H is a maximal triangle-free graph.

Due to Fact 5.2, it suffices to consider the case

$$\Delta(H) \leqslant \alpha(H) \leqslant s < \frac{k-1}{3k-4}n .$$

Since $\delta(H) > (k+1)n/(3k+2) > n/3$, Theorem 5.1 tells us that H is a proper blow-up of some graph J, which is either an Andrásfai graph Γ_ℓ, or a Vega graph $\Upsilon_i^{\mu\nu}$. In the latter case we set $\ell = 9i - (6 + \mu + \nu)$. Now in both cases J has a proper ℓ-regular blow-up on $3\ell - 1$ vertices and Lemma 2.2 yields $k = \ell$. If $J = \Gamma_k$ is an Andrásfai graph the assertion follows from Lemma 3.3 and if J is a Vega graph we use Lemma 4.1 instead. $\square$

The other main result follows by means of a vertex deletion argument.

Proof of Theorem 1.2. Define

$$\zeta = \frac{1}{8k^2} \quad \text{and} \quad \gamma = \frac{1}{600k^6} .$$

Consider a triangle-free graph G on n vertices with $\alpha(G) \leqslant s$ and $e(G) = \mathrm{ex}(n, s)$, where

$$\frac{k}{3k-1}n \leqslant s \leqslant \left(\frac{k}{3k-1} + \gamma \right) n . \tag{5.1}$$

Since γ is sufficiently small, we have $s < (k-1)n/(3k-4)$ and Fact 5.2 implies $e(G) \leqslant g_\ell(n, s)$ for every $\ell \neq k$. On the other hand, Lemma 3.5 yields

$$e(G) = \mathrm{ex}(n, s) \geqslant g_k(n, s) \tag{5.2}$$

and it remains to prove that this holds with equality. It will be convenient to rewrite (5.2) as

$$e(G) \geqslant g_k(n, s) = \tfrac{1}{2} \big[ns - \big((k-1)n - (3k-4)s \big) \big((3k-1)s - kn \big) \big] .$$

Since

$$0 \leqslant (3k-1)s - kn < 3k\gamma n$$

and

$$(k-1)n - (3k-4)s \leqslant \frac{(3k-1)(k-1)}{k}s - (3k-4)s = \frac{s}{k} ,$$

we have

$$\big((k-1)n - (3k-4)s \big) \big((3k-1)s - kn \big) < 3\gamma ns ,$$

and thus

$$ns - 2e(G) < 3\gamma ns \,.$$

Therefore, the set

$$A = \{v \in V(G) \colon d(v) < (1 - \zeta)\, s\} \tag{5.3}$$

satisfies

$$\zeta s |A| \leqslant \sum_{v \in V} (s - d(v)) = ns - 2e(G) < 3\gamma ns$$

and, consequently,

$$|A| < \frac{3\gamma n}{\zeta} = \frac{n}{25k^4} \,. \tag{5.4}$$

Now our argument will proceed as follows. We shall verify that the graph $G' = G - A$ satisfies the assumptions of Theorem 1.3 and, hence, $e(G')$ is bounded from above by $g_k(n - |A|, s)$. Then, using the fact that all vertices in A are of small degree, we derive an upper bound of $g_k(n, s)$ on the number of edges in G.

For the minimum degree of G' we obtain

$$\delta(G') \geqslant (1 - \zeta)\, s - |A| > \frac{k}{3k - 1} n - \frac{n/2}{8k^2} - \frac{n}{25k^4}$$

$$> \left(\frac{k + 1}{3k + 2} + \frac{1}{12k^2} - \frac{1}{16k^2} - \frac{1}{100k^2} \right) n$$

$$> \frac{k + 1}{3k + 2} n \geqslant \frac{k + 1}{3k + 2} |V(G')| \,.$$

As the graph G' is triangle-free and satisfies $\alpha(G') \leqslant s$, this shows that Theorem 1.3 applies to G' and we are lead to

$$e(G') \leqslant g_k(n - |A|, s) = g_k(n, s) - |A| \left(k(k - 1)n - \tfrac{1}{2}k(k - 1)|A| - k(3k - 4)s \right) . \tag{5.5}$$

Now our choice of ζ and γ yields

$$\left((k - 1)(3k - 1) - \zeta \right) s \leqslant \left((k - 1)(3k - 1) - \frac{1}{8k^2} \right) \left(\frac{k}{3k - 1} + \frac{1}{600k^6} \right) n$$

$$< \left(k(k - 1) + \frac{1}{200k^4} - \frac{1}{8k(3k - 1)} \right) n$$

$$< \left(k(k - 1) + \frac{1}{50k^2} - \frac{1}{25k^2} \right) n < k(k - 1) \left(1 - \frac{1}{50k^4} \right) n$$

$$\overset{(5.4)}{<} k(k - 1)(n - \tfrac{1}{2}|A|) \,,$$

and for this reason (5.5) can be continued to

$$e(G') \leqslant g_k(n, s) - \left((k - 1)(3k - 1) - \zeta - k(3k - 4) \right) |A|s$$

$$= g_k(n, s) - (1 - \zeta)|A|s \,.$$

Every vertex in A has degree at most $(1 - \zeta)s$ in G, so we arrive at

$$e(G) \leqslant e(G') + (1 - \zeta)|A|s \leqslant g_k(n, s)\,, \tag{5.6}$$

which together with (5.2) concludes the proof of Theorem 1.2. □

Finally, let us remark that our results allow to determine the extremal graphs for Theorem 1.2.

Corollary 5.3. *Suppose that $k \geqslant 2$ and that G denotes a triangle-free graph on n vertices with $\alpha(G) \leqslant s$ for some integer s satisfying*

$$\frac{k}{3k - 1}n \leqslant s \leqslant \left(\frac{k}{3k - 1} + \frac{1}{600k^6} \right) n\,.$$

If $e(G) = \mathrm{ex}(n, s)$, then G is isomorphic to a graph in $\mathscr{G}_k^n(s) \cup \mathscr{H}_k^n(s)$.

Proof. Following the proof of Theorem 1.2, we see that (5.6) holds with equality, for which reason the set A defined in (5.3) has to be empty. In other words, $G' = G$ and the proof of Theorem 1.3 discloses that G' is a blow-up of either the Andrásfai graph Γ_k, or of a Vega graph $\Upsilon_i^{\mu\nu}$ with $k = 9i - (6 + \mu + \nu)$. In the former case, Lemma 3.5 shows that G is isomorphic to a graph in $\mathscr{G}_k^n(s)$ and in the latter case we apply Lemma 4.4. □

In fact, we strongly suspect that these are the only extremal graphs for the whole range of s, i.e., that the following stronger version of our initial conjecture holds.

Conjecture 5.4. *If $n/3 < s \leqslant n/2$, then*

$$\mathrm{ex}(n, s) = \min_k g_k(n, s)\,,$$

where $g_k(n, s)$ is defined by (1.2). Moreover, each extremal graph with $\mathrm{ex}(n, s)$ edges is isomorphic to one of the graphs from the families $\mathscr{G}_k^n(s)$ and $\mathscr{H}_k^n(s)$ described in the Definitions 3.4 and 4.3, respectively.

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