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The $\mathbb{R}$-Local Homotopy Theory of Smooth Spaces

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Abstract

Simplicial presheaves on cartesian spaces provide a general notion of smooth spaces. We define a corresponding smooth version of the singular complex functor, which maps smooth spaces to simplicial sets. We exhibit this functor as one of several Quillen equivalences between the Kan-Quillen model category of simplicial sets and a motivic-style $\mathbb{R}$-localisation of the (projective or injective) model category of smooth spaces. These Quillen equivalences and their interrelations are powerful tools: for instance, they allow us to give a purely homotopy-theoretic proof of a Whitehead Approximation Theorem for manifolds. Further, we provide a functorial fibrant replacement in the $\mathbb{R}$-local model category of smooth spaces. This allows us to compute the homotopy types of mapping spaces in this model category in terms of smooth singular complexes. We explain the relation of our fibrant replacement functor to the concordance sheaves introduced recently by Berwick-Evans, Boavida de Brito, and Pavlov. Finally, we show how the $\mathbb{R}$-local model category of smooth spaces formalises the homotopy theory on sheaves used by Galatius, Madsen, Tillmann, and Weiss in their seminal paper on the homotopy type of the cobordism category.

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1 Introduction and overview

Topological spaces and simplicial sets can be used to construct the same homotopy theory. This is made rigorous by the fact that the singular complex and the geometric realisation functors form a Quillen equivalence between the standard model structure on the category $\mathcal{T}op$ of topological spaces and the Kan-Quillen model structure on the category $\mathcal{S}et_\Delta$ of simplicial sets. Both of these model categories formalise what is often called the homotopy theory of spaces, or ∞-groupoids (these are the same according to Grothendieck's homotopy hypothesis). The two models differ significantly in their features, though, in that topological spaces encode homotopies via the notion of continuity, while simplicial sets are inherently combinatorial. Consequently, each of these two models for the homotopy theory of spaces has its own merits in different contexts and applications.

Apart from continuity or combinatorics, another important feature that spaces can possess and that is relevant in many problems in mathematics is smoothness. The prime example of a category of smooth spaces is the category $\mathcal{M}fd$ of manifolds and smooth maps, which underlies much of geometry. There exists a notion of smooth homotopy within the category $\mathcal{M}fd$, and one can find smooth versions of many topological concepts, such as cohomology, which are invariant under these smooth homotopies. It would therefore be desirable to have a homotopy theory on the category of manifolds. However, the category $\mathcal{M}fd$ is poorly behaved in many ways. For instance, it is far from being complete or cocomplete, thus making it unable to admit a model structure in the sense of Quillen.

The way to cure this is to weaken—and therefore to generalise—the concept of a manifold. Here, we take the following approach to smooth spaces, with the main goal of simultaneously capturing the notions of a manifold and of a (higher) stack. We start from the category $\mathcal{C}art$ of cartesian spaces: its objects are all smooth manifolds that are diffeomorphic to $\mathbb{R}^n$ for any $n \in \mathbb{N}_0$, and its morphisms are all smooth maps between these manifolds. We define a smooth space to be a simplicial presheaf on $\mathcal{C}art$—informally, we understand the sections of a simplicial presheaf over $c \in \mathcal{C}art$ as c-parameterised families of simplices in a space. We denote the category of simplicial presheaves on $\mathcal{C}art$ by $\mathcal{H}_\infty$. It contains many geometrically interesting objects that are not manifolds or even diffeological spaces [IZ13] (for instance the presheaf of k-forms, or the simplicial presheaf of G-bundles with connection, for any Lie group G). The category of manifolds, the category of diffeological spaces, and the category of simplicial sets each include fully faithfully into $\mathcal{H}_\infty$.

The category $\mathcal{H}_\infty$ carries two natural model structures, namely the projective and the injective model structures on functors $\mathcal{C}art^{\mathrm{op}} \to \mathcal{S}et_\Delta$, where $\mathcal{S}et_\Delta$ carries the Kan-Quillen model structure. We denote the projective and injective model categories by $\mathcal{H}_\infty^p$ and $\mathcal{H}_\infty^i$, respectively, and we write $\mathcal{H}_\infty^{p/i}$ to refer to either of these model structures simultaneously. The projective and injective model structures are canonically Quillen equivalent via the identity functors $\mathcal{H}_\infty^p \rightleftarrows \mathcal{H}_\infty^i$, but they are not Quillen equivalent to $\mathcal{S}et_\Delta$. In that sense, the model structures $\mathcal{H}_\infty^{p/i}$ do not yet define smooth versions of the homotopy theory of spaces. To achieve that, one needs a weaker notion of equivalence in $\mathcal{H}_\infty$.

There exist (at least) two candidates for such weakened versions of equivalences in $\mathcal{H}_\infty$. First, in [MW07, GTMW09] a notion of weak equivalence has been introduced on sheaves on $\mathcal{M}fd$ as follows: let $\Delta_e^k \cong \mathbb{R}^k$ denote the smooth extended (affine) k-simplex. Extending the usual face and degeneracies of the topological standard simplices, this gives rise to a cosimplicial cartesian space $\Delta_e \colon \mathbb{\Delta} \to \mathcal{C}art \subset \mathcal{M}fd$. Via precomposition, this induces a functor from (pre)sheaves on manifolds to simplicial sets. In [MW07, GTMW09], a morphism of sheaves is considered a weak equivalence of (pre)sheaves whenever it becomes a weak equivalence of simplicial sets under this functor. We adapt this to our set-up as follows: for technical reasons, we work with presheaves on cartesian spaces rather than manifolds, and we work with simplicial (pre)sheaves instead of ordinary (pre)sheaves. Let $\delta\colon \mathbb{\Delta} \to \mathbb{\Delta} \times \mathbb{\Delta}$ denote the diagonal functor. We define the *smooth singular complex functor*

$$S_e\colon \mathcal{H}_\infty \xrightarrow{\Delta_e^*} s\mathcal{S}\mathrm{et}_\Delta \xrightarrow{\delta^*} \mathcal{S}\mathrm{et}_\Delta\,, \tag{1.1}$$

where the first functor evaluates $F \in \mathcal{H}_\infty$ on the extended simplices to obtain a bisimplicial set, of which the second functor then takes the diagonal. Let $S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})$ denote the class of morphisms in $\mathcal{H}_\infty$ that are mapped to a weak equivalence by S_e. The formal generalisation of the homotopy theory from [MW07, GTMW09] is then the localisation

$$L_{S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})}\mathcal{H}_\infty^{p/i}\,.$$

The second notion of weak equivalence in $\mathcal{H}_\infty$ is motivated by motivic homotopy theory (see [Voe98, MV99, DLØ+07], for instance). Let I denote the class of all morphisms in $\mathcal{H}_\infty$ of the form $c \times \mathbb{R} \to c$, where c ranges over all objects in $\mathcal{C}\mathrm{art}$, and where the morphism is the identity on c and collapses $\mathbb{R}$ to the point. The localisation

$$\mathcal{H}_\infty^{p/i\,I} := L_I \mathcal{H}_\infty^{p/i}$$

is then a version in smooth geometry of motivic localisation. We call $\mathcal{H}_\infty^{p/i\,I}$ the $\mathbb{R}$*-local model category* of simplicial presheaves on $\mathcal{C}\mathrm{art}$, or equivalently of smooth spaces. This localisation has appeared before in [Sch, Dug01b] and other works of these authors. Our first main result is

Theorem 1.2 *Let $S_e\colon \mathcal{H}_\infty \to \mathcal{S}\mathrm{et}_\Delta$ be as in* (1.1).

(1) The functor $S_e\colon \mathcal{H}_\infty^{pI} \to \mathcal{S}\mathrm{et}_\Delta$ is a left Quillen equivalence.

(2) The functor $S_e\colon \mathcal{H}_\infty^{iI} \to \mathcal{S}\mathrm{et}_\Delta$ is both a left and a right Quillen equivalence.

In particular, the localised model structures $\mathcal{H}_\infty^{p/i\,I}$ define homotopy theories of smooth spaces that are equivalent to the usual homotopy theories of spaces. The key to proving this is to relate the functor S_e to other functors that extract spaces from objects of $\mathcal{H}_\infty$. One of these functors is the left Quillen functor

$$Re\colon \mathcal{H}_\infty^p \to \Delta\mathcal{T}\mathrm{op}\,, \qquad F \longmapsto \int^{c \in \mathcal{C}\mathrm{art}^{\mathrm{op}}} |F(c)| \times \mathrm{D}c\,,$$

where $\Delta\mathcal{T}\mathrm{op}$ is the category of Δ-generated topological spaces (see Section 2.3 for details). Further, $|{-}|\colon \mathcal{S}\mathrm{et}_\Delta \to \Delta\mathcal{T}\mathrm{op}$ is the usual geometric realisation functor, and where $\mathrm{D}c \in \Delta\mathcal{T}\mathrm{op}$ is the topological space underlying $c \in \mathcal{C}\mathrm{art}$. We show:

Theorem 1.3 *Let $Q^p\colon \mathcal{H}_\infty^p \to \mathcal{H}_\infty^p$ be a cofibrant replacement functor for the projective model structure. There is a zig-zag of natural weak equivalences*

$$|{-}| \circ S_e \xleftarrow{\sim} |{-}| \circ S_e \circ Q^p \xrightarrow{\sim} Re \circ Q^p\,.$$

In particular, given any $F \in \mathcal{H}_\infty$, we can identify $|S_e(F)| \in \Delta\mathcal{T}\mathrm{op}$ with the homotopy colimit of the diagram $|F|\colon \mathcal{C}\mathrm{art}^{\mathrm{op}} \to \Delta\mathcal{T}\mathrm{op}$.

This comparison result has several implications. First, on a very formal level, it allows us to identify S_e as a presentation of the left adjoint in the cohesive structure on the ∞-topos of presheaves of spaces on $\mathcal{C}\mathrm{art}$. This has been indicated recently in [BEBdBP]; here we prove this formally (see the end of Section 4, and see [Buna] for an ∞-categorical treatment of this fact).

Further, we can apply Theorem 1.3 to give a purely homotopy-theoretic proof of the following result, which is sometimes referred to as the Whitehead Approximation Theorem:

Theorem 1.4 *Let M be any manifold. The smooth singular complex of M—that is, the simplicial set $\mathcal{M}\mathrm{fd}(\Delta_e, M)$—and the singular complex $\mathrm{Sing}(M)$ of the topological space underlying M are weakly equivalent in $\mathcal{S}\mathrm{et}_\Delta$.*

In our proof, at no point do we need to approximate a continuous map by smooth maps. Instead, the proof relies on a result about Čech nerves of open coverings from [DHI04], a version of the Nerve Theorem, see e.g. [Bor48, Ler50, Seg74, DI04] (concretely, we use [Lur17, Thm. A.3.1], which is sometimes also called Lurie's Seifert van Kampen Theorem), and a modified two-sided simplicial bar construction for simplicial presheaves, which we introduce in Appendix C. This illustrates that the model categories $\mathcal{H}_\infty^{p/i\,I}$ are of interest beyond their abstract properties: they provide useful tools for doing smooth homotopy theory.

Next, we construct a fibrant replacement functor $\mathrm{Cc}^{p/i}$ for the localisation $\mathcal{H}_\infty^{p/i\,I}$; this construction is motivated by the *concordance sheaves* introduced recently in [BEBdBP]. Thereby, we obtain explicit access to the mapping spaces in $\mathcal{H}_\infty^{p/i\,I}$. Proving that $\mathrm{Cc}^{p/i}$ is indeed a fibrant replacement relies on the properties of the functor S_e that we prove throughout this text, and in particular on Theorem 1.2. Combining this with Theorems 1.3 and 1.4, we can compute the mapping spaces in the localisation $\mathcal{H}_\infty^{p/i\,I}$:

Theorem 1.5 *Let $F, G \in \mathcal{H}_\infty$ be any simplicial presheaves on $\mathcal{C}\mathrm{art}$. Let $M \in \mathcal{M}\mathrm{fd}$ be any manifold, and define $\underline{M} \in \mathcal{H}_\infty$ by setting $\underline{M}(c) = \mathcal{M}\mathrm{fd}(c, M)$ for any cartesian space $c \in \mathcal{C}\mathrm{art}$. There are canonical isomorphisms*

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(S_e F, S_e G) \cong \mathrm{Map}_{\mathcal{H}_\infty^{p/i\,I}}(F, G)\,,$$
$$\mathrm{Map}_{\Delta\mathcal{T}\mathrm{op}}(M, |S_e G|) \cong \mathrm{Map}_{\mathcal{H}_\infty^{p/i\,I}}(\underline{M}, G)$$

in $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$, the homotopy category of spaces.

Here, the injective case is more easy to treat since every object in $\mathcal{H}_\infty^{iI}$ is cofibrant, so we prove this case first. We then show the projective case, based on the injective case and on an explicit Quillen equivalence $Q' : \mathcal{H}_\infty^i \rightleftarrows \mathcal{H}_\infty^p : R'$ which we construct in Appendix A. (Note that this Quillen equivalence goes in the opposite direction of the immediate Quillen equivalence $\mathcal{H}_\infty^p \rightleftarrows \mathcal{H}_\infty^i$ induced by the identity functors.) Further, we relate Theorem 1.5 to [BEBdBP] explicitly. This relation relies on results of [BEBdBP] and on a comparison between homotopy sheaves on $\mathcal{C}\mathrm{art}$ with respect to good open coverings and homotopy sheaves on $\mathcal{M}\mathrm{fd}$ with respect to open coverings: in Appendix B, we provide an explicit Quillen equivalence between model categories whose fibrant objects are these two classes of homotopy sheaves.

Finally, we apply Theorem 1.5 to identify the homotopy theory on $\mathcal{H}_\infty$ motivated by the ideas of [MW07, GTMW09] with the $\mathbb{R}$-local homotopy theory in the following very strong sense:

Theorem 1.6 *There is an identity of localisations of $\mathcal{H}_\infty^{p/i}$:*

$$\mathcal{H}_\infty^{p/i\,I} = L_{S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})} \mathcal{H}_\infty^{p/i}\,.$$

This extends and formalises the homotopy theory used in [MW07, GTMW09] and provides further interpretation to each of these model structures.

Outline. This paper is organised as follows: We begin in Section 2 by defining the $\mathbb{R}$-localisations $\mathcal{H}_\infty^{p/i\,I}$ of $\mathcal{H}_\infty^{p/i}$. We show that $\mathcal{H}_\infty^{pI}$ and $\mathcal{H}_\infty^{iI}$ have the same weak equivalences and that they can also be seen as further localisations of the local model structures with respect to differentiably good open coverings. We define the functor $Re\colon \mathcal{H}_\infty^{pI} \to \Delta\mathcal{T}\mathrm{op}$ and show that it is a left Quillen equivalence.

In Section 3, we study the smooth singular complex functor $S_e\colon \mathcal{H}_\infty \to \mathcal{S}\mathrm{et}_\Delta$. We first show that $S_e\colon \mathcal{H}_\infty^{p/i\,I} \to \mathcal{S}\mathrm{et}_\Delta$ is a left Quillen equivalence. Subsequently, we establish $S_e\colon \mathcal{H}_\infty^{iI} \to \mathcal{S}\mathrm{et}_\Delta$ as a right Quillen equivalence. As an intermediate step, we relate the model categories $\mathcal{H}_\infty^{iI}$ and $\mathcal{S}\mathrm{et}_\Delta$ to localisations of the model category of complete Segal spaces. This sheds additional light on the interpretation of the functor S_e. Section 3 contains the proof of Theorem 1.2.

Section 4 is concerned with the comparison of different ways of extracting spaces from simplicial presheaves on $\mathcal{C}\mathrm{art}$. The key concept is to extend the homotopy equivalence that embeds the topological standard simplices into the smooth extended simplices to obtain natural weak equivalences between functors from $\mathcal{H}_\infty$ to $\mathcal{S}\mathrm{et}_\Delta$ and $\Delta\mathcal{T}\mathrm{op}$. Here we prove Theorem 1.3.

The Whitehead Approximation Theorem, Theorem 1.4, is proven in Section 5. We start from an open covering of a manifold and manipulate it via the modified two-sided bar construction from Appendix C until we can apply [Lur17, Thm. A.3.1].

In Section 6 we construct a fibrant replacement functor for $\mathcal{H}_\infty^{p/i\,I}$ and use it to prove Theorem 1.5. We spell out the relation of this theorem to [BEBdBP] and works of Dugger. Then we apply Theorem 6.6 to prove the coincidence of model structures from Theorem 1.6.

Finally, we include four appendices; Appendix A contains the explicit construction of a fibrant replacement functor for the injective model structure on $\mathcal{H}_\infty$, which features in the proof of Theorem 1.5. Building on this, we provide a Quillen equivalence between model categories for homotopy sheaves on $\mathcal{C}\mathrm{art}$ and homotopy sheaves on $\mathcal{M}\mathrm{fd}$ in Appendix B. In Appendix C, we develop our modified two-sided bar construction, which we use to prove Theorem 1.4, and in Appendix D we include some rather standard material on recognising Quillen equivalences.

Notation and conventions We briefly summarise some notational conventions that will be used throughout this text.

- We let $\mathcal{C}\mathrm{at}$ denote the category of categories, and we let $\mathbb{\Delta}$ denote the simplex category.
- Given two categories $\mathcal{C}, \mathcal{I}$, we write $\mathcal{C}\mathrm{at}(\mathcal{I}, \mathcal{C})$ or $\mathcal{C}^\mathcal{I}$ for the categories of functors $\mathcal{I} \to \mathcal{C}$.
- We let $\mathcal{S}\mathrm{et}_\Delta = \mathcal{C}\mathrm{at}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ denote the category of simplicial sets. In this article, when viewing $\mathcal{S}\mathrm{et}_\Delta$ as a model category, we will always use the cartesian closed Kan-Quillen mode structure on $\mathcal{S}\mathrm{et}_\Delta$, that is, the model structure for ∞-groupoids.
- We will also be working with the category $s\mathcal{S}\mathrm{et}_\Delta = \mathcal{C}\mathrm{at}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta)$ of bisimplicial sets. Our convention is always to write a bisimplicial set as a functor
$$X\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta\,, \quad [n] \mapsto X_n\,, \quad \text{with} \quad X_{n,k} \coloneqq \big(X[n]\big)[k]\,.$$
- If $\mathcal{C}$ is a simplicial category, we write $\underline{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{S}\mathrm{et}_\Delta$ for the simplicially enriched hom-functor.
- We let $\Delta\mathcal{T}\mathrm{op}$ denote the symmetric monoidal, simplicial model category of Δ-generated topological spaces.

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2 $\mathbb{R}$-local model structures and smooth spaces

We start by setting up the model-categorical background used throughout this paper. Partially following [Sch] and [Dugb], we consider the category of simplicial presheaves on cartesian spaces with its canonical projective and injective model structures. In analogy with $\mathbb{A}^1$-local homotopy theory, we localise this category at all the morphisms $c \times \mathbb{R} \to c$, where c is any cartesian space and where the morphism collapses the real line. Extending ideas from [Dugb], we establish several Quillen equivalences of this localised model category with the categories of simplicial sets and topological spaces.

2.1 $\mathbb{R}$-local model structures on simplicial presheaves

In this section we start by setting up the various model structures on simplicial presheaves that will play a role in this article. Let $\mathcal{C}\mathrm{art}$ denote the (small) category of submanifolds of $\mathbb{R}^\infty$ that are diffeomorphic to $\mathbb{R}^n$, for any $n \in \mathbb{N}_0$. These manifolds are called *cartesian spaces*. The morphisms $c \to d$ in $\mathcal{C}\mathrm{art}$ are the smooth maps $c \to d$ between these manifolds; in other words, $\mathcal{C}\mathrm{art}$ is the full subcategory of the category $\mathcal{M}\mathrm{fd}$ of smooth manifolds and smooth maps on the cartesian spaces.

Let $\mathcal{H}_\infty := \mathcal{C}\mathrm{at}(\mathcal{C}\mathrm{art}^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta)$ denote the category of simplicial presheaves on $\mathcal{C}\mathrm{art}$. There is a fully faithful inclusion

$$\underline{(-)} \colon \mathcal{M}\mathrm{fd} \to \mathcal{H}_\infty\,, \qquad \underline{M}(c) = \mathcal{M}\mathrm{fd}(c, M)\,.$$

We view $\mathcal{S}\mathrm{et}_\Delta$ as endowed with the Kan-Quillen model structure. The category $\mathcal{C}\mathrm{art}$ carries a Grothendieck pretopology τ of differentiably good open coverings—see [Bunb, FSS12] for details. A covering of $c \in \mathcal{C}\mathrm{art}$ in this pretopology is a collection of morphisms $\{\iota_i \colon c_i \to c\}_{i \in \Lambda}$ in $\mathcal{C}\mathrm{art}$ such that each ι_i is an embedding of an open subset, the images of the maps ι_i cover c (i.e. each $x \in c$ lies in the image of some ι_i), and every finite intersection

$$C_{i_0 \dots i_n} := \bigcap_{a=0}^{n} \iota_{i_a}(c_{i_a}) \subset c \tag{2.1}$$

with $i_0, \dots, i_n \in \Lambda$ is either empty or a cartesian space. (For $F \in \mathcal{H}_\infty$ we set $F(\emptyset) = *$, in accordance with the Yoneda Lemma and $\underline{\mathcal{H}}_\infty(\emptyset, F) = *$, where here $\emptyset \in \mathcal{H}_\infty$ is the initial object.) We let ℓ denote the class of Čech coverings in $\mathcal{H}_\infty$ with respect to the Grothendieck pretopology τ. Given a simplicial model category $\mathcal{M}$ and a class S of morphisms in $\mathcal{M}$, we denote by $L_S\mathcal{M}$ the simplicially enriched left Bousfield localisation of $\mathcal{M}$ at the morphisms in S (see [Bar10] for more background).

Definition 2.2 *We define the following model categories:*

(1) $\mathcal{H}_\infty^{p/i}$ *is the projective (resp. injective) model structure on* $\mathcal{H}_\infty$*. We also refer to* $\mathcal{H}_\infty^{p/i}$ *as the model categories of* smooth spaces.

(2) We define the $\mathcal{S}\mathrm{et}_\Delta$*-enriched left Bousfield localisations*

$$\mathcal{H}_\infty^{p/i\,\ell} := L_\ell \mathcal{H}_\infty^{p/i}\,.$$

This is the projective (resp. injective) model structure for sheaves of ∞-groupoids on $\mathcal{C}\mathrm{art}$.

(3) Let $I := \{\mathcal{Y}_c \times \mathcal{Y}_\mathbb{R} \longrightarrow \mathcal{Y}_c\}_{c \in \mathcal{C}\mathrm{art}}$ be the set of morphisms obtained by taking the product of the collapse map $\mathbb{R} \to *$ with all identities $\{1_c\}_{c\in\mathcal{C}\mathrm{art}}$. We define the projective (resp. injective) $\mathbb{R}$-local model category of smooth spaces *as the enriched left Bousfield localisation*

$$\mathcal{H}_\infty^{p/i\, I} := L_I \mathcal{H}_\infty^{p/i}\,.$$

(4) We can further define the model categories

$$\mathcal{H}_\infty^{p/i\,\ell I} := L_I \mathcal{H}_\infty^{p/i\,\ell}\,, \qquad \mathcal{H}_\infty^{p/i\,I\ell} := L_\ell \mathcal{H}_\infty^{p/i\,I}\,.$$

Remark 2.3 We interpret the localisation $\mathcal{H}_\infty^{p/i\,I}$ as an $\mathbb{R}$-localisation of $\mathcal{H}_\infty^{p/i}$ akin to motivic localisation (see, for example, [Voe98, MV99, DLØ$^+$07]). Though thinking of objects in $\mathcal{H}_\infty$ as smooth spaces, we will mostly refer objects in $\mathcal{H}_\infty$ by the more technically precise term of simplicial presheaves, and we will sometimes refer to fibrant objects in $\mathcal{H}_\infty^{p/i\,I}$ as *locally constant* simplicial presheaves. ◁

Proposition 2.4 *All model structures in Definition 2.2 are simplicial, left proper, tractable, and symmetric monoidal.*

Proof. Except for the claim that the model structures are symmetric monoidal, all assertions follow from [Bar10, Thm. 4.46]. The model structures for sheaves of ∞-groupoids are symmetric monoidal by [Bar10, Thm. 4.58]. To see that $\mathcal{H}_\infty^{p/i\,I}$ is symmetric monoidal, observe that the objects $\mathcal{Y}_c$, $c \in \mathcal{C}\mathrm{art}$, form a set of homotopy generators for $\mathcal{H}_\infty^{p/i}$. Let $F \in \mathcal{H}_\infty^{p/i\,I}$ be a local object, and consider the internal hom object $F^{\mathcal{Y}_d}$ for any $d \in \mathcal{C}\mathrm{art}$. For any of the morphisms $\mathcal{Y}_c \times \mathcal{Y}_\mathbb{R} \to \mathcal{Y}_c$ in I, the internal hom adjunction yields a commutative diagram of simplicially enriched hom spaces

$$\begin{array}{ccc} \underline{\mathcal{H}}_\infty(\mathcal{Y}_c, F^{\mathcal{Y}_d}) & \longrightarrow & \underline{\mathcal{H}}_\infty(\mathcal{Y}_c \times \mathcal{Y}_\mathbb{R}, F^{\mathcal{Y}_d}) \\ \cong\downarrow & & \downarrow\cong \\ \underline{\mathcal{H}}_\infty(\mathcal{Y}_{c\times d}, F) & \longrightarrow & \underline{\mathcal{H}}_\infty(\mathcal{Y}_{c\times d} \times \mathcal{Y}_\mathbb{R}, F) \end{array}$$

Here we have used that $\mathcal{C}\mathrm{art}$ has finite products. The bottom horizontal morphism is induced by the morphism $c \times d \times \mathbb{R} \to c \times d$, which is an element of I. Hence, the bottom morphism is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$. Therefore, it follows from [Bar10, Prop. 4.47] that $\mathcal{H}_\infty^{p/i\,I}$ is symmetric monoidal. The exact same proof shows that $\mathcal{H}_\infty^{p/i\,\ell I}$ is symmetric monoidal as well. The fact that $\mathcal{H}_\infty^{p/i\,I\ell}$ is symmetric monoidal will be seen in Corollary 2.12. □

Proposition 2.5 *There are commutative diagrams of simplicial Quillen adjunctions:*

$$\begin{array}{ccc} \mathcal{H}_\infty^{p} & \rightleftarrows & \mathcal{H}_\infty^{i} \\ \downarrow\uparrow & & \downarrow\uparrow \\ \mathcal{H}_\infty^{p\ell} & \rightleftarrows & \mathcal{H}_\infty^{i\ell} \\ \downarrow\uparrow & & \downarrow\uparrow \\ \mathcal{H}_\infty^{p\ell I} & \rightleftarrows & \mathcal{H}_\infty^{i\ell I} \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathcal{H}_\infty^{p} & \rightleftarrows & \mathcal{H}_\infty^{i} \\ \downarrow\uparrow & & \downarrow\uparrow \\ \mathcal{H}_\infty^{pI} & \rightleftarrows & \mathcal{H}_\infty^{iI} \\ \downarrow\uparrow & & \downarrow\uparrow \\ \mathcal{H}_\infty^{pI\ell} & \rightleftarrows & \mathcal{H}_\infty^{iI\ell} \end{array}$$

where the rightwards and downwards arrows are the left adjoints. All arrows are identity functors, and all horizontal arrows are Quillen equivalences.

Proof. This follows directly from [Bunb, Prop. 3.13] and the well-known fact that the identity functor on $\mathcal{H}_\infty$ induces a Quillen equivalence $\mathcal{H}_\infty^p \rightleftarrows \mathcal{H}_\infty^i$. ◻

Proposition 2.6 *Each pair of model categories defined in Definition 2.2(1)–(4) (based on either the projective or the injective model structure), respectively, has the same weak equivalences.*

Proof. For the model structures in (1) this is clear by definition of the projective and injective model structures. Regarding the pairs of model categories in (2), let $Q\colon \mathcal{H}_\infty \to \mathcal{H}_\infty$ be a cofibrant replacement functor for the projective model structure $\mathcal{H}_\infty^p$. A morphism $f\colon F \to G$ is a weak equivalence in $\mathcal{H}_\infty^{p\ell}$ if and only if $Qf\colon QF \to QG$ is a weak equivalence in $\mathcal{H}_\infty^{p\ell}$. Since the identity functor on $\mathcal{H}_\infty$ induces a Quillen equivalence $\mathcal{H}_\infty^{p\ell} \rightleftarrows \mathcal{H}_\infty^{i\ell}$ (Proposition 2.5), and since the left adjoint of a Quillen equivalence preserves and reflects weak equivalences between cofibrant objects, Qf is a weak equivalence in $\mathcal{H}_\infty^{p\ell}$ if and only if it is a weak equivalence in $\mathcal{H}_\infty^{i\ell}$. Since the canonical natural transformation $Q \xrightarrow{\sim} 1$ is an objectwise weak equivalence, that is equivalent to f being a weak equivalence in $\mathcal{H}_\infty^{p\ell}$ itself. The proofs for (3) and (4) are analogous. ◻

The reason why we also refer to fibrant objects in $\mathcal{H}_\infty^{p/i\,I}$ as *locally constant* simplicial presheaves is the following fact (the second statement is a generalisation of [Dugb, Lemma 3.4.2]):

Proposition 2.7 *Let $F \in \mathcal{H}_\infty$. The following statements hold true:*

(1) The canonical morphism $F \otimes \mathcal{Y}_c \to F$ is a weak equivalence in $\mathcal{H}_\infty^{p/i\,I}$, for every $c \in \mathcal{C}\mathrm{art}$.

(2) The object F is fibrant in $\mathcal{H}_\infty^{p/i\,I}$ if and only if it is fibrant in $\mathcal{H}_\infty^{p/i}$ and the canonical map $F(*) \to F(c)$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$ for every $c \in \mathcal{C}\mathrm{art}$

Proof. Ad (1): By Proposition 2.5, it suffices to show this for $\mathcal{H}_\infty^{iI}$. There, every object is cofibrant (since every object in $\mathcal{S}\mathrm{et}_\Delta$ is cofibrant), so that the functor $F \otimes (-)\colon \mathcal{H}_\infty^{iI} \to \mathcal{H}_\infty^{iI}$ is left Quillen. Thus, it suffices to show that the morphism $\mathcal{Y}_c \to *$ is a weak equivalence. Since $c \cong \mathbb{R}^n$ for some $n \in \mathbb{N}_0$, we can reduce to the case where $c = \mathbb{R}^n$.

We can write the collapse morphism $\mathbb{R}^n \to *$ as a composition

$$\mathbb{R}^n \cong \mathbb{R}^{n-1} \times \mathbb{R} \longrightarrow \mathbb{R}^{n-1} \cong \mathbb{R}^{n-2} \times \mathbb{R} \longrightarrow \cdots \longrightarrow *\,,$$

where each arrow is an element of I. Thus, the claim follows.

Ad (2): The second condition implies that F is fibrant in $\mathcal{H}_\infty^{p/i\,I}$: since $\mathcal{C}\mathrm{art}$ has finite products, we have a commutative triangle

$$\begin{array}{ccc} & F(*) & \\ {\scriptstyle\sim}\swarrow & & \searrow{\scriptstyle\sim} \\ F(c) & \longrightarrow & F(c \times \mathbb{R}) \end{array}$$

for any $c \in \mathcal{C}\mathrm{art}$. The fact that F is I-local thus follows from the two-out-of-three property of weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$.

It remains to check the other implication. By part (1) we know that for each $c \in \mathcal{C}\mathrm{art}$, the morphism $\mathcal{Y}_c \to *$ is a weak equivalence in $\mathcal{H}_\infty^{p/i\,I}$. The claim then follows from the enriched Yoneda Lemma: the top arrow in the commutative diagram

$$\begin{array}{ccc} \underline{\mathcal{H}}_\infty(*, F) & \longrightarrow & \underline{\mathcal{H}}_\infty(\mathcal{Y}_c, F) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ F(*) & \longrightarrow & F(c) \end{array}$$

is a weak equivalence since $\mathcal{H}_\infty^{p/i\,I}$ is symmetric monoidal (Proposition 2.4), so that $\underline{\mathcal{H}}_\infty(-,F)$ is a right Quillen functor. As representables are cofibrant in $\mathcal{H}_\infty^{p/i\,I}$, the functor $\underline{\mathcal{H}}_\infty(-,F)$ thus preserves the weak equivalence $\mathcal{Y}_c \xrightarrow{\sim} *$. □

We can show that our definitions of model structures on $\mathcal{H}_\infty$ are redundant. This uses the following strong theorem, which goes back to Joyal.

Theorem 2.8 [Rie14, Thm. 15.3.1] *Let $\mathcal{M}$ and $\mathcal{M}'$ be two model categories with the same underlying category. Then, $\mathcal{M}$ and $\mathcal{M}'$ coincide as model categories if and only if they have the same cofibrations and the same fibrant objects.*

Corollary 2.9 *We have to following identities of model categories:*

$$\mathcal{H}_\infty^{p/i\,\ell I} = \mathcal{H}_\infty^{p/i\,I} = \mathcal{H}_\infty^{p/i\,I\ell}\,.$$

In particular, every Čech-local weak equivalence is an I-local weak equivalence.

Proof. By their construction as left Bousfield localisations, all of the above three model categories have the same cofibrations. Thus, it suffices to check that their fibrant objects coincide.

We first show that $\mathcal{H}_\infty^{p/i\,\ell I} = \mathcal{H}_\infty^{p/i\,I}$. An object $F \in \mathcal{H}_\infty^{p/i\,\ell I}$ is fibrant if and only if it is fibrant in $\mathcal{H}_\infty^{p/i\,\ell}$ and satisfies that the canonical map

$$F(c) \cong \underline{\mathcal{H}}_\infty(\mathcal{Y}_c, F) \longrightarrow \underline{\mathcal{H}}_\infty(\mathcal{Y}_c \times \mathcal{Y}_\mathbb{R}, F) \cong F(c \times \mathbb{R})$$

is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$, for every $c \in \mathcal{C}\mathrm{art}$. That is, an object in $\mathcal{H}_\infty^{p/i\,\ell I}$ is fibrant precisely if it is fibrant in both $\mathcal{H}_\infty^{p/i\,\ell}$ and in $\mathcal{H}_\infty^{p/i\,I}$. In particular, this implies that F is fibrant also in $\mathcal{H}_\infty^{p/i\,I}$.

Conversely, let $F \in \mathcal{H}_\infty^{p/i\,I}$ be fibrant. We need to check that F satisfies descent with respect to the Grothendieck pretopology τ of differentiably good open coverings on $\mathcal{C}\mathrm{art}$. To that end, let $c \in \mathcal{C}\mathrm{art}$, let $\mathcal{U} = \{c_i \to c\}_{i\in\Lambda}$ be a covering of c in the site $(\mathcal{C}\mathrm{art}, \tau)$, and consider the commutative diagram

$$\begin{aligned} F(c) \longrightarrow &\operatorname*{holim}_{\mathbb{\Delta}}{}^{\mathcal{S}\mathrm{et}_\Delta}\Big(\cdots \prod_{i_0,\ldots,i_n} \underline{\mathcal{H}}_\infty(QC_{i_0\ldots i_n}, F)\cdots\Big) \\ &\simeq \operatorname*{holim}_{\mathbb{\Delta}}{}^{\mathcal{S}\mathrm{et}_\Delta}\Big(\cdots \prod_{i_0,\ldots,i_n} F(C_{i_0\ldots i_n})\cdots\Big)\,, \end{aligned}$$

where $C_{i_0\ldots i_n}$ are as in (2.1). (The superscript on holim denotes the category in which the homotopy limit is formed.) Since F is locally constant (i.e. fibrant in $F \in \mathcal{H}_\infty^{p/i\,I}$), by Proposition 2.7 the collapse maps $c \to *$ induce weak equivalences $F(*) \xrightarrow{\sim} F(c)$. We thus have a commutative diagram

$$\begin{array}{ccc} F(*) & \longrightarrow & \operatorname*{holim}_{\mathbb{\Delta}}{}^{\mathcal{S}\mathrm{et}_\Delta}\Big(\cdots \prod_{i_0,\ldots,i_n\in\Lambda} F(*)\cdots\Big) \\ \Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} \\ F(c) & \longrightarrow & \operatorname*{holim}_{\mathbb{\Delta}}{}^{\mathcal{S}\mathrm{et}_\Delta}\Big(\cdots \prod_{i_0,\ldots,i_n\in\Lambda} F(C_{i_0\ldots i_n})\cdots\Big) \end{array}$$

in $\mathcal{S}\mathrm{et}_\Delta$. We claim that the top morphism in this diagram is an equivalence: to see this, we first note that, by assumption on the covering $\mathcal{U}$, the collapse morphism $\mathrm{Sing}(C_{i_0\ldots i_n}) \to *$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$ for any $i_0, \dots, i_n \in \Lambda$ such that $C_{i_0 \dots i_n}$ is non-empty. For any fibrant $K \in \mathcal{S}\mathrm{et}_\Delta$, we thus obtain a weak equivalence

$$K \cong \underline{\mathcal{S}\mathrm{et}_\Delta}(*, K) \xrightarrow{\sim} \underline{\mathcal{S}\mathrm{et}_\Delta}\big(\mathrm{Sing}(C_{i_0 \dots i_n}), K\big)\,.$$

Since $F(*) \in \mathcal{S}\mathrm{et}_\Delta$ is fibrant, the product of these morphisms indexed by $i_0, \dots, i_n \in \Lambda$ is still a weak equivalence, so we obtain a commutative diagram

$$\begin{array}{ccc} F(*) & \longrightarrow & \underset{\mathbb{\Delta}}{\mathrm{holim}}^{\mathcal{S}\mathrm{et}_\Delta}\Big(\cdots \prod\limits_{i_0,\dots,i_n \in \Lambda} F(*) \cdots\Big) \\ {\scriptstyle\sim}\downarrow & & \downarrow{\scriptstyle\sim} \\ \underline{\mathcal{S}\mathrm{et}_\Delta}\big(\mathrm{Sing}(c), F(*)\big) & \longrightarrow & \underline{\mathcal{S}\mathrm{et}_\Delta}\Big(\underset{\mathbb{\Delta}^{\mathrm{op}}}{\mathrm{hocolim}}^{\mathcal{S}\mathrm{et}_\Delta}\Big(\cdots \coprod\limits_{i_0,\dots,i_n \in \Lambda} \mathrm{Sing}(C_{i_0 \dots i_n}) \cdots\Big), F(*)\Big) \end{array} \tag{2.10}$$

It follows from an application of [Lur09, Thm. A.3.1], or by [DI04, Thm. 1.1] that the morphism

$$\underset{\mathbb{\Delta}^{\mathrm{op}}}{\mathrm{hocolim}}^{\mathcal{S}\mathrm{et}_\Delta}\Big(\cdots \coprod_{i_0,\dots,i_n \in \Lambda} \mathrm{Sing}(C_{i_0 \dots i_n}) \cdots\Big) \longrightarrow \mathrm{Sing}(c)$$

is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$. This is preserved by the right Quillen functor $\underline{\mathcal{S}\mathrm{et}_\Delta}(-, F(*))$, and thus the claim follows. The proof for $\mathcal{H}_\infty^{p/i\,I\ell} = \mathcal{H}_\infty^{p/i\,I}$ works entirely in parallel. □

Remark 2.11 Corollary 2.9 fails for simplicial (pre)sheaves on manifolds: the proof we give above relies on the fact that the left-hand vertical morphism in Diagram (2.10) is a weak equivalence. This is true because every $c \in \mathcal{C}\mathrm{art}$ has an underlying topological space which is contractible. In contrast, consider a simplicial presheaf on manifolds, $G \colon \mathcal{M}\mathrm{fd}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$, which is projectively fibrant and I-local, i.e. it satisfies that the canonical morphism $G(M) \to G(M \times \mathbb{R})$ is a weak equivalence for every $M \in \mathcal{M}\mathrm{fd}$. Then, G does not necessarily satisfy descent with respect to open coverings of manifolds. For instance, consider the presheaf $[-, \mathbb{S}^1]$, sending $M \in \mathcal{M}\mathrm{fd}$ to the set of homotopy classes of continuous (or smooth) maps from M to $\mathbb{S}^1$. This is projectively fibrant and I-local. However, let $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ be an open covering of $M = \mathbb{S}^1$ such that each finite intersection $U_{i_0 \dots i_n}$ is empty or a cartesian space. Then,

$$\mathrm{holim}_{\mathbb{\Delta}}^{\mathcal{S}\mathrm{et}_\Delta} \prod_{i_0,\dots,i_n \in \Lambda} [U_{i_0 \dots i_n}, \mathbb{S}^1] \simeq *\,,$$

but $[\mathbb{S}^1, \mathbb{S}^1] \not\simeq *$. For more details on the relation between sheaves on $\mathcal{C}\mathrm{art}$ and sheaves on $\mathcal{M}\mathrm{fd}$, see Appendix B. ◁

Corollary 2.12 *The model category $\mathcal{H}_\infty^{p/i\,I\ell}$ is symmetric monoidal.*

Proposition 2.13 *Let $L_{\mathbb{R}^\bullet}\mathcal{H}_\infty^{p/i}$ denote the simplicial left Bousfield localisation of $\mathcal{H}_\infty^{p/i}$ at the collapse morphisms $\{c \to *\}_{c \in \mathcal{C}\mathrm{art}}$. Further, let $L_{\mathcal{C}\mathrm{art}}\mathcal{H}_\infty^{p/i}$ denote the left Bousfield localisation of $\mathcal{H}_\infty^{p/i}$ at all morphisms in $\mathcal{C}\mathrm{art}$. We have the following identities of model categories:*

$$\mathcal{H}_\infty^{p/i\,I} \overset{(1)}{=} L_{\mathbb{R}^\bullet}\mathcal{H}_\infty^{p/i} \overset{(2)}{=} L_{\mathcal{C}\mathrm{art}}\mathcal{H}_\infty^{p/i}\,.$$

Proof. The first identity follows from Proposition 2.7 and Theorem 2.8. The second identity holds true since for any morphism $c \to d$ in $\mathcal{C}\mathrm{art}$ there exists a commutative triangle

$$\begin{array}{ccc} c & \longrightarrow & d \\ & \searrow \quad \swarrow & \\ & * & \end{array}$$

Therefore, the weak equivalences and the cofibrations in $L_{\mathbb{R}^\bullet}\mathcal{H}_\infty^{p/i}$ and in $L_{\mathcal{C}\mathrm{art}}\mathcal{H}_\infty^{p/i}$ coincide. $\square$

2.2 Evaluation on the point

Here we present the first of several ways of extracting a space from an object $F \in \mathcal{H}_\infty$ and show that it provides a Quillen equivalence between $\mathcal{H}_\infty^{p/i\,I}$ and the Kan-Quillen model category $\mathcal{S}\mathrm{et}_\Delta$.

Consider the adjunction

$$\tilde{\mathsf{c}} : \mathcal{S}\mathrm{et}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}_\infty^{p/i} : \mathrm{ev}_* \,,$$

whose left adjoint $\tilde{\mathsf{c}}$ sends a simplicial set K to the constant simplicial presheaf with value K, and whose right adjoint evaluates a simplicial presheaf at the final object $* \in \mathcal{C}\mathrm{art}$. (Indeed, the adjunction is Quillen for both targets $\mathcal{H}_\infty^p$ and $\mathcal{H}_\infty^i$; in the projective case, we readily see that ev_* is right Quillen, and in the injective case we see that $\tilde{\mathsf{c}}$ is left Quillen.) Composing this with the localisation adjunction $\mathcal{H}_\infty^{p/i} \rightleftarrows \mathcal{H}_\infty^{p/i\,I}$, we obtain Quillen adjunctions

$$\tilde{\mathsf{c}} : \mathcal{S}\mathrm{et}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}_\infty^{p/i\,I} : \mathrm{ev}_* \,. \tag{2.14}$$

Lemma 2.15 *Let $e\colon \tilde{\mathsf{c}} \circ \mathrm{ev}_* \to 1_{\mathcal{H}_\infty}$ denote the evaluation natural transformation of the adjunction* (2.14)*. The morphism $e_{|F}\colon \tilde{\mathsf{c}}\circ\mathrm{ev}_*(F) \longrightarrow F$ is an objectwise weak equivalence whenever $F \in \mathcal{H}_\infty^{p/i\,I}$ is fibrant.*

Proof. For any $F \in \mathcal{H}_\infty$, the morphism $e_{|F}$ of simplicial presheaves is the morphisms $F(c) \to F(*)$ in $\mathcal{S}\mathrm{et}_\Delta$ induced by the collapse maps $c \to *$. It readily follows from Proposition 2.7 that $e_{|F}$ is an objectwise weak equivalence whenever F is fibrant. $\square$

Lemma 2.16 *Let $K \in \mathcal{S}\mathrm{et}_\Delta$ be any simplicial set. Let $R^i\colon \mathcal{H}_\infty^i \to \mathcal{H}_\infty^i$ denote a fibrant replacement functor in $\mathcal{H}_\infty^i$ (see Appendix A for an explicit construction). Then, the simplicial presheaf $R^i\tilde{\mathsf{c}}(K)$ is fibrant in $\mathcal{H}_\infty^{iI}$.*

Proof. The simplicial presheaf $R^i \circ \tilde{\mathsf{c}}(K)$ is fibrant in $\mathcal{H}_\infty^i$ by construction. It further comes with an *objectwise* weak equivalence $r^i_{\tilde{\mathsf{c}}K}\colon \tilde{\mathsf{c}}K \overset{\sim}{\longrightarrow} R^i\tilde{\mathsf{c}}(K)$. Thus, it follows by the two-out-of-three property of weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$ that, for any $c \in \mathcal{C}\mathrm{art}$, the canonical map $(R^i\tilde{\mathsf{c}}(K))(*) \longrightarrow (R^i\tilde{\mathsf{c}}(K))(c)$ is a weak equivalence. The claim then follows from Proposition 2.13. $\square$

We can now prove a version of [Dugb, Thm. 3.4.3] in the context of simplicial presheaves on cartesian spaces rather than on manifolds. (There, the proof is outlined for simplicial presheaves on manifolds, where several additional steps are necessary. Since we work over cartesian spaces, we can employ a slightly different strategy in our proof that allows us to avoid these additional steps.)

Theorem 2.17 *The Quillen adjunction $\tilde{\mathsf{c}} \dashv \mathrm{ev}_*$ from* (2.14) *is a Quillen equivalence.*

Proof. It is evident that $\tilde{\mathsf{c}}$ both preserves and reflects weak equivalences as a functor $\tilde{\mathsf{c}}\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{H}_\infty^{p/i}$. We claim that it still has that property as a functor $\tilde{\mathsf{c}}\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{H}_\infty^{p/i\,I}$. First, since every weak equivalence in $\mathcal{H}_\infty^{p/i}$ is also a weak equivalence in $\mathcal{H}_\infty^{p/i\,I}$, it follows that $\tilde{\mathsf{c}}\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{H}_\infty^{p/i\,I}$ still preserves weak equivalences. It thus remains to check that it still reflects weak equivalences. To see this, let $\psi\colon K \to L$ be an arbitrary morphism in $\mathcal{S}\mathrm{et}_\Delta$. Since $\tilde{\mathsf{c}}$ takes values in cofibrant objects (in both the projective and the injective situation), it follows that the morphism $\tilde{\mathsf{c}}\psi \colon \tilde{\mathsf{c}}K \to \tilde{\mathsf{c}}L$ is a weak equivalence in $\mathcal{H}_\infty^{p/i\, I}$ if and only if, for every fibrant object $G \in \mathcal{H}_\infty^{p/i\, I}$, the induced morphism

$$\underline{\mathcal{H}_\infty}(\tilde{\mathsf{c}}L, G) \xrightarrow{(\tilde{\mathsf{c}}\psi)^*} \underline{\mathcal{H}_\infty}(\tilde{\mathsf{c}}K, G)$$

is a weak equivalence of simplicial sets. By adjointness, this is equivalent to requiring that the morphism

$$\underline{\mathcal{S}\mathrm{et}_\Delta}(L, \mathrm{ev}_* G) \xrightarrow{\psi^*} \underline{\mathcal{S}\mathrm{et}_\Delta}(K, \mathrm{ev}_* G)$$

be a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$ whenever $G \in \mathcal{H}_\infty^{p/i\, I}$ is fibrant. Suppose that this is the case, i.e. suppose that $\tilde{\mathsf{c}}(\psi)$ is a weak equivalence in $\mathcal{H}_\infty^{p/i\, I}$.

We briefly need to treat the projective and injective cases separately: for a model category $\mathcal{C}$, let $\mathcal{C}_f$ denote the full subcategory of $\mathcal{C}$ on the fibrant objects. In the projective case, the functor $\mathrm{ev}_* \colon (\mathcal{H}_\infty^{pI})_f \to (\mathcal{S}\mathrm{et}_\Delta)_f$ is surjective; it hits every Kan complex. Therefore, it follows that the map

$$\underline{\mathcal{S}\mathrm{et}_\Delta}(L, T) \xrightarrow{\psi^*} \underline{\mathcal{S}\mathrm{et}_\Delta}(K, T)$$

is a weak equivalence for every Kan complex $T \in \mathcal{S}\mathrm{et}_\Delta$. Hence, ψ is a weak equivalence. For the injective case, one may now invoke the fact that $\mathcal{H}_\infty^{pI}$ and $\mathcal{H}_\infty^{iI}$ have the same weak equivalences (Proposition 2.6); that already proves the claim in the injective case as well.

However, for later purposes, we briefly give the following alternative argument: the construction in Lemma 2.16 shows that every Kan complex is weakly equivalent to one in the image of the functor $\mathrm{ev}_* \colon (\mathcal{H}_\infty^{iI})_f \to (\mathcal{S}\mathrm{et}_\Delta)_f$. By the two-out-of-three property and the fact that every simplicial set is cofibrant, it then follows that the map

$$\underline{\mathcal{S}\mathrm{et}_\Delta}(L, T) \xrightarrow{\psi^*} \underline{\mathcal{S}\mathrm{et}_\Delta}(K, T)$$

is a weak equivalence for *every* Kan complex T, and thus that ψ is a weak equivalence.

To complete the proof that $\tilde{\mathsf{c}} \dashv \mathrm{ev}_*$ provides a Quillen equivalence $\mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathcal{H}_\infty^{p/i\, I}$, let $F \in \mathcal{H}_\infty^{p/i\, I}$ be fibrant, and consider the composition

$$\tilde{\mathsf{c}} \circ Q^{\mathcal{S}\mathrm{et}_\Delta} \circ \mathrm{ev}_*(F) \xrightarrow{\tilde{\mathsf{c}}(q_{|\, \mathrm{ev}_* F})} \tilde{\mathsf{c}} \circ \mathrm{ev}_*(F) \xrightarrow{e_{|F}} F\,,$$

where $Q^{\mathcal{S}\mathrm{et}_\Delta}$ is a cofibrant replacement functor in $\mathcal{S}\mathrm{et}_\Delta$ with natural weak equivalence $q \colon Q^{\mathcal{S}\mathrm{et}_\Delta} \to 1$, and where the second morphism $e_{|F}$ is the component at F of the evaluation of the adjunction $\tilde{\mathsf{c}} \dashv \mathrm{ev}_*$. Since every object in $\mathcal{S}\mathrm{et}_\Delta$ is cofibrant and since $\tilde{\mathsf{c}}$ is left Quillen, we have that $\tilde{\mathsf{c}}(q_{|\, \mathrm{ev}_* F})$ is a weak equivalence. Further, the morphism $e_{|F}$ is an equivalence by Lemma 2.15. The claim now follows by [Hov99, Cor. 1.3.6]. □

Example 2.18 Let G be a Lie group with Lie algebra $\mathfrak{g}$. Consider the object $\mathcal{B}\mathrm{un}^\nabla_{G,0} \in \mathcal{H}_\infty^p$, whose value on $c \in \mathcal{C}\mathrm{art}$ is the nerve of the following groupoid (in particular, $\mathcal{B}\mathrm{un}^\nabla_{G,0}$ is fibrant in $\mathcal{H}_\infty^p$ by construction): its objects are $\mathfrak{g}$-valued 1-forms $A \in \Omega^1(c, \mathfrak{g})$ such that $\mathrm{d}A + \frac{1}{2}[A, A] = 0$, and its morphisms $A \to A'$ are smooth maps $g \colon c \to G$ such that $A' = \mathrm{Ad}(g^{-1}) \circ A + g^*\mu_G$, where μ_G is the Maurer-Cartan form on G. In other words, A is a flat G-connection on a trivial principal G-bundle on c, and g is equivalently a morphism of flat principal G-bundles on c. In particular, any such morphism g is actually a *constant* map $g \colon c \to G$. Observe that $\mathcal{B}\mathrm{un}^\nabla_{G,0}(*)$ is the nerve of the groupoid with one object and the group underlying G as its morphisms. It hence follows that the functor $\mathcal{B}\mathrm{un}^{\nabla}_{G,0}(*) \longrightarrow \mathcal{B}\mathrm{un}^{\nabla}_{G,0}(c)$ is fully faithful (on the underlying groupoids), for any $c \in \mathcal{C}\mathrm{art}$. Since any flat G-bundle on c is isomorphic to the trivial flat G-bundle (because $c \cong \mathbb{R}^n$ for some $n \in \mathbb{N}_0$), the functor $\mathcal{B}\mathrm{un}^{\nabla}_{G,0}(*) \longrightarrow \mathcal{B}\mathrm{un}^{\nabla}_{G,0}(c)$ is also essentially surjective. Since the nerve of an equivalence of groupoids is an equivalence of Kan complexes, it follows that $\mathcal{B}\mathrm{un}^{\nabla}_{G,0}$ is a fibrant object in $\mathcal{H}^{pI}_{\infty}$. ◁

2.3 Topological realisation

In this subsection we further build on and extend ideas from [Dugb] to investigate a second way of obtaining a space from a simplicial presheaf on $\mathcal{C}\mathrm{art}$. This time, we send a simplicial presheaf to a certain coend valued in topological spaces.

More precisely, we let $\Delta\mathcal{T}\mathrm{op}$ denote the category of Δ-generated topological spaces (see [Duga, Vog71] for background). We will be working with $\Delta\mathcal{T}\mathrm{op}$ as our choice of category of topological spaces throughout; however, most of the theory in this paper also works with the category of Kelley spaces (see, for instance, [Hov99]), except for where we work explicitly with diffeological spaces (Lemma 2.27, Remark 4.12).

We provide some very compact background on Δ-generated topological spaces. A topological space X is Δ-generated precisely if its topology coincides with the final topology induced by all continuous maps $|\Delta^n| \to X$, for all $n \in \mathbb{N}_0$. (Here, $|\Delta^n|$ is the standard topological n-simplex.) In particular, the category of Δ-generated topological spaces and continuous maps, denoted $\Delta\mathcal{T}\mathrm{op}$, is symmetric monoidal and cartesian closed [Vog71]. Note, however, that the product in $\Delta\mathcal{T}\mathrm{op}$ is not the usual product of topological spaces—one has to pass to the Δ-generated topology after taking the usual product of topological spaces.

The category $\Delta\mathcal{T}\mathrm{op}$ carries a cofibrantly generated model structure, having the same generating cofibrations and generating trivial cofibrations as the standard model structure on topological spaces [Duga]. The geometric realisation functor $|{-}|\colon \mathcal{S}\mathrm{et}_\Delta \to \Delta\mathcal{T}\mathrm{op}$ takes values in Δ-generated spaces, since $\Delta\mathcal{T}\mathrm{op}$ is closed under colimits of topological spaces and contains $|\Delta^n|$ for each $n \in \mathbb{N}_0$. By construction of the model structure on $\Delta\mathcal{T}\mathrm{op}$, the induced adjunction $|{-}| : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \Delta\mathcal{T}\mathrm{op} : \mathrm{Sing}$ is a Quillen adjunction ($|{-}|$ sends generating (trivial) cofibrations to (trivial) cofibrations). Further, by the same proof as in [Hov99, Lemma 3.18], it follows that $|{-}|$ preserves finite products. Then, the proof of [Hov99, Prop. 4.2.11] applies as well, showing that $\Delta\mathcal{T}\mathrm{op}$ is a symmetric monoidal model category. It also follows that $\Delta\mathcal{T}\mathrm{op}$ is a simplicial model category and that $|{-}|$ is a monoidal left Quillen functor. Finally, the Quillen adjunction $|{-}| \dashv \mathrm{Sing}$ is even a Quillen equivalence, since the inclusion of $\Delta\mathcal{T}\mathrm{op}$ into Kelley spaces (or all topological spaces) is a Quillen equivalence [Duga]; the claim then follows from the two-out-of-three property of Quillen equivalences.

In this section, we provide a left Quillen equivalence $\mathcal{H}^{pI}_{\infty} \to \Delta\mathcal{T}\mathrm{op}$. The main ideas for this section stem from [Dugb]; there the full proof is technically rather involved. Again, we circumvent these problems here by working over cartesian spaces rather than over the category of manifolds.

Let $\mathcal{D}\mathrm{fg}$ denote the category of diffeological spaces as defined in [Bunb]—this is the full subcategory of $\mathcal{C}\mathrm{at}(\mathcal{C}\mathrm{art}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ on the concrete sheaves with respect to the Grothendieck pretopology τ (see the beginning of Section 2.1). Concretely, a diffeological space can be defined as a pair (X, Plot_X), where $X \in \mathcal{S}\mathrm{et}$, and where Plot_X assigns to every cartesian space $c \in \mathcal{C}\mathrm{art}$ a subset $\mathrm{Plot}_X(c) \subset \mathcal{S}\mathrm{et}(c, X)$ of the maps from the underlying set of c to X. These maps are called *plots of* X and have to satisfy that

(1) $\mathrm{Plot}_X(*) = X$ (every constant map is a plot),

(2) for every $f \in \mathcal{C}\mathrm{art}(c,d)$ and every $g \in \mathrm{Plot}_X(d)$, we have that $g \circ f \in \mathrm{Plot}_X(c)$ (i.e. Plot_X is a presheaf on $\mathcal{C}\mathrm{art}$), and

(3) the presheaf Plot_X is a sheaf with respect to τ.

We will often identify a diffeological space (X, Plot_X) with the sheaf it defines (see [Bunb] for more background), and we will denote this simply by X.

Example 2.19 For any manifold $M \in \mathcal{M}\mathrm{fd}$, the presheaf $\underline{M}$, given by $c \mapsto \mathcal{M}\mathrm{fd}(c, M)$, is a diffeological space. ◁

Definition 2.20 *Let* $\mathrm{D}\colon \mathcal{D}\mathrm{fg} \to \Delta\mathcal{T}\mathrm{op}$ *be the functor defined as follows: for* $X \in \mathcal{D}\mathrm{fg}$*, we let* $\mathrm{D}X$ *be the underlying set of the diffeological space* $X \in \mathcal{D}\mathrm{fg}$*, endowed with the final topology defined by its plots* $c \to X$*, where* c *ranges over all cartesian spaces. A morphism* $f \in \mathcal{D}\mathrm{fg}(X, Y)$ *is sent to the map it defines on the sets underlying* X *and* Y*. We call* D *the* diffeological topology *functor and* $\mathrm{D}(X)$ *the* underlying topological space of X*.*

The Δ-generated topological spaces are in fact precisely those topological spaces that arise as the underlying topological spaces of diffeological spaces [SYH, CSW14].

Proposition 2.21 [CW14, SYH] *There exists an adjunction*

$$\mathrm{D} : \mathcal{D}\mathrm{fg} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \Delta\mathcal{T}\mathrm{op} : \mathrm{C}\,,$$

where (under the embedding of $\mathcal{D}\mathrm{fg}$ *into presheaves on* $\mathcal{C}\mathrm{art}$*) we have* $\mathrm{C}(T)(c) = \Delta\mathcal{T}\mathrm{op}(c, T)$ *for any topological space* T *and any cartesian space* c*.*

The following proposition consists of results that can already be found in [CSW14]; we only include the proofs here for completeness.

Proposition 2.22 *Let* $(-) \times (-)$ *denote the product in* $\Delta\mathcal{T}\mathrm{op}$*, and let* $(-) \times^t (-)$ *denote the usual product of topological spaces. The functor* D *has the following properties:*

(1) For any manifold M*, the space* $\mathrm{D}(\underline{M})$ *coincides with the underlying topological space of* M*.*

(2) For any manifolds $M, N \in \mathcal{M}\mathrm{fd}$*, the canonical maps* $\mathrm{D}(M \times N) \to \mathrm{D}M \times \mathrm{D}N \to \mathrm{D}M \times^t \mathrm{D}N$ *are homeomorphisms.*

(3) $\mathrm{D}\colon \mathcal{D}\mathrm{fg} \to \Delta\mathcal{T}\mathrm{op}$ *preserves finite products.*

Proof. Part (1) is [CW14, E.g. 3.7]: it is clear that any subset $U \subset M$ which is open in the manifold topology is also open in the diffeological topology. Conversely, if U is open in $\mathrm{D}(\underline{M})$, then its intersection with all images of charts of M must be open. As these images form a basis for the manifold topology, U is open in the manifold topology.

Part (2) follows readily from Part (1) together with the fact that $M \times N$ is again a manifold.

Part (3) is merely [CSW14, Lemma 4.1] and the remarks following that lemma. For completeness, we fill in the details omitted there. In [CSW14] it is proven that the natural map $\mathrm{D}(X \times Y) \to \mathrm{D}X \times \mathrm{D}Y$ is a homeomorphism whenever $\mathrm{D}X$ is locally compact Hausdorff. Since $\mathrm{D}\underline{c}$ is locally compact Hausdorff for any $c \in \mathcal{C}\mathrm{art}$, and since D preserves colimits, we have the following canonical isomorphisms in $\Delta\mathcal{T}\mathrm{op}$: let $X, Y \in \mathcal{D}\mathrm{fg}$ be arbitrary. Using that $\mathcal{D}\mathrm{fg}$ and $\Delta\mathcal{T}\mathrm{op}$ are cartesian closed, we compute

$$\begin{aligned} \mathrm{D}(X \times Y) &\cong \mathrm{D}\big((\operatorname*{colim}_{\mathcal{C}\mathrm{art}/X}{}^{\mathcal{D}\mathrm{fg}} \underline{c}) \times Y\big) \\ &\cong \operatorname*{colim}_{\mathcal{C}\mathrm{art}/X}{}^{\Delta\mathcal{T}\mathrm{op}} \mathrm{D}(\underline{c} \times Y) \end{aligned}$$

$$\cong (\operatorname*{colim}_{\mathcal{C}\mathrm{art}/X}{}^{\Delta\mathcal{T}\mathrm{op}} \mathrm{D}\underline{c}) \times Y$$
$$\cong \mathrm{D}X \times \mathrm{D}Y\,.$$

In the third isomorphism we have used the above-mentioned result [CSW14, Lemma 4.1]. ☐

We point out that we only use manifolds without boundary or corners here. For manifolds with boundary, part (1) of Proposition 2.22 fails—see, for instance, [CW14, Cor. 4.47] and [OT, Warning 2.22]. Since each cartesian space $c \in \mathcal{C}\mathrm{art}$ is diffeomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}_0$, and since $\mathbb{R}^n$ is (isomorphic to) a CW complex for any $n \in \mathbb{N}_0$, it follows that $\mathrm{D}\underline{c}$ is cofibrant in $\Delta\mathcal{T}\mathrm{op}$ for every $c \in \mathcal{C}\mathrm{art}$. We have the following version of [Dug01b, Prop. 2.3]:

Theorem 2.23 *There exists a Quillen adjunction $Re \dashv S$, sitting inside a weakly commutative diagram*

$$\begin{array}{ccc} \mathcal{C}\mathrm{art} & \xrightarrow{\ \mathrm{D}\ } & \Delta\mathcal{T}\mathrm{op} \\ {\scriptstyle \mathcal{Y}}\downarrow & \overset{Re}{\nearrow}\ \underset{S}{\swarrow} & \\ \mathcal{H}_\infty^p & & \end{array} \tag{2.24}$$

Further, there is a canonical natural isomorphism $Re \circ \mathcal{Y} \cong \mathrm{D}$.

Proof. The functor Re is defined as the (enriched) left Kan extension of D along $\mathcal{Y}$ in digram (2.24). Explicitly, we can write

$$\begin{aligned} Re(F) &= \int^{c\in\mathcal{C}\mathrm{art}} F(c) \otimes \mathrm{D}\underline{c} \\ &= \int^{c\in\mathcal{C}\mathrm{art}} |F(c)| \times \mathrm{D}\underline{c}\,, \end{aligned} \tag{2.25}$$

$$\begin{aligned} S(T)(c) &= \Delta\mathcal{T}\mathrm{op}(|\Delta^\bullet| \times \mathrm{D}\underline{c}, T) \\ &\cong \mathrm{Sing}(T^{\mathrm{D}\underline{c}})\,. \end{aligned}$$

Since $\mathrm{D}c$ is cofibrant in $\Delta\mathcal{T}\mathrm{op}$ and $\mathrm{Sing}\colon \mathcal{S}\mathrm{et}_\Delta \to \Delta\mathcal{T}\mathrm{op}$ is right Quillen, it follows that S maps fibrations (resp. trivial fibrations) in $\Delta\mathcal{T}\mathrm{op}$ to objectwise fibrations (resp. trivial fibrations) in $\mathcal{H}_\infty$. Thus, S is right Quillen.

The second claim follow from the canonical isomorphisms

$$Re(\mathcal{Y}_d) = \int^{c\in\mathcal{C}\mathrm{art}} \mathcal{Y}_d(c) \otimes \mathrm{D}\underline{c} \cong \int^{c\in\mathcal{C}\mathrm{art}} \mathcal{C}\mathrm{art}(c,d) \times \mathrm{D}\underline{c} \cong \mathrm{D}\underline{d}\,.$$

The statement now follows from Proposition 2.22(1). ☐

Lemma 2.26 *The adjunction $Re \dashv S$ has the following properties:*

(1) It is a simplicial adjunction.

(2) S is monoidal.

Proof. Part (1) holds true since geometric realisation preserves finite products of simplicial sets and since the functor $K \otimes (-)\colon \Delta\mathcal{T}\mathrm{op} \to \Delta\mathcal{T}\mathrm{op}$ is a left adjoint, for any $K \in \mathcal{S}\mathrm{et}_\Delta$. Part (2) holds true since S is right adjoint and $\Delta\mathcal{T}\mathrm{op}$ is cartesian monoidal. ☐

Lemma 2.27 *Consider the fully faithful inclusion* $\iota\colon \mathcal{D}\mathrm{fg} \hookrightarrow \mathcal{C}\mathrm{at}(\mathcal{C}\mathrm{art}^{\mathrm{op}}, \mathcal{S}\mathrm{et}) \hookrightarrow \mathcal{H}_\infty$. *The diagram*

$$\begin{array}{ccc} \mathcal{D}\mathrm{fg} & \overset{\iota}{\hookrightarrow} & \mathcal{H}_\infty \\ & {\scriptstyle \mathrm{D}}\searrow & \downarrow{\scriptstyle Re} \\ & & \Delta\mathcal{T}\mathrm{op} \end{array}$$

commutes up to natural isomorphism. In particular, for any manifold $M \in \mathcal{M}\mathrm{fd}$, $Re\underline{M}$ *is homeomorphic to the underlying topological space of* M.

Proof. For $X \in \mathcal{D}\mathrm{fg}$, we have canonical natural isomorphisms

$$Re \circ \iota(X) \cong Re \int^{c \in \mathcal{C}\mathrm{art}} \iota(X)(c) \otimes \mathcal{Y}_c \cong \int^{c \in \mathcal{C}\mathrm{art}} \iota(X)(c) \otimes \mathrm{D}\underline{c}$$

and

$$\mathrm{D}X \cong \mathrm{D} \int^{c \in \mathcal{C}\mathrm{art}} \iota(X)(c) \otimes \underline{c} \cong \int^{c \in \mathcal{C}\mathrm{art}} \iota(X)(c) \otimes \mathrm{D}\underline{c}\,.$$

Combining this with Proposition 2.22 completes the proof. □

Proposition 2.28 *The pair* $Re \dashv S$ *induces a Quillen adjunction*

$$Re : \mathcal{H}_\infty^{pI} \rightleftarrows \Delta\mathcal{T}\mathrm{op} : S\,.$$

Proof. By [Hir03, Prop. 3.1.6, Prop. 3.3.18], it suffices to show that S preserves fibrant objects as a functor $\Delta\mathcal{T}\mathrm{op} \to \mathcal{H}_\infty^{pI}$. By Proposition 2.13 we are thus left to check that, for every $T \in \Delta\mathcal{T}\mathrm{op}$ and any $c \in \mathcal{C}\mathrm{art}$, the canonical morphism $S(T)(*) \to S(T)(c)$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$. However, recalling the canonical isomorphism $S(T)(c) \cong \mathrm{Sing}(T^{\mathrm{D}\underline{c}})$, this follows readily from the fact that both $*$ and c are cofibrant in $\Delta\mathcal{T}\mathrm{op}$, that T is fibrant, and that Sing is a right Quillen functor. Consequently, the functor $c \mapsto S(T)(c)$ maps the weak equivalence $c \to *$ in $\Delta\mathcal{T}\mathrm{op}$ to a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$. □

Proposition 2.29 *The functor* Re *from diagram* (2.24) *has the following properties:*

(1) Re *sends the morphism* $F \times \mathcal{Y}_\mathbb{R} \to F$ *to a weak equivalence in* $\Delta\mathcal{T}\mathrm{op}$, *for every cofibrant* $F \in \mathcal{H}_\infty^p$.

(2) Re *sends every Čech nerve* $\check{C}\mathcal{U} \to \mathcal{Y}_c$ *to a weak equivalence in* $\Delta\mathcal{T}\mathrm{op}$, *for every differentiably good open covering* $\mathcal{U} = \{c_a \to c\}_{a \in A}$ *in* $\mathcal{C}\mathrm{art}$.

Proof. Ad (1): The morphism is a weak equivalence in $\mathcal{H}_\infty^{pI}$ between cofibrant objects by Proposition 2.7. Therefore, the claim follows from Proposition 2.28.

Ad (2): Let $\check{C}\mathcal{U} \to \mathcal{Y}_c$ denote Čech nerve of the covering $\mathcal{U}$. We view this as a morphism from a simplicial presheaf $\check{C}\mathcal{U}$ to a simplicially constant presheaf $\mathcal{Y}_c$. Since $\mathcal{U}$ is a differentiably good open covering, $\check{C}\mathcal{U}$ is levelwise a coproduct of representable presheaves on $\mathcal{C}\mathrm{art}$; hence, $\check{C}\mathcal{U}$ is cofibrant in $\mathcal{H}_\infty^p$. By construction of the Čech model structure $\mathcal{H}_\infty^{p\ell}$, we have that the morphism $\check{C}\mathcal{U} \to \mathcal{Y}_c$ is a weak equivalence in $\mathcal{H}_\infty^{p\ell}$. By Corollary 2.9, this is also a weak equivalence in $\mathcal{H}_\infty^{pI}$. The result now follows from Proposition 2.28 and since both $\check{C}\mathcal{U}$ and $\mathcal{Y}_c$ are cofibrant. □

We now prove an important property of the model categories $\mathcal{H}_\infty^{p/i\,I}$ that allows us to detect I-local weak equivalences. Dugger calls this property *rigidity* in [Dugb].

Proposition 2.30 [Dugb, Lemma 3.4.4] *If $F, G \in \mathcal{H}_\infty^{p/i\,I}$ are fibrant, then a morphism $\psi\colon F \to G$ is an I-local weak equivalence if and only if the morphism $\psi_{|*}\colon F(*) \to G(*)$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$.*

Proof of Proposition 2.30. Since ψ is a morphism between local objects in a left Bousfield localisation of $\mathcal{H}_\infty^{p/i}$, it is an equivalence in $\mathcal{H}_\infty^{p/i\,I}$ if and only if it is a weak equivalence in $\mathcal{H}_\infty^{p/i}$. That is, ψ is an I-local weak equivalence if and only if it is an objectwise weak equivalence.

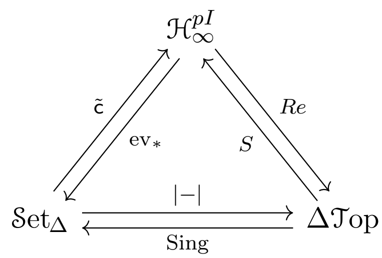
Therefore, it immediately follows that if ψ is an I-local weak equivalence, then $\psi_{|*}$ is a weak equivalence of simplicial sets.

Conversely, suppose that $\psi_{|*}$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$. For any $c \in \mathcal{C}\mathrm{art}$, we have the following commutative square in $\mathcal{S}\mathrm{et}_\Delta$:

$$
\begin{array}{ccc}
F(*) = \big(\tilde{\mathsf{c}}\,\mathrm{ev}_*(F)\big) & \xrightarrow{\;e_{|F}\;} & F(c) \\
{\scriptstyle \psi_{|*}}\big\downarrow & & \big\downarrow{\scriptstyle \psi_{|c}} \\
G(*) = \big(\tilde{\mathsf{c}}\,\mathrm{ev}_*(G)\big) & \xrightarrow[\;e_{|G}\;]{} & G(c)
\end{array}
$$

As both F and G are I-local objects, it follows from Lemma 2.15 that the horizontal morphisms are weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$. Together with the assumption that $\psi_{|*}$ is a weak equivalence on simplicial sets, it now follows that ψ is, in fact, an objectwise weak equivalence. □

Theorem 2.31 *There is a commutative diagram of simplicial Quillen equivalences*



(2.32)

where $\tilde{\mathsf{c}}$, Re, and $|-|$ are the left adjoints.

Proof. It is well-established that the pair $|-| \dashv \mathrm{Sing}$ is a simplicial Quillen equivalence (see e.g. [Hov99]). We have also seen in Theorem 2.17 that the adjunction $\tilde{\mathsf{c}} \dashv \mathrm{ev}_*$ is a simplicial Quillen equivalence. The commutativity of (2.32) follows from the definitions (2.25) of the functors Re and S, which use $|-|$ and Sing, respectively. The fact that $Re \dashv S$ is a Quillen equivalence then follows from the two-out-of-three property of Quillen equivalences and Corollary D.8. □

Remark 2.33 A slightly different version of Theorem 2.31 has been found previously in [Dug01b, Dugb], working over $\mathcal{M}\mathrm{fd}$ instead of $\mathcal{C}\mathrm{art}$. We found that $\mathcal{C}\mathrm{art}$ has several technical advantages (in particular, we do not need to additionally consider stalk-wise weak equivalences) and provides a sufficiently large category of parameter spaces to describe geometric and topological structures, as Theorem 2.31 shows (see also [Bunb] for a treatment of geometric structures). ◁

3 The smooth singular complex of a simplicial presheaf

In this section we introduce the *smooth singular complex*, sometimes also called the *concordance space*, of a simplicial presheaf on $\mathcal{C}\mathrm{art}$. We investigate its homotopical properties—for instance, it sends smooth homotopies to simplicial homotopies—and we establish it both as a left Quillen equivalence $\mathcal{H}_\infty^{p/i\,I} \to \mathcal{S}\mathrm{et}_\Delta$ and as a right Quillen equivalence $\mathcal{H}_\infty^{iI} \to \mathcal{S}\mathrm{et}_\Delta$.

3.1 Extended simplices and the smooth singular complex

In a fashion similar to motivic homotopy theory (see e.g. [MV99, Voe98, DLØ+07]), we consider the extended affine simplices in order to build our smooth singular complex functor. However, we purely rely on the smooth manifold structure of the affine cartesian simplices rather than on their function algebras.

Definition 3.1 *The* extended n-simplex *is the cartesian space*

$$\Delta_e^n \coloneqq \big\{(t^0, \ldots, t^n) \in \mathbb{R}^{n+1} \,\big|\, \sum_{i=0}^n t^i = 1\big\} \subset \mathbb{R}^{n+1}\,.$$

Face and degeneracy maps are defined as the affine linear extensions of the face and degeneracy maps of the standard simplices $|\Delta^n|$. The extended simplices thus define a functor $\Delta_e \colon \mathbb{\Delta} \to \mathcal{C}\mathrm{art}$.

By construction, the topological standard simplex

$$|\Delta^n| = \{t \in \mathbb{R}^{n+1} \,|\, \sum_{i=0}^n t^i = 1,\ 0 \leq t^i \leq 1\ \forall i = 0, \ldots, n\}$$

is a subset of the extended simplex Δ_e^n, for any $n \in \mathbb{N}_0$. This inclusion $|\Delta^n| \hookrightarrow \Delta_e^n$ is compatible with the face and degeneracy maps. Recalling the functor $\mathrm{D} \colon \mathcal{D}\mathrm{fg} \to \Delta\mathcal{T}\mathrm{op}$ from Definition 2.20, we see that there is a morphism

$$\iota \colon |\Delta| \to \mathrm{D}\Delta_e$$

of functors $\mathbb{\Delta} \to \Delta\mathcal{T}\mathrm{op}$. In particular, the diagram

$$\begin{array}{ccc} |\Delta^n| & \xrightarrow{\iota^n} & \mathrm{D}\Delta_e^n \\ {\scriptstyle |\Delta|(\sigma)}\downarrow & & \downarrow{\scriptstyle \mathrm{D}\Delta_e(\sigma)} \\ |\Delta^k| & \xrightarrow[\iota^k]{} & \mathrm{D}\Delta_e^k \end{array} \tag{3.2}$$

in $\Delta\mathcal{T}\mathrm{op}$ commutes for every morphism $\sigma \in \mathbb{\Delta}([n],[k])$.

The extended simplices functor Δ_e induces a Quillen adjunction

$$\mathcal{H}_\infty^p \underset{1}{\overset{1}{\rightleftarrows}} \mathcal{H}_\infty^i \underset{(\Delta_e)_!}{\overset{\Delta_e^*}{\rightleftarrows}} (\mathcal{S}\mathrm{et}_\Delta{}^{\mathbb{\Delta}^{\mathrm{op}}})^i = (\mathcal{S}\mathrm{et}_\Delta{}^{\mathbb{\Delta}^{\mathrm{op}}})^{Reedy}\,.$$

Here we have made use of [Hir03, Thm. 15.8.7], which implies that the injective model structure on bisimplicial sets agrees with the Reedy model structure.

Theorem 3.3 [Rie14, Thm. 5.2.3] *Let $\mathcal{M}$ be a simplicial model category. Then, the realisation functor*

$$|-|_{\mathcal{M}} \colon \mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}} \to \mathcal{M}\,, \qquad X_\bullet \longmapsto \int^{[n]\in\mathbb{\Delta}^{\mathrm{op}}} \Delta^n \otimes X_n$$

is a left Quillen functor with respect to the Reedy model category structure on $\mathcal{M}^{\mathbb{\Delta}^{\mathrm{op}}}$.

Proposition 3.4 *Let $\delta \colon \mathbb{\Delta} \to \mathbb{\Delta} \times \mathbb{\Delta}$ be the diagonal functor. There exists a canonical isomorphism*

$$|-|_{\mathcal{S}\mathrm{et}_\Delta} \cong \delta^*\,,$$

of functors $s\mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$, where $\delta^*(X)_n = X_{n,n}$ is the pullback along the diagonal functor.

Proof. This is a straightforward application of the Yoneda Lemma in the (co)end calculus. □

Corollary 3.5 *The diagonal functor is a left Quillen functor*

$$\delta^* \colon \big(\mathcal{S}\mathrm{et}_\Delta{}^{(\mathbb{\Delta}^{\mathrm{op}})}\big)^i \longrightarrow \mathcal{S}\mathrm{et}_\Delta\,.$$

In particular, it is homotopical, i.e. it preserves all weak equivalences.

Consequently, we can define a left Quillen functor as the composition

$$S_e \coloneqq \delta^* \circ \Delta_e^* \colon \mathcal{H}_\infty^{p/i} \longrightarrow \mathcal{S}\mathrm{et}_\Delta\,.$$

Consider a complete and cocomplete category $\mathcal{E}$, two categories $\mathcal{C}, \mathcal{D}$, and a functor $F \colon \mathcal{C} \to \mathcal{D}$. Recall that, in this situation, the functor $F^* \colon \mathcal{C}\mathrm{at}(\mathcal{D},\mathcal{E}) \longrightarrow \mathcal{C}\mathrm{at}(\mathcal{C},\mathcal{E})$ has a left adjoint $F_!$ and a right adjoint F_*, which are given by the left and the right Kan extension along F. By the construction of S_e as a composition of pullback functors which act on categories of simplicial presheaves, we infer:

Proposition 3.6 *The functor $S_e = \delta^* \circ \Delta_e^*$ has both adjoints. We thus obtain a triple of adjunctions $L_e \dashv S_e \dashv R_e$, where L_e and R_e are given by the compositions*

$$L_e = \Delta_{e!} \circ \delta_!\,, \qquad \text{and} \qquad R_e = \Delta_{e*} \circ \delta_*\,.$$

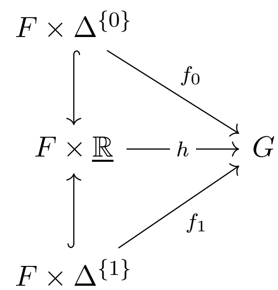
The adjunction $S_e \dashv R_e$ is a simplicial Quillen adjunction.

Definition 3.7 *We call the functor $S_e \colon \mathcal{H}_\infty^{p/i} \to \mathcal{S}\mathrm{et}_\Delta$ the* smooth singular complex functor*. For $F \in \mathcal{H}_\infty$, the simplicial set $S_e F$ is called the* smooth singular complex of F.

3.2 S_e as a left Quillen equivalence

We further investigate the homotopical properties of the smooth singular complex functor S_e. So far, we know that the adjunction $S_e : \mathcal{H}_\infty^{p/i} \rightleftarrows \mathcal{S}\mathrm{et}_\Delta : R_e$ is Quillen. Our goal here is to show that this Quillen adjunction descends to the localisation $\mathcal{H}_\infty^{p/i\,I}$ and that there it even forms a Quillen equivalence.

Definition 3.8 *Let $F, G \in \mathcal{H}_\infty$ be two simplicial presheaves on $\mathcal{C}\mathrm{art}$, and let $f_0, f_1 \colon F \to G$ be a pair* *of morphisms. A* smooth homotopy from f_0 to f_1 *is a commutative diagram*



(3.9)

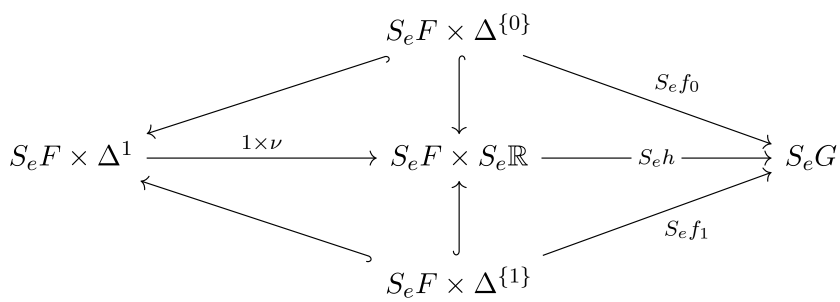
in $\mathcal{H}_\infty$*, where the vertical inclusions are induced by the maps* $* \to \mathbb{R}$*, given by* $* \mapsto 0$ *and* $* \mapsto 1$*.*

Lemma 3.10 *The functor* $S_e \colon \mathcal{H}_\infty \to \mathcal{S}\mathrm{et}_\Delta$ *maps smoothly homotopic morphisms to simplicially homotopic morphisms.*

Proof. The projection $(t^0, t^1) \mapsto t^0$ yields a diffeomorphism $\psi \colon \Delta^1_e \to \mathbb{R}$ of cartesian spaces. Observe that there is a morphism of simplicial sets

$$\nu \colon \Delta^1 \longrightarrow S_e\underline{\mathbb{R}} = \mathcal{C}\mathrm{art}(\Delta^\bullet_e, \mathbb{R})\,,$$

defined by sending the generating non-degenerate 1-simplex of Δ^1 to the 1-simplex ψ. Hence, using the fact that S_e preserves products, we apply S_e to diagram (3.9) and augment it using ν to obtain a commutative diagram



This establishes a simplicial homotopy $S_e h \circ (1_X \times \nu)$ from $S_e f_0$ to $S_e f_1$. □

Lemma 3.10 can be seen as a generalisation of [CW14, Lemma 4.10] away from diffeological spaces to simplicial presheaves. Indeed, the composition

$$\mathcal{D}\mathrm{fg} \xhookrightarrow{\ \iota\ } \mathcal{H}_\infty \xrightarrow{\ S_e\ } \mathcal{S}\mathrm{et}_\Delta$$

is precisely the *smooth singular functor* from [CW14].

Proposition 3.11 *For any* $c \in \mathcal{C}\mathrm{art}$*, the functor* S_e *sends the collapse morphism* $\mathsf{c} \colon \mathcal{Y}_c \to *$ *to a weak equivalence in* $\mathcal{S}\mathrm{et}_\Delta$*.*

Proof. Let $c \in \mathcal{C}\mathrm{art}$, and let $x \in c$ be any point. The inclusion $x \colon * \to c$ induces a smooth homotopy equivalence $* \rightleftarrows c$. The functor S_e maps this to a simplicial homotopy equivalence according to Lemma 3.10. □

Corollary 3.12 *The functor S_e induces Quillen adjunctions*

$$S_e : \mathcal{H}_\infty^{p/i\,I} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{S}\mathrm{et}_\Delta : R_e\,.$$

Proof. Each morphism $\mathcal{Y}_c \times \mathcal{Y}_\mathbb{R} \to \mathcal{Y}_c$ in I is a morphism between cofibrant objects in $\mathcal{H}_\infty^{p/i}$. Therefore, by [Hir03, Prop. 3.3.18] it suffices to show that S_e sends each morphism in I to a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$. One way to see that this holds true is by observing that S_e preserves finite products (it is a monoidal functor). Therefore, we have that

$$\begin{aligned} S_e(\mathcal{Y}_c \times \mathcal{Y}_\mathbb{R} \to \mathcal{Y}_c) &= S_e\big(\mathcal{Y}_c \xrightarrow{1_{\mathcal{Y}_c}} \mathcal{Y}_c\big) \times S_e\big(\mathcal{Y}_\mathbb{R} \xrightarrow{\mathsf{c}} *\big) \\ &= S_e(1_{\mathcal{Y}_c}) \otimes S_e\big(\mathcal{Y}_\mathbb{R} \xrightarrow{\mathsf{c}} *\big)\,, \end{aligned}$$

where $\mathsf{c}\colon \mathcal{Y}_\mathbb{R} \to *$ is the collapse morphism. By Proposition 3.11, the morphism $S_e(\mathsf{c})$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$, and since $\mathcal{S}\mathrm{et}_\Delta$ is a monoidal model category in which every object is cofibrant, it follows that $S_e(1_{\mathcal{Y}_c}) \otimes S_e(\mathsf{c})$ is a weak equivalence as well. □

Proposition 3.13 *The functors $S_e\colon \mathcal{H}_\infty^{p/i\,I} \to \mathcal{S}\mathrm{et}_\Delta$ are homotopical.*

Proof. $S_e\colon \mathcal{H}_\infty^{iI} \to \mathcal{S}\mathrm{et}_\Delta$ is homotopical because it is left Quillen and every object in $\mathcal{H}_\infty^{iI}$ is cofibrant. The corresponding statement for the projective model structure now follows from Proposition 2.6. □

Note, in particular, that by Proposition 2.9 the functor S_e also sends weak equivalences in the Čech local model structures $\mathcal{H}_\infty^{p/i\,\ell}$ to weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$.

Theorem 3.14 *The Quillen adjunctions*

$$S_e : \mathcal{H}_\infty^{p/i\,I} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{S}\mathrm{et}_\Delta : R_e$$

are Quillen equivalences.

Proof. We have Quillen adjunctions

$$\mathcal{S}\mathrm{et}_\Delta \underset{\mathrm{ev}_*}{\overset{\tilde{\mathsf{c}}}{\rightleftarrows}} \mathcal{H}_\infty^{p/i\,I} \underset{R_e}{\overset{S_e}{\rightleftarrows}} \mathcal{S}\mathrm{et}_\Delta$$

and we know from Theorem 2.17 that the Quillen adjunction $\tilde{\mathsf{c}} \dashv \mathrm{ev}_*$ is even a Quillen equivalence. We readily see that $S_e \circ \tilde{\mathsf{c}}$ is the identity functor on $\mathcal{S}\mathrm{et}_\Delta$. Therefore, $S_e \dashv R_e$ is a Quillen equivalence by Corollary D.8. □

Corollary 3.15 *The functor S_e both preserves and reflects weak equivalences in $\mathcal{H}_\infty^{p/i\,I}$.*

Proof. In the injective case, this follows from Theorem 3.14 since every object in $\mathcal{H}_\infty^{iI}$ is cofibrant and S_e is a left Quillen equivalence (see e.g. [Hov99, Prop. 1.3.16]). The projective case then follows from Proposition 2.6. □

Corollary 3.16 *Any smooth homotopy equivalence in $\mathcal{H}_\infty$ is a weak equivalence in $\mathcal{H}_\infty^{p/i\,I}$.*

Proof. By Proposition 3.10, S_e sends smooth homotopy equivalences to simplicial homotopy equivalences, which are, in particular, weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$. Thus, the claim follows from Corollary 3.15. □

Remark 3.17 Let $W_{\mathcal{S}\mathrm{et}_\Delta}$ denote the class of weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$, and let $S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})$ denote the class of morphisms in $\mathcal{H}_\infty$ whose image under S_e is in $W_{\mathcal{S}\mathrm{et}_\Delta}$. Corollary 3.15 lets us suspect that there is an equivalence of model categories

$$\mathcal{H}_\infty^{p/i\,I} \simeq L_{S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})}\mathcal{H}_\infty^{p/i}\,.$$

Using properties of local weak equivalences in Bousfield localisation should allow us to prove that conjecture here already, but instead we give a very direct proof later in Theorem 6.10. ◁

3.3 S_e as a right Quillen equivalence

The goal of this subsection is to establish the smooth singular functor as a right Quillen functor $S_e\colon \mathcal{H}_\infty^{iI} \to \mathcal{S}\mathrm{et}_\Delta$. Apart from having convenient technical implications on the functor $S_e\colon \mathcal{H}_\infty^{iI} \to \mathcal{S}\mathrm{et}_\Delta$, the appearance of several intermediate model structures of bisimplicial sets sheds additional light on the way that the functor S_e works. We already know from Proposition 3.6 that $S_e = \delta^* \circ \Delta_e^*$ has a left adjoint $L_e = \Delta_{e!} \circ \delta_!$. We will show that both its constituting functors $\Delta_{e!}$ and $\delta_!$ are left Quillen functors.

3.3.1 Model structures for ∞-groupoids on the category of bisimplicial sets

We start by analysing the functor $\delta_!$ in more detail. Let

$$\iota_n\colon \mathrm{Sp}_n \coloneqq \Delta^1 \underset{\Delta^0}{\sqcup} \cdots \underset{\Delta^0}{\sqcup} \Delta^1 \lhook\joinrel\longrightarrow \Delta^n$$

denote the *spine-inclusion* of the n-simplex Δ^n, for $n \geq 1$. (Note that for $n = 1$ the morphism ι_1 is an isomorphism.) Let $\mathrm{Sp} \coloneqq \{\iota_n\colon \mathrm{Sp}_n \hookrightarrow \Delta^n \,|\, n \geq 1\}$ denote the set of all spine inclusions.

We write $s\mathcal{S}\mathrm{et}_\Delta = \mathcal{C}\mathrm{at}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta)$ for the category of bisimplicial sets. There exists a bifunctor

$$\boxtimes\colon \mathcal{S}\mathrm{et}_\Delta \times \mathcal{S}\mathrm{et}_\Delta \longrightarrow s\mathcal{S}\mathrm{et}_\Delta\,, \qquad (K \boxtimes L)_{m,n} \coloneqq K_m \times L_n.$$

We view a bisimplicial set X as a simplicial diagram $m \mapsto X_{m,\bullet}$ in $\mathcal{S}\mathrm{et}_\Delta$. Let $J \in \mathcal{S}\mathrm{et}_\Delta$ denote the nerve of the groupoid with two objects and a unique isomorphism between them. The following definitions are taken from [Rez01, Bar05, Hor15].

Definition 3.18 *We define the following model structures on the category $s\mathcal{S}\mathrm{et}_\Delta$ of bisimplicial sets:*

(1) We view $s\mathcal{S}\mathrm{et}_\Delta = \mathcal{C}\mathrm{at}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta)$ as endowed with the injective model structure. Recall that this coincides with the Reedy model structure [Hir03].

(2) We let $\mathcal{S}\mathcal{S}\mathrm{p} \coloneqq L_{\mathrm{Sp}} s\mathcal{S}\mathrm{et}_\Delta$ be the left Bousfield localisation of $s\mathcal{S}\mathrm{et}_\Delta$ at the spine inclusions. This is the model category for Segal spaces.

(3) The model category for complete Segal spaces *is the localisation $\mathcal{C}\mathcal{S}\mathcal{S} \coloneqq L_{J\boxtimes\Delta^0}\mathcal{S}\mathcal{S}\mathrm{p}$.*

Let $L_{\Delta^\bullet\boxtimes\Delta^0} s\mathcal{S}\mathrm{et}_\Delta$ denote the left Bousfield localisation of the injective model category of bisimplicial sets at all collapse morphisms $\{\Delta^n\boxtimes\Delta^0 \to \Delta^0\boxtimes\Delta^0 \,|\, n \in \mathbb{N}_0\}$. Let $L_{\mathbb{\Delta}\boxtimes\Delta^0} s\mathcal{S}\mathrm{et}_\Delta$ denote the left Bousfield localisation of $s\mathcal{S}\mathrm{et}_\Delta$ at all morphisms $\{\Delta^n \boxtimes \Delta^0 \to \Delta^m \boxtimes \Delta^0 \,|\, n, m \in \mathbb{N}_0\}$. (Compare these localisations to those in Proposition 2.13.) We will mostly be using the model category $L_{\Delta^\bullet\boxtimes\Delta^0} s\mathcal{S}\mathrm{et}_\Delta$. However, for conceptual clarity and for an interpretation as model categories for ∞-groupoids, we include the following proposition.

Proposition 3.19 *The following left Bousfield localisations yield identical model categories:*

$$L_{\mathbb{\Delta}\boxtimes\Delta^0} s\mathcal{S}\mathrm{et}_\Delta \overset{(1)}{=} L_{\Delta^\bullet\boxtimes\Delta^0} s\mathcal{S}\mathrm{et}_\Delta \overset{(2)}{=} L_{\Delta^1\boxtimes\Delta^0} \mathcal{S}\mathcal{S}\mathrm{p} \overset{(3)}{=} L_{\Delta^1\boxtimes\Delta^0} \mathcal{C}\mathcal{S}\mathcal{S}\,.$$

Proof. By Theorem 2.8 it suffices to check that all four model categories have the same cofibrations and fibrant objects. For cofibrations, this is trivial since each of the model categories is a left Bousfield localisation of $s\mathcal{S}\mathrm{et}_\Delta$. It thus remains to check that the fibrant objects of the three model categories coincide.

Identity (1) is a direct consequence of the two-out-of-three property of weak equivalences.

For identity (2), let $X \in L_{\Delta^\bullet\boxtimes\Delta^0} s\mathcal{S}\mathrm{et}_\Delta$ be fibrant. That is, X is injective fibrant in $s\mathcal{S}\mathrm{et}_\Delta$ and the canonical map $X_0 \to X_n$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$ for any $n \in \mathbb{N}_0$. We have to show that X satisfies the Segal condition, i.e. that for every $n \geq 2$ the spine inclusion $\mathrm{Sp}_n \hookrightarrow \Delta^n$ induces a weak equivalence

$$X_n \longrightarrow X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1\,.$$

(As pointed out in [Rez01], the strict pullback is a homotopy pullback here because X is Reedy fibrant.) Consider the commutative diagram

$$\begin{array}{ccc} X_n & \longrightarrow & X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1 \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s_0\times\cdots\times s_0} \\ X_0 & \xrightarrow[(1_{X_0},\ldots,1_{X_0})]{\cong} & X_0 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_0 \end{array} \tag{3.20}$$

Since X is Reedy fibrant, the pullbacks on the right-hand side are homotopy pullbacks. Therefore, both vertical maps in (3.20) are weak equivalence. It follows by the commutativity of the diagram that X satisfies the Segal condition. Then, X is fibrant in $L_{\Delta^1\boxtimes\Delta^0}\mathcal{S}\mathcal{S}\mathrm{p}$ since, by assumption, the morphism $X_0 \to X_1$ is a weak equivalence.

Conversely, if X is fibrant in $L_{\Delta^1\boxtimes\Delta^0}\mathcal{S}\mathcal{S}\mathrm{p}$, then the top horizontal morphism in diagram (3.20) is a weak equivalence because X satisfies the Segal condition, and the right-hand vertical morphism is a weak equivalence because X is injective fibrant and X is local with respect to $\Delta^1 \boxtimes \Delta^0 \to \Delta^0 \boxtimes \Delta^0$. It thus follows by the commutativity of the diagram that also the left vertical morphism is an equivalence, for any $n \geq 2$, so that X is fibrant in $L_{\Delta^\bullet\boxtimes\Delta^0} s\mathcal{S}\mathrm{et}_\Delta$.

For identity (3), recall that in any Segal space X there is a notion of when a morphism $f \in X_1$ is invertible (or a 'homotopy equivalence' in the language of [Rez01]). One defines the *space X_{weq} of homotopy equivalences* in X to be the union of those connected components of X_1 that contain invertible morphisms (by [Rez01, Lemma 5.8], if $X \in \mathcal{S}\mathcal{S}\mathrm{p}$ is fibrant, then a connected component of X_1 contains a homotopy equivalence if and only if it consists purely of homotopy equivalences). For any Segal space, the degeneracy morphism $s_0 \colon X_0 \to X_1$ factors as

$$\begin{array}{ccc} & & X_{weq} \\ & {\scriptstyle \widehat{s_0}}\nearrow & \downarrow{\scriptstyle \iota_X} \\ X_0 & \xrightarrow[s_0]{} & X_1 \end{array} \tag{3.21}$$

Let $X \in \mathcal{C}\mathcal{S}\mathcal{S}$ be fibrant. In other words, X is a fibrant object in $\mathcal{S}\mathcal{S}\mathrm{p}$ and the morphism $\widehat{s_0} \colon X_0 \to X_{weq}$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$. Then, X is local in $L_{\Delta^1\boxtimes\Delta^0}\mathcal{C}\mathcal{S}\mathcal{S}$ precisely if, additionally, the morphism $s_0\colon X_0 \to X_1$ is a weak equivalence of simplicial sets. Since every fibrant object in $\mathcal{CSS}$ is also fibrant in $\mathcal{SSp}$, this implies that every fibrant object in $L_{\Delta^1 \boxtimes \Delta^0}\mathcal{CSS}$ is fibrant in $L_{\Delta^1 \boxtimes \Delta^0}\mathcal{SSp}$.

Conversely, let $Y \in \mathcal{SSp}$ be fibrant. Then, Y is local in $L_{\Delta^1 \boxtimes \Delta^0}\mathcal{SSp}$ precisely if the morphism $s_0\colon Y_0 \to Y_1$ is a weak equivalence. We need to show that if this criterion is satisfied, then Y satisfies that $\widehat{s_0}\colon Y_0 \to Y_{weq}$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$. However, since $\iota_Y\colon Y_{weq} \to Y$ is the inclusion of a union of connected components of Y_1, diagram (3.21) and the fact that s_0 is a weak equivalence imply that ι_Y hits every connected component of Y_1. Therefore, ι_Y is a weak equivalence; it follows from the two-out-of-three property that $\widehat{s_0}\colon Y_0 \to Y_{weq}$ is a weak equivalence as well. □

Remark 3.22 Let $X \in \mathcal{SSp}$ be fibrant. Since $\iota_X\colon X_{weq} \to X_1$ is the inclusion of a union of connected components of X_1, it follows that ι_X is a weak equivalence precisely if it is an isomorphism, i.e. precisely if $X_{weq} = X_1$. In other words, a fibrant object in both $L_{\Delta^1 \boxtimes \Delta^0}\mathcal{SSp}$ and $L_{\Delta^1 \boxtimes \Delta^0}\mathcal{CSS}$ is a complete Segal space with all 1-morphisms invertible. In that sense, the fibrant objects in these model categories are ∞-groupoids. The model category $L_{\mathbb{\Delta} \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta$ can be seen as the model category of essentially constant simplicial diagrams of spaces, in analogy to how $L_{\mathcal{C}\mathrm{art}}\mathcal{H}_\infty^{p/i}$ describes locally constant simplicial presheaves. ◁

Remark 3.23 The model structures in Proposition 3.19 agree with the *diagonal model structure* on bisimplicial sets: these model categories have the same underlying categories, the same cofibrations, an, by [Ras, Thm. 2.1], they also have the same fibrant objects. Thus, the claimed equality follows from Theorem 2.8. ◁

Proposition 3.24 *The diagonal $\delta^*\colon s\mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ induces a Quillen adjunction*

$$\delta_! : \mathcal{S}\mathrm{et}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta : \delta^* .$$

Proof. This is a direct consequence of Remark 3.23 and [Ras, Thm. 2.4]. □

From now on, we will understand the adjunction $\delta_! \dashv \delta^*$ as the above Quillen adjunction. There is another Quillen adjunction that relates $L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta$ to the model category of simplicial sets, in analogy with Theorem 2.17.

Proposition 3.25 *Consider the adjoint pair $\mathsf{c}_{\mathbb{\Delta}} : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows s\mathcal{S}\mathrm{et}_\Delta : \mathrm{ev}_{[0]}$, where $\mathsf{c}_{\mathbb{\Delta}} = \Delta^0 \boxtimes (-)$, and where $\mathrm{ev}_{[0]}(X) = X_{0,\bullet}$. This satisfies:*

(1) $\mathsf{c}_{\mathbb{\Delta}} \dashv \mathrm{ev}_{[0]}$ is a Quillen adjunction $\mathcal{S}\mathrm{et}_\Delta \rightleftarrows s\mathcal{S}\mathrm{et}_\Delta$.

(2) Composing the Quillen adjunction from (1) with the localisation adjunction $s\mathcal{S}\mathrm{et}_\Delta \rightleftarrows L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta$ yields a Quillen equivalence

$$\mathsf{c}_{\mathbb{\Delta}} : \mathcal{S}\mathrm{et}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta : \mathrm{ev}_{[0]} .$$

Proof. It is straightforward to see that $\mathsf{c}_{\mathbb{\Delta}}\colon \mathcal{S}\mathrm{et}_\Delta \to s\mathcal{S}\mathrm{et}_\Delta$ preserves cofibrations and further preserves as well as reflects weak equivalences. This proves claim (1).

To see part (2), we first show that $\mathsf{c}_{\mathbb{\Delta}}\colon \mathcal{S}\mathrm{et}_\Delta \to L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta$ still reflects weak equivalences. The logic of the proof is entirely parallel to the proof of Theorem 2.17 for the injective case: consider any morphism $\psi\colon K \to L$ of simplicial sets and suppose that $\mathsf{c}_{\mathbb{\Delta}}\psi\colon \mathsf{c}_{\mathbb{\Delta}}K \to \mathsf{c}_{\mathbb{\Delta}}L$ is a weak equivalence in $L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta$. This is the case if and only if, for any fibrant $X \in L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta$, the induced map

$$\underline{s\mathcal{S}\mathrm{et}_\Delta}(\mathsf{c}_{\mathbb{\Delta}}L, X) \xrightarrow{(\mathsf{c}_{\mathbb{\Delta}}\psi)^*} \underline{s\mathcal{S}\mathrm{et}_\Delta}(\mathsf{c}_{\mathbb{\Delta}}K, X)$$

is an equivalence in $\mathcal{S}\mathrm{et}_\Delta$. By adjointness, that is equivalent to the map

$$\underline{\mathcal{S}\mathrm{et}_\Delta}(L, \mathrm{ev}_{[0]} X) \xrightarrow{\psi^*} \underline{\mathcal{S}\mathrm{et}_\Delta}(K, \mathrm{ev}_{[0]} X)$$

being a weak equivalence of simplicial sets for every $X \in (L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta)_f$. Using a fibrant replacement in the injective model category $s\mathcal{S}\mathrm{et}_\Delta$, we see in analogy with Lemma 2.16 and the proof of Theorem 2.17 that every Kan complex is weakly equivalent to an object in the image of the (Kan-complex-valued) functor $\mathrm{ev}_{[0]} \colon (L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta)_f \longrightarrow (\mathcal{S}\mathrm{et}_\Delta)_f$. Thus, it follows that ψ^* is a weak equivalence.

Now consider a fibrant object $X \in L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta$ and the composition

$$\mathsf{c}_{\mathbb{\Delta}} \circ Q^{\mathcal{S}\mathrm{et}_\Delta} \circ \mathrm{ev}_{[0]}(X) \xrightarrow{\mathsf{c}_{\mathbb{\Delta}}(q_{|\mathrm{ev}_{[0]} X})} \mathsf{c}_{\mathbb{\Delta}} \circ \mathrm{ev}_{[0]}(X) \xrightarrow{e_{|X}} X\,,$$

where $e \colon \mathsf{c}_{\mathbb{\Delta}} \circ \mathrm{ev}_{[0]} \to 1_{s\mathcal{S}\mathrm{et}_\Delta}$ is the evaluation natural transformation of the adjunction $\mathsf{c}_{\mathbb{\Delta}} \dashv \mathrm{ev}_{[0]}$. Further, $q \colon Q^{\mathcal{S}\mathrm{et}_\Delta} \xrightarrow{\sim} 1_{\mathcal{S}\mathrm{et}_\Delta}$ is a cofibrant replacement functor in $\mathcal{S}\mathrm{et}_\Delta$ (which we can take to be the identity, since $(\mathcal{S}\mathrm{et}_\Delta)_c = \mathcal{S}\mathrm{et}_\Delta$). The first morphism is a weak equivalence since $\mathsf{c}_{\mathbb{\Delta}}$ preserves weak equivalences, and the second morphism is a weak equivalence as a consequence of the fibrancy of X; compare to the proof of Theorem 2.17. This proves the claim by [Hov99, Cor. 1.3.16]. □

3.3.2 The functors $\Delta_{e!}$ and L_e

Next, we show that $\Delta_{e!} \colon s\mathcal{S}\mathrm{et}_\Delta \to \mathcal{H}_\infty^{iI}$ is left Quillen, and that the Quillen adjunction $\Delta_{e!} \dashv \Delta_{e*}$ descends to the localisation $L_{\Delta^1 \boxtimes \Delta^0} \mathcal{S}\mathrm{Sp}$ of $s\mathcal{S}\mathrm{et}_\Delta$. The functor $\Delta_{e!}$ acts as

$$\Delta_{e!}(X) = \int^n X_{n,\bullet} \otimes \Delta_e^n\,.$$

In particular, for bisimplicial sets in the image of $(-) \boxtimes (-)$ we find

$$\begin{aligned} \Delta_{e!}(K \boxtimes L) &= \int^n K_n \otimes L \otimes \Delta_e^n \\ &\cong L \otimes \int^n K_n \otimes \Delta_e^n \\ &\cong L \otimes \Delta_{e!}(K \boxtimes \Delta^0)\,. \end{aligned} \tag{3.26}$$

It follows that

$$\Delta_{e!}(\Delta^0 \boxtimes L) \cong \tilde{\mathsf{c}} L \tag{3.27}$$

for any $L \in \mathcal{S}\mathrm{et}_\Delta$, where $\tilde{\mathsf{c}} \colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{H}_\infty$ is the constant-presheaf functor.

Lemma 3.28 *For any $n \in \mathbb{N}_0$, the morphism $\Delta_{e!}(\partial\Delta^n \boxtimes \Delta^0 \longrightarrow \Delta^n \boxtimes \Delta^0)$ is a cofibration in $\mathcal{H}_\infty^i$.*

Proof. For $n = 0, 1$ this is straightforward. Consider the presentation of $\partial\Delta^n$ as a coequaliser,

$$\partial\Delta^n \cong \mathrm{coeq}\bigg(\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{0 \leq k \leq n} \Delta^{n-1} \bigg)\,.$$

Since $\Delta_{e!}$ preserves colimits, we obtain

$$\partial\Delta_e^n \coloneqq \Delta_{e!}(\partial\Delta^n \boxtimes \Delta^0) \cong \mathrm{coeq}\bigg(\coprod_{0 \leq i < j \leq n} \Delta_e^{n-2} \rightrightarrows \coprod_{0 \leq k \leq n} \Delta_e^{n-1} \bigg)\,. \tag{3.29}$$

The colimit is taken in $\mathcal{H}_\infty$ (not in $\mathcal{D}$fg, even though all Δ_e^k are diffeological spaces), and so we have

$$\partial\Delta_e^n(c) \cong \mathrm{coeq}\Big(\coprod_{0 \leq i < j \leq n} \Delta_e^{n-2}(c) \rightrightarrows \coprod_{0 \leq k \leq n} \Delta_e^{n-1}(c) \Big)$$

for any $c \in \mathcal{C}$art. In particular, any section of $\partial\Delta_e^n$ over $c \in \mathcal{C}$art comes from some section $f \in \Delta_e^{n-1}(c)$ of a face of Δ_e^n. Two such sections $f, g \in \Delta_e^{n-1}(c)$ are identified precisely if they factor through the copy of Δ_e^{n-2} that joins the respective faces of Δ_e^n and if, further, f and g agree as maps $c \to \Delta_e^{n-2}$.

Let $f, g \in \partial\Delta_e^n(c)$ be any two elements, and assume that $\iota_e^n \circ f = \iota_e^n \circ g$, where $\iota_e^n \colon \partial\Delta_e^n \longrightarrow \Delta_e^n$ is the canonical morphism. Observe that ι_e^n is injective as a map on the underlying sets $\partial\Delta_e^n(*) \hookrightarrow \Delta_e^n(*)$. Since every section $f \colon \mathcal{Y}_c \to \partial\Delta_e^n$ is, in particular, a map $\mathcal{Y}_c(*) \to \partial\Delta_e^n(*)$ of the underlying sets, and analogously a section $\mathcal{Y}_c \to \Delta_e^n$ is, in particular, a map $\mathcal{Y}_c(*) \to \Delta_e^n(*)$ of underlying sets, it follows that $\iota_e^n \colon \partial\Delta_e^n \to \Delta_e^n$ is an objectwise monomorphism. □

Remark 3.30 We point out that $\partial\Delta_e^n$, as defined in (3.29), is not a diffeological space for $n \geq 2$. For instance, consider a differentiably good open covering $c = c_0 \cup c_1$ of a cartesian space c. We denote the intersection $c_0 \cap c_1$ by $c_{01} \in \mathcal{C}$art. Let $f_i \colon c_i \to \Delta_e^{n-1}$ be smooth maps, for $i = 0, 1$, to adjacent faces of Δ_e^n, such that $f_{i|c_{01}} \colon c_{01} \to \Delta_e^{n-2}$ factors through the $n-2$-simplex which joins the two faces. These data to not lift to a section $f \in \partial\Delta_e^n(c)$, since such an f must factor through one of the faces $\partial\Delta_e^{n-1}$. That is, $\partial\Delta_e^n$ does not satisfy the sheaf condition. ◁

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories. Recall the notion of an *adjunction of two variables* $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ (see, for example, [Hov99, Def. 4.1.12]). We will denote an adjunction of two variables only by its tensor functor $\otimes \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$. If $\mathcal{E}$ has pushouts, then there is an induced *pushout product*, or *box product* on morphisms: given morphisms $f \colon A \to B$ in $\mathcal{C}$ and $g \colon X \to Y$ in $\mathcal{D}$, their pushout product (relative to $\otimes$) is the induced morphism in $\mathcal{E}$ given by

$$A \otimes Y \underset{A \otimes B}{\sqcup} B \otimes X \xrightarrow{f \Box g} B \otimes Y\,.$$

We recall following definitions:

Definition 3.31 [Hov99, Def. 4.2.1] *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be model categories, and let $\otimes \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be an adjunction of two variables. Then, $\otimes$ is a* Quillen adjunction of two variables *if the induced pushout product $f, g \mapsto f \Box g$ satisfies the* pushout-product axiom*:*

(1) if both f and g are cofibrations, then so is $f \Box g$, and

(2) if, in addition, *f or g is a weak equivalence, then so is $f \Box g$.*

Definition 3.32 [Hov99, Def. 4.2.6] *A* (symmetric) monoidal model category *is a closed (symmetric) monoidal category $(\mathcal{C}, \otimes)$ together with a model structure on the underlying category $\mathcal{C}$ such that:*

(1) the closed monoidal structure is a Quillen adjunction of two variables $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

(2) Let $u \in \mathcal{C}$ be the monoidal unit, and let $q_u \colon Q^{\mathcal{C}} u \xrightarrow{\sim} u$ be a cofibrant replacement. Then, tensoring with any cofibrant object from the left or the right sends q_u to a weak equivalence.

Example 3.33 The model category $s\mathcal{S}\mathrm{et}_\Delta$ with the injective model structure is symmetric monoidal. Similarly, each of the model categories $\mathcal{H}_\infty^{p/i}$, $\mathcal{H}_\infty^{p/i\,\ell}$, and $\mathcal{H}_\infty^{p/i\,I}$ is symmetric monoidal by Proposition 2.4. In each of the monoidal model structures we encounter here, the monoidal unit is already cofibrant, so that the second axiom of Definition 3.32 is trivially satisfied. ◁

The injective model structure on $\mathcal{S}\mathrm{et}_\Delta$ is cofibrantly generated (see e.g. [Rez01]), with generating cofibrations

$$\mathcal{I} = \big\{ (\partial\Delta^n \boxtimes \Delta^0 \hookrightarrow \Delta^n \boxtimes \Delta^0) \square (\Delta^0 \boxtimes \partial\Delta^m \hookrightarrow \Delta^0 \boxtimes \Delta^m) \,\big|\, n, m \in \mathbb{N}_0 \big\}$$

and generating trivial cofibrations

$$\mathcal{J} = \big\{ (\partial\Delta^n \boxtimes \Delta^0 \hookrightarrow \Delta^n \boxtimes \Delta^0) \square (\Delta^0 \boxtimes \Lambda^m_k \hookrightarrow \Delta^0 \boxtimes \Delta^m) \,\big|\, m, n \in \mathbb{N}_0,\, 0 \leq k \leq m \big\} .$$

Proposition 3.34 *There is a Quillen adjunction*

$$\Delta_{e!} : s\mathcal{S}\mathrm{et}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}^i_\infty : \Delta_e^* .$$

Proof. We have already seen in Lemma 3.28 that $\Delta_{e!}$ sends the morphism $\partial\Delta^n \boxtimes \Delta^0 \hookrightarrow \Delta^n \boxtimes \Delta^0$ to the injective cofibration $\partial\Delta^n_e \hookrightarrow \Delta^n_e$ in $\mathcal{H}^i_\infty$. Further, it follows from Equations (3.26) and (3.27) that

$$\Delta_{e!}(\Delta^0 \boxtimes \partial\Delta^m \hookrightarrow \Delta^0 \boxtimes \Delta^m) = \big(\tilde{\mathsf{c}}\partial\Delta^m \longrightarrow \tilde{\mathsf{c}}\Delta^m \big) ,$$

which is an injective cofibration, and that

$$\Delta_{e!}(\Delta^0 \boxtimes \Lambda^m_k \hookrightarrow \Delta^0 \boxtimes \Delta^m) = \big(\tilde{\mathsf{c}}\Lambda^m_k \longrightarrow \tilde{\mathsf{c}}\Delta^m \big) ,$$

which is an injective trivial cofibration. Since $\mathcal{H}^i_\infty$ is a symmetric monoidal model category, and since $\Delta_{e!}$ preserves pushouts, it now follows that $\Delta_{e!}$ sends the generating (trivial) cofibrations of $\mathcal{S}\mathrm{et}_\Delta$ to (trivial) cofibrations in $\mathcal{H}^i_\infty$. Thus, $\Delta_{e!}$ is a left Quillen functor by [Hov99, Lemma 2.1.20]. $\square$

Corollary 3.35 *There is a Quillen adjunction*

$$\Delta_{e!} : s\mathcal{S}\mathrm{et}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}^{iI}_\infty : \Delta_e^* .$$

Proposition 3.36 *The Quillen adjunction $\Delta_{e!} \dashv \Delta_e^*$ descends to a Quillen adjunction*

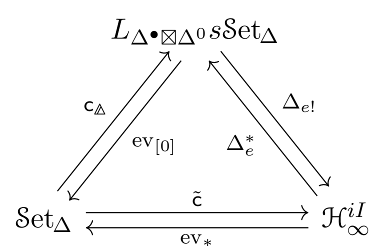
$$\Delta_{e!} : L_{\Delta^\bullet \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}^{iI}_\infty : \Delta_e^* .$$

Proof. This is a direct consequence of the fact that $\Delta_{e!}(\Delta^n \boxtimes \Delta^0) \cong \Delta^n_e$. The collapse morphism $\Delta^n_e \to *$ is a weak equivalence in $\mathcal{H}^{iI}_\infty$ by Proposition 2.7. $\square$

Corollary 3.37 *The adjunction $L_e : \mathcal{S}\mathrm{et}_\Delta \rightleftarrows \mathcal{H}^{iI}_\infty : S_e$ is a Quillen adjunction.*

Proof. This is a direct consequence of Proposition 3.24 and Proposition 3.36. $\square$

Theorem 3.38 *There is a commutative triangle of Quillen equivalences*



where c_Δ, $\Delta_{e!}$, and $\tilde{\mathsf{c}}$ are the left adjoints.

Proof. We know from Theorem 2.17 that the bottom adjunction is a Quillen equivalence, and we know from Proposition 3.25 that the left diagonal adjunction is a Quillen equivalence. Further, it is evident that the diagram of right adjoints commutes strictly. Thus, the claim follows from the two-out-of-three property of Quillen equivalences. □

Lemma 3.39 *Consider the functors* $\mathrm{ev}_*, S_e \colon \mathcal{H}_\infty \to \mathcal{S}\mathrm{et}_\Delta$.

(1) There is a canonical natural transformation $\gamma \colon \mathrm{ev}_* \to S_e$.

(2) The restriction of γ *to a morphism between functors* $(\mathcal{H}_\infty^{p/i\,I})_f \longrightarrow (\mathcal{S}\mathrm{et}_\Delta)_f$ *is a natural weak equivalence.*

Proof. Consider first the functors $\Delta_e^*, \mathsf{c}_{\mathbb{\Delta}} \circ \mathrm{ev}_{[0]} \colon \mathcal{H}_\infty^{p/i\,I} \longrightarrow s\mathcal{S}\mathrm{et}_\Delta$. For any $F \in \mathcal{H}_\infty$, the collapse map $\Delta_e^n \to *$ induces a morphism $\widehat{\gamma}_{|F,n} \colon F(*) \to F(\Delta_e^n)$ of simplicial sets. Since $* \in \mathcal{C}\mathrm{art}$ is final, this induces a natural transformation $\widehat{\gamma} \colon \mathsf{c}_{\mathbb{\Delta}} \circ \mathrm{ev}_* \longrightarrow \Delta_e^*$. Applying the diagonal functor δ^* to this natural transformation, we obtain a natural transformation

$$\gamma \colon \mathrm{ev}_* = \delta^* \circ \mathsf{c}_{\mathbb{\Delta}} \circ \mathrm{ev}_* \longrightarrow \delta^* \circ \Delta_e^* = S_e\,.$$

This shows part (1). Part (2) then follows from the fact that, whenever $F \in \mathcal{H}_\infty^{p/i\,I}$ is fibrant, the morphism $F(*) \to F(\Delta_e^n)$ is a weak equivalence for every $n \in \mathbb{N}_0$. Therefore, if F is fibrant, then $\widehat{\gamma}_{|F} \colon \mathsf{c}_{\mathbb{\Delta}}(F(*)) \longrightarrow \Delta_e^* F$ is an objectwise weak equivalence in $s\mathcal{S}\mathrm{et}_\Delta$. The claim now follows from the fact that the diagonal functor δ^* is homotopical. □

Theorem 3.40 *The Quillen adjunction*

$$L_e : \mathcal{S}\mathrm{et}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}_\infty^{iI} : S_e\,.$$

is a Quillen equivalence.

Proof. We will show that the total right derived functor $\mathbb{R}S_e \colon \mathrm{h}\mathcal{H}_\infty^{iI} \longrightarrow \mathrm{h}\mathcal{S}\mathrm{et}_\Delta$ is an equivalence of categories. If $R^{iI} \colon \mathcal{H}_\infty^{iI} \to \mathcal{H}_\infty^{iI}$ is a fibrant replacement functor in $\mathcal{H}_\infty^{iI}$, we can write $\mathbb{R}S_e$ as the composition [Hov99, Def. 1.3.6]

$$\mathrm{h}\mathcal{H}_\infty^{iI} \xrightarrow{\mathrm{h}R^{iI}} \mathrm{h}(\mathcal{H}_\infty^{iI})_f \xrightarrow{\mathrm{h}S_e} \mathcal{S}\mathrm{et}_\Delta\,.$$

Consider the natural transformation $\gamma \colon \mathrm{ev}_* \to S_e$ from Lemma 3.39. Since its component on each fibrant object $F \in \mathcal{H}_\infty^{iI}$ is a weak equivalence, γ induces a natural isomorphism

$$\mathbb{R}\gamma = \mathrm{h}R^{iI}(\mathrm{h}\gamma) \colon \mathbb{R}\,\mathrm{ev}_* \xrightarrow{\cong} \mathbb{R}S_e$$

of total right derived functors. Since ev_* is a right Quillen equivalence by Theorem 2.17, its total right derived functor $\mathbb{R}\,\mathrm{ev}_*$ is an equivalence of categories. Therefore, it now follows that also $\mathbb{R}S_e$ is an equivalence of categories. Thus, it follows from Proposition D.3 that $L_e \dashv S_e$ is a Quillen equivalence. □

Corollary 3.41 *The Quillen adjunction*

$$\delta_! : \mathcal{S}\mathrm{et}_\Delta \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} L_{\Delta^1 \boxtimes \Delta^0} s\mathcal{S}\mathrm{et}_\Delta : \delta^*\,.$$

is a Quillen equivalence.

Proof. This follows from the fact that $L_e = \Delta_{e!} \circ \delta_!$, together with Theorem 3.38 and the two-out-of-three property for Quillen equivalences. □

Corollary 3.41 becomes particularly interesting in light of Proposition 3.19: it establishes a Quillen equivalence between the Kan-Quillen model structure on simplicial sets and each of the model structures in Proposition 3.19. In other words, Corollary 3.41 shows that each of the model categories from Proposition 3.19 is a model category for ∞-groupoids.

4 Comparison of spaces constructed from simplicial presheaves

In Sections 2 and 3 we have seen several ways of extracting a space from a simplicial presheaf on $\mathcal{C}\mathrm{art}$. The main goal of this section is to establish comparisons between the resulting spaces. In particular, these comparisons are useful tools in applications of the $\mathbb{R}$-local homotopy theory of simplicial presheaves, such as in Section 5.

The right adjoint of $S_e = \delta^* \circ \Delta_e^*$ is given as $R_e = \Delta_{e*} \circ \delta_*$. We start by making this functor more explicit: consider a simplicial set $K \in \mathcal{S}\mathrm{et}_\Delta$ and a cartesian space $c \in \mathcal{C}\mathrm{art}$. Since the adjunction $S_e \dashv R_e$ is simplicial, there are natural isomorphisms

$$\begin{aligned}(R_e K)(c) &\cong \underline{\mathcal{H}_\infty}(\mathcal{Y}_c, R_e K) \\ &\cong \underline{\mathcal{S}\mathrm{et}_\Delta}(S_e \mathcal{Y}_c, K) \\ &\cong K^{S_e(\mathcal{Y}_c)}\,.\end{aligned} \tag{4.1}$$

The following lemma is then immediate:

Lemma 4.2 *There exist canonical natural isomorphisms*

$$S_e \circ \tilde{\mathsf{c}} \cong 1_{\mathcal{S}\mathrm{et}_\Delta}\,, \qquad \textit{and} \qquad \mathrm{ev}_* \circ R_e \cong 1_{\mathcal{S}\mathrm{et}_\Delta}\,.$$

It follows that there exists a natural isomorphism $\mathrm{ev}_* \circ R_e \circ \mathrm{Sing} \cong \mathrm{Sing}$.

Lemma 4.3 *There is an isomorphism*

$$\mathrm{Sing}(T)^K \cong \mathrm{Sing}\left(T^{|K|}\right),$$

natural in both $T \in \Delta\mathcal{T}\mathrm{op}$ and in $K \in \mathcal{S}\mathrm{et}_\Delta$.

Proof. Since the adjunction $|-| \dashv \mathrm{Sing}$ is simplicial (because $|-|$ preserves finite products), we have binatural isomorphisms

$$\begin{aligned}\mathrm{Sing}(T)^K &= \underline{\mathcal{S}\mathrm{et}_\Delta}\big(K, \mathrm{Sing}(T)\big) \\ &\cong \underline{\Delta\mathcal{T}\mathrm{op}}(|K|, T) \\ &= \mathrm{Sing}(T^{|K|})\,.\end{aligned}$$

Here we have used that $\Delta\mathcal{T}\mathrm{op}$ is cartesian closed and simplicially enriched. □

Lemma 4.4 *For every manifold $M \in \mathcal{M}\mathrm{fd}$ there exist morphisms*

$$\varphi_M \colon S_e \underline{M} \longrightarrow \mathrm{Sing}(\mathrm{D}M) \qquad \textit{and} \qquad \psi_M \colon |-| \circ S_e(\underline{M}) \longrightarrow \mathrm{D}M$$

of simplicial sets and of topological spaces, respectively. These assemble into natural transformations

$$\varphi\colon S_e \longrightarrow \operatorname{Sing} \circ \mathrm{D} \qquad \textit{and} \qquad \psi\colon |-| \circ S_e \longrightarrow \mathrm{D}$$

of functors $\mathcal{M}\mathrm{fd} \to \mathcal{S}\mathrm{et}_\Delta$ *and* $\mathcal{M}\mathrm{fd} \to \Delta\mathcal{T}\mathrm{op}$*, respectively.*

Proof. Let $n \in \mathbb{N}_0$ and consider the set $\underline{M}(\Delta_e^n) = \mathcal{M}\mathrm{fd}(\Delta_e^n, M)$; it is the set of all smooth maps $\Delta_e^n \to M$ of manifolds. Recall the morphism $\iota^\bullet\colon |\Delta^\bullet| \longrightarrow \mathrm{D}\Delta_e^\bullet$ of cosimplicial topological spaces from (3.2). If $f\colon \Delta_e^n \to M$ is any smooth map, then the composition $f \circ \iota^n\colon |\Delta^n| \to \mathrm{D}M$ is continuous. Here we have used that $\mathrm{D}M$ coincides with the underlying topological space of the manifold M (see Proposition 2.22). This provides a map

$$\varphi_{M|n}\colon \underline{M}(\Delta_e^n) = \mathcal{M}\mathrm{fd}(\Delta_e^n, M) \xrightarrow{\mathrm{D}} \Delta\mathcal{T}\mathrm{op}(\mathrm{D}\Delta_e^n, \mathrm{D}M) \xrightarrow{(\iota_n)^*} \Delta\mathcal{T}\mathrm{op}(|\Delta^n|, \mathrm{D}M)\,.$$

Since $\iota^\bullet$ is a morphism of cosimplicial topological spaces, and since the maps $\varphi_{M|n}$ are defined by precomposition by ι^n, it readily follows that $\varphi_{M|n}$ is natural in both $M \in \mathcal{M}\mathrm{fd}$ and $n \in \mathbb{\Delta}$. Thus, we obtain the desired morphism of simplicial sets

$$\varphi_M\colon S_e\underline{M} \longrightarrow \operatorname{Sing}(\mathrm{D}M)\,.$$

The composition

$$\psi_M\colon |S_e\underline{M}| \xrightarrow{|\varphi_M|} |\operatorname{Sing}(\mathrm{D}M)| \underset{\sim}{\xrightarrow{e_M}} \mathrm{D}M$$

then defines the morphism ψ_M, where $e\colon |-| \circ \operatorname{Sing} \xrightarrow{\sim} 1_{\Delta\mathcal{T}\mathrm{op}}$ is the evaluation morphism of the adjunction $|-| \dashv \operatorname{Sing}$. □

Lemma 4.5 *The restrictions of* φ *and* ψ *to* $\mathcal{C}\mathrm{art} \subset \mathcal{M}\mathrm{fd}$ *are natural weak equivalences of functors* $\mathcal{C}\mathrm{art} \to \mathcal{S}\mathrm{et}_\Delta$ *and* $\mathcal{C}\mathrm{art} \to \Delta\mathcal{T}\mathrm{op}$*, respectively.*

Proof. This follows readily from the observation that, for any cartesian space $c \in \mathcal{C}\mathrm{art}$, both $|S_e\mathcal{Y}_c|$ and $\mathrm{D}(\mathcal{Y}_c)$ are weakly equivalent to $* \in \Delta\mathcal{T}\mathrm{op}$. Hence, by the two-out-of-three property of weak equivalences any morphism $|S_e\mathcal{Y}_c| \longrightarrow \mathrm{D}(\mathcal{Y}_c)$ is a weak equivalence. □

Proposition 4.6 *There exists a natural weak equivalence*

$$\Delta\mathcal{T}\mathrm{op} \underset{S}{\overset{R_e \circ \operatorname{Sing}}{\rightrightarrows}} \mathcal{H}_\infty^{p/i\,I}\,, \qquad \eta\colon S \overset{\sim}{\Longrightarrow} R_e \circ \operatorname{Sing}$$

Proof. We consider the projective case; the injective case then follows since $\mathcal{H}_\infty^{pI}$ and $\mathcal{H}_\infty^{iI}$ have the same weak equivalences (Proposition 2.6). Given a topological space $T \in \Delta\mathcal{T}\mathrm{op}$, by Equation (4.1) and Lemma 4.3 we have

$$R_e \circ \operatorname{Sing}(T)(c) \cong \operatorname{Sing}\big(T^{|S_e\mathcal{Y}_c|}\big) \qquad \text{and} \qquad S(T)(c) \cong \operatorname{Sing}\big(T^{\mathrm{D}c}\big)\,.$$

The natural morphisms ψ from Lemma 4.4 induce a morphism

$$\eta_{|T} \coloneqq \operatorname{Sing}\big(T^\psi\big)\colon S(T) \longrightarrow R_e \circ \operatorname{Sing}(T)$$

in $\mathcal{H}_\infty$, which is natural in $T \in \Delta\mathcal{T}\mathrm{op}$.

In order to see that the morphism $\mathrm{Sing}(T^\psi)$ is a weak equivalence in $\mathcal{H}_\infty^{pI}$, we first observe that both $S(T)$ and $R_e \circ \mathrm{Sing}(T)$ are fibrant in $\mathcal{H}_\infty^{pI}$. We have that

$$\mathrm{ev}_* \big(S(T)\big) = \mathrm{Sing}(T) \qquad \text{and} \qquad \mathrm{ev}_* \big(R_e \circ \mathrm{Sing}(T)\big) = \mathrm{Sing}\left(T^{|S_e(*)|}\right) = \mathrm{Sing}(T)\,.$$

Further, the morphism $\psi_{|*} \colon |S_e(*)| = * \to * = \mathrm{D}(*)$ is the identity, so that also the morphism

$$\Big(\mathrm{Sing}\left(T^\psi\right)\Big)_{|*} \colon S(T)(*) = \mathrm{Sing}(T) \longrightarrow \mathrm{Sing}(T) = R_e \circ \mathrm{Sing}(T)(*)$$

is the identity. The fact that η is a natural weak equivalence now follows from the fact that the right Quillen equivalence $\mathrm{ev}_* \colon \mathcal{H}_\infty^{pI} \to \mathcal{S}\mathrm{et}_\Delta$ reflects weak equivalences between fibrant objects (which was also the content of Proposition 2.30). $\square$

Corollary 4.7 *There is a natural weak equivalence*

$$\Delta\mathcal{T}\mathrm{op} \underset{S_e \circ S}{\overset{\mathrm{Sing}}{\rightrightarrows}} \mathcal{S}\mathrm{et}_\Delta\,, \qquad \eta' \colon S_e \circ S \overset{\sim}{\Longrightarrow} \mathrm{Sing}\,.$$

Proof. Since the functor $S_e \colon \mathcal{H}_\infty^{p/i\,I} \to \mathcal{S}\mathrm{et}_\Delta$ is homotopical, we obtain a natural weak equivalence

$$S_e\eta \colon S_e \circ S \overset{\sim}{\longrightarrow} S_e \circ R_e \circ \mathrm{Sing}\,.$$

Let $e \colon S_e \circ R_e \longrightarrow 1_{\mathcal{S}\mathrm{et}_\Delta}$ denote the evaluation morphism of the adjunction $S_e \dashv R_e$. The fact that every object in $\mathcal{S}\mathrm{et}_\Delta$ is cofibrant, together with the fact that $S_e \dashv R_e$ is a Quillen equivalence imply that the morphism $e_{|K} \colon S_e \circ R_e(K) \longrightarrow K$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$ for every fibrant simplicial set K. Since $\mathrm{Sing} \colon \Delta\mathcal{T}\mathrm{op} \to \mathcal{S}\mathrm{et}_\Delta$ takes values in fibrant simplicial sets, it follows that the composition

$$\eta' \colon S_e \circ S \xrightarrow[\sim]{S_e\eta} S_e \circ R_e \circ \mathrm{Sing} \xrightarrow[\sim]{e_{\mathrm{Sing}}} \mathrm{Sing}$$

is a natural weak equivalence. $\square$

Corollary 4.8 *Let $Q^p \colon \mathcal{H}_\infty^{pI} \to \mathcal{H}_\infty^{pI}$ be a cofibrant replacement functor, with associated natural weak equivalence $q^p \colon Q^p \to 1_{\mathcal{H}_\infty}$. There is a zig-zag of natural weak equivalences*

$$S_e \xleftarrow[\sim]{S_e q^p} S_e \circ Q^p \xrightarrow[\sim]{\eta''} \mathrm{Sing} \circ Re \circ Q^p \tag{4.9}$$

of functors $\mathcal{H}_\infty^{pI} \longrightarrow \Delta\mathcal{T}\mathrm{op}$.

Proof. The left-facing natural transformation is a weak equivalence since S_e is homotopical. The right-facing morphism is the composition

$$\eta'' \colon S_e \circ Q^p \xrightarrow{S_e co_{Q^p}} S_e \circ S \circ Re \circ Q^p \xrightarrow[\sim]{(\eta')_{ReQ^p}} \mathrm{Sing} \circ Re \circ Q^p\,,$$

where $co \colon 1_{\Delta\mathcal{T}\mathrm{op}} \longrightarrow S \circ Re$ is the coevaluation morphism of the adjunction $Re \dashv S$. Since this adjunction is a Quillen equivalence and since every object in $\Delta\mathcal{T}\mathrm{op}$ is fibrant, it follows that $co_F \colon F \to S \circ Re(F)$ is a weak equivalence for every cofibrant object $F \in \mathcal{H}_\infty^{pI}$. $\square$

Proposition 4.10 *There exists a zig-zag of natural weak equivalences*

$$|-| \circ S_e \xleftarrow[\sim]{|S_e(q^p)|} |-| \circ S_e \circ Q^p \xrightarrow[\sim]{\eta'''} Re \circ Q^p$$

of functors $\mathcal{H}_\infty^{pI} \longrightarrow \Delta\mathcal{T}\mathrm{op}$. *In particular, there exists a natural isomorphism of total left derived functors*

$$\mathrm{h}\mathcal{H}_\infty^{pI} \underset{\mathrm{h}|-|\circ \mathrm{h}S_e}{\overset{\mathbb{L}Re}{\rightrightarrows}} \mathrm{h}\Delta\mathcal{T}\mathrm{op}\,. \quad (\eta''' \Uparrow \cong) \tag{4.11}$$

Observe that S_e and $|-|$ are already homotopical, so that we do not need to precompose them by a cofibrant replacement in order to obtain their total left derived functors.

Proof. We readily obtain a zig-zag as in (4.11) by applying the functor $|-|$ to the zig-zag (4.9) and then postcomposing by the evaluation transformation $e\colon |-| \circ \mathrm{Sing} \xrightarrow{\sim} 1_{\Delta\mathcal{T}\mathrm{op}}$. However, there is an alternative way of obtaining a zig-zag as in (4.11) directly and explicitly, which we think is worth showing: let Q^p be Dugger's cofibrant replacement functor for $\mathcal{H}_\infty^p$ [Dug01b]. Explicitly, it sends a simplicial presheaf F to the two-sided bar construction

$$Q^p F = B^{\mathcal{H}_\infty}(F, \mathcal{C}\mathrm{art}, \mathcal{Y})\,,$$

in the notation of [Rie14]. (The superscript indicates in which simplicial category we are forming the bar construction.) Using that Re is simplicial and commutes with colimits, and that there is a natural isomorphism $Re \circ \mathcal{Y}_c \cong \mathrm{D}c$ for any cartesian space $c \in \mathcal{C}\mathrm{art}$ (cf. Lemma 2.27), we obtain a canonical isomorphism

$$Re \circ Q^p(F) \cong B^{\Delta\mathcal{T}\mathrm{op}}(F, \mathcal{C}\mathrm{art}, \mathrm{D})\,.$$

Now we use that the morphism ψ from Lemma 4.4 induces a natural weak equivalence $\psi\colon |-| \circ S_e \xrightarrow{\sim} \mathrm{D}$ of functors $\mathcal{C}\mathrm{art} \to \Delta\mathcal{T}\mathrm{op}$ (cf. Lemma 4.5). Since each of the functors $F\colon \mathcal{C}\mathrm{art}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$ and $\mathrm{D}, |-| \circ S_e\colon \mathcal{C}\mathrm{art} \to \Delta\mathcal{T}\mathrm{op}$ are objectwise cofibrant, [Rie14, Cor. 5.2.5] implies that ψ induces a weak equivalence

$$B^{\Delta\mathcal{T}\mathrm{op}}(-, \mathcal{C}\mathrm{art}, \psi)\colon B^{\Delta\mathcal{T}\mathrm{op}}(-, \mathcal{C}\mathrm{art}, |-| \circ S_e) \xrightarrow{\sim} B^{\Delta\mathcal{T}\mathrm{op}}(-, \mathcal{C}\mathrm{art}, \mathrm{D}) = Re \circ Q^p$$

of functors $\mathcal{H}_\infty \to \Delta\mathcal{T}\mathrm{op}$.

On the other hand, since both $|-|$ and S_e are left adjoints, we have a natural isomorphism

$$\begin{aligned} B^{\Delta\mathcal{T}\mathrm{op}}(F, \mathcal{C}\mathrm{art}, |-| \circ S_e) &\cong |-| \circ S_e\big(B^{\mathcal{H}_\infty}(F, \mathcal{C}\mathrm{art}, \mathcal{Y})\big) \\ &= |-| \circ S_e \circ Q^p(F)\,. \end{aligned}$$

Now, the morphism $q^p\colon Q^p \xrightarrow{\sim} 1_{\mathcal{H}_\infty}$, together with the fact that both $|-|$ and S_e preserve weak equivalences, yield the claim. □

Remark 4.12 Recall the embedding $\iota\colon \mathcal{D}\mathrm{fg} \hookrightarrow \mathcal{H}_\infty$ of diffeological spaces into simplicial presheaves. By Lemma 2.27, the composition $Re \circ \iota$ agrees with the functor $\mathrm{D}\colon \mathcal{D}\mathrm{fg} \to \Delta\mathcal{T}\mathrm{op}$ that sends a diffeological space to its underlying topological space, whose topology is induced by its plots. It is interesting to ask whether the homotopy type of $\mathrm{D}X$ agrees with that of the smooth singular complex $S_e\iota(X)$ of

X, for any diffeological space $X \in \mathcal{D}\mathrm{fg}$. So far, however, we only see that the homotopy type of $S_e \iota X$ agrees with the cobar construction

$$\begin{aligned} S_e\iota(X) &\simeq B^{\Delta\mathcal{T}\mathrm{op}}(X, \mathcal{C}\mathrm{art}, \mathrm{D}) \\ &= \int^n |\Delta^n| \times \Big(\coprod_{c_0,\ldots,c_n \in \mathcal{C}\mathrm{art}} \mathrm{D}c_0 \times \mathcal{C}\mathrm{art}(c_0, c_1) \times \mathcal{C}\mathrm{art}(c_{n-1}, c_n) \times X(c_n) \Big) \end{aligned}$$

rather than with the underlying topological space $\mathrm{D}X$ of X. This is in accordance with—and maybe provides some further insight to—results from [CSW14, OT] that the smooth singular complex of a diffeological space X is not in general equivalent to the smooth singular complex of $\mathrm{D}X$. $\lhd$

To conclude this section, we can interpret the functor Re—and because of Proposition 4.10 also the functor S_e—in the context of the cohesive ∞-topos $\mathbf{H}$ of presheaves of spaces on $\mathcal{C}\mathrm{art}$ as follows (see [Sch] for more background). From the proof of Proposition 4.10 we see that there are canonical weak equivalences

$$\begin{aligned} Re \circ Q^p(F) &\cong B^{\Delta\mathcal{T}\mathrm{op}}(F, \mathcal{C}\mathrm{art}, \mathrm{D}) \\ &\simeq B^{\Delta\mathcal{T}\mathrm{op}}(F, \mathcal{C}\mathrm{art}, *) \\ &\cong \big| B^{\mathcal{S}\mathrm{et}_\Delta}(F, \mathcal{C}\mathrm{art}, *) \big| \,. \end{aligned}$$

Using the fact that the topological realisation of a bisimplicial set is independent of which simplicial direction one realises first (up to canonical isomorphism), we obtain canonical weak equivalences

$$\begin{aligned} \big| B^{\mathcal{S}\mathrm{et}_\Delta}(F, \mathcal{C}\mathrm{art}, *) \big| &\cong \big| B^{\mathcal{S}\mathrm{et}_\Delta}(*, \mathcal{C}\mathrm{art}^{\mathrm{op}}, F) \big| \\ &\simeq \mathrm{hocolim}^{\Delta\mathcal{T}\mathrm{op}} \big(|-| \circ F \colon \mathcal{C}\mathrm{art}^{\mathrm{op}} \longrightarrow \Delta\mathcal{T}\mathrm{op} \big) \,. \end{aligned}$$

Consequently, from Proposition 4.10 we obtain that each of the functors

$$Re \circ Q^p \simeq |-| \circ S_e \circ Q^p \simeq |-| \circ S_e$$

models the homotopy colimit of the diagram $|F| \colon \mathcal{C}\mathrm{art}^{\mathrm{op}} \to \Delta\mathcal{T}\mathrm{op}$, for any $F \in \mathcal{H}_\infty$. Therefore, the functors they present on the ∞-categories underlying $\mathcal{H}_\infty^p$ and $\Delta\mathcal{T}\mathrm{op}$ are equivalent, and they are further equivalent to the ∞-colimit functor. Consequently, on the level of the underlying ∞-categories they each present left-adjoints to the functor that sends a space to the constant presheaf whose value is that space. This means that both Re and S_e provide explicit presentations for the left-adjoint Π in the three-fold adjunction which implements the cohesive structure on $\mathbf{H}$ (see [Sch]). This appears to have been known for Re, but the functor S_e has not been formally identified as a model for the cohesion functor Π (although this has been indicated in [BEBdBP]).

5 A Whitehead Approximation Theorem

In the last section we established several comparison results for the various simplicial sets and topological spaces that we can naturally extract from an object $F \in \mathcal{H}_\infty$. It does not seem to be true, however, that the functors Re and $|-| \circ S_e$ are generally weakly equivalent. We were only able to relate these functors on the level of homotopy categories and their derived functors (Proposition 4.10). The results about diffeological spaces referred to in Remark 4.12 show that, in general, we cannot expect anything better for general simplicial presheaves on $\mathcal{C}\mathrm{art}$.

In this section, we show that if a simplicial presheaf comes from a smooth manifold M, then the smooth singular complex $S_e\underline{M}$ and the underlying topological space $\mathrm{D}\underline{M}$ are canonically weakly equivalent. This is a classical result, sometimes referred to as Whitehead's Approximation Theorem for manifolds (see [OT], for instance). Here, we employ our results thus far to give a purely homotopy-theoretic proof of this theorem, which avoids having to approximate continuous maps by smooth ones.

Theorem 5.1 *The natural transformations*

$$\mathcal{M}\mathrm{fd} \underset{S_e\circ\underline{(-)}}{\overset{\mathrm{Sing}\circ\mathrm{D}}{\rightrightarrows}} \mathcal{S}\mathrm{et}_\Delta \ (\varphi \Uparrow) . \qquad \mathcal{M}\mathrm{fd} \underset{|-|\circ S_e\circ\underline{(-)}}{\overset{\mathrm{D}}{\rightrightarrows}} \Delta\mathcal{T}\mathrm{op} \ (\psi \Uparrow) .$$

introduced in Lemma 4.4 are natural weak equivalences. In particular, the smooth singular complex of M has the same homotopy type as $\mathrm{Sing}\, M$*.*

The proof of Theorem 5.1 requires a couple of steps, which will occupy the remainder of this section. To start with, let $\mathcal{U} = \{U_a\}_{a\in A}$ be a differentiably good open covering of M (see Section 2.1), and let $\check{C}\mathcal{U} \to \underline{M}$ be the associated Čech covering in $\mathcal{H}_\infty$.

Lemma 5.2 *The augmentation map $\check{C}\mathcal{U} \to \underline{M}$ is a weak equivalence in $\mathcal{H}_\infty^{p/i\,\ell}$.*

Proof. The morphism $\check{C}\mathcal{U}_0 = \coprod_{a\in A} \underline{U}_a \longrightarrow \underline{M}$ is a local epimorphism, or generalised cover (in the sense of [DHI04, p. 7]) with respect to the Grothendieck topology of differentiably good open coverings on $\mathcal{C}\mathrm{art}$ [FSS12, Bunb]. Hence, the result follows from [DHI04, Cor. A.3]. □

For a differentiably good open covering $\mathcal{U} = \{U_a\}_{a\in A}$ of M, every non-empty finite intersection $\underline{U}_{a_0\ldots a_n}$ is representable in $\mathcal{H}_\infty$; in particular, each $\underline{U}_{a_0\ldots a_n}$, as well as the Čech nerve $\check{C}\mathcal{U}$, is cofibrant in $\mathcal{H}_\infty^p$. We let sA denote the partially ordered set of non-empty finite subsets of A, ordered by inclusion. By abuse of notation, we denote the category associated to a partially ordered set P also by P. For a partially ordered set P, we write $s_<P$ for the partially ordered set of totally ordered finite subsets of P. The assignment $\{a_0,\ldots,a_n\} \mapsto \underline{U}_{a_0\ldots a_n}$ defines a functor $\underline{U}_{(-)} \colon sA^{\mathrm{op}} \to \mathcal{H}_\infty$, which takes values in cofibrant objects (where $\underline{U}_{a_0\ldots a_n}$ is either the presheaf represented by the cartesian space $U_{a_0} \cap \cdots \cap U_{a_n}$ if this is non-empty, or it is the initial presheaf $\emptyset \in \mathcal{H}_\infty$). We will show that the Čech nerve of a differentiably good open covering $\mathcal{U}$ of M is equivalent (more precisely, isomorphic in $\mathrm{h}\mathcal{H}_\infty^{p/i}$) to the homotopy colimit of the functor $\underline{U}_{(-)} \colon sA^{\mathrm{op}} \to \mathcal{H}_\infty^{p/i}$. To that end, we use the modified two-sided simplicial bar construction $B^{\mathcal{H}_\infty}_{\mathrm{Ex}}$, which is introduced in Appendix C.

Proposition 5.3 *For any differentiably good open covering $\mathcal{U} = \{U_a\}_{a\in A}$ of a manifold M, there is an objectwise weak equivalence in $\mathcal{H}_\infty^p$,*

$$\mathrm{hocolim}^{\mathcal{H}_\infty^{p/i}}\big(sA^{\mathrm{op}} \xrightarrow{\underline{U}_{(-)}} \mathcal{H}_\infty^p\big) \simeq B^{\mathcal{H}_\infty}(*, sA^{\mathrm{op}}, \underline{U}_{(-)}) \xrightarrow{\sim} B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)}) \,.$$

Proof. This is a direct application of Proposition C.7 and Corollary C.8 for objectwise cofibrant diagrams. □

Example 5.4 A 1-simplex of $B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)})(c)$ consists of a finite subset $\alpha_{0,1} \subset A$, together with a choice of two non-empty subsets $\alpha_0, \alpha_1 \subset \alpha_{0,1}$ and a smooth map $c \to U_{\alpha_{0,1}}$. Here we have used that c is connected, so that each map $\mathcal{Y}_c \to B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)})_1$ must factor through exactly one component of the coproduct on the right-hand side. Note, however, that each of the subsets $\alpha_0, \alpha_1, \alpha_{0,1} \subset A$ may have any finite (non-zero) number of elements. ◁

Definition 5.5 *We say a covering $\mathcal{U} = \{U_a\}_{a \in A}$ is* closed under finite intersections *if it satisfies the following property: for any finite subset $\alpha = \{a_0, \ldots, a_n\} \subset A$ such that $U_\alpha \neq \emptyset$ there exists an element $a \in A$ such that $U_\alpha = U_a$.*

Lemma 5.6 *Given any differentiably good open covering $\mathcal{U} = \{U_a\}_{a \in A}$ of M, there exists a differentiably good open covering $\mathcal{U}^{cl}$ of M which is closed under finite intersections, and such that there is a weak equivalence $\check{C}\mathcal{U} \xrightarrow{\sim} \check{C}\mathcal{U}^{cl}$ in $\mathcal{H}_\infty^{p/i\,\ell}$ over $\underline{M}$.*

Proof. Let $\mathcal{U} = \{U_a\}_{a \in A}$ be a differentiably good open covering of M. As before, given a finite subset $\alpha = \{a_0, \ldots, a_n\} \subset A$, we set $U_\alpha \coloneqq \bigcap_{i=0}^n U_{a_i}$. Let

$$\mathcal{U}^{cl} \coloneqq \{U_\alpha \,|\, \alpha \in sA,\, U_\alpha \neq \emptyset\}$$

be the open covering of M consisting of all non-empty finite intersections of the patches of the covering $\mathcal{U}$. It is indexed over sA, and it is differentiably good if $\mathcal{U}$ is so. (Any finite intersection of elements of $\mathcal{U}^{cl}$ can be written as a finite intersection of elements of $\mathcal{U}$.) The canonical inclusion $\mathcal{U} \hookrightarrow \mathcal{U}^{cl}$ induces a commutative triangle

$$\begin{array}{ccc} \check{C}\mathcal{U} & \hookrightarrow & \check{C}\mathcal{U}^{cl} \\ & {\scriptstyle\sim}\searrow \quad \swarrow{\scriptstyle\sim} & \\ & \underline{M} & \end{array}$$

in $\mathcal{H}_\infty$. The diagonal arrows are weak equivalences in the Čech model structures $\mathcal{H}_\infty^{p/i\,\ell}$ by Lemma 5.2. Therefore, it follows that the horizontal morphism is a Čech weak equivalence as well. □

Let A be a set, and let $A^{[\bullet]}$ denote the simplicial set with $(A^{[\bullet]})_n = A^{n+1}$, and whose i-th face map forgets the i-th entry of a tuple. In other words, $A^{[\bullet]}$ is the Čech nerve of the collapse map $A \to *$ in $\mathcal{S}\mathrm{et}$. Let $\mathrm{Ex}\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ be the right adjoint to the simplicial subdivision functor (for more background, see Appendix C and [Cis19]). There exists a morphism of simplicial sets,

$$\sigma_A\colon A^{[\bullet]} \longrightarrow \mathrm{Ex}N(sA)\,,$$

which can be described as follows: By adjointness and [Cis19, Lemma 3.1.25], we have

$$\begin{aligned} \big(\mathrm{Ex}N(sA)\big)_n &\cong \mathcal{S}\mathrm{et}_\Delta\big(\mathrm{Sd}\,\Delta^n, N(sA)\big) \\ &\cong \mathcal{S}\mathrm{et}_\Delta\big(N(s_<[n]), N(sA)\big) \\ &\cong \mathcal{P}\mathrm{o}\mathcal{S}\mathrm{et}\big(s_<[n], sA\big)\,, \end{aligned} \tag{5.7}$$

where $\mathcal{P}\mathrm{o}\mathcal{S}\mathrm{et}$ is the category of partially ordered sets, and where in the last step we have used that the nerve functor N is fully faithful. Given an n-simplex $(a_0, \ldots, a_n) \in A^{[n]} = A^{n+1}$, we thus need to describe a map of partially ordered sets from $s_<[n]$ to sA. Recall that the elements of $s_<[n]$ are the totally ordered finite subsets of the partially ordered set $[n]$. For any $0 \leq k \leq n$ and any element $\{i_0, \ldots, i_k\} \in s_<[n]$, we set

$$\sigma_A(a_0, \ldots, a_n)\big(\{i_0, \ldots, i_k\}\big) \coloneqq \{a_{i_0}, \ldots, a_{i_k}\}\,.$$

Further, there exists a morphism of simplicial sets

$$\varrho_A\colon \mathrm{Ex} \circ N(sA) \longrightarrow (sA)^{[\bullet]}\,, \qquad \big(\alpha\colon s_<[n] \to sA\big) \longmapsto \big(\alpha(\{0\}), \ldots, \alpha(\{n\})\big)\,.$$

The n-simplices of $(sA)^{[\bullet]}$ are nothing but $(n+1)$-tuples of finite subsets of A. For $(a_0, \ldots, a_n) \in A^{[n]}$, we have

$$\varrho_A \circ \sigma_A(a_0, \ldots, a_n) = \big(\{a_0\}, \ldots, \{a_n\}\big) \quad \in (sA)^{[n]} \,. \tag{5.8}$$

Let $\mathcal{U} = \{U_a\}_{a\in A}$ be an open covering of a manifold M. Consider the morphism

$$\psi\colon \check{C}\mathcal{U} \longrightarrow B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)})$$

induced by σ_A; on simplicial level n it reads as

$$\psi_n\colon \coprod_{a_0,\ldots,a_n\in A} \underline{U}_{a_0\ldots a_n} \longrightarrow \coprod_{\alpha\in \mathrm{Ex}N(sA)} \underline{U}_{\alpha(\{0,\ldots,n\})} \,,$$

and it maps the component labelled by $a_0, \ldots, a_n \in A$ to the component labelled by $\sigma_A(a_0, \ldots, a_n)$ using the identity on $\underline{U}_{a_0\ldots a_n}$. Further, consider the morphism

$$\phi\colon B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)}) \longrightarrow \check{C}\mathcal{U}^{cl}$$

defined as follows. The morphism ϕ maps the component labelled by $\alpha \in (\mathrm{Ex}N(sA))_n$ to the component labelled by $\varrho_A(\alpha) = (\alpha(\{0\}), \ldots, \alpha(\{n\}))$ using the canonical inclusion

$$\underline{U}_{\alpha\{0,\ldots,n\}} \hookrightarrow \underline{U}_{\bigcup_{i=0}^n \alpha(\{i\})} \,.$$

Proposition 5.9 *Let $\mathcal{U} = \{U_a\}_{a\in A}$ be a differentiably good open covering of a manifold M. Then, the morphism*

$$\phi\colon B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)}) \longrightarrow \check{C}\mathcal{U}^{cl}$$

induced by ϱ_A is a trivial fibration in $\mathcal{H}_\infty^p$. Further, the canonical inclusion $\check{C}\mathcal{U} \hookrightarrow \check{C}\mathcal{U}^{cl}$ factors as

$$\check{C}\mathcal{U} \xrightarrow{\psi} B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)}) \xrightarrow{\phi} \check{C}\mathcal{U}^{cl}$$

in the slice category $(\mathcal{H}_\infty)_{/\underline{M}}$. In particular, the morphism ψ is a weak equivalence in $\mathcal{H}_\infty^{p/i\,\ell}$.

Proof. The second part of the claim is straightforward from (5.8); we thus have to prove that ϕ is a projective trivial fibration. To that end, we check that ϕ has the right lifting property with respect to the morphisms $\mathcal{Y}_c \otimes \partial\Delta^n \to \mathcal{Y}_c \otimes \Delta^n$ for $c \in \mathcal{C}\mathrm{art}$ and $n \in \mathbb{N}_0$. These morphisms form a set of generating cofibrations for the projective model structure $\mathcal{H}_\infty^p$ (see e.g. [Bar10, Proof of Prop. 4.52]). For any morphism $p\colon Y \to X$ of simplicial presheaves, there is a bijection between solutions to the lifting problems

$$\begin{array}{ccc} \mathcal{Y}_c \otimes \partial\Delta^n & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathcal{Y}_c \otimes \Delta^n & \longrightarrow & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \partial\Delta^n & \longrightarrow & Y(c) \\ \big\downarrow & & \big\downarrow{\scriptstyle p_{|c}} \\ \Delta^n & \longrightarrow & X(c) \end{array}$$

in $\mathcal{H}_\infty$ and in $\mathcal{S}\mathrm{et}_\Delta$, respectively.

In the case $n = 0$, the right-hand side amounts to a commutative diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)})(c) \\ \big\downarrow & & \big\downarrow{\scriptstyle \phi_{|c}} \\ \Delta^0 & \longrightarrow & \check{C}\mathcal{U}^{cl}(c) \end{array}$$

in $\mathcal{S}\mathrm{et}_\Delta$. Since c is connected, a vertex in $\check{C}\mathcal{U}^{cl}(c)$ consists of an element $\alpha \in sA$ and a smooth map $c \to U_\alpha$, which is precisely the same as the data for a vertex of $B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)})(c)$. In other words, the map $\phi_{|c}$ is a bijection on vertices, for every $c \in \mathcal{C}\mathrm{art}$.

For $n = 1$, we need to consider diagrams in $\mathcal{S}\mathrm{et}_\Delta$ of the form

$$
\begin{array}{ccc}
\partial\Delta^1 & \xrightarrow{(\beta,g)} & B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)})(c) \\
\big\downarrow & & \big\downarrow{\scriptstyle \phi_{|c}} \\
\Delta^1 & \xrightarrow[(\alpha,f)]{} & \check{C}\mathcal{U}^{cl}(c)
\end{array}
$$

A 1-simplex in $\check{C}\mathcal{U}^{cl}(c)$ is a pair $(\alpha_0, \alpha_1) \in (sA)^2$ (i.e. a pair of finite subsets of A), together with a smooth map $f \colon c \to U_{\alpha_0} \cap U_{\alpha_1} = U_{\alpha_0 \cup \alpha_1}$. The top morphism corresponds to elements $\beta_i \in sA$ and smooth maps $g_i \colon c \to U_{\beta_i}$ for $i = 0, 1$. The commutativity of the diagram is precisely the condition that $\beta_i = \alpha_i$ and that we have commutative diagrams

$$
\begin{array}{ccc}
c & \xrightarrow{f} & U_{\alpha_0 \cup \alpha_1} \\
& {\scriptstyle g_i}\searrow & \big\downarrow \\
& & U_{\alpha_i}
\end{array}
$$

of smooth maps, for $i = 0, 1$. Thus, there exists a lift in the diagram, provided by the 1-simplex

$$
(\alpha_0 \hookrightarrow \alpha_0 \cup \alpha_1 \hookleftarrow \alpha_1, f \colon c \to U_{\alpha_0 \cup \alpha_1}) \quad \in B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)})_1(c)\,.
$$

For $n > 1$, we need to consider commutative diagrams

$$
\begin{array}{ccc}
\partial\Delta^n & \xrightarrow{(\beta,g)} & B^{\mathcal{H}_\infty}_{\mathrm{Ex}}(*, sA^{\mathrm{op}}, \underline{U}_{(-)})(c) \\
\big\downarrow & & \big\downarrow{\scriptstyle \phi_{|c}} \\
\Delta^n & \xrightarrow[(\alpha,f)]{} & \check{C}\mathcal{U}^{cl}(c)
\end{array}
\tag{5.10}
$$

An n-simplex in $\check{C}\mathcal{U}^{cl}(c)$ is an $(n{+}1)$-tuple $(\alpha_0, \ldots, \alpha_n) \in (sA)^{n+1}$, together with a smooth map $f \colon c \to U_{\bigcup_{i=0}^n \alpha_i}$. The top morphism is equivalently the following data: we have $(n{-}1)$-simplices $\beta_j \in (\mathrm{Ex}N(sA))_{n-1}$ and smooth maps $g_j \colon c \to U_{\beta_j(\{0,\ldots,n-1\})}$. Via (5.7) we can rephrase β_j as a morphism of posets $\beta_j \colon s_<[n-1] \to sA$. For notational purposes, we use the canonical identification of the poset $[n-1]$ with the poset $\{0, \ldots, \hat{j}, \ldots, n\}$, where the hat means that we are omitting the respective index. Then, we can write the smooth maps g_j as

$$
g_j \colon c \longrightarrow U_{\beta_j(\{0,\ldots,\hat{j},\ldots,n\})} = U_{\beta_j(d_j\{0,\ldots,n\})}\,.
$$

Here, d_j is the j-th face map in the simplicial set $Ns_<[n]$. The fact that these $(n{-}1)$-simplices assemble into a map from $\partial\Delta^n$ amounts to $\{\beta_j\}_{j=0,\ldots,n}$ forming a map $\partial\Delta^n \to \mathrm{Ex}N(sA)$ and to the commutativity of the diagrams

$$
\begin{array}{cccc}
& U_{\beta_j(d_j\{0,\ldots,n\})} & \hookrightarrow & U_{\beta_j(d_i d_j\{0,\ldots,n\})} \\
{\scriptstyle g_j}\nearrow & & & \big\| \\
c & & & \\
{\scriptstyle g_i}\searrow & & & \big\| \\
& U_{\beta_i(d_i\{0,\ldots,n\})} & \hookrightarrow & U_{\beta_i(d_{j-1} d_i\{0,\ldots,n\})}
\end{array}
\qquad \forall\, 0 \leq i < j \leq n\,.
$$

This implies that each map g_i factors through a unique map $\hat{g}\colon c \to U_{\bigcup_{j=0}^n \beta_j[n-1]} \subset U_{\beta_i[n-1]}$.

The commutativity of diagram (5.10) means that

$$\beta_j\{i\} = \begin{cases} \alpha_i\,, & i < j\,, \\ \alpha_{i+1}\,, & i \geq j \end{cases}$$

and that the map $f\colon c \to U_{\bigcup_{i=0}^n \alpha_i}$ factors through the map

$$\hat{g}\colon c \to U_{\bigcup_{j=0}^n \beta_j(d_j\{0,\ldots,n\})} \subset U_{\bigcup_{i=0}^n \alpha_i}\,.$$

Consequently, there exists an n-simplex $\hat{\beta} \in (\mathrm{Ex}N(sA))_n$, with

$$\hat{\beta}(\{0,\ldots,n\}) = \bigcup_{j=0}^n \beta_j(d_j\{0,\ldots,n\})$$

and whose boundary is β. The pair $(\hat{\beta}, \hat{g})$ then provides a lift in diagram (5.10). □

Corollary 5.11 *Let $\mathcal{U} = \{U_a\}_{a\in A}$ be a differentiably good open covering of a manifold M. The canonical morphism*

$$\operatorname*{hocolim}_{sA}{}^{\mathcal{H}_\infty}(\underline{U}_{(-)}) = B^{\mathcal{H}_\infty}(*, sA^{\mathrm{op}}, \underline{U}_{(-)}) \longrightarrow \underline{M}$$

is a weak equivalence in $\mathcal{H}_\infty^{p/i\,\ell}$.

Proof. This is a combination of Lemma 5.2, Proposition 5.3, and Proposition 5.9. □

Proof of Theorem 5.1. Choose a differentiably good open covering $\mathcal{U} = \{U_a\}_{a\in A}$ of M and denote by $\pi_{\mathcal{U}}\colon B^{\mathcal{H}_\infty}(*, sA^{\mathrm{op}}, \underline{U}_{(-)}) \longrightarrow \underline{M}$ the canonical Čech weak equivalence from Corollary 5.11. By Propositions 2.5 and 2.6, this is a weak equivalence in $\mathcal{H}_\infty^{iI}$. Since each of the functors $|-|\colon \mathcal{S}\mathrm{et}_\Delta \to \Delta\mathcal{T}\mathrm{op}$, $S_e\colon \mathcal{H}_\infty \to \mathcal{S}\mathrm{et}_\Delta$, and $Re\colon \mathcal{H}_\infty \to \Delta\mathcal{T}\mathrm{op}$ is left adjoint and simplicial, each of them preserves two-sided bar constructions. Therefore, applying $|-| \circ S_e$ to the morphism $\pi_{\mathcal{U}}$, we obtain a morphism

$$|S_e\pi_{\mathcal{U}}|\colon B^{\Delta\mathcal{T}\mathrm{op}}\big(*, sA^{\mathrm{op}}, |S_e\underline{U}_{(-)}|\big) \longrightarrow |S_e\underline{M}|\,,$$

which is a weak equivalence since both $|-|$ and S_e are homotopical. On the other hand, applying the functor Re to $\pi_{\mathcal{U}}$ yields a morphism

$$Re(\pi_{\mathcal{U}})\colon B^{\Delta\mathcal{T}\mathrm{op}}\big(*, sA^{\mathrm{op}}, Re\underline{U}_{(-)}\big) \longrightarrow Re\underline{M}\,.$$

Observing that the functor $\underline{U}_{(-)}\colon sA^{\mathrm{op}} \to \mathcal{H}_\infty$ factors through the category of manifolds, we can use Proposition 2.22 to replace Re by the functor D from Definition 2.20; we can thus equivalently (up to canonical isomorphism) write $Re(\pi_{\mathcal{U}})$ as a morphism

$$Re(\pi_{\mathcal{U}})\colon B^{\Delta\mathcal{T}\mathrm{op}}\big(*, sA^{\mathrm{op}}, \mathrm{D}\underline{U}_{(-)}\big) \longrightarrow \mathrm{D}\underline{M}\,.$$

By Lemma 4.4, we obtain a diagram

$$\begin{array}{ccc} B^{\Delta\mathcal{T}\mathrm{op}}\big(*, sA^{\mathrm{op}}, |S_e\underline{U}_{(-)}|\big) & \xrightarrow{B^{\Delta\mathcal{T}\mathrm{op}}(*, sA^{\mathrm{op}}, \psi)} & B^{\Delta\mathcal{T}\mathrm{op}}\big(*, sA^{\mathrm{op}}, \mathrm{D}\underline{U}_{(-)}\big) \\ {\scriptstyle |S_e\pi_{\mathcal{U}}|}\downarrow\wr & & \downarrow{\scriptstyle Re(\pi_{\mathcal{U}})} \\ |S_e\underline{M}| & \xrightarrow[\psi_M]{} & \mathrm{D}\underline{M} \end{array} \tag{5.12}$$

This diagram commutes by the naturality of ψ.

Both functors $|S_e \underline{U}_{(-)}|$ and $\mathrm{D}\underline{U}_{(-)}$ take values in cofibrant topological spaces. This has two implications: first, by Lemma 4.5 and by the homotopical properties of the bar construction [Rie14, Sec. 5] the top morphism in diagram (5.12) is a weak equivalence. Second, both bar constructions model the homotopy colimits in $\Delta\mathcal{T}\mathrm{op}$ of the diagrams $|S_e \underline{U}_{(-)}| \colon sA^{\mathrm{op}} \to \Delta\mathcal{T}\mathrm{op}$ and $\mathrm{D} \colon sA^{\mathrm{op}} \to \Delta\mathcal{T}\mathrm{op}$, respectively.

We now aim to show that the right-hand vertical map is a weak equivalence. To that end, we consider the diagram $\mathrm{D}\underline{U}_{(-)} \colon sA^{\mathrm{op}} \to \Delta\mathcal{T}\mathrm{op}$. For a topological space T, let $\mathcal{O}(T)$ denote its partially ordered set of open subsets. Then, we can write $\mathrm{D}\underline{U}_{(-)}$ as a diagram $\mathrm{D}\underline{U}_{(-)} \colon sA^{\mathrm{op}} \to \mathcal{O}(\mathrm{D}M)$. For $x \in \mathrm{D}\underline{M}$, let $(sA^{\mathrm{op}})_x \subset sA^{\mathrm{op}}$ denote the full subcategory of sA^{op} on those objects $\alpha \in sA$ satisfying that $x \in U_\alpha \subset M$. Since $\mathcal{U}$ is an open covering of M, the category $(sA^{\mathrm{op}})_x$ is non-empty and filtered, for each $x \in M$. We can therefore apply Lurie's Seifert-van Kampen Theorem [Lur17, Thm. A.3.1] (which, in this particular incarnation is often called the Nerve Theorem): the map

$$\mathrm{hocolim}^{\mathcal{S}\mathrm{et}_\Delta}\big(sA^{\mathrm{op}} \xrightarrow{\mathrm{D}\underline{U}_{(-)}} \Delta\mathcal{T}\mathrm{op} \xrightarrow{\mathrm{Sing}} \mathcal{S}\mathrm{et}_\Delta\big) \longrightarrow \mathrm{Sing}\,\mathrm{D}M$$

is a weak equivalence of simplicial sets. Applying the realisation functor and using that the evaluation morphism $|-| \circ \mathrm{Sing} \xrightarrow{\sim} 1_{\Delta\mathcal{T}\mathrm{op}}$ is a natural weak equivalence (every object in $\Delta\mathcal{T}\mathrm{op}$ is fibrant, and every object in $\mathcal{S}\mathrm{et}_\Delta$ is cofibrant), we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{hocolim}^{\Delta\mathcal{T}\mathrm{op}}\big(sA^{\mathrm{op}} \xrightarrow{\mathrm{D}\underline{U}_{(-)}} \Delta\mathcal{T}\mathrm{op} \xrightarrow{|-|\circ\mathrm{Sing}} \Delta\mathcal{T}\mathrm{op}\big) & \xrightarrow{\quad\sim\quad} & |\,\mathrm{Sing}\,\mathrm{D}M| \\
\sim\big\downarrow & & \big\downarrow\sim \\
\mathrm{hocolim}^{\Delta\mathcal{T}\mathrm{op}}\big(sA^{\mathrm{op}} \xrightarrow{\mathrm{D}\underline{U}_{(-)}} \Delta\mathcal{T}\mathrm{op}\big) & \xrightarrow{\qquad\qquad} & \mathrm{D}M
\end{array}$$

It follows that the morphism $Re(\pi_{\mathcal{U}}) \colon \mathrm{hocolim}^{\Delta\mathcal{T}\mathrm{op}}(sA^{\mathrm{op}} \xrightarrow{\mathrm{D}\underline{U}_{(-)}} \Delta\mathcal{T}\mathrm{op}) \longrightarrow \mathrm{D}M$ is a weak equivalence of topological spaces. Thus, in the commutative diagram (5.12), the vertical morphisms and the top morphism are weak equivalences of topological spaces. It follows that the bottom morphism ψ_M is a weak equivalence as well. □

6 Local fibrant replacement, concordance, and mapping spaces

In this section, we present a fibrant replacement functor in the model structures $\mathcal{H}_\infty^{p/i\,I}$. Its construction is motivated by the *concordance sheaf* construction from [BEBdBP]. This functor allows us to compute mapping spaces in $\mathcal{H}_\infty^{p/i\,I}$ in Theorem 6.6. We start by presenting the fibrant replacement functor:

Lemma 6.1 *Suppose that $F_0, F_1 \in \mathcal{H}_\infty$ and that $h \colon F_0 \times \underline{\mathbb{R}} \to F_1$ is a smooth homotopy between morphisms $f, g \colon F_0 \to F_1$. Then, for any $G \in \mathcal{H}_\infty$, there is a smooth homotopy $\tilde{h} \colon G^{F_1} \times \underline{\mathbb{R}} \to G^{F_0}$ from G^f to $G^g \colon G^{F_1} \to G^{F_0}$.*

Proof. Applying the exponential functor $G^{(-)}$ to the morphism h, we obtain a morphism

$$G^h \in \mathcal{H}_\infty(G^{F_1}, G^{F_0 \times \underline{\mathbb{R}}}) \cong \mathcal{H}_\infty(G^{F_1}, (G^{F_0})^{\underline{\mathbb{R}}})\,.$$

Using the internal-hom adjunction of $\mathcal{H}_\infty$ then yields a morphism $\tilde{h} = (G^h)^{\dashv} \in \mathcal{H}_\infty(G^{F_1} \times \underline{\mathbb{R}}, G^{F_0})$ as desired. □

We consider the following construction: let $\underline{\Delta}_e^k \in \mathcal{H}_\infty$ denote the simplicial presheaf represented by the extended affine simplex $\Delta_e^k \in \mathcal{C}\mathrm{art}$. This provides a functor

$$\underline{\Delta}_e \colon \mathbb{\Delta} \to \mathcal{H}_\infty\,, \qquad [k] \longmapsto \underline{\Delta}_e^k = \mathcal{Y}_{\Delta_e^k}\,.$$

Given an object $F \in \mathcal{H}_\infty$, we can compose this functor by the functor $F^{(-)} \colon \mathcal{H}_\infty \to \mathcal{H}_\infty$, obtaining an object $F^{\underline{\Delta}_e} \in \mathcal{C}\mathrm{at}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{H}_\infty)$. Equivalently, we can view this as a bisimplicial presheaf

$$F^{\underline{\Delta}_e} \colon \mathcal{C}\mathrm{art}^{\mathrm{op}} \longrightarrow s\mathcal{S}\mathrm{et}_\Delta\,, \qquad c \longmapsto F(\Delta_e \times c)\,,$$

which we can now compose by the diagonal functor $\delta^* \colon s\mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ to obtain a new simplicial presheaf on $\mathcal{C}\mathrm{art}$. Putting everything together, this defines a functor

$$\delta^* \circ (-)^{\underline{\Delta}_e} \colon \mathcal{H}_\infty \to \mathcal{H}_\infty\,, \qquad F \longmapsto \delta^* \circ F^{\underline{\Delta}_e}\,.$$

The collapse morphisms $\Delta_e^k \to *$ induce a natural transformation $\underline{\Delta}_e \to *$ of functors $\mathbb{\Delta} \to \mathcal{H}_\infty$ (this even consists of I-local equivalences by Proposition 2.7). From this we obtain a natural transformation

$$\gamma \colon 1_{\mathcal{H}_\infty} \longrightarrow \delta^* \circ (-)^{\underline{\Delta}_e}\,.$$

Now, let $R^{p/i} \colon \mathcal{H}_\infty \to \mathcal{H}_\infty$ be a fibrant replacement functor for the projective (resp. injective) model structure, with natural objectwise weak equivalence $r^{p/i} \colon 1_{\mathcal{H}_\infty} \xrightarrow{\sim} R^{p/i}$. (Observe, in particular, that a fibrant replacement functor R^p for the projective model structure can be obtained by postcomposition with a fibrant replacement functor in $\mathcal{S}\mathrm{et}_\Delta$, i.e. we can use $R^p(F) = R^{\mathcal{S}\mathrm{et}_\Delta} \circ F$ for $F \in \mathcal{H}_\infty$. An explicit model for an injective fibrant replacement functor is given in Appendix A.) We define functors

$$\mathrm{Cc}^{p/i} \colon \mathcal{H}_\infty \to \mathcal{H}_\infty\,, \qquad F \longmapsto R^{p/i}\big(\delta^* \circ F^{\underline{\Delta}_e}\big)\,,$$

for the projective and for the injective model structure, respectively. Further, we define the natural transformation

$$\mathrm{cc}^{p/i}_{|F} \colon F \xrightarrow{\gamma_{|F}} \delta^* \circ F^{\underline{\Delta}_e} \underset{\sim}{\xrightarrow{r^{p/i}}} \mathrm{Cc}^{p/i} F\,.$$

Proposition 6.2 *The functors* $\mathrm{Cc}^{p/i}$*, together with the natural morphisms* $\mathrm{cc}^{p/i}$ *provide a functorial fibrant replacement in* $\mathcal{H}_\infty^{p/i\,I}$*.*

Proof. First, we show that $\mathrm{Cc}^{p/i}F$ is indeed fibrant in the I-local model structure $\mathcal{H}_\infty^{p/i\,I}$, for every $F \in \mathcal{H}_\infty$. By construction, $\mathrm{Cc}^{p/i}F$ is a fibrant object in $\mathcal{H}_\infty^{p/i}$. It thus remains to show that it is $\mathbb{R}$-local. Given any $c \in \mathcal{C}\mathrm{art}$, we have

$$\big(\delta^* \circ F^{\underline{\Delta}_e}\big)(c) = \delta^*\big(F(c \times \Delta_e)\big) = \delta^*\big(F^{\mathcal{Y}_c}(\Delta_e)\big) = S_e(F^{\mathcal{Y}_c})\,.$$

By Lemma 6.1, the smooth homotopy equivalence $\mathcal{Y}_c \to *$ induces a smooth homotopy equivalence $F \to F^{\mathcal{Y}_c}$, which is a weak equivalence in $\mathcal{H}_\infty^{p/i\,I}$ by Corollary 3.16. Since $r^{p/i}$ is a natural weak equivalence of functors valued in $\mathcal{H}_\infty^{p/i}$, its component

$$\delta^* \circ F^{\underline{\Delta}_e} \underset{\sim}{\xrightarrow{r^{p/i}}} R^{p/i}\big(\delta^* \circ F^{\underline{\Delta}_e}\big)$$

is an objectwise weak equivalence. Consequently, for every $c \in \mathcal{C}\mathrm{art}$, we have a commutative square

$$\begin{array}{ccc}
\big(\delta^* \circ F^{\underline{\Delta}_e}\big)(c) & \underset{\sim}{\xrightarrow{r^{p/i}}} & R^{p/i}\big(\delta^* \circ F^{\underline{\Delta}_e}\big)(c) \\
\sim\big\uparrow & & \big\uparrow \\
\big(\delta^* \circ F^{\underline{\Delta}_e}\big)(*) & \overset{\sim}{\underset{r^{p/i}}{\longrightarrow}} & R^{p/i}\big(\delta^* \circ F^{\underline{\Delta}_e}\big)(*)
\end{array}$$

whose vertical morphisms are induced from the collapse morphism $c \to *$. It follows that $\mathrm{Cc}^{p/i}F$ is $\mathbb{R}$-local and hence a fibrant object in $\mathcal{H}_\infty^{p/i\,I}$.

Finally, we need to show that the morphism $\gamma_{|F} \colon F \longrightarrow \delta^* \circ F^{\underline{\Delta}_e}$, induced by the collapse $\Delta_e \to *$, is a weak equivalence in $\mathcal{H}_\infty^{p/i\,I}$. Since the functor $S_e \colon \mathcal{H}_\infty^{p/i\,I} \to \mathcal{S}\mathrm{et}_\Delta$ preserves as well as reflects weak equivalences, that is equivalent to showing that the induced morphism

$$S_e(\gamma_{|F}) \colon S_e F \longrightarrow S_e\big(\delta^* \circ F^{\underline{\Delta}_e}\big)$$

is a weak equivalence of simplicial sets. More explicitly, $S_e(\gamma_{|F})$ is the morphism

$$\tilde{\delta}^*\big(F(\tilde{\Delta}_e)\big) \longrightarrow \tilde{\delta}^*\big(\delta^* F(\tilde{\Delta}_e \times \Delta_e)\big)$$

induced by collapsing the extended simplices Δ_e without the tilde. Note that we have only added the tilde in order to keep track of which diagonal functor refers to which copy of Δ_e. We have canonical isomorphisms

$$S_e\big(\delta^* \circ F^{\underline{\Delta}_e}\big) \cong \tilde{\delta}^*\big(\delta^* F(\tilde{\Delta}_e \times \Delta_e)\big) \cong \delta^*\big((\tilde{\delta}^* \circ F^{\tilde{\Delta}_e})(\Delta_e)\big)$$

and we know from the first part of this proof that the morphism

$$\tilde{\delta}^*\big(F(\tilde{\Delta}_e)\big) = (\tilde{\delta}^* \circ F^{\tilde{\Delta}_e})(*) \longrightarrow (\tilde{\delta}^* \circ F^{\tilde{\Delta}_e})(\Delta_e^k)$$

is a weak equivalence, for every $k \in \mathbb{N}_0$. This induces a (levelwise) weak equivalence of bisimplicial sets

$$\Delta^0 \boxtimes \tilde{\delta}^*\big(F(\tilde{\Delta}_e)\big) \xrightarrow{\sim} (\tilde{\delta}^* \circ F^{\tilde{\Delta}_e})(\Delta_e)\,,$$

which under δ^* maps to the morphism $S_e(\gamma_{|F})$. Since the diagonal functor $\delta^* \colon s\mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ is homotopical, we obtain that $S_e(\gamma_{|F})$, and thus also $\gamma_{|F}$, is indeed a weak equivalence. □

Corollary 6.3 *Mapping spaces in $\mathcal{H}_\infty^{p/i\,I}$ can be computed (up to isomorphism in* $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$*) as the simplicially enriched hom spaces*

$$\begin{aligned} \mathrm{Map}_{\mathcal{H}_\infty^{iI}}(F,G) &\simeq \underline{\mathcal{H}}_\infty\big(F, R^i(\delta^* \circ G^{\underline{\Delta}_e})\big)\,, \\ \mathrm{Map}_{\mathcal{H}_\infty^{pI}}(F,G) &\simeq \underline{\mathcal{H}}_\infty\big(Q^p F, R^{\mathcal{S}\mathrm{et}_\Delta} \circ \delta^* \circ G^{\underline{\Delta}_e}\big)\,, \end{aligned} \tag{6.4}$$

where Q^p is a cofibrant replacement functor for the projective model structure $\mathcal{H}_\infty^p$.

Definition 6.5 *Given an object $F \in \mathcal{H}_\infty$, we call* $\mathrm{Cc}^{p/i}F$ *its* (projective/injective) derived concordance sheaf. *For $G \in \mathcal{H}_\infty$ we refer to the spaces in* (6.4) *as the spaces of* derived concordances of morphisms *from F to G.*

We can apply these insights to describe the mapping spaces in the model categories $\mathcal{H}_\infty^{p/i\,I}$. In particular, given any $G \in \mathcal{H}_\infty$, part (3) of the following theorem shows that the derived sections of the derived concordance sheaf of G (in the sense of Definition 6.5) on manifolds are represented by maps from the space underlying M to the smooth singular complex of G.

Theorem 6.6 *There are natural isomorphisms in* $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$ *as follows:*

(1) For $F, G \in \mathcal{H}_\infty$, we have

$$\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(S_e F, S_e G) \cong \mathrm{Map}_{\mathcal{H}_\infty^{p/i\,I}}(F,G)\,.$$

(2) For $F, G \in \mathcal{H}_\infty$, we have

$$\mathrm{Map}_{\Delta\mathcal{T}\mathrm{op}}(|S_e F|, |S_e G|) \cong \mathrm{Map}_{\mathcal{H}_\infty^{p/i\,I}}(F, G)\,.$$

(3) For any manifold M and $G \in \mathcal{H}_\infty$, we have

$$\mathrm{Map}_{\Delta\mathcal{T}\mathrm{op}}(M, |S_e G|) \cong \mathrm{Map}_{\mathcal{H}_\infty^{p/i\,I}}(\underline{M}, G)\,.$$

Proof. For claim (1), we consider the injective case first. The projective case then follows immediately from Corollary A.10. Using that every object in $\mathcal{H}_\infty^{iI}$ is cofibrant, we have the following isomorphisms in $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$:

$$\begin{aligned}
\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(S_e F, S_e G) &\cong \underline{\mathcal{S}\mathrm{et}_\Delta}(S_e F, R^{\mathcal{S}\mathrm{et}_\Delta} S_e G) \\
&\cong \underline{\mathcal{H}}_\infty\big(F, R_e R^{\mathcal{S}\mathrm{et}_\Delta} S_e(G)\big) \\
&\cong \underline{\mathcal{H}}_\infty\big(F, R_e R^{\mathcal{S}\mathrm{et}_\Delta} S_e \mathrm{Cc}^i(G)\big) \\
&\cong \underline{\mathcal{H}}_\infty\big(F, \mathrm{Cc}^i(G)\big) \\
&\cong \mathrm{Map}_{\mathcal{H}_\infty^{iI}}(F, G)\,.
\end{aligned}$$

The first isomorphism is merely the fact that $\mathcal{S}\mathrm{et}_\Delta$ is a simplicial model category in which every object is cofibrant. In the second isomorphism, we use the fact that $S_e \dashv R_e$ is a simplicial adjunction. To see the third isomorphism, we use the weak equivalence $\mathrm{cc}^i_{|G}\colon G \xrightarrow{\sim} \mathrm{Cc}^i(G)$ from Proposition 6.2. Since both S_e and $R^{\mathcal{S}\mathrm{et}_\Delta}$ are homotopical functors, and since R_e preserves fibrant objects, the morphism $R_e R^{\mathcal{S}\mathrm{et}_\Delta} S_e(\mathrm{cc}^i_{|G})$ is a weak equivalence between fibrant objects in $\mathcal{H}_\infty^{iI}$, which is thus preserved by $\underline{\mathcal{H}}_\infty(F, -)$. The fourth isomorphism stems from the fact that $S_e \dashv R_e$ is a Quillen equivalence, so that the canonical natural transformation $1_{\mathcal{H}_\infty} \longrightarrow R_e R^{\mathcal{S}\mathrm{et}_\Delta} S_e$ is a weak equivalence in $\mathcal{H}_\infty^{iI}$ on every cofibrant object—that is, it is a weak equivalence on every object, since all objects in $\mathcal{H}_\infty^{iI}$ are cofibrant. Further, its component at the object $\mathrm{Cc}^i(G)$ is a weak equivalence between fibrant objects in $\mathcal{H}_\infty^{iI}$, which is again preserved by $\underline{\mathcal{H}}_\infty(F, -)$. The final isomorphism directly follows from the insight that Cc^i is a fibrant replacement functor for $\mathcal{H}_\infty^{iI}$ (Proposition 6.2).

Claim (2) then follows from the fact that every object in $\Delta\mathcal{T}\mathrm{op}$ is fibrant and that every object of $\mathcal{S}\mathrm{et}_\Delta$ is cofibrant: if $K, L \in \mathcal{S}\mathrm{et}_\Delta$, then there are canonical isomorphisms in $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$

$$\begin{aligned}
\mathrm{Map}_{\Delta\mathcal{T}\mathrm{op}}\big(|K|, |L|\big) &\cong \underline{\Delta\mathcal{T}\mathrm{op}}\big(|K|, |L|\big) \\
&\cong \underline{\mathcal{S}\mathrm{et}_\Delta}\big(K, \mathrm{Sing}\,|L|\big) \\
&\cong \underline{\mathcal{S}\mathrm{et}_\Delta}\big(K, \mathrm{Sing}\,|R^{\mathcal{S}\mathrm{et}_\Delta}(L)|\big) \\
&\cong \underline{\mathcal{S}\mathrm{et}_\Delta}\big(K, \mathrm{Sing}\,|R^{\mathcal{S}\mathrm{et}_\Delta}(L)|\big) \\
&\cong \underline{\mathcal{S}\mathrm{et}_\Delta}\big(K, R^{\mathcal{S}\mathrm{et}_\Delta}(L)\big) \\
&\cong \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(K, L)\,.
\end{aligned}$$

Claim (3) now follows from combining part (2) with Theorem 5.1. □

Remark 6.7 Part (3) of Theorem 6.6 is related to recent results by Berwick-Evans, Boavida de Brito, and Pavlov from [BEBdBP]: in that paper, the authors work with the category $\widetilde{\mathcal{H}}_\infty$ of simplicial presheaves on $\mathcal{M}\mathrm{fd}$. Given $\widetilde{G} \in \widetilde{\mathcal{H}}_\infty$, they consider the concordance sheaf

$$\mathrm{B}\widetilde{G} = \delta^* \widetilde{G}^{\underline{\Delta}_e}\,, \tag{6.8}$$

where the exponential is taken in $\widetilde{\mathcal{H}}_\infty$. The inclusion functor $\iota\colon \mathcal{C}\mathrm{art} \hookrightarrow \mathcal{M}\mathrm{fd}$ induces a homotopy right Kan extension $\mathrm{hoRan}'_\iota\colon \mathcal{H}_\infty \to \widetilde{\mathcal{H}}_\infty$ (see Appendix B for details). Explicitly, working with projective model structures, we have

$$\mathrm{hoRan}_\iota(F)(M) \cong \underline{\mathcal{H}}_\infty(Q'M, F)$$

for any $M \in \mathcal{M}\mathrm{fd}$, where $Q'\colon \mathcal{H}^p_\infty \to \mathcal{H}^p_\infty$ is the cofibrant replacement functor introduced in Appendix A. The functor $\mathrm{hoRan}'_\iota\colon \mathcal{H}^{p\ell}_\infty \to \widetilde{\mathcal{H}}^{p\ell}_\infty$ is a right Quillen functor, where on the target side we consider the Čech localisation with respect to open coverings—this is a consequence of [Bunb, Prop. 3.16, Thm. 3.18]. It provides a concrete way of comparing sheaves on $\mathcal{C}\mathrm{art}$ to sheaves on $\mathcal{M}\mathrm{fd}$; the functor

$$\mathrm{hoRan}'_\iota\colon \mathcal{H}^{p\ell}_\infty \to \widetilde{\mathcal{H}}^{p\ell}_\infty$$

is a right Quillen equivalence by Theorem B.8.

We can now compare the derived concordance sheaves from Definition 6.5 to the concordance sheaf construction (6.8) from [BEBdBP]: let $F \in \mathcal{H}_\infty$ be any simplicial presheaf on $\mathcal{C}\mathrm{art}$. On the one hand, we have

$$\mathrm{hoRan}'_\iota(\mathrm{Cc}^p F)(M) = \underline{\mathcal{H}}_\infty(Q'M, \mathrm{Cc}^p F)\,,$$

and on the other hand, we have

$$\big(\mathrm{B}\,\mathrm{hoRan}'_\iota(F)\big)(M) = \delta^*\big(\underline{\mathcal{H}}_\infty(Q'(M \times \Delta_e), F)\big)\,.$$

Given that $\mathrm{hoRan}'_\iota(F)$ is a sheaf on $\mathcal{M}\mathrm{fd}$, it now follows from [BEBdBP, Thm. 1.1, Thm. 1.2] that $\mathrm{B}\,\mathrm{hoRan}'_\iota(F)$ is a sheaf on $\mathcal{M}\mathrm{fd}$ as well. Then, the results in Appendix B imply that $\mathrm{hoRan}'_\iota(\mathrm{Cc}^p F)$ and $\mathrm{B}\,\mathrm{hoRan}'_\iota(F)$ are isomorphic in $\mathrm{h}\widetilde{\mathcal{H}}^{p\ell}_\infty$ if and only if their images under the restriction functor $\iota^*\colon \widetilde{\mathcal{H}}^{p\ell}_\infty \to \mathcal{H}^{p\ell}_\infty$ are isomorphic in $\mathrm{h}\mathcal{H}^{p\ell}_\infty$.

If $M = c$ is a cartesian space, we readily obtain a natural weak equivalence

$$\mathrm{hoRan}'_\iota(\mathrm{Cc}^p F)(c) = \underline{\mathcal{H}}_\infty(Q'c, \mathrm{Cc}^p F) \xleftarrow{\sim} \mathrm{Cc}^p(F)(c)\,,$$

and we further obtain natural weak equivalences

$$\begin{aligned}\big(\mathrm{B}\,\mathrm{hoRan}'_\iota(F)\big)(M) &= \delta^*\big(\underline{\mathcal{H}}_\infty(Q'(c \times \Delta_e), F)\big)\\ &\xleftarrow{\sim} \delta^*\big(\underline{\mathcal{H}}_\infty(c \times \Delta_e, F)\big)\\ &\cong \delta^* F(c \times \Delta_e)\\ &\xrightarrow{\sim} \mathrm{Cc}^p(F)(c)\,.\end{aligned}$$

The first weak equivalence uses that Δ^k_e is a cartesian space, for each $k \in \mathbb{N}_0$, so that $c \times \Delta^k_e$ is representable in $\mathcal{H}_\infty$, and the last morphism arises from postcomposing with a fibrant replacement functor in $\mathcal{S}\mathrm{et}_\Delta$. This establishes a natural zig-zag of weak equivalences between $\mathrm{hoRan}'_\iota(\mathrm{Cc}^p F)$ and $\mathrm{B}\,\mathrm{hoRan}'_\iota(F)$. Note that the results [BEBdBP, Thm. 1.1, Thm. 1.2] enter crucially in the construction of this zig-zag because we use the fact that $\mathrm{B}\,\mathrm{hoRan}'_\iota(F)$ is a sheaf on $\mathcal{M}\mathrm{fd}$. ◁

Remark 6.9 We outline an alternative proof of Theorem 6.6, based on Proposition 6.2 and Theorem 2.17 (which we recall goes back to [Dugb]). Let $F, G \in \mathcal{H}_\infty$, and let $Q\colon \mathcal{H}^p_\infty \to \mathcal{H}^p_\infty$ denote a cofibrant replacement functor. In the homotopy category $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$ of spaces, we have natural isomorphisms

$$\begin{aligned}\mathrm{Map}_{\mathcal{H}^{pI}_\infty}(F, G) &\cong \underline{\mathcal{H}}_\infty(QF, \mathrm{Cc}^p G)\\ &\cong \underline{\mathcal{H}}_\infty(QF, \tilde{c} R^{\mathrm{Set}_\Delta} S_e G)\end{aligned}$$

$$
\begin{aligned}
&\cong \underline{\mathcal{S}\mathrm{et}_\Delta}\big(\operatorname*{colim}_{\mathcal{C}\mathrm{art}^{\mathrm{op}}} QF, R^{\mathcal{S}\mathrm{et}_\Delta} S_e G\big) \\
&\cong \underline{\mathcal{S}\mathrm{et}_\Delta}\big(\operatorname*{hocolim}_{\mathcal{C}\mathrm{art}^{\mathrm{op}}} F, R^{\mathcal{S}\mathrm{et}_\Delta} S_e G\big) \\
&\cong \underline{\mathcal{S}\mathrm{et}_\Delta}\big(S_e F, R^{\mathcal{S}\mathrm{et}_\Delta} S_e G\big) \\
&\cong \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(S_e F, S_e G)\,.
\end{aligned}
$$

In the second isomorphism, we have used that $\mathrm{Cc}^p G$ is locally constant and that $\mathrm{Cc}^p G(*) = R^{\mathcal{S}\mathrm{et}_\Delta} \circ S_e G$. The third isomorphism arises from the Quillen adjunction $\mathrm{colim} : \mathcal{H}_\infty^p \rightleftarrows \mathcal{S}\mathrm{et}_\Delta : \tilde{\mathtt{c}}$, and the fourth isomorphism stems from fact that $\mathrm{colim} \circ Q$ models the homotopy colimit. Finally, the fifth isomorphism arises from our observation at the end of Section 4 that $S_e F$ is a model for the homotopy colimit of the diagram $F\colon \mathcal{C}\mathrm{art}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$. Parts (2) and (3) of Theorem 6.6 then follow as in our proof above. $\triangleleft$

We can now give a direct proof of the relation between model categories from Remark 3.17; that is, we identify the homotopy theory induced on $\mathcal{H}_\infty$ by the smooth singular complex functor S_e:

Theorem 6.10 *Let $W_{\mathcal{S}\mathrm{et}_\Delta}$ denote the class of weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$, and let $S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})$ denote the class of morphisms in $\mathcal{H}_\infty$ whose image under S_e is in $W_{\mathcal{S}\mathrm{et}_\Delta}$. There is an identity of model categories*

$$
\mathcal{H}_\infty^{p/i\, I} = L_{S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})} \mathcal{H}_\infty^{p/i}\,.
$$

Proof. We set

$$
\mathcal{M}^{p/i} := L_{S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})} \mathcal{H}_\infty^{p/i}\,.
$$

The model categories $\mathcal{H}_\infty^{p/i\, I}$ and $\mathcal{M}^{p/i}$ have the same cofibrations, since they are left Bousfield localisations of the same model category. (Here we use either the projective or the injective model structure on *both* sides.) By Theorem 2.8 it now suffices to show that they have the same fibrant objects. Corollary 3.15 implies that $I \subset S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})$; thus, any fibrant object in $\mathcal{M}^{p/i}$ is also fibrant in $\mathcal{H}_\infty^{p/i\, I}$.

To see that any fibrant object of $\mathcal{H}_\infty^{p/i\, I}$ is also fibrant in $\mathcal{M}^{p/i}$, consider a fibrant object $G \in \mathcal{H}_\infty^{p/i\, I}$ and a morphism $f\colon F_0 \to F_1$ in $S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})$. By Theorem 6.6, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathcal{H}_\infty^{p/i\, I}}(F_1, G) & \xrightarrow{\ f^*\ } & \mathrm{Map}_{\mathcal{H}_\infty^{p/i\, I}}(F_0, G) \\
\cong \big\downarrow & & \big\downarrow \cong \\
\mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(S_e F_1, S_e G) & \xrightarrow[(S_e f)^*]{} & \mathrm{Map}_{\mathcal{S}\mathrm{et}_\Delta}(S_e F_0, S_e G)
\end{array}
$$

in $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$. By assumption on f, the morphism $S_e f$ is a weak equivalence in $\mathcal{S}\mathrm{et}_\Delta$. Hence, it induces a weak equivalence on mapping spaces; that is, the bottom morphism in the diagram is an isomorphism in $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$. From that, it follows that also the top morphism is an isomorphism in $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$, which implies that G is $S_e^{-1}(W_{\mathcal{S}\mathrm{et}_\Delta})$-local, and therefore fibrant in $\mathcal{M}^{p/i}$. $\square$

A An injective fibrant replacement of simplicial presheaves

From the definition of the projective and the injective model structure on $\mathcal{H}_\infty$, it follows directly that there is a Quillen equivalence

$$
\mathcal{H}_\infty^p \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}_\infty^i\,.
$$

Both of the functors in this adjunction are the identity on $\mathcal{H}_\infty$. Here, we will construct a Quillen equivalence in the opposite direction, i.e.

$$Q' : \mathcal{H}^i_\infty \xrightarrow[\longleftarrow]{\perp} \mathcal{H}^p_\infty : R' .$$

We start by defining the functor Q'. Its construction is not specific to simplicial presheaves on $\mathcal{C}\mathrm{art}$, but it works for simplicial presheaves over any small category. Thus, let $\mathcal{C}$ be a small category, let $\mathcal{K}_\infty$ denote the category of simplicial presheaves on $\mathcal{C}$, and let $\mathcal{Y}\colon \mathcal{C} \to \mathcal{K}_\infty$ denote the Yoneda embedding. We denote the projective and the injective model structures on $\mathcal{K}_\infty$ by $\mathcal{K}^p_\infty$ and by $\mathcal{K}^i_\infty$, respectively. The conventional two-sided simplicial bar construction provides a functor [Rie14]

$$B_\bullet\big((-), \mathcal{C}, \mathcal{Y}\big) \colon \mathcal{K}_\infty \longrightarrow (\mathcal{K}_\infty)^{\mathbb{\Delta}^{\mathrm{op}}} , \qquad F \longmapsto B_\bullet(F, \mathcal{C}, \mathcal{Y}) .$$

Lemma A.1 *The functor $B_\bullet((-), \mathcal{C}, \mathcal{Y})$ sends injective cofibrations in $\mathcal{K}_\infty$ to injective cofibrations in $(\mathcal{K}^p_\infty)^{\mathbb{\Delta}^{\mathrm{op}}}$. That is, if $f\colon F \to G$ is an objectwise cofibration of simplicial presheaves on $\mathcal{C}$, then $B_n(f, \mathcal{C}, \mathcal{Y})$ is a projective cofibration of simplicial presheaves, for each $n \in \mathbb{N}_0$.*

Proof. We have that

$$B_n(F, \mathcal{C}, \mathcal{Y}) = \coprod_{c_0,\ldots,c_n \in \mathcal{C}} F(c_n) \otimes \mathcal{C}(c_{n-1}, c_n) \otimes \cdots \otimes \mathcal{C}(c_0, c_1) \otimes \mathcal{Y}_{c_0} .$$

Observing that $\mathcal{Y}_c \in \mathcal{K}^p_\infty$ is cofibrant for every $c \in \mathcal{C}$ and recalling that $\mathcal{K}^p_\infty$ is a simplicial model category, we see that, in each part of the coproduct, the functor

$$(-) \otimes \mathcal{C}(c_{n-1}, c_n) \otimes \cdots \otimes \mathcal{C}(c_0, c_1) \otimes \mathcal{Y}_{c_0} \colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{K}_\infty$$

preserves cofibrations. Since the objects under the coproduct are each cofibrant, these cofibrations induce a cofibration between the coproducts. □

Let $\mathcal{I}$ be a small category, $\mathcal{V}$ a symmetric monoidal category, and $\mathcal{M}$ a model category category enriched, tensored and cotensored over $\mathcal{V}$ (i.e. a model $\mathcal{V}$-category in the terminology of [Bar10]). Following the notation in [Rie14], we let

$$(-) \underset{\mathcal{I}}{\otimes} (-) \colon \mathcal{V}^{\mathcal{I}^{\mathrm{op}}} \times \mathcal{M}^{\mathcal{I}} \longrightarrow \mathcal{M} \qquad \text{and} \qquad \{-,-\}^{\mathcal{I}} \colon \mathcal{V}^{\mathcal{I}} \times \mathcal{M}^{\mathcal{I}} \longrightarrow \mathcal{M}$$

denote the functor tensor product and the functor hom, respectively. Let $\mathbb{\Delta}_{/(-)} \colon \mathbb{\Delta} \to \mathcal{C}\mathrm{at}$ denote the functor that sends $[k] \in \mathbb{\Delta}$ to the slice category $\mathbb{\Delta}_{/[k]}$, and let N denote the nerve functor. We now define the functor

$$Q' \colon \mathcal{K}_\infty \longrightarrow \mathcal{K}_\infty , \qquad Q' \coloneqq N(\mathbb{\Delta}_{/(-)})^{\mathrm{op}} \underset{\mathbb{\Delta}^{\mathrm{op}}}{\otimes} B_\bullet\big((-), \mathcal{C}, \mathcal{Y}\big) . \tag{A.2}$$

Lemma A.3 *The functor $Q' \colon \mathcal{K}^i_\infty \to \mathcal{K}^p_\infty$ preserves cofibrations.*

Proof. We view the functor tensor product in the definition of Q' as

$$(-) \underset{\mathbb{\Delta}^{\mathrm{op}}}{\otimes} (-) \colon (\mathcal{S}\mathrm{et}_\Delta)^{\mathbb{\Delta}} \times (\mathcal{K}^p_\infty)^{\mathbb{\Delta}^{\mathrm{op}}} \longrightarrow \mathcal{K}^p_\infty .$$

Further, the functor tensor product is a left Quillen bifunctor when endowing the two source categories with any of the pairs of model structures (projective, injective), (Reedy, Reedy), or (injective, projective) [Rie14, Thms. 11.5.9, 14.3.1]. Further, the functor $N(\mathbb{\Delta}_{/(-)})^{\mathrm{op}} \colon \mathbb{\Delta} \longrightarrow \mathcal{S}\mathrm{et}_\Delta$ is projectively cofibrant [Hir03, Prop. 14.8.8]. (Note that it is then also Reedy cofibrant.) Consequently, the functor

$$N(\mathbb{\Delta}_{/(-)})^{\mathrm{op}} \underset{\mathbb{\Delta}^{\mathrm{op}}}{\otimes} (-) \colon (\mathcal{K}^p_\infty)^{\mathbb{\Delta}^{\mathrm{op}}} \longrightarrow \mathcal{K}^p_\infty$$

is left Quillen with respect to the injective model structure on $(\mathcal{K}^p_\infty)^{\mathbb{\Delta}^{\mathrm{op}}}$ (and hence also with respect to the Reedy model structure). We can write Q' as the composition

$$\mathcal{K}^i_\infty \xrightarrow{B_\bullet((-),\mathcal{C},\mathcal{Y})} \big((\mathcal{K}^p_\infty)^{\mathbb{\Delta}^{\mathrm{op}}}\big)_{inj} \xrightarrow{N(\mathbb{\Delta}_{/(-)})^{\mathrm{op}} \otimes_{\mathbb{\Delta}^{\mathrm{op}}} (-)} \mathcal{K}^p_\infty\,.$$

The model category of simplicial diagrams in the middle carries the injective model structure. We have shown in Lemma A.1 that the first functor preserves cofibrations, and it follows from our arguments above that also the second functor preserves cofibrations. □

Next, we are going to employ the Bousfield-Kan map to show that Q' can also be seen as a cofibrant replacement functor on $\mathcal{K}^p_\infty$. The Bousfield-Kan map is a morphism

$$bk \colon N(\mathbb{\Delta}_{/(-)})^{\mathrm{op}} \longrightarrow \Delta^\bullet$$

of cosimplicial simplicial sets. We will use the following two statements:

Proposition A.4 [Hir03, Prop. 18.7.2] *The Bousfield-Kan map is a Reedy weak equivalence between Reedy cofibrant cosimplicial simplicial sets.*

Proposition A.5 *For any $F \in \mathcal{K}_\infty$, the simplicial object in $\mathcal{K}_\infty$ given as $B_\bullet(F, \mathcal{C}, \mathcal{Y})$ is Reedy cofibrant.*

Proof. This follows directly from [Rie14, Rmk. 5.2.2], applied to the functor $\mathcal{Y} \colon \mathcal{C} \to \mathcal{K}^p_\infty$. □

We now consider the induced natural transformation

$$bk \underset{\mathbb{\Delta}^{\mathrm{op}}}{\otimes} B_\bullet\big((-), \mathcal{C}, \mathcal{Y}\big) \colon Q' \longrightarrow \Delta^\bullet \underset{\mathbb{\Delta}^{\mathrm{op}}}{\otimes} B_\bullet\big((-), \mathcal{C}, \mathcal{Y}\big) = B\big((-), \mathcal{C}, \mathcal{Y}\big)\,. \tag{A.6}$$

By Proposition A.5, the functor

$$(-) \underset{\mathbb{\Delta}^{\mathrm{op}}}{\otimes} B_\bullet\big(F, \mathcal{C}, \mathcal{Y}\big) \colon \big((\mathcal{S}\mathrm{et}_\Delta)^{\mathbb{\Delta}}\big)_{Reedy} \longrightarrow \mathcal{K}^p_\infty$$

is a left Quillen functor for every $F \in \mathcal{K}$. It then follows from Proposition A.4 and the arguments in the proof of Proposition A.3 that the natural transformation (A.6) is a natural weak equivalence of functors $\mathcal{K}_\infty \to \mathcal{K}^p_\infty$. Finally, we use that the functor $B((-), \mathcal{C}, \mathcal{Y})$ agrees with Dugger's cofibrant replacement functor Q^p for $\mathcal{K}^p_\infty$ from [Dug01b]. In particular, it comes with a natural weak equivalence $q^p \colon Q^p \to 1_{\mathcal{K}_\infty}$. Composing q^p with the morphism (A.6), we obtain a natural weak equivalence $q' \colon Q' \to 1_{\mathcal{K}_\infty}$. Putting everything together, we have proven

Proposition A.7 *The functor Q' from* (A.2)*, together with the natural weak equivalence q' provide a cofibrant replacement functor for $\mathcal{K}^p_\infty$. In particular, Q' preserves objectwise weak equivalences.*

Finally, we observe that Q' has a right adjoint, which is explicitly given by

$$R' \colon \mathcal{K}_\infty \longrightarrow \mathcal{K}_\infty\,, \qquad R'(G) = \big\{N(\mathbb{\Delta}_{/(-)})^{\mathrm{op}}, C^\bullet(G, \mathcal{C}^{\mathrm{op}}, \mathcal{Y}^{(-)})\big\}^{\mathbb{\Delta}}\,, \tag{A.8}$$

where $\mathcal{Y}^{(-)} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}(\mathcal{C}, \mathcal{S}\mathrm{et}_\Delta)$ denotes the co-Yoneda embedding of $\mathcal{C}$.

Theorem A.9 *The functors Q' and R' satisfy the following properties:*

(1) The adjunction $Q' \dashv R'$ is a Quillen equivalence

$$Q' : \mathcal{K}^i_\infty \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{K}^p_\infty : R' .$$

(2) There is a natural transformation $r' \colon 1_{\mathcal{K}_\infty} \to R'$ such that $r'_{|G} \colon G \to R'G$ is a weak equivalence in $\mathcal{K}^i_\infty$ for every projectively fibrant $G \in \mathcal{K}_\infty$.

(3) Let $R^{\mathcal{S}\mathrm{et}_\Delta}$ be a fibrant replacement functor for simplicial sets. Then, $G \mapsto R'(R^{\mathcal{S}\mathrm{et}_\Delta} \circ G)$ is a fibrant replacement functor on $\mathcal{K}^i_\infty$.

Proof. Ad (1): Proposition A.7, together with the observation that Q' is a left adjoint, readily implies that Q' is a left Quillen functor. The fact that this is a Quillen equivalence follows from the existence of the natural weak equivalence $q' \colon Q' \xrightarrow{\sim} 1_{\mathcal{K}_\infty}$. Formally, this implies that the composition of Q' by the left Quillen equivalence $1_{\mathcal{K}_\infty} \colon \mathcal{K}^p_\infty \to \mathcal{K}^i_\infty$ is weakly equivalent to the identity functor on $\mathcal{K}_\infty$. Thus, the statement follows from the two-out-of-three property of Quillen equivalences [Hov99] and Corollary D.8.

Ad (2): Let

$$\tau_{F,G} \colon \mathcal{K}_\infty(Q'F, G) \longrightarrow \mathcal{K}_\infty(F, R'G)$$

be the natural isomorphism that establishes the adjunction $Q' \dashv R'$. We define $r' \colon 1_{\mathcal{K}_\infty} \to R'$ to be the image under τ of the natural transformation q'.

Let $G \in \mathcal{K}_\infty$ be a projectively fibrant object. Since every object in $\mathcal{K}^i_\infty$ is cofibrant and since $Q' \dashv R'$ is a Quillen equivalence, a morphism $\varphi \colon Q'F \to G$ is a weak equivalence (in $\mathcal{K}^p_\infty$) if and only if $\tau_{F,G}(\varphi) \colon F \to R'G$ is a weak equivalence (in $\mathcal{K}^i_\infty$).

Now consider the weak equivalence $q'_{|G} \colon Q'G \to G$, for $G \in \mathcal{K}^p_\infty$ fibrant. This is a weak equivalence of the form considered above; thus, the morphism $r'_{|G} \colon G \to R'G$ is a weak equivalence whenever G is projectively fibrant.

Ad (3): Let $R^{\mathcal{S}\mathrm{et}_\Delta}$ be a fibrant replacement functor in $\mathcal{S}\mathrm{et}_\Delta$, with associated natural weak equivalence $r^{\mathcal{S}\mathrm{et}_\Delta} \colon 1_{\mathcal{S}\mathrm{et}_\Delta} \xrightarrow{\sim} R^{\mathcal{S}\mathrm{et}_\Delta}$. Let $G \in \mathcal{K}_\infty$ be arbitrary and consider the composition

$$G \xrightarrow{r^{\mathcal{S}\mathrm{et}_\Delta} \circ 1_G} R^{\mathcal{S}\mathrm{et}_\Delta} \circ G \xrightarrow{r'|R^{\mathcal{S}\mathrm{et}_\Delta} G} R'(R^{\mathcal{S}\mathrm{et}_\Delta} \circ G) .$$

The first morphism is an objectwise weak equivalence by definition. Since $R^{\mathcal{S}\mathrm{et}_\Delta} \circ G$ is projectively fibrant, the second morphism is a weak equivalence as well by part (2). $\square$

Corollary A.10 *Let F, G be any two objects in $\mathcal{K}_\infty$. There is a canonical isomorphism between (the homotopy types of) the mapping spaces of the projective and the injective model structures*

$$\mathrm{Map}_{\mathcal{K}^p_\infty}(F, G) \cong \mathrm{Map}_{\mathcal{K}^i_\infty}(F, G)$$

in the homotopy category $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$ of spaces.

Proof. Since both $\mathcal{K}^p_\infty$ and $\mathcal{K}^i_\infty$ are simplicial model categories, we have the following isomorphisms in $\mathrm{h}\mathcal{S}\mathrm{et}_\Delta$:

$$\begin{aligned}\mathrm{Map}_{\mathcal{K}^p_\infty}(F, G) &\cong \underline{\mathcal{K}}_\infty(Q'F, R^{\mathcal{S}\mathrm{et}_\Delta} \circ G) \\ &\cong \underline{\mathcal{K}}_\infty(F, R'R^{\mathcal{S}\mathrm{et}_\Delta} \circ G)\end{aligned}$$

$$\cong \mathrm{Map}_{\mathcal{K}^i_\infty}(F, G)\,.$$

The first isomorphism arises from the fact that $\mathcal{K}^p_\infty$ is simplicial and that Q' is a cofibrant replacement functor in $\mathcal{K}^p_\infty$ (Proposition A.7). The second isomorphism is the adjointness $Q' \dashv R'$, and the third isomorphism stems from the facts that every object in $\mathcal{K}^i_\infty$ is cofibrant and that the functor $G \mapsto R'(R^{\mathcal{S}\mathrm{et}_\Delta} \circ G)$ is a fibrant replacement functor in $\mathcal{K}^i_\infty$ (Theorem A.9). □

B Sheaves on manifolds and sheaves on cartesian spaces

This appendix is devoted to the comparison of two Čech localisations: on the one hand, we consider the localisation $\mathcal{H}^{p\ell}_\infty$ of the projective model structure $\mathcal{H}^p_\infty$ of simplicial presheaves on cartesian spaces at the differentiably good open coverings (see Section 2.1). On the other hand, we have the Čech localisation $\widetilde{\mathcal{H}}^{p\ell}_\infty$ of the projective model structure $\widetilde{\mathcal{H}}^p_\infty$ of simplicial presheaves on manifolds at the open coverings. (The same arguments also apply to the localisation of $\widetilde{\mathcal{H}}^p_\infty$ at the surjective submersions.)

We will compare these localisations by means of the functors

$$\begin{aligned} \mathrm{hoRan}_\iota \colon \mathcal{H}_\infty &\longrightarrow \widetilde{\mathcal{H}}_\infty\,, \qquad F \mapsto \underline{\mathcal{H}}_\infty\big(Q(-), F\big)\,,\\ \mathrm{hoRan}'_\iota \colon \mathcal{H}_\infty &\longrightarrow \widetilde{\mathcal{H}}_\infty\,, \qquad F \mapsto \underline{\mathcal{H}}_\infty\big(Q'(-), F\big)\,, \end{aligned}$$

where $\iota\colon \mathcal{C}\mathrm{art} \hookrightarrow \mathcal{M}\mathrm{fd}$ is the canonical inclusion, and where $Q\colon \mathcal{H}^p_\infty \to \mathcal{H}^p_\infty$ is Dugger's cofibrant replacement functor for the projective model structure on simplicial presheaves (see Appendix A). Further, $Q' \dashv R'$ is the Quillen equivalence from Theorem A.9. The natural transformation (A.6) induces a natural transformation $\eta\colon\ \mathrm{hoRan}_\iota \longrightarrow \mathrm{hoRan}'_\iota$ whose component $\eta_{|F}$ on every fibrant object $F \in \mathcal{H}^p_\infty$ is a projective weak equivalence.

The functor $\mathrm{hoRan}_\iota\colon \mathcal{H}^p_\infty \to \widetilde{\mathcal{H}}^p_\infty$ is a right Quillen functor with left adjoint $Q \circ \iota^*$, and this Quillen adjunction descends to a Quillen adjunction on Čech localisations,

$$Q \circ \iota^* : \widetilde{\mathcal{H}}^{p\ell}_\infty \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}^{p\ell}_\infty : \mathrm{hoRan}_\iota\,,$$

which follows from [Bunb, Thm. 3.18]. Analogously, $\mathrm{hoRan}'_\iota\colon \mathcal{H}^p_\infty \to \widetilde{\mathcal{H}}^p_\infty$ is a right Quillen functor (since Q' is homotopical and valued in projectively cofibrant simplicial presheaves). Further, the natural transformation $\eta\colon\ \mathrm{hoRan}_\iota \longrightarrow \mathrm{hoRan}'_\iota$ establishes that hoRan'_ι maps local objects in $\mathcal{H}^{p\ell}_\infty$ to local objects in $\widetilde{\mathcal{H}}^{p\ell}_\infty$, and hence that we also have a Quillen adjunction

$$Q' \circ \iota^* : \widetilde{\mathcal{H}}^{p\ell}_\infty \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}^{p\ell}_\infty : \mathrm{hoRan}'_\iota\,.$$

Note that we can also write the right adjoint as

$$\mathrm{hoRan}'_\iota \cong \iota_* \circ R'\,,$$

with $R'\colon \mathcal{H}_\infty \to \mathcal{H}_\infty$ defined as in (A.8), and where ι_* is the right Kan extension along ι.

Lemma B.1 *The derived counit of the Quillen adjunction $Q' \circ \iota^* \dashv \mathrm{hoRan}'_\iota$ is a projective weak equivalence on every fibrant $F \in \mathcal{H}^p_\infty$.*

Proof. Let $\widetilde{Q}\colon \widetilde{\mathcal{H}}_\infty^p \to \widetilde{\mathcal{H}}_\infty^p$ be a cofibrant replacement functor with associated natural weak equivalence $\widetilde{q}\colon \widetilde{Q} \xrightarrow{\sim} 1_{\widetilde{\mathcal{H}}_\infty}$. Consider a fibrant object $F \in \mathcal{H}_\infty^p$ and the diagram

$$\begin{array}{ccccc} (Q'\circ\iota^*)\circ\widetilde{Q}\circ(\iota_*\circ R')(F) & & \dashrightarrow & & F \\ \big\downarrow{\scriptstyle (Q'\circ\iota^*)\widetilde{q}_{|\iota_*\circ R'(F)}} & & & & \\ (Q'\circ\iota^*)\circ(\iota_*\circ R')(F) & \xrightarrow{\cong} & (Q'\circ R')(F) & \longrightarrow & F \end{array}$$

The left-hand vertical morphism is a weak equivalence in $\mathcal{H}_\infty^p$ because $\widetilde{q}_{|\iota_*\circ R'(F)}$ is a projective weak equivalence and $\iota^*\colon \widetilde{\mathcal{H}}_\infty^p \to \mathcal{H}_\infty^p$ is homotopical, as is Q'. The left-hand bottom morphism is the counit of the adjunction $\iota^* \dashv \iota_*$, which is an isomorphism by the Yoneda Lemma. The right-hand bottom morphism is a weak equivalence by Theorem A.9 and the fact that every object in $\mathcal{H}_\infty^i$ is cofibrant. □

It follows that the total derived functor $\mathbb{R}\,\mathrm{hoRan}'_\iota\colon \mathrm{h}\mathcal{H}_\infty^p \to \mathrm{h}\widetilde{\mathcal{H}}_\infty^p$ is fully faithful, and hence that also $\mathbb{R}\,\mathrm{hoRan}'_\iota\colon \mathrm{h}\mathcal{H}_\infty^{p\ell} \to \mathrm{h}\widetilde{\mathcal{H}}_\infty^{p\ell}$ is fully faithful.

Next, we would like to show that $\mathbb{R}\,\mathrm{hoRan}'_\iota$ is also essentially surjective. We start by recalling that, for any open covering $\mathcal{U} = \{U_a\}_{a\in A}$ of a manifold M, the induced Čech nerve $\check{C}\mathcal{U} \to \underline{M}$ is a weak equivalence in both $\mathcal{H}_\infty^{p\ell}$ and $\widetilde{\mathcal{H}}_\infty^{p\ell}$ (this follows directly from [DHI04, Cor. A.3]). If $\mathcal{U}$ is even a differentiably good open covering of M (i.e. each finite intersection of patches is either empty or a cartesian space), then $\check{C}\mathcal{U}$ is levelwise a coproduct of representables in $\mathcal{H}_\infty$, and hence it is cofibrant in $\mathcal{H}_\infty^{p\ell}$.

Suppose that $G \in \widetilde{\mathcal{H}}_\infty^{p\ell}$ is fibrant. Given a manifold M with a differentiably good open covering $\mathcal{U}$, we have weak equivalences

$$G(M) \cong \underline{\widetilde{\mathcal{H}}}_\infty(\underline{M}, G) \xrightarrow{\sim} \underline{\widetilde{\mathcal{H}}}_\infty(\check{C}\mathcal{U}, G) \cong \underline{\mathcal{H}}_\infty(\check{C}\mathcal{U}, \iota^* G)\,. \tag{B.2}$$

In this sense, the value of a sheaf G on manifolds is determined by its values on cartesian spaces. However, the above weak equivalence is by no means functorial in M, so to make this statement precise, we need to improve on (B.2).

For $M \in \mathcal{M}\mathrm{fd}$, let $\mathrm{Cov}(M)$ denote the following category: an object of $\mathrm{Cov}(M)$ is an open covering $\mathcal{U} = \{U_a\}_{a\in A}$ of M, and given another open covering $\mathcal{V}$ of M there is a unique morphism $\mathcal{V} \to \mathcal{U}$ precisely if $\mathcal{V}$ refines $\mathcal{U}$. Let $\mathrm{GCov}(M)$ denote the full subcategory of $\mathrm{Cov}(M)$ on the differentiably good open coverings. Note that $\mathrm{Cov}(M)$ is cofiltered since every pair $(\mathcal{U}, \mathcal{V})$ of open coverings has a common refinement $\mathcal{U} \times_M \mathcal{V}$. Moreover, any open covering of M has a differentiably good refinement [FSS12, App. A], which implies that $\mathrm{GCov}(M)$ is cofiltered and that the canonical inclusion $j_M\colon \mathrm{GCov}(M) \hookrightarrow \mathrm{Cov}(M)$ is final, since for any $\mathcal{U} \in \mathrm{Cov}(M)$ the slice category $j/\mathcal{U}$ is cofiltered as well.

Given $G \in \widetilde{\mathcal{H}}_\infty$, we consider the diagrams in simplicial sets

$$\mathrm{Cov}(M)^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}_\Delta\,, \qquad \mathcal{U} \longmapsto \underline{\widetilde{\mathcal{H}}}_\infty(\check{C}\mathcal{U}, G) \quad \text{and} \quad \mathcal{U} \longmapsto G(M)\,.$$

Since $\mathrm{Cov}(M)^{\mathrm{op}}$ is filtered and $\mathcal{S}\mathrm{et}_\Delta$ is combinatorial, the ordinary colimits of these diagrams model their homotopy colimits [Dug01a, Prop. 7.3], so that we obtain a morphism

$$G(M) \longrightarrow \underset{\mathcal{U}\in\mathrm{Cov}(M)^{\mathrm{op}}}{\mathrm{colim}^{\mathcal{S}\mathrm{et}_\Delta}} \Big(\underline{\widetilde{\mathcal{H}}}_\infty(\check{C}\mathcal{U}, G)\Big) \simeq \underset{\mathcal{U}\in\mathrm{Cov}(M)^{\mathrm{op}}}{\mathrm{hocolim}^{\mathcal{S}\mathrm{et}_\Delta}} \Big(\underline{\widetilde{\mathcal{H}}}_\infty(\check{C}\mathcal{U}, G)\Big)\,. \tag{B.3}$$

(The right-hand side of (B.3) is Grothendieck's Plus Construction, applied to M and G.) If $G \in \widetilde{\mathcal{H}}_\infty^{p\ell}$ is fibrant, then the morphism in (B.3) is a weak equivalence, for each manifold $M \in \mathcal{M}\mathrm{fd}$. Since $j_M^{\mathrm{op}} \colon \mathrm{GCov}(M)^{\mathrm{op}} \hookrightarrow \mathrm{Cov}(M)^{\mathrm{op}}$ is cofinal, we further obtain a canonical isomorphism

$$\operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{U} \in \mathrm{Cov}(M)^{\mathrm{op}}} \Big(\underline{\widetilde{\mathcal{H}}_\infty}(\check{C}\mathcal{U}, G) \Big) \cong \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{U} \in \mathrm{GCov}(M)^{\mathrm{op}}} \Big(\underline{\widetilde{\mathcal{H}}_\infty}(\check{C}\mathcal{U}, G) \Big) .$$

If $\mathcal{U} \in \mathrm{GCov}(M)$, then $\check{C}\mathcal{U}$ is levelwise a coproduct of presheaves represented by cartesian spaces. Hence, there is a further canonical isomorphism

$$\operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{U} \in \mathrm{GCov}(M)^{\mathrm{op}}} \Big(\underline{\widetilde{\mathcal{H}}_\infty}(\check{C}\mathcal{U}, G) \Big) \cong \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{U} \in \mathrm{GCov}(M)^{\mathrm{op}}} \Big(\underline{\mathcal{H}_\infty}(\check{C}\mathcal{U}, \iota^* G) \Big) , \tag{B.4}$$

where $\iota \colon \mathcal{C}\mathrm{art} \hookrightarrow \mathcal{M}\mathrm{fd}$ is the canonical inclusion functor. Thus, for any $G \in \widetilde{\mathcal{H}}_\infty$ and any $M \in \mathcal{M}\mathrm{fd}$ we obtain a morphism

$$\gamma_{G,M} \colon G(M) \longrightarrow \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{U} \in \mathrm{GCov}(M)^{\mathrm{op}}} \Big(\underline{\mathcal{H}_\infty}(\check{C}\mathcal{U}, \iota^* G) \Big)$$

which is natural in G and which is a weak equivalence whenever $G \in \widetilde{\mathcal{H}}_\infty^{p\ell}$ is fibrant.

Next we show that $\gamma_{G,M}$ is also natural in M. Let $f \colon M \to N$ be a smooth map. This induces a functor $f^* \colon \mathrm{Cov}(N) \to \mathrm{Cov}(M)$, acting via $f^*(\{U_a\}_{a \in A}) = \{f^{-1}(U_a)\}_{a \in A}$. This establishes the assignment $M \mapsto \mathrm{Cov}(M)$ as a (strict) functor $\mathrm{Cov} \colon \mathcal{M}\mathrm{fd}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$. Given an open covering $\mathcal{U}$ of N, the map f induces a canonical morphism $\widehat{f} \colon \check{C}(f^*\mathcal{U}) \to \check{C}\mathcal{U}$ in $\widetilde{\mathcal{H}}_\infty$. This induces a natural transformation

$$\begin{array}{ccc} \mathrm{Cov}(N)^{\mathrm{op}} & \xrightarrow{\quad f^* \quad} & \mathrm{Cov}(M)^{\mathrm{op}} \\ & \overset{\varphi}{\Longrightarrow} & \\ {\scriptstyle \widetilde{\mathcal{H}}_\infty(\check{C}(-), G)} \searrow & & \swarrow {\scriptstyle \widetilde{\mathcal{H}}_\infty(\check{C}(-), G)} \\ & \mathcal{S}\mathrm{et}_\Delta & \end{array}$$

Further, recall that for any composition of functors $\mathcal{I} \xrightarrow{F} \mathcal{J} \xrightarrow{D} \mathcal{C}$ there is a canonical morphism $\operatorname{colim}(D \circ F) \to \operatorname{colim}(D)$. Thus, we have a canonical morphism

$$\operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{U} \in \mathrm{Cov}(N)^{\mathrm{op}}} \Big(\underline{\widetilde{\mathcal{H}}_\infty}(\check{C}(f^*\mathcal{U}), G) \Big) \longrightarrow \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{U} \in \mathrm{Cov}(M)^{\mathrm{op}}} \Big(\underline{\widetilde{\mathcal{H}}_\infty}(\check{C}\mathcal{U}, G) \Big) .$$

Combining this morphism with the natural transformation φ yields a morphism

$$\operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{U} \in \mathrm{Cov}(N)^{\mathrm{op}}} \Big(\underline{\widetilde{\mathcal{H}}_\infty}(\check{C}\mathcal{U}, G) \Big) \longrightarrow \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{V} \in \mathrm{Cov}(M)^{\mathrm{op}}} \Big(\underline{\widetilde{\mathcal{H}}_\infty}(\check{C}\mathcal{V}, G) \Big)$$

in $\mathcal{S}\mathrm{et}_\Delta$ which is compatible with composition of smooth maps of manifolds. Finally, we use the isomorphisms (B.4) to define the top morphism in the diagram

$$\begin{array}{ccc} \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{U} \in \mathrm{GCov}(N)^{\mathrm{op}}} \Big(\underline{\mathcal{H}_\infty}(\check{C}\mathcal{U}, \iota^* G) \Big) & \dashrightarrow & \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{V} \in \mathrm{GCov}(M)^{\mathrm{op}}} \Big(\underline{\mathcal{H}_\infty}(\check{C}\mathcal{V}, \iota^* G) \Big) \\ \Big\downarrow \cong & & \Big\downarrow \cong \\ \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{U} \in \mathrm{Cov}(N)^{\mathrm{op}}} \Big(\underline{\widetilde{\mathcal{H}}_\infty}(\check{C}\mathcal{U}, G) \Big) & \longrightarrow & \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{V} \in \mathrm{Cov}(M)^{\mathrm{op}}} \Big(\underline{\widetilde{\mathcal{H}}_\infty}(\check{C}\mathcal{V}, G) \Big) \end{array}$$

It follows that the assignment

$$M \longmapsto \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{V} \in \mathrm{GCov}(M)^{\mathrm{op}}} \Big(\underline{\mathcal{H}_\infty}(\check{C}\mathcal{V}, \iota^* G) \Big)$$

defines a functor $\mathcal{M}\mathrm{fd}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$. That is, we obtain a functor

$$\mathrm{Pl}\colon \widetilde{\mathcal{H}}_\infty \to \widetilde{\mathcal{H}}_\infty\,, \qquad \mathrm{Pl}\,G(M) = \operatorname*{colim}^{\mathcal{S}\mathrm{et}_\Delta}_{\mathcal{V}\in \mathrm{GCov}(M)^{\mathrm{op}}} \Big(\underline{\mathcal{H}_\infty}(\check{C}\mathcal{V}, \iota^*G) \Big)\,, \tag{B.5}$$

coming with a natural transformation $\gamma\colon 1_{\widetilde{\mathcal{H}}_\infty} \to \mathrm{Pl}$ which is an objectwise weak equivalence on every fibrant $G \in \widetilde{\mathcal{H}}^{p\ell}_\infty$. In particular, we have shown:

Proposition B.6 *Let $G, G' \in \widetilde{\mathcal{H}}^{p\ell}_\infty$ be fibrant. A morphism $g\colon G \to G'$ is a weak equivalence in $\widetilde{\mathcal{H}}^{p\ell}_\infty$ if and only if $\iota^*g\colon \iota^*G \to \iota^*G'$ is a weak equivalence in $\mathcal{H}^{p\ell}_\infty$.*

Note that for $G, G' \in \widetilde{\mathcal{H}}^{p\ell}_\infty$ fibrant, a morphism $g\colon G \to G'$ is a weak equivalence in $\widetilde{\mathcal{H}}^{p\ell}_\infty$ precisely if it is an objectwise equivalence (since both G and G' are local objects), and analogously $\iota^*g\colon \iota^*G \to \iota^*G'$ is a weak equivalence if and only if it is an objectwise equivalence (since $\iota^*\colon \widetilde{\mathcal{H}}^{p\ell}_\infty \to \mathcal{H}^{p\ell}_\infty$ preserves fibrant objects).

Lemma B.7 *The functor $\mathbb{R}\,\mathrm{hoRan}'_\iota\colon \mathrm{h}\mathcal{H}^{p\ell}_\infty \longrightarrow \mathrm{h}\widetilde{\mathcal{H}}^{p\ell}_\infty$ is essentially surjective.*

Proof. Let $G \in \widetilde{\mathcal{H}}^{p\ell}_\infty$ be fibrant. We show that there is a zig-zag of weak equivalences in $\widetilde{\mathcal{H}}^{p\ell}_\infty$ linking G to $\mathrm{hoRan}'_\iota \circ \iota^*(G)$. First, since G and $\mathrm{hoRan}'_\iota \circ \iota^*(G)$ are both sheaves on $\mathcal{M}\mathrm{fd}$, there are local weak equivalences

$$G \overset{\sim}{\longrightarrow} \mathrm{Pl}\,G \qquad \text{and} \qquad \mathrm{hoRan}'_\iota \circ \iota^*(G) \overset{\sim}{\longrightarrow} \mathrm{Pl}\big(\mathrm{hoRan}'_\iota \circ \iota^*(G)\big)\,,$$

where $\mathrm{Pl}\colon \widetilde{\mathcal{H}}^{p\ell}_\infty \to \widetilde{\mathcal{H}}^{p\ell}_\infty$ is the functor from (B.5). Further, for $c \in \mathcal{C}\mathrm{art}$ we have that

$$\iota^*\big(\mathrm{hoRan}'_\iota \circ \iota^*(G)\big)(c) = \underline{\mathcal{H}}\big(Q'(c), \iota^*(G)\big)\,,$$

so the natural weak equivalence $Q' \overset{\sim}{\longrightarrow} 1_{\mathcal{H}_\infty}$ induces a projective weak equivalence

$$\iota^*(G) \overset{\sim}{\longrightarrow} \iota^*\big(\mathrm{hoRan}'_\iota \circ \iota^*(G)\big)\,.$$

For any manifold M and any differentiably good open covering $\mathcal{U}$ of M, this induces a weak equivalence

$$\underline{\mathcal{H}_\infty}(\check{C}\mathcal{U}, \iota^*G) \overset{\sim}{\longrightarrow} \underline{\mathcal{H}_\infty}\big(\check{C}\mathcal{U}, \iota^*(\mathrm{hoRan}'_\iota \circ \iota^*(G))\big)\,.$$

Since $\mathrm{GCov}(M)^{\mathrm{op}}$ is filtered, this yields a weak equivalence $\mathrm{Pl}\,G \overset{\sim}{\longrightarrow} \mathrm{Pl}(\mathrm{hoRan}'_\iota \circ \iota^*(G))$, thus establishing the desired zig-zag of weak equivalences. □

Combining Lemmas B.1 and B.7, we obtain that the total derived functor $\mathbb{R}\,\mathrm{hoRan}'_\iota\colon \mathrm{h}\mathcal{H}^{p\ell}_\infty \to \mathrm{h}\widetilde{\mathcal{H}}^{p\ell}_\infty$ is an equivalence. Thus, we have shown

Theorem B.8 *The adjunction*

$$Q' \circ \iota^* : \widetilde{\mathcal{H}}^{p\ell}_\infty \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathcal{H}^{p\ell}_\infty : \mathrm{hoRan}'_\iota$$

is a Quillen equivalence.

C A modified two-sided bar construction

A very efficient and "unreasonably effective" [Rie14, Sec. 4] tool for the computation of homotopy colimits in simplicial model categories is given by the two-sided (simplicial) bar construction. We recommend the book [Rie14] as an introduction and as a reference. Let $\mathcal{I}$ be a small category, and consider $\mathcal{J}$-shaped diagrams in a simplicial category $\mathcal{M}$. Heuristically, the homotopical meaningfulness of the bar construction stems from the fact that it introduces coherence data into diagrams $D\colon \mathcal{J} \to \mathcal{M}$ by keeping track of composable sequences of morphisms in $\mathcal{J}$. That is, it takes into account all n-simplices of the nerve $N\mathcal{J}$.

For the purposes of Section 5 of this paper, however, we need a bar construction for diagrams of simplicial presheaves that builds on cospans in $\mathcal{J}$ rather than on ordinary morphisms in $\mathcal{J}$. More concretely, in Section 5 we consider the category whose objects are the open sets in an open covering of a manifold and all finite intersections of these open sets. The morphisms in this category are inclusions of open subsets. In the ordinary bar construction, we obtain a morphism for every inclusion. However, geometrically, it is often more useful to view an overlap U_{ab} of two elements U_a and U_b of the cover as a morphism $U_a \to U_b$. Including higher overlaps naturally leads us to considering subdivisions of simplicial sets and the associated Ex functor (explained in more detail below). This can be seen as a generalisation of constructions in [DI04, Sec. 4].

The purpose of this appendix is to introduce a modified two-sided (simplicial) bar construction whose coherence data is encoded not by the nerve $N\mathcal{J}$, but by the simplicial set $\mathrm{Ex}N\mathcal{J}$. Again heuristically, since $N\mathcal{J}$ and $\mathrm{Ex}N\mathcal{J}$ are weakly equivalent in $\mathcal{S}\mathrm{et}_\Delta$, we should expect the modified bar construction to be equivalent to the original version—we indeed prove this in Proposition C.7.

Let $\mathcal{C}$ be a small category, and let $\mathcal{K}_\infty$ denote the category of simplicial presheaves on $\mathcal{C}$. We denote by $\mathcal{K}_\infty^{p/i}$ the category $\mathcal{K}_\infty$ endowed with the projective or the injective model structure, respectively. Let $\mathcal{J}$ be a small category, and let $E\colon \mathcal{J} \to \mathcal{K}_\infty$ be a diagram. We can equivalently view E as a functor $E\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{S}\mathrm{et}_\Delta$. Further, let $F\colon \mathcal{J}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$ be a functor. Given these data, there is an associated *two-sided bar construction* [Rie14]

$$B^{\mathcal{K}_\infty}(F, \mathcal{J}, E) = \int^{n \in \mathbb{\Delta}} \Delta^n \otimes B_n^{\mathcal{K}_\infty}(F, \mathcal{J}, E)\,,$$

where the n-th level of the *two-sided simplicial bar construction* in $\mathcal{K}_\infty$ reads as

$$B_n^{\mathcal{K}_\infty}(F, \mathcal{J}, E) = \coprod_{i_0, \ldots, i_n \in \mathcal{J}} E(-, i_0) \times \mathcal{J}(i_0, i_1) \times \cdots \times \mathcal{J}(i_{n-1}, i_n) \times F(i_n)\,. \tag{C.1}$$

We will refer to the two-sided bar construction as the bar construction, for short. We can use the bar construction to model the homotopy colimit in $\mathcal{K}_\infty$ as follows [Rie14, Sec. 5]:

$$\operatorname*{hocolim}_{\mathcal{J}}{}^{\mathcal{K}_\infty}(E) \simeq B^{\mathcal{K}_\infty}(*, \mathcal{J}, Q^{\mathcal{K}_\infty} \circ E)\,,$$

where $\mathcal{K}_\infty$ is endowed with some simplicial model structure, and where $Q^{\mathcal{K}_\infty}$ is a cofibrant replacement functor for that model structure. In particular, if $E\colon \mathcal{J} \to \mathcal{K}_\infty$ is objectwise cofibrant with respect to that model structure, then we have

$$\operatorname*{hocolim}_{\mathcal{J}}{}^{\mathcal{K}_\infty}(E) \simeq B_n^{\mathcal{K}_\infty}(*, \mathcal{J}, E)\,.$$

We record the following immediate results:

Lemma C.2 *Let $F \in \mathcal{K}_\infty$, $E\colon \mathcal{J} \to \mathcal{K}_\infty$, and view $\mathcal{K}_\infty$ as a simplicial category.*

(1) For any $c \in \mathcal{C}$, there is a canonical isomorphism, natural in c,

$$B^{\mathcal{K}_\infty}(F, \mathcal{J}, E)(c) \cong B^{\mathcal{S}\mathrm{et}_\Delta}\big(F, \mathcal{J}, E(c, -)\big)\,.$$

(2) The bar construction is naturally isomorphic to the composition

$$(\mathcal{S}\mathrm{et}_\Delta)^{\mathcal{J}^{\mathrm{op}}} \times (\mathcal{K}_\infty)^{\mathcal{J}} \xrightarrow{B^{\mathcal{K}_\infty}_\bullet(-,\mathcal{J},-)} (\mathcal{K}_\infty)^{\mathbb{\Delta}^{\mathrm{op}}} \cong (s\mathcal{S}\mathrm{et}_\Delta)^{\mathcal{C}^{\mathrm{op}}} \xrightarrow{\delta^* \circ (-)} \mathcal{K}_\infty\,.$$

Let $\mathcal{Y}_{(-)} \colon \mathcal{J} \to \mathcal{C}\mathrm{at}(\mathcal{J}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$ denote the Yoneda embedding of $\mathcal{J}$, and let $\mathcal{Y}^{(-)} \colon \mathcal{J}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}(\mathcal{J}, \mathcal{S}\mathrm{et})$ denote the co-Yoneda embedding of $\mathcal{J}$, i.e. for $i \in \mathcal{J}$ we have $\mathcal{Y}^i = \mathcal{J}(i,-)$. Recall that there are canonical natural isomorphisms

$$\mathcal{S}\mathrm{et}^{\mathcal{J}}(\mathcal{Y}^i, Z) \xrightarrow{\cong} Z(i)$$

for any $i \in \mathcal{J}$ and $Z \in \mathcal{S}\mathrm{et}^{\mathcal{J}}$.

Definition C.3 *Consider a pair of functors $X \colon \mathcal{J}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ and $Y \colon \mathcal{J} \to \mathcal{S}\mathrm{et}$. We define the following categories:*

(1) The category $\mathcal{J}_{/X}$ has as objects the morphisms $\chi \colon \mathcal{Y}_i \to X$ in $\mathcal{S}\mathrm{et}^{\mathcal{J}^{\mathrm{op}}}$. A morphism $\chi_0 \to \chi_1$ consists of a morphism $f \colon i_0 \to i_1$ such that the following triangle commutes:

$$\begin{array}{ccc} \mathcal{Y}_{i_0} & \xrightarrow{\chi_0} & \\ {\scriptstyle \mathcal{Y}_f}\downarrow & & X \\ \mathcal{Y}_{i_1} & \xrightarrow{\chi_1} & \end{array}$$

(2) The category $\mathcal{J}_{Y/}$ has as objects the morphisms $\nu \colon \mathcal{Y}^i \to Y$ in $\mathcal{S}\mathrm{et}^{\mathcal{J}}$. A morphism $\nu_0 \to \nu_1$ consists of a morphism $f \colon i_0 \to i_1$ such that the following triangle commutes:

$$\begin{array}{ccc} \mathcal{Y}^{i_0} & \xrightarrow{\nu_0} & \\ {\scriptstyle \mathcal{Y}^f}\uparrow & & Y \\ \mathcal{Y}^{i_1} & \xrightarrow{\nu_1} & \end{array}$$

(3) We define the category $\mathcal{J}_{Y/\!\!/X}$ as the strict pullback of categories

$$\mathcal{J}_{Y/\!\!/X} \coloneqq \mathcal{J}_{Y/} \underset{\mathcal{J}}{\times} \mathcal{J}_{/X}\,.$$

Example C.4 Let $i \in \mathcal{J}$ and $Y = \mathcal{Y}^i$. Then, the category $\mathcal{J}_{\mathcal{Y}^i/}$ can be described as follows: an object $\nu \colon \mathcal{Y}^{i_0} \to \mathcal{Y}^i$ is equivalent to a morphism $\nu \colon i \to i_0$, and a morphism $f \colon \nu_0 \to \nu_1$ is equivalent to a morphism $f \colon i_0 \to i_1$ such that $f \circ \nu_0 = \nu_1$. In other words, there is an isomorphism of categories

$$\mathcal{J}_{\mathcal{Y}^i/} \cong \mathcal{J}_{i/}\,,$$

where the category on the right-hand side is the usual slice category under the object $i \in \mathcal{J}$. This also justifies our notation $\mathcal{J}_{Y/}$. ◁

We obtain the following lemma directly from Definition C.3:

Lemma C.5 *Let $\mathcal{C}\mathrm{at}_1$ be the strict (1-)category of categories and functors. The constructions in Definition C.3 give rise to functors*

$$\mathcal{J}_{/(-)} \colon \mathcal{S}\mathrm{et}^{\mathcal{J}^{\mathrm{op}}} \longrightarrow \mathcal{C}\mathrm{at}_1\,, \qquad \mathcal{J}_{(-)/} \colon \mathcal{S}\mathrm{et}^{\mathcal{J}} \longrightarrow \mathcal{C}\mathrm{at}_1\,, \qquad \textit{and} \qquad \mathcal{J}_{(-)/\!\!/(-)} \colon \mathcal{S}\mathrm{et}^{\mathcal{J}} \times \mathcal{S}\mathrm{et}^{\mathcal{J}^{\mathrm{op}}} \longrightarrow \mathcal{C}\mathrm{at}_1\,.$$

Now consider again the functors $F\colon \mathcal{J} \to \mathcal{S}\mathrm{et}_\Delta$ and $E\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{S}\mathrm{et}_\Delta$. We can equivalently view these as functors

$$F\colon \mathbb{\Delta}^{\mathrm{op}} \times \mathcal{J} \longrightarrow \mathcal{S}\mathrm{et} \qquad \text{and} \qquad E\colon \mathbb{\Delta}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{J} \longrightarrow \mathcal{S}\mathrm{et}\,.$$

Given any $[k] \in \mathbb{\Delta}$ and $c \in \mathcal{C}$, we see from (C.1) and Definition C.3 that there is a canonical isomorphism of sets (or, more formally, of discrete simplicial sets)

$$B_n^{\mathcal{S}\mathrm{et}_\Delta}\big(F_k, \mathcal{J}, E_k(c,-)\big) \cong N(\mathcal{J}_{E_k(c,-)/\!\!/ F_k})_n\,,$$

which extends to an isomorphism of simplicial sets

$$B_\bullet^{\mathcal{S}\mathrm{et}_\Delta}\big(F_k, \mathcal{J}, E_k(c,-)\big) \cong N(\mathcal{J}_{E_k(c,-)/\!\!/ F_k})\,.$$

Because of the functoriality of $\mathcal{J}_{(-)/\!\!/(-)}$, we even obtain a natural isomorphism

$$B_\bullet^{\mathcal{K}_\infty}\big(F_k, \mathcal{J}, E_k\big) \cong N(\mathcal{J}_{E_k/\!\!/ F_k})\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}_\Delta$$

of functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$, i.e. of simplicial presheaves. Letting $[k] \in \mathbb{\Delta}$ vary, and again using the functoriality of $\mathcal{J}_{(-)/\!\!/(-)}$, we obtain an isomorphism

$$B_\bullet^{\mathcal{K}_\infty}\big(F_\star, \mathcal{J}, E_\star\big) \cong N(\mathcal{J}_{E_\star/\!\!/ F_\star})\colon \mathbb{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{K}_\infty$$

of simplicial objects in $\mathcal{K}_\infty$.

We introduce the following auxiliary construction, which is interesting in its own right. Let $\mathrm{Ex}\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ denote the right adjoint to the simplicial subdivision functor Sd—recall that Sd sends Δ^n to the nerve of the category of totally ordered subsets of $[n]$. For instance, $\mathrm{Sd}\,\Delta^1$ can be sketched as the cospan $\{0\} \to \{0,1\} \leftarrow \{1\}$. Our main reference on Ex is [Cis19, Sec. 3.1]. The functor Ex comes with a natural weak equivalence $b\colon 1_{\mathcal{S}\mathrm{et}_\Delta} \xrightarrow{\sim} \mathrm{Ex}$ (see, for instance, [Cis19, Prop. 3.1.21]).

Definition C.6 *Let $F\colon \mathcal{J} \to \mathcal{S}\mathrm{et}_\Delta$ and $E\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{S}\mathrm{et}_\Delta$ be functors. Given any $[k] \in \mathbb{\Delta}$ and $c \in \mathcal{C}$, we set*

$$B_{\mathrm{Ex},\bullet,k}^{\mathcal{S}\mathrm{et}_\Delta}\big(F, \mathcal{J}, E(c,-)\big) \coloneqq \mathrm{Ex} \circ N(\mathcal{J}_{E_k(c,-)/\!\!/ F_k})\,.$$

By the functoriality of Ex *and $\mathcal{J}_{(-)/\!\!/(-)}$, we obtain a functor*

$$B_{\mathrm{Ex},\bullet,\star}^{\mathcal{K}_\infty}(F, \mathcal{J}, E) \coloneqq \mathrm{Ex} \circ N(\mathcal{J}_{E_\star/\!\!/ F_\star})\colon \mathbb{\Delta}^{\mathrm{op}} \times \mathbb{\Delta}^{\mathrm{op}} \longrightarrow \mathcal{S}\mathrm{et}^{\mathcal{C}^{\mathrm{op}}}\,.$$

Finally, we set

$$B_{\mathrm{Ex}}^{\mathcal{K}_\infty}(F, \mathcal{J}, E) \coloneqq \int^n \Delta^n \otimes B_{\mathrm{Ex},n,\star}^{\mathcal{K}_\infty}(F, \mathcal{J}, E) \cong \int^n \Delta^n \otimes \big(\mathrm{Ex} \circ N(\mathcal{J}_{E_\star/\!\!/ F_\star})\big)_n\,.$$

This defines a functor

$$B_{\mathrm{Ex}}^{\mathcal{K}_\infty}(-, \mathcal{J}, -)\colon \mathcal{S}\mathrm{et}^{\mathcal{J}^{\mathrm{op}}} \times (\mathcal{K}_\infty)^{\mathcal{J}} \longrightarrow \mathcal{K}_\infty\,.$$

Proposition C.7 *The natural weak equivalence $b\colon 1_{\mathcal{S}\mathrm{et}_\Delta} \xrightarrow{\sim} \mathrm{Ex}$ indues a natural transformation*

$$b^{\mathcal{K}_\infty}\colon B^{\mathcal{K}_\infty}(-, \mathcal{J}, -) \longrightarrow B_{\mathrm{Ex}}^{\mathcal{K}_\infty}(-, \mathcal{J}, -)\,,$$

all of whose components $b_{F,E}^{\mathcal{K}_\infty}\colon B^{\mathcal{K}_\infty}(F, \mathcal{J}, E) \xrightarrow{\sim} B_{\mathrm{Ex}}^{\mathcal{K}_\infty}(F, \mathcal{J}, E)$ are objectwise weak equivalences in $\mathcal{K}_\infty$.

Proof. The naturality of $b\colon 1_{\mathcal{S}\mathrm{et}_\Delta} \to \mathrm{Ex}$ readily implies that the morphisms

$$b_{|N(\mathcal{I}_{E_k(c,-)/\!\!/ F_k})}\colon N(\mathcal{I}_{E_k(c,-)/\!\!/ F_k}) \longrightarrow \mathrm{Ex} \circ N(\mathcal{I}_{E_k(c,-)/\!\!/ F_k})$$

are weak equivalences of simplicial sets and that they are natural in $[k] \in \mathbb{\Delta}$ and $c \in \mathcal{C}$, as well as in F and E. Letting $[k]$ vary, we obtain a morphism

$$b_{|N(\mathcal{I}_{E_\star(c,-)/\!\!/ F_\star})}\colon N(\mathcal{I}_{E_\star(c,-)/\!\!/ F_\star}) \longrightarrow \mathrm{Ex} \circ N(\mathcal{I}_{E_\star(c,-)/\!\!/ F_\star})$$

of bisimplicial sets which is natural in c, and which is a *horizontal* weak equivalence. We know that the diagonal $\delta^*\colon s\mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ is homotopical, i.e. that it sends all *vertical* weak equivalences of bisimplicial sets to weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$. However, since $\delta^*(X_{\bullet,\star}) = \delta^*(X_{\star,\bullet})$, it follows that δ^* also sends horizontal weak equivalences in $s\mathcal{S}\mathrm{et}_\Delta$ to weak equivalences in $\mathcal{S}\mathrm{et}_\Delta$. Therefore, the natural isomorphisms

$$B^{\mathcal{K}_\infty}_{\mathrm{Ex}}(F, \mathcal{I}, E)(c) \cong B^{\mathcal{S}\mathrm{et}_\Delta}_{\mathrm{Ex}}\big(F, \mathcal{I}, E(c,-)\big) \cong \delta^*\big(B^{\mathcal{S}\mathrm{et}_\Delta}_{\mathrm{Ex},\bullet,\star}\big(F, \mathcal{I}, E(c,-)\big)\big)$$

complete the proof. $\square$

Corollary C.8 *Let $E\colon \mathcal{I} \to \mathcal{K}_\infty$ be an $\mathcal{I}$-shaped diagram in $\mathcal{K}_\infty$. The modified bar construction models the homotopy colimit:*

$$\operatorname*{hocolim}_{\mathcal{I}}{}^{\mathcal{K}_\infty}(E) \simeq B^{\mathcal{K}_\infty}_{\mathrm{Ex}}(*, \mathcal{I}, Q^{\mathcal{K}_\infty} \circ E)\,.$$

If E is pointwise cofibrant in $\mathcal{K}_\infty$, then we have

$$\operatorname*{hocolim}_{\mathcal{I}}{}^{\mathcal{K}_\infty}(E) \simeq B^{\mathcal{K}_\infty}_{\mathrm{Ex}}(*, \mathcal{I}, E)\,.$$

We conclude this section by considering the case $\mathcal{I} = \mathcal{C}$. Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$ be an object in $\mathcal{K}_\infty$. We recall Dugger's cofibrant replacement functor for the projective model structure $\mathcal{K}^p_\infty$: it reads as

$$Q^p F = B^{\mathcal{K}_\infty}(F, \mathcal{C}, \mathcal{Y})\,.$$

More explicitly, for $c \in \mathcal{C}$, and $[n] \in \mathbb{\Delta}$ we have

$$\begin{aligned} Q^p F(c)_n &= \coprod_{\vec{c} \in (N\mathcal{C})_n} \mathcal{Y}_{c_0}(c) \times F_n(c_n) \\ &\cong \big(N(\mathcal{C}_{c/\!\!/ F_n})\big)_n\,. \end{aligned}$$

Here, $\vec{c} = (c_0 \to \cdots \to c_n)$ is equivalently a functor $[n] \to \mathcal{C}$ (or, equivalently, a morphism $\Delta^n \to N\mathcal{C}$). Using the modified two-sided bar construction, we define

$$Q^p_{\mathrm{Ex}} F \coloneqq B^{\mathcal{K}_\infty}_{\mathrm{Ex}}(F, \mathcal{C}, \mathcal{Y})\,.$$

That is, for $c \in \mathcal{C}$, and $[n] \in \mathbb{\Delta}$ we have

$$\big((Q^p_{\mathrm{Ex}} F)(c)\big)_n = \mathrm{Ex} \circ N(\mathcal{C}_{\mathcal{Y}_{(-)}(c)/\!\!/ F_n})_n \cong \big(\mathrm{Ex} \circ N(\mathcal{C}_{c/\!\!/ F_n})\big)_n\,.$$

The map $N(\mathcal{C}_{c/\!\!/ F_n})_n \longrightarrow F_n(c)$, which sends an element of $N(\mathcal{C}_{c/\!\!/ F_n})_n$ to the unique composition $\mathcal{Y}_c \to F_n$, induces a natural augmentation map $q^p_{\mathrm{Ex}}\colon Q^p_{\mathrm{Ex}} \to 1_{\mathcal{K}_\infty}$ and we obtain a commutative diagram

$$\begin{array}{ccc} Q^p F & \xrightarrow[\sim]{q^p_F} & F \\ {\scriptstyle \sim}\big\downarrow & \nearrow_{q^p_{\mathrm{Ex},F}} & \\ Q^p_{\mathrm{Ex}} F & & \end{array}$$

Hence, $q^p_{\mathrm{Ex}}\colon Q^p_{\mathrm{Ex}} \xrightarrow{\sim} 1_{\mathcal{K}_\infty}$ is a natural weak equivalence. Since every level of $Q^p_{\mathrm{Ex}}F$ is a coproduct of representables, we infer

Proposition C.9 *$(Q^p_{\mathrm{Ex}}, q^p_{\mathrm{Ex}})$ is a cofibrant replacement functor in $\mathcal{K}^p_\infty$.*

D Comparing adjunctions and detecting Quillen equivalences

Here we recall and collect some basic background on how to detect Quillen equivalences. We first recall the following well-known fact and definition:

Proposition D.1 *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjoint pair. The following are equivalent:*

(1) The counit and unit of the adjunction are isomorphisms.

(2) F is an equivalence.

(3) G is an equivalence.

Proof. It is clear that (1) implies both (2) and (3). We will show that (2) implies (1)—the proof that (3) implies (1) is analogous. Thus, suppose that F is an equivalence, i.e. suppose that it is fully faithful and essentially surjective. It follows that the coevaluation morphism $co\colon 1_{\mathcal{C}} \to GF$ is a natural isomorphism. The triangle identity then implies that the component $e_{|Fc}\colon FG(Fc) \to F(c)$ of the evaluation morphism $e\colon FG \to 1_{\mathcal{D}}$ is an isomorphism for every $c \in \mathcal{C}$. Since F is essentially surjective, it follows from the naturality of e that e is a natural isomorphism as well. □

Definition D.2 *In any of the equivalent cases of Proposition D.1, the adjunction $F \dashv G$ is called an* adjoint equivalence.

A similar characterisation exists for Quillen equivalences:

Proposition D.3 [Lur09, A.2.5.1] *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction between model categories. The following are equivalent:*

(1) The total left derived functor $\mathbb{L}F\colon \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is an equivalence of categories.

(2) The total right derived functor $\mathbb{R}G\colon \mathrm{h}\mathcal{D} \to \mathrm{h}\mathcal{C}$ is an equivalence of categories.

(3) The pair $F \dashv G$ is a Quillen equivalence.

Let (F, G, φ) be an adjunction $\mathcal{C} \rightleftarrows \mathcal{D}$, where $F\colon \mathcal{C} \to \mathcal{D}$ is the left adjoint, $G\colon \mathcal{D} \to \mathcal{C}$ is the right adjoint, and $\varphi\colon \mathcal{D}(F(-), -) \longrightarrow \mathcal{C}(-, G(-))$ is the binatural isomorphism that establishes the adjunction. A morphism of adjunctions $(F, G, \varphi) \to (F', G', \varphi')$ is a pair (f, g) of natural transformations $f\colon F \to F'$, $g\colon G' \to G$, such that, for every $c \in \mathcal{C}$ and $d \in \mathcal{D}$, the diagram

$$\begin{array}{ccc} \mathcal{D}(F'c, d) & \xrightarrow{f^*_{|c}} & \mathcal{D}(Fc, d) \\ {\scriptstyle \varphi'_{c,d}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \varphi_{c,d}} \\ \mathcal{C}(c, G'd) & \xrightarrow[(g_{|d})_*]{} & \mathcal{C}(c, Gd) \end{array} \tag{D.4}$$

commutes. As an equation, this amounts to demanding that

$$\varphi_{c,d}(\chi \circ f_{|c}) = g_{|d} \circ \varphi'_{c,d}(\chi)$$

for every morphism $\chi\colon F'c \to d$ in $\mathcal{D}$.

Definition D.5 *Let $\mathcal{C}, \mathcal{D}$ be categories. We define the following categories:*

(1) Let $\mathcal{F}\mathrm{un}^L(\mathcal{C}, \mathcal{D}) \subset \mathcal{C}\mathrm{at}(\mathcal{C}, \mathcal{D})$ be the full subcategory on the left adjoint functors.

(2) Let $\mathcal{A}\mathrm{dj}(\mathcal{C}, \mathcal{D})$ be the category whose objects are adjunctions (F, G, φ), and whose morphisms are morphisms (f, g) of adjunctions.

(3) Let $\mathcal{F}\mathrm{un}^{L,AE}(\mathcal{C}, \mathcal{D}) \subset \mathcal{F}\mathrm{un}^L(\mathcal{C}, \mathcal{D})$ be the full subcategory on the left adjoint functors which are equivalences of categories.

(4) Let $\mathcal{A}\mathrm{d}\mathcal{E}\mathrm{q}(\mathcal{C}, \mathcal{D}) \subset \mathcal{A}\mathrm{dj}(\mathcal{C}, \mathcal{D})$ be the full subcategory on the adjoint equivalences, i.e. on those adjunctions (F, G, φ) where both F and G are equivalences.

(5) Suppose both $\mathcal{C}$ and $\mathcal{D}$ are endowed with model structures. Let $\mathcal{F}\mathrm{un}^{LQ}(\mathcal{C}, \mathcal{D}) \subset \mathcal{F}\mathrm{un}^L(\mathcal{C}, \mathcal{D})$ be the full subcategory on the left Quillen functors.

(6) Finally, let $\mathcal{Q}\mathcal{A}\mathrm{dj}(\mathcal{C}, \mathcal{D}) \subset \mathcal{A}\mathrm{dj}(\mathcal{C}, \mathcal{D})$ be the full subcategory on the Quillen adjunctions.

There are canonical projection functors

$$\begin{aligned} \pi \colon \mathcal{A}\mathrm{dj}(\mathcal{C}, \mathcal{D}) &\longrightarrow \mathcal{F}\mathrm{un}^L(\mathcal{C}, \mathcal{D})\,, \\ \pi_{AE} \colon \mathcal{A}\mathrm{d}\mathcal{E}\mathrm{q}(\mathcal{C}, \mathcal{D}) &\longrightarrow \mathcal{F}\mathrm{un}^{L,AE}(\mathcal{C}, \mathcal{D})\,, \\ \pi_Q \colon \mathcal{Q}\mathcal{A}\mathrm{dj}(\mathcal{C}, \mathcal{D}) &\longrightarrow \mathcal{F}\mathrm{un}^{LQ}(\mathcal{C}, \mathcal{D})\,. \end{aligned}$$

Proposition D.6 *The functors π, π_Q, and π_{AE} are fibred and cofibred in groupoids, with contractible fibres. They are equivalences and surjective on objects.*

Proof. Consider the functor $\pi \colon \mathcal{A}\mathrm{dj}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathcal{F}\mathrm{un}^L(\mathcal{C}, \mathcal{D})$, $(F, G, \varphi) \mapsto F$. This is surjective on objects. Let $F_0, F_1 \in \mathcal{F}\mathrm{un}^L(\mathcal{C}, \mathcal{D})$, and let $f \colon F_0 \to F_1$ be a natural transformation. Suppose we are given lifts (F_i, G_i, φ_i) of F_i to $\mathcal{A}\mathrm{dj}(\mathcal{C}, \mathcal{D})$, for $i = 0, 1$. Then, for any $d \in \mathcal{D}$, we define a morphism $g_{|d} \colon G_1 d \to G_0 d$ by the Yoneda Lemma and by demanding commutativity of diagram (D.4). Thus, once the lifts (F_i, G_i, φ_i) are specified, there exists a unique lift of f to a morphism $(f, g) \colon (F_0, G_0, \varphi_0) \longrightarrow (F_1, G_1, \varphi_1)$ of adjunctions. In particular, π is an equivalence of categories. This also shows that $\mathcal{A}\mathrm{dj}(\mathcal{C}, \mathcal{D})$ is both fibred and cofibred in groupoids over $\mathcal{F}\mathrm{un}^L(\mathcal{C}, \mathcal{D})$.

The functors π_{AE} and π_Q can be seen as restrictions of π to full subcategories; the same reasoning applies. □

Consequently, given adjunctions (F_0, G_0, φ_0) and (F_1, G_1, φ_1), specifying a morphism of adjunctions is equivalent to specifying a natural transformation $f \colon F_0 \to F_1$. Recalling that any functor which is naturally isomorphic to an equivalence is an equivalence itself, and combining this with Proposition D.1, we obtain

Proposition D.7 *Let (F_0, G_0, φ_0) and (F_1, G_1, φ_1) be two adjunctions $\mathcal{C} \rightleftarrows \mathcal{D}$ such that there exists a natural isomorphism $f \colon F_0 \to F_1$. Then, (F_0, G_0, φ_0) is an adjoint equivalence if and only if (F_1, G_1, φ_1) is so.*

Corollary D.8 *Let $F_0, F_1 \colon \mathcal{C} \to \mathcal{D}$ be left Quillen functors such that there exists a natural weak equivalence $f \colon \mathbb{L}F_0 \to \mathbb{L}F_1$ as functors $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$. Then, F_0 is a Quillen equivalence if and only if F_1 is a Quillen equivalence.*

Proof. Let (F_i, G_i, φ_i) be lifts of F_i to Quillen adjunctions $\mathcal{C} \rightleftarrows \mathcal{D}$. We can apply Proposition D.7 to the induced adjunctions $(\mathbb{L}F_i, \mathbb{R}G_i, \mathrm{h}\varphi)$ on homotopy categories $\mathrm{h}\mathcal{C} \rightleftarrows \mathrm{h}\mathcal{D}$. The claim then follows from Proposition D.3. □

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