SHORT PROOF THAT KNESER GRAPHS ARE HAMILTONIAN FOR $n \geqslant 4k$

JOHANN BELLMANN AND BJARNE SCHÜLKE

Abstract. For integers $n \geqslant k \geqslant 1$, the Kneser graph $K(n, k)$ is the graph with vertex set $V = [n]^{(k)}$ and edge set $E = \{\{x, y\} \in V^{(2)} : x \cap y = \varnothing\}$. Chen proved that for $n \geqslant 3k$, Kneser graphs are Hamiltonian and later improved this to $n \geqslant 2.62k + 1$. Furthermore, Chen and Füredi gave a short proof that if $k|n$, Kneser graphs are Hamiltonian for $n \geqslant 3k$. In this note, we present a short proof that does not need the divisibility condition, i.e., we give a short proof that $K(n, k)$ is Hamiltonian for $n \geqslant 4k$.

§1. Introduction

Throughout the paper, let $n \geqslant k \geqslant 1$ be integers and set $[n] = \{1, \dots, n\}$. For a set A, define $A^{(k)}$ to be the set of all k-element subsets (or k-subsets) of A. The *Kneser graph* $K(n, k)$ has vertex set $[n]^{(k)}$ and two vertices form an edge if and only if they are disjoint (as subsets of $[n]$). With rather involved proofs Chen [2] showed that Kneser graphs with n linear in k, and even triangle-free Kneser graphs [3] contain Hamiltonian cycles. More precisely, in [3] she showed the following.

Theorem 1.1. *If $n \geqslant 2.62k + 1$, then $K(n, k)$ is Hamiltonian.*

Chen and Füredi [4] simplified the proof for the case when $k|n$ and $n \geqslant 3k$. Katona [5] conjectured that, apart from finitely many exceptions, $K(n, k)$ is Hamiltonian if $n \geqslant 2k+1$. Recently Mütze, Nummenpalo, and Walczak [6] showed that for $k \geqslant 3$, the Kneser graph $K(2k + 1, k)$ is Hamiltonian (and they also provide a more exhaustive coverage of the previous work in this area).

In this note, we elaborate the short proof due to Chen and Füredi to work for the general case by removing the divisibility condition. More precisely, we give a short proof that

Theorem 1.2. *$K(n, k)$ is Hamiltonian for $n \geqslant 4k$.*

A *Gray-Code* is an enumeration $x_1 \dots x_m$ of all sets in $[n]^{(k)}$ such that each two consecutive sets and in addition x_m, x_1 differ by exactly one element (of $[n]$; in general, we say two k-sets $x, y \in [n]^{(k)}$ *differ by i elements* if $|x \smallsetminus y| = i$). The existence of Gray-Codes follows

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easily by induction (see e.g., [7]) and they were also used in [4]. Observing that the edges of $K_n^{(k)}$ correspond to the vertices of $K(n,k)$ and a matching of size s in $K_n^{(k)}$ corresponds to a clique of size s in $K(n,k)$, we get the following corollary to Baranyai's theorem [1].

Theorem 1.3. *Let $n \geqslant k$ and $a_1, \dots, a_t \leqslant \frac{n}{k}$ be integers such that $\sum_{i=1}^t a_i = \binom{n}{k}$. Then $K(n,k)$ can be partitioned into cliques A_i with $|A_i| = a_i$ for $i \in [t]$.*

§2. Short proof of Theorem 1.2

For clarity, we first give the proof for $n \geqslant 5k$ and afterwards, in Remark 2.1, we will go through the proof again inserting the additional arguments for $n \geqslant 4k$.

Proof of Theorem 1.2 if $n \geqslant 5k$. Let $p, r, m, q \in \mathbb{Z}$ such that $n = pk + r$ with $0 \leqslant r \leqslant k-1$ and $\binom{n}{k} = (m-1)p + q$ with $1 \leqslant q \leqslant p$. Set $b = b(q) = \max\{4-q, 0\}$ and define

$$a_i = \begin{cases} p & \text{for } i \in [m-1-b] \\ p-1 & \text{for } i \in \{m-b, \dots, m-1\} \\ q+b & \text{for } i = m \end{cases}. \tag{2.1}$$

Then we have $\sum_{i=1}^m a_i = \binom{n}{k}$ and Theorem 1.3 provides a partition of $K(n,k)$ into cliques $A_1, \dots, A_m$ with $|A_i| = a_i \geqslant 4$ for all $i \in [m]$. For $i \in [m]$, define *marking vertices* as follows. If there is an $x_i \in A_i$ with $n \in x_i$, set x_i to be the *marking vertex* of A_i. If there is no vertex containing n in A_i, we choose an arbitrary vertex $x_i \in A_i$ as *marking vertex* of A_i. We call the set of marking vertices M and note that M contains $M' = \{z \cup \{n\} : z \in [n-1]^{(k-1)}\}$. Next, we use a Gray-Code on $[n-1]^{(k-1)}$ to obtain an enumeration $x'_1 \dots x'_{m'}$ of $M' \subseteq M$ with $|x'_i \smallsetminus x'_{i+1}| = 1$ for all $i \in \mathbb{Z}/m'\mathbb{Z}$. Further, we consider a map $\varphi : M \smallsetminus M' \to M'$, so that for each $x \in M \smallsetminus M'$, we have $|\varphi(x) \smallsetminus x| = 1$ (this is possible since for all $a \in x \in M \smallsetminus M'$, the vertex $x \smallsetminus \{a\} \cup \{n\}$ is in M'). Thus, the enumeration

$$x'_1 \varphi^{-1}(x'_1) x'_2 \varphi^{-1}(x'_2) \dots x'_{m'} \varphi^{-1}(x'_{m'}) = y_1 \dots y_m$$

of M (here $\varphi^{-1}(x'_i)$ stands for an arbitrary enumeration of $\varphi^{-1}(x'_i)$) has the property that y_i and y_{i+1} differ by at most two elements (for $i \in \mathbb{Z}/m\mathbb{Z}$). Since $|A_i| \geqslant 4$, this yields that there is a vertex $z_i \in A(y_i)$ which is disjoint to y_{i+1}, where $A(y_i)$ is the clique among $A_1, \dots, A_m$ that contains y_i. Thus, denoting by α_i an enumeration of $A(y_i)$ that starts with y_i and ends with z_i, we get that $\alpha_1 \dots \alpha_m$ is a Hamiltonian cycle.

□

Remark 2.1. Here we mention the modifications that let the proof above work for all $n \geqslant 4k$. Note that for $k = 1$ the result is trivial, so assume $k \geqslant 2$. First, using 0 instead of b in (2.1) yields that $|A_i| \geqslant 4$ for $i \in [m-1]$ and $|A_m| = q$. If $q \geqslant 4$, the same proof as above still works. So we can assume that $q \leqslant 3$ and hence, there is an element of $[n]$, w.l.o.g. n, that is not contained in any vertex of A_m. For $i \in [m-1]$, we define marking vertices as before and for A_m, we do not define a marking vertex. Proceeding as above gives an enumeration $y_1 \dots y_{m-1}$ of the marking vertices with the property that y_i and y_{i+1} differ by at most two elements (for $i \in \mathbb{Z}/(m-1)\mathbb{Z}$) and so we still know that the vertices z_i exist as before (for $i \in [m-1]$).

Thus, we get as above that $\alpha_1 \dots \alpha_{m-1}$ is a cycle C which covers all but at most three vertices $v_1, v_2, v_3 \in A_m$. Note that for each v_i, we can choose a marking vertex $y_{j(i)} \in M'$ with $|v_i \setminus y_{j(i)}| = 1$. Since $k \geqslant 2$ and A_m is a clique, $y_{j(i)} \neq y_{j(i')}$ whenever $v_i \neq v_{i'}$. Further, $|A(y_{j(i)})| \geqslant 4$ implies that there are two vertices $u_i^1, u_i^2 \in A(y_{j(i)})$ which are disjoint to v_i. Note, that the enumeration $\alpha_{j(i)}$ of $A(y_{j(i)})$ was arbitrary apart from the start $(y_{j(i)})$ and the end (z_i). So we can additionally request that u_i^1 and u_i^2 are next to each other in this enumeration and insert v_i into C between u_i^1 and u_i^2, obtaining a Hamiltonian cycle.

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Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany
E-mail address: `johann.bellmann@studium.uni-hamburg.de`

Fachbereich Mathematik, Universität Hamburg, Hamburg, Germany
E-mail address: `bjarne.schuelke@uni-hamburg.de`