A NOTE ON MINOR ANTICHAINS OF UNCOUNTABLE GRAPHS

MAX PITZ

Abstract. A simplified construction is presented for Komjáth's result that for every uncountable cardinal κ, there are 2^κ graphs of size κ none of them being a minor of another.

§1. Introduction

The famous Robertson-Seymour Theorem asserts that the class of finite graphs is well-quasi-ordered under the minor relation $\preccurlyeq$: For every sequence $G_1, G_2, \dots$ of finite graphs there are indices $i < j$ such that $G_i \preccurlyeq G_j$.[1] This is no longer true for arbitrary infinite graphs. Thomas [8] has constructed a sequence $G_1, G_2, \dots$ of *binary trees with tops* of size of size continuum, such that $G_i \not\preccurlyeq G_j$ whenever $i < j$. Here, *binary tree with tops* describes the class of graphs where one selects in the rooted infinite binary tree T_2 a collection $\mathcal{R}$ of rays all starting at the root, adds for each $R \in \mathcal{R}$ a new vertex v_R, and makes v_R adjacent to all vertices on R. Let us write $G(\mathcal{R})$ for the resulting graph. In his proof, Thomas carefully selects continuum-sized collections of rays $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \dots$ such that $G_i = G(\mathcal{R}_i)$ form the desired bad sequence.

Thomas's result raises the question whether infinite graphs smaller than size continuum are well-quasi ordered. While this question for countable graphs is arguably the most important open problem in infinite graph theory, Komjáth [3] has established that for all other (uncountable) cardinals κ, there are in fact 2^κ pairwise minor-incomparable graphs of size κ.

The purpose of this note is to give an alternative construction for Komjáth's result which is simpler than the original, and also more integrated with other problems in the area:

First, our construction reinstates a pleasant similarity to Thomas's original strategy: The desired minor-incomparable graphs can already be found amongst the κ-regular trees with κ many tops. Second, our construction bears a surprising similarity to a family of rays considered in the 60's by A.H. Stone in his work on Borel isomorphisms [6]. Third, our examples allow for a considerable sharpening of a result by Thomas and Kriz [4] on graphs without uncountable

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[1] Recall that a graph H is a *minor* of another graph G, written $H \preccurlyeq G$, if to every vertex $x \in H$ we can assign a (possibly infinite) connected set $V_x \subseteq V(G)$, called the *branch set* of x, so that these sets V_x are pairwise disjoint and G contains a $V_x - V_y$ edge whenever xy is an edge of H.

clique minors but arbitrarily large tree width. And finally, a very similar family of graphs had recent applications for results about normal spanning trees in infinite graphs [5].

§2. Trees with tops and Stone's example

Consider the order tree $(T, \leqslant)$ where the nodes of T are all sequences of elements of κ of length $\leqslant \omega$ including the empty sequence, and let $t \leqslant t'$ if t is a proper initial segment of t'. The graph on T where any two comparable vertices are connected by an edge was considered by Kriz and Thomas in [4] where they showed that any tree-decomposition of this graph must have a part of size κ, despite not containing a subdivision of an uncountable clique.

For our purposes, however, it suffices to consider a graph G on T such that any node represented by finite sequences of length n is connected to all its successors of length $n+1$ in the tree order $\leqslant$, and any node represented by an ω-sequence is connected to all elements below in the tree order $\leqslant$. Clearly, G is connected. We later use the simple fact that

(i) every connected subgraph $H \subseteq G$ has a unique minimal node t_H in $(T, \leqslant)$.

Now given a set $S \subseteq \kappa$ consisting just of cofinality ω ordinals, choose for each $s \in S$ a cofinal sequence $f_s \colon \omega \to s$, and let $F = F(S) := \{f_s \colon s \in S\}$ be the corresponding collection of sequences in κ. Let T^S denote the subtree of T given by all finite sequences in T together with $F(S)$, and let $G(S)$ denote the corresponding induced subgraph of G. We will refer to $G(S)$ as a 'κ-regular tree with tops', where the elements of $F(S)$ are of course the 'tops'.

To the author's best knowledge, such a collection of tree branches $F(S) = \{f_s \colon s \in S\}$ for S the set of *all* cofinality ω ordinals was first considered by Stone in [6, §5] for the case $\kappa = \omega_1$ and in [7, §3.5] for the general case of uncountable regular κ.

We consider below graphs $G(S)$ where $S \subseteq \kappa$ is stationary. Recall that a subset $A \subseteq \kappa$ is *unbounded* if $\sup A = \kappa$, and *closed* if $\sup(A \cap \ell) = \ell$ implies $\ell \in A$ for all limits $\ell < \kappa$. The set A is a *club* in κ if it is both closed and unbounded. A subset $S \subseteq \kappa$ is *stationary* (in κ) if S meets every club of κ. Below, we use the following two elementary properties of stationary sets of regular uncountable cardinals κ (for details see e.g. [2, §8]):

(ii) If $S \subseteq \kappa$ is stationary and $S = \bigcup \{S_n \colon n \in \mathbb{N}\}$, then some S_n is stationary.

(iii) *Fodor's lemma*: If $S \subseteq \kappa$ is stationary and $f \colon S \to \kappa$ is such that $f(s) < s$ for all $s \in S$, then there is $i < \kappa$ such that $f^{-1}(i)$ is stationary.

§3. Constructing families of minor-incomparable graphs

At the heart of Komjáth's proof lies the construction, for regular uncountable κ, of κ pairwise minor-incomparable connected graphs of cardinality κ. From this, the singular case follows, and by considering disjoint unions of these graphs, one obtains an antichain of size 2^κ, see [3, Lemma 2]. Hence, it will be enough to prove:

Theorem 1. *For regular uncountable κ, the class of κ-regular trees with κ many tops contains a minor-antichain of size κ.*

Proof. As the set of cofinality ω ordinals of a regular uncountable κ splits into κ many disjoint stationary subsets [2, Lemma 8.8], it suffices to show: If S, R are disjoint stationary subsets consisting of cofinality ω ordinals, then $G(S) \not\preccurlyeq G(R)$.

Suppose for a contradiction that $G(S) \preccurlyeq G(R)$. For ease of notation, we identify s with f_s for all $s \in S$, and similarly for R. For $v \in T^S$ write $t_v \in T^R$ for the by (i) unique minimal node of the branch set of v in $G(R)$. Note that if v, w are adjacent in $G(S)$, then t_v and t_w are comparable in $(T^R, \leqslant)$. Since T^R has countable height, by (ii) there is a stationary subset $S' \subseteq S$ such that all t_s for $s \in S'$ belong to the same level of T^R. Suppose for a contradiction this level has finite height n. By applying Fodor's lemma (iii) iteratively $n+1$ times, we obtain a stationary subset $S'' \subseteq S'$ such that all f_s for $s \in S''$ agree on $f_s(i)$ for $i \leqslant n$. So distinct t_s for $s \in S''$ have at least $n+1$ common neighbours below them in $(T^R, \leqslant)$, a contradiction.

Thus, we may assume that $t_s \in R$ for all $s \in S$, giving rise an injective function $f \colon S \to R,\ s \to t_s$. Since f is injective, we cannot have $x < f(x)$ on a stationary subset of S by Fodor's lemma (iii). Hence, we may further assume that $f(x) \geqslant x$ for all $x \in S$.

For $i < \kappa$ let T^S_i be the subtree of T^S of all elements whose coordinates are strictly less than i, and consider the function $g \colon \kappa \to \kappa,\ i \mapsto \min \left\{ j < \kappa \colon t_v \in T^R_j \text{ for all } v \in T^S_i \right\}$. Since κ is regular, the function g is well-defined. And clearly, g is increasing. The function g is also continuous. Indeed, for a limit $\ell < \kappa$ consider any $v \in T^S_\ell \setminus \bigcup_{i<\ell} T^S_i$. Clearly, v is a top, and so all its neighbours belong to $\bigcup_{i<\ell} T^S_i$. Hence, t_v must be comparable to infinitely many nodes in $\bigcup_{i<\ell} T^R_{g(i)}$, implying that $t_v \in \bigcup_{i<\ell} T^R_{g(i)}$, too.

Hence, the set of fixed points C of g forms a club in κ, see [2, Exercise 8.1]. But any $s \in S \cap C$ satisfies $s \leqslant f(s) \leqslant g(s) = s$, showing that $s = f(s) \in S \cap R$, a contradiction. □

§4. A sharpening of Kriz and Thomas's result

Kriz and Thomas have used the graph on T where any two comparable vertices are connected by an edge in [4, Theorem 4.2] as an example of a graph without a subdivision of an uncountable clique, but where any tree-decomposition must have a part of size κ. For background on tree-decompositions see [1, §12].

In this section we establish that if κ is regular, then already any graph $G(S)$ from above – for $S \subseteq \kappa$ stationary and consisting just of cofinality ω ordinals – has the same property that any tree-decomposition of G must have a part of size κ.

Theorem 2. *For regular uncountable κ and $S \subseteq \kappa$ stationary and consisting just of cofinality ω ordinals, any tree-decomposition of $G(S)$ has a part of size κ.*

We remark that this result is sharp: If $\kappa = \omega_1$ and $S \subseteq \omega_1$ is non-stationary, then $G(S)$ has a normal spanning tree [5], and hence a tree-decomposition into finite parts.

In the proof, Q^n denotes the nth level of a rooted tree Q, and $Q^{<n} = \bigcup_{m<n} Q^m$.

Proof. We start with an observation: Given any stationary subset $S' \subseteq S$ and any set X of vertices of $G(S)$ with $|X| < \kappa$, at least one component of $G(S) - X$ contains a stationary subset of S'. Indeed, using the notation as in the previous proof, since X is small, we have $X \subseteq T_i^S$ for some $i < \kappa$, and $S' \setminus T_i^S$ is still stationary. Now for every $s \in S' \setminus T_i^S$, let n_s be minimal such that $f_s(n_s) \notin T_i^S$. By (ii) there is a stationary subset $S'' \subseteq S' \setminus T_i^S$ whose elements agree on $n = n_s$. By applying Fodor's lemma (iii) $n+1$ times, there is a stationary subset $S''' \subseteq S''$ such that the maps f_s agree on their first coordinates up to n for all $s \in S'''$. Then all $s \in S'''$ have a common neighbour outside of T_i^S, so belong to the same component.

Now suppose for a contradiction that $G(S)$ has a tree-decomposition $(Q, (V_q)_{q \in Q})$ such that $|V_q| < \kappa$ for all $q \in Q$. Fix an arbitrary root q_0 of Q. Every $s \in S$ is contained in some part; let $q_s \in Q$ be minimal in the tree order of Q such that $s \in V_{q_s}$. By (ii) there is a minimal $n \in \mathbb{N}$ and a stationary subset $S' \subseteq S$ such that $q_s \in Q^n$ for all $s \in S'$.

Then no component of $G - \bigcup_{q \in Q^{<n}} V_q$ contains a stationary subset of S', as the intersection of any such component with S' is contained in some less than κ sized V_{q_s}. However, by iteratively using our earlier observation, there is a path $q_0 q_1 \dots q_{n-1}$ starting at the root of Q, a decreasing sequence of components C_i of $G - (V_{q_0} \cup \dots \cup V_{q_i})$ and stationary subsets S_i of S' such that each $S_i \subseteq C_i$. But $S_{n-1} \subseteq C_{n-1} \subseteq G - \bigcup_{q \in Q^{<n}} V_q$, a contradiction. □

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Hamburg University, Department of Mathematics, Bundesstrasse 55 (Geomatikum), 20146 Hamburg, Germany

E-mail address: max.pitz@uni-hamburg.de