A PAIR-DEGREE CONDITION FOR HAMILTONIAN CYCLES IN 3-UNIFORM HYPERGRAPHS

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Abstract. We prove a new sufficient pair-degree condition for Hamiltonian cycles in 3-uniform hypergraphs that asymptotically improves the best known pair-degree condition due to Rödl, Ruciński, and Szemerédi. For graphs, Chvátal improved on Dirac's tight condition on the minimum degree of a graph ensuring a Hamiltonian cycle by characterising all degree sequences that guarantee the existence of a Hamiltonian cycle. A step towards Chvátal's theorem was taken by Pósa who showed that a graph on at least 3 vertices whose degree sequence $d(1) \leqslant \cdots \leqslant d(n)$ satisfies $d(i) \geqslant i+1$, for all $i < (n-1)/2$, and furthermore $d(\lceil n/2 \rceil) \geqslant \lceil n/2 \rceil$, when n is odd, contains a Hamiltonian cycle.

More recently, there has been some progress on generalising Dirac's theorem to hypergraphs. Rödl, Ruciński, and Szemerédi obtained an asymptotically tight minimum pair-degree condition for 3-uniform hypergraphs (and generalised this result to k-graphs).

In this work, we will take a step towards a full characterisation of all pair-degree matrices that ensure the existence of Hamiltonian cycles in 3-uniform hypergraphs by proving a 3-uniform analogue of Pósa's result. In particular, our result strengthens the asymptotic version of the result by Rödl, Ruciński, and Szemerédi.

§1. Introduction

The search for conditions ensuring the existence of Hamiltonian cycles in graphs has been one of the main themes in graph theory. For graphs, several classic results exist, starting with the necessary condition by Dirac [5] stating that every graph $G = (V, E)$ on at least 3 vertices and with minimum degree $\delta(G) \geqslant |V|/2$ contains a Hamiltonian cycle. Pósa [12] improved this result to a condition on the degree sequence:

Theorem 1.1. *Let $G = ([n], E)$ be a graph on $n \geqslant 3$ vertices with $d(1) \leqslant \cdots \leqslant d(n)$. If $d(i) \geqslant i+1$ for all $i < (n-1)/2$ and if furthermore $d(\lceil n/2 \rceil) \geqslant \lceil n/2 \rceil$ when n is odd, then G contains a Hamiltonian cycle.*

Finally, Chvátal [3] achieved an even stronger result: A graph $G = ([n], E)$ on $n \geqslant 3$ vertices with degree sequence $d(1) \leqslant \cdots \leqslant d(n)$ contains a Hamiltonian cycle if for all $i < \frac{n}{2}$ we have: $d(i) \leqslant i \Rightarrow d(n-i) \geqslant n-i$. On the other hand, for any sequence $a_1 \leqslant \cdots \leqslant a_n < n$

2010 *Mathematics Subject Classification.* Primary: 05C65. Secondary: 05C35.

Key words and phrases. Hamiltonian cycles, hypergraphs.

not satisfying this condition there exists a graph on vertex set $[n]$ with $a_i \leqslant d(i)$ for all $i \in [n]$ that does not contain a Hamiltonian cycle.

One can also investigate Hamiltonian cycles in more general structures: A *k-uniform hypergraph* (or *k-graph*) is a pair (V, E) consisting of a (vertex) set V and an (edge) set $E \subseteq V^{(k)}$. In the following let $H = (V, E)$ be a 3-graph. We write $E(H) := E$ for the edge set and for $U \subseteq V$ we define $H[U] := (U, E(U))$ with $E(U) := \{e \in E : e \subseteq U\}$. For vertices $v, w \in V$ we denote by $d(v, w) := |\{x \in V : vwx \in E\}|$ the *pair-degree*, where for convenience we write an edge as vwx instead of $\{v, w, x\}$. In addition, it is also common to study the *vertex degree* $d(v) := |\{e \in E : v \in e\}|$. The minimum pair-degree is $\delta_2(H) := \min_{vw \in V^{(2)}} d(v, w)$ and the minimum vertex degree is $\delta(H) := \min_{v \in V} d(v)$. Often it is useful to consider something like a 2-uniform projection of H with respect to a vertex $v \in V$: We define the *link graph* L_v of v as the graph $(V, \{xy : xyv \in E\})$.

We will follow the definition of paths and cycles in [13], suggested by Katona and Kierstead in [9]. A hypergraph P is a *tight path* of length ℓ, if $|V(P)| = \ell + 2$ and there is an ordering of the vertices $V(P) = \{x_1, \ldots, x_{\ell+2}\}$ such that a triple e forms a hyperedge of P if and only if $e = \{x_i, x_{i+1}, x_{i+2}\}$ for some $i \in [\ell]$. The ordered pairs (x_1, x_2) and $(x_{\ell+1}, x_{\ell+2})$ are the *end-pairs* of P and we say that P is a tight (x_1, x_2)-$(x_{\ell+1}, x_{\ell+2})$-path. All other vertices of P are called *internal*. We might identify a path with the sequence of its vertices $x_1, \ldots, x_{\ell+2}$. Accordingly, a *tight cycle C* of length $\ell \geqslant 4$ consists of a path $x_1, \ldots, x_\ell$ of length $\ell - 2$ together with the two hyperedges $x_{\ell-1}x_\ell x_1$ and $x_\ell x_1 x_2$. A *tight walk* of length ℓ is a hypergraph W with $V(W) = \{x_1, \ldots, x_{\ell+2}\}$, where the x_i are not necessarily distinct, and $E(W) = \{x_i x_{i+1} x_{i+2}, i \in [\ell]\}$. Note that the length of a path, a cycle or a walk is the number of its edges and we will use this convention for cycles, paths and walks in graphs as well.

So tight paths are hypergraphs with ordered vertex set and all consecutive triples appearing as edges. Since consecutive edges intersect in two vertices, these paths are called tight. One can also consider degree conditions for loose Hamiltonian cycles, looking at loose paths and cycles in which consecutive edges only intersect in one vertex. Loose Hamiltonian cycles were for instance studied in [1, 4, 7, 10].

From now on we only consider tight paths and cycles and consequently we may omit the addition of "tight".

In recent years there has been some progress to achieve Dirac-like results on hypergraphs. Rödl, Ruciński, and Szemerédi [15] started by showing that for given $\alpha > 0$, there is $n_0 \in \mathbb{N}$ such that every 3-graph on $n \geqslant n_0$ vertices with minimum pair-degree at least $(\frac{1}{2} + \alpha)n$ contains a Hamiltonian cycle. Actually, in [16] they improved the result to the following, which is optimal (as minimum pair-degree condition):

Theorem 1.2. *Let H be a 3-graph on n vertices, where n is sufficiently large. If H satisfies $\delta_2(H) \geqslant \lfloor n/2 \rfloor$, then H has a (tight) Hamiltonian cycle. Moreover, for every n there exists an n-vertex 3-graph H_n such that $\delta_2(H_n) = \lfloor n/2 \rfloor - 1$ and H_n does not have a (tight) Hamiltonian cycle.*

Some important ideas of the present work originate in the proof of an asymptotically optimal sufficient condition for the vertex degree of hypergraphs that Reiher, Rödl, Ruciński, Schacht, and Szemerédi achieved in [13]:

Theorem 1.3. *For every $\alpha > 0$, there exists an integer n_0 such that every 3-graph H with $n \geqslant n_0$ vertices and with minimum vertex degree $\delta(H) \geqslant \left(\frac{5}{9} + \alpha\right)\frac{n^2}{2}$ contains a (tight) Hamiltonian cycle.*

In this work, we study a new asymptotically optimal pair-degree condition that forces large 3-graphs to contain a Hamiltonian cycle. It would be very desirable to get a result for 3-graphs similar to the one by Chvátal: A condition on the matrix of pair-degrees $D(H) = (d(i,j))_{ij \in [n]^2}$ of a 3-graph $H = ([n], E)$ that forces H to contain a Hamiltonian cycle, while for each matrix $A \in M(n \times n, \mathbb{N})$ not satisfying that condition, there exists a 3-graph H such that $D(H) \geqslant A$ (pointwise) and H does not contain a Hamiltonian cycle. For the graph case, Pósa's result (Theorem 1.1) was a step towards the characterisation by Chvátal. In a sense, our main result can be seen as a 3-uniform (asymptotic) analogue of the theorem by Pósa.

Theorem 1.4 (Main result)**.** *For $\alpha > 0$, there exists an $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ the following holds: If $H = ([n], E)$ is a 3-graph with $d(i,j) \geqslant \min\left(\frac{i+j}{2}, \frac{n}{2}\right) + \alpha n$ for all $ij \in [n]^{(2)}$, then H contains a (tight) Hamiltonian cycle.*

This result strengthens the asymptotic version of Theorem 1.2 achieved in [15].

Let us remark that recently there have also been related results on degree sequences in graphs. Namely, Treglown [19] gave a degree sequence condition that forces the graph to contain a clique factor and Staden and Treglown [17] proved a degree sequence condition that forces the graph to contain the square of a Hamiltonian cycle.

Note that in the proof (and the proofs of the Lemmas) we can always assume $\alpha \ll 1$. Before we start with the outline of the proof of Theorem 1.4 in the next section, we give the following examples showing that our result is asymptotically optimal.

Example 1.5. *(i) Consider the partition $X \dot\cup Y = [n]$ with $X = \left[\left\lceil \frac{n+1}{3} \right\rceil\right]$ and let H be the hypergraph on $[n]$ containing all triples $e \in V^{(3)}$ such that $|e \cap X| \neq 2$.*

Then for $\gamma < 1/2$ and $\frac{1}{n} \ll \alpha \ll \gamma$, we have $d(i,j) \geqslant \min\left(\gamma\,(i+j)\,, \frac{n}{2}\right) + \alpha n$ for all $ij \in [n]^{(2)}$. On the other hand, if there was a Hamiltonian cycle C in H, it would *contain at least one edge with two vertices from X. But such an edge can only lie in a cycle in which all vertices are from $X \subsetneq [n]$. Hence, H does not contain a Hamiltonian cycle.*

(ii) Next, look at the partition $X \dot\cup Y = [n]$ with $X = \left[\left\lfloor \frac{n}{2} \right\rfloor\right]$ and let H be the hypergraph on $[n]$ containing all triples $e \in V^{(3)}$ such that $|e \cap Y| \neq 2$.

Then for all $ij \in [n]^{(2)}$, we have $d(i,j) \geqslant \frac{n}{2} - 2$. But an analogous argument as above shows that H does not contain a Hamiltonian cycle.

The two examples show that Theorem 1.4 does not hold when replacing the degree condition with $d(i,j) \geqslant \min\left(\gamma(i+j), \frac{n}{2}\right) + \alpha n$, where $\gamma < 1/2$, and neither when replacing it with $d(i,j) \geqslant \min\left(\frac{i+j}{2}, \gamma n\right) + \alpha n$, where again $\gamma < 1/2$.

Organisation. In the next section we give an overview over the proof, state the auxiliary results for each step and finally deduce the main result Theorem 1.4 from these. Sections 3-6 are devoted to the proofs of the auxiliary results. In the end, we collect some interesting related problems in Section 7.

§2. Overview and Final Proof

Our proof follows the strategy of Rödl, Ruciński, and Szemerédi in [15]. Since a Hamiltonian cycle is a substructure that includes every vertex of a hypergraph H, it seems suitable to attack the problem via the absorption method: We find a cycle that uses almost all vertices of H and show that we can "absorb" the remaining vertices, meaning we can integrate them into the large cycle. For that, we use that for every vertex $v \in V(H)$, there exist many absorbers in H, path-like structures into which we can insert v. Then, utilising the probabilistic method, we can construct an absorbing path, a path containing many absorbers for every vertex. Lastly, we build a long cycle containing this path and almost all vertices. Consequently, we can then absorb the small set of remaining vertices into that cycle and obtain a Hamiltonian cycle in H.

For these constructions we often need to connect two paths, that is, find a path between their end-pairs. Hence, we will begin by showing that we can connect every pair of pairs of vertices by a large number of paths with a fixed length.

Lemma 2.1 (Connecting Lemma)**.** *Let $\alpha > 0$. There exist $n_0, L \in \mathbb{N}, \vartheta > 0$ such that for all $n \geqslant n_0$ the following holds: If $H = ([n], E)$ is a 3-graph with*

$$d(i,j) \geqslant \min\left(\frac{i+j}{2}, \frac{n}{2}\right) + \alpha n$$

for all $ij \in [n]^{(2)}$, then for all disjoint ordered pairs of distinct vertices $(x,y), (w,z) \in [n]^2$, there exist at least ϑn^{L-2} many (x,y)-(w,z)-paths of length L in H.

See Section 3 for the proof of Lemma 2.1. Note that of course the conclusion still holds for a $\vartheta' \ll \vartheta, 1/L$ in place of ϑ. For simplicity, we might say "we choose $\vartheta \ll 1/L$ as in the Connecting Lemma" (or something similar) instead of introducing this new ϑ'.

Later, we will use this result whenever we need to connect different paths that have been constructed before. In fact, we will take a special selection of vertices - the reservoir - aside, with the property that for every pair of pairs of vertices, we still have many paths of fixed length connecting them, where all internal vertices of those paths are vertices of the reservoir. The existence of such a set will be shown by the probabilistic method.

Lemma 2.2 (Reservoir Lemma)**.** *For every $\alpha > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ the following holds: If $H = ([n], E)$ is a 3-graph satisfying*

$$d(i,j) \geqslant \min\left(\frac{i+j}{2}, \frac{n}{2}\right) + \alpha n$$

for all $ij \in [n]^{(2)}$ and $1/L \gg \vartheta$ are given by the Connecting Lemma (Lemma 2.1), then there exists a reservoir set $\mathcal{R} \subseteq [n]$ with $\frac{\vartheta^2}{2} n \leqslant |\mathcal{R}| \leqslant \vartheta^2 n$ such that for all disjoint ordered pairs of distinct vertices (x,y) and (w,z), there are at least $\vartheta \, |\mathcal{R}|^{L-2}/2$ (tight) (x,y)-(w,z)-paths of length L in H whose internal vertices all belong to $\mathcal{R}$.

We further show that removing a few vertices from the reservoir will not destroy its connectability property.

Lemma 2.3 (Preservation of the Reservoir)**.** *For every $\alpha > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ the following holds: If $H = ([n], E)$ is a 3-graph satisfying*

$$d(i,j) \geqslant \min\left(\frac{i+j}{2}, \frac{n}{2}\right) + \alpha n$$

for all $ij \in [n]^{(2)}$, and $1/L \gg \vartheta$ are given by the Connecting Lemma 2.1, and $\mathcal{R}$ is a reservoir set given by the previous Lemma 2.2, and $\mathcal{R}' \subseteq \mathcal{R}$ is an arbitrary subset of size at most $2\vartheta^4 n$, then for all disjoint ordered pairs of distinct vertices (x,y) and (w,z), there is a (tight) (x,y)-(w,z)-path of length L in H with all internal vertices belonging to $\mathcal{R} \setminus \mathcal{R}'$.

See Section 4 for the proof of Lemma 2.2 and Lemma 2.3.

The proof will continue with the definition of the absorbers and we will show that for each vertex there are many absorbers. We make use of this fact when we show that a small random selection of small paths still contains many absorbers for every $v \in V(H)$. With the Connecting Lemma we can afterwards connect all the small paths in that selection to a path that can absorb any small set of vertices.

Lemma 2.4 (Absorbing Path)**.** *For every $\alpha > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ the following holds: If $H = ([n], E)$ is a 3-graph satisfying*

$$d(i,j) \geqslant \min\left(\frac{i+j}{2}, \frac{n}{2}\right) + \alpha n$$

for all $ij \in [n]^{(2)}$, $\vartheta \ll \alpha$ is given by the Connecting Lemma 2.1 and $\mathcal{R}$ by the Reservoir Lemma 2.2, then there exists a path $P_A \subseteq H \setminus \mathcal{R}$ with $v(P_A) \leqslant \vartheta n$ and with the (absorbing) property that for each $X \subseteq [n]$ with $|X| \leqslant 2\vartheta^2 n$, there is a path with vertex set $X \cup V(P_A)$ and the same end-pairs as P_A.

See Section 5 for the proof of Lemma 2.4.

At last, we find a path in H containing almost all vertices using hypergraph regularity, similarly as Rödl, Ruciński, and Szemerédi did in [15], though we use a simpler version of regularity. We will regularise H and in that way reduce the problem to finding an almost perfect matching in a reduced hypergraph that almost obeys the same degree condition as H.

Proposition 2.5 (Long Path)**.** *For all $\alpha > 0$, there is an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, the following holds: Let $H = ([n], V)$ be a 3-graph with $d(i,j) \geqslant \min\left(\frac{i+j}{2}, \frac{n}{2}\right) + \alpha n$ for all $ij \in [n]^{(2)}$. Let $\vartheta \ll \alpha$ be given by the Connecting Lemma 2.1, let $\mathcal{R}$ be the reservoir from the Reservoir Lemma 2.2, and P_A the absorbing path from the Absorbing Path Lemma 2.4. Then there exists a path Q in H with $V(Q) \subseteq [n] \setminus P_A$ such that*

$$v(Q) \geqslant \left(1 - 2\vartheta^2\right) n - v\left(P_A\right)$$

and $|V(Q) \cap \mathcal{R}| \leqslant \vartheta^4 n$.

See Section 6 for the proof of Proposition 2.5.

Now we are ready to prove our main result, Theorem 1.4.

Proof of Theorem 1.4. Let $1 \gg \alpha$ be given and let L be as in the Connecting Lemma. Choose an integer $n \gg 1/\vartheta \gg L$ such that all the Lemmas and Propositions above hold. Now let $H = ([n], E)$ be a 3-graph satisfying the degree condition $d(i,j) \geqslant \min\left(\frac{i+j}{2}, \frac{n}{2}\right) + \alpha n$ for all $ij \in [n]^{(2)}$. Lemmas 2.2, 2.4, and Proposition 2.5 provide a reservoir $\mathcal{R}$, an absorbing path P_A and a long path Q, respectively. Let $(a,b), (c,d)$ be the end-pairs of P_A and let $(r,s), (t,u)$ be the end-pairs of Q (note that they are disjoint since we have $Q \subseteq H - P_A$). Since $|\mathcal{R} \cap V(Q)| \leqslant \vartheta^4 n$ and $V\left(P_A\right) \cap \mathcal{R} = \varnothing$, by Lemma 2.3 we can choose a path P_1 of length L connecting (t,u) and (a,b) with all internal vertices in $\mathcal{R} \setminus (V(Q) \cup V\left(P_A\right))$ and, since $\vartheta^4 n \gg L - 2$, we also find a path P_2 of length L connecting (c,d) and (r,s) with all internal vertices in $\mathcal{R} \setminus (V(Q) \cup V\left(P_A\right) \cup V\left(P_1\right))$. That leaves us with a cycle C in H

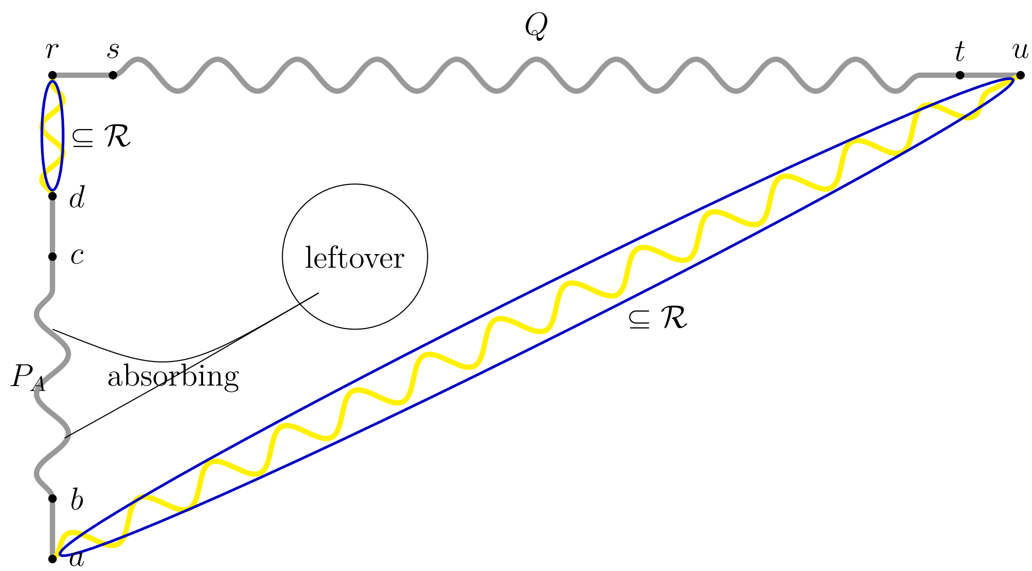

Figure 2.1. Overview of the proof

which satisfies $v(C) \geqslant (1-2\vartheta^2)\, n$ and $P_A \subseteq C$. The absorbing property of P_A guarantees that for $X := [n] \smallsetminus V(C)$, there exists a path P'_A with $V(P'_A) = V(P_A) \cup X$ that has the same end-pairs as P_A (which are connected to Q) and hence there is a Hamiltonian cycle in H. □

§3. Connecting Lemma

Before we start with the actual proof of Lemma 2.1, let us take a look at some other strategies. Say, we want to connect two pairs (x, y) and (w, z). One can easily reduce the case of both pairs being arbitrary to that of both having pair-degree of at least $\frac{n}{2} + \alpha n$ by "climbing up" in the degree sequence (see the beginning of the proof). For the results on the minimum pair-degree conditions for 3-graphs [15, 16] the Connecting Lemma was proved by a cascade-like method: In each step the "front" of reachable vertices with a good "backward connectivity" grows, one of those cascades growing from each of the two pairs that are to be connected. In the end one can find vertices v_1, v_2 in the last front of each starting pair, respectively, that have high "backward degree". Thus, utilising the minimum pair-degree condition for v_1v_2 gives us a neighbour of those v_1v_2 that is also a backward neighbour of v_1 or v_2 in their respective cascades, allowing us to find a path back to each (x, y) and (w, z).

However, with our pair-degree condition we cannot guarantee that the vertices v_1, v_2 with high backward degree in the cascade (which we might still find) have a high pair-degree, since v_1, v_2 may be small. On the other hand, it might be possible to connect (x, y) with some pairs (v_1, v_2) of low pair-degree that we can also connect with (w, z) if we construct the connecting walks more flexibly than with cascades. We therefore omit the approach via cascades and change our strategy to one similar to the one used in [13].

In the simplest form, the strategy goes as follows: We connect two pairs by considering $N((x, y), (w, z))$, the common neighbours of (x, y) and (w, z), which exist because of the high pair-degrees, and then find many connections between y and w in the common link graphs of those neighbours. In a second step, one can then insert the elements of $N((x, y), (w, z))$ at every third position, thereby obtaining a 3-uniform walk.

So we could connect two pairs if the link graphs of vertices in $N((x, y), (w, z))$ inherit the right degree condition, i.e., if the vertices are large (regarded as elements of $\mathbb{N}$).

However, we cannot control how large the elements of $N((x, y), (w, z))$ are. Therefore, the degree condition that the link graphs of vertices in $N((x, y), (w, z))$ inherit may not be strong enough to let us connect two vertices by "climbing up" the degree sequence. The idea to insert a middle pair ab, as done in [13], overcomes this problem. If ab has some large common neighbours with (x, y) and some with (w, z), then we can find some (x, y)-(w, z) walks passing through ab by applying the strategy explained above (now we can connect vertices in the link graphs by "climbing up" the degree sequence). The number of those walks will depend on the number of large common neighbours that ab has with each xy and wz. So roughly speaking, if the sum over all ab of large common neighbours of ab and xy and of ab and wz is large, we can indeed prove the Connecting Lemma 2.1. This last point (in its accurate form) will follow from the observation that each two link graphs have many common edges.

Note that this strategy can be used in the seemingly different settings of our pair-degree condition and the minimum vertex degree condition in [13], since in both cases we have "well connected" subgraphs in every link graph and each two of these subgraphs intersect in many edges: In [13] those subgraphs are the *robust subgraphs* and in our case we can just consider the complete link graphs.

Proof of Lemma 2.1. For $n \gg \frac{1}{\alpha} \gg 1$, let H be given as described, let $(x, y), (w, z) \in [n]^2$ be two disjoint ordered pairs of distinct vertices.

First we will show that it is possible to "climb up" along the degree sequence in (compared to n) few steps, starting from the pairs (x, y) and (w, z) and ending with pairs of vertices $\geqslant \frac{n}{2}$.

In the second step we will connect these two by utilising an analogous "climb up" argument in the link graphs of neighbours of a pair and slipping in an additional connective pair, similarly as in [13]. We first look for walks rather than paths and conclude by remarking that many of them will actually be paths.

First Step. By induction on $\ell \geqslant 3$, we will prove the following statement: There exist at least $\left(\frac{\alpha}{3}\right)^{\ell-2} n^{\ell-2}$ walks $x_1 = x, x_2 = y, x_3, \ldots, x_\ell$ such that for $i \geqslant 3$ we have:

$$x_i \geqslant \min\left(\frac{\alpha}{4}n(i-2), \frac{n}{2}\right) + \frac{\alpha}{4}n \tag{3.1}$$

We will first show the statement for $\ell = 3$ and $\ell = 4$ and then deduce it for any $\ell \geqslant 5$ given that it holds for $\ell - 1$.

$\ell = 3$: By the degree condition on H we have $d(x, y) \geqslant \min\left(\frac{1+2}{2}, \frac{n}{2}\right) + \alpha n$. Hence, there exist at least $\frac{\alpha}{3}n$ possible vertices x_3 such that x_1, x_2, x_3 build a walk and $x_3 \geqslant \frac{\alpha}{4}n + \frac{\alpha}{4}n$.

$\ell = 4$: Let x_1, x_2, x_3 be one of those $\left(\frac{\alpha}{3}\right) n$ walks satisfying the condition (3.1) that we get by the precious case. We then have $d(x_2, x_3) \geqslant \min\left(\frac{1+\frac{\alpha}{2}n}{2}, \frac{n}{2}\right) + \alpha n$ so there exist at least $\frac{\alpha}{3}n$ possible vertices x_4 such that x_1, x_2, x_3, x_4 build a walk and $x_i \geqslant \frac{\alpha}{4}n(i-2) + \frac{\alpha}{4}n$ for $i = 3, 4$.

$\ell \geqslant 5$: Let $x_1, x_2, x_3 \ldots x_{\ell-1}$ be one of the $\left(\frac{\alpha}{3}\right)^{\ell-3} n^{\ell-3}$ walks satisfying

$$x_i \geqslant \min\left(\frac{\alpha}{4}n(i-2), \frac{n}{2}\right) + \frac{\alpha}{4}n$$

for $i \geqslant 3$ that we get by induction. Then our pair-degree condition

$$d(x_{\ell-2}, x_{\ell-1}) \geqslant \min\left(\frac{2\ell-7}{2} \cdot \frac{\alpha}{4}n + \frac{\alpha}{4}n, \frac{n}{2}\right) + \alpha n \geqslant \min\left((\ell-2)\frac{\alpha}{4}n, \frac{n}{2}\right) + \frac{\alpha}{4}n + \frac{\alpha n}{2}$$

gives rise to at least $\frac{\alpha}{3}n$ possible vertices x_ℓ such that $x_1, x_2, \ldots x_\ell$ build a walk and we have $x_i \geqslant \min\left(\frac{\alpha}{4}n(i-2), \frac{n}{2}\right) + \frac{\alpha}{4}n$ for all $i \in [\ell], i \geqslant 3$.

This leaves us with $\left(\frac{\alpha}{3}\right)^{\lceil \frac{2}{\alpha} \rceil} n^{\lceil \frac{2}{\alpha} \rceil}$ possibilities for walks

$$x_1 = x, x_2 = y, x_3, \ldots, x_{\lceil \frac{2}{\alpha} \rceil + 2}$$

with $x_{\lceil \frac{2}{\alpha} \rceil + 1}, x_{\lceil \frac{2}{\alpha} \rceil + 2} \geqslant \frac{n}{2}$ and an analogous argument for (w, z) with just as many possibilities for walks

$$z_1 = z, z_2 = w, z_3, \ldots, z_{\lceil \frac{2}{\alpha} \rceil + 2}$$

with $z_{\lceil \frac{2}{\alpha} \rceil + 1}, z_{\lceil \frac{2}{\alpha} \rceil + 2} \geqslant \frac{n}{2}$.

Second Step. Let m be the smallest even number $\geqslant \frac{1}{\alpha} + 1$. It now suffices to show that for all ordered pairs $(x', y'), (w', z')$ of vertices, where the vertices within each pair are distinct and $x', y', w', z' \geqslant \frac{n}{2}$, there are at least $\vartheta' n^{3m+4}$ many (x', y')-(w', z') walks with $3m + 4$ internal vertices, where $\vartheta' \ll \alpha$ does not depend on n.

Since x', y' have at least $\frac{n}{2} + \alpha n$ neighbours, there exists

$$U_{x'y'} = \left\{u_1, \ldots, u_{\lfloor \alpha n \rfloor}\right\} \subseteq \left[\lceil n/2 \rceil\right]$$

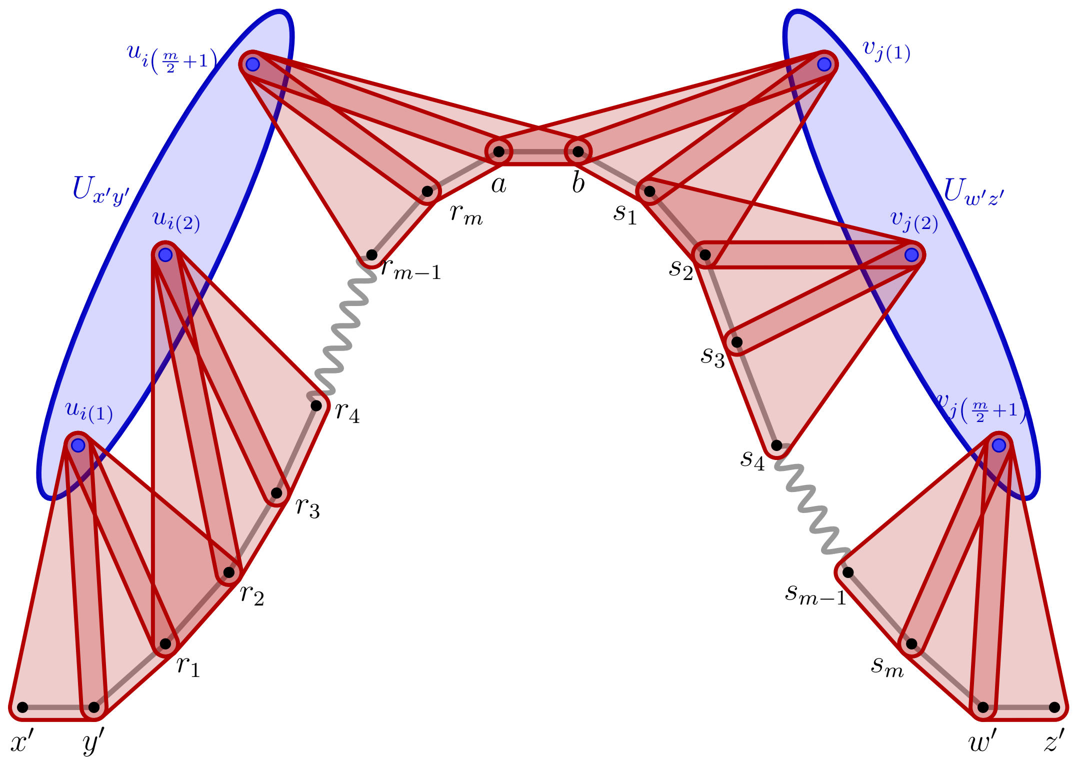

FIGURE 3.1. Idea of the second step, the picture is similar to [13, Fig. 4.1]

such that $x'y' \in E(L_{u_i})$ for all $i \in \left[\lfloor \alpha n \rfloor\right]$ (recall that L_{u_i} denotes the link graph of u_i). Similarly, there exists

$$U_{w'z'} = \left\{v_1, \ldots, v_{\lfloor \alpha n \rfloor}\right\} \subseteq \left[\lceil n/2 \rceil\right]$$

such that $w'z' \in E(L_{v_i})$ for all $i \in \left[\lfloor \alpha n \rfloor\right]$.

For $(a,b) \in [n]^2$ let $I_{ab} = \{i \in [\lfloor \alpha n \rfloor] : ab \in E(L_{u_i}) \cap E(L_{v_i})\}$. Since all vertices $\geqslant \frac{n}{2}$ (apart from u_i, v_i) have in both L_{u_i} and L_{v_i} at least $\frac{n}{2} + \alpha n$ many neighbours, and therefore $2\alpha n$ vertices that they are adjacent to in both L_{u_i} and L_{v_i}, there are at least $\frac{1}{2} 2\alpha n \left(\frac{n}{2} - 3\right) \geqslant \frac{\alpha n^2}{4}$ common edges of L_{v_i} and L_{u_i}. Thus, by double counting we have

$$\sum_{(a,b)\in[n]^2} |I_{ab}| \geqslant \sum_{i \in [\lfloor \alpha n \rfloor]} |E(L_{v_i}) \cap E(L_{u_i})| \geqslant \frac{\alpha n^2}{4} \lfloor \alpha n \rfloor .$$

Next, for fixed $(a,b) \in [n]^2$ we find a lower bound for the number L_{ab} of 3-uniform walks of the form

$$x'y'u_{i(1)}r_1r_2u_{i(2)} \ldots u_{i\left(\frac{m}{2}\right)}r_{m-1}r_m u_{i\left(\frac{m}{2}+1\right)}ab$$

where $y'r_1r_2 \ldots r_{m-1}r_m a$ is a 2-uniform walk in $L_{u_{i(k)}}$ and $i(k) \in I_{ab}$ for all $k \in \left[\frac{m}{2} + 1\right]$.

To this goal, first observe that for all $i \in \left[\lfloor \alpha n \rfloor\right]$ there are $\left(\frac{\alpha}{3}\right)^m n^m$ many $y'a$-walks of length $m+1$ in L_{u_i}: In L_{u_i} any vertex j has degree $\geqslant \min\left(j, \frac{n}{2}\right) + \alpha n$ because $u_i \geqslant n/2$. Therefore, there are at least $\left(\left\lfloor \frac{\alpha n}{2} \right\rfloor\right)^{m-1}$ many walks of length $m-1$ starting in a, in which each vertex is either $\geqslant \frac{n}{2} + \frac{\alpha n}{2}$ or at least $\frac{\alpha n}{2}$ larger than the preceding vertex. Since we set $m \geqslant 1/\alpha + 1$, each of these walks ends in a vertex $\geqslant \frac{n}{2}$ and for at least $\left(\frac{\alpha n}{3}\right)^{m-1}$ of them the ending vertex is not y'. For each such walk T with its last vertex $a'_T \neq y'$ there are $2\alpha n$ possibilities for common neighbours of y' and a'_T (note that the degrees in L_{u_i} of both y' and a'_T are at least $\frac{n}{2} + \alpha n$). In total, that gives us at least $\left(\frac{\alpha n}{3}\right)^m$ many $y'a$-walks of length $m+1$ in L_{u_i}.

Now, for $\vec{r} \in [n]^m$ we set $D_{ab}(\vec{r}) := \{i \in I_{ab} : y'\vec{r}a \text{ is a walk in } L_{u_i}\}$.

Again we get by double counting that

$$\sum_{\vec{r}\in[n]^m} |D_{ab}(\vec{r})| = \sum_{i\in I_{ab}} \left|\left\{\vec{r} \in [n]^m : y'\vec{r}a \text{ is a walk in } L_{u(i)}\right\}\right| \geqslant |I_{ab}| \left(\frac{\alpha}{3}\right)^m n^m.$$

For each $\vec{r} \in [n]^m$ we have that $x'y'u_{i(1)}r_1r_2u_{i(2)}\dots u_{i\left(\frac{m}{2}\right)}r_{m-1}r_mu_{i\left(\frac{m}{2}+1\right)}ab$, with $\vec{r}$ being a $y'a$ walk in $L_{u_{i(k)}}$ for every $k \in \left[\frac{m}{2}+1\right]$, is a 3-uniform $(x'y')$-(ab)-walk of length $m + \frac{m}{2} + 3$ in H. Hence, by Jensen's inequality we derive:

$$L_{ab} \geqslant \sum_{\vec{r}\in[n]^m} |D_{ab}(\vec{r})|^{\frac{m}{2}+1} \geqslant n^m \left(\sum \frac{1}{n^m} |D_{ab}(\vec{r})|\right)^{\frac{m}{2}+1} \geqslant n^m \left(|I_{ab}| \left(\frac{\alpha}{3}\right)^m\right)^{\frac{m}{2}+1}.$$

We define R_{ab} analogously as the number of 3-uniform walks of the form

$$abv_{j(1)}s_1s_2v_{j(2)}\dots v_{j\left(\frac{m}{2}\right)}s_{m-1}s_mv_{j\left(\frac{m}{2}+1\right)}w'z'$$

where $bs_1s_2\dots s_{m-1}s_mw'$ is a 2-uniform walk in $L_{v_{j(k)}}$ and $j(k) \in I_{ab}$ for all $k \in \left[\frac{m}{2}+1\right]$ and get the same lower bound by an analogous argument.

At last, let W be the number of $(x'y')$-$(w'z')$-walks of length $3m+6$ in H. We apply Jensen's inequality a second time to obtain:

$$\begin{aligned}
W \geqslant & \sum_{(a,b)\in[n]^2} L_{ab}R_{ab} \\
\geqslant & n^{2m} \left(\frac{\alpha}{3}\right)^{m^2+2m} \sum_{(a,b)\in[n]^2} |I_{ab}|^{m+2} \\
\geqslant & n^{2m} \left(\frac{\alpha}{3}\right)^{m^2+2m} n^2 \left(\frac{1}{n^2}\frac{\alpha n^2}{4}\lfloor \alpha n \rfloor\right)^{m+2} \\
\geqslant & \left(\frac{\alpha}{3}\right)^{m^2+2m} \left(\frac{\alpha^2}{5}\right)^{m+2} n^{3m+4} \\
\geqslant & \left(\frac{\alpha^2}{5}\right)^{m^2+3m+2} n^{3m+4}.
\end{aligned}$$

In total, putting together the walks connecting (x,y) and (x',y'), (x',y') and (w',z') and (w',z') and (w,z) we have

$$\left(\left(\frac{\alpha}{3}\right)^{\left\lceil\frac{2}{\alpha}\right\rceil} n^{\left\lceil\frac{2}{\alpha}\right\rceil}\right)^2 \times \left(\left(\frac{\alpha^2}{5}\right)^{m^2+3m+2} n^{3m+4}\right)$$

many possibilities for (x,y)-(w,z) walks of length $2\cdot\left\lceil\frac{2}{\alpha}\right\rceil + 3m + 6$ in H.

Only $\mathcal{O}\left(n^{2\left\lceil\frac{\alpha}{2}\right\rceil+3m+3}\right)$ of these fail to be a path. Therefore, recalling $1 \gg \alpha \gg \frac{1}{n}$, we have at least $\left(\frac{\alpha^2}{9}\right)^{\left\lceil\frac{2}{\alpha}\right\rceil+m^2+3m+2} n^{2\left\lceil\frac{\alpha}{2}\right\rceil+3m+4}$ many (x,y)-(w,z) paths of length $2\cdot\left\lceil\frac{2}{\alpha}\right\rceil + 3m + 6$ in H. □

§4. Reservoir

In this section we will prove the existence of a small set, the reservoir, such that any two pairs of vertices can be connected by paths with all internal vertices lying in the reservoir. The probabilistic proof of this Lemma as done in [13] works in almost the same way with different conditions as soon as the Connecting Lemma is provided. We will state two important inequalities first that we will need for the probabilistic method.

Lemma 4.1 (Chernoff, see for instance Cor. 2.3 in [8])**.** *Let $X_1, X_2, \dots, X_m$ be a sequence of m independent random variables $X_i :\to \{0,1\}$ with $\mathbb{P}(X_i = 1) = p$ and $\mathbb{P}(X_i = 0) = 1 - p$. Then we have for $\delta \in (0,1)$:*

- $\mathbb{P}\left(\sum_{i\in[m]} X_i \geqslant (1+\delta)\,pm\right) \leqslant \exp\left(-\frac{\delta^2}{3}pm\right)$
- $\mathbb{P}\left(\sum_{i\in[m]} X_i \leqslant (1-\delta)\,pm\right) \leqslant \exp\left(-\frac{\delta^2}{2}pm\right)$

Lemma 4.2 (Azuma-Hoeffding, McDiarmid, Cor. 2.27 in [8] and Thm. 1 in [11])**.** *Suppose that $X_1, \dots, X_m$ are independent random variables taking values in $\Lambda_1, \dots, \Lambda_m$ and let $f : \Lambda_1 \times \dots \times \Lambda_m \to \mathbb{R}$ be a measurable function. Moreover, suppose that for certain real numbers $c_1, \dots, c_m \geqslant 0$, we have that if $J, J' \in \prod \Lambda_i$ differ only in the k-th coordinate, then $|f(J) - f(J')| \leqslant c_k$. Then the random variable $X := f(X_1, \dots, X_m)$ satisfies*

$$\mathbb{P}(|X - \mathbb{E}(X)| \geqslant t) \leqslant 2\exp\left(-\frac{2t^2}{\sum c_i^2}\right)$$

We are now ready to prove Lemma 2.2.

Proof of Lemma 2.2. Let α, L, ϑ be given as in the statement. Then we choose $n \in \mathbb{N}$ such that we have the following hierarchy: $1 \gg \alpha, \frac{1}{L} \gg \vartheta \gg 1/n$. Let H be given as described in the Lemma. We choose a random subset $\mathcal{R} \subseteq [n]$, where we select each vertex independently with probability $p = \left(1 - \frac{1}{10L}\right)\vartheta^2$. Since $|\mathcal{R}|$ is now binomially distributed, we can apply Chernoff's inequality (Lemma 4.1) and utilise the hierarchy to obtain

$$\mathbb{P}\left(|\mathcal{R}| < \vartheta^2 n/2\right) \leqslant \mathbb{P}\left(|\mathcal{R}| < \frac{2}{3}\mathbb{E}\left(\mathcal{R}\right)\right) \leqslant \exp\left(-\frac{(1/3)^2}{2}pn\right) < 1/3. \tag{4.1}$$

We also have $\vartheta^2 n \geqslant (1 + c(L))\,\mathbb{E}\left(|\mathcal{R}|\right)$ for some small $c(L) \in (0,1)$ not depending on n and therefore, again by Chernoff (Lemma 4.1) we get for large n:

$$\mathbb{P}\left(|\mathcal{R}| > \vartheta^2 n\right) \leqslant \mathbb{P}\left(|\mathcal{R}| \geqslant (1 + c(L))\,\mathbb{E}\left(\mathcal{R}\right)\right) \leqslant \exp\left(-\frac{c(L)^2}{3}pn\right) < 1/3 \tag{4.2}$$

By the Connecting Lemma 2.1 we have that for all disjoint ordered pairs of distinct vertices (x, y) and (w, z), there exist ϑn^{L-2} many (x, y)-(w, z)-paths of length L in H. Let $X = X\left((x, y), (w, z)\right)$ denote the random variable counting the number of those (x, y)-(w, z)-paths in H that are of length L and have all internal vertices in $\mathcal{R}$. We then have $\mathbb{E}(X) \geqslant p^{L-2}\vartheta n^{L-2}$.

Now we apply the Azuma-Hoeffding inequality (Lemma 4.2) (with $X_1, \ldots, X_n$ being the indicator variables for the events "$1 \in \mathcal{R}$",... ,"$n \in \mathcal{R}$") which gives us, since the presence or absence of one particular vertex in $\mathcal{R}$ affects X by at most $(L-2)\,n^{L-3}$, that

$$\begin{aligned}\mathbb{P}\left(X \leqslant \frac{2}{3}\vartheta(pn)^{L-2}\right) \leqslant & \mathbb{P}\left(X \leqslant \frac{2}{3}\mathbb{E}(X)\right) \\ \leqslant & 2\exp\left(-\frac{2\left(p^{L-2}\vartheta n^{L-2}\right)^2}{9n\left((L-2)n^{L-3}\right)^2}\right) \\ = & \exp\left(-\Omega(n)\right).\end{aligned}$$

By the union bound, also the probability, that for one of the at most n^4 pairs of pairs that we have to consider the respective number of connecting paths with all internal vertices in $\mathcal{R}$ is less than $\frac{2}{3}\vartheta(pn)^{L-2}$, can be bounded from above by

$$\exp\left(-\Omega(n)\right) \times n^4 < 1/3 \tag{4.3}$$

for n large. Moreover, recalling our hierarchy we have

$$\frac{2}{3}\vartheta p^{L-2} n^{L-2} = \left(1 - \frac{1}{10L}\right)^{L-2} \frac{2}{3}\vartheta\left(\vartheta^2 n\right)^{L-2} \geqslant \frac{\vartheta}{2}\left(\vartheta^2 n\right)^{L-2}$$

which together with (4.2) and (4.3) implies the following: With probability $> 1/3$ the chosen set $\mathcal{R}$ satisfies $|\mathcal{R}| \leqslant \vartheta^2 n$ and has the property that for all disjoint ordered pairs of distinct vertices (x, y) and (w, z) there exist at least $\frac{\vartheta}{2}\left|\mathcal{R}\right|^{L-2}$ paths of length L in H that connect those pairs and have all their internal vertices in $\mathcal{R}$. Therefore, combining this with (4.1) ensures that there indeed exists a version of $\mathcal{R}$ that has all the required properties of our reservoir set. $\square$

It is not hard now to show the preservation of the reservoir, Lemma 2.3.

Proof of Lemma 2.3. Let α, L, ϑ be as in the statement. Choose $n \in \mathbb{N}$ such that we have the hierarchy $1 \gg \alpha, \frac{1}{L} \gg \vartheta \gg \frac{1}{n}$. Let $H, \mathcal{R}, \mathcal{R}'$ be as in the statement of the Lemma. Consider any two disjoint ordered pairs of distinct vertices (x, y) and (w, z). We have

$$|\mathcal{R}'| \leqslant 2\vartheta^4 n \leqslant \vartheta^{3/2}\frac{\vartheta^2}{2}n \leqslant \vartheta^{3/2}\,|\mathcal{R}|$$

by the lower bound we get from Lemma 2.2. Since every particular vertex in $\mathcal{R}'$ is an internal vertex of at most $(L-2)|\mathcal{R}|^{L-3}$ many (x, y)-(w, z)-paths of length L in H with all internal vertices from $\mathcal{R}$, the Reservoir Lemma tells us that there are at least

$$\frac{\vartheta}{2}|\mathcal{R}|^{L-2} - |\mathcal{R}'|\,(L-2)|\mathcal{R}|^{L-3} \geqslant \frac{\vartheta}{2}|\mathcal{R}|^{L-2} - \vartheta^{3/2}(L-2)|\mathcal{R}|^{L-2} > 0$$

such (x, y)-(w, z)-paths with all internal vertices in $\mathcal{R} \smallsetminus \mathcal{R}'$. □

§5. Absorbing Path

In this section we will construct a short absorbing path P_A that can "absorb" every small set of arbitrary vertices: For each such set, we can build a path P containing each vertex in the set and all the vertices from the path P_A while keeping the same end-pairs as P_A. Later, it will then suffice to find a cycle containing P_A and almost all vertices and then absorb the remaining vertices into P_A. Since we already have a Connecting Lemma, actually the only step left will be to find a long path. So this approach allows us to reduce the problem of finding a certain subgraph containing *all* vertices (in this case a Hamiltonian cycle) to the easier problem of finding a subgraph (in our case a cycle or actually a path) that contains only *almost all* vertices - this is the core idea of the absorption method.

In order to construct such an absorbing path, one first has to find many *absorbers* for each vertex v: In our case, those are two small paths that allow us to build two new small paths with the same end-pairs, containing all vertices of the first two paths and in addition the "absorbed" vertex v. This makes sure that we can maintain the path structure when absorbing single vertices, since the linking pairs remain unchanged. Once we know that for every vertex v, there exist many such v-absorbers in H, the probabilistic method provides a small set of disjoint paths with the property that for every vertex v, this set contains many v-absorbers. We simply connect all these paths via the Connecting Lemma and then we can absorb a small set of vertices by greedily inserting each vertex into a different absorber.

The difficult part is to find the right structure for those absorbers, namely one that ensures the existence of many absorbers in H. Here we can use an idea similar to one in the proof of a minimum vertex degree condition in [13]: It seems difficult to directly

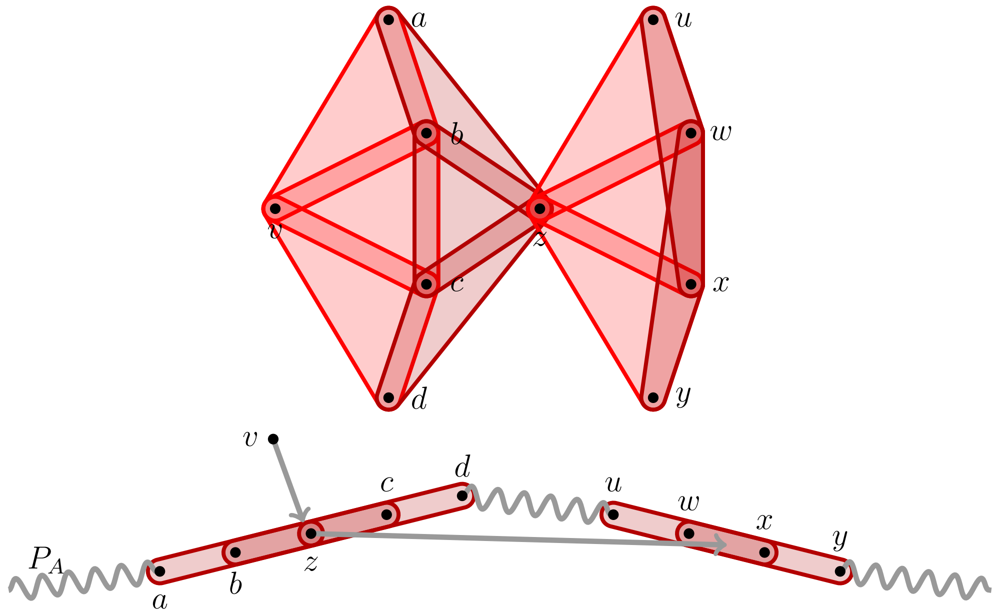

FIGURE 5.1. Structure of the absorbers with hyperedges used before absorption of v in dark red and hyperedges used after absorption of v in light red; the pictures are similar to [13, Fig. 6.1]

absorb an arbitrary vertex into a path with four vertices, as it would be possible with a minimum pair-degree of $(\frac{1}{2} + \alpha)n$. However, for certain vertices, namely large ones, we can do just this. We can thus absorb an arbitrary vertex by inserting it into a short path at the position of a large vertex (here we utilize the concept of link graphs once again). In a second step, we then easily absorb that large vertex.

Definition 5.1. *Let $H = ([n], E)$. For $v \in [n]$, a 9-tuple $(a, b, c, d, z, u, w, x, y) \in [n]^9$ of distinct vertices is called v-absorber (in H) if*

$$vab, vbc, vcd, abz, bcz, cdz, zuw, zwx, zxy, uwx, wxy \in E$$

Note that an absorber can be seen as two paths in one of which v can take the place of z and z can be inserted into the second (both without changing the end-pairs), see also Figure 5.1.

Lemma 5.2 (Many Absorbers)**.** *For every $\alpha > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ the following holds: If $H = ([n], E)$ is a 3-graph that satisfies*

$$d(i, j) \geqslant \min\left(\frac{i+j}{2}, \frac{n}{2}\right) + \alpha n$$

for all $ij \in [n]^{(2)}$ and $\mathcal{R}$ is a reservoir set given by the Reservoir Lemma 2.2, then for every $v \in [n]$, there exist at least $\alpha^{14} n^9$ many v-absorbers in $([n] \setminus \mathcal{R})^9$.

Proof of Lemma 5.2. Let $n \gg \frac{1}{\vartheta} \gg \frac{1}{\alpha} \gg 1$, where ϑ is as in the Reservoir Lemma. Let H be as in the statement, $v \in [n]$. There are at least $\frac{\alpha n}{2}$ possibilities to choose a vertex $z \geqslant n - \alpha n$ with $z \neq v$. Also, there are at least $\frac{\alpha n}{2}$ many vertices $\geqslant n - \alpha n$ that are neither v nor z and those have a degree of at least $\frac{n}{2} + \frac{\alpha}{2} n$ in the link graph of v and in the link graph of z. Hence, L_v and L_z share at least $\frac{\alpha^2 n^2}{4}$ edges. The following claim ensures many short walks in $L_v \cap L_z$.

Claim 5.3. *If two graphs $G_1 = (V, E_1)$, $G_2 = (V, E_2)$ share at least $\frac{\alpha^2 n^2}{4}$ edges, that is, $|E_1 \cap E_2| \geqslant \frac{\alpha^2 n^2}{4}$, then there are at least $\left(\frac{\alpha^2}{8}\right)^4 n^4$ quadruples $(a, b, c, d) \in V^4$ that form a walk of length four in $G_1 \cap G_2$.*

Proof. Start with $F_1 := G_1 \cap G_2$ and for $i \geqslant 2$, let F_i be the graph on vertex set V obtained from F_{i-1} by deleting all edges incident to those vertices $s \in V$ with $d_{F_{i-1}}(s) < \frac{\alpha^2}{8} n$. This procedure ends with some F_j in which for every vertex t with $d_{F_j}(t) > 0$, we actually have $d_{F_j}(t) \geqslant \frac{\alpha^2}{8} n$. Since $e(F_j) > e(F_1) - n\frac{\alpha^2 n}{8} \geqslant \frac{\alpha^2 n^2}{8}$ and noting that all neighbours (in F_j) of a vertex t with $d_{F_j}(t) \geqslant \frac{\alpha^2}{8} n$ also have degree $\geqslant \frac{\alpha^2}{8} n$ in F_j, there are indeed at least $\frac{\alpha^2}{8} n$ vertices in F_j that have degree > 0, therefore $\geqslant \frac{\alpha^2}{8} n$. Now, just choose an arbitrary vertex a with $d_{F_j}(a) \geqslant \frac{\alpha^2}{8} n$ and then $b \in N_{F_j}(a), c \in N_{F_j}(b)$ (not necessarily different from a), $d \in N_{F_j}(c)$ (not necessarily different from a or b). That gives us a walk of length four in $G_1 \cap G_2$ and since for each of the choices there are at least $\frac{\alpha^2}{8} n$ possibilities this proves the claim. $\square$

Thus, we have at least $\left(\frac{\alpha^2}{8}\right)^4 n^4$ quadruples $(a, b, c, d) \in [n]^4$ that form a walk of length 4 in $L_v \cap L_z$. There are at least $\frac{\alpha n}{2}$ possible choices for $w \neq z$ as a vertex $\geqslant n - \alpha n$. Then $d(w, z) \geqslant \frac{n}{2} + \alpha n$ implies that there are $\geqslant \lfloor \alpha n \rfloor$ choices for x such that $zwx \in E$ and $x \geqslant \frac{n}{2}$. So we have $d(z, w) \geqslant \frac{n}{2} + \alpha n$, $d(z, x) \geqslant \frac{n}{2} + \alpha n$ and $d(w, x) \geqslant \frac{n}{2} + \alpha n$ which means there exist at least $2\alpha n$ vertices $u \in N(z, w) \cap N(w, x)$ and there are at least $2\alpha n$ vertices $y \in N(z, x) \cap N(w, x)$.

In total, there are at least $\lfloor \alpha n \rfloor^5 \cdot \frac{\alpha^8}{2^{12}} n^4 \geqslant \frac{\alpha^{13}}{2^{17}} n^9$ choices for $(abcdzuwxy) \in V^9$ such that

$$vab, vbc, vcd, abz, bcz, cdz, zuw, zwx, zxy, uwx, wxy \in E.$$

We only have $\mathcal{O}(n^8)$ 9-tuples in which not all vertices are distinct and at most $9\vartheta^2 n^9$ 9-tuples contain a vertex from $\mathcal{R}$. Thus, with $1 \gg \alpha \gg \vartheta \gg \frac{1}{n}$ there are at least $\alpha^{14} n^9$ v-absorbers $(a, b, c, d, z, u, w, x, y) \in ([n] \smallsetminus \mathcal{R})^9$. $\square$

We are now ready to prove Lemma 2.4.

Proof of Lemma 2.4. The proof proceeds in two steps. First, we will use the probabilistic method, showing that a randomly chosen set of 9-tuples contains with probability > 0

many absorbers, while being not too large and each tuple consisting (just as the absorber) of two paths. In the second part we connect all those paths using the Connecting Lemma.

Let α, ϑ with $\vartheta \ll \alpha$ be given as in the statement and let $L \ll 1/\vartheta$ be given by the Connecting Lemma. Choose $n \gg \frac{1}{\vartheta}$ and let $H, \mathcal{R}$ be given as in the statement.
Select each 9-tuple from $([n] \setminus \mathcal{R})^9$ independently with probability $p := \frac{4\vartheta^2}{\alpha^{14} n^8}$ to be in a random selection $\mathcal{X} \subseteq ([n] \setminus \mathcal{R})^9$. Then $\mathbb{E}[|\mathcal{X}|] \leqslant pn^9 = \frac{4\vartheta^2}{\alpha^{14}} n$ and by Markov's inequality we get

$$\mathbb{P}\left(|\mathcal{X}| > 2\frac{4\vartheta^2}{\alpha^{14}} n\right) \leqslant \frac{1}{2}. \tag{5.1}$$

Calling two distinct 9-tuples overlapping if they contain a common vertex, we observe that there are at most $81n^{17}$ ordered pairs of overlapping 9-tuples. Let us denote the number of overlapping pairs with both of their tuples occurring in $\mathcal{X}$ by D. We then get $\mathbb{E}[D] \leqslant 81 n^{17} p^2 = 81 \left(\frac{4\vartheta^2}{\alpha^{14}}\right)^2 n$ and Markov yields

$$\mathbb{P}\left[D > \vartheta^2 n\right] \leqslant \mathbb{P}\left[D > 324 \left(\frac{4\vartheta^2}{\alpha^{14}}\right)^2 n\right] \leqslant \frac{1}{4} \tag{5.2}$$

since $\alpha \gg \vartheta \gg 1/n$.

Next, we focus on the number of absorbers contained in $\mathcal{X}$. For $v \in [n]$ let A_v denote the set of all v-absorbers. Lemma 5.2 gives us for every $v \in [n]$ that

$$\mathbb{E}[|A_v \cap \mathcal{X}|] \geqslant \alpha^{14} n^9 p = 4\vartheta^2 n.$$

Since $|A_v \cap \mathcal{X}|$ is binomially distributed, we may apply Chernoff's inequality to get for all $v \in [n]$:

$$\mathbb{P}\left(|A_v \cap \mathcal{X}| \leqslant 3\vartheta^2 n\right) \leqslant \exp\left(-\frac{\left(\frac{1}{4}\right)^2}{2} 4\vartheta^2 n\right) < \frac{1}{5n} \tag{5.3}$$

Hence, by (5.1), (5.2) and (5.3) there exists a selection $\mathcal{F}_*$ of 9-tuples from $([n] \setminus \mathcal{R})^9$ with:

- $|\mathcal{F}_*| \leqslant \frac{8\vartheta^2}{\alpha^{14}} n$
- $\mathcal{F}_*$ contains at most $\vartheta^2 n$ overlapping pairs
- $\mathcal{F}_*$ contains at least $3\vartheta^2 n$ many v-absorbers for all $v \in [n]$

For each overlapping pair, we delete one of its 9-tuples and thus lose at most $\vartheta^2 n$ many absorbers. Furthermore, we delete all those 9-tuples $(a, b, c, d, z, u, w, x, y) \in \mathcal{F}_*$ for which we do not have $abz, bzc, zcd, uwx, wxy \in E$ and those whose vertices are not all distinct. Note that we do not lose any absorbers by the last two steps since absorbers satisfy both conditions. This deletion process gives rise to an $\mathcal{F} \subseteq ([n] \setminus \mathcal{R})^9$ satisfying:

- $|\mathcal{F}| \leqslant \frac{8\vartheta^2}{\alpha^{14}} n$

- all 9-tuples in $\mathcal{F}$ are pairwise disjoint and all the vertices in each 9-tuple are distinct
- if $(a, b, c, d, z, u, w, x, y) \in \mathcal{F}$, then $abz, bzc, zcd, uwx, wxy \in E$
- for every $v \in [n]$, there are at least $2\vartheta^2 n$ many v-absorbers in $\mathcal{F}$

Next, we want to connect the elements in $\mathcal{F}$ to a path utilising the Connecting Lemma 2.1. Let $\mathcal{G}$ be the set consisting of all the paths $abzcd$ and $uwxy$ for each $(abcdzuwxy) \in \mathcal{F}$:

$$\mathcal{G} = \bigcup_{(abcdzuwxy)\in\mathcal{F}} \{abzcd, uwxy\}$$

We then have $|\mathcal{G}| = 2|\mathcal{F}| \leqslant \frac{16\vartheta^2}{\alpha^{14}} n$. Let $\mathcal{G}^* \subseteq \mathcal{G}$ be a maximal subset such that there exists a path $P^* \subseteq H - \mathcal{R}$ with:

- P^* contains all paths in $\mathcal{G}^*$ as subpaths
- $V(P^*) \cap \bigcup_{P\in\mathcal{G}\smallsetminus\mathcal{G}^*} V(P) = \varnothing$
- P^* satisfies $v(P^*) \leqslant (L - 2 + 5)\,|\mathcal{G}^*| + 2$.

First assume $\mathcal{G}^* \subsetneq \mathcal{G}$, and let $Q^* \in \mathcal{G} \smallsetminus \mathcal{G}^*$. Notice that by $1 \gg \alpha, \frac{1}{L} \gg \vartheta \gg \frac{1}{n}$ we have

$$v(P^*) + \left|\bigcup_{P\in\mathcal{G}\smallsetminus\mathcal{G}^*} V(P)\right| + |\mathcal{R}| \leqslant 2 + (L+3)\,\frac{16\vartheta^2}{\alpha^{14}} n + \vartheta^2 n \leqslant 2\vartheta^{3/2} n \leqslant \frac{\vartheta n}{2\,(L-2)}. \tag{5.4}$$

The Connecting Lemma 2.1 now tells us that there are at least ϑn^{L-2} paths of length L connecting an end-pair (x, y) of P^* with an end-pair (w, z) of Q^* (which are disjoint by the choice of P^*). By (5.4) at least half of those are disjoint to $\mathcal{R} \cup \bigcup_{P\in\mathcal{G}\smallsetminus\mathcal{G}^*\smallsetminus\{Q^*\}} V(P)$ and (apart from the end-pairs) disjoint to $V(P^*)$ and $V(Q^*)$. Hence, there exists a path P^{**}, starting with P^* and ending with Q^* whose vertex set is disjoint to $\bigcup_{P\in\mathcal{G}\smallsetminus\mathcal{G}^*\smallsetminus\{Q^*\}} V(P)$ and for which we further have

$$v(P^{**}) = v(P^*) + L - 2 + v(Q^*) \leqslant 2 + (L - 2 + 5)\,|\mathcal{G}^* \cup \{Q^*\}|\,.$$

Therefore $\mathcal{G}^* \cup \{Q^*\}$ contradicts the maximality of $\mathcal{G}^*$.

Thus $\mathcal{G}^* = \mathcal{G}$. Also, for $P_A := P^*$, the hierarchy $1 \gg \alpha, \frac{1}{L} \gg \vartheta \gg \frac{1}{n}$ gives us the required bound on $v(P_A)$:

$$v(P_A) \leqslant 2 + (L - 2 + 5)\frac{16\vartheta^2}{\alpha^{14}} n \leqslant \vartheta n.$$

And lastly, the structure and the number of the absorbers in P_A ensures the absorbing property: Let $X \subseteq [n]$ with $|X| \leqslant 2\vartheta^2 n$. For each $v \in X$, we can choose one v-absorber $(a, b, c, d, z, u, w, x, y)$ from $\mathcal{F}$ such that all chosen absorbers are distinct, since for every $v \in V$, there are at least $2\vartheta^2 n$ many v-absorbers appearing as two subpaths in P_A. For every $v \in X$, we then replace the two subpaths $abzcd$ and $uwxy$ of P_A with the two new paths $abvcd$, $uwzxy$ (those are paths because of the edge requirements on v-absorbers) that have the same end-pairs as the former two paths. That leaves us with a path P' that satisfies $V(P') = V(P_A) \cup X$ and has the same end-pairs as P_A. □

§6. Long Path

In this section we will prove the existence of a path that contains almost all vertices. To do so, we will need a weak form of the hypergraph regularity method which we will therefore introduce briefly.

Let $H = (V, E)$ be a 3-graph and $V_1, V_2, V_3 \subseteq V$; we write

$$E(V_1, V_2, V_3) = \{(v_1, v_2, v_3) \in V_1 \times V_2 \times V_3 : v_1 v_2 v_3 \in E\}$$

and $e(V_1, V_2, V_3) = |E(V_1, V_2, V_3)|$. Further, we write

$$H(V_1, V_2, V_3) = (V_1 \dot\cup V_2 \dot\cup V_3, E(V_1, V_2, V_3)).$$

A central definition for the regularity method is that of quasirandomness: The property of three partition classes having roughly the same edge density on each selection of subsets of the partition classes: For $\delta > 0, d \geqslant 0$ and $V_1, V_2, V_3 \subseteq V$, we say that $H(V_1, V_2, V_3)$ is *weakly* (δ, d)*-quasirandom* if for all $U_1 \subseteq V_1, U_2 \subseteq V_2, U_3 \subseteq V_3$, we have that

$$|e(U_1, U_2, U_3) - d|U_1||U_2||U_3|| \leqslant \delta |V_1||V_2||V_3|.$$

We say that $H(V_1, V_2, V_3)$ is weakly δ-quasirandom if it is weakly (δ, d)-quasirandom for some $d \geqslant 0$. For brevity, we might also say that V_1, V_2, V_3 are weakly (δ, d)-quasirandom (or δ-quasirandom) (in H). Lastly, since we only look at weak quasirandomness in this section, we may omit the addition of "weakly".

The regularity lemma is a strong tool in extremal combinatorics. While the full generalisation to hypergraphs is more involved than the version for graphs, there is also a light version for hypergraphs that can already be useful and indeed it is for us:

Lemma 6.1 (Weak Hypergraph Regularity Lemma)**.** *For $\delta > 0, t_0 \in \mathbb{N}$, there exists a $T_0 \in \mathbb{N}$ such that for every 3-graph $H = ([n], E)$ with $n \geqslant t_0$ there exist an integer t with $t_0 \leqslant t \leqslant T_0$ and a partition $[n] = V_0 \dot\cup V_1 \dot\cup \ldots \dot\cup V_t$ such that:*

- $|V_0| \leqslant \delta n$ *and* $|V_1| = \cdots = |V_t|$
- *for* $i \geqslant 1$, *we have* $\max(V_i) \leqslant \max(V_{i+1})$ *and* $\max(V_i) - \min(V_i) \leqslant \frac{n}{t_0}$
- *there are at most δt^3 many $ijk \in [t]^{(3)}$ such that the "triplet" V_i, V_j, V_k, also written as V^{ijk}, is not weakly δ-quasirandom in H.*

For a proof of Lemma 6.1 see for instance [2, 6, 18]. One can get the slight extra requirement on the ordering of the vertices by dividing the vertex set in intervals of length $\left\lfloor \frac{n}{t_0} \right\rfloor$ and afterwards going on with the proof refining those sets. This has been remarked before, e.g. by Reiher, Rödl, and Schacht in [14].

We will now proceed in the following way that roughly imitates the strategy of Rödl, Ruciński, and Szemerédi in [15]: We regularise H and then observe that a triplet V^{ijk} that is quasirandom and has a certain density can almost be covered with not too short disjoint paths. Now we can think of the situation as a reduced hypergraph with the partition classes as vertices and edges encoding those "good triplets" that in H we can almost cover with paths. Then we notice that we can almost transfer the degree condition to that reduced hypergraph. This degree condition will ensure an almost perfect matching in the reduced hypergraph. But that means that in H almost all vertices can be covered with paths, which we can then connect through the reservoir to a long path in H.

Lemma 6.2 (Good Triplets)**.** *For $\xi > 0, d > 0, \delta > 0, n \in \mathbb{N}$ with $\frac{d\xi^3 - \delta}{2} n \geqslant 3$, the following holds. Let $H = (U \dot\cup V \dot\cup W, E)$ with $|U|, |V|, |W| = n$ be a 3-graph and suppose that U, V, W are weakly (δ, d)-quasirandom in H. Then at least $3n - 3\xi n$ vertices of H can be covered by vertex disjoint paths of length at least $\frac{d\xi^3 - \delta}{2} n - 2$.*

Proof of Lemma 6.2. For convenience set $c = \frac{d\xi^3 - \delta}{6} n$. Let $\mathcal{P}$ be a maximal set of vertex disjoint paths of length $3\lceil c \rceil - 2$ in H, where each path takes alternatingly vertices from each partition class, i.e., each path is of the form

$$u_1 v_1 w_1 u_2 v_2 w_2 \dots u_{\lceil c \rceil} v_{\lceil c \rceil} w_{\lceil c \rceil}$$

with $u_i \in U, v_i \in V, w_i \in W$.

Assume, $|V| - |\bigcup_{P \in \mathcal{P}} V(P)| > 3\xi n$. Then the sets

$$U' := U \smallsetminus \bigcup_{P \in \mathcal{P}} V(P), V' := V \smallsetminus \bigcup_{P \in \mathcal{P}} V(P), W' := W \smallsetminus \bigcup_{P \in \mathcal{P}} V(P)$$

satisfy $|U'|, |V'|, |W'| > \xi n$.

Next we will use a similar argument as in the proof of Claim 5.3: We delete all the edges that contain vertex pairs of small pair-degree. With the edges that still remain after this process we can build a path of the required length.

We start with $F_1 = H[U', V', W']$ and set F_{i+1} for $i \geqslant 1$ as the hypergraph obtained from F_i by deleting all edges containing a vertex pair xy with $d^{\times}_{F_i}(x, y) \leqslant c$, where $d^{\times}_{F_i}(x, y) = |\{e \in E(F_i) : x, y \in e, |e \cap U'| = |e \cap V'| = |e \cap W'| = 1\}|$. This process stops with a hypergraph F_j in which for all $x, y \in V(F_j)$ we either have $d^{\times}_{F_j}(x, y) = 0$ or $d^{\times}_{F_j}(x, y) \geqslant c$. The deletion condition guarantees

$$e^{\times}(F_1) - e^{\times}(F_j) \leqslant c \cdot 3n^2,$$

with $e^{\times}(F_i) = |\{e \in E(F_i) : |e \cap U'| = |e \cap V'| = |e \cap W'| = 1\}|$, and the quasirandomness of U, V, W gives that $e^{\times}(F_1) = e(U', V', W') \geqslant (d\xi^3 - \delta) n^3$. Thus, there still exists an edge $u_1v_1w_1$ in F_j with $u_1 \in U'$, $v_1 \in V'$ and $w_1 \in W'$. But this means that there is a path of length $3\lceil c \rceil - 2$ in F_j: Let $P^* = u_1v_1w_1 \ldots u_k, v_k, w_k$ be a maximal path in F_j with $u_i \in U', v_i \in V'$ and $w_i \in W'$ for all $i \in [k]$ (note that $k \geqslant 1$). Assuming $k < \lceil c \rceil$ for a contradiction, less than c vertices of U' appear in P^*. But since v_kw_k is contained in the edge $u_kv_kw_k \in E^{\times}(F_j)$, we actually have that $d^{\times}_{F_j}(v_k, w_k) \geqslant c$, whence there is a $u_{k+1} \in U' \setminus V(P^*)$ such that P^*u_{k+1} is a path in F_j.

The same argument applied to w_ku_{k+1} gives a $v_{k+1} \in V'$ such that $P^*u_{k+1}v_{k+1}$ is a path in F_j and finally applying the argument to $u_{k+1}v_{k+1}$ gives rise to a $w_{k+1} \in W'$ such that the path $P^*u_{k+1}v_{k+1}w_{k+1}$ exists in F_j and thus contradicts the maximality of P^*, telling us that P^* actually contains an alternating path of length $3\lceil c \rceil - 2$. That, on the other hand, gives us another alternating path of length at least $3\lceil c \rceil - 2$ that is vertex disjoint to all paths in $\mathcal{P}$ and therefore contradicts the maximality of $\mathcal{P}$. So we indeed have $|V| - |\bigcup_{P \in \mathcal{P}} V(P)| \leqslant 3\xi n$. □

As mentioned before, we later want to find an almost perfect matching in a reduced hypergraph whose edges represent "good" triplets like in the Lemma before. Then translating back those edges in the matching will give us a set of (not too many) paths in H that almost covers all vertices. To find an almost perfect matching in a hypergraph satisfying our pair degree condition for almost all pairs, we look at a maximal matching in which the sum of the vertices not contained in it is also maximal. This should give us the best chance to enlarge the matching if too many vertices would be left over, deriving a contradiction. A similar maximisation idea has also been used in [19] when a degree sequence condition was given for a graph. The following Lemma will later guarantee the existence of an almost perfect matching in the reduced hypergraph.

Lemma 6.3 (Matching)**.** *For all $\alpha, \beta > 0$, there exists an $n_0 \in \mathbb{N}$ such that the following holds: If $H = ([n], E)$ is a 3-graph, G_H a graph on vertex set $[n]$ with $\Delta(G_H) \leqslant \beta n$ and H satisfies $d(i,j) \geqslant \min\left(\frac{i+j}{2}, \frac{n}{2}\right) + \alpha n$ for all $ij \in [n]^{(2)}$ such that $ij \notin E(G_H)$, then H has a matching M with $v(M) \geqslant (1 - 3\beta) n$.*

Proof of Lemma 6.3. Without restriction let $\alpha \ll 1$ and $\beta < 1/3$. Given α and β we then choose $n \gg \frac{1}{\alpha}, \frac{1}{\beta}$. For this n let H, G_H be given as in the statement. Now, let M be a maximal matching in H and subject to that such that $\sum_{v \in [n] \setminus V(M)} v$ is maximised. Assuming the claim is false gives an $A \subseteq [n] \setminus V(M)$ with $|A| \geqslant 3\beta n$. Let us call a pair *true* if it is not an edge in G_H. Since $\Delta(G_H) \leqslant \beta n$ we can find $2\lfloor \beta n \rfloor$ distinct vertices $v_1, \ldots, v_{\lfloor \beta n \rfloor}, w_1, \ldots, w_{\lfloor \beta n \rfloor} \in A$ such that all the pairs v_iw_i are true. Notice that all the neighbours of each such pair lie inside $V(M)$, otherwise adding the respective edge to M would lead to a larger matching. In the following we will show two properties and afterwards deduce the statement from them.

Firstly, we have that for each v_iw_i there are at least $\frac{\alpha n}{3}$ edges in M in which v_iw_i has at least two neighbours: Let us first consider a pair v_iw_i with $\frac{v_i+w_i}{2} \leqslant \frac{n}{2}$. For any edge abc of the matching with $a \in N(v_i, w_i)$ we have that $b + c \leqslant v_i + w_i$ as otherwise $E(M) \setminus \{abc\} \cup \{av_iw_i\}$ would be the edge set of a matching M' with the same size as M, but $\sum_{v\in[n]\setminus V(M')} v$ being larger than the respective sum for M. Since in each edge of M that contains only one neighbour of v_iw_i there are two vertices $\leqslant v_i + w_i$ (and all those edges are disjoint), at most $\frac{v_i+w_i}{2}$ many neighbours of v_iw_i can lie in edges that contain no further neighbour of v_iw_i. Hence, recalling $d(v_i, w_i) \geqslant \frac{v_i+w_i}{2} + \alpha n$, at least $\frac{\alpha n}{3}$ edges in M contain at least two neighbours of v_iw_i.

For a pair v_iw_i with $\frac{v_i+w_i}{2} \geqslant n/2$, there exist at least $\frac{\alpha n}{3}$ edges in M containing more than one neighbour of v_iw_i as well, since $d(v_i, w_i) \geqslant \frac{n}{2} + \alpha n$ but $e(M) \leqslant n/3$.

Secondly, note that any edge of M that contains at least two neighbours of one true pair cannot contain a neighbour of any other true pair: Assume for contradiction there were true pairs v_iw_i and v_jw_j together with an edge $abc \in E(M)$ such that $a \in N(v_i, w_i)$ and $|\{abc\} \cap N(v_j, w_j)| \geqslant 2$. Then b or c, without restriction b, is a neighbour of v_jw_j and $E(M) \setminus \{abc\} \cup \{av_iw_i, bv_jw_j\}$ is the edge set of a matching in H contradicting the maximality of M.

Summarised, for each of the $\lfloor \beta n \rfloor$ true pairs v_iw_i in $[n] \setminus V(M)$, we get a set of at least $\frac{\alpha n}{3}$ edges in M that contain more than one neighbour of the respective pair and thus all those sets of edges are pairwise disjoint. Therefore, we have $\frac{\alpha n}{3} \times \lfloor \beta n \rfloor$ distinct edges in M which is a contradiction for $\alpha\beta \gg \frac{1}{n}$. So M was indeed a matching satisfying $v(M) \geqslant (1 - 3\beta)\, n$. □

We are now ready to prove Proposition 2.5. For that we will apply the Weak Regularity Lemma to H (actually to a slightly smaller subgraph), obtain a pair-degree condition for the reduced hypergraph and hence find a matching in it by the previous Lemma. Lastly, we will "unfold" the edges of that matching to paths in H by Lemma 6.2 and connect these to a long path by the Connecting Lemma.

Proof of Proposition 2.5. Let α, ϑ be given as in the Proposition and set $\alpha' = \alpha - \vartheta - \vartheta^2$. Next choose ξ, δ, t_0 such that we have $\alpha' \gg \vartheta \gg \xi \gg \delta \gg \frac{1}{t_0}$. Applying the Weak Regularity Lemma 6.1 to δ, t_0 gives us a T_0 and we choose $n \gg T_0$. Now let $H, \mathcal{R}, P_A$ be given as in the statement. Notice that $H' = H\left[[n] \setminus (\mathcal{R} \cup V(P_A))\right]$ after a renaming of the vertices can be seen as a 3-graph $H' = ([n'], E')$ with $n' \geqslant (1 - \vartheta^2 - \vartheta)\, n$ and satisfying the usual degree condition: $d(i, j) \geqslant \min\left(\frac{i+j}{2}, \frac{n'}{2}\right) + \alpha' n'$ for all $ij \in [n']^{(2)}$.

For H', the statement of the Weak Regularity Lemma provides an integer $t \in [t_0, T_0]$ and a partition $V = V_0 \dot\cup V_1 \dot\cup V_2 \dot\cup \ldots \dot\cup V_t$ satisfying all three points of Lemma 6.1. Setting $m = |V_1| = \cdots = |V_t|$, we have that $\frac{n'}{t} \geqslant m \geqslant \frac{1-\delta}{t} n'$ and recall that $|V_0| \leqslant \delta n'$. Note that for $v_i \in V_i$, we have $v_i \geqslant i \cdot m - \frac{n'}{t_0}$. Summarised, we have the following hierarchy:

$$1 \gg \alpha' \gg \vartheta \gg \xi \gg \delta \gg \frac{1}{t_0}, \frac{1}{t}, \frac{1}{T_0} \gg \frac{1}{n'}. \tag{6.1}$$

Let us write $e^{\times}\left(V^{ijk}\right) = |\{e \in E' : |e \cap V_i| = |e \cap V_j| = |e \cap V_k| = 1\}|$ for the number of crossing edges in V^{ijk} and we call a triplet V^{ijk} dense, if $e^{\times}\left(V^{ijk}\right) \geqslant \frac{\alpha' m^3}{2}$.

Now we will show that we can almost "transfer" the pair-degree condition to a reduced hypergraph. We will do this in two steps: First we show that every pair V_iV_j belongs to many dense triplets V^{ijk}, and second we show that we can almost keep that up when restricting ourselves to quasirandom triplets.

Claim 6.4. *For every $ij \in [t]^{(2)}$, there are at least $\min\left(\frac{i+j}{2}, \frac{t}{2}\right) + \frac{\alpha' t}{3}$ many $k \in [t] - \{i, j\}$ such that V^{ijk} is a dense triplet.*

Proof. Suppose there is a pair V_iV_j, $ij \in [t]^{(2)}$, belonging to less than $\min\left(\frac{i+j}{2}, \frac{t}{2}\right) + \frac{\alpha' t}{3}$ dense triplets V^{ijk}. Let S be the set of hyperedges in H' that contain one vertex in V_i, one in V_j and a third vertex outside of $V_i \dot\cup V_j$. By applying the pair-degree condition of H' and with the hierarchy (6.1) we get that

$$\begin{aligned} |S| \geqslant & m^2 \left[\min\left(\frac{(i+j)\,m}{2} - \frac{n'}{t_0}, \frac{n'}{2}\right) + \alpha' n' - 2m\right] \\ > & \frac{n'^3}{t^2}(1-\delta)^3 \left(\min\left(\frac{i+j}{2t}, \frac{1}{2(1-\delta)}\right) + \frac{6}{7}\frac{\alpha'}{1-\delta}\right) \end{aligned}$$

We will derive a contradiction by finding a smaller upper bound on $|S|$. For that we split S into two parts. By S_1 let us denote the set of those edges in S that lie in a dense triplet V^{ijk} for some $k \in [t] \smallsetminus \{i, j\}$ (we say an edge e lies or is in V^{ijk} if we have $|e \cap V_i| = |e \cap V_j| = |e \cap V_k| = 1$). Since in one triplet there are at most m^3 edges and by assumption V_iV_j does not belong to many dense triplets, we get

$$|S_1| < \left(\min\left(\frac{i+j}{2}, \frac{t}{2}\right) + \frac{\alpha' t}{3}\right) m^3 \leqslant \frac{n'^3}{t^2}\left(\min\left(\frac{i+j}{2t}, \frac{1}{2}\right) + \frac{\alpha'}{3}\right)$$

Let $S_2 = S \smallsetminus S_1$ be the set of edges in S lying in triplets that are not dense. There are less than $\frac{\alpha'}{2} m^3$ crossing edges in each triplet that is not dense and V_iV_j belongs to at most t triplets. Hence

$$|S_2| < \frac{\alpha'}{2} m^3 \times t \leqslant \frac{n'^3}{t^2}\frac{\alpha'}{2}.$$

Summarised, we have

$$\begin{aligned}
&\frac{n'^3}{t^2}(1-\delta)^3\left(\min\left(\frac{i+j}{2t},\frac{1}{2(1-\delta)}\right)+\frac{6}{7}\frac{\alpha'}{1-\delta}\right)\\
<&|S|=|S_1|+|S_2|\\
<&\frac{n'^3}{t^2}\left(\min\left(\frac{i+j}{2t},\frac{1}{2}\right)+\frac{5\alpha'}{6}\right)
\end{aligned}$$

which is a contradiction when considering (6.1). Thus, the assumption that V_iV_j is not contained in many triplets is wrong and the claim is proven. □

From the Weak Regularity Lemma we also get that in total at most δt^3 triplets V^{ijk} are not δ-quasirandom.

Let us now complete the "reduction" of the hypergraph and notice that we can find an almost perfect matching in the reduced hypergraph. Denote with D the hypergraph on the vertex set $[t]$ with ijk being an edge if and only if the triplet V^{ijk} is dense. Let, on the other hand, IR be the hypergraph on the vertex set $[t]$ with ijk being an edge if and only if V^{ijk} is not weakly δ-quasirandom in H'. In the following we will remove a few vertices in such a way that $D - IR$ induced on the remaining vertices satisfies our pair-degree condition for almost all pairs.

We call a pair $ij \in [t]^2$ *malicious pair* if it belongs to more than $\sqrt{\delta}t$ edges of IR. Since $e(IR) \leqslant \delta t^3$, there are at most $3\sqrt{\delta}t^2$ malicious pairs. Let B be the graph on vertex set $[t]$ in which the edges are given by the malicious pairs. We call a vertex i *malicious vertex* if $d_B(i) > \delta^{1/4}t$, i.e., if it belongs to many malicious pairs. The upper bound on the number of malicious pairs implies that there are at most $6\delta^{1/4}t$ malicious vertices. Now we remove these malicious vertices and set $D' := D\left[[t] \smallsetminus \{v \in [t] : v \text{ malicious}\}\right]$ and $B' = B\left[[t] \smallsetminus \{v \in [t] : v \text{ malicious}\}\right]$.

The reduced hypergraph we looked for is now $K = D' - IR$, in which edges encode dense, δ-quasirandom triplets. In K every pair $ij \in V(K)^{(2)}$ with $ij \notin E\left[B'\right]$ satisfies

$$d_K(i,j) \geqslant \min\left(\frac{i+j}{2},\frac{t}{2}\right)+\left(\frac{\alpha'}{3}-6\delta^{1/4}-\sqrt{\delta}\right)t \geqslant \min\left(\frac{i+j}{2},\frac{t}{2}\right)+\frac{\alpha'}{4}t.$$

Thus, we have that the graph G_K on vertex set $V(K)$ with ij being an edge if and only if ij does not satisfy the degree condition $d_K(i,j) \geqslant \min\left(\frac{i+j}{2},\frac{t}{2}\right)+\frac{\alpha'}{4}t$ is a subgraph of B'. Therefore $\Delta\left(G_K\right) \leqslant \Delta\left(B'\right) \leqslant \delta^{1/4}t$ and we can apply Lemma 6.3 to K with $\frac{\alpha'}{4}$ in place of α and $\delta^{1/4}$ instead of β and obtain a matching M in K covering all apart from at most $3\delta^{1/4}t$ vertices of K.

Finally, notice that each triplet V^{ijk} with ijk being an edge in K is (δ, d_{ijk})-quasirandom with $d_{ijk} \geqslant \frac{\alpha'}{2} - \delta \geqslant \frac{\alpha'}{3}$. Hence, we may apply Lemma 6.2 (with ξ as in (6.1)), $d_{ijk} \geqslant \frac{\alpha'}{3}$ in place of d and δ as δ) to each of the triplets V^{ijk} that corresponds to an edge in M. Doing so and recalling the definition of H', we notice that in H we can cover at least

$$n - \left(\left(\delta + 3\delta^{1/4} + 6\delta^{1/4} + \xi\right) n' + \vartheta^2 n + v\left(P_A\right)\right) \geqslant n - \left(2\vartheta^2 n + v\left(P_A\right)\right)$$

vertices with paths of length at least $\frac{\frac{\alpha'}{3}\xi^3 - \delta}{2}m - 2$ that are all disjoint to $\mathcal{R}$ and $V\left(P_A\right)$. We can connect all those at most $\frac{3t}{\left(\frac{\alpha'}{3}\xi^3-\delta\right)}$ paths in H through $\mathcal{R}$ to a path Q by Lemma 2.3, since until we connect the last one we have still only used at most

$$(L-2) \cdot \frac{3t}{\frac{\alpha'}{3}\xi^3 - \delta} < \vartheta^4 n$$

vertices from $\mathcal{R}$ (recall the hierarchy (6.1)). In fact, we have that Q has at most a small intersection with $\mathcal{R}$, that is, $|V(Q) \cap \mathcal{R}| \leqslant \vartheta^4 n$ and it covers many vertices, i.e., $v(Q) \geqslant (1 - 2\vartheta^2)\, n - v\left(P_A\right)$. Hence, Q is a path satisfying the claims in the statement. □

§7. Concluding Remarks

We would like to finish by pointing to some related problems. Firstly, as mentioned in the introduction, our result can be seen as a stepping stone towards a complete characterisation of those pair-degree matrices that force a 3-graph to contain a Hamiltonian cycle.

Further, it seems possible to generalise our proof without too much effort for k-uniform hypergraphs $H = ([n], E)$ with n large satisfying the degree condition

$$d_{k-1}(i_1, \ldots, i_{k-1}) \geqslant \min\left(\frac{1}{k-1}\sum_{j=1}^{k-1} i_j, \frac{n}{2}\right) + \alpha n,$$

where $d_{k-1}(i_1, \ldots, i_{k-1}) = |\{\, e \in E : i_1, \ldots, i_{k-1} \in e \,\}|$ is the (k-uniform) *codegree*. However, it is not clear whether common counter examples can show that the factor $\frac{1}{k-1}$ is optimal as it is for $k = 3$. It would be interesting both if there is another kind of counter examples or if "averaging" as in the condition above is not optimal.

Another very interesting problem is to get a similar result for the vertex degree, strengthening the result by Reiher, Rödl, Ruciński, Schacht, and Szemerédi in [13]: Does every 3-graph $H = ([n], E)$ with $d(i) \geqslant \min\left(\max\left(i, \gamma n\right), \frac{5}{9}n\right) + \alpha n$ for some $\gamma < 5/9$ contain a Hamiltonian cycle if n is large? The proof of Theorem 1.3 in [13] depends on the existence of *robust subgraphs* for every vertex, for which one needs the factor 5/9.

Lastly, one could try to improve Theorem 1.4 by weakening the pair-degree condition to $d(i,j) \geqslant \min\left(\frac{i+j}{2}, \frac{n}{2}\right)$, i.e. without the additional αn term, as Rödl, Ruciński, and Szemerédi did for the minimum pair-degree condition in [16].

§8. Acknowledgement

Up to minor changes, this present article is my master thesis from summer 2018 which was supervised by Christian Reiher. I would like to thank him for introducing me to the absorption method and to this problem.

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