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Principal ∞-Bundles and Smooth String Group Models

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Abstract

We provide a general, homotopy-theoretic definition of string group models within an ∞-category of smooth spaces, and we present new smooth models for the string group. Here, a smooth space is a presheaf of ∞-groupoids on the category of cartesian spaces. The key to our definition and construction of smooth string group models is a version of the singular complex functor, which assigns to a smooth space an underlying ordinary space. We provide new characterisations of principal ∞-bundles and group extensions in ∞-topoi, building on work of Nikolaus, Schreiber, and Stevenson. These insights allow us to transfer the definition of string group extensions from the ∞-category of spaces to the ∞-category of smooth spaces. Finally, we consider smooth higher-categorical group extensions that arise as obstructions to the existence of equivariant structures on gerbes. These extensions give rise to new smooth models for the string group, as recently conjectured in joint work with Müller and Szabo.

Contents

1 Introduction and overview

The most direct way to define the string group is via the Whitehead tower of $\mathsf{O}(n)$,

$$\cdots \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathsf{SO}(n) \longrightarrow \mathsf{O}(n)\,. \tag{1.1}$$

By this approach, $\mathrm{String}(n)$ is defined as a 3-connected topological space with a continuous map $\mathrm{String}(n) \to \mathrm{Spin}(n)$ which induces an isomorphism on all homotopy groups except for in degree three. So far, this defines String(n) only as a space, but in [Sto96] Stolz constructed String(n) as a topological group and the map String(n) $\to$ Spin(n) as a morphism of topological groups. In fact, he presented a construction that produces, for any compact, connected, and simply connected Lie group H, a morphism String(H) $\to H$ of topological groups whose underlying continuous map is a three-connected covering. A covering of this type is also called a *string group extension of* H. In these conventions, we write String(n) := String(Spin(n)).

The string group is important in geometry and topology in several ways. Originally, Killingback [Kil87] and Witten [Wit] investigated the two-dimensional supersymmetric σ-model on background manifolds M and found that this is well-defined only if the free loop space LM admits a spin structure. Witten, moreover, computed the index of a hypothetical Dirac operator on LM based on physical arguments, leading to the definition of the Witten genus. By now, it has been understood that the Witten genus is related to the cohomology theory of topological modular forms (TMF). The string group enters in this story, for example by defining orientations in TMF [AHR, DHH11], analogously to how the spin group underlies orientations in real K-theory.

Since the free loop space LM is less tractable than the manifold M itself, it is an important question whether the condition that LM admit a spin structure can be recast as a condition on the manifold M itself. This is indeed the case: *spin* structures on LM correspond to *string* structures on M [ST, ST04, Wal15]. Topologically, a string structure on M is a lift of the classifying map $M \to \mathrm{BO}(n)$ of the tangent bundle $TM \to M$ to a map $M \to$ BString(n). That is, a string structure is a reduction of the structure group of TM to String(n). From a geometric perspective, the interest ultimately is in identifying consequences and constructions that are facilitated by a string structure on a manifold. Concrete examples include the Höhn-Stolz conjecture [Höh, Sto96] that the Witten genus is trivial for any Riemannian $4k$-manifold with positive Ricci curvature which admits a string structure, or the long-standing goal to define a Dirac operator on the loop space LM.

In order to study the differential geometric, rather than topological, implications of string structures, it is paramount to have models for String(n) not just as a topological group, but as a group object in some geometric category. For instance, given a Riemannian manifold M, the construction of the Dirac operator associated with a spin structure on M depends on the ability to glue the tangent bundle TM from *smooth* Spin(n)-valued functions. Technically, one also needs to find local frames for TM in which the Levi-Civita connection of M is represented by 1-forms valued in the Lie algebra $\mathfrak{spin}(n)$ rather than $\mathfrak{so}(n)$; however, since the fibre of the map Spin(n) $\to \mathsf{SO}(n)$ is discrete, these Lie algebras happen to be canonically isomorphic (for more background on spin geometry and Dirac operators, we recommend [LM89]). Analogously to how spin structures on LM stem from string structures on M, a hypothetical Dirac operator on LM may well stem from a geometric operator on M itself (e.g. via some transgression procedure), obtained from a further lift of the Levi-Civita connection to the Lie algebra $\mathfrak{string}(n)$. However, for this to make sense, one must work with a *smooth*, rather than topological, model for String(n).

Classical results on cohomology readily imply that it is impossible to construct String(H) as a finite-dimensional Lie group (for any compact, connected, simply connected Lie group H). Thus, to find geometric models for String(H), one needs to look beyond the category of smooth, finite-dimensional manifolds. Indeed, a number of models for String(H) have been found in (higher) categories of smooth spaces that generalise the notion of a manifold in various ways [BSCS07, Hen08, SP11, Wal12, NSW13, FRS16].

In each of these constructions, an extension

$$A \longrightarrow \mathrm{String}(H) \longrightarrow H$$

of a compact, simple, simply connected Lie group H is constructed within the chosen ambient category of smooth spaces. It is then argued that on the underlying ordinary spaces (meaning topological spaces or simplicial sets) one obtains a string group extension in the sense of (1.1). However, so far there is no general definition of $\mathrm{String}(H)$ in a smooth context that formalises this procedure. Consequently, in geometric models for $\mathrm{String}(H)$ the extending group A currently has to be chosen ad hoc as an explicit delooping of the Lie group $\mathsf{U}(1)$ in a rather strict sense. This obscures the homotopy-theoretic nature of $\mathrm{String}(H)$, since from a homotopical point of view, not A is fixed, but only its homotopy type.

In [BMS], studying symmetries of gerbes, we came across extensions of Lie groups H not by a delooping of the Lie group $\mathsf{U}(1)$, but by the delooping of the diffeological group $\mathsf{U}(1)^H$ of smooth maps from H to $\mathsf{U}(1)$. However, if H is simply connected, then the smooth group $\mathsf{U}(1)^H$ is homotopy equivalent to $\mathsf{U}(1)$. Therefore, extensions of H by the delooping $\mathrm{B}(\mathsf{U}(1)^H)$ potentially have the correct homotopy type to produce smooth string group extensions of H. Nevertheless, we could not make this rigorous due to the lack of a homotopy-theoretic notion of smooth string group extensions that does not fix the extending group, but only its homotopy type.

Here, we provide such a general definition of smooth string group extensions, and we prove that the string group models proposed in [BMS] fit within this definition. Let $\mathcal{M}\mathrm{fd}$ denote the category of manifolds and smooth maps, and let $\mathcal{C}\mathrm{art} \subset \mathcal{M}\mathrm{fd}$ be the full subcategory on those manifolds that are diffeomorphic to $\mathbb{R}^n$ for any $n \in \mathbb{N}_0$. As our ambient category of smooth spaces, we choose the ∞-category $\mathbf{H}_\infty := \mathcal{F}\mathrm{un}(\mathcal{C}\mathrm{art}^{\mathrm{op}}, \mathbf{S})$ of presheaves of spaces on $\mathcal{C}\mathrm{art}$. This provides a very general notion of smooth space: for instance, $\mathbf{H}_\infty$ contains the categories of manifolds, diffeological spaces, and Lie groupoids. We write $\underline{M}$ for the image of a manifold M under the fully faithful inclusion $\mathcal{M}\mathrm{fd} \hookrightarrow \mathbf{H}_\infty$.

The ∞-category $\mathbf{H}_\infty$ is even an ∞-topos, and there exists an established theory of group objects in ∞-topoi [Lur09]. Moreover, there exists a notion of principal ∞-bundles and extensions of group objects in ∞-topoi due to [NSS15]. A large part of this paper is devoted to developing this theory further. In particular, we show that group actions in ∞-topoi automatically form groupoid objects (Theorem 3.19) and that principal ∞-bundles essentially consists of an effective epimorphism and a principal group action (Theorem 3.31); this is analogous to the definition of principal bundles of topological spaces as a locally trivial map and a principal group action. Then, we provide the following characterisation of extensions of group objects:

Theorem 1.2 *Let $\mathbf{H}$ be an ∞-topos. Given a group object $\widehat{A}$ in $\mathbf{H}$, denote its underlying object in $\mathbf{H}$ by A. Let $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{p}} \widehat{H}$ be a sequence of morphisms of group objects in $\mathbf{H}$. The following are equivalent:*

(1) $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{p}} \widehat{H}$ is an extension of group objects in $\mathbf{H}$, i.e. the sequence $\mathrm{B}A \to \mathrm{B}G \to \mathrm{B}H$ is a fibre sequence of pointed connected objects in $\mathbf{H}$.

(2) The sequence $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{p}} \widehat{H}$ is a fibre sequence of group objects in $\mathbf{H}$.

(3) The sequence $A \xrightarrow{\iota} G \xrightarrow{p} H$ is a fibre sequence in $\mathbf{H}$.

(4) The map $p\colon G \to H$ together with the action of A on G induced by ι define a principal A-bundle over H.

Point (1) in Theorem 1.2 is the definition of extensions of group objects in ∞-topoi from [NSS15]. In order to give a general homotopy-theoretic definition of string group extensions within $\mathbf{H}_\infty$, we need to associate an underlying space to an object in $\mathbf{H}_\infty$. In [Bunb] we investigated (a model categorical presentation of) a functor $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$ from $\mathbf{H}_\infty$ to the ∞-category $\mathbf{S}$ of spaces. It evaluates a smooth space $X \in \mathbf{H}_\infty$ on the extended affine simplices $\Delta^k_e \in \mathcal{C}\mathrm{art}$ and then takes the geometric realisation of the resulting simplicial object in $\mathbf{S}$—that is, S_e is a smooth version of the singular complex functor. Here, we give further interpretation and context to this functor: the adjunction $\widetilde{\mathsf{c}} \dashv \Gamma$, where $\Gamma \colon \mathbf{H}_\infty \to \mathbf{S}$ is the global-section functor, fits into a triple adjunction $\Pi \dashv \widetilde{\mathsf{c}} \dashv \Gamma \dashv \mathit{codisc}$, where codisc is fully faithful and where Π preserves finite products. That is, the ∞-topos $\mathbf{H}_\infty$ is cohesive.

Theorem 1.3 *The functor $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$ is part of the cohesion of $\mathbf{H}_\infty$: there is a canonical equivalence*

$$\Pi \simeq \mathrm{S}_e \,.$$

This has already been indicated in [BEBdBP] and proven on the level of model categories of simplicial presheaves in [Bunb]; here we provide an ∞-categorical proof based on findings from [Bunb]. Let $\mathrm{L} \colon \mathbf{H} \to \mathbf{H}'$ be a functor between ∞-topoi which preserves finite products and geometric realisations of simplicial objects. We show that L maps principal ∞-bundles in $\mathbf{H}$ to principal ∞-bundles in $\mathbf{H}'$ and group extensions in $\mathbf{H}$ to group extensions in $\mathbf{H}'$. (This relies on Theorem 3.19.) In particular, the functor $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$ has these properties. In $\mathbf{S}$, a string group extension of a compact, connected, simply connected Lie group H can be defined as usual: it is an extension $A \to \mathrm{String}(H) \to H$ of group objects in $\mathbf{S}$ such that $\mathrm{String}(H)$ is 3-connected and such that the morphism $\mathrm{String}(H) \to H$ induces an isomorphism on all homotopy groups except for in degree three. Using that $\mathrm{S}_e\underline{M} \simeq M$ for any manifold M [Bunb] and that S_e preserves principal ∞-bundles and group extensions, we are now able to transfer this definition to $\mathbf{H}_\infty$:

Definition 1.4 *Let H be a compact, simple, and simply connected Lie group, and let $\widehat{\underline{H}}$ denote the induced group object in $\mathbf{H}_\infty$. An extension $\widehat{A} \longrightarrow \widehat{\mathrm{String}(\underline{H})} \longrightarrow \widehat{\underline{H}}$ of group objects in $\mathbf{H}_\infty$ is called a* smooth string group extension of H *if its image under S_e is a string group extension in $\mathbf{S}$.*

Finally, we show that the string group models conjectured in [BMS] fit within Definition 1.4. Let M be a manifold endowed with a bundle gerbe $\mathcal{G}$ (a categorified hermitean line bundle). In [BMS], we addressed the question of when an action of a Lie group H on M lifts to an equivariant structure on $\mathcal{G}$. We found that the obstruction to such a lift is an extension

$$\mathrm{HLB}^M \xrightarrow{i} \mathrm{Sym}(\mathcal{G}) \xrightarrow{p} H \tag{1.5}$$

of H by the smooth 2-group HLB^M of hermitean line bundles on M. Each of the above objects can be interpreted as a group object in $\mathbf{H}_\infty$ via the nerve functor, and so (1.5) provides an extension of H as a group object in $\mathbf{H}_\infty$. The case relevant for string group extensions is $M = H$, where H is a compact, connected and simply connected Lie group, acting on itself via left multiplication. Since H is 2-connected, there is an objectwise equivalence $\mathrm{HLB}^H \simeq \mathrm{B}(\mathsf{U}(1)^H)$, and since H is 1-connected, there is a smooth homotopy equivalence $\mathsf{U}(1)^H \simeq \mathsf{U}(1)$. Therefore, the extending group in (1.5) has the correct homotopy type for a string group extension. We prove:

Theorem 1.6 *Let H be a compact, simple, simply connected Lie group, and let N be the nerve functor. We consider the left-action of H on itself via left multiplication. Let $\mathcal{G} \in \mathcal{G}\mathrm{rb}(H)$ be a gerbe on H whose class in $\mathrm{H}^3(H;\mathbb{Z}) \cong \mathbb{Z}$ is a generator. The sequence*

$$\widehat{N\mathrm{HLB}^H} \xrightarrow{\widehat{Ni}} \widehat{N\mathrm{Sym}(\mathcal{G})} \xrightarrow{\widehat{Np}} \widehat{\underline{H}}$$

is a smooth string group extension of H.

This string group model is somewhat similar to the model in [FRS16], which is obtained by studying symmetries of gerbes with connection. However, the presence of connections forces the extending group to be the delooping $\mathsf{BU}(1)$ in this case. It is interesting that the connection does not change the homotopy type of the extension. A similar observation has been made in [BMS], where a second extension of H, equivalent to (1.5), was constructed with a connection on the gerbe $\mathcal{G}$ acting as crucial auxiliary data. Since that second extension is equivalent to the one in (1.5) [BMS], it gives rise to a second smooth string group extension of H. Finally, we expect that most (or possibly all) of the aforementioned smooth string group models fit within Definition 1.4.

Organisation. In Section 2 we investigate the functor $\mathrm{S}_e\colon \mathbf{H}_\infty \to \mathbf{S}$. Further, we recall some basic notions and facts about ∞-topoi and prove Theorem 1.3.

Section 3 is devoted to the theory of group objects, group extensions, and principal ∞-bundles in ∞-topoi. We recall the definitions of these notions from [NSS15] and provide new characterisations of principal ∞-bundles and group extensions. In particular, we prove Theorem 1.2.

In Section 4, we use the results obtained thus far to transfer the definition of string group extensions in $\mathbf{S}$ to the ∞-topos $\mathbf{H}_\infty$. After recalling from [BMS] the smooth 2-group extensions which obstruct the existence of equivariant structures on gerbes, we show that these extensions give rise to new smooth models for string group, thus proving Theorem 1.6.

Finally, in Appendix A we prove Theorem 3.19: we show that group actions in an ∞-topos give rise to groupoid objects.

Notation. We usually make no notational distinction between ordinary categories and ∞-categories; the nerve functor will be used implicitly where necessary.

We write $\mathbb{\Delta}$ for the simplex category, and $\mathcal{S}\mathrm{et}_\Delta$ for the category of simplicial sets. In a simplicial category $\mathcal{C}$, we denote the simplicially enriched hom-functor by $\underline{\mathcal{C}}(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta$.

We write $|-| = \mathrm{colim}^{\mathcal{C}}_{\mathbb{\Delta}^{\mathrm{op}}}$ for the colimit of simplicial objects in an ∞-category $\mathcal{C}$. Moreover, we also refer to $|X|$ (if it exists) as the geometric realisation of a simplicial object X in $\mathcal{C}$.

Usually, we denote ∞-categories by letters $\mathcal{C}, \mathcal{D}, \dots$, but for ∞-topoi we use bold-face letters $\mathbf{H}$. In particular, the ∞-topos of spaces is denoted by $\mathbf{S}$. We write $\underline{\mathcal{D}}(-,-)\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathbf{S}$ for the mapping spaces in an ∞-category $\mathcal{D}$.

We model ∞-categories by quasi-categories. Given an ∞-category $\mathcal{C}$ and a simplicial set $K \in \mathcal{S}\mathrm{et}_\Delta$, we write $\mathcal{F}\mathrm{un}(K, \mathcal{C}) = \underline{\mathcal{S}\mathrm{et}_\Delta}(K, \mathcal{C}) = \mathcal{C}^K$ for the ∞-category of functors from K to $\mathcal{C}$.

We let $\mathbb{\Delta}_+$ denote the augmented simplex category, i.e. the category $\mathbb{\Delta}$ with an initial object adjoined. We usually do not distinguish notationally between augmented simplicial objects $X \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}_+^{\mathrm{op}}, \mathcal{C})$ in an ∞-category $\mathcal{C}$ and their underlying simplicial objects. If we wish to make this distinction explicit for clarity, we will denote the latter by the restriction $X_{|\mathbb{\Delta}^{\mathrm{op}}}$.

If $\mathcal{M}$ is a simplicial model category, then $\mathcal{M}^\circ$ is the full simplicial subcategory on the cofibrant-fibrant objects of $\mathcal{M}$. Recall from [Lur09] that the coherent nerve $N(\mathcal{M}^\circ)$ is an ∞-category.

If $\mathcal{C}$ is a (small) ∞-category, we write $\mathcal{P}(\mathcal{C}) = \mathcal{F}\mathrm{un}(\mathcal{C}^{\mathrm{op}}, \mathbf{S})$ for the ∞-category of presheaves of spaces on $\mathcal{C}$.

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2 Smooth spaces and ∞-topoi

In this section we recall and develop some background on the ∞-categories most relevant in this paper. Most importantly, we consider a presheaf ∞-category $\mathbf{H}_\infty$, whose objects can be interpreted as a general notion of smooth spaces. We study an ∞-categorical version $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$ of a Quillen functor considered in [Bunb], which provides a version of the singular complex functor for smooth spaces. Subsequently, we briefly recall the definition of an ∞-topos and of cohesion of ∞-topoi, and we show that S_e is part of the cohesion of $\mathbf{H}_\infty$.

2.1 Presheaves on cartesian spaces and the smooth singular complex

We let $\mathcal{C}\mathrm{art}$ denote the (small) category whose objects are submanifolds of $\mathbb{R}^\infty$ that are diffeomorphic to $\mathbb{R}^n$ for any $n \in \mathbb{N}_0$, and whose morphisms are the smooth maps between these manifolds. We let

$$\mathbf{H}_\infty \coloneqq \mathcal{P}(\mathcal{C}\mathrm{art}) = \mathcal{F}\mathrm{un}(\mathcal{C}\mathrm{art}^{\mathrm{op}}, \mathbf{S})$$

denote the ∞-category of presheaves of spaces on $\mathcal{C}\mathrm{art}$. The ∞-category $\mathbf{H}_\infty$ is presented by several model categories of simplicial presheaves on $\mathcal{C}\mathrm{art}$—for example, there is a canonical equivalence [Lur09]

$$\mathbf{H}_\infty \simeq N\big((\mathcal{H}^i_\infty)^\circ\big)\,,$$

where $\mathcal{H}^i_\infty$ is the category of simplicial presheaves on $\mathcal{C}\mathrm{art}$, endowed with the injective model structure.

Let $I \coloneqq \{c \times \mathbb{R} \to c \,|\, c \in \mathcal{C}\mathrm{art}\}$ denote the set of morphisms in $\mathcal{C}\mathrm{art}$ of the form $1_c \times c_{\mathbb{R}}$, where $c_{\mathbb{R}} \colon \mathbb{R} \to *$ is the map that collapses the real line to the point. We can localise both $\mathcal{H}^i_\infty$ and $\mathbf{H}_\infty$ at this set of morphisms (or rather at its image under the Yoneda embedding), and there is still a canonical equivalence between the localisations [Lur09],

$$N\big((L_I\mathcal{H}^i_\infty)^\circ\big) \simeq L_I\mathbf{H}_\infty\,.$$

The simplicial model categories $\mathcal{H}^i_\infty$ and $L_I\mathcal{H}^i_\infty$ were the subject of [Bunb]. On the level of their underlying ∞-categories, one of the main results of that paper can be phrased as follows. For $k \in \mathbb{N}_0$, we let $\Delta^k_e \coloneqq \{t \in \mathbb{R}^{k+1} \,|\, \sum_{i=0}^k t^i = 1\}$ denote the extended (affine) k-simplex. This is a k-dimensional affine subspace of $\mathbb{R}^{k+1}$, and hence forms a cartesian space. The face and degeneracy maps of the standard topological simplices $|\Delta^k|$ extend to the extended affine simplices, turning them into a functor

$$\Delta_e \colon \mathbb{\Delta} \to \mathcal{C}\mathrm{art}\,, \quad [k] \mapsto \Delta^k_e\,.$$

We let $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$ denote the composition of functors

$$\mathrm{S}_e \colon \mathbf{H}_\infty \xrightarrow{\Delta_e^*} \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{S}) \xrightarrow{\mathrm{colim}} \mathbf{S}\,. \tag{2.1}$$

We refer to this functor as the *smooth singular complex functor*; viewing the ∞-category $\mathbf{H}_\infty$ as an ∞-category of smooth spaces, S_e thus assigns an underlying ordinary space to a smooth space.

Theorem 2.2 [Bunb] *There exist adjunctions of ∞-categories*

$$\begin{tikzcd}
\mathbf{H}_\infty \arrow[rr, shift left, "\mathrm{Loc}"] \arrow[rr, phantom, "\perp"] \arrow[d, shift right, "\mathrm{S}_e"'] \arrow[d, phantom, "\dashv"] & & L_I\mathbf{H}_\infty \arrow[ll, shift left, "\iota"] \arrow[d, "\mathrm{S}^I_e" description] \\
\mathbf{S} \arrow[u, shift right, "\mathrm{R}_e"'] \arrow[rr, equal] & & \mathbf{S} \arrow[u, bend left=60, "\mathrm{L}^I_e"] \arrow[u, bend right=60, "\mathrm{R}^I_e"']
\end{tikzcd}$$

where S_e^I *is the restriction of* S_e *to* $L_I\mathbf{H}_\infty \subset \mathbf{H}_\infty$*. Furthermore, the following statements hold true:*

(1) The functor $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$ *preserves and reflects* I*-local equivalences.*

(2) The morphism ι *is fully faithful, i.e.* Loc *is a reflective localisation.*

(3) The three right-hand vertical functors are equivalences of ∞*-categories.*

(4) The diagram obtained by omitting the morphism L_e^I *is (weakly) commutative.*

Proof. The first claim follows readily from Proposition 3.6, Corollary 3.12, and Corollary 3.37 of [Bunb]. (Note that model categorical presentations of $\mathbf{H}_\infty$, $L_I\mathbf{H}_\infty$, and $\mathbf{S}$ are used in [Bunb], and the functors in the statement are presented by Quillen functors.)

Further, claim (1) follows readily from [Bunb, Cor. 3.15]. Claim (2) follows from general properties of ∞-categories underlying simplicial model categories and their Bousfield localisations [Lur09]. Claim (3) is the version on the underlying ∞-categories of Theorems 3.14 and 3.40 of [Bunb]. Claim (4) holds true because the diagram of the right-adjoints clearly commutes (ι is an inclusion, and R_e simply factors through $L_I\mathbf{H}_\infty \subset \mathbf{H}_\infty$ [Bunb]). ☐

Remark 2.3 There is a fully faithful embedding $\mathcal{M}\mathrm{fd} \hookrightarrow \mathbf{H}_\infty$ from the category of manifolds into $\mathbf{H}_\infty$: it sends a manifold M to the presheaf $\underline{M}$ of discrete spaces that maps a cartesian space c to the set $\mathcal{M}\mathrm{fd}(c, M)$ of smooth maps from c to M. By [Bunb, Thm. 5.1] there is a canonical equivalence of spaces $M \simeq \mathrm{S}_e\underline{M}$ for any $M \in \mathcal{M}\mathrm{fd}$, which is natural in M. ◁

Proposition 2.4 *The localisation functor* $\mathrm{Loc} \colon \mathbf{H}_\infty \to L_I\mathbf{H}_\infty$ *preserves finite products. The class* W_I *of* I*-local equivalences in* $\mathbf{H}_\infty$ *is closed under finite products.*

Proof. By [Bunb, Prop. 2.13], the localisation $L_I\mathbf{H}_\infty$ agrees with the localisation $L_W\mathbf{H}_\infty$ of $\mathbf{H}_\infty$ at all collapse morphisms $c \to *$, for $c \in \mathcal{C}\mathrm{art}$. The class W is stable under finite products in $\mathbf{H}_\infty$, since $\mathcal{C}\mathrm{art}$ has finite products. Therefore, the first claim follows from [Cis19, Cor. 7.1.16]. The second claim then follows since a morphism in $\mathbf{H}_\infty$ is in W_I precisely if its image under Loc is an equivalence [Lur09, Prop. 5.5.4.15]. ☐

Proposition 2.5 *For* $X, Y \in \mathbf{H}_\infty$*, let* $Y^X \in \mathbf{H}_\infty$ *denote their internal hom object in* $\mathbf{H}_\infty$*. The localisation functor* $\mathrm{Loc} \colon \mathbf{H}_\infty \to L_I\mathbf{H}_\infty$ *is given (up to equivalence) by*

$$\mathrm{Loc} \simeq \operatorname*{colim}_{\mathbb{\Delta}^{\mathrm{op}}}{}^{\mathbf{H}_\infty} \big((-)^{\Delta_e} \big) \,.$$

Proof. By Theorem 2.2(4), there is a canonical equivalence $S_e^I \circ \mathrm{Loc} \simeq \mathrm{S}_e$. Combining this with Theorem 2.2(3), we obtain canonical equivalences

$$\mathrm{Loc} \simeq \mathrm{L}_e^I \circ \mathrm{S}_e^I \circ \mathrm{Loc} \simeq \mathrm{L}_e^I \circ \mathrm{S}_e \,.$$

Consider the adjunction $\tilde{\mathsf{c}} : \mathbf{S} \rightleftarrows \mathbf{H}_\infty : \mathrm{ev}_*$, where $\tilde{\mathsf{c}}$ assigns to a space K the constant presheaf with value K, and where ev_* evaluates a presheaf on the final object $* \in \mathcal{C}\mathrm{art}$. These functors induce an equivalence $\tilde{\mathsf{c}} : \mathbf{S} \rightleftarrows L_I\mathbf{H}_\infty : \mathrm{ev}_*$ [Bunb, Thm. 2.17], and there is a canonical equivalence $\mathrm{ev}_* \simeq \mathrm{S}_e^I$ of functors $L_I\mathbf{H}_\infty \to \mathbf{S}$ by [Bunb, Prop. 2.7, Cor. 3.15]. By adjointness, we also obtain a canonical equivalence $\tilde{\mathsf{c}} \simeq \mathrm{L}_e^I$. Consequently, there is a canonical equivalence

$$\mathrm{Loc} \simeq \tilde{\mathsf{c}} \circ \mathrm{S}_e \,.$$

We observe that there exists a canonical equivalence

$$\mathrm{S}_e = \operatorname*{colim}_{\mathbb{\Delta}^{\mathrm{op}}}^{\mathbf{S}} \big(\Delta_e^*(-)\big) \simeq \mathrm{ev}_* \circ \operatorname*{colim}_{\mathbb{\Delta}^{\mathrm{op}}}^{\mathbf{H}_\infty} \big((-)^{\Delta_e}\big)\,.$$

By [Bunb, Prop. 6.2], we have that $\operatorname{colim}^{\mathbf{H}_\infty}_{\mathbb{\Delta}^{\mathrm{op}}}((-)^{\Delta_e})$ is a functor $\mathbf{H}_\infty \to L_I\mathbf{H}_\infty$; that is, it takes values in I-local objects. It follows that there are canonical equivalences

$$\mathrm{Loc} \simeq \tilde{\mathfrak{c}} \circ \mathrm{S}_e \simeq \tilde{\mathfrak{c}} \circ \mathrm{ev}_* \circ \operatorname*{colim}_{\mathbb{\Delta}^{\mathrm{op}}}^{\mathbf{H}_\infty} \big((-)^{\Delta_e}\big) \simeq \operatorname*{colim}_{\mathbb{\Delta}^{\mathrm{op}}}^{\mathbf{H}_\infty} \big((-)^{\Delta_e}\big)\,.$$

This completes the proof. □

2.2 Background on ∞-topoi

In this section, we briefly recall some background on ∞-topoi. Most of the material in this section can be found in [Lur09, Sch, NSS15]. For $n \in \mathbb{N}_0$ and a subset $S \subset [n]$, let $\Delta^S \subset \Delta^n$ be the full ∞-subcategory on the vertices that lie in S. There is a canonical isomorphism $\Delta^S \cong \Delta^{|S|}$ as simplicial sets, where $|S|$ is the cardinality of S. The simplicial set Δ^S can equivalently be seen as the image of an inclusion $\Delta^{|S|} \hookrightarrow \Delta^n$ that sends the i-th vertex of $\Delta^{|S|}$ to the vertex of Δ^n which corresponds to the i-th element of S (with the order induced from the inclusion $S \subset [n]$). Given a simplicial object in an ∞-category $\mathcal{C}$, i.e. $\widehat{X} \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$, we set $\widehat{X}(S) := \widehat{X}(\Delta^{|S|})$. This comes with a canonical morphism $\widehat{X}_n \to \widehat{X}(S)$, induced by the inclusion $S \subset [n]$.

Definition 2.6 *Let $\mathcal{C}$ be an ∞-category. A* groupoid object *in $\mathcal{C}$ is a simplicial object $\widehat{X} \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ such that, for every $n \in \mathbb{N}_0$ and every partition $[n] = S \cup S'$ (as finite sets) with $S \cap S' \cong \{*\}$ consisting of a single element, the diagram*

$$\begin{tikzcd} \widehat{X}_n \arrow[r] \arrow[d] & \widehat{X}(S') \arrow[d] \\ \widehat{X}(S) \arrow[r] & \widehat{X}_0 \end{tikzcd}$$

is a pullback diagram in $\mathcal{C}$.

We denote the full subcategory of $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ on the groupoid objects by

$$\mathcal{G}\mathrm{pd}(\mathcal{C}) \subset \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})\,.$$

Let $\mathbb{\Delta}_+$ denote the simplex category with an initial object $[-1]$ adjoined. For $n \in \mathbb{N}_0$, let $\mathbb{\Delta}_{+,\leq n} \subset \mathbb{\Delta}_+$ be the full subcategory on the objects $[-1], \ldots, [n]$. In particular, $\mathbb{\Delta}^{\mathrm{op}}_{+,\leq 0}$ is the category with two objects and one non-trivial morphism $[0] \to [-1]$. Therefore, any morphism $p\colon P \to X$ in an ∞-category $\mathcal{C}$ defines an object $\{p\} \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}_{+,\leq 0}, \mathcal{C})$.

Definition 2.7 *Given a morphism $p\colon P \to X$ in an ∞-category $\mathcal{C}$, its* Čech nerve *$\check{C}p$ (if it exists) is the augmented simplicial object obtained as the right Kan extension*

$$\begin{tikzcd} \mathbb{\Delta}^{\mathrm{op}}_{+,\leq 0} \arrow[r, "\{p\}"] \arrow[d, hook, "\imath"'] & \mathcal{C} \\ \mathbb{\Delta}^{\mathrm{op}}_{+} \arrow[ur, dashed, "\check{C}p"'] & \end{tikzcd}$$

That is, $\check{C}p = \mathrm{Ran}_\imath\{p\}$, where $\imath$ is the inclusion $\mathbb{\Delta}^{\mathrm{op}}_{+,\leq 0} \hookrightarrow \mathbb{\Delta}^{\mathrm{op}}_{+}$.

For later use, we record:

Proposition 2.8 [Lur09, Prop. 6.1.2.11] *Let $\mathcal{C}$ be an ∞-category, and let $\widehat{X}\colon \mathbb{\Delta}_+^{\mathrm{op}} \to \mathcal{C}$ be an augmented simplicial object. The following are equivalent:*

(1) $\widehat{X}$ is a right Kan extension of $\widehat{X}_{|\mathbb{\Delta}^{\mathrm{op}}_{+,\leq 0}}$.

(2) The underlying simplicial object $\widehat{X}_{|\mathbb{\Delta}^{\mathrm{op}}}$ is a groupoid object in $\mathcal{C}$ and the diagram

$$\widehat{X}_{|\mathbb{\Delta}^{\mathrm{op}}_{+,\leq 1}} = \begin{tikzcd} \widehat{X}_1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & \widehat{X}_0 \arrow[d] \\ \widehat{X}_0 \arrow[r] & \widehat{X}_{-1} \end{tikzcd}$$

is a pullback square in $\mathcal{C}$.

In this situation, it follows that, for every $n \geq 1$, the spine decomposition $[n] = [1] \sqcup_{[0]} \cdots \sqcup_{[0]} [1]$ induces a canonical equivalence

$$(\check{C}p)_n \simeq \underbrace{P \times_X \cdots \times_X P}_{n+1 \text{ factors}}.$$

Definition 2.9 *Let $\mathcal{C}$ be an ∞-category, and let $p\colon P \to X$ be a morphism in $\mathcal{C}$. Then, p is an* effective epimorphism *if $\check{C}p \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}_+^{\mathrm{op}}, \mathcal{C}) \cong \mathcal{F}\mathrm{un}((\mathbb{\Delta}^{\mathrm{op}})^{\triangleright}, \mathcal{C})$ is a colimiting cocone in $\mathcal{C}$. That is, the morphism $p\colon P \to X$ is an effective epimorphism precisely if the induced morphism $|\check{C}p| \to X$ is an equivalence.*

Let $\widehat{X}\colon \mathbb{\Delta}_+^{\mathrm{op}} \to \mathcal{C}$ be an augmented simplicial object in an ∞-category $\mathcal{C}$. We denote the morphism $\widehat{X}_0 \to \widehat{X}_{-1}$ by p. Suppose that its Čech nerve $\check{C}p$ exists. Observe that $\{p\} = \imath^*\widehat{X}$ as objects in $\mathcal{F}\mathrm{un}(\Delta^1, \mathcal{C}) \cong \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}_{+,\leq 0}, \mathcal{C})$. By the adjointness property of the right Kan extension, there is a canonical equivalence

$$\underline{\mathcal{F}\mathrm{un}}(\mathbb{\Delta}^{\mathrm{op}}_{+,\leq 0}, \mathcal{C})(\imath^*\widehat{X}, \{p\}) \simeq \underline{\mathcal{F}\mathrm{un}}(\mathbb{\Delta}^{\mathrm{op}}_{+}, \mathcal{C})(\widehat{X}, \check{C}p).$$

The identity $\imath^*\widehat{X} = \{p\}$ thus induces a canonical morphism

$$\eta\colon \widehat{X} \longrightarrow \check{C}p. \tag{2.10}$$

We define ∞-topoi in terms of the Giraud-Lurie-Rezk axioms [Lur09, Def. 6.1.0.4, Thm. 6.1.0.6]:

Definition 2.11 *An* ∞-topos *is an ∞-category $\mathbf{H}$ satisfying the following axioms:*

(1) $\mathbf{H}$ is presentable. In particular, $\mathbf{H}$ has all limits and colimits. We denote its initial object by $\emptyset \in \mathbf{H}$ and its final object by $* \in \mathbf{H}$.

(2) Colimits in $\mathbf{H}$ are universal: for any diagram $D\colon K \to \mathbf{H}$, any cocone $\overline{D}\colon K^{\triangleright} \to \mathbf{H}$ under D with apex $Y \in \mathbf{H}$, and any morphism $f\colon X \to Y$ in $\mathbf{H}$, the induced morphism

$$\operatorname*{colim}_K{}^{\mathbf{H}}(D \underset{\mathsf{c}Y}{\times} \mathsf{c}X) \longrightarrow \big(\operatorname*{colim}_K{}^{\mathbf{H}} D\big) \underset{Y}{\times} X$$

is an equivalence (on the left-hand side, $\mathsf{c}X, \mathsf{c}Y : K \to \mathbf{H}$ are the constant diagrams with values X and Y, respectively, and the pullback is formed in $\mathcal{F}\mathrm{un}(K, \mathbf{H})$).

(3) Coproducts in $\mathbf{H}$ are disjoint: for every pair of objects $X, Y \in \mathbf{H}$, the pushout diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \sqcup Y \end{tikzcd}$$

is also a pullback diagram.

(4) *Groupoids in* $\mathbf{H}$ *are effective: given any groupoid object* $\widehat{X} \in \mathcal{G}\mathrm{pd}(\mathbf{H})$*, let* $p\colon \widehat{X}_0 \to |\widehat{X}|$ *denote the canonical morphism which is part of the colimiting cocone. Then, the comparison morphism* $\eta\colon \widehat{X} \to \check{C}p$ *constructed in* (2.10) *is an equivalence of simplicial objects in* $\mathbf{H}$*. In particular,* p *is an effective epimorphism.*

Example 2.12 We list some examples of ∞-topoi; we will mostly be using the first two cases.

(1) The ∞-category of spaces $\mathbf{S}$ is an ∞-topos.

(2) Any ∞-category $\mathcal{P}(\mathcal{C})$ of presheaves of spaces on a (small) ∞-category $\mathcal{C}$ is an ∞-topos.

(3) Any accessible, left-exact, reflective localisation of an ∞-category $\mathcal{P}(\mathcal{C})$ of presheaves on a small ∞-category $\mathcal{C}$ is an ∞-topos; in fact, every ∞-topos is equivalent to an ∞-topos of this form [Lur09, Thm. 6.1.0.6, Prop. 6.1.5.3]. $\triangleleft$

An important notion of morphism between ∞-topoi is that of a geometric morphism, which is more adapted to the additional structure on ∞-topoi than a mere functor of ∞-categories:

Definition 2.13 *Let* $\mathbf{X}, \mathbf{Y}$ *be* ∞*-topoi. A* geometric morphism *of* ∞*-topoi from* $\mathbf{X}$ *to* $\mathbf{Y}$ *is a functor* $\mathrm{F}_*\colon \mathbf{X} \to \mathbf{Y}$ *admitting a left exact left adjoint* $\mathrm{F}^*\colon \mathbf{Y} \to \mathbf{X}$.

One can show that the ∞-category $\mathbf{S}$ of spaces is final in the ∞-category of ∞-topoi and geometric morphisms [Lur09, Prop. 6.3.4.1]. That is, for every ∞-topos $\mathbf{H}$ there exists an essentially unique geometric morphism $\mathbf{H} \to \mathbf{S}$. We will denote the corresponding adjunction by $\widetilde{\mathsf{c}} : \mathbf{S} \rightleftarrows \mathbf{H} : \Gamma$ and refer to Γ as the *global-section functor*.

Example 2.14 Consider a Grothendieck ∞-site, i.e. a small ∞-category $\mathcal{C}$ with a Grothendieck (pre)topology. Suppose $\mathcal{C}$ additionally has a final object. If $\mathbf{H}$ is the ∞-category of sheaves of spaces on $\mathcal{C}$, then the global section functor Γ of $\mathbf{H}$ agrees with the evaluation of sheaves at the final object of $\mathcal{C}$. In particular, this applies to $\mathbf{H}_\infty$, the ∞-topos of presheaves of spaces on $\mathcal{C}\mathrm{art}$ from Section 2.1. $\triangleleft$

Definition 2.15 *An* ∞*-topos* $\mathbf{H}$ *is called* cohesive *if the adjunction* $\widetilde{\mathsf{c}} : \mathbf{H} \rightleftarrows \mathbf{S} : \Gamma$ *can be extended to a triple adjunction* $\Pi \dashv \mathsf{c} \dashv \Gamma \dashv$ *codisc, in which the left adjoint* Π *preserves finite products and in which the right adjoint codisc is fully faithful.*

Cohesive ∞-topoi have been studied extensively in [Sch] and related works.

Theorem 2.16 *The* ∞*-topos* $\mathbf{H}_\infty$ *is cohesive, i.e. there exists a triple adjunction* $\Pi \dashv \widetilde{\mathsf{c}} \dashv \Gamma \dashv$ *codisc as in Definition 2.15, and there is a canonical equivalence*

$$\Pi \simeq \mathrm{S}_e\,.$$

Remark 2.17 The fact that $\mathbf{H}_\infty$ is cohesive is not new, see [Sch]. The second statement has been indicated in [BEBdBP] and has been worked out in detail in a model categorical presentation in [Bunb]. Here, we give an ∞-categorical proof of this fact for completeness. $\triangleleft$

Proof. The ∞-topos $\mathbf{H}_\infty = \mathcal{P}(\mathcal{C}\mathrm{art})$ admits a right-adjoint to its global-section functor Γ by abstract arguments: evaluation of a presheaf at any object preserves colimits, and since both $\mathbf{H}_\infty$ and $\mathbf{S}$ are presentable, Γ must admit a further right adjoint. It is well-known that this can actually be extended into a triple adjunction which establishes that $\mathbf{H}_\infty$ is cohesive [Sch].

For the second part of the statement, we show that S_e is left-adjoint to the functor $\widetilde{\mathsf{c}}$. Recall from Section 2.1 that, in this situation, $\widetilde{\mathsf{c}}$ simply sends a space $K \in \mathbf{S}$ to the constant presheaf on $\mathcal{C}\mathrm{art}$ with value K. Further, recall from the proof of Proposition 2.4 (and [Bunb, Prop. 2.13]) that the

I-local objects in $\mathbf{H}_\infty$ are precisely the essentially constant presheaves, i.e. those $X \in \mathbf{H}_\infty$ for which the canonical morphism $X(*) \to X(c)$ is an equivalence for every $c \in \mathcal{C}\mathrm{art}$. Equivalently, X is I-local if and only if the canonical morphism $\widetilde{\mathsf{c}}\Gamma X \to X$ is an equivalence in $\mathbf{H}_\infty$. Further, by Theorem 2.2 the right adjoint R_e to S_e factors through the localisation $L_I\mathbf{H}_\infty \subset \mathbf{H}_\infty$; this is precisely the full ∞-subcategory of $\mathbf{H}_\infty$ on the I-local objects.

Consider the two adjunctions $\mathrm{S}_e : \mathbf{H}_\infty \rightleftarrows \mathbf{S} : \mathrm{R}_e$ and $\widetilde{\mathsf{c}} : \mathbf{S} \rightleftarrows \mathbf{H}_\infty : \Gamma$. They induce an adjunction

$$\mathrm{S}_e \circ \widetilde{\mathsf{c}} : \mathbf{S} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{S} : \Gamma \circ \mathrm{R}_e\,.$$

By the definition (2.1) of S_e, for any space $K \in \mathbf{S}$ we have a canonical natural equivalence

$$\mathrm{S}_e \circ \widetilde{\mathsf{c}}(K) = \operatorname*{colim}_{\mathbb{\Delta}^{\mathrm{op}}}^{\mathbf{S}}\big(\widetilde{\mathsf{c}}(K)(\Delta_e)\big) \simeq K\,,$$

because left-hand side is the colimit of a constant diagram over an indexing category whose nerve is contractible in the Kan-Quillen model structure on $\mathcal{S}\mathrm{et}_\Delta$ (see Lemma A.7, Example A.8). In other words, there is a canonical natural equivalence $\mathrm{S}_e \circ \widetilde{\mathsf{c}} \simeq 1_{\mathbf{S}}$. Consequently, there is also a canonical equivalence on the right adjoints, $\Gamma \circ \mathrm{R}_e \simeq 1_{\mathbf{S}}$. We obtain natural equivalences

$$\widetilde{\mathsf{c}} \simeq \widetilde{\mathsf{c}} \circ \Gamma \circ \mathrm{R}_e \simeq \mathrm{R}_e\,.$$

In the second equivalence we have used that R_e takes values in $L_I\mathbf{H}_\infty \subset \mathbf{H}_\infty$ and that on objects in $L_I\mathbf{H}_\infty$ the morphism $\widetilde{\mathsf{c}} \circ \Gamma \to 1_{\mathbf{H}_\infty}$ is an equivalence. From the equivalence $\mathrm{R}_e \simeq \widetilde{\mathsf{c}}$ and the adjunction $\mathrm{S}_e \dashv \mathrm{R}_e$ we infer that S_e is a further left adjoint to $\widetilde{\mathsf{c}}$. Hence, it is equivalent to the functor Π. $\square$

Theorem 2.16 shows that the smooth singular complex functor $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$ has a deep homotopical meaning for assigning homotopy types to objects in $\mathbf{H}_\infty$ and for studying these homotopy types. It also provides an additional, refined, perspective on the good homotopical properties of the functor S_e that were found and studied in [Bunb]. Finally, note that it also follows that there is a natural equivalence

$$\mathrm{S}_e(F) = \operatorname*{colim}_{\mathbb{\Delta}^{\mathrm{op}}}^{\mathbf{S}}\big(F(\Delta_e)\big) \simeq \operatorname*{colim}_{\mathcal{C}\mathrm{art}^{\mathrm{op}}}^{\mathbf{S}}(F)\,.$$

That is, S_e computes the colimit of $\mathcal{C}\mathrm{art}^{\mathrm{op}}$-shaped diagrams of spaces.

3 Principal ∞-bundles and group extensions in ∞-topoi

In this section, starting from the theory introduced in [NSS15], we develop characterisations of principal ∞-bundles and of extensions of group objects in ∞-topoi. These characterisations are interesting already in their own right, but in Section 4 they will also allow us to transfer the definition of string group extensions from $\mathbf{S}$ to $\mathbf{H}_\infty$ and to construct explicit smooth models for the string group.

3.1 Groups and group extensions

Here, we recall the definitions of group objects and their extensions in ∞-topoi [NSS15]. We investigate how to compute limits of group and groupoid objects in ∞-topoi, and how group objects and their classifying objects behave under functors between ∞-topoi that preserve finite products and geometric realisations.

Let $\mathbf{H}$ be an ∞-topos, and let $\mathcal{G}\mathrm{pd}(\mathbf{H})$ be the ∞-category of groupoid objects in $\mathbf{H}$. Further, let $\mathrm{EEpi}(\mathbf{H}) \subset \mathcal{F}\mathrm{un}(\Delta^1, \mathbf{H})$ denote the full ∞-subcategory on the effective epimorphisms in $\mathbf{H}$. Recall that by Definition 2.11(4) there is a canonical equivalence

$$\mathcal{G}\mathrm{pd}(\mathbf{H}) \simeq \mathrm{EEpi}(\mathbf{H})\,, \tag{3.1}$$

given by forming colimits and Čech nerves, respectively.

Lemma 3.2 *In an ∞-topos $\mathbf{H}$, effective epimorphisms are stable under pullback and under pushout.*

Proof. The fact that effective epimorphisms are stable under pullback is [Lur09, Prop. 6.2.3.15]. The stability under pushouts follows from the facts that the effective epimorphisms in $\mathbf{H}$ are precisely the 0-connective[1] morphisms [Lur09, Def. 6.5.1.10], and that n-connected morphisms in an ∞-topos are stable under pushout for any $n \leq -1$ [Lur09, Prop. 6.5.1.17]. (The n-connected morphisms are even stable under colimits, since they form the left class of an orthogonal factorisation system on $\mathbf{H}$.) □

Definition 3.3 *Let $\mathcal{C}$ be an ∞-category. Let $\mathcal{G}\mathrm{rp}(\mathcal{C}) \subset \mathcal{G}\mathrm{pd}(\mathcal{C})$ denote the full ∞-subcategory on those groupoid objects where X_0 is a final object of $\mathcal{C}$. We call $\mathcal{G}\mathrm{rp}(\mathcal{C})$ the ∞-category of* group objects *in $\mathcal{C}$.*

Proposition 3.4 *For any ∞-topos $\mathbf{H}$, there are reflective localisations*

$$\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H}) \underset{\hookleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{G}\mathrm{pd}(\mathbf{H}) \underset{\hookleftarrow}{\overset{\perp}{\longrightarrow}} \mathcal{G}\mathrm{rp}(\mathbf{H})\,.$$

Proof. First, the right adjoints in the above sequence of adjunctions are fully faithful by definition. The first morphism has a left adjoint by [Lur09, Prop. 6.1.2.9]. For the second left adjoint, we use the equivalence (3.1)[2]: this equivalence induces a commutative square

$$\begin{tikzcd}
\mathcal{G}\mathrm{pd}(\mathbf{H}) \arrow[d, "\simeq"'] & \mathcal{G}\mathrm{rp}(\mathbf{H}) \arrow[l, hook'] \arrow[d, "\simeq"] \\
\mathrm{EEpi}(\mathbf{H}) & \mathrm{EEpi}_*(\mathbf{H}) \arrow[l, hook']
\end{tikzcd}$$

where $\mathrm{EEpi}_*(\mathbf{H}) \subset \mathrm{EEpi}(\mathbf{H})$ is the full ∞-subcategory on those effective epimorphisms $f \colon X_0 \to X_{-1}$ where X_0 is a final object. A left adjoint to the bottom morphism is given by the functor that sends an effective epimorphism $f \colon X_0 \to X_{-1}$ to the morphism $g \colon * \to X_{-1} \sqcup_{X_0} *$ induced by the pushout. Since f is an effective epimorphism, Lemma 3.2 implies that so is g. □

For a group object $\widehat{G} \in \mathcal{G}\mathrm{rp}(\mathbf{H})$ in an ∞-topos $\mathbf{H}$, we set

$$G \coloneqq \widehat{G}_1 \in \mathbf{H} \quad \text{and} \quad \mathrm{B}G \coloneqq \operatorname*{colim}^{\mathbf{H}}_{\mathbb{\Delta}^{\mathrm{op}}} \widehat{G} = |\widehat{G}| \in \mathbf{H}\,.$$

Note that in an ∞-topos $\mathbf{H}$, the map $X_0 \to \mathrm{colim}^{\mathbf{H}}_{\mathbb{\Delta}^{\mathrm{op}}} \widehat{X}$ is an effective epimorphism for any groupoid object $\widehat{X} \in \mathcal{G}\mathrm{pd}(\mathbf{H})$. Hence, given a group object $\widehat{G}$ in $\mathbf{H}$, the morphism $* \simeq G_0 \to \mathrm{B}G$ is an effective epimorphism. Moreover, the functor B is part of an equivalence [Lur09, Lemma 7.2.2.11] (see also [NSS15, Thm. 2.19])

$$\mathbf{H}^{*/}_{\geq 1} \underset{\mathrm{B}}{\overset{\Omega}{\underset{\longleftarrow}{\overset{\perp}{\longrightarrow}}}} \mathcal{G}\mathrm{rp}(\mathbf{H})\,,$$

[1] Note that there is a shift in counting between [Lur09] and the nLab: A morphism f in $\mathbf{H}$ is n-connect*ive* in the conventions of [Lur09] if and only if it is $(n{-}1)$-connect*ed* in the conventions used on the nLab.

[2] This proof goes back to a mathoverflow answer by Jacob Lurie, see `https://mathoverflow.net/questions/140639/is-the-category`

where $\mathbf{H}^{*/}_{\geq 1}$ is the ∞-category of pointed, connected objects in $\mathbf{H}$.

Unravelling the definition, we obtain that a group object in an ∞-category $\mathcal{C}$ with a final object $* \in \mathcal{C}$ is a equivalently simplicial object $\widehat{G}$ in $\mathcal{C}$ such that $\widehat{G}_0 \simeq *$ and, for any $[n] \in \mathbb{\Delta}$ and any partition $[n] = S \cup S'$ as finite sets with $S \cap S' \cong \{*\}$ consisting of a single element, the diagram

$$\begin{tikzcd}
\widehat{G}_n \arrow[r] \arrow[d] & \widehat{G}(S) \arrow[d] \\
\widehat{G}(S') \arrow[r] & \widehat{G}_0 \simeq *
\end{tikzcd}$$

is a pullback diagram in $\mathcal{C}$. That is, there is a canonical equivalence $\widehat{G}_n \xrightarrow{\simeq} \widehat{G}(S) \times \widehat{G}(S')$. In particular, iterating this for the spine partition $[n] = [1] \sqcup_{[0]} \cdots \sqcup_{[0]} [1]$, we obtain a canonical equivalence

$$\widehat{G}_n \xrightarrow{\simeq} G^n\,.$$

Proposition 3.5 *Let* $\mathrm{L} \colon \mathbf{H} \to \mathbf{H}'$ *be a functor of* ∞*-topoi.*

(1) If L *preserves finite products, then it preserves group objects.*

(2) If L *additionally preserves geometric realisations, then, for any group object* $\widehat{G}$ *in* $\mathbf{H}$*, there is a canonical equivalence*

$$\mathrm{B}(\mathrm{L}G) \simeq \mathrm{L}(\mathrm{B}G)\,.$$

Proof. The first part of the Proposition is known [Lur09]; we include its proof only for completeness. Any functor $F \colon \mathcal{C} \to \mathcal{D}$ between ∞-categories preserves simplicial objects, i.e. it induces a functor $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C}) \longrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{D})$. Suppose that $\widehat{A} \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ is a group object in $\mathbf{H}$. Since L preserves finite products, it preserves final objects, so that $(\mathrm{L}\widehat{A})_0 \simeq *$ is final in $\mathbf{H}'$. For $n \neq 0$ and any partition $[n] = S \cup S'$ with $S \cap S' \cong \{*\}$, we obtain a commutative diagram

$$\begin{tikzcd}
(\mathrm{L}\widehat{A})_n = \mathrm{L}(\widehat{A}_n) \arrow[r, "\simeq"] \arrow[dr] & \mathrm{L}\big(\widehat{A}(S) \times \widehat{A}(S')\big) \arrow[d, "\simeq"] \\
& \mathrm{L}\widehat{A}(S) \times \mathrm{L}\widehat{A}(S')
\end{tikzcd}$$

The top morphism is an equivalence since $\widehat{A}$ is a group object in $\mathbf{H}$ and the vertical morphism is an equivalence since L preserves products. This proves claim (1). Using that $\mathrm{B}A = \mathrm{colim}^{\mathbf{H}}_{\mathbb{\Delta}^{\mathrm{op}}} \widehat{A} = |\widehat{A}|$, the second part is now immediate. $\square$

Remark 3.6 We will prove a number of statements about functors as in Proposition 3.5(2), i.e. functors between ∞-topoi which preserve geometric realisations and finite products. An important class of such functors arises is given by the additional left-adjoints of cohesive ∞-topoi—see Definition 2.15. In particular, the functor $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$ from Section 2.1 is of this type by Theorem 2.16. $\triangleleft$

Lemma 3.7 *Let* $\mathbf{H}$ *be an* ∞*-topos.*

(1) A morphism $\widehat{X} \to \widehat{Y}$ *in* $\mathcal{G}\mathrm{pd}(\mathbf{H})$ *is an equivalence if and only if* $X_i \to Y_i$ *is an equivalence in* $\mathbf{H}$ *for* $i = 0, 1$.

(2) A morphism $\widehat{G} \to \widehat{H}$ *in* $\mathcal{G}\mathrm{rp}(\mathbf{H})$ *is an equivalence if and only if* $G \to H$ *is an equivalence in* $\mathbf{H}$.

Proof. Proposition 3.4 implies that an equivalence of groupoid objects $\widehat{X} \xrightarrow{\simeq} \widehat{Y}$ in $\mathbf{H}$ is the same as an objectwise equivalence of the underlying simplicial objects in $\mathbf{H}$: $\widehat{X}$ and $\widehat{Y}$ are local objects in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ with respect to the localisation $\mathcal{G}\mathrm{pd}(\mathbf{H}) \subset \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$, so that the local equivalences between them are precisely the original, i.e. the levelwise, equivalences. In particular, this implies the 'only if' part of claim (1).

Conversely, if we are given a morphism $\widehat{X} \xrightarrow{\simeq} \widehat{Y}$ of groupoid objects in $\mathbf{H}$ such that $\widehat{X}_i \to \widehat{Y}_i$ is an equivalence for $i = 0, 1$, then it follows that $\widehat{X} \xrightarrow{\simeq} \widehat{Y}$ is a levelwise equivalence of simplicial objects; this is because there are canonical equivalences $\widehat{X}_n \simeq \widehat{X}_1 \times_{\widehat{X}_0} \cdots \times_{\widehat{X}_0} \widehat{X}_1$, natural in $\widehat{X} \in \mathcal{G}\mathrm{pd}(\mathbf{H})$. It then follows that the morphism $\widehat{X} \to \widehat{Y}$ is also an equivalence in $\mathcal{G}\mathrm{pd}(\mathbf{H})$.

The same line of argument shows the second claim. ☐

Lemma 3.8 *Let $\mathbf{H}$ be an ∞-topos, and let $K \in \mathcal{S}\mathrm{et}_\Delta$ be a simplicial set.*

(1) A diagram $\widehat{X}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{pd}(\mathbf{H})$ of groupoid objects in $\mathbf{H}$ is a limit diagram if and only if the composition $\iota\widehat{X}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{pd}(\mathbf{H}) \hookrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ is a limit diagram.

(2) A diagram $\widehat{X}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{pd}(\mathbf{H})$ of groupoid objects in $\mathbf{H}$ is a limit diagram if and only if the induced diagrams $\widehat{X}_i\colon K^{\triangleleft} \to \mathbf{H}$ are limit diagrams for $i = 0, 1$.

(3) A diagram $\widehat{G}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{rp}(\mathbf{H})$ of group objects in $\mathbf{H}$ is a limit diagram if and only if the the composition $\jmath\widehat{G}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{rp}(\mathbf{H}) \hookrightarrow \mathcal{G}\mathrm{pd}(\mathbf{H})$ is a limit diagram.

(4) A diagram $\widehat{G}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{rp}(\mathbf{H})$ of group objects in $\mathbf{H}$ is a limit diagram if and only if the induced diagram $\widehat{G}_1 = G\colon K^{\triangleleft} \to \mathbf{H}$ is a limit diagram.

Proof. The 'only if' direction of claims (1) and (2) is readily seen as follows: we first note that since the inclusion $\mathcal{G}\mathrm{pd}(\mathbf{H}) \subset \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ is a right adjoint, we have that if $\widehat{X}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{pd}(\mathbf{H})$ is a limit diagram, then so is $\widehat{X}\colon K^{\triangleleft} \to \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. Further, since limits in presheaf (or diagram) categories are computed pointwise, this is equivalent to the functor $\widehat{X}_i\colon K^{\triangleleft} \to \mathbf{H}$ being a limit diagram in $\mathbf{H}$ for every $[i] \in \mathbb{\Delta}$.

For the converse direction in claim (1), we first show that limits of diagrams in $\mathcal{G}\mathrm{pd}(\mathbf{H})$ can be computed in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. More precisely, a functor $\widehat{X}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{pd}(\mathbf{H})$ is a limit diagram whenever its composition with the inclusion $\iota\colon \mathcal{G}\mathrm{pd}(\mathbf{H}) \hookrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ is so, i.e. the inclusion reflects limits. Equivalently, the ∞-subcategory $\mathcal{G}\mathrm{pd}(\mathbf{H}) \hookrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ is closed under limits in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. This is seen as follows: consider a functor $\widehat{X}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{pd}(\mathbf{H})$ and a decomposition $[n] = S \cup S'$ with $S \cap S' = \{*\}$. This induces an equivalence

$$\widehat{X}_n \xrightarrow{\simeq} \widehat{X}(S) \underset{\widehat{X}_0}{\times} \widehat{X}(S')$$

in $\mathcal{F}\mathrm{un}(K^{\triangleleft}, \mathbf{H})$. Setting $\widehat{Y} \coloneqq \lim_K^{\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})}(\iota\widehat{X})$ and using that limits commute with limits, we have

$$\begin{aligned} \widehat{Y}_n &\simeq \big(\lim_K{}^{\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})}(\iota\widehat{X})\big)_n \\ &\simeq \lim_K{}^{\mathbf{H}}(\iota\widehat{X}_n) \\ &\simeq \lim_K{}^{\mathbf{H}}\big(\iota\widehat{X}(S) \underset{\iota\widehat{X}_0}{\times} \iota\widehat{X}(S')\big) \\ &\simeq \big(\lim_K{}^{\mathbf{H}}\iota\widehat{X}(S)\big) \underset{(\lim_K^{\mathbf{H}}\iota\widehat{X}_0)}{\times} \big(\lim_K{}^{\mathbf{H}}\iota\widehat{X}(S')\big) \\ &\simeq \widehat{Y}(S) \times_{\widehat{Y}_0} \widehat{Y}(S'), \end{aligned} \tag{3.9}$$

which shows that $\widehat{Y} \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ is local with respect to the localisation $\mathcal{G}\mathrm{pd}(\mathbf{H}) \hookrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$, i.e. that $\widehat{Y} \in \mathcal{G}\mathrm{pd}(\mathbf{H})$. Since the inclusion $\iota\colon \mathcal{G}\mathrm{pd}(\mathbf{H}) \hookrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ is fully faithful, $\widehat{Y}$ is also a limit of the diagram $\widehat{X}\colon K \to \mathcal{G}\mathrm{pd}(\mathbf{H})$. Consequently, if $\iota\widehat{X}\colon K^{\triangleleft} \to \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ is a limit diagram, then $\widehat{X}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{pd}(\mathbf{H})$ is a limit diagram.

For the converse direction in claim (2), suppose that $\widehat{X}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{pd}(\mathbf{H})$ is a diagram such that the functors $\widehat{X}_i\colon K^{\triangleleft} \to \mathbf{H}$ are limit diagrams for $i = 0, 1$. By part (1) it suffices to show that the composition $\iota\widehat{X}\colon K^{\triangleleft} \to \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ is a limit diagram; that is, it suffices to show that $\widehat{X}_i\colon K^{\triangleleft} \to \mathbf{H}$ is a limit diagram for every $[i] \in \mathbb{\Delta}$.

Since $\widehat{X}\colon K^{\triangleleft} \to \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ is valued in groupoid objects, and since limits in $\mathbf{H}$ commute [Lur09, Lemma 5.5.2.3], it follows from (3.9) that for every $[n] \in \mathbb{\Delta}$ the diagram $\widehat{X}_n\colon K^{\triangleleft} \to \mathbf{H}$ is equivalent to a limit diagram $\widehat{X}_1 \times_{\widehat{X}_0} \cdots \times_{\widehat{X}_0} \widehat{X}_1\colon K^{\triangleleft} \to \mathbf{H}$, and is hence a limit diagram itself.

The proof of claim (3) proceeds along the exact same line as the proof of part (2): the key insight is the fact that if $\widehat{G}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{rp}(\mathbf{H})$ is a diagram such that the composition $\jmath\widehat{G}\colon K^{\triangleleft} \to \mathcal{G}\mathrm{pd}(\mathbf{H})$ is a limit diagram, then $\lim_K^{\mathcal{G}\mathrm{pd}(\mathbf{H})}(\jmath\widehat{G})$ is still local with respect to the localisation $\mathcal{G}\mathrm{rp}(\mathbf{H}) \subset \mathcal{G}\mathrm{pd}(\mathbf{H})$.

Claim (4) is then the combination of claims (2) and (3). $\square$

Having established several properties of the ∞-category of group objects in $\mathbf{H}$, we now define extensions of group objects:

Definition 3.10 [NSS15, Def. 4.26] *Let $\widehat{A}$ and $\widehat{H}$ be group objects in an ∞-topos $\mathbf{H}$. An* extension of group objects of $\widehat{H}$ by $\widehat{A}$ *is a sequence $\widehat{A} \to \widehat{G} \to \widehat{H}$ in the ∞-category $\mathcal{G}\mathrm{rp}(\mathbf{H})$ such that the sequence $\mathrm{B}A \to \mathrm{B}G \to \mathrm{B}H$ is a fibre sequence in $\mathbf{H}^{*/}_{\geq 1}$.*

Remark 3.11 This definition of a group extension has advantages from a theoretical perspective. Nevertheless, it appears that there should be a simpler definition that more directly generalises extensions of groups in $\mathcal{S}\mathrm{et}$ to the ∞-categorical setting. For group objects in $\mathcal{S}\mathrm{et}$, a group extension is a sequence $A \to G \to H$ of group homomorphisms such that A is the fibre of the morphism $G \to H$ at the identity element of H. We will prove in Theorem 3.49 that one can indeed define group extensions in ∞-topoi along these principles. $\triangleleft$

3.2 Group actions in ∞-categories

We now investigate actions of group objects in ∞-topoi. For a simplicial set $K \in \mathcal{S}\mathrm{et}_\Delta$, we let $\mathrm{obj}(K)$ be the set K_0 of vertices of K, seen as a discrete simplicial set. Let $J \coloneqq \Delta^1[f^{-1}]$ be the localisation of Δ^1 at its non-trivial edge (see e.g. [Cis19, Sec. 3.3]).

Lemma 3.12 *Let $\mathcal{C}$ be an ∞-category, and let K be a simplicial set.*

(1) The inclusion $\iota\colon \mathrm{obj}(K) \hookrightarrow K$ induces a morphism

$$\iota^*\colon \mathcal{C}^K = \mathcal{F}\mathrm{un}(K, \mathcal{C}) \longrightarrow \mathcal{F}\mathrm{un}\big(\mathrm{obj}(K), \mathcal{C}\big) = \mathcal{C}^{\mathrm{obj}(K)}$$

of simplicial sets, which is a fibration between fibrant objects in the Joyal model structure.

(2) Consider either of the inclusions $\Delta^{\{i\}} \hookrightarrow J$, where $i = 0, 1$. The induced morphism

$$\mathcal{F}\mathrm{un}(J, \mathcal{C}^K) \longrightarrow \mathcal{C}^K \underset{\mathcal{C}^{\mathrm{obj}(K)}}{\times} \mathcal{F}\mathrm{un}\big(J, \mathcal{C}^{\mathrm{obj}(K)}\big)$$

is a trivial Kan fibration.

(3) Let $g\colon K \to \mathcal{C}$ and $g'\colon \mathrm{obj}(K) \to \mathcal{C}$ be functors. For any equivalence $\eta\colon \iota^* g \xrightarrow{\simeq} g'$, consider the space of pairs $(\hat{g}', \hat{\eta})$, where $\hat{g}'$ is a lift of g' to a functor $\hat{g}'\colon K \to \mathcal{C}$, and where $\hat{\eta}$ is an equivalence $g \xrightarrow{\simeq} \hat{g}'$ such that $\iota^*\hat{\eta} = \eta$. This space is a contractible Kan complex.

Proof. Part (1) follows since $\mathrm{obj}(K) \hookrightarrow K$ is a cofibration in the Joyal model category $\mathcal{S}\mathrm{et}_{\Delta J}$, $\mathcal{C}$ is a fibrant object in $\mathcal{S}\mathrm{et}_{\Delta J}$, and $\mathcal{S}\mathrm{et}_{\Delta J}$ is a (closed) symmetric monoidal model category.

For part (2), we apply [Cis19, Cor. 3.6.4] to the categorical anodyne extension $\Delta^{\{i\}} \hookrightarrow J = \Delta^1[f^{-1}]$ and the Joyal fibration (i.e. isofibration) from part (1).

Part (3) is obtained by taking the fibre of the morphism from part (2), which is a contractible Kan complex since it is the fibre of a trivial Kan fibration. This fibre is equivalently described as the space of lifts in the commutative diagram

$$\begin{tikzcd} \Delta^{\{i\}} \arrow[r, "g"] \arrow[d, hook] & \mathcal{C}^K \arrow[d, "\iota^*"] \\ J \arrow[r, "\eta"'] & \mathcal{C}^{\mathrm{obj}(K)} \end{tikzcd}$$

which is precisely the space of pairs $(\hat{g}', \hat{\eta})$ of lifts $\hat{g}'\colon K \to \mathcal{C}$ of g' and equivalences $\hat{\eta}\colon g \xrightarrow{\simeq} \hat{g}'$ such that $\iota^*\hat{\eta} = \eta$. $\square$

Example 3.13 Let $\widehat{G}$ be a group object in an ∞-category $\mathcal{C}$ with a final object. This is, in particular, a simplicial object $\widehat{G}\colon \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$ (we suppress the canonical inclusion functors $\mathcal{G}\mathrm{rp}(\mathcal{C}) \hookrightarrow \mathcal{G}\mathrm{pd}(\mathcal{C}) \hookrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$). Consider the functor

$$[0] \star (-)\colon \mathbb{\Delta} \longrightarrow \mathbb{\Delta}\,, \qquad [n] \longmapsto [0] \star [n] \cong [n+1]\,,$$

where $\star$ denotes the join of categories (and where we view partially ordered sets as categories). The induced pullback functor

$$\mathrm{Dec}^0 := \big([0] \star (-)\big)^*\colon \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C}) \longrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$$

is also called the *decalage* functor; see [Ste12] for more background. For any $n \geq 1$, the partition $[n] = \{0, 1\} \sqcup_{\{1\}} \{1, \ldots, n\}$ induces an equivalence

$$\gamma_n\colon (\mathrm{Dec}^0 \widehat{G})_n = \widehat{G}_{n+1} \simeq G \times \widehat{G}_n\,. \tag{3.14}$$

We can phrase this as an equivalence of functors $\mathrm{Dec}^0 \widehat{G} \simeq G \times \widehat{G}\colon \mathrm{obj}(\mathbb{\Delta}^{\mathrm{op}}) \to \mathcal{C}$. From Lemma 3.12 we obtain that there exists an essentially unique way to lift these data to a functor $\mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}$, which we denote by $G/\!\!/G$, and an equivalence $\gamma\colon \mathrm{Dec}^0 \widehat{G} \xrightarrow{\simeq} G/\!\!/G$ in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$, whose components are exactly the equivalences γ_n from (3.14). One can now check that $G/\!\!/G$ is the simplicial object in $\mathcal{C}$ that describes the right action of $\widehat{G}$ on itself via the group multiplication in $\widehat{G}$. $\triangleleft$

Definition 3.15 *Let $\mathcal{C}$ be an ∞-category with pullbacks and a final object, let $\widehat{G}$ be a group object in $\mathcal{C}$, and let $P \in \mathcal{C}$ be an object in $\mathcal{C}$. An* action of $\widehat{G}$ on P *is a simplicial object $P/\!\!/G \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ such that*

(1) for each $n \in \mathbb{N}_0$, we have $(P/\!\!/G)_n = P \times G^{n-1}$,

(2) the morphism $d_1\colon P \times G \to P$ agrees with the canonical projection onto P, the morphism $s_0\colon P \to P \times G$ agrees with the morphism $1_P \times (* \to G)$, and

(3) the collapse morphism $P \to *$ induces a morphism $P/\!\!/G \to \widehat{G}$ in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$.

Given a group action $P/\!/G$, we set $a \coloneqq d_0 \colon P \times G \to P$. It follows by the pasting law for pullbacks that there are canonical equivalences of morphisms between $d_0 \colon P \times G^n \to P \times G^{n-1}$ and $a \times 1_{G^{n-1}} \colon P \times G^n \to P \times G^{n-1}$, and similarly between $d_n \colon P \times G^n \to P \times G^{n-1}$ and the projection onto the first n factors.

Remark 3.16 Definition 3.15 is taken from [NSS15, Def. 3.1] almost verbatim, but it differs from that source in that we do not *require* group actions to be groupoid objects. Instead, we show in Theorem 3.19 that this is a *consequence* of the axioms in Definition 3.15. A second (minor) difference is that we also fix the level-zero degeneracy map $s_0 \colon P \to P \times G$. ◁

Example 3.17 For any group object $\widehat{G} \in \mathcal{G}\mathrm{rp}(\mathcal{C})$ there is a canonical trivial action $*/\!/G$ on the final object $* \in \mathcal{C}$, coming from the canonical equivalence $* \times \widehat{G} \simeq \widehat{G}$ of simplicial objects. That is, there is a canonical equivalence $\widehat{G} \simeq */\!/G$ in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$. ◁

Example 3.18 We can now give a precise meaning to the last sentence of Example 3.13: the object $G/\!/G \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ is an action of G on itself via right multiplication. ◁

Given an action of a group object $\widehat{G}$ on an object P in $\mathcal{C}$, we would like to think of the simplicial object $P/\!/G$ as the *action groupoid* associated with this action. This is indeed justified:

Theorem 3.19 *Let $\mathcal{C}$ be an ∞-category with finite limits, let $\widehat{G} \in \mathcal{G}\mathrm{rp}(\mathcal{C})$ be a group object in $\mathcal{C}$, and let $P/\!/G \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ be an action of $\widehat{G}$ on an object $P \in \mathcal{C}$. Then, $P/\!/G$ is a groupoid object in $\mathcal{C}$.*

Remark 3.20 Theorem 3.19 is important for us since will need to show that functors $\mathrm{L} \colon \mathbf{H} \to \mathbf{H}'$ between ∞-topoi that preserve finite products and geometric realisations map group actions to group actions (see Theorem 3.32). In [NSS15], group actions are *defined* to be groupoid objects, but functors $\mathrm{L} \colon \mathbf{H} \to \mathbf{H}'$ as above do not preserve groupoid objects in general. However, Theorem 3.19 shows that, as in the classical case of (set-theoretic) group actions, actions of group objects in ∞-topoi are *automatically* a groupoid objects. ◁

We prove Theorem 3.19 in Appendix A. For the remainder of this section, let $\mathbf{H}$ be an ∞-topos.

Definition 3.21 *Let $\widehat{G} \in \mathcal{G}\mathrm{rp}(\mathbf{H})$ be a group object. A* G-action over an object $X \in \mathbf{H}$ *is an augmented simplicial object $\widehat{X} \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}_+^{\mathrm{op}}, \mathbf{H})$ whose underlying simplicial object is a G-action $P/\!/G$ on some object $P \in \mathbf{H}$, and whose augmenting object is X, i.e. $\widehat{X}_{-1} = X$. Writing $p \colon P \to X$ for the morphism $\widehat{X}_{|\mathbb{\Delta}_{+,\leq 0}}$, we also denote a G-action over X by*

$$P/\!/G \xrightarrow{p} X \qquad \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}_+^{\mathrm{op}}, \mathbf{H})\,.$$

A morphism of G-actions over $X \in \mathbf{H}$,

$$(P/\!/G \to X) \xrightarrow{f} (Q/\!/G \to X)\,,$$

is a morphism f in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}_+^{\mathrm{op}}, \mathbf{H})$ as above such that

(1) $f_{-1} = 1_X$ is the identity on X, and

(2) the collapse morphisms $P \to *$ and $Q \to *$ induce a (weakly) commutative diagram

$$\begin{tikzcd} P/\!/G \arrow[rr, "f_{|\mathbb{\Delta}^{\mathrm{op}}}"] \arrow[dr] & & Q/\!/G \arrow[dl] \\ & */\!/G & \end{tikzcd}$$

of simplicial objects in $\mathbf{H}$.

The ∞-category of G-actions over $X \in \mathbf{H}$ is the full ∞-subcategory of $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})_{/(X \times (*/\!/G))}$ on those objects whose underlying simplicial object is a G-action.

Observe that an ordinary G-action is equivalent to a G-action over the final object $* \in \mathbf{H}$.

Example 3.22 For a group object $\widehat{G} \in \mathcal{G}\mathrm{rp}(\mathbf{H})$ and an action $P/\!/G$ of $\widehat{G}$ on an object $P \in \mathbf{H}$, let $q\colon (\mathbb{\Delta}^{\mathrm{op}})^{\triangleright} \to \mathbf{H}$ be a colimiting cocone of the simplicial diagram $P/\!/G$ in $\mathbf{H}$. Observing that $(\mathbb{\Delta}^{\mathrm{op}})^{\triangleright} \cong (\mathbb{\Delta}^{\triangleleft})^{\mathrm{op}} \cong (\mathbb{\Delta}_+)^{\mathrm{op}}$, this defines an augmented simplicial object in $\mathbf{H}$, which we denote as

$$q\colon P/\!/G \longrightarrow \operatorname*{colim}_{\mathbb{\Delta}^{\mathrm{op}}}{}^{\mathbf{H}}(P/\!/G) = |P/\!/G|\,.$$

Therefore, the data $P/\!/G \to |P/\!/G|$ form a G-action over $|P/\!/G|$. In particular, the canonical morphism $*/\!/G \to \mathrm{B}G$ is of this form. $\triangleleft$

Another example of a morphism of this type is the collapse morphism $G/\!/G \to *$, as we show now:

Proposition 3.23 *If $\widehat{G}$ is a group object in $\mathbf{H}$, then the canonical morphism*

$$|G/\!/G| \xrightarrow{\simeq} *$$

is an equivalence.

Proof. Since $\mathbf{H}$ is presentable, there exists a combinatorial simplicial model category $\mathcal{M}$ and an equivalence of ∞-categories $\mathbf{H} \simeq N(\mathcal{M}^{\circ})$ [Lur09, Prop. A.3.7.6]. Under this equivalence, colimits in $\mathbf{H}$ over diagrams indexed by ordinary categories correspond to homotopy colimits in $\mathcal{M}$ [Lur09, Cor. 4.2.4.8]. It now suffices to observe that any simplicial object in $\mathcal{M}$ obtained as the decalage of another simplicial object has an augmentation and extra degeneracies [Rie14, Ste12]. $\square$

Any morphism $\widehat{A} \to \widehat{G}$ of group objects induces an action of $\widehat{A}$ on G by the following construction:

Proposition 3.24 *Let $\widehat{f}\colon \widehat{A} \to \widehat{G}$ be a morphism in $\mathcal{G}\mathrm{rp}(\mathbf{H})$. Define a simplicial object $G/\!/A$ as the pullback*

$$\begin{tikzcd} G/\!/A \arrow[r] \arrow[d] & G/\!/G \arrow[d] \\ */\!/A \arrow[r] & */\!/G \end{tikzcd} \tag{3.25}$$

in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. Then, $G/\!/A$ is an action of $\widehat{A}$ on G.

Proof. We check the axioms in Definition 3.15: axiom (1) follows from the pasting law for pullbacks and the diagram

$$\begin{tikzcd} (G/\!/A)_n \arrow[r] \arrow[d] & (G/\!/G)_n \arrow[r] \arrow[d] & G \arrow[d] \\ (*/\!/A)_n \arrow[r] & (*/\!/G)_n \arrow[r] & * \end{tikzcd}$$

in which the right-hand square is a pullback for any $n \in \mathbb{N}_0$ by construction of $G/\!/G$.

Axiom (2) is readily seen from applying the maps d_1 and s_0 to the diagram (3.25), for $n = 0, 1$. Axiom (3) follows since the morphism $G/\!/A \longrightarrow */\!/A$ induced by the above diagram agrees with the morphism obtained by collapsing the first factor G. $\square$

3.3 Principal ∞-bundles

In this subsection, we characterise principal ∞-bundles and group extensions in ∞-topoi. Throughout this section, let $\mathbf{H}$ be an ∞-topos and let $\widehat{G} \in \mathcal{G}\mathrm{rp}(\mathbf{H})$ be a group object in $\mathbf{H}$.

Definition 3.26 [NSS15, Def. 3.4] *A G-*principal ∞-bundle on $X \in \mathbf{H}$ *is a G-action $P/\!/G \to X$ over X such that the augmented simplicial object $P/\!/G \to X$ is a colimiting cocone for the simplicial diagram $P/\!/G \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. In other words, the augmenting map $p\colon P \to X$ induces an equivalence* $\mathrm{colim}^{\mathbf{H}}_{\mathbb{\Delta}^{\mathrm{op}}}(P/\!/G) \xrightarrow{\simeq} X$ *in* $\mathbf{H}$.

A morphism of G-principal ∞-bundles on X, *denoted $(P/\!/G \to X) \longrightarrow (Q/\!/G \to X)$, is a morphism of the underlying G-actions over X. The ∞-category $\mathcal{B}\mathrm{un}_G(X)$ of G-principal ∞-bundles over X is the full ∞-subcategory of $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})_{/(X \times (*/\!/G))}$ (cf. Definition 3.21) on the G-principal ∞-bundles on X.*

Example 3.27 Let $\widehat{G}$ be a group object in $\mathbf{H}$. For any G-action $P/\!/G$ in $\mathbf{H}$, the morphism $P/\!/G \to |P/\!/G|$ is a principal G-bundle in $\mathbf{H}$ over $|P/\!/G|$. As concrete examples of this type, we have already seen that $G/\!/G$ turns G into a principal G-bundle over $* \in \mathbf{H}$ (Proposition (3.23)), and that $*/\!/G$ turns $*$ into a principal G-bundle over $\mathrm{B}G$ (by the definition of $\mathrm{B}G$). $\triangleleft$

We now provide an alternative characterisation of principal ∞-bundles in an ∞-topoi. Let $\widehat{G} \in \mathcal{G}\mathrm{rp}(\mathbf{H})$ be a group object in $\mathbf{H}$, and let $p\colon P/\!/G \to X$ be a G-action over an object $X \in \mathbf{H}$. Let $\imath\colon \mathbb{\Delta}^{\mathrm{op}}_{+,\leq 0} \hookrightarrow \mathbb{\Delta}^{\mathrm{op}}_{+}$ be the inclusion. The identity provides a canonical equivalence

$$\eta\colon \{p\} = \imath^*(P/\!/G \to X) \xrightarrow{\simeq} \imath^*(\check{C}p) = \{p\}$$

in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}_{+,\leq 0}, \mathbf{H}) \simeq \mathcal{F}\mathrm{un}(\Delta^1, \mathbf{H})$. Since right Kan extension is a right adjoint, there is an equivalence

$$\underline{\mathcal{F}\mathrm{un}}(\mathbb{\Delta}^{\mathrm{op}}_{+,\leq 0}, \mathbf{H})\big(\imath^*(P/\!/G \to X), \{p\}\big) \simeq \underline{\mathcal{F}\mathrm{un}}(\mathbb{\Delta}^{\mathrm{op}}_{+}, \mathbf{H})\big((P/\!/G \to X), \check{C}p\big)$$

of mapping spaces (compare also (2.10)). We denote the image of η under this equivalence by

$$\alpha\colon (P/\!/G \to X) \longrightarrow \check{C}p\,.$$

Observe that, by construction, the restriction of α along $\imath$ is η. We will not distinguish notationally between α as defined here and its restriction along the inclusion $\mathbb{\Delta}^{\mathrm{op}} \subset \mathbb{\Delta}^{\mathrm{op}}_{+}$ (since $\alpha_{-1} = 1_X$).

Definition 3.28 *A G-action $P/\!/G \longrightarrow X$ over $X \in \mathbf{H}$ is called* principal *if the canonical morphism $\alpha\colon P/\!/G \longrightarrow \check{C}p$ is an equivalence in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$.*

This is an ∞-categorical version of the principality condition for a group action. It is, in fact, equivalent to the usual principality condition—that the morphism $P \times G \to P \times_X P$ is an equivalence—in the following sense (in particular, this implies the converse to [NSS15, Prop. 3.7]):

Lemma 3.29 *Let $P/\!/G \to X$ be a G-action over $X \in \mathbf{H}$. The following are equivalent:*

(1) The G-action is principal.

(2) The diagram

$$\begin{tikzcd} P \times G \arrow[r, "d_1 = \mathrm{pr}_P"] \arrow[d, "a = d_0"'] & P \arrow[d, "p"] \\ P \arrow[r, "p"'] & X \end{tikzcd} \tag{3.30}$$

is a pullback diagram in $\mathbf{H}$.

Proof. (1) implies (2) since the action $P/\!\!/G \xrightarrow{p} X$ is principal precisely if it is equivalent, as an augmented simplicial object in $\mathbf{H}$, to the Čech nerve $\check{C}p = \mathrm{Ran}_\iota\{p\}$. Thus, the implication follows from Proposition 2.8.

Conversely, (2) also implies (1): we know from Theorem 3.19 that $P/\!\!/G$ is a groupoid object. If we additionally have that (3.30) is a pullback diagram, then we can again apply Proposition 2.8 to obtain the claim. □

We can use Lemma 3.29 to give a characterisation of principal ∞-bundles which can be understood as encoding directly the classical criteria for principal bundles: a locally trivial map $p\colon P \to X$ and a principal G-action over X.

Proposition 3.31 *Let $P/\!\!/G \xrightarrow{p} X$ be a G-action over an object $X \in \mathbf{H}$. The following are equivalent:*

(1) $P/\!\!/G \xrightarrow{p} X$ is a principal ∞-bundle (in the sense of Definition 3.26).

(2) The morphism p is an effective epimorphism and the action $P/\!\!/G$ is principal.

Proof. First, observe that since $P/\!\!/G$ is a groupoid object in $\mathbf{H}$, and since by assumption the canonical morphism $|P/\!\!/G| \to X$ is an equivalence, it follows from Definition 2.11(4) that the canonical morphism $\alpha\colon P/\!\!/G \to \check{C}p$ is an equivalence in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. In particular, p is an effective epimorphism. Further, it has been shown in [NSS15, Prop. 3.7] that if $P/\!\!/G \to X$ is a principal bundle, then the action $P/\!\!/G$ satisfies condition (2) of Lemma 3.29, and so the action is principal.

To see the other direction, consider the commutative diagram

$$\begin{tikzcd} {|P/\!\!/G|} \arrow[rr, "|\alpha|"] \arrow[dr] & & {|\check{C}p|} \arrow[dl] \\ & X & \end{tikzcd}$$

In this case, both the top and the right-hand morphisms in diagram are equivalences. It thus follows that also the left-hand morphism is an equivalence, which amounts to the fact that $P/\!\!/G \to X$ is a principal G-bundle in the sense of Definition 3.26. □

Theorem 3.32 *Let $\mathrm{L}\colon \mathbf{H} \to \mathbf{H}'$ be a functor between ∞-topoi which preserves geometric realisations and finite products. Suppose $\widehat{G}$ is any group object in $\mathbf{H}$.*

(1) L maps G-actions $P/\!\!/G \xrightarrow{p} X$ over $X \in \mathbf{H}$ to $\mathrm{L}G$-actions $\mathrm{L}P/\!\!/\mathrm{L}G \xrightarrow{\mathrm{L}p} \mathrm{L}X$ over $\mathrm{L}X \in \mathbf{H}'$.

(2) If the action $P/\!\!/G \to X$ is a principal G-bundle, then the action $\mathrm{L}P/\!\!/\mathrm{L}G \xrightarrow{\mathrm{L}p} \mathrm{L}X$ is a principal $\mathrm{L}G$-bundle.

Proof. Since L preserves finite products, the first claim follows readily from Definition 3.15.

For the second claim, recall that $P/\!\!/G \to X$ is a principal G-bundle precisely if the map $|P/\!\!/G| \to X$ is an equivalence. Applying the functor L to this morphism, we obtain an equivalence $\mathrm{L}|P/\!\!/G| \xrightarrow{\simeq} \mathrm{L}X$. Since L preserves geometric realisations, and using claim (1), we obtain canonical equivalences

$$|\mathrm{L}P/\!\!/\mathrm{L}G| \xrightarrow{\simeq} \mathrm{L}|P/\!\!/G| \xrightarrow{\simeq} \mathrm{L}X\,,$$

which establishes the action $\mathrm{L}P/\!\!/\mathrm{L}G \xrightarrow{\mathrm{L}p} \mathrm{L}X$ as a principal $\mathrm{L}G$-bundle over X. □

Remark 3.33 The proof of Theorem 3.32 would fail if it were not automatic that group actions are groupoid objects (Theorem 3.19), since L does not preserve groupoid objects in general. ◁

Proposition 3.34 *Let $\widehat{G}$ be a group object in $\mathbf{H}$, and let $P/\!\!/G \to Y$ be a G-principal ∞-bundle in $\mathbf{H}$. For any morphism $f\colon X \to Y$ in $\mathbf{H}$, there is a canonical G-action over X on the pullback $Q := X \times_Y P$ that makes $Q/\!\!/G \to X$ into a G-principal ∞-bundle on X.*

Proof. Let $\mathsf{c}\colon \mathbf{H} \longrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ be the constant-diagram functor. Consider the pullback diagram

$$\begin{tikzcd} \mathsf{c}X \times_{\mathsf{c}Y} (P/\!\!/G) \arrow[r, "\hat{f}"] \arrow[d, "f^*p"'] & P/\!\!/G \arrow[d, "p"] \\ \mathsf{c}X \arrow[r, "\mathsf{c}f"'] & \mathsf{c}Y \end{tikzcd} \tag{3.35}$$

in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ (or, equivalently, in $\mathcal{G}\mathrm{pd}(\mathbf{H})$). For any $[n] \in \mathbb{\Delta}$ there exists a canonical equivalence

$$\big(\mathsf{c}X \times_{\mathsf{c}Y} (P/\!\!/G)\big)_n \simeq X \times_Y (P \times G^{n-1}) \simeq (X \times_Y P) \times G^{n-1}\,.$$

We use Lemma 3.12 to obtain from these equivalences a canonical pair (up to contractible choices) of an object $(X \times_Y P)/\!\!/G \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$, with $((X \times_Y P)/\!\!/G)_n = (X \times_Y P) \times G^n$ for all $n \in \mathbb{N}_0$, together with an equivalence

$$(X \times_Y P)/\!\!/G \xrightarrow{\simeq} \mathsf{c}X \times_{\mathsf{c}Y} (P/\!\!/G) \tag{3.36}$$

of simplicial objects in $\mathbf{H}$. By a slight abuse of notation, we also denote the composition

$$(X \times_Y P)/\!\!/G \xrightarrow{\simeq} \mathsf{c}X \times_{\mathsf{c}Y} (P/\!\!/G) \longrightarrow \mathsf{c}X$$

by f^*p. It follows by construction that $(X \times_Y P)/\!\!/G \xrightarrow{f^*p} X$ is a G-action over X. We are hence left to show that it is a principal ∞-bundle.

To that end, we will show that the morphism

$$|(X \times_Y P)/\!\!/G| \longrightarrow X$$

is an equivalence (compare Definition 3.26). Diagram (3.35) is a diagram of the form $\Delta^1 \times \Delta^1 \longrightarrow \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. Composing with the functor $\mathrm{colim}^{\mathbf{H}}_{\mathbb{\Delta}^{\mathrm{op}}} = |{-}|\colon \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H}) \longrightarrow \mathbf{H}$ we obtain a diagram

$$\begin{tikzcd} {|(X \times_Y P)/\!\!/G|} \arrow[r, "|\hat{f}|"] \arrow[d, "f^*p"'] & {|P/\!\!/G|} \arrow[d, "\simeq"', "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{3.37}$$

in $\mathbf{H}$. The right-hand morphism is an equivalence since $P/\!\!/G \to Y$ is assumed to be a principal ∞-bundle. Using the equivalence (3.36), diagram (3.37) is equivalent to the diagram

$$\begin{tikzcd} {|\mathsf{c}X \times_{\mathsf{c}Y} (P/\!\!/G)|} \arrow[r, "|\hat{f}|"] \arrow[d, "f^*p"'] & {|P/\!\!/G|} \arrow[d, "\simeq"', "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{3.38}$$

By the universality of colimits in $\mathbf{H}$, we have a canonical equivalence

$$|\mathsf{c}X \times_{\mathsf{c}Y} (P/\!\!/G)| \simeq X \times_Y |P/\!\!/G| \,.$$

This establishes that the morphism f^*p in diagram (3.38) is the pullback of an equivalence in $\mathbf{H}$, and hence that f^*p is an equivalence itself. $\square$

One can now even show that every G-principal ∞-bundle arises as a pullback of the bundle $(*/\!\!/G) \to \mathrm{B}G$. This insight is not new, but has been observed in [NSS15, Prop. 3.13, Thm. 3.17] already. However, in Section 4.3 it will be important to have a good understanding of the classifying map of a principal ∞-bundle, and so we include a brief treatment of these maps. We start with two short technical lemmas, before constructing for each G-principal ∞-bundle in $\mathbf{H}$ its classifying map.

Lemma 3.39 *Let $\widehat{G}$ be a group object in $\mathbf{H}$, and let $f\colon * \to \mathrm{B}G$ be the base point of $\mathrm{B}G$. The pullback of the canonical bundle $(*/\!\!/G) \to \mathrm{B}G$ along f agrees with the bundle $G/\!\!/G \to *$.*

Proof. Consider the commutative square

$$\begin{tikzcd}
G/\!\!/G \arrow[r] \arrow[d] & */\!\!/G \arrow[d] \\
* \arrow[r] & \mathsf{c}\mathrm{B}G
\end{tikzcd}$$

in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. By the canonical equivalence $G \simeq \Omega\mathrm{B}G$, this diagram is level-wise a pullback, i.e. it is a pullback diagram in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. That proves the claim by Proposition 3.34. $\square$

Definition 3.40 *A G-principal ∞-bundle $P/\!\!/G \to X$ is* trivial *if it is equivalent in $\mathcal{B}\mathrm{un}_G(X)$ to the* trivial G-principal ∞-bundle $X \times (G/\!\!/G) \to X$, *i.e. if there is an equivalence of simplicial objects in $\mathbf{H}_{/X}$ between $P/\!\!/G$ and $X \times (G/\!\!/G)$ that commutes with the canonical morphisms to $*/\!\!/G$.*

Lemma 3.41 [NSS15, Prop. 3.12] *For every G-principal ∞-bundle $P/\!\!/G \to X$ in $\mathbf{H}$, there exists an effective epimorphism $U \to X$ such that the pullback bundle $U \times_X (P/\!\!/G)$ is trivial.*

Proof. We give an alternative proof to [NSS15]. Given a G-principal ∞-bundle $P/\!\!/G \to X$ in $\mathbf{H}$, consider the effective epimorphism $P \to X$ and the pullback bundle $P \times_X (P/\!\!/G)$. We have a commutative diagram

$$\begin{tikzcd}
P \times (G/\!\!/G) \arrow[dr, dashed, "\psi"] \arrow[drr, bend left, "a\times 1"] \arrow[ddr, bend right, "\mathrm{pr}"'] & & \\
& P \times_X (P/\!\!/G) \arrow[r] \arrow[d] & P/\!\!/G \arrow[d] \\
& \mathsf{c}P \arrow[r] & \mathsf{c}X
\end{tikzcd}$$

in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$, where $a \times 1$ acts on P with the first copy of G and as the identity on the remaining copies of G. The induced morphism ψ is a morphism of G-principal ∞-bundles (since the triangles in the diagram commute and since $a \times 1$ is a morphism of G-actions). It is thus equivalent to a morphism

$$\psi'\colon \check{C}(P \times G \to P) \longrightarrow \check{C}\big((P \times_X (P/\!\!/G)) \longrightarrow P\big)$$

of Čech nerves over P. The level-zero component of ψ' is precisely the equivalence $P \times G \to P \times_X P$ which establishes that $P/\!\!/G \to X$ is principal (cf. Proposition 3.29). Since ψ' is the image of ψ under the right Kan extension Ran_ι (compare Definition 2.7), it follows that ψ', and hence ψ, is an equivalence. □

Proposition 3.42 *For every G-principal ∞-bundle $P/\!\!/G \to X$ in $\mathbf{H}$, the diagram*

$$\begin{tikzcd} P/\!\!/G \arrow[r, "p"] \arrow[d] & */\!\!/G \arrow[d] \\ \mathsf{c}X \arrow[r, "|p|"'] & \mathsf{c}\mathrm{B}G \end{tikzcd} \tag{3.43}$$

is a pullback diagram in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$: there is an equivalence $(P/\!\!/G \to X) \simeq \times_{\mathrm{B}G}(*/\!\!/G)$ of G-principal ∞-bundles over X. In particular, every G-principal ∞-bundle is a pullback of the bundle $*/\!\!/G \to \mathrm{B}G$.

This is a refinement of [NSS15, Prop. 3.13] to a statement on the level of simplicial objects, rather than only on their zeroth level.

Proof. Consider the diagram

$$\begin{tikzcd} & G/\!\!/G \arrow[rr] \arrow[dd] & & */\!\!/G \arrow[dd] \\ P \times (G/\!\!/G) \arrow[ur] \arrow[rr, crossing over] \arrow[dd] & & P/\!\!/G \arrow[ur] & \\ & * \arrow[rr] & & \mathsf{c}\mathrm{B}G \\ \mathsf{c}P \arrow[ur] \arrow[rr] & & \mathsf{c}X \arrow[ur] \arrow[from=uu, crossing over] & \end{tikzcd} \tag{3.44}$$

in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. Here, the front and back squares are pullbacks (by Lemmas 3.41 and 3.39), and the diagram is obtained as a morphism of pullback diagrams. We need to show that the right-hand face is a pullback square in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$.

First, we show that the top square of (3.44) is a pullback. By Lemma 3.8 it suffices to check this level-wise: at simplicial level $n = 0$, it is trivial. For $n \in \mathbb{N}$, the square consists of the the image under the functor $(-) \times G^{n-1}$ of the diagram

$$\begin{tikzcd} P \times G \arrow[r, "a"] \arrow[d, "\mathrm{pr}_1"'] & P \arrow[d] \\ G \arrow[r] & * \end{tikzcd} \tag{3.45}$$

This a pullback diagram: there is a commutative diagram

$$\begin{tikzcd} P \times G \arrow[dr, dashed, "g"] \arrow[drr, bend left, "\mathrm{pr}_0"] \arrow[ddr, bend right, "\mathrm{pr}_1"'] & & \\ & P \times G \arrow[r, "a"] \arrow[d, "\mathrm{pr}_1"'] & P \arrow[d] \\ & G \arrow[r] & * \end{tikzcd}$$

in which the dashed morphism is given by $g = (a \times 1_G) \circ (1_P \times \mathrm{inv} \times 1_G) \circ (1_P \circ \Delta_G)$, where $\Delta_G \colon G \to G^2$ is the diagonal morphism, and where $\mathrm{inv}\colon G \to G$ is the choice of an inverse in G: since the group object $\widehat{G} \in \mathcal{G}\mathrm{rp}(\mathbf{H})$ is in particular a groupoid object, we have a diagram

$$\widehat{G}_1 \simeq \widehat{G}_1 \times * \longrightarrow \widehat{G}_1 \times \widehat{G}_1 \simeq \widehat{G}(\Lambda^2_0) \xleftarrow{\ \simeq\ } \widehat{G}_2 \xrightarrow{\ d_0\ } \widehat{G}_1\,,$$

where we have used the characterisation of groupoid objects as certain category objects from Proposition A.5. Choosing an inverse for the right-facing morphism defines the morphism inv.

Since g is an equivalence (because $\widehat{G}$ is a group object), diagram (3.45) is a pullback in $\mathbf{H}$, and since the span category $\{0,1\} \leftarrow \{0\} \rightarrow \{0,2\}$ has contractible nerve, the pullback (3.45) is preserved by $(-) \times G^{n-1}$ (see Lemma A.9). We thus obtain that the top square in diagram (3.44) is a pullback.

Next, we prove that the bottom square of (3.44) is a pullback. We define $Y := * \times_{\mathrm{B}G} X \in \mathbf{H}$, and we consider the diagram (omitting constant-diagram functors)

$$\begin{tikzcd}
Y \times_X (P/\!\!/G) \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & * \arrow[d] \\
P/\!\!/G \arrow[r] & X \arrow[r] & \mathrm{B}G
\end{tikzcd}$$

Both squares in this diagram are pullbacks in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$, so that the pasting law yields a canonical equivalence of simplicial objects

$$Y \underset{X}{\times} (P/\!\!/G) \simeq * \underset{\mathrm{B}G}{\times} (P/\!\!/G)\,.$$

Observe that $Y \times_X (P/\!\!/G) \to Y$ is a G-principal ∞-bundle by Proposition 3.34, so that

$$Y \simeq \big|Y \underset{X}{\times} (P/\!\!/G)\big| \simeq \big| * \underset{\mathrm{B}G}{\times} (P/\!\!/G)\big|\,.$$

Now we use that the morphism $P/\!\!/G \to \mathrm{B}G$ factors through $*/\!\!/G$ (by Definitions 3.15 and 3.26) and that $* \times_{\mathrm{B}G} (*/\!\!/G) \simeq G/\!\!/G$ (by Lemma 3.39). Applying the pasting law to the diagram

$$\begin{tikzcd}
(P/\!\!/G) \underset{(*/\!\!/G)}{\times} (G/\!\!/G) \arrow[r] \arrow[d] & G/\!\!/G \arrow[r] \arrow[d] & * \arrow[d] \\
P/\!\!/G \arrow[r] & */\!\!/G \arrow[r] & \mathrm{B}G
\end{tikzcd}$$

in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$, in which both squares are pullbacks, we obtain a canonical equivalence

$$* \underset{\mathrm{B}G}{\times} (P/\!\!/G) \simeq (P/\!\!/G) \underset{(*/\!\!/G)}{\times} (G/\!\!/G)\,.$$

The right-hand side is precisely the pullback described by the top square in diagram (3.44), and is hence canonically equivalent to $P \times (G/\!\!/G)$ in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. Thus, it follows that

$$Y \simeq \big| * \underset{\mathrm{B}G}{\times} (P/\!\!/G)\big| \simeq \big|P \times (G/\!\!/G)\big| \simeq P\,.$$

The last equivalence can be seen either by combining Proposition 3.23 with the fact that $|-|$ preserves finite products (because $\mathbb{\Delta}^{\mathrm{op}}$ is sifted [Lur09]), or simply by recalling that $P \times (G/\!\!/G) \to P$ is a G-principal ∞-bundle on P. This shows that the bottom square in (3.44) is a pullback.

Finally, we prove that the right-hand square in (3.44) is a pullback as well. Consider the commutative diagram of solid arrows

$$\begin{tikzcd} P/\!\!/G \arrow[dr, dashed, "\varphi"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X \underset{\mathrm{B}G}{\times} (*/\!\!/G) \arrow[r] \arrow[d] & */\!\!/G \arrow[d] \\ & X \arrow[r] & \mathrm{B}G \end{tikzcd}$$

which induces an essentially unique morphism φ of simplicial objects in $\mathbf{H}$. By the commutativity of the right-hand triangle in this diagram φ is even a morphism of G-actions, and by the commutativity of the left-hand triangle it is a morphism of G-actions over X. Since its source and target are G-principal ∞-bundles, φ is equivalent to a morphism of Čech nerves

$$\varphi' \colon \check{C}(P \to X) \longrightarrow \check{C}\big((X \underset{\mathrm{B}G}{\times} *) \to X\big)\,.$$

That is, φ' is the image under Ran_ι (compare Definition 2.7) of the square

$$\begin{tikzcd} P \arrow[r, "\varphi'_0"] \arrow[d] & X \underset{\mathrm{B}G}{\times} * \arrow[d] \\ X \arrow[r, equal] & X \end{tikzcd}$$

This φ'_0 is an equivalence precisely because the bottom square of (3.44) is a pullback. Consequently, the morphism φ is an equivalence in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$, and thus the right-hand face in (3.44) is a pullback. $\square$

Corollary 3.46 *Let $P/\!\!/G \to X$ be a G-principal ∞-bundle in $\mathbf{H}$. For any morphism $x \colon * \to X$, we have a pullback diagram*

$$\begin{tikzcd} G/\!\!/G \arrow[r] \arrow[d] & P/\!\!/G \arrow[d] \\ * \arrow[r, "x"'] & X \end{tikzcd}$$

in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. In particular, any fibre of $P \to X$ is canonically equivalent to G in $\mathbf{H}$.

Remark 3.47 In fact, for any group object $\widehat{G}$ in $\mathbf{H}$ and any object $X \in \mathbf{H}$, there is an equivalence

$$\mathcal{B}\mathrm{un}_G(X) \simeq \underline{\mathbf{H}}(X, \mathrm{B}G)$$

between the ∞-category of G-principal ∞-bundles on X and the mapping space $\underline{\mathbf{H}}(X, \mathrm{B}G)$ [NSS15, Thm. 3.17]. This implies that every morphism of principal G-bundles on X is an equivalence. Proposition 3.42 feeds into the proof of this equivalence by showing that the functor $\underline{\mathbf{H}}(X, \mathrm{B}G) \to \mathcal{B}\mathrm{un}_G(X)$, sending a morphism $X \to \mathrm{B}G$ to the principal ∞-bundle $X \times_{\mathrm{B}G} (*/\!\!/G)$, is fully faithful. $\triangleleft$

In particular, under the equivalence of Remark 3.47, the morphism $|p| \colon X \to \mathrm{B}G$ in diagram (3.43) is a classifying morphism for the bundle $P/\!\!/G \to X$.

Proposition 3.48 *Let $\mathrm{L} \colon \mathbf{H} \to \mathbf{H}'$ be a functor of ∞-topoi that preserves finite products and geometric realisations. If $P/\!\!/G \to X$ is a G-principal ∞-bundle in $\mathbf{H}$, classified (up to canonical equivalence) by a morphism $|p| \colon X \to \mathrm{B}G$, then the $\mathrm{L}G$-principal ∞-bundle $\mathrm{L}P/\!\!/\mathrm{L}G \longrightarrow \mathrm{L}X$ (compare Theorem 3.32) in $\mathbf{H}'$ is classified by the morphism $|\mathrm{L}p| \simeq \mathrm{L}|p|$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
& \mathrm{L}(P/\!\!/G) \arrow[rr, "\mathrm{L}p"] \arrow[dd] & & \mathrm{L}(*/\!\!/G) \arrow[dd] \\
\mathrm{L}P/\!\!/\mathrm{L}G \arrow[ur, "\simeq"] \arrow[rr, "q"'] \arrow[dd] & & */\!\!/\mathrm{L}G \arrow[ur, "\simeq"'] \arrow[dd] & \\
& \mathrm{L}|P/\!\!/G| \arrow[rr, "\mathrm{L}|p|"] & & \mathrm{LB}G \\
|\mathrm{L}P/\!\!/\mathrm{L}G| \arrow[ur, "\simeq"] \arrow[rr, "|q|"'] & & \mathrm{BL}G \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

The morphism q is the canonical morphism induced from the collapse morphism $\mathrm{L}P \to *$. The front face of this diagram is a pullback in $\mathbf{H}'$, witnessing $|q|$ as the classifying morphism $\mathrm{L}X \to \mathrm{BL}G$ of the bundle $\mathrm{L}P/\!\!/\mathrm{L}G \longrightarrow \mathrm{L}X$. Since all diagonal morphisms are equivalences, the back face of the diagram is a pullback as well, showing that $\mathrm{L}|p|$ is a classifying morphism of the $\mathrm{L}G$-principal ∞-bundle $\mathrm{L}(P/\!\!/G) \longrightarrow \mathrm{L}X$, which is equivalent to the bundle $\mathrm{L}P/\!\!/\mathrm{L}G \longrightarrow \mathrm{L}X$. Finally, since the diagonal morphisms arise from the natural equivalences $\mathrm{L} \circ |-| \simeq |-| \circ \mathrm{L}$, it follows that $|q| \simeq |\mathrm{L}p|$. $\square$

We now state several alternative characterisations of group extensions in ∞-topoi. These clarify the relation between the original notion of an extension of group objects from Definition 3.10 and more direct categorifications of several perspectives on group extensions in $\mathcal{S}\mathrm{et}$. The last of these alternative characterisations will be important in Section 4.3.

Theorem 3.49 *Let $\mathbf{H}$ be an ∞-topos, and let $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{p}} \widehat{H}$ be a sequence of morphisms in $\mathcal{G}\mathrm{rp}(\mathbf{H})$. The following are equivalent:*

(1) $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{p}} \widehat{H}$ is an extension of group objects in $\mathbf{H}$ (see Definition 3.10).

(2) The sequence $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{p}} \widehat{H}$ is a fibre sequence in $\mathcal{G}\mathrm{rp}(\mathbf{H})$.

(3) The sequence $A \xrightarrow{\iota} G \xrightarrow{p} H$ is a fibre sequence in $\mathbf{H}$.

(4) The map $p\colon G \to H$ together with the action $G/\!\!/A$ of A on G induced by ι define a principal A-bundle over H.

Proof. (1) $\Rightarrow$ (3): This implication has been proven in [NSS15] already. We import the proof for completeness: consider the diagram

$$\begin{tikzcd}
A \arrow[r, "\iota'"] \arrow[d] & G \arrow[r] \arrow[d, "p'"] & * \arrow[d] & \\
* \arrow[r] & H \arrow[r] \arrow[d] & \mathrm{B}A \arrow[r] \arrow[d, "\mathrm{B}\iota"'] & * \arrow[d] \\
& * \arrow[r] & \mathrm{B}G \arrow[r, "\mathrm{B}p"'] & \mathrm{B}H
\end{tikzcd} \tag{3.50}$$

in $\mathbf{H}$. Every square in diagram (3.50) is a pullback square (this assumes (1)). It thus follows that the sequence $A \xrightarrow{\iota'} G \xrightarrow{p'} H$ is a fibre sequence in $\mathbf{H}$. By this construction, the morphisms ι' and p' coincide with the morphisms $\Omega \circ \mathrm{B}(\iota)$ and $\Omega \circ \mathrm{B}(p)$, respectively. The natural equivalence $\Omega \circ \mathrm{B} \simeq \mathrm{id}_{\mathbf{H}}$ then yields that also $A \xrightarrow{\iota} G \xrightarrow{p} H$ is a fibre sequence in $\mathbf{H}$. Observe that every vertical morphism in diagram (3.50) is an effective epimorphism since $* \to \mathrm{B}G$ is an effective epimorphism for every group object $\widehat{G} \in \mathcal{G}\mathrm{rp}(\mathbf{H})$ and since effective epimorphisms are stable under pullback.

(3) $\Rightarrow$ (2): This follows from Proposition 3.8.

(2) $\Rightarrow$ (1): This implication holds because the delooping functor B is a right adjoint.

(4) $\Rightarrow$ (3): Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & H \arrow[r] & \mathrm{B}A \end{tikzcd}$$

The outer rectangle is a pullback diagram by construction of $\mathrm{B}A$, and the right-hand square is a pullback by Corollary 3.46. Thus, the implication follows by applying the pasting law for pullbacks.

(1) $\Rightarrow$ (4): Recall that the morphism $* \to \mathrm{B}H$ is an effective epimorphism and that effective epimorphisms are stable under pullback. It then readily follows from the pasting diagram (3.50) that $p\colon G \to H$ is an effective epimorphism. By Proposition 3.34, the morphism $H \to \mathrm{B}A$ induces an A-principal ∞-bundle over H as the pullback of $*/\!\!/A \to \mathrm{B}A$. We know from the pasting construction (3.50) that the level-zero object of the induced pullback principal ∞-bundle is (canonically) equivalent to G. Further, we have that the map $G \to H$ induced from the pullback construction coincides with $p\colon G \to H$ under this equivalence (as in the proof of "(1) $\Rightarrow$ (3)").

It thus remains to show that the action of A on G obtained via the upper central pullback square in (3.50) and the pullback construction in Proposition 3.34 coincides with the action $G/\!\!/A$ induced from the morphism $\widehat{\iota}\colon \widehat{A} \to \widehat{G}$ in $\mathcal{G}\mathrm{rp}(\mathbf{H})$ (cf. Proposition 3.24).

To that end, we consider the following diagram in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$:

$$\begin{tikzcd} & G/\!\!/G \arrow[rr] \arrow[dd] & & */\!\!/G \arrow[dd] \\ (H \times_{\mathrm{B}A} *)/\!\!/A \arrow[ur, "g"] \arrow[rr, crossing over] \arrow[dd, "\mathrm{pr}_H"'] & & */\!\!/A \arrow[ur, "\iota"'] & \\ & * \arrow[rr] & & \mathsf{c}\mathrm{B}G \\ \mathsf{c}H \arrow[ur] \arrow[rr] & & \mathsf{c}\mathrm{B}A \arrow[ur, "\mathsf{c}\mathrm{B}\iota"'] \arrow[from=uu, crossing over] & \end{tikzcd} \tag{3.51}$$

The simplicial object in the front, upper-left corner is obtained from the pullback construction for A-principal ∞-bundles (Proposition 3.34). Therefore, the front face of the cube in diagram (3.51) is a pullback square in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. Further, the bottom square is a pullback by assuming (1) and by (3.50), and the back square is a pullback diagram because it is so objectwise (see also Lemma 3.39). Forgetting for a moment about the two upper left objects, the remaining diagram can be viewed as a morphism of cospans in $\mathbf{H}$. This induces an essentially unique morphism of the left and right pullback squares, and this is how we define the upper-left diagonal morphism, labelled g, and the remaining coherence data of the cube (3.51).

Applying the pasting law to the front and bottom square, we deduce that the diagonal square

$$\begin{tikzcd} (H \times_{\mathrm{B}A} *)/\!\!/A \arrow[r] \arrow[d] & */\!\!/A \arrow[d] \\ * \arrow[r] & \mathsf{c}\mathrm{B}G \end{tikzcd}$$

is a pullback diagram in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$. Its vertical morphisms are the diagonals of the left and right faces of the cube in diagram (3.51). This, in turn, yields a commutative diagram

$$
\begin{tikzcd}
(H \times_{\mathrm{B}A} *)/\!\!/ A \arrow[r] \arrow[d] & G/\!\!/ G \arrow[r] \arrow[d] & * \arrow[d] \\
*/\!\!/ A \arrow[r] & */\!\!/ G \arrow[r] & \mathsf{c}\mathrm{B}G
\end{tikzcd}
$$

in which the outer square and the right-hand square are pullbacks. (These squares are precisely the top and the back squares of diagram (3.51).) The pasting law for pullbacks thus implies that the left-hand square is a pullback diagram in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$ as well. By construction of the action $G/\!\!/ A$ induced from the morphism $\widehat{\iota}\colon \widehat{A} \to \widehat{G}$ in $\mathcal{G}\mathrm{rp}(\mathbf{H})$ (Proposition 3.24) as a pullback in $\mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H})$, we thus obtain a canonical equivalence

$$(H \times_{\mathrm{B}A} *)/\!\!/ A \simeq G/\!\!/ A$$

in $\mathcal{G}\mathrm{pd}(\mathbf{H})$. That is, the two actions of A on G agree, as desired. Since the action $(H \times_{\mathrm{B}A} *)/\!\!/ A$ is principal by Proposition 3.34, it follows that the action $G/\!\!/ A$ is principal as well. This proves the claim. □

Corollary 3.52 *Suppose $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{p}} \widehat{H}$ is an extension of group objects in $\mathbf{H}$. Then, there is a canonical equivalence in $\mathbf{H}$,*

$$|G/\!\!/ A| \simeq H\,.$$

This is the ∞-categorical analogue of the canonical isomorphism $G/A \cong H$ for ordinary (set-theoretic) group extensions $A \to G \to H$.

Corollary 3.53 *Let $\mathrm{L}\colon \mathbf{H} \to \mathbf{H}'$ be a functor between ∞-topoi that preserves geometric realisations and finite products. Suppose $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{G} \xrightarrow{\widehat{p}} \widehat{H}$ is an extension of group objects in $\mathbf{H}$. Then, the sequence $\mathrm{L}\widehat{A} \xrightarrow{\mathrm{L}\widehat{\iota}} \mathrm{L}\widehat{G} \xrightarrow{\mathrm{L}\widehat{p}} \mathrm{L}\widehat{H}$ is an extension of group objects in $\mathbf{H}'$.*

Proof. This statement now follows from combining Theorem 3.32 and Theorem 3.49. □

4 Homotopy-theoretic smooth string group models

In this section, we present a definition of string group extensions within the ∞-category $\mathbf{H}_\infty$ of smooth spaces. This relies on the singular complex functor $\mathrm{S}_e\colon \mathbf{H}_\infty \to \mathbf{S}$ for smooth spaces from Section 2 and the theory of group extensions in ∞-topoi from Section 3. We begin by recalling the definition of a string group extension in the ∞-category $\mathbf{S}$ of spaces. Then, we use our results thus far to transfer this definition to $\mathbf{H}_\infty$ along the functor S_e, leading to a homotopy-theoretic definition of smooth string group extensions (Definition 4.2).

After recalling some background on bundle gerbes in Section 4.2, we provide new smooth models for the string group in Section 4.3, building on recent constructions of smooth 2-group extensions in [BMS]. (Already in that paper, evidence was given that these smooth 2-group extensions can model the string group; here we provide a full formal framework and proof for that conjecture.)

4.1 The definition of smooth string groups

The definition of a string group via the Whitehead tower (see Section 1) is originally purely homotopy-theoretic. In particular, in a string group extension $A \to \mathrm{String}(H) \to H$ the extending group A is not fixed, but only its homotopy type. So far, to our knowledge there does not exist a definition of string group extensions in a smooth context that contains this flexibility—the extending group A is usually chosen ad hoc to be some smooth version of $\mathsf{B}\mathrm{U}(1)$. Here, we provide a smooth version of the original homotopy-theoretic definition (see Definition 4.2). In particular, only the underlying homotopy type of the extending smooth group A is fixed in this definition.

Recall that any compact, simple, and simply connected Lie group H is also 2-connected and satisfies $\mathrm{H}^3(H;\mathbb{Z}) \cong \mathbb{Z}$. Any Lie group H defines a group object $\widehat{H}$ in the ∞-topos of spaces $\mathbf{S}$. We start by reformulating the definition of a string group extension of topological groups within the ∞-category of spaces:

Definition 4.1 *Let H be a compact, simple, and simply connected Lie group, and denote by $\widehat{H} \in \mathcal{G}\mathrm{rp}(\mathbf{S})$ its associated group object in $\mathbf{S}$. A* string group extension of H *is an extension of group objects $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{\mathrm{String}(H)} \xrightarrow{\widehat{p}} \widehat{H}$ in $\mathbf{S}$ such that*

(1) A is an Eilenberg-MacLane space $K(\mathbb{Z},2)$, and

(2) under the isomorphism

$$\pi_0\underline{\mathbf{S}}(H, \mathrm{B}A) \cong \pi_0\underline{\mathbf{S}}\big(H, K(\mathbb{Z},3)\big) \cong \mathrm{H}^3(H;\mathbb{Z}) \cong \mathbb{Z}\,,$$

the classifying morphism $H \to \mathrm{B}A$ of the A-principal ∞-bundle $\mathrm{String}(H)/\!\!/A \to H$ (compare Remark 3.47 and Theorem 3.49(4)) represents a generator of $\mathbb{Z}$.

Given condition (1), condition (2) is equivalent to saying that the map $G \to H$ of spaces induces an isomorphism $\pi_i(G) \to \pi_i(H)$ for $i \neq 3$ and that $\pi_3(G) \cong 0$. This is a consequence of the Hurewicz Theorem, the Universal Coefficient Theorem, and the long exact sequence of homotopy groups associated to a (homotopy) fibre sequence of spaces. That is, G is a 3-connected approximation to H.

Recall the ∞-topos $\mathbf{H}_\infty = \mathcal{P}(\mathcal{C}\mathrm{art})$ from Section 2.1. There we also introduced the localisation $L_I\mathbf{H}_\infty$ of $\mathbf{H}_\infty$ at the set $I = \{c \times \mathbb{R} \to c \,|\, c \in \mathcal{C}\mathrm{art}\}$ and the smooth singular complex functor $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$. Further, recall the fully faithful embedding $\underline{(-)} \colon \mathcal{M}\mathrm{fd} \hookrightarrow \mathbf{H}_\infty$, with $\underline{M}(c) = \mathcal{M}\mathrm{fd}(c, M)$; under this embedding, any Lie group H gives rise to a group object $\widehat{\underline{H}}$ in $\mathbf{H}_\infty$. We can now use our results from Section 3 to transfer the definition of a string group extension to the ∞-topos $\mathbf{H}_\infty$:

Definition 4.2 *Let H be a compact, simple, and simply connected Lie group. A* smooth string group extension of H *is an extension $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{\mathrm{String}(\underline{H})} \xrightarrow{\widehat{p}} \widehat{\underline{H}}$ of group objects in $\mathbf{H}_\infty$ such that its image under S_e is a string group extension in $\mathbf{S}$.*

Note that by Theorem 3.53 the functor S_e maps group extensions in $\mathbf{H}_\infty$ to group extensions in $\mathbf{S}$. Further, even though S_e induces an equivalence between $\mathbf{S}$ and the localisation $L_I\mathbf{H}_\infty$ rather than the full ∞-category $\mathbf{H}_\infty$, we do not need to demand that A, $\mathrm{String}(H)$ and H are local objects, because S_e sends all I-local equivalences in $\mathbf{H}_\infty$ to equivalences in $\mathbf{S}$ (Theorem 2.2(1)).

Definition 4.2 is a generalisation as well as a weakening of the following approach to smooth string group extensions (see, for instance, [FRS16]): there, one works in the localisation $L_\tau\mathbf{H}_\infty$ of $\mathbf{H}_\infty$ at the differentiably good open coverings $\{c_a \to c\}_{a\in\Lambda}$ of cartesian spaces $c \in \mathcal{C}\mathrm{art}$, and one defines a string group extension of H via the pullback

$$\begin{tikzcd} \mathrm{BString}(\underline{H}) \arrow[r] \arrow[d] & * \arrow[d] \\ \mathrm{B}\underline{H} \arrow[r, "\frac{1}{2}p_1"'] & \mathrm{B}^3\underline{\mathsf{U}(1)} \end{tikzcd} \tag{4.3}$$

Here, $\frac{1}{2}p_1$ denotes the fractional first Pontryagin class, which is a generator of $\mathrm{H}^4(\mathrm{B}\underline{H};\mathbb{Z}) \cong \mathbb{Z}$. However, this definition of $\mathrm{String}(\underline{H})$ is considerably stricter than the original perception of $\mathrm{String}(H)$ as a 3-connected covering of H by another group object (Definition 4.1). For instance, the definition of a string group extension based on (4.3) enforces that $\mathrm{String}(\underline{H}) \to \underline{H}$ is a $\mathrm{B}\underline{\mathsf{U}(1)}$-principal ∞-bundle. (Note that if $\mathbf{H}$ is an ∞-topos and $\widehat{A} \in \mathcal{G}\mathrm{rp}(\mathbf{H})$ is a group object whose multiplication lifts to an $\mathbb{E}_2$-algebra structure, then $\mathrm{B}A$ is canonically the level-zero object of a group object in $\mathbf{H}$ [NSS15].) However, from the purely homotopy-theoretic point of view, not the actual fibre of this map should be fixed, but the homotopy type of its underlying space (which must be a $K(\mathbb{Z};2)$). Definition 4.2 emphasises this latter, homotopy-theoretic aspect of string group extensions.

More concretely, for smooth string group extensions $\widehat{A} \xrightarrow{\widehat{\iota}} \widehat{\mathrm{String}(\underline{H})} \xrightarrow{\widehat{p}} \widehat{\underline{H}}$ in the sense of Definition 4.2 it is enough if there is an I-local equivalence $A \simeq \underline{\mathsf{U}(1)}$ in $\mathbf{H}_\infty$. Therefore, this setup is considerably more general than working with the pullback (4.3); in particular, two different smooth string group extensions of a Lie group H need not be equivalent in $\mathbf{H}_\infty$, but only in $L_I\mathbf{H}_\infty$.

Remark 4.4 It will be interesting to see a Lie-algebra version of Definition 4.2. The ∞-groups $\widehat{A} \in \mathcal{G}\mathrm{rp}(\mathbf{H}_\infty)$ that we allow to appear in string group extensions can have much larger Lie algebras than those which appear in the stricter definition via (4.3). This is true, in particular, for the smooth string group extension we present in Section 4.3 below. There might hence be a Lie-algebra version of I-local equivalences of group objects in $\mathbf{H}_\infty$. $\triangleleft$

We now work towards establishing a new string group model that fits Definition 4.2, but does not fit into the pullback (4.3). (The reason will be that the extension is not by $\mathrm{B}\underline{\mathsf{U}(1)}$, but by a much larger, but also homotopically correct, ∞-group.)

4.2 Bundle gerbes and their symmetries

Before we can present our smooth string group extension, we need to recall some background on bundle gerbes. We will not give full definitions or details here; for these, we refer the reader to [Wal07, Bun17, BS17, BSS18]. Bundle gerbes provide an explicit, geometric model for categorified line bundles. We point out that there also exists a notion of connection on a bundle gerbe, but here we will only be working with bundle gerbes *without* connection. (This is the main technical cause for the distinction between our smooth string group model and that in [FRS16].) Further, bundle gerbes are a very geometric, 2-categorical model for higher line bundles, so that our smooth string group model will arise as an explicitly defined smooth 2-group, making it particularly tangible.

To any manifold M, we can assign a symmetric monoidal 2-groupoid $(\mathcal{G}\mathrm{rb}(M), \otimes)$ of bundle gerbes on M. Given a bundle gerbe $\mathcal{G} \in \mathcal{G}\mathrm{rb}(M)$, the monoidal groupoid $\mathcal{G}\mathrm{rb}(M)(\mathcal{G},\mathcal{G})$ of automorphisms of $\mathcal{G}$ is canonically equivalent to the symmetric monoidal groupoid $(\mathrm{HLB}(M), \otimes)$ of hermitean line bundles on M with the usual tensor product (which we also denote by $\otimes$). Note that $(\mathrm{HLB}(M), \otimes)$ is even a *2-group*; that is, it is a symmetric monoidal groupoid in which every object has an inverse with respect to the monoidal product. Every smooth map $f\colon N \to M$ of manifolds induces a monoidal 2-functor

$$f^*\colon \mathcal{G}\mathrm{rb}(M) \longrightarrow \mathcal{G}\mathrm{rb}(N)\,.$$

Isomorphism classes of gerbes are in canonical bijection with the third integer cohomology of M: there is an isomorphism of abelian groups

$$\pi_0\big(\mathcal{G}\mathrm{rb}(M), \otimes\big) \cong \mathrm{H}^3(M;\mathbb{Z})\,. \tag{4.5}$$

The class associated to a gerbe $\mathcal{G}$ under this isomorphism is called the *Dixmier-Douady class of* $\mathcal{G}$.

We let $\mathbf{H}_{\leq 1}$ denote the following 2-category: its objects are functors $\pi\colon \mathcal{C} \to \mathcal{C}\mathrm{art}$ that are Grothendieck fibrations in groupoids (that is, π is a Grothendieck fibration and all its fibres are groupoids). Its morphisms $(\pi\colon \mathcal{C} \to \mathcal{C}\mathrm{art}) \longrightarrow (\pi'\colon \mathcal{C}' \to \mathcal{C}\mathrm{art})$ are functors $F\colon \mathcal{C} \to \mathcal{C}'$ such that $\pi' \circ F = \pi$, and its 2-morphisms are natural transformations $\eta\colon F \to F'$ such that $\pi'\eta$ is the identity natural transformation $1_{\mathcal{C}\mathrm{art}} \to 1_{\mathcal{C}\mathrm{art}}$. Note that the 2-category $\mathbf{H}_{\leq 1}$ is canonically equivalent to the 2-category of pseudo-functors $\mathcal{C}\mathrm{art}^{\mathrm{op}} \to \mathcal{G}\mathrm{pd}$ from $\mathcal{C}\mathrm{art}^{\mathrm{op}}$ to the 2-category of groupoids via the Grothendieck construction. We make the following definitions; for more background, see [BMS, SP11].

Definition 4.6 [BMS] *The* 2-category of smooth 2-groups *is the 2-category of group objects in the 2-category* $\mathbf{H}_{\leq 1}$.

Example 4.7 Let H be a Lie group. We associate to it the following category, denoted by $\int \underline{H}$: its objects are pairs (c, h) of a cartesian space $c \in \mathcal{C}\mathrm{art}$ and a smooth map $h\colon c \to H$. A morphism $(c, h) \to (c', h')$ is a smooth map $f\colon c \to c'$ such that $h' \circ f = h$, and the category $\int \underline{H}$ comes with a canonical projection functor $\int \underline{H} \to \mathcal{C}\mathrm{art}$. The product on H-valued maps turns $\int \underline{H}$ into a smooth 2-group in the sense of Definition 4.6. Note that $\int \underline{H}$ is simply the Grothendieck construction of the presheaf of sets $\underline{H}$ on $\mathcal{C}\mathrm{art}$. $\triangleleft$

Example 4.8 Let M be a manifold, and define a category HLB^M as follows: its objects are pairs (c, L) of a cartesian space $c \in \mathcal{C}\mathrm{art}$ and a hermitean line bundle $L \in \mathrm{HLB}(c \times M)$. A morphism $(c, L) \to (c', L')$ is a pair (f, ψ) of a smooth map $f\colon c \to c'$ and an isomorphism $\psi\colon L \to (f \times 1_M)^*L'$ of hermitean line bundles over c. This category comes with a projection functor $\mathrm{HLB}^M \to \mathcal{C}\mathrm{art}$. The tensor product of hermitean line bundles turns HLB^M into a smooth 2-group. $\triangleleft$

Let M be a manifold, and let $\mathcal{G} \in \mathcal{G}\mathrm{rb}(M)$ be a gerbe on M. Further, let H be a connected Lie group acting smoothly on M from the left; we denote the action by $\Phi\colon H \times M \to M$. Given these data, we define a category $\mathrm{Sym}(\mathcal{G})$ as follows: an object in $\mathrm{Sym}(\mathcal{G})$ is a triple $(c, h, \mathcal{A})$, where $c \in \mathcal{C}\mathrm{art}$ is a cartesian space and where $h\colon c \to H$ is a smooth map. These give rise to a smooth map $\Phi_h\colon c \times M \to c \times M$, defined as the composition

$$\Phi_h\colon c \times M \xrightarrow{\Delta \times 1_M} c \times c \times M \xrightarrow{1_c \times h \times 1_M} c \times H \times M \xrightarrow{1_c \times \Phi} c \times M\,,$$

where $\Delta\colon c \to c \times c$ is the diagonal map. Then, $\mathcal{A}$ is a 1-isomorphism

$$\mathcal{A}\colon \mathrm{pr}_M^*\mathcal{G} \longrightarrow \Phi_h^*\mathcal{G}$$

of gerbes on the manifold $c \times M$. A morphism $(c, h, \mathcal{A}) \to (c', h', \mathcal{A}')$ is a pair (f, ψ), where f is a smooth map $f\colon c \to c'$ such that $h' \circ f = h$, and where ψ is a 2-isomorphism $\psi\colon \mathcal{A} \longrightarrow (f \times 1_M)^*\mathcal{A}'$. (Here we have implicitly used that there is a canonical 1-isomorphism $(f \times 1_M)^*\Phi_{h'}^*\mathcal{G} \cong \Phi_h^*\mathcal{G}$.) Observe that there is a projection functor $p\colon \mathrm{Sym}(\mathcal{G}) \to \int \underline{H}$, acting as $(c, h, \mathcal{A}) \mapsto (c, h)$ and $(f, \psi) \mapsto f$.

Remark 4.9 In this set-up, the following statements hold true:

(1) The connectedness of H ensures that the functor p is surjective on objects. It is an essentially surjective Grothendieck fibration in groupoids [BMS, Thm. 5.27].

(2) The equivalence $\mathcal{G}\mathrm{rb}(N)(\mathcal{G}', \mathcal{G}') \simeq (\mathrm{HLB}(N), \otimes)$ for any gerbe $\mathcal{G}'$ on any manifold N implies that the diagram

$$\begin{tikzcd} \mathrm{HLB}^M \arrow[r] \arrow[d] & \mathrm{Sym}(\mathcal{G}) \arrow[d] \\ * \arrow[r, "e_H"'] & \textstyle\int \underline{H} \end{tikzcd}$$

is a pullback in $\mathbf{H}_{\leq 1}$, where e_H is the functor that sends $c \in \mathcal{C}\mathrm{art}$ to the constant map $c \to H$ with value the unit element of H. Since p is a Grothendieck fibration in groupoids, this pullback is even a homotopy pullback [BMS, App. A.1]. $\triangleleft$

Theorem 4.10 [BMS, Thms. 5.23, 5.27] *Let $\Phi \colon H \times M \to M$ be a smooth action of a connected Lie group H on a manifold M. Let $\mathcal{G} \in \mathcal{G}\mathrm{rb}(M)$ be a bundle gerbe on M.*

(1) $\mathrm{Sym}(\mathcal{G})$ *is a smooth 2-group.*

(2) The functor p fits into a sequence

$$\mathrm{HLB}^M \xrightarrow{i} \mathrm{Sym}(\mathcal{G}) \xrightarrow{p} \textstyle\int \underline{H} \tag{4.11}$$

of smooth 2-groups. Further, p is a Grothendieck fibration in groupoids and surjective on objects.

The nerve functor $N \colon \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_\infty$ induces a functor $N \colon \mathbf{H}_{\leq 1} \to \mathbf{H}_\infty$ (where we have used the canonical equivalence between $\mathbf{H}_{\leq 1}$ and the 2-category of pseudo-functors $\mathcal{C}\mathrm{art}^{\mathrm{op}} \to \mathcal{G}\mathrm{pd}$ from $\mathcal{C}\mathrm{art}^{\mathrm{op}}$ to the 2-category of groupoids). This functor, in particular, preserves final objects and products, so that it maps smooth 2-groups to group objects in $\mathbf{H}_\infty$. Our smooth string group model will be obtained by applying this functor to the sequence (4.11).

4.3 A smooth string group model

We can now state the main theorem of this section. It provides a new smooth model for smooth string group extensions which fits Definition 4.2, but which lies outside the scope of the stricter definition via the pullback (4.3). Note that applying the nerve functor N to $\int \underline{H} \in \mathbf{H}_{\leq 1}$ yields the familiar presheaf of spaces $\underline{H} \in \mathbf{H}_\infty$, defined via $\underline{H}(c) = \mathcal{M}\mathrm{fd}(c, H)$ for cartesian spaces $c \in \mathcal{C}\mathrm{art}$.

Theorem 4.12 *Let H be a compact, simple, simply connected Lie group. We consider the left-action of H on itself via left multiplication. Let $\mathcal{G} \in \mathcal{G}\mathrm{rb}(H)$ be a gerbe on H whose class in* $\mathrm{H}^3(H; \mathbb{Z}) \cong \mathbb{Z}$ *is a generator (see* (4.5)*). Then, the sequence*

$$\widehat{N\,\mathrm{HLB}^H} \xrightarrow{\widehat{Ni}} \widehat{N\mathrm{Sym}(\mathcal{G})} \xrightarrow{\widehat{Np}} \widehat{\underline{H}} \tag{4.13}$$

is a smooth string group extension of H.

The proof of Theorem 4.12 will occupy the remainder of this section. By Definition 4.2 we have to show that the sequence (4.13) is an extension of group objects in $\mathbf{H}_\infty$ and that its image under the functor $\mathrm{S}_e \colon \mathbf{H}_\infty \to \mathbf{S}$ is a string group extension in $\mathbf{S}$ in the sense of Definition 4.1.

Proposition 4.14 *The sequence* (4.13) *is an extension of group objects in the ∞-topos $\mathbf{H}_\infty$.*

Proof. The nerve functor $N\colon \mathcal{C}\mathrm{at} \to \mathcal{C}\mathrm{at}_\infty$ is a right adjoint and hence maps products in $\mathbf{H}_{\leq 1}$ to products in $\mathbf{H}_\infty$, and it maps the final object in $\mathbf{H}_{\leq 1}$ to the final object in $\mathbf{H}_\infty$. Consequently, it preserves group objects and group actions.

We will now use the characterisation of group extensions from Theorem 3.49(4) to show that the sequence (4.13) of group objects in $\mathbf{H}_\infty$ is an extension of group objects. That is, we have to show that $N\mathrm{Sym}(\mathcal{G})$ with the $N\mathrm{HLB}^H$-action induced by the morphism $\widehat{Ni}$ (cf. Proposition 3.24) is an $N\,\mathrm{HLB}^M$-principal ∞-bundle over $\underline{H}$. According to the characterisation of principal ∞-bundles in Proposition 3.31, it suffices to prove that the morphism Np is an effective epimorphism and that the action of $N\,\mathrm{HLB}^M$ on $N\mathrm{Sym}(\mathcal{G})$ is principal.

We start by showing that the morphism Np is an effective epimorphism: by [BMS, Sec. 5.1] the restriction $p_{|c}$ of p to any fibre is essentially surjective, hence $Np_{|c}$ is surjective on connected components. Since $\mathbf{H}_\infty$ is a presheaf ∞-topos (in which limits and colimits are computed objectwise), a morphism in $\mathbf{H}_\infty$ is an effective epimorphism if and only if it is objectwise an effective epimorphism in $\mathbf{S}$. The effective epimorphisms in $\mathbf{S}$, however, are exactly those morphisms which are surjective on connected components [Lur09, Cor. 7.2.1.15]. Therefore, Np is an effective epimorphism in $\mathbf{H}_\infty$.

The action of $N\,\mathrm{HLB}^H$ on $N\mathrm{Sym}(\mathcal{G})$ is principal with respect to Np as was shown in [BMS, Thm. 5.27]. (There, the principality condition was shown on the level of the sequence (4.11) of smooth 2-groups—this suffices for the ∞-categorical context used here because of Lemma 3.29 and because the nerve functor is a right adjoint.) Therefore, the sequence (4.13) is a group extension in $\mathbf{H}_\infty$. $\square$

It thus remains to show that the image of the sequence (4.13) under S_e is a string group extension in $\mathbf{S}$. To that end, we first show the following lemma:

Lemma 4.15 *Let X be a connected manifold with $\mathrm{H}^2(X;\mathbb{Z}) \cong 0$.*

(1) The object $N\,\mathrm{HLB}^X \in \mathbf{H}_\infty$ is equivalent to the object $\mathrm{B}(\underline{\mathsf{U}(1)^X})$.

(2) Suppose that also $\pi_1(N) = 0$. Then, the object $N\,\mathrm{HLB}^X \in \mathbf{H}_\infty$ is I-locally equivalent to $\mathrm{B}\underline{\mathsf{U}(1)} \in \mathbf{H}_\infty$.

Both equivalences are even established by morphisms of group objects in $\mathbf{H}_\infty$.

Since S_e maps I-local equivalences in $\mathbf{H}_\infty$ to equivalences of spaces, applying Lemma 4.15 to $X = H$ establishes axiom (1) of Definition 4.1 for the image of the sequence (4.13) under the functor S_e.

Proof. We proceed in parallel to the proof of [BMS, Thm. 8.7]: since any $c \in \mathcal{C}\mathrm{art}$ is contractible and since $\mathrm{H}^2(X;\mathbb{Z}) \cong 0$, it follows that any hermitean line bundle on $c \times N$ is trivialisable. Consequently, $\mathrm{HLB}^X(c)$ is equivalent to the groupoid with one object and morphisms given by the group $\underline{\mathsf{U}(1)}(c)$ of smooth maps from c to $\mathsf{U}(1)$. This induces an equivalence $N\,\mathrm{HLB}^X \simeq \mathrm{B}(\underline{\mathsf{U}(1)^X})$ in $\mathbf{H}_\infty$, which extends to a morphism of group objects in $\mathbf{H}_\infty$. This proves (1).

Next, since $\pi_1(X)$ is trivial, there exists a smooth homotopy equivalence $\mathrm{ev}_e\colon \underline{\mathsf{U}(1)^X} \to \underline{\mathsf{U}(1)}$, given by restricting a smooth map $c \times X \to \mathsf{U}(1)$ to $c \times \{x\}$, where $x \in X$ is any point. A homotopy inverse to ev_x is given by pulling a smooth map $c \to \mathsf{U}(1)$ back along the projection $c \times X \to c$ [BMS, Lemma 8.9]. In particular, ev_x is an I-local equivalence [Bunb, Cor. 3.16].

Observe that ev_x induces a morphism of group objects

$$\widehat{\mathrm{ev}}_x\colon \big(\underline{\mathsf{U}(1)^X}\big)^\frown \longrightarrow \widehat{\underline{\mathsf{U}(1)}}\,.$$

Since ev_x is an I-local equivalence in $\mathbf{H}_\infty$ and I-local equivalences are closed under finite products (Proposition 2.4), the morphism $\widehat{\mathrm{ev}}_x$ is a levelwise I-local equivalence of simplicial objects in $\mathbf{H}_\infty$.

Further, the class W_I of I-local equivalences in $\mathbf{H}_\infty$ is strongly saturated [Lur09, Lemma 5.5.4.11]. In particular, the full subcategory of $\mathcal{F}\mathrm{un}(\Delta^1, \mathbf{H}_\infty)$ on the I-local equivalences is stable under colimits. Therefore, taking the colimit in $\mathbf{H}_\infty$ of simplicial objects (i.e. taking geometric realisations), we obtain an I-local equivalence

$$\mathrm{B}\,\mathrm{ev}_x \colon \mathrm{B}\big(\underline{\mathsf{U}(1)}^{\underline{X}}\big) \longrightarrow \mathrm{B}\underline{\mathsf{U}(1)}$$

in $\mathbf{H}_\infty$. Composing with the morphism constructed in part (1), we now obtain the desired I-local equivalence $N\,\mathrm{HLB}^X \longrightarrow \mathrm{B}\underline{\mathsf{U}(1)}$ in $\mathbf{H}_\infty$. $\square$

We are thus left to show that the sequence of group objects in $\mathbf{S}$ obtained by applying the functor S_e to the sequence (4.13) of group objects in $\mathbf{H}_\infty$ satisfies axiom (2) of Definition 4.1. That is, we have to show that the principal ∞-bundle of spaces

$$\big(\mathrm{S}_e N\mathrm{Sym}(\mathcal{G})\big) /\!\!/ \big(\mathrm{S}_e N\,\mathrm{HLB}^H\big) \longrightarrow \mathrm{S}_e\underline{H}$$

represents a generator of $\mathrm{H}^3(H;\mathbb{Z}) \cong \mathbb{Z}$. This is best checked using Čech cohomology.

Recall that a differentiably good open covering of $c \in \mathcal{C}\mathrm{art}$ is an open covering $\{c_a \hookrightarrow c\}_{a\in\Lambda}$ such that every finite non-empty intersection of the images of the patches c_a is again a cartesian space. The differentiably good open coverings endow $\mathcal{C}\mathrm{art}$ with a Grothendieck pretopology τ [FSS12, Sch].

Lemma 4.16 *Let $X \in \mathcal{M}\mathrm{fd}$ be a connected manifold, and let $k \in \mathbb{N}_0$. Then,*

(1) In $\mathbf{H}_\infty$, there is an I-local equivalence

$$\mathrm{B}^k\big(\underline{\mathsf{U}(1)}^{\underline{X}}\big) \simeq \mathrm{B}^k\underline{\mathsf{U}(1)}\,.$$

(2) Suppose X is simply connected. Then, the presheaf $\mathrm{B}^k\big(\underline{\mathsf{U}(1)}^{\underline{X}}\big)$ satisfies descent with respect to the Grothendieck pretopology τ.

Proof. Ad (1): This is an iteration of the argument in the proof of Lemma 4.15(2).

Ad (2): We prove this claim by induction. For $k = 0$, we have to check that the functor $\mathcal{C}\mathrm{art}^{\mathrm{op}} \to \mathbf{S}$, $c \mapsto \mathcal{M}\mathrm{fd}(c \times X, \mathsf{U}(1))$ satisfies descent with respect to good open coverings of c. However, this follows directly from the fact that, for any manifold Y, the assignment functor

$$\mathcal{O}\mathrm{p}(Y)^{\mathrm{op}} \to \mathcal{S}\mathrm{et}\,, \qquad U \mapsto \mathcal{M}\mathrm{fd}\big(U, \mathsf{U}(1)\big)$$

defines a sheaf on Y, where $\mathcal{O}\mathrm{p}(Y)$ is the category of open subsets of Y and their inclusions.

Suppose that $\mathrm{B}^l\big(\underline{\mathsf{U}(1)}^{\underline{X}}\big)$ is a sheaf on $\mathcal{C}\mathrm{art}$ for all $l = 0, \ldots, k$. Let $\mathcal{U} = \{c_a \hookrightarrow c\}_{a\in\Lambda}$ be a differentiably good open covering of c. We have to show that the canonical morphism

$$q^* \colon \mathrm{B}^{k+1}\big(\underline{\mathsf{U}(1)}^{\underline{X}}\big)(c) \longrightarrow \lim_{n\in\mathbb{\Delta}}{}^{\mathbf{S}}\, \underline{\mathbf{H}}_\infty\big(\check{C}\mathcal{U}_n, \mathrm{B}^{k+1}\underline{\mathsf{U}(1)}^{\underline{X}}\big) \tag{4.17}$$

is an equivalence of spaces. Here, $\check{C}\mathcal{U} \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{H}_\infty)$ is the Čech nerve of the covering $\mathcal{U}$.

We first show that q^* is essentially surjective; that is, it induces a bijection on isomorphism classes of objects. Since limits and colimits in $\mathbf{H}_\infty = \mathcal{F}\mathrm{un}(\mathcal{C}\mathrm{art}^{\mathrm{op}}, \mathbf{S})$ are computed pointwise, we have isomorphisms

$$\pi_0\,\mathrm{B}^{k+1}\big(\underline{\mathsf{U}(1)}^{\underline{X}}\big)(c) \cong \pi_0\,\mathrm{B}^{k+1}\big(\underline{\mathsf{U}(1)}^{\underline{X}}(c)\big) = *\,.$$

On the other hand, we have that

$$\pi_0 \lim_{n\in\mathbb{\Delta}}{}^{\mathbf{S}}\, \underline{\mathbf{H}}_\infty\big(\check{C}\mathcal{U}_n, \mathrm{B}^{k+1}\underline{\mathsf{U}(1)}^{\underline{X}}\big) \cong \check{\mathrm{H}}^{k+1}\big(\mathcal{U}; \underline{\mathsf{U}(1)}^{\underline{X}}\big) \cong \mathrm{H}^{k+2}(c;\mathbb{Z}) \cong *\,, \tag{4.18}$$

where on the right-hand side we have the usual Čech cohomology group with respect to the covering $\mathcal{U}$ of the sheaf of abelian groups on c given by

$$\mathcal{O}\mathrm{p}(c)^{\mathrm{op}} \to \mathcal{A}\mathrm{b}\,, \qquad U \mapsto \mathcal{M}\mathrm{fd}\big(U \times X, \mathsf{U}(1)\big)\,.$$

By a slight abuse of notation, we also denote this sheaf by $\underline{\mathsf{U}(1)}^{\underline{X}}$.

This Čech cohomology is indeed isomorphic to the full sheaf cohomology of $\underline{\mathsf{U}(1)}^{\underline{X}}$: first, we observe that since X is simply connected, there is a short exact sequence

$$\mathbb{Z} \longrightarrow \underline{\mathbb{R}}^{\underline{X}} \longrightarrow \underline{\mathsf{U}(1)}^{\underline{X}}\,.$$

We further observe that the sheaf $\underline{\mathbb{R}}^{\underline{X}}$ is fine (it admits partitions of unity, for instance those induced from the canonical map $\underline{\mathbb{R}} \to \underline{\mathbb{R}}^{\underline{X}}$) when seen as a sheaf on the open subsets of any manifold. Therefore, for any manifold Y, we have a canonical isomorphism

$$\mathrm{H}^l\big(Y; \underline{\mathsf{U}(1)}^{\underline{X}}\big) \cong \mathrm{H}^{l+1}(Y;\mathbb{Z})$$

for every $l \geq 1$. From this, we see that

$$\mathrm{H}^l\big(d; \underline{\mathsf{U}(1)}^{\underline{X}}\big) \cong *$$

for every cartesian space $d \in \mathcal{C}\mathrm{art}$ and $l \geq 1$. Since the covering $\mathcal{U}$ of c is differentiably good, it follows from [Bry08, Thm. 1.3.6] that there is a canonical isomorphism

$$\check{\mathrm{H}}^k\big(\mathcal{U}; \underline{\mathsf{U}(1)}^{\underline{X}}\big) \cong \mathrm{H}^k\big(c; \underline{\mathsf{U}(1)}^{\underline{X}}\big)\,.$$

This proves the isomorphisms in (4.18), and hence that the morphism q^* from (4.17) is bijective on connected components.

It remains to check that q^* is an isomorphism on all higher homotopy groups. We will achieve this by comparing the automorphisms of the unique object in the source and target space of q^*. On the source side, this automorphism space is given as the pullback of spaces

$$\begin{tikzcd}
\Omega \mathrm{B}^{k+1}\big(\underline{\mathsf{U}(1)}^{\underline{X}}\big)(c) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \mathrm{B}^{k+1}\big(\underline{\mathsf{U}(1)}^{\underline{X}}\big)(c)
\end{tikzcd}$$

and there is a canonical equivalence

$$\Omega \mathrm{B}^{k+1}\big(\underline{\mathsf{U}(1)}^{\underline{X}}\big)(c) \simeq \mathrm{B}^k\big(\underline{\mathsf{U}(1)}^{\underline{X}}(c)\big)\,.$$

On the target side of q^*, the automorphism space of the (essentially) unique object is the pullback

$$\begin{tikzcd}
\Omega \lim^{\mathbf{S}}_{n \in \mathbb{\Delta}} \underline{\mathbf{H}}_\infty\big(\check{C}\mathcal{U}_n, \mathrm{B}^{k+1}\underline{\mathsf{U}(1)}^{\underline{X}}\big) \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \lim^{\mathbf{S}}_{n \in \mathbb{\Delta}} \underline{\mathbf{H}}_\infty\big(\check{C}\mathcal{U}_n, \mathrm{B}^{k+1}\underline{\mathsf{U}(1)}^{\underline{X}}\big)
\end{tikzcd}$$

Since limits in $\mathbf{H}_\infty$ are computed objectwise, there are canonical equivalences

$$\begin{aligned}\Omega \lim_{n\in\mathbb{\Delta}}{}^{\mathbf{S}} \underline{\mathbf{H}}_\infty\big(\check{C}\mathcal{U}_n, \mathrm{B}^{k+1}\underline{\mathsf{U}(1)}^{\underline{X}}\big) &\simeq \lim_{n\in\mathbb{\Delta}}{}^{\mathbf{S}} \underline{\mathbf{H}}_\infty\big(\check{C}\mathcal{U}_n, \Omega\, \mathrm{B}^{k+1}\underline{\mathsf{U}(1)}^{\underline{X}}\big)\\ &\simeq \lim_{n\in\mathbb{\Delta}}{}^{\mathbf{S}} \underline{\mathbf{H}}_\infty\big(\check{C}\mathcal{U}_n, \mathrm{B}^{k}\underline{\mathsf{U}(1)}^{\underline{X}}\big)\,.\end{aligned}$$

However, by the induction hypothesis, the presheaf $\mathrm{B}^k\underline{\mathsf{U}(1)}^{\underline{X}}$ is a sheaf, so that q^* induces an equivalence between the automorphism spaces. This proves that q^* is indeed an equivalence. $\square$

Another application of [Buna, Thm. 1.2] now implies that the presheaf of spaces

$$\underline{\mathbf{H}}_\infty\big(-, \mathrm{B}^n\big(\underline{\mathsf{U}(1)}^{\underline{H}}\big)\big) \colon \mathcal{M}\mathrm{fd}^{\mathrm{op}} \longrightarrow \mathbf{S}$$

satisfies descent with respect to open coverings (and even surjective submersions). Consequently, given any open covering $\{c_a \hookrightarrow H\}_{a\in\Lambda}$, whose Čech nerve we denote by $\mathcal{V} \to \underline{H}$, the canonical morphism

$$\underline{\mathbf{H}}_\infty\big(\underline{H}, \mathrm{B}^n\big(\underline{\mathsf{U}(1)}^{\underline{H}}\big)\big) \cong \lim_{\mathbb{\Delta}}{}^{\mathbf{S}} \underline{\mathbf{H}}_\infty\big(\mathcal{V}, \mathrm{B}^n\big(\underline{\mathsf{U}(1)}^{\underline{H}}\big)\big)\,.$$

is an equivalence of spaces because $\mathrm{B}^n\big(\underline{\mathsf{U}(1)}^{\underline{H}}\big)$ is local with respect to the class of morphisms consisting of the Čech nerves of coverings. Therefore, there is an isomorphism

$$\pi_0\underline{\mathbf{H}}_\infty\big(\underline{H}, \mathrm{B}^n\big(\underline{\mathsf{U}(1)}^{\underline{H}}\big)\big) \cong \check{\mathrm{H}}^n\big(H; \underline{\mathsf{U}(1)}^{\underline{H}}\big)\,, \tag{4.19}$$

which can be represented explicitly by composing any morphism $\underline{H} \to \mathrm{B}^n\big(\underline{\mathsf{U}(1)}^{\underline{H}}\big)$ with any Čech nerve $\mathcal{V} \to \underline{H}$ of an open covering of H. (Alternatively, this can be seen directly in the presentation of $\mathbf{H}_\infty$ by the projective model structure on simplicial presheaves on $\mathcal{C}\mathrm{art}$.) Let $\mathrm{ev}_e \colon \underline{\mathsf{U}(1)}^{\underline{H}} \longrightarrow \underline{\mathsf{U}(1)}$ be the morphism induced by pullback along the base-point inclusion $* \hookrightarrow \underline{H}$. We obtain a commutative diagram

$$\begin{tikzcd} \pi_0\underline{\mathbf{H}}_\infty\big(\underline{H}, \mathrm{B}^n\big(\underline{\mathsf{U}(1)}^{\underline{H}}\big)\big) \arrow[r, "(\mathrm{B}^n \mathrm{ev}_e)_*"] \arrow[d, "\cong"] & \pi_0\underline{\mathbf{H}}_\infty\big(\underline{H}, \mathrm{B}^n\underline{\mathsf{U}(1)}\big) \arrow[d, "\cong"] \\ \check{\mathrm{H}}^n\big(H; \underline{\mathsf{U}(1)}^{\underline{H}}\big) \arrow[r, "(\mathrm{ev}_e)_*"'] & \check{\mathrm{H}}^n\big(H; \underline{\mathsf{U}(1)}\big) \end{tikzcd} \tag{4.20}$$

It was shown in [BMS, Prop. 8.11] that the bottom horizontal morphism is an isomorphism for all $n \in \mathbb{N}$ (with $n > 0$). Consider the morphisms

$$\begin{aligned}\pi_0\underline{\mathbf{H}}_\infty\big(\underline{H}, \mathrm{B}^n\big(\underline{\mathsf{U}(1)}^{\underline{H}}\big)\big) &\longrightarrow \pi_0\underline{\mathbf{S}}\big(\mathrm{S}_e\underline{H}, \mathrm{S}_e\mathrm{B}^n\big(\underline{\mathsf{U}(1)}^{\underline{H}}\big)\big)\\ &\cong \pi_0\underline{\mathbf{S}}\big(H, \mathrm{B}^n\big(\mathrm{S}_e\underline{\mathsf{U}(1)}^{\underline{H}}\big)\big)\\ &\cong \pi_0\underline{\mathbf{S}}\big(H, \mathrm{B}^n\mathrm{S}_e\underline{\mathsf{U}(1)}\big)\\ &\cong \pi_0\underline{\mathbf{S}}\big(H, \mathrm{B}^n\mathsf{U}(1)\big)\,.\end{aligned} \tag{4.21}$$

The first morphism is applying the functor S_e. For the second morphism we have used [Bunb, Thm. 5.1]: for every manifold $X \in \mathcal{M}\mathrm{fd}$, there is a canonical equivalence $\mathrm{S}_e\underline{X} \simeq X$ in $\mathbf{S}$. Further, here we have used that S_e commutes with B (Proposition 3.5). For the third morphism, we have used that the inclusion $\underline{\mathsf{U}(1)} \hookrightarrow \underline{\mathsf{U}(1)}^{\underline{H}}$ is an I-local equivalence in $\mathbf{H}_\infty$: since H is connected and simply connected, this morphism is a smooth homotopy equivalence by [BMS, Lemma 8.9], and by [Bunb, Cor. 3.16] any smooth homotopy equivalence is an I-local equivalence. The last morphism again uses [Bunb, Thm. 5.1]. Since S_e preserves finite products, the equivalence $\mathrm{S}_e\underline{\mathsf{U}(1)} \simeq \mathsf{U}(1)$ in $\mathbf{S}$ is compatible with the group structure[3] on $\mathsf{U}(1)$.

We can describe the map (4.21) more explicitly as follows: we have already seen above that any element in $\pi_0\underline{\mathbf{H}}_\infty(\underline{H}, \mathrm{B}^n(\underline{\mathsf{U}(1)^H}))$ can be described as a smooth $\mathsf{U}(1)^H$-valued Čech cocycle with respect to a (differentiably good) open cover $\mathcal{V}$ of H. Under the map (4.21), these data are sent first to the same Čech cocycle, but seen as a map of spaces, and then this resulting Čech cocycle is composed with the evaluation $\mathsf{U}(1)^H \to \mathsf{U}(1)$ at the unit element in H. Therefore, using the canonical isomorphism $\pi_0\underline{\mathbf{S}}(H, \mathrm{B}^n\mathsf{U}(1)) \cong \check{\mathrm{H}}^n(H; \mathsf{U}(1))$ and combining this with the maps (4.20) and (4.21) we obtain a commutative diagram of abelian groups

$$\begin{tikzcd} \pi_0\underline{\mathbf{H}}_\infty\big(\underline{H}, \mathrm{B}^n\big(\underline{\mathsf{U}(1)^H}\big)\big) \arrow[r] \arrow[d] & \pi_0\underline{\mathbf{S}}\big(H, \mathrm{B}^n\big(\mathrm{S}_e\underline{\mathsf{U}(1)^H}\big)\big) \arrow[d, "(\mathrm{ev}_e)_*"] \\ \check{\mathrm{H}}^n\big(H; \underline{\mathsf{U}(1)^H}\big) \arrow[r, "(\mathrm{ev}_e)_*"'] & \check{\mathrm{H}}^n\big(H; \mathsf{U}(1)\big) \end{tikzcd} \tag{4.22}$$

In this diagram, the left-hand vertical morphism is invertible as argued before (4.19). The bottom morphism is an isomorphism by [BMS, Prop. 8.11] and the fact that Čech cohomology and abelian sheaf cohomology are isomorphic on manifolds. The right-hand vertical morphism is invertible as a consequence of the isomorphisms in (4.20) and the fact that the map $\mathrm{ev}_e \colon \underline{\mathsf{U}(1)^H} \to \underline{\mathsf{U}(1)}$ is an I-local equivalence.

Combining diagram (4.22) with Proposition 3.48 and Lemma 4.15, we obtain that the class in $\mathrm{H}^3(H;\mathbb{Z}) \cong \check{\mathrm{H}}^2(H, \mathsf{U}(1))$ defined by the $N\mathrm{HLB}^H$-principal ∞-bundle

$$\big(\mathrm{Sym}(\mathcal{G})\big) /\!\!/ N\mathrm{HLB}^H \longrightarrow \underline{H} \tag{4.23}$$

in $\mathbf{H}_\infty$ agrees with the class defined by the principal ∞-bundle

$$\big(\mathrm{S}_e N \mathrm{Sym}(\mathcal{G})\big) /\!\!/ \big(\mathrm{S}_e N\, \mathrm{HLB}^H\big) \longrightarrow \mathrm{S}_e\underline{H} \simeq H$$

in $\mathbf{S}$. Here we have used that there is an equivalence $\widehat{N\,\mathrm{HLB}}^H \simeq \mathrm{B}\underline{\mathsf{U}(1)^H}$ in $\mathcal{G}\mathrm{rp}(\mathbf{H}_\infty)$, so that

$$\pi_0\underline{\mathbf{H}}_\infty\big(\underline{H}, \mathrm{B}\, N\, \mathrm{HLB}^H\big) \simeq \pi_0\underline{\mathbf{H}}_\infty\big(\underline{H}, \mathrm{B}^2\underline{\mathsf{U}(1)^H}\big)\,.$$

(Again, one can alternatively see the coincidence of the cohomology classes more explicitly on the level of Čech cocycles in the presentation of $\mathbf{H}_\infty$ by the simplicial model category $\mathcal{H}^p_\infty$: a smooth bundle represented by a smooth $\mathsf{U}(1)^H$-valued cocycle on H gets sent to the topological bundle represented by the same Čech cocycle interpreted as a collection of continuous maps.) It thus remains to compute the cohomology class associated to these bundles. In [BMS, Sec. 8] it has been shown that the class in $\mathrm{H}^3(H;\mathbb{Z})$ of the bundle (4.23) agrees with the class in $\mathrm{H}^3(H;\mathbb{Z})$ that classifies the gerbe $\mathcal{G}$ under the isomorphism (4.5). Since we started our construction from a so-called *basic* gerbe, i.e. one whose Dixmier-Douady class is a generator of $\mathrm{H}^3(H;\mathbb{Z})$, this concludes the proof of Theorem 4.12.

Remark 4.24 We conclude with the following remarks:

(1) In [BMS], we suggested the smooth 2-group extension (4.13) as a model for the string group extension of H. However, the necessary formalism to make this precise was not available then—its development was the main goal of the present paper.

[3] This can also be seen directly: the comparison map $\mathrm{S}_e M \to M$ sends a smooth map $\Delta^k_e \to M$ to the restriction $|\Delta^k| \to M$, which is a k-simplex in $\mathrm{Sing}(M)$—see [Bunb, Secs. 4, 5] for details.

(2) Moreover, in [BMS, Sec. 5.5] we also presented a second smooth 2-group extension

$$\mathrm{HLB}^H \xrightarrow{\ i\ } \mathfrak{Des}_{\mathsf{L}} \xrightarrow{\ p\ } \textstyle\int \underline{H} \tag{4.25}$$

of H; its construction uses a connection on $\mathcal{G}$ as auxiliary data and relies heavily on a notion of parallel transport on a gerbe $\mathcal{G}$ with connection, as developed in [BMS]. The extension (4.25) is then obtained as an explicit homotopy-coherent version of an associated bundle construction. By [BMS, Thm. 5.33], there is an equivalence (in $\mathbf{H}_{\leq 1}$) between the smooth 2-group extension in (4.25) and (4.11), so that we automatically obtain an equivalence between the ∞-bundles in $\mathbf{H}_\infty$ they induce under the nerve functor. Therefore, given the input of a basic gerbe $\mathcal{G}$ on H, by Theorem 4.12 the extension (4.25) also gives rise to a second (but equivalent) smooth string group extension

$$\widehat{N\mathrm{HLB}^H} \longrightarrow \widehat{N\mathfrak{Des}_{\mathsf{L}}} \longrightarrow \widehat{\underline{H}}$$

of H, for any compact, connected and simply connected Lie group H. ◁

A Actions and category objects

In this appendix, we prove Theorem 3.19; that is, we show that group actions $P/\!/G$ in ∞-topoi (as in Definition 3.15) are automatically groupoid objects.

Definition A.1 *Let $\mathcal{C}$ be an ∞-category. A* category object *in $\mathcal{C}$ is a simplicial object $X \in \mathfrak{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ such that for every $n \in \mathbb{N}_0$ the pullback $X_1 \times_{X_0} \cdots \times_{X_0} X_1$ exists in $\mathcal{C}$ and the morphism*

$$X_n \longrightarrow X_1 \underset{X_0}{\times} \cdots \underset{X_0}{\times} X_1\,,$$

induced by the spine decomposition $[n] \cong [1] \sqcup_{[0]} \cdots \sqcup_{[0]} [1]$ of finite ordered sets, is an equivalence.

Suppose $\mathcal{C}$ has a final object. In analogy with Definition 3.3, a *monoid object* in $\mathcal{C}$ is a category object $*/\!/M \in \mathfrak{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ such that $(*/\!/M)_0 \simeq *$ is a final object in $\mathcal{C}$. As for group objects, we set $M := (*/\!/M)_1$, and it follows that there are canonical natural equivalences $(*/\!/M)_n \simeq M^{n-1}$. Therefore, by Lemma 3.12 we may assume, without loss of generality, that we have $(*/\!/M)_n = M^{n-1}$ for any $n \in \mathbb{N}_0$. We set $M/\!/M := \mathrm{Dec}^0(*/\!/M) \in \mathfrak{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$. Monoid objects can act on objects in their ambient ∞-category. A monoid action is defined precisely like a group action (Definition 3.15), but for the reader's convenience, we spell out the definition:

Definition A.2 *Let $\mathcal{C}$ be an ∞-category with a final object, and let $*/\!/M$ be a monoid object in $\mathcal{C}$. Let $P \in \mathcal{C}$ be an object in $\mathcal{C}$. An* action of $*/\!/M$ on P *is a simplicial object $P/\!/M \in \mathfrak{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ such that*

(1) for each $n \in \mathbb{N}_0$, we have $(P/\!/M)_n = P \times M^n$,

(2) the morphism $d_1 \colon P \times M \to P$ agrees with the canonical projection onto P, the morphism $s_0 \colon P \to P \times M$ agrees with the morphism $1_P \times (* \to M)$, and

(3) the collapse morphism $P \to *$ induces a morphism $P/\!/M \to */\!/M$ in $\mathfrak{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$.

We set $a := d_0 \colon P \times M \to P$. The pasting law for pullbacks implies that there are canonical equivalences of morphisms between $d_0 \colon P \times M^n \to P \times M^{n-1}$ and $a \times 1_{M^{n-1}} \colon P \times M^n \to P \times M^{n-1}$, and similarly between $d_n \colon P \times M^n \to P \times M^{n-1}$ and the projection onto the first n factors.

Proposition A.3 *Let $\mathcal{C}$ be an ∞-category with pullbacks and a final object, let $*/\!\!/M \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ be a monoid object in $\mathcal{C}$, and let $P/\!\!/M \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ be an action of $*/\!\!/M$ on an object $P \in \mathcal{C}$. Then, $P/\!\!/M$ is a category object in $\mathcal{C}$.*

Proof. Consider the diagram

$$\begin{tikzcd}[column sep=small] (P/\!\!/M)_1 \arrow[dr, "d_0"'] & & (P/\!\!/M)_1 \arrow[dl, "d_1"] & = & P \times M \arrow[dr, "a"'] & & P \times M \arrow[dl, "\mathrm{pr}_0"] \\ & (P/\!\!/M)_0 & & & & P & \end{tikzcd}$$

We use the following notational convention: let I be a set, and consider a product $\prod_{i \in I} C_i$ of objects in $\mathcal{C}$. For a subset $J \subset I$, we let $\mathrm{pr}_J \colon \prod_{i \in I} C_i \to \prod_{j \in J} C_j$ denote the canonical projection. If $J = \{i_0, \ldots, i_n\}$ is finite, we also write $\mathrm{pr}_{i_0 \ldots i_n}$ instead of $\mathrm{pr}_{\{i_0, \ldots, i_n\}}$.

We can augment the above diagram to a diagram

$$\begin{tikzcd} P \times M \times M \arrow[r, "a \times 1_M"] \arrow[d, "\mathrm{pr}_{01}"'] & P \times M \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_0"] & M \arrow[d] \\ P \times M \arrow[r, "a"'] & P \arrow[r] & * \end{tikzcd}$$

Here, the right and the outer rectangle are pullback diagrams, and hence the left square is a pullback diagram as well by the pasting law. It follows that the canonical morphism

$$(P/\!\!/M)_2 \longrightarrow (P/\!\!/M)_1 \underset{(P/\!\!/M)_0}{\times} (P/\!\!/M)_1$$

is an equivalence in $\mathcal{C}$.

We now proceed by induction: suppose that the canonical morphism

$$(P/\!\!/M)_k \longrightarrow (P/\!\!/M)_1 \underset{(P/\!\!/M)_0}{\times} \cdots \underset{(P/\!\!/M)_0}{\times} (P/\!\!/M)_1$$

is an equivalence, for each $2 \leq k \leq n$. By this assumption, it now suffices to show that the morphism

$$(P/\!\!/M)_{n+1} \longrightarrow (P/\!\!/M)_n \underset{(P/\!\!/M)_0}{\times} (P/\!\!/M)_1 \tag{A.4}$$

induced by the partition $[n+1] = [n] \sqcup_{[0]} [1]$ is an equivalence. We again have an augmented diagram

$$\begin{tikzcd} P \times M^{n+1} \arrow[r, "a^{(n)} \times 1_M"] \arrow[d, "\mathrm{pr}_{01}"'] & P \times M \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_0"] & M \arrow[d] \\ P \times M^n \arrow[r, "a^{(n)}"'] & P \arrow[r] & * \end{tikzcd}$$

where the morphism $a^{(n)}$ is, up to canonical equivalence, the morphism

$$a \circ (a \times 1_M) \circ \cdots \circ (a \times 1_{M^n}) \colon P \times M^n \to P\,.$$

Again, the right-hand square in this diagram is a pullback square, and the top left object is constructed as the pullback of $P \times M^n \to * \leftarrow M$. It follows by the pasting law that the left-hand square is a pullback as well, which then implies that the top left morphism is (canonically equivalent to) $a^{(n)} \times 1_M$. Since $(P/\!\!/M)_{n+1} = P \times M^{n+1}$, and the morphisms $P \times M^{n+1} \to P \times M^n$ and $P \times M^{n+1} \to P \times M$ in (A.4) are canonically equivalent to the morphisms induced from the partition $[n+1] = [n] \sqcup_{[0]} [1]$ this completes the proof. $\square$

We recall a criterion from (the proof of) [Lur, Prop. 1.1.8] for when a category object is a groupoid object. Given a simplicial object $X \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ in an ∞-category $\mathcal{C}$ and a simplicial set $K \in \mathcal{S}\mathrm{et}_\Delta$, we define an object $X(K) \in \mathcal{C}$ as the limit (if it exists) of the diagram

$$N(\mathbb{\Delta}_{/K})^{\mathrm{op}} \longrightarrow N\mathbb{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathcal{C}\,.$$

Proposition A.5 [Lur] *Let $\mathcal{C}$ be an ∞-category with finite limits. A category object X in $\mathcal{C}$ is a groupoid object in $\mathcal{C}$ if and only if the inclusion $\Lambda^2_0 \hookrightarrow \Delta^2$ induces an equivalence $X_2 \xrightarrow{\simeq} X(\Lambda^2_0)$.*

Let $\mathcal{I}$ be the span category, depicted as $\{0,1\} \leftarrow \{0\} \to \{0,2\}$. Consider the functor $D\colon \mathcal{I} \to \mathbb{\Delta}_{/\Lambda^2_0}$, which sends the object $\{0\} \in \mathcal{I}$ to the canonical inclusion $\Delta^{\{0\}} \hookrightarrow \Lambda^2_0$ and the object $\{0,i\}$ to the canonical inclusion $\Delta^{\{0,i\}} \hookrightarrow \Lambda^2_0$, for $i = 0, 2$.

Lemma A.6 *Let $D\colon \mathcal{I} \to \mathbb{\Delta}_{/\Lambda^2_0}$ be defined as above, and let $\mathcal{C}$ be an ∞-category with finite limits. The following statements hold true:*

(1) The functor $D\colon \mathcal{I} \to \mathbb{\Delta}_{/\Lambda^2_0}$ is cofinal.

(2) For any $X \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$, the diagram

$$\begin{tikzcd} X(\Lambda^2_0) \arrow[r, "\iota^*_{0,1}"] \arrow[d, "\iota^*_{0,2}"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_1"'] & X_0 \end{tikzcd}$$

is a pullback diagram in $\mathcal{C}$, where $\iota^*_{0,i}$ denotes the morphism $X(\Delta^{\{0,i\}} \hookrightarrow \Lambda^2_0)$.

(3) A category object $X \in \mathcal{F}\mathrm{un}(\mathbb{\Delta}^{\mathrm{op}}, \mathcal{C})$ in $\mathcal{C}$ is a groupoid object precisely if the diagram

$$\begin{tikzcd} X_2 \arrow[r, "d_2"] \arrow[d, "d_1"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_1"'] & X_0 \end{tikzcd}$$

is a pullback diagram in $\mathcal{C}$.

Proof. For claim (1), note that an object of $\mathbb{\Delta}_{/\Lambda^2_0}$ is a pair $([n], \varphi)$ of an object $[n] \in \mathbb{\Delta}$ and a morphism of simplicial sets $\varphi\colon \Delta^n \to \Lambda^2_0$. We show that, for each object $([n], \varphi)$ of $\mathbb{\Delta}_{/\Lambda^2_0}$, the slice category $([n], \varphi)_{/D}$ is contractible.

Since the horn Λ^2_0 fits into a pushout diagram

$$\begin{tikzcd} \Delta^{\{0\}} \arrow[r, hook, "d^1"] \arrow[d, hook, "d^1"'] & \Delta^{\{0,1\}} \arrow[d, hook] \\ \Delta^{\{0,2\}} \arrow[r, hook] & \Lambda^2_0 \end{tikzcd}$$

in $\mathcal{S}\mathrm{et}_\Delta$, the morphism $\varphi\colon \Delta^n \to \Lambda^2_0$ is either the constant map at the apex of the horn, i.e. φ factors as $\varphi\colon \Delta^n \to \Delta^{\{0\}} \hookrightarrow \Lambda^2_0$, or it factors through a unique map $\Delta^n \to \Delta^{\{0,i\}}$, for $i = 0$ or $i = 2$, but not through the apex $\Delta^{\{0\}} \hookrightarrow \Lambda^2_0$. (One can see this either by writing $\Delta^n = N[n]$ and $\Lambda^2_0 = N\mathcal{I}$ and using that the nerve functor is fully faithful, or by using that $\mathcal{S}\mathrm{et}_\Delta(\Delta^n, -) = (-)_n$ preserves colimits.)

In the first case, the slice category $([n], \varphi)_{/D}$ is the category describing spans; in other words, it is isomorphic to $\mathcal{I}$, and we have $|N\mathcal{I}| \cong |\Delta^1 \sqcup_{\Delta^0} \Delta^1| \cong |\Delta^1| \sqcup_{|\Delta^0|} |\Delta^1| \simeq *$. In the other cases, the slice category $([n], \varphi)_{/D}$ is the final category, and hence contractible as well.

Claim (2) now follows directly from the definition of $X(K)$, for $K \in \mathcal{S}\mathrm{et}_\Delta$, together with part (1) (after taking opposites), and claim (3) then follows by combining claim (2) with Proposition A.5. □

Lemma A.7 *Let K be a simplicial set, let $\mathcal{C}$ be an ∞-category, and let $C \in \mathcal{C}$ be an object. Let $\mathsf{c}\colon \mathcal{C} \to \mathcal{F}\mathrm{un}(K, \mathcal{C})$ denote the constant-diagram functor.*

(1) If K is contractible, i.e. $K \simeq *$ in $\mathcal{S}\mathrm{et}_\Delta$ with the Kan-Quillen model structure, and $\mathrm{colim}^{\mathcal{C}}_K(\mathsf{c}C)$ exists in $\mathcal{C}$, then the canonical morphism $\mathrm{colim}^{\mathcal{C}}_K(\mathsf{c}C) \to C$ in $\mathcal{C}$ is an equivalence.

(2) Dually, if K is contractible and $\lim^{\mathcal{C}}_K(\mathsf{c}C)$ exists in $\mathcal{C}$, then the canonical morphism $C \to \lim^{\mathcal{C}}_K(\mathsf{c}C)$ in $\mathcal{C}$ is an equivalence.

Proof. By the definition of c, there is a commutative diagram

$$\begin{tikzcd} K \arrow[r, "\mathsf{c}C"] \arrow[d, "coll"'] & \mathcal{C} \\ * \arrow[ur, "C"'] & \end{tikzcd}$$

in $\mathbf{S}$. By [Lur09, Cor. 4.1.2.6, Thm. 4.1.3.1], the morphism *coll* is cofinal if and only if the simplicial set $K \times * \cong K$ is contractible, i.e. precisely if $coll\colon K \to *$ is an equivalence in $\mathcal{S}\mathrm{et}_\Delta$ (in the Kan-Quillen model structure). The first claim then follows from the fact that cofinal morphisms preserve colimits [Lur09, Prop. 4.1.1.8]. The second statement follows by duality. □

Example A.8 We need the following two specific cases in which Lemma A.7 applies:

(1) The nerve $N\mathcal{I} \in \mathcal{S}\mathrm{et}_\Delta$ is contractible, as already seen in the proof of Lemma A.6.

(2) The inclusion $\{[0]\} \hookrightarrow \mathbb{\Delta}$ is the inclusion of a final object. Thus, the nerve $N\mathbb{\Delta} \in \mathcal{S}\mathrm{et}_\Delta$ is contractible in $\mathcal{S}\mathrm{et}_\Delta$. ◁

Lemma A.9 *Let $K \in \mathcal{S}\mathrm{et}_\Delta$ be contractible (in the Kan-Quillen model structure) and let $\mathcal{C}$ be an ∞-category admitting limits of shape K. Let $P \in \mathcal{C}$ be any object.*

(1) The constant diagram functor $\mathsf{c}\colon \mathcal{C} \to \mathcal{F}\mathrm{un}(K, \mathcal{C})$ is fully faithful.

(2) If $\mathcal{C}$ admits finite products, then the functor $P \times (-)\colon \mathcal{C} \to \mathcal{C}$ preserves limits of shape K.

Proof. Let $C, D \in \mathcal{C}$ be any objects. To see (1), we use the adjunction $\mathsf{c} \dashv \lim^{\mathcal{C}}_K$ and Lemma A.7, which yield canonical equivalences

$$\underline{\mathcal{C}^K}(\mathsf{c}C, \mathsf{c}D) \simeq \underline{\mathcal{C}}(C, \lim\nolimits^{\mathcal{C}}_K \mathsf{c}D) \simeq \underline{\mathcal{C}}(C, D)\,.$$

For claim (2), let $C, P \in \mathcal{C}$ be objects, and let $F\colon K \to \mathcal{C}$ be a diagram. We now have canonical equivalences

$$\begin{aligned} \underline{\mathcal{C}}\big(C, \lim\nolimits^{\mathcal{C}}_K(\mathsf{c}P \times F)\big) &\simeq \underline{\mathcal{C}^K}(\mathsf{c}C, \mathsf{c}P \times F) \\ &\simeq \underline{\mathcal{C}^K}(\mathsf{c}C, \mathsf{c}P) \times \underline{\mathcal{C}^K}(\mathsf{c}C, F) \end{aligned}$$

$$\begin{aligned} &\simeq \underline{\mathcal{C}}(C, P) \times \underline{\mathcal{C}}(C, \lim\nolimits_K^{\mathcal{C}} F) \\ &\simeq \underline{\mathcal{C}}(C, P \times \lim\nolimits_K^{\mathcal{C}} F)\,. \end{aligned}$$

In the third equivalence we have used part (1), i.e. that c is fully faithful here. Then, the statement follows from the Yoneda Lemma. ☐

Remark A.10 If we were working with categories instead of ∞-categories, it would suffice to have an indexing category $\mathcal{J}$ (instead of K) which is connected, rather than contractible, to prove an analogue of Lemma A.9. ◁

We can now prove Theorem 3.19:

Proof of Theorem 3.19. Since every group object in $\mathcal{C}$ is in particular a monoid object in $\mathcal{C}$, it follows from Proposition A.3 that $P/\!\!/G$ is a category object in $\mathcal{C}$. We now use Lemma A.6 to show that it is even a groupoid object. By that lemma, it suffices to check that the diagram

$$\begin{tikzcd} (P/\!\!/G)_2 \arrow[r, "d_2"] \arrow[d, "d_1"'] & (P/\!\!/G)_1 \arrow[d, "d_1"] \\ (P/\!\!/G)_1 \arrow[r, "d_1"'] & (P/\!\!/G)_0 \end{tikzcd} \quad = \quad \begin{tikzcd} P \times G^2 \arrow[r, "d_2 = \mathrm{pr}_{01}"] \arrow[d, "d_1"'] & P \times G \arrow[d, "d_1 = \mathrm{pr}_0"] \\ P \times G \arrow[r, "d_1 = \mathrm{pr}_0"'] & P \end{tikzcd} \tag{A.11}$$

is a pullback diagram in $\mathcal{C}$, where we have used axioms (1) and (2) of Definition A.2 and their consequences pointed out after Definition 3.15.

Our goal now is to split off the factor P in diagram (A.11). To that end, consider the diagram

$$\begin{tikzcd} & P \arrow[rr, equal] & & P \\ P \times G^2 \arrow[ur, "\mathrm{pr}_0 \simeq d_1 d_2"] \arrow[rr, "d_1"] \arrow[dr, "\mathrm{pr}_{12}"'] & & P \times G \arrow[ur, "\mathrm{pr}_0 = d_1"'] \arrow[dr, "\mathrm{pr}_1"] & \\ & G^2 \arrow[rr, "d_1^G"'] & & G \end{tikzcd} \tag{A.12}$$

The bottom rectangle in diagram (A.12) commutes by axiom (3) of Definition 3.15. The top rectangle commutes because $P/\!\!/G$ is a simplicial object in $\mathcal{C}$, so we have a canonical equivalence $d_1 d_2 \simeq d_1 d_1$. This establishes the morphism d_1 as a morphism of binary products in $\mathcal{C}$. As such, it is induced by the morphisms $1_P \colon P \to P$ and $d_1^G \colon G^2 \to G$. Thus, there is a canonical equivalence

$$d_1 \simeq 1_P \times d_1^G\,,$$

of morphisms $(P/\!\!/G)_2 \to (P/\!\!/G)_1$ in $\mathbf{H}$. We thus have an equivalence of diagrams

$$\begin{tikzcd} P \times G^2 \arrow[r, "d_2 = \mathrm{pr}_{01}"] \arrow[d, "d_1 \simeq 1_P \times d_1^G"'] & P \times G \arrow[d, "d_1 = \mathrm{pr}_0"] \\ P \times G \arrow[r, "d_1 = \mathrm{pr}_0"'] & P \end{tikzcd} \quad \simeq \quad P \times \left(\begin{tikzcd} G^2 \arrow[r, "d_2^G"] \arrow[d, "d_1^G"'] & G \arrow[d] \\ G \arrow[r] & * \end{tikzcd} \right)$$

By the definition of $\widehat{G}$ as a groupoid object with $G_0 \simeq *$, we have that the diagram

$$\begin{tikzcd} G^2 \arrow[r, "d_2^G"] \arrow[d, "d_1^G"'] & G \arrow[d] \\ G \arrow[r] & * \end{tikzcd}$$

is a pullback diagram in $\mathcal{C}$ by Lemma A.5. It now follows from Lemma A.9 that the functor $P \times (-)$ sends this pullback diagram to a pullback diagram. Consequently, the square (A.11) is a pullback diagram in $\mathbf{H}$. □

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