INTERSECTION OF A PARTITIONAL AND A GENERAL INFINITE MATROID

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ABSTRACT. Let E be a possibly infinite set and let M and N be matroids defined on E. We say that the pair $\{M, N\}$ has the Intersection property if M and N share an independent set I admitting a bipartition $I_M \cup I_N$ such that $\mathsf{span}_M(I_M) \cup \mathsf{span}_N(I_N) = E$. The Matroid Intersection Conjecture of Nash-Williams says that every matroid pair has the Intersection property.

It was shown by N. Bowler and J. Carmesin that the conjecture can be reduced to the special case where one of the matroids is a partitional matroid. We prove that if M is an arbitrary matroid and N is a partitional matroid of finitely many components, then $\{M, N\}$ has the Intersection property.

1. INTRODUCTION

Some of the motivating examples of matroids are vector-systems with the linear independence and graphs with graph theoretic cycles as circuits. Both type of structures can be infinite in which case the resulting matroid is infinite as well. An axiomatization of matroids (in the language of circuits) that allows infinite ground sets can be obtained from the axiomatization of finite matroids in a natural way: $\mathcal{C}$ is the set of the circuits of a matroid if $\mathcal{C}$ is a family of finite nonempty and pairwise $\subseteq$-incomparable subsets of a possible infinite set E satisfying the Circuit elimination axiom[1]. Working with this definition, Nash-Williams proposed around 1990 his Matroid Intersection Conjecture [1] which has been the most important open problem in infinite matroid theory for decades. It generalizes the Matroid Intersection Theorem of Edmonds [2] to infinite matroids capturing the combinatorial structure corresponding to the largest common independent sets instead of dealing with infinite quantities (cardinality is usually turn out to be an overly rough measure for problems in infinite combinatorics). Adapting a terminology of Bowler and Carmesin, for a pair $\{M, N\}$ of matroids defined on the same edge set E we say that it has the Intersection property if M and N has a common independent set I admitting a bipartition $I_M \cup I_N$ such that $\mathsf{span}_M(I_M) \cup \mathsf{span}_N(I_N) = E$

Conjecture 1.1 (Matroid Intersection Conjecture, [1])**.** *Every pair $\{M, N\}$ of matroids defined on the same (potentially infinite) ground set has the Intersection property.*

The infinite matroid concept given in the first paragraph was not entirely satisfying for the experts in matroid theory. Under that definition matroids may fail to have a dual

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[1]If $C_0, C_1 \in \mathcal{C}$ are distinct and $e \in C_0 \cap C_1$, then $\exists C_2 \in \mathcal{C}$ with $C_2 \subseteq C_0 \cup C_1 - e$

although that is a key phenomenon of the finite theory. Rado asked in 1966 for a more general concept of infinite matroids having duality while preserving the minor operations. Among other attempts Higgs introduced [3] a class of structures he called "B-matroids" that solves Rado's problem. Oxley gave an axiomatization of B-matroids and showed that under some reasonable assumptions they are the largest class of structures that we may get (see [4] and [5]). Despite these discoveries of Higgs and Oxley, the systematic investigation of B-matroids started only around 2010 when Bruhn, Diestel, Kriesell, Pendavingh and Wollan found a set of cryptographic axioms for them, generalising the usual independent set-, bases-, circuit-, closure- and rank-axioms for finite matroids (see [6]). They also showed that several well-known facts of the theory of finite matroids are preserved. They axiomatization in the language of independent sets is the following:

An $M=(E,\mathcal{I})$ is a B-matroid (or simply matroid) if $\mathcal{I}\subseteq\mathcal{P}(E)$ with

(i) $\varnothing\in\mathcal{I}$;
(ii) $\mathcal{I}$ is downward closed;
(iii) For every $I,J\in\mathcal{I}$ where J is $\subseteq$-maximal in $\mathcal{I}$ but I is not, there exists an $e\in J\setminus I$ such that $I+e\in\mathcal{I}$;
(iv) For every $X\subseteq E$, any $I\in\mathcal{I}\cap\mathcal{P}(X)$ can be extended to a $\subseteq$-maximal element of $\mathcal{I}\cap\mathcal{P}(X)$.

After this success of Rado's program the name 'Matroid Intersection Conjecture' gained a new interpretation by applying the definition above instead of the more restrictive infinite matroid concept given at the beginning of the Introduction. The latter class consists of exactly those matroids that have only finite circuits which matroids are called finitary. Although several partial results have been obtained about the Matroid Intersection Conjecture (see [1],[10], [11], [12], [13], [14]), even the original finitary version is remained wide open.

Bowler and Carmesin discovered (Corollary 3.9 (b) in [11]) that the general form of Conjecture 1.1 is implied by its special case where one of the two matroids is a partitional matroid[2]. Our main result is the following:

Theorem. *Let M and N be matroids on the common edge set E where N is a partitional matroid of finitely many components. Then $\{M,N\}$ has the Intersection property.*

We do not presuppose any background about finite or infinite matroids but give in the next section all the basic facts we need.

2. Notation and Preliminaries

We apply the standard set theoretic convention that natural numbers are identified with the set of smaller natural numbers, i.e., $n=\{0,\dots,n-1\}$. By abusing the notation we write simply $X-y+z$ instead of $X\setminus\{y\}\cup\{z\}$.

2.1. **General matroidal terms.** A pair $M=(E,\mathcal{I})$ is a *matroid* if $\mathcal{I}\subseteq\mathcal{P}(E)$ satisfies the axioms (i)-(iv). The sets in $\mathcal{I}$ are called *independent* and the maximal independent sets are the *bases*. For an $X\subseteq E$, $\boldsymbol{M\upharpoonright X}:=(X,\mathcal{I}\cap\mathcal{P}(X))$ is a matroid and it is called

[2]They proved actually more: all components of the partitional matroid is 1-uniform on a 2-element set.

the *restriction* of M to X. We write $\boldsymbol{M - X}$ for $M \upharpoonright (E \setminus X)$ and call it the minor obtained by the *deletion* of X. Let B_X be a maximal independent subset of X. The *contraction* $\boldsymbol{M/X}$ of X in M is the matroid on $E \setminus X$ where $I \subseteq E \setminus X$ is independent if $I \cup B_X$ is independent in M. One can show that the definition does not depend on the choice of B_X. Contraction and deletion commute, i.e., for disjoint $X, Y \subseteq E$, we have $(M/X) - Y = (M - Y)/X$. Matroids of this form are the *minors* of M. The set $\mathsf{span}_M(X)$ of edges spanned by X in M consists of the edges in X and of those $e \in E \setminus X$ for which $\{e\}$ fails to be independent in M/X. From the well-definedness of contraction minors the following simple fact can be obtained:

Fact 2.1. *Suppose that I, I_0, I_1 are independent where $I_0 \subseteq I$ and $\mathsf{span}(I_0) = \mathsf{span}(I_1)$. Then $(I \setminus I_0) \cup I_1$ is independent and $\mathsf{span}\left[(I \setminus I_0) \cup I_1\right] = \mathsf{span}(I)$.*

By abusing the notation we will write simply M for the set $\mathcal{I}_M$ of M-independent sets. For $i \in K$ let M_i be a matroid on E_i such that $E_i \cap E_j = \varnothing$ for $i \neq j$. Then the *direct sum* $\boldsymbol{\bigoplus_{i\in K} M_i}$ is the matroid defined on $E := \bigcup_{i\in K} E_i$ where $I \in \bigoplus_{i\in K} M_i$ if $I \cap E_i \in M_i$ for every $i \in K$. For a detailed introduction to the theory of infinite matroids we refer to [7].

2.2. **Uniform matroids.** A matroid U is called *uniform* if for any $I \in U$, $e \in I$ and $f \in E \setminus I$ we have $I - e + f \in U$. Note that the class of uniform matroids are closed under taking minors. On a finite E, the bases of an uniform matroids are the n-element subsets of E for some $n \leq |E|$. If E is infinite and $n \in \mathbb{N}$, then the n-element subsets of E as well as the complements of those sets is the family of bases of a uniform matroid. A natural question is if there are uniform matroids beyond these trivial ones.

It was discovered by Bowler and Geschke in [8] that (under appropriate set theoretic assumption) the answer is yes, actually most of the uniform matroids on a given infinite ground set are non-trivial. In contrast of the finite case, uniform matroids are interesting themself, they led to important relative consistency results. Indeed, Bowler and Geschke showed for example using a uniform matroid that the existence of a matroid admitting two bases with different infinite sizes is consistent with set theory ZFC (actually independent of it). In order to mention another such an application, let us tell that for finitary matroids: if M has an N-independent base and visa versa then they actually share a base (see Corollary 1.4 of [9]). By constructing a suitable pair of uniform matroids, the unprovability of the generalization of the previous statement for general matroids was demonstrated (take U and its dual in Theorem 5.1 of [9]).

We will apply often the following characteristic property of uniform matroids:

Proposition 2.2. *Matroid U on E is uniform if and only if each $F \subseteq E$ either contains a base of U or contained in a base of U.*

Direct sums of uniform matroids are called *partitional* matroids.

3. Main result

Theorem 3.1. *Let M and N be matroids on the common edge set E such that $N = \bigoplus_{i<n} U_i$ where U_i is a uniform matroid on E_i and $n \in \mathbb{N}$. Then $\{M, N\}$ has the Intersection property.*

Proof. We use induction on n.

Observation 3.2. *If M' and N' are arbitrary matroids and $B_{M'} \subseteq B_{N'}$ for some base $B_{M'}$ of M' and base $B_{N'}$ of N', then $B_{M'}$ is a witness of the Intersection property ensured by the trivial bipartition $B_{M'} \cup \varnothing$.*

For $n = 1$, we take an arbitrary base B of M. Since in this case $N = U_0$ itself is uniform, Proposition 2.2 tells that either B contained in a base N or it contains a base N both of which cases we are done by Observation 3.2.

Let $n > 1$. Suppose first that there is a nonempty $W \subseteq E$ which is the union of some of the sets E_i such that $M \upharpoonright W$ admits an N-independent base B. Note that $N - W = N/W$. We apply the induction hypothesis with M/W and $N - W$ to obtain a witness $I_{M/W} \cup I_{N-W}$ that $\{M/W, N - W\}$ has the Intersection property. It is easy to check that for $I_M := I_{M/W} \cup B$ and $I_N := I_{N-W}$ the set $I_M \cup I_N$ shows the Intersection property of $\{M, N\}$. Let us now assume the following condition:

Condition 3.3. *There is no nonempty $W \subseteq E$ which is the union of some of the sets E_i such that $M \upharpoonright W$ admits an N-independent base B.*

In order to prove Theorem 3.1 it is (more than) enough to show the following theorem:

Theorem 3.4. *Let M and N be matroids on the common edge set E such that $N = \bigoplus_{i<n} U_i$ where U_i is a uniform matroid on E_i and $n \in \mathbb{N}$. Assume that Condition 3.3 holds. Then for every $J \in M \cap N$ there exists an M-independent N-base B with $\mathsf{span}_M(J) \subseteq \mathsf{span}_M(B)$.*

Indeed, Observation 3.2 guarantees that the B in Theorem 3.4 witnesses the Intersection property of $\{M, N\}$.

Proof of Theorem 3.4. Note that both in M and in N the union of a $\subseteq$-increasing sequence of independent sets may fail to be independent. Therefore Zorn's Lemma cannot be used to extend a common independent set to a maximal one. Even so, this extension always can be done without much effort:

Lemma 3.5. *Every common independent set I of M and N can be extended to a maximal common independent set.*

Proof. Let $i < n$ be arbitrary. By applying Proposition 2.2 with an arbitrary base of $M/I \upharpoonright E_i$ and uniform matroid $U_i/(I \cap E_i)$, we obtain $B_0, B_1 \subseteq E_i$ with $B_0 \subseteq B_1$, where either B_0 is a base of $M/I \upharpoonright E_i$ and B_1 is a base of $U_i/(I \cap E_i)$ or the other way around. It follows directly from the construction that $I \cup B_0 \in M \cap N$, furthermore, $I \cup B_0$ either M-spans or N-spans E_i depending on whose base was B_0. Iterating this with all indices yields a maximal common independent set. □

One may observed that in the previous proof we obtained a seemingly stronger property than maximality. Let us point out that it is not really stronger:

Lemma 3.6. *If I is a maximal element of $M \cap N$, then every E_i is spanned by I in at least one of the matroids*

Proof. Let I be a maximal common independent set and let $i < n$. On the one hand, every $e \in E_i \setminus I$ is spanned by I in at least one of the matroids because $I + e \notin M \cap N$. On the other hand, either $I \cap E_i$ is a base of U_i in which case $E_i \subseteq \mathsf{span}_N(I)$, or $I \cap E_i$ is not a base of U_i but then (by the uniformity of U_i) I does not N-span any edge from $E \setminus I$ and hence we must have $E_i \subseteq \mathsf{span}_M(I)$. $\square$

For $I \in M \cap N$ let

$$\begin{aligned}
\boldsymbol{P(I)} &:= \{i < n : E_i \subseteq \mathsf{span}_M(I) \cap \mathsf{span}_N(I)\}\\
\boldsymbol{Q(I)} &:= \{i \in n \setminus P(I) : E_i \subseteq \mathsf{span}_M(I)\}\\
\boldsymbol{R(I)} &:= n \setminus (P(I) \cup Q(I)).
\end{aligned}$$

Furthermore, for $K \subseteq n$ let $\boldsymbol{E_K} := \cup\{E_i : i \in K\}$ and $\boldsymbol{I_K} := I \cap E_K$.

Lemma 3.7. *For a maximal $I \in M \cap N$ we must have $E_i \subseteq \mathsf{span}_N(I)$ for $i \in R(I)$.*

Proof. We cannot have $E_i \subseteq \mathsf{span}_M(I)$ because $R(I) = \{i < n : E_i \nsubseteq \mathsf{span}_M(I)\}$ by definition thus it follows from Lemma 3.6. $\square$

Lemma 3.8. *There exists a maximal $I \in M \cap N$ with $\mathsf{span}_M(I) \supseteq \mathsf{span}_M(J)$ such that $|R(I)|$ is minimal among such I's and $E_j \subseteq \mathsf{span}_M(I_{P(I)\cup\{k\in Q(I):k\geq j\}})$ for every $j \in Q(I)$.*

Proof. Suppose for a contradiction that it is not the case and take a maximal $I \in M \cap N$ with $\mathsf{span}_M(I) \supseteq \mathsf{span}_M(J)$ minimizing $|R(I)|$ in which the largest violating index $j \in Q(I)$ is minimal. We pick an $F \subseteq E_j \setminus I_j$ such that $I_{P(I)\cup\{k\in Q(I):k\geq j\}} \cup F \in M \cap N$ and either $I_j \cup F$ is a base of U_j or $E_j \subseteq \mathsf{span}_M(I_{P(I)\cup\{k\in Q(I):k\geq j\}} \cup F)$. Then we choose a $G \subseteq E \setminus (I_{P(I)\cup\{k\in Q(I):k\geq j\}} \cup F)$ such that $(I \cup F) \setminus G$ is M-independent again and M-spans G. Finally, we extend $(I \cup F) \setminus G$ to a maximal element I' of $M \cap N$.

On the one hand, either $j \in P(I')$ or $j \in Q(I') \wedge E_j \subseteq \mathsf{span}_M(I_{P(I')\cup\{k\in Q(I'):k\geq j\}} \cup F)$ depending on if $I_j \cup F$ is a base of U_j or not. Note that $\mathsf{span}_M(I') \supseteq \mathsf{span}_M(I)$. It is enough to show that $Q(I') \subseteq Q(I)$ because then the largest violating index for I' is strictly smaller than j contradicting the choice of I. Since I minimized $|R(I)|$ and $\mathsf{span}_M(I') \supseteq \mathsf{span}_M(I)$ we must have $R(I') = R(I)$. By the construction $P(I') \supseteq P(I)$ also holds. Since $P(I') \cup Q(I') \cup R(I')$ is a partition of n, the inclusion $Q(I') \subseteq Q(I)$ follows. $\square$

Let $\boldsymbol{I}$ be as in Lemma 3.8 fixed. If I is a base of N then $B := I$ is as demanded by Theorem 3.4 and we are done. Suppose for a contradiction that it is not the case. It means that $Q(I) \neq \varnothing$. We define a sequence $(I^t)_{t\in\mathbb{N}}$ of maximal common independent sets of M and N. Let us write simply P_t and Q_t for $P(I^t)$ and $Q(I^t)$ respectively. We also maintain an ordering $<_{P_t}$ of P_t and an ordering $<_{Q_t}$ of Q_t.

Let $I^0 = I$ and let $<_{P_0}$ and $<_{Q_0}$ be the usual ordering of $\mathbb{N}$ restricted to P_0 and Q_0 respectively. Suppose that I^t together with the orderings $<_{P_t}$ and $<_{Q_t}$ is already defined for some $t \in \mathbb{N}$ and the following conditions hold:

(1) I^t is a maximal element of $M \cap N$,
(2) $\mathsf{span}_M(I^t) = \mathsf{span}_M(I^0)$,
(3) $E_j \subseteq \mathsf{span}_M(I^t_{P_t\cup\{k\in Q_t:k\geq_{Q_t} j\}})$ for every $j \in Q_t$,

(4) $Q_t \neq \varnothing$.

Let $\boldsymbol{j}$ be the $<_{Q_t}$-largest element of Q_t (exists by (4)). Then by property (3), $E_j \subseteq \mathsf{span}_M(I^t_{P_t+j})$. Let $\boldsymbol{S_t}$ be the smallest $<_{P_t}$-upward closed subset of P_t for which $E_j \subseteq \mathsf{span}_M(I^t_{S_t+j})$. Note that $S(t) \neq \varnothing$ since otherwise I^t_j witnesses that $W := E_j$ violates Condition 3.3. Let $\boldsymbol{i}$ be the $<_{P_t}$-smallest element of S_t. We take an $\boldsymbol{F} \subseteq E_j \setminus I^t_j$ in such a way that $I^t_{S_t-i+j} \cup F \in M \cap N$ and either $I^t_j \cup F$ is a base of U_j or $E_j \subseteq \mathsf{span}_M(I^t_{S_t-i+j} \cup F)$. Then we pick a $\boldsymbol{G} \subseteq I^t_i$ such that $(I^t_{S_t+j} \cup F) \setminus G$ is M-independent again and M-spans G. Note that $F, G \neq \varnothing$. Then Fact 2.1 guarantees that $\boldsymbol{I^{t+1}} := (I^t \cup F) \setminus G$ satisfies properties (1) and (2) while $i \in Q_{t+1}$ ensures property (4). On the sets $P_t \cap P_{t+1}$ and $Q_t \cap Q_{t+1}$ we define $<_{P_{t+1}}$ and $<_{Q_{t+1}}$ to be identical to $<_{P_t}$ and $<_{Q_t}$ respectively. If $I^t_j \cup F$ is a base of U_j, then $P_{t+1} \setminus P_t = \{j\}$ and j is defined to be the $<_{P_{t+1}}$-largest element of P_{t+1}. Furthermore, the unique element i of $Q_{t+1} \setminus Q_t$ is defined to be the $<_{Q_{t+1}}$-largest element of Q_{t+1}. If $I^t_j \cup F$ is not a base of U_j (and hence $E_j \subseteq \mathsf{span}_M(I^t_{S_t-i+j} \cup F)$), then only index i changes position. In this case we define i to be the second $<_{Q_{t+1}}$-largest element of Q_{t+1} (right bellow j). It follows directly from the construction via Fact 2.1 that property (3) is also kept. The recursion is done.

Let $\boldsymbol{O}$ be the set of those indices i that are moving infinitely often. More precisely $i \in O$ if and only if there are infinitely many t with $i \in P_t$ and there are also infinitely many t with $i \in Q_t$. We are going to show that $W := E_O$ contradicts Condition 3.3. Let us choose $\boldsymbol{t_0} \in \mathbb{N}$ in such a way that the position of indices in $n \setminus O$ are stabilized already, i.e., for every $i \in n \setminus O$ either $i \in P_t$ for every $t \geq t_0$ or $i \in Q_t$ for every $t \geq t_0$.

Lemma 3.9. *For every $k \in n \setminus O$ there are only finitely many t for which $k \in S_t$.*

Proof. Suppose for a contradiction that index k is a counterexample. Then necessarily $k \in P_t$ for $t \geq t_0$ since $S_t \subseteq P_t$. On the one hand, if $P_{t+1} \setminus P_t \neq \varnothing$, then for its unique element j we have $k <_{P_{t+1}} j$ because j is the $<_{P_{t+1}}$-largest element of P_{t+1}. On the other hand, whenever $k \in S_t$ for some $t \geq t_0$, then for $i := (\min_{<_{P_t}} S_t) <_{P_t} k$ we have $i \in Q_{t+1}$. Combining these we see that as t goes from t_0 to ∞, index k never gets a new element $<_{P_t}$-bellow itself but loses such an element infinitely many times. Since there are only finitely many indices it is a contradiction. □

By Lemma 3.9 we can find some $\boldsymbol{t_1} \geq t_0$ such that $S_t \subseteq O$ for $t \geq t_1$.

Lemma 3.10. $\mathsf{span}_M(I_{t,O}) = \mathsf{span}_M(I_{t_1,O})$ *for every $t \geq t_1$.*

Proof. We use induction on t. Suppose that we know the statement for some $t \geq t_1$. Let j, F, G be as in the description of the general step of the recursion. Let us point out that necessarily $j \in O$ since otherwise $|P_t|$ would be eventually strictly decreasing in t, since P_t looses one element and get no new in each step after j is stacked on the top of Q_t, which is impossible.

This together with Lemma 3.9 ensures that $S_t + j \subseteq O$ for every $t \geq t_1$. It is enough to show that $\mathsf{span}_M(I^t_{S_t+j}) = \mathsf{span}_M(I^{t+1}_{S_t+j})$ since I^t and I^{t+1} are identical out of E_{S_t+j} (use Fact 2.1). But we know from the recursion that $F \subseteq E_j \subseteq \mathsf{span}_M(I^t_{S_t+j})$ and we chose G in such a way that $G \subseteq \mathsf{span}_M\left[(I^t_{S_t+j} \cup F) \setminus G\right]$, thus $\mathsf{span}_M(I^t_{S_t+j}) = \mathsf{span}_M(I^{t+1}_{S_t+j})$ holds. □

Finally, we show that $E_O \subseteq \mathsf{span}_M(I_O^{t_1})$. Let $j \in O$ be arbitrary and choose some $t \geq t_1$ for which $j \in Q_t$ but $j \in P_{t+1}$. Then by the definition of the recursion j must be the $<_{Q_t}$-largest element of Q_t and we have $E_j \subseteq \mathsf{span}_M(I_{S_t+j}^t)$. Lemma 3.9 ensures $S_t \subseteq O$, thus $E_j \subseteq \mathsf{span}_M(I_O^t)$ follows. But then by Lemma 3.10, $E_j \subseteq \mathsf{span}_M(I_O^{t_1})$. Since $j \in O$ was arbitrary it means $E_O \subseteq \mathsf{span}_M(I_O^{t_1})$. Therefore $W := E_O$ contradicts Condition 3.3 witnessed by $I_O^{t_1}$ which completes the proof of Theorem 3.4.

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