GREEDOIDS FROM FLAMES

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Abstract. A digraph D with $r \in V(D)$ is an r-flame if for every $v \in V(D) - r$, the in-degree of v is equal to the local edge-connectivity $\lambda_D(r, v)$. We show that for every digraph D and $r \in V(D)$, the edge sets of the r-flame subgraphs of D form a greedoid. Our method yields a new proof of Lovász' theorem stating: for every digraph D and $r \in V(D)$, there is an r-flame subdigraph F of D such that $\lambda_F(r, v) = \lambda_D(r, v)$ for $v \in V(D) - r$. We also give a strongly polynomial algorithm to find such an F working with a fractional generalization of Lovász' theorem.

1. Introduction

Subgraphs preserving some connectivity properties while having as few edges as possible is a subject of interest since the beginning of graph theory. Suppose that D is a digraph with $r \in V(D)$ and we are looking for a spanning subgraph H of D with the smallest possible number of edges in which all the local edge-connectivities outwards from the root r are the same as in D, i.e., $\lambda_H(r, v) = \lambda_D(r, v)$ for all $v \in V(D) - r$. In order to have $\lambda_D(r, v)$ many pairwise edge-disjoint paths from r to v in H, it is obviously necessary for H to contain at least $\lambda_D(r, v)$ ingoing edges of v. Writing this with a formula: $\varrho_H(v) \geq \lambda_D(r, v)$ for all $v \in V(D) - r$, from which the estimation $|E(H)| \geq \sum_{v \in V(D)-r} \lambda_D(r, v)$ follows. It was shown by Lovász that, maybe surprisingly, this trivial lower bound is always sharp.

Theorem 1.1 (Lovász, Theorem 2 of [1])**.** *For every digraph D and $r \in V(D)$, there is a spanning subdigraph H of D such that for every $v \in V(D) - r$*

$$\lambda_D(r, v) = \lambda_H(r, v) = \varrho_H(v).$$

Calvillo-Vives rediscovered Theorem 1.1 independently in [2] and named 'r-flame' the rooted digraphs F with $\lambda_F(r, v) = \varrho_F(v)$ for all $v \in V(F) - r$.

We establish a direct connection between the extremal problem above and the theory of greedoids. The latter were introduced by Korte and Lovász as a generalization of matroids to capture greedy solvability in problems where the matroid concept turned out to be too restrictive. The field is actively investigated since the '80s, for a survey we refer to [3].

We show that the subflames of a rooted digraph always form a greedoid whose bases are exactly the subdigraphs described in Theorem 1.1.

Theorem 1.2. *Let D be a digraph and $r \in V(D)$. Then*

$$\mathcal{F}_{D,r} := \{E(F) \mid F \subseteq D \text{ is an } r\text{-flame}\}$$

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is a greedoid on $E(D)$*. Furthermore, for each* $\subseteq$*-maximal element* F^* *of* $\mathcal{F}(D,r)$ *we have* $\lambda_{F^*}(r,v)=\lambda_D(r,v)$ *for all* $v\in V(D)-r$*.*

The proof of Theorem 1.1 by Lovász is algorithmic but only for simple digraphs polynomial. We prove a fractional generalization of Lovász' theorem considering digraphs with non-negative edge-capacities and replacing 'edge-connectivity' by 'flow-connectivity'. Our proof provides a simple strongly polynomial algorithm to find an H with properties given in Theorem 1.1.

It is worth to mention that one can formulate a structural infinite generalization of Theorem 1.1 in the same manner as Erdős conjectured such an extension of Menger's theorem (see [4]). As in the case of Menger's theorem, the problem is getting much harder by dropping finiteness. The "vertex-variant" of this generalization was proved for countably infinite digraphs in [5] which was then further developed in [6].

2. Notation

In this paper we deal only with finite combinatorial structures. An $\mathcal{F}\subseteq 2^E$ is a greedoid on E if $\varnothing\in\mathcal{F}$ and $\mathcal{F}$ has the *Augmentation property*, i.e., whenever $F,F'\in\mathcal{F}$ with $|F|<|F'|$, there is some $e\in F'\setminus F$ such that $F+e\in\mathcal{F}$. Let a vertex set V and a "root vertex" $r\in V$ be fixed through the paper. A digraph D is a set of directed edges with their endpoints in V where parallel edges are allowed but loops are not. For $U\subseteq V$, $\mathsf{in}_D(U)$ and $\mathsf{out}_D(U)$ stand for the set of ingoing and outgoing edges of U respectively, furthermore, let $\varrho_D(U):=|\mathsf{in}_D(U)|$ and $\delta_D(U):=|\mathsf{out}_D(U)|$. For simplicity we always assume that $\mathsf{in}_D(r)=\varnothing$. For $v\in V-r$, we write $\lambda_D(v)$ for the local edge-connectivity (i.e., the maximal number of pairwise disjoint paths) from r to v . We define $\mathcal{G}_D(v)$ to be the set of those $I\subseteq\mathsf{in}_D(v)$ for which there exists a system $\mathcal{P}$ of edge-disjoint $r\to v$ paths where the set of the last edges of the paths in $\mathcal{P}$ is I. Recall that set $\mathcal{G}_D(v)$ is the family of independent sets of a matroid and matroids representable this way are called gammoids. A digraph F is a flame if $\mathcal{G}_F(v)$ is a free matroid[1] for every $v\in V-r$, equivalently $\lambda_F(v)=\varrho_F(v)$ for every $v\neq r$.

3. The flame greedoid of a rooted digraph

The core of the proof of Theorem 1.2 is the following lemma.

Lemma 3.1. *Let* H *and* D *be digraphs and assume that* $\lambda_H(u)<\lambda_D(u)$ *for some* $u\in V-r$*. Then there is an* $e\in D\setminus H$ *with head, say* v*, such that* e *is a coloop*[2] *of* $\mathcal{G}_{H+e}(v)$*, i.e.,*

$$\mathcal{G}_{H+e}(v)=\{I+e:I\in\mathcal{G}_H(v)\}.$$

Proof. Let $\mathcal{U}:=\{U\subseteq V-r:u\in U\text{ and }\varrho_H(u)=\lambda_H(u)\}$. By Menger's theorem $\mathcal{U}\neq\varnothing$ and the submodularity of the map $X\mapsto\varrho_H(X)$ ensures that $\mathcal{U}$ is closed under union and intersection. Let U be the $\subseteq$-largest element of $\mathcal{U}$. Since $\lambda_H(u)<\lambda_D(u)$, there exists some edge $e\in\mathsf{in}_D(U)\setminus\mathsf{in}_H(U)$. Note that in $H+e$ every $X\subseteq V-r$ with $X\supseteq U$ has at least $\lambda_H(u)+1=\varrho_{H+e}(U)$ many ingoing edges because of the maximality of U. By

[1] matroid where all sets are independent

[2] edge of a matroid which can be added to any independent set without ruin independence

applying Menger's theorem in $H+e$ with r and U, we find a system $\mathcal{P}$ of edge-disjoint $r \to U$ paths of size $\lambda_H(u)+1$ (see Figure 1). The set of the last edges of the paths in $\mathcal{P}$ is necessarily the whole $\mathsf{in}_{H+e}(U)$. Let the head of e be v and let $I \in \mathcal{G}_H(v)$ witnessed by the path-system $\mathcal{Q}$. Clearly each $Q \in \mathcal{Q}$ enters U at least once. For $Q \in \mathcal{Q}$, we define f_Q as the last meeting of Q with $\mathsf{in}_H(U)$. Finally, we build a path-system $\mathcal{R}$ witnessing $I+e \in \mathcal{G}_H(v)$ as follows. For $Q \in \mathcal{Q}$, we consider the unique $P_Q \in \mathcal{P}$ with last edge f_Q and concatenate it with the terminal segment of Q from f_Q to obtain R_Q. Moreover, let R_e be the unique path in $\mathcal{P}$ with last edge e. Then $\mathcal{R} := \{R_Q : Q \in \mathcal{Q}\} \cup \{R_e\}$ is as desired.

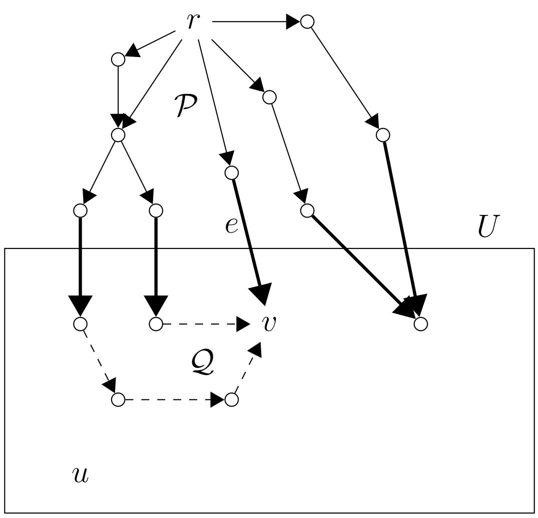

FIGURE 1. $\mathsf{in}_{H+e}(U)$ consists of the thick edges, the terminal segments of the paths in $\mathcal{Q}$ are dashed.

□

Proof of Theorem 1.2. Suppose that $F_0, F_1 \subseteq D$ are flames with $|F_0| < |F_1|$. Then there must be some $u \in V - r$ for which $\varrho_{F_0}(u) < \varrho_{F_1}(u)$. Since F_0 and F_1 are flames

$$\lambda_{F_0}(u) = \varrho_{F_0}(u) < \varrho_{F_1}(u) = \lambda_{F_1}(u).$$

By applying Lemma 3.1 with F_0, F_1 and u, we find an $e \in F_1 \setminus F_0$ with head v where e is a coloop of $\mathcal{G}_{F_0+e}(v)$. On the one hand, $\mathcal{G}_{F_0}(v)$ is a free matroid and the previous sentence ensures that $\mathcal{G}_{F_0+e}(v)$ is free as well. On the other hand, for $w \in V \setminus \{r, v\}$ any path-system witnessing that $\mathcal{G}_{F_0}(w)$ is the free matroid showing the same for $\mathcal{G}_{F_0+e}(w)$. By combining these we may conclude that $F_0 + e$ is a flame.

In order to prove the last sentence of Theorem 1.2, let F^* be a maximal flame in D and suppose for a contradiction that $\lambda_{F^*}(u) < \lambda_D(u)$ for some $u \in V - r$. Applying Lemma 3.1 gives again some $e \in D \setminus F^*$ for which $F^* + e$ is a flame contradicting the maximality of F^*. □

4. Fractional generalization and algorithmic aspects

In this section we define a fractional version of Lovász's theorem and prove it by giving a strongly polynomial algorithm that finds a desired optimal substructure. We consider non-negative vectors defined on (the edge set of) a fixed digraph D. This time we assume without loss of generality that D has no parallel edges because replacing a bunch of parallel edges by a single edge whose capacity is defined to be the sum of the capacities of those will be a meaningful reduction step in all the results we discuss. For $x, y \in \mathbb{R}_+^D$, we write $x \leq y$ if $x(e) \leq y(e)$ for every $e \in D$ and for $U \subseteq V$ let $\varrho_x(U) := \sum_{e \in \mathsf{in}_D(U)} x(e)$ and $\delta_x(U) := \sum_{e \in \mathsf{out}_D(U)} x(e)$. An $x \in \mathbb{R}_+^D$ is an $r \to v$ flow if $\varrho_x(u) = \delta_x(u)$ holds for all $u \in V \setminus \{r, v\}$ and $\varrho_x(r) = \delta_x(v) = 0$. The *amount* of the flown x is defined to be $\delta_x(r)$ which is equal to $\varrho_x(W) - \delta_x(W)$ for every choice of $W \subseteq V - r$ containing v. Recall that x can be written as the non-negative combination of directed cycles and $r \to v$ paths (more precisely of their characteristic vectors). The sum of the coefficients of the paths in any such a decomposition is again $\delta_x(r)$. For $v \in V - r$ and $c \in \mathbb{R}_+^D$, the *flow-connectivity* of c from r to v is

$$\lambda_c(v) := \max\{\delta_x(r) : x \text{ is an } r \to v \text{ flow with } x \leq c.\}$$

The Max flow min cut theorem (see [7]) guarantees that $\lambda_c(v)$ is well-defined and equals to

$$\min\{\varrho_c(W) : W \subseteq V - r \text{ with } v \in W\}.$$

For $v \in V - r$ and $c \in \mathbb{R}_+^D$, we write $\mathcal{G}_c(v)$ for the set of those vectors in $\mathbb{R}_+^{\mathsf{in}_D(v)}$ that can be obtained as a restriction of an $r \to v$ flow $x \leq c$ to $\mathsf{in}_D(v)$ that we denote by $x \upharpoonright \mathsf{in}_D(v)$. It is not too hard to prove that $\mathcal{G}_c(v)$ is a polymatroid and it is natural to call it a *polygammoid*. An $f \in \mathbb{R}_+^D$ is a *fractional flame* if $f \upharpoonright \mathsf{in}_D(v) \in \mathcal{G}_f(v)$ (equivalently $\lambda_f(v) = \varrho_f(v)$) for all $v \in V - r$. For $e \in D$, let $\chi_e \in \mathbb{R}_+^D$ be the vector where $\chi_e(e')$ is 1 if $e = e'$ and 0 otherwise. We call a vector *integral* if all of its coordinates are integers.

The fractional version of Lovász' theorem can be formulated in the following way.

Theorem 4.1. *For every $c \in \mathbb{R}_+^D$ there is an $f \leq c$ such that for every $v \in V - r$*

$$\lambda_c(v) = \lambda_f(v) = \varrho_f(v),$$

moreover, f can be chosen to be integral if so is c. Such an f can be found in strongly polynomial time.

Proof. In the contrast of Theorem 1.1, the following fractional analogue of Lemma 3.1 is not sufficient itself to provide the existence part of Theorem 4.1 but will be an important tool later.

Lemma 4.2. *Let $x, y \in \mathbb{R}_+^D$ such that $\lambda_y(u) < \lambda_x(u)$ for some $u \in V - r$. Then there is an $e \in D$ with $\mathsf{head}(e) =: v$ and an $\varepsilon > 0$ such that $x(e) - y(e) \geq \varepsilon$ and*

$$\mathcal{G}_{y+\varepsilon\chi_e}(v) = \{s + \delta\chi_e : s \in \mathcal{G}_y(v) \wedge 0 \leq \delta \leq \varepsilon\}.$$

Proof. The proof goes similarly as for Lemma 3.1. By applying the Max flow min cut theorem and the submodularity of the function $X \mapsto \varrho_y(X)$, we take the maximal $U \subseteq V - r$ with $u \in U$ and $\varrho_y(U) = \lambda_y(u)$. We pick some $e \in \mathsf{in}_D(U)$ with $x(e) > y(e)$ and let

$$\varepsilon := \min\{x(e) - y(e),\ \varrho_y(W) - \varrho_y(U) : U \subsetneq W \subseteq V - r\}.$$

The Max flow min cut theorem ensures (applying it after contracting U to u) that there is a $p \leq y + \varepsilon\chi_e$, which can be chosen as a non-negative combination of $r \to U$ paths, such that

$$p \restriction \mathsf{in}_D(U) = (y + \varepsilon\chi_e) \restriction \mathsf{in}_D(U).$$

Let $s \in \mathcal{G}_y(u)$ witnessed by the $r \to u$ flow q which is a non-negative combination of $r \to u$ paths. Take the sum of the terminal segments of these weighted paths from the last common edge with $\mathsf{in}_D(U)$ together with the trivial path e with a given weight δ to obtain a vector q'. Starting with p one can construct a $p' \leq p$ which is a non-negative combination of $r \to U$ paths and for which $p' \restriction \mathsf{in}_D(U) = q' \restriction \mathsf{in}_D(U)$. It is easy to see that the coordinate-wise maximum of p' and q' witnessing $s + \delta\chi_e \in \mathcal{G}_{y+\varepsilon\chi_e}(v)$. □

Now we turn to the description of the algorithm. Let $V = \{v_0, \dots, v_n\}$ where $v_0 = r$. The algorithm starts with $f_0 := c$. If $f_k \in \mathbb{R}_+^D$ is already constructed and $k < n$, then we take an $r \to v_{k+1}$ flow $z_{k+1} \leq f_k$ of amount $\lambda_{f_k}(v_{k+1})$, which we choose to be integral if so is f_k, and define

$$f_{k+1}(e) := \begin{cases} z_{k+1}(e) & \text{if } e \in \mathsf{in}_D(v_{k+1}) \\ f_k(e) & \text{otherwise.} \end{cases}$$

Since the flow problem can be solved in strongly polynomial time, the algorithm described above is strongly polynomial with a suitable flow-subroutine. We claim that f_n satisfies the demands of Theorem 4.1. Since we start with c and lower some values in each step, $f_n \leq c$ holds. If $c \in \mathbb{Z}_+^D$, then a straightforward induction shows that $f_n \in \mathbb{Z}_+^D$.

Lemma 4.3. *If $z \leq x$ is an $r \to v$ flow of amount $\lambda_x(v)$ and* $y(e) := \begin{cases} z(e) & \textit{if } e \in \mathsf{in}_D(v) \\ x(e) & \textit{otherwise} \end{cases}$ *then $\lambda_y(u) = \lambda_x(u)$ for every $u \in V - r$.*

Proof. Suppose for a contradiction that there exists a $u \in V - r$ with $\lambda_y(u) < \lambda_x(u)$. Note that $u \neq v$ because $\lambda_x(v) = \lambda_y(v)$ is witnessed by z. By Lemma 4.2, there is an $e \in D$ and an ε such that $x(e) - y(e) > \varepsilon > 0$ (which implies that $\mathsf{head}(e)$ must be v) and $\mathcal{G}_{y+\varepsilon\chi_e}(v) = \{s + \delta\chi_e : s \in \mathcal{G}_y(v) \wedge 0 \leq \delta \leq \varepsilon\}$. Let $s_0 := z \restriction \mathsf{in}_D(v)$.

$$\lambda_x(v) \geq \lambda_{y+\varepsilon\chi_e}(v) \geq ||s_0||_1 + \varepsilon = \lambda_x(v) + \varepsilon$$

which is a contradiction. □

By applying Lemma 4.3 with $x = f_k$, $y = f_{k+1}$ and $z = z_{k+1}$ we obtain the following.

Corollary 4.4. $\lambda_{f_k}(v) = \lambda_{f_{k+1}}(v)$ *for every $k < n$ and $v \in V - r$.*

It follows by induction on k that $\lambda_{f_k}(v) = \lambda_c(v)$ for every $v \in V - r$ and $k \leq n$. In particular $\lambda_{f_n}(v) = \lambda_c(v)$ for all $v \in V - r$. Let $1 \leq k \leq n$ be arbitrary. Then $\varrho_{f_k}(v_k) = \lambda_{f_k}(v_k)$ follows directly from the algorithm (the common value is $\varrho_{z_k}(v_k)$). On the one hand, the left side is equal to $\varrho_{f_n}(v_k)$ since the algorithm does not modify anymore the relevant coordinates. On the other hand, we have seen that $\lambda_{f_k}(v_k) = \lambda_{f_n}(v_k) = \lambda_c(v_k)$. By combines these we have $\varrho_{f_n}(v) = \lambda_{f_n}(v)$ which completes the proof of Theorem 4.1. □

Finally, let us point out a special case of Lemma 4.3.

Corollary 4.5. *Let D be a directed graph and let $\mathcal{P}$ be a maximal sized family of pairwise edge-disjoint $r \to v$ paths in D. Then the deletion of those ingoing edges of v that are unused by the path-family $\mathcal{P}$ does not reduce any local edge-connectivities of the form $\lambda_D(r,u)$ with $u \in V(D) - r$.*

5. Outlook

By Theorem 4.1, finding a spanning subdigraph of a given digraph D that preserves all the local edge-connectivities from a prescribed root vertex r and has the fewest possible edges with respect to this property can be done in polynomial time. It is natural to ask the complexity of the weighted version:

Question 5.1. *What is the complexity of the following combinatorial optimization problem?*
Input: digraph D, $r \in V(D)$ and cost function $c : E(D) \to \mathbb{R}_+$
Output: spanning subdigraph F of D with $\lambda_F(r,v) = \lambda_D(r,v)$ for every $v \in V(D) - r$ for which $\sum_{e \in E(F)} c(e)$ is minimal with respect to this property.

The special case where $\lambda_D(r,v)$ is the same for every $v \in V(D) - r$ can be solved in polynomial time by using weighted matroid intersection (see [8]).

There are more general flow models involving polymatroidal bounding functions (see for example [9] and [10]). The Max flow min cut theorem is preserved under these models.

Question 5.2. *Is it possible to generalize Theorem 4.1 by using the polymatroidal flow model of Lavler and Martel in [9]?*

The relation between matroids and polymatroids motivates the following concept of polygreedoids: a *polygreedoid* is a compact $\mathcal{P} \subseteq \mathbb{R}_+^E$ such that

- PG1 $\underline{0} \in \mathcal{P}$,
- PG2 whenever $x, y \in \mathcal{P}$ with $||x||_1 < ||y||_1$, there is some $e \in E$ with $y(e) > x(e)$ such that $x + \varepsilon\chi_e \in \mathcal{P}$ for all small enough $\varepsilon > 0$.

It follows directly from Lemma 4.2 that fractional flames under a given bounding vector form a polygreedoid. Greedoids has the property called *accesibility* which can be considered as a weakening of the downward closedness of matroids. It tells that every $F \in \mathcal{F}$ can be enumerated in such a way that each initial segment belongs to $\mathcal{F}$, i.e., $F = \{e_1, \dots, e_n\}$ such that $\{e_1, \dots, e_k\} \in \mathcal{F}$ for every $k \leq n$. Accessibility tends to be a part of the axiomatization of greedoids via the restriction the Augmentation axiom for pairs with $|F'| = |F| + 1$. It is not too hard to prove that polygreedoids satisfy the following analogous propery: for every $x \in \mathcal{P}$ there is a continues strictly increasing[3] function $g : [0,1] \to \mathcal{P}$ with $g(0) = \underline{0}$ and $g(1) = x$.

Question 5.3. *How much of the theory of greedoids is preserved for polygreedoids?*

[3]with respect to the coordinate-wise partial ordering of $\mathbb{R}_+^E$

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