ON A LINKING PROPERTY OF INFINITE MATROIDS

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Abstract. Let M_0 and M_1 be matroids on E having only finitary and cofinitary components and let $X_i \subseteq E$ for $i \in \{0,1\}$. We show that if X_i can be spanned in M_i by an M_{1-i}-independent set for $i \in \{0,1\}$, then there is a common independent set I with $X_i \subseteq \mathsf{span}_{M_i}(I)$ for $i \in \{0,1\}$. As an application we derive an analogue of Pym's theorem in compact graph-like spaces. We also prove a packing-covering-partitioning type of result for matroid families that generalizes the base partitioning theorem [1] of Erde et al.

1. Introduction

Linking property attracts a lot of attention in combinatorics and optimization. Roughly speaking it says that whenever there exists an object satisfying a property A and there is also one satisfying property B, then one can find a single object satisfying both. It was discovered for example in the '50s that matchings of bipartite graphs have the linking property with respect to covering vertices in the two vertex classes:

Theorem 1.1 (N. S. Mendelsohn and A. L. Dulmage, Theorem 1 in [3]). *Let $G = (V_0, V_1; E)$ be a bipartite graph and let $I_0, I_1 \subseteq E$ be matchings in G. Then there exists a matching I such that $V(I) \cap V_i \supseteq V(I_i) \cap V_i$ for $i \in \{0,1\}$.*[1]

An important special case is the following classical theorem in set theory:

Theorem 1.2 (F. Bernstein, G. Cantor, R. Dedekind, E. Schröder). *If there are injections $f_i : V_i \to V_{1-i}$ for $i \in \{0,1\}$ then there exists a bijection f between V_0 and V_1.*

A bipartite graph can be represented by a pair of matroids on E each of which is a direct sum of 1-uniform matroids. Indeed, let U_v be the 1-uniform matroid on the edges incident with v in G and let $M_i := \bigoplus_{v \in V_i} U_v$ for $i \in \{0,1\}$. Note that the common independent sets of M_0 and M_1 are exactly the matchings in G. It led to the following matroidal generalization of Theorem 1.1:

Theorem 1.3 (S. Kundu and E. L. Lawler, [2]). *Let M_i be a matroid on the finite edge set E for $i \in \{0,1\}$. Then for every $I_0, I_1 \in M_0 \cap M_1$ there exists an $I \in M_0 \cap M_1$ with $\mathsf{span}_{M_i}(I) \supseteq I_i$ for $i \in \{0,1\}$.*

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[1]The authors proved it originally only for finite graphs, the general version was discovered later.

The analogue of Theorem 1.3 for arbitrary infinite matroids fails under the Continuum Hypothesis even if M_i is uniform and I_i is a base of it (take U and U^* in Theorem 5.1 of [1]). We have also shown that if M_i is a finitary[2] or cofinitary matroid and $I_i \in M_0 \cap M_1$ is a base of M_i for $i \in \{0,1\}$, then the conclusion of Theorem 1.3 is true, i.e., there exists a common base (see Corollary 1.4 of [1]). The condition that I_i is a base of M_i turned out to be too restrictive in the sense of applicability which motivated further investigation.

Eventually we found an entirely different approach based on a relatively simple but powerful method that led us to the main result of this paper:

Theorem 1.4. *For $i \in \{0,1\}$, let M_i be a matroid on E such that each of its components is either finitary of cofinitary and let $F_i \subseteq E$. Then there exists an $F \subseteq E$ such that $\mathsf{span}_{M_i}(F) \supseteq F_i$ and $\mathsf{span}_{M_i^*}(E \setminus F) \supseteq E \setminus F_{1-i}$ for $i \in \{0,1\}$.*

Note that if $F_{1-i} \in M_i$ for $i \in \{0,1\}$, then the dual conditions mean $F \in M_0 \cap M_1$ thus it really generalizes Theorem 1.3.

In order to introduce an application of Theorem 1.4, we need to mention Pym's theorem and graph-like spaces. Pym's theorem (for undirected graphs) is a generalization of Theorem 1.1 in which disjoint paths are used to connect two vertex classes (instead of independent edges as in the Mendelsohn-Dulmage theorem). A V_0V_1-path-system $\mathcal{P}$ in graph G is a set of pairwise disjoint V_0V_1-paths (i.e., finite paths meeting V_0 and V_1 without having internal vertex in $V_0 \cup V_1$).

Theorem 1.5 (Pym's theorem, [4])**.** *Assume that $G=(V,E)$ is a graph, $V_0, V_1 \subseteq V$ and $\mathcal{P}_i$ are V_0V_1-path-systems in G for $i \in \{0,1\}$. Then there exists a V_0V_1-path-system $\mathcal{P}$ with $V(\mathcal{P}) \cap V_i \supseteq V(\mathcal{P}_i) \cap V_i$ for $i \in \{0,1\}$.*

End compactification of infinite graphs gave rise to new research directions in group theory and infinite graph theory. The central idea is to consider a graph as a cell complex and take the Freudenthal compactification of this space calling the new points ends (for more details see [5]). An even more general phenomenon, the graph-like space, was introduced by Thomassen and Vella in [6]. Roughly speaking, we have a graph $G=(V,E)$ with a totally separated topology on V and for every $e \in E$ we take a copy of $[0,1]$ and identify 0 and 1 with the endpoints of e respectively.

We prove that an analogue of Pym's theorem is true in compact graph-like spaces. To do so, we will define V_0V_1-pseudo-arc systems in a similar way as V_0V_1-path-systems were defined (the precise definition is given later).

Theorem 1.6. *Assume that $\Gamma = (V,E)$ is a compact graph-like space, $V_0, V_1 \subseteq V$ are closed sets and $\mathcal{A}_i$ are V_0V_1-pseudo-arc systems in G for $i \in \{0,1\}$. Then there exists a V_0V_1-pseudo-arc system $\mathcal{A}$ with $V(\mathcal{A}) \cap V_i \supseteq V(\mathcal{A}_i) \cap V_i$ for $i \in \{0,1\}$.*

It will be shown that Theorems 1.5 and 1.6 can be obtained as special instances of our Theorem 1.4.

To prove Theorem 1.4 we will show first the following packing-covering-partitioning type of statement:

[2] A matroid is called (co)fininary if it has only finite (co)circuits.

Theorem 1.7. *Let $P_i, R_i \subseteq E$ for $i \in \Theta$ such that $P_i \cap P_j = \varnothing$ for $i \neq j$ and $\bigcup_{i\in\Theta} R_i = E$. For $i \in \Theta$, let M_i be a matroid on E such that each of its components is either finitary of cofinitary. Then there are $T_i \subseteq P_i \cup R_i$ for $i \in \Theta$ forming a partition of E such that $\mathsf{span}_{M_i}(T_i) \supseteq P_i$ and $\mathsf{span}_{M_i^*}(E \setminus T_i) \supseteq E \setminus R_i$.*

If the sets P_i and R_i are bases of M_i for each $i \in \Theta$, then we get back the main result Theorem 1.2 of [1]. However our new approach yields to a significantly shorter proof.

The paper is structured as follows. In the next section we give a brief introduction on matroids and graph-like spaces. Our main results Theorems 1.4 and 1.7 are proved in the third section. Finally, Pym's theorem in compact graph-like spaces is shown in the last section.

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2. Preliminaries

2.1. **Infinite matroids.** Rado asked in 1966 if there is an infinite generalisation of matroids preserving the key concepts (like duality and minors) of the finite theory. Based on some early results of Higgs [7] and Oxley [8], Bruhn, Diestel, Kriesell, Pendavingh and Wollan answered the question affirmatively and gave a set of cryptomorphic axioms for infinite matroids, generalising the usual independent set-, bases-, circuit-, closure- and rank-axioms for finite matroids (see [9]). They showed that several fundamental facts of the theory of finite matroids are preserved in the infinite case. It opened the door for a more systematic investigation of infinite matroids. An $M = (E, \mathcal{I})$ is a matroid if $\mathcal{I} \subseteq \mathcal{P}(E)$ with

(I) $\varnothing \in \mathcal{I}$;
(II) $\mathcal{I}$ is downward closed;
(III) For every $I, J \in \mathcal{I}$ where J is $\subseteq$-maximal in $\mathcal{I}$ but I is not, there exists an $e \in J \setminus I$ such that $I + e \in \mathcal{I}$;
(IV) For every $X \subseteq E$, any $I \in \mathcal{I} \cap \mathcal{P}(X)$ can be extended to a $\subseteq$-maximal element of $\mathcal{I} \cap \mathcal{P}(X)$.

For a finite E, axioms (I)-(III) are equivalent to the usual axiomatization of matroids in terms of independent sets (while (IV) is automatically true).

The terminology and basic facts we will use are well-known for finite matroids. The elements of $\mathcal{I}$ are called *independent* while the sets in $\mathcal{P}(E) \setminus \mathcal{I}$ are *dependent*. The maximal independent sets are the *bases* and the minimal dependent sets are the *circuits* of the matroid. Every dependent set contains a circuit (which fact is not obvious if E is infinite). A singleton circuit is called a *loop*. The *components* of a matroid are the connected components of the hypergraph of its circuits on E. The *dual* of matroid M is the matroid M^* on the same edge set whose bases are the complements of the bases of M. By the deletion of an $X \subseteq E$ we obtain the matroid $M - X := (E \setminus X, \{Y \in \mathcal{I} : Y \subseteq E \setminus X\})$ and the contraction of X gives $M/X := (M^* - X)^*$. If I is independent in M but $I + e$

is dependent for some $e \in E \setminus I$ then there is a unique circuit $C_M(e, I)$ of M through e contained in $I + e$. We say $X \subseteq E$ spans $e \in E$ in matroid M if either $e \in X$ or there exists a circuit $C \ni e$ with $C - e \subseteq X$. We denote the set of edges spanned by X in M by $\mathsf{span}_M(X)$. A matroid is called *finitary* if all of its circuits are finite. A matroid is *cofinitary* if its dual is finitary. A family $\mathcal{C}$ of subsets of E is the set of the circuits of a cofinitary matroid if and only if the following axioms hold:

($\mathcal{C}1$) $\varnothing \notin \mathcal{C}$;
($\mathcal{C}2$) There are no $C, D \in \mathcal{C}$ with $C \subsetneq D$;
($\mathcal{C}3$) *Strong circuit elimination*: Whenever $e \in C \in \mathcal{C},\ X \subseteq C - e$ and $\{C_x : x \in X\}$ is a subfamily of $\mathcal{C}$ with $C_x \cap X = \{e\}$ and $e \notin C_x$, there is a $D \in \mathcal{C}$ with $e \in D \subseteq [C \cup \bigcup_{x \in X} C_x] \setminus X$.
(cF) If $\mathcal{F}$ is a nested family of subsets of E and $e \in E$ such that each $X \in \mathcal{F}$ contains some $C \in \mathcal{C}$ through e, then $\cap\{X : X \in \mathcal{F}\}$ also contains such a C.

Note that strong circuit elimination implies that if C_1 and C_2 are circuits with $e \in C_1 \setminus C_2$ and $f \in C_1 \cap C_2$, then that there is a circuit C_3 with $e \in C_3 \subseteq C_1 \cup C_2 - f$. For finitary matroids ($\mathcal{C}3$) is actually equivalent with this simpler statement.

For more information about infinite matroids we refer to [10]. We abuse the notation and write simply $M \cap N$ instead of $\mathcal{I}_M \cap \mathcal{I}_N$ for the set of common independent sets of matroids M and N, similarly $I \in M$ is short for $I \in \mathcal{I}_M$.

2.2. **Graph-like spaces.** Graph-like spaces were introduced by Thomassen and Vella in [6]. A strong connection with the theory of infinite matroids was discovered by N. Bowler, J. Carmesin and R. Christian in [11]. A graph-like space is a topological space Γ together with *vertex set* V, *edge set* E and functions $\iota_e : [0, 1] \to \Gamma$ for $e \in E$ satisfying the following:

(I) The underlying set of Γ[3] is the disjoint union of V and $E \times (0, 1)$;
(II) For every $e \in E$:
  (a) $\iota_e(x) = (e, x)$ for $x \in (0, 1)$
  (b) $\iota_e(0), \iota_e(1) \in V$,
  (c) ι_e is continues,
  (d) ι_e is a closed map,
  (e) $\iota_e \restriction (0, 1)$ is an open map;
(III) The subspace V is totally separated.

It follows from the axioms that graph-like spaces are Hausdorff. The set $\{e\} \times (0, 1)$ is the *inner points* of edge e while $\iota_e(0)$ and $\iota_e(1)$ are its *end-vertices* (or just end-vertex if $\iota_e(0) = \iota_e(1)$, in which case e is a *loop*). If $V(\Gamma) = U \cup W$ is a bipartition where U and W are clopen in the subspace $V(\Gamma)$, then the set of edges with one end-vertex in U and the other in W is called a *topological cut* of Γ. It is easy to see that U and W can be extended to disjoint open sets of Γ and the topological connectedness of Γ is equivalent with the non-existence of an empty topological cut. A *graph-like subspace* H of Γ is a graph-like space where H is a subspace of Γ in the topological sense, $V(\Gamma) \supseteq V(H),\ E(\Gamma) \supseteq E(H)$ and $\iota_e^H = \iota_e^\Gamma$ for $e \in E(H)$. The deletion of $F \subseteq E(\Gamma)$ from Γ is the graph-like subspace

[3] we abuse the notation and denote the underlying set also by Γ

$\Gamma - F := \Gamma \setminus (F \times (0,1))$ on the same vertex set $V(\Gamma)$ and with edge set $E(\Gamma) \setminus F$ and let $\Gamma(F) := \Gamma - (E(\Gamma) \setminus F)$. Note that for a compact Γ the deletion of edges preserves compactness. The *contraction* of a closed vertex set $W \subseteq V$ in a compact graph-like space Γ is a compact graph-like space Γ/W obtained by identifying the vertices W, i.e., we take the quotient topology with respect to the equivalence of the elements in W, $V(\Gamma/W)$ consists of the vertices $V \setminus W$ together with the equivalence class w corresponding to W and for $e \in E(\Gamma/W) := E(\Gamma)$ and $x \in [0,1]$ we have $\iota_e^{\Gamma/W}(e,x) := \iota_e^{\Gamma}(e,x)$ if the right side is not a vertex in W and w otherwise.

A *pseudo-arc* between u and v is a compact connected graph-like space A with $u, v \in V(A)$ in which every $e \in E(A)$ separates u and v (i.e., u and v are in different connected components of $A - e$). We call a pseudo-arc *trivial* if it consists of a single vertex. A *pseudo-circle* is a compact connected graph-like space C with $E(C) \neq \varnothing$ where

- $C - e$ is connected for each $e \in E(C)$ but the deletion of any pair of edges disconnects C,
- any vertex pair of C can be disconnected by the deletion of a suitable edge pair.

A graph-like space Γ is called *pseudo-arc-connected* if for any $u \neq v \in V(\Gamma)$ there is a graph-like subspace A of Γ which is pseudo-arc between u and v.

Theorem 2.1 (P. J. Gollin and J. Kneip, Theorem 4.6 in [13])**.** *A compact graph-like space is (topologically) connected if and only if it is pseudo-arc connected.*

Finally, we will use the following fundamental facts where the analogous graph theoretic statements are trivial.

Fact 2.2 (Lemma 4.4 in [13])**.** *If C is a pseudo-circle and $e \in E(C)$, then $C - e$ is a pseudo-arc between the end-vertices of e.*

If A is a pseudo-arc between u and v in a graph-like space Γ and the end-vertices of $e \in E(\Gamma)$ are u and v, then $A \cup [\{e\} \times (0,1)]$ is a pseudo-circle.

A *graph-like tree* is a connected graph-like space without pseudo-circles.

Fact 2.3 (Proposition 4.9 of [13])**.** *A compact loop-free graph-like space is a graph-like-tree if and only if each vertex pair is connected by a unique pseudo-arc.*

3. The proof of the main results

Let us start with the packing-covering-partitioning variant of our main result. We repeat it here for convenience.

Theorem. *Let $P_i, R_i \subseteq E$ for $i \in \Theta$ such that $P_i \cap P_j = \varnothing$ for $i \neq j$ and $\bigcup_{i\in\Theta} R_i = E$. For $i \in \Theta$, let M_i be a matroid on E such that each of its components is either finitary of cofinitary. Then there are $T_i \subseteq P_i \cup R_i$ for $i \in \Theta$ forming a partition of E such that $\mathsf{span}_{M_i}(T_i) \supseteq P_i$ and $\mathsf{span}_{M_i^*}(E \setminus T_i) \supseteq E \setminus R_i$.*

Proof. We may assume without loss of generality by "trimming" the sets R_i that they form a partition of E. We can also assume that $P_i \in M_i$ since otherwise we replace P_i with a maximal M_i-independent subset of it. It is enough to consider the case where $P_i \cap R_i = \varnothing$ for $i \in \Theta$ since if it is not the case we contract $P_i \cap R_i$ and delete $P_j \cap R_j$ for $j \neq i$ in M_i.

Finally, by dividing each M_i into a finitary and a cofinitary matroid (which we extend to E by loops) and bipartition the sets R_i and P_i according to this, it is enough to deal with matroid families where each M_i is either finitary or cofinitary.

Let $<_i$ be a well-order on $P_i \cup R_i$ where $r <_i p$ for every $p \in P_i$ and $r \in R_i$. We define a well-order $\prec_i$ on the set $[P_i \cup R_i]^{<\aleph_0}$ of finite subsets of $P_i \cup R_i$. For $X \neq Y \in [P_i \cup R_i]^{<\aleph_0}$ let $X \prec_i Y$ if one of the following holds:

- $X = \varnothing$,
- $\max X <_i \max Y$,
- $\max X = \max Y =: z$ and $X - z \prec_i Y - z$.

It is easy to check that $\prec_i$ is indeed a well-order.

Observation 3.1. *If $X \prec_i Y$ then $X + z \prec_i Y + z$ for every $z \in P_i \cup R_i$.*

Let $\langle E_\beta : \beta < \alpha \rangle$ be a sequence of subsets of E where α is a limit ordinal. If

$$\bigcup_{\gamma<\alpha} \bigcap_{\beta>\gamma} E_\beta = \bigcap_{\gamma<\alpha} \bigcup_{\beta>\gamma} E_\beta$$

then we call this set the limit of the sequence and denote it by $\lim \langle E_\beta : \beta < \alpha \rangle$. Since a finite subset of the limit is a subset of all the members with large enough index, we obtain the following.

Observation 3.2. *Suppose that E_α is the limit of $\langle E_\beta : \beta < \alpha \rangle$.*

(i) If E_α contains an M_i-circuit $C \nsubseteq R_i$ where M_i is finitary, then so does E_β for every large enough $\beta < \alpha$;

(ii) If $g \in \mathsf{span}_{M_i}(E_\beta)$ for $\beta < \alpha$ where M_i is cofinitrary then $g \in \mathsf{span}_{M_i}(E_\alpha)$.

To construct the desired partition $(T_i : i \in \Theta)$, we apply transfinite recursion. Let $T_i^0 := P_i \in M_i$ for $i \in \Theta$. Suppose that T_i^β is defined for $\beta < \alpha$ and $i \in \Theta$ satisfying the following properties:

(1) $T_i^\beta \cap T_j^\beta = \varnothing$ for $i \neq j \in \Theta$,

(2) $T_i^\beta \subseteq P_i \cup R_i$,

(3) $T_i^\beta \cap P_i$ is $\subseteq$-decreasing and $T_i^\beta \cap R_i$ is $\subseteq$-increasing in β,

(4) T_i^β is the limit of $\left\langle T_i^\delta : \delta < \beta \right\rangle$ if β is a limit ordinal,

(5) $\mathsf{span}_{M_i}(T_i^\beta) \supseteq P_i$,

(6) for every finitary M_i each M_i-circuit $C \subseteq T_i^\beta$ is a subset of R_i,

(7) for every finitary M_i and $g \in P_i$ the $\prec_i$-smallest finite set S_g^β which is witnessing $g \in \mathsf{span}_{M_i}(T_i^\beta)$ is a $\preceq_i$-decreasing function of β,

(8) $(T_i^\delta : i \in \Theta) \neq (T_i^{\delta+1} : i \in \Theta)$ for $\delta + 1 < \alpha$.

Note that condition (6) is a rephrasing of "$\mathsf{span}_{M_i^*}(E \setminus T_i^\beta) \supseteq E \setminus R_i$ for finitary M_i". Assume first that α is a limit ordinal. Then conditions (2) and (3) guarantee that $T_i^\alpha := \lim \left\langle T_i^\beta : \beta < \alpha \right\rangle$ is well-defined. Preservation of conditions (1)-(4) is straightforward. Condition (5) restricted to cofinitary matroids and condition (6) are kept by Observation 3.2. To check condition (5) for a finitary M_i, let $g \in P_i$ be arbitrary. Since $\preceq_i$ is a well-order, it follows from condition (7) that there is an S_g such that $S_g^\beta = S_g$ for all large enough $\beta < \alpha$. But then $S_g \subseteq T_i^\alpha$ from which $g \in \mathsf{span}_{M_i}(T_i^\alpha)$ follows. Furthermore, clearly $S_g^\alpha = S_g$ since a finite set which is $\prec_i$-smaller than S_g and M_i-spans g would have appeared already before the limit.

Suppose now that $\alpha = \beta + 1$. If $\bigcup_{i\in\Theta} T_i^\beta \supseteq E$ and the analogue of condition (6) for the cofinitary M_i holds, then $(T_i^\beta : i \in \Theta)$ is a desired partition of E and we are done. Suppose it is not the case. If there is some T_j^β that contains an M_j-circuit C with $C \nsubseteq R_j$, then we take an $e \in P_j \cap C$ (see property (2)) and define $T_j^{\beta+1} := T_j^\beta - e$ and $T_i^{\beta+1} := T_i^\beta$ for $i \neq j$. The preservation of the conditions (1)-(8) is trivial. If there is no such a T_j^β, then there must be some $e \in E$ which is not covered by the sets T_i^β. Then there is a unique $k \in \Theta$ with $e \in R_k$. If M_k is cofinitary then let $T_k^{\beta+1} := T_k^\beta + e$ and $T_i^{\beta+1} := T_i^\beta$ for $i \neq k$. We proceed the same way if M_k is finitary and $T_k^\beta + e$ does not contain any M_k-circuit C with $C \nsubseteq R_k$. The preservation of the conditions is again straightforward in both cases.

Finally assume that M_k is finitary and $T_k^\beta + e$ contains an M_k-circuit C with $C \subsetneq R_k$. Let f be the $<_k$-maximal element of C and we define $T_k^{\beta+1} := T_k^\beta + e - f$ and $T_i^{\beta+1} := T_i^\beta$ for $i \neq k$. Since $C \cap P_k \neq \varnothing$ (because $C \nsubseteq R_k$) and the elements of P_k are $<_k$-larger then the elements of R_k, we have $f \in P_k$. Conditions (1)-(5) remain true for obvious reasons. Suppose for a contradiction that condition (6) fails and C' is an M_k-circuit in $T_k^{\beta+1}$ with $C' \nsubseteq R_k$. Then $f \notin C'$ and we must have $e \in C'$ since otherwise $C' \subseteq T_k^\beta$ and therefore this condition would have been already violated with respect to T_k^β. By applying strong circuit elimination with the M_k-circuits C and C' we obtain a circuit $C'' \subseteq C \cup C' - e$ through f. But then $C'' \subseteq T_k^\beta$ is an M_k-circuit with $C'' \nsubseteq R_k$ violating condition (6) for β which is a contradiction. To check (7), we may assume that $f \in S_g^\beta$ since otherwise $S_g^\beta \subseteq T_k^{\beta+1}$ and thus $S_g^{\beta+1} \preceq_k S_g^\beta$. If $S_g^\beta = \{g\}$, then by the previous sentence we have $f = g$ and by the choice of f we have $S_f^{\beta+1} \preceq_k C - f \prec_k \{f\}$. Otherwise there is an M_k-circuit $C' \ni f, g$ such that $S_g^\beta = C' - g \subseteq T_k^\beta$. By applying strong circuit elimination with C and C' we obtain a circuit $C'' \subseteq C \cup C' - f$ through g. Since $f \in C' \setminus C''$ and each element of $C'' \setminus C'$ is $\prec_k$-smaller than f (because $f = \max_{\prec_k} C$) we may conclude that $C'' \setminus C' \prec_k C' \setminus C''$ and hence by applying Observation 3.1 repeatedly with the edges $C' \cap C''$ we get $C'' - g \prec_k C' - g$. Therefore

$$S_g^{\beta+1} \preceq_k C'' - g \prec_k C' - g = S_g^\beta.$$

The recursion is done and it terminates at some ordinal since the constructed set families $(T_i^\beta : i \in \Theta)$ are pairwise distinct by conditions (2), (3) and (8). $\square$

Note that if each M_i is cofinitary then the proof above can be shorten significantly. Indeed, we do not need the well-orders $<_i$ and $\prec_i$ and the transfinite recursion becomes essentially a "greedy" approach. Let us sketch a similarly simple proof for the special case where all M_i are finitary. We apply transfinite recursion starting with the set family $\{R_i : i \in \Theta\}$ and "going towards" $\{P_i : i \in \Theta\}$. In the general step we have a partition $E = \bigcup_{i\in\Theta} T_i$ where for $i \in \Theta$, each M_i-circuit C with $C \subseteq T_i$ is a subset of R_i. Then we pick a j with $P_j \nsubseteq \mathsf{span}_{M_j}(T_j)$ and add an $e \in P_j \setminus \mathsf{span}_{M_j}(T_j)$ to T_j and remove it from the unique $T_k \cap R_k$ that contained it. Limit steps are defined the limits of sequences so far. Since cycles are finite, a violating C cannot appear in a limit step.

We proceed with the Mendelsohn-Dulmage type of formulation which was our original goal.

Theorem. *For $i \in \{0,1\}$, let M_i be a matroid on E that such that each of its components is either finitary of cofinitary and let $F_i \subseteq E$. Then there exists an $F \subseteq E$ such that $\mathsf{span}_{M_i}(F) \supseteq F_i$ and $\mathsf{span}_{M_i^*}(E \setminus F) \supseteq E \setminus F_{1-i}$ for $i \in \{0,1\}$.*

Proof. We can assume by "trimming" that $F_i \in M_i$ for $i \in \{0,1\}$. Furthermore, we may suppose by contracting $F_0 \cap F_1$ and deleting $E \setminus (F_0 \cup F_1)$ in both matroids that the sets F_i form a bipartition of E. We apply Theorem 1.7 with the matroids M_0 and M_1^* and sets $P_0 := R_1 := F_0$ and $P_1 := R_0 := F_1$. From the resulting bipartition $E = T_0 \cup T_1$ we take $F := T_0$. Then

(1) $\mathsf{span}_{M_0}(F) \supseteq F_0$,
(2) $\mathsf{span}_{M_1^*}(E \setminus F) \supseteq F_1$,
(3) $\mathsf{span}_{M_0^*}(E \setminus F) \supseteq F_0$,
(4) $\mathsf{span}_{M_1}(F) \supseteq F_1$.

□

It is worth to mention that Theorems 1.7 and 1.4 are actually equivalent. On the one hand, the special case of Theorem 1.7 where $|\Theta| = 2$ has a direct connection with Theorem 1.4 through the dualization of one of the matroids. On the other hand, a technique of N. Bowler and J. Carmesin makes possible to reduce Theorem 1.7 to this special case (see Proposition 3.8 in [12])

4. Pym's Theorem in compact graph-like spaces

In this section we derive Theorem 1.6 from Theorem 1.4. First we illustrate our proof method by giving a new proof for Theorem 1.5 that we restate here for convenience.

Theorem. *Assume that $G = (V, E)$ is a graph, $V_0, V_1 \subseteq V$ and $\mathcal{P}_i$ are V_0V_1-path-systems in G for $i \in \{0,1\}$. Then there exists a V_0V_1-path-system $\mathcal{P}$ with $V(\mathcal{P}) \cap V_i \supseteq V(\mathcal{P}_i) \cap V_i$ for $i \in \{0,1\}$.*

Proof. For $i \in \{0,1\}$, we define M_i to be the finite cycle matroid[4] of the graph we obtain from G by contracting V_i to a single vertex. Then $E(\mathcal{P}_i) \in M_0 \cap M_1$. By applying Theorem 1.4 with $F_i := E(\mathcal{P}_i)$ and M_{1-i}, we can find an $F \in M_0 \cap M_1$ with $E(\mathcal{P}_{1-i}) \subseteq \mathsf{span}_{M_i}(F)$ for $i \in \{0,1\}$. Then F is a forest in which every tree meets each V_i at most once. Each connected component of F which meets both V_i contains a unique V_0V_1-path. We define $\mathcal{P}$ to be the set of these paths. It remains to show that $\mathcal{P}$ satisfies the requirements. Suppose that $P \in \mathcal{P}_i$ with vertices $v_0, \dots, v_n$ enumerated in the path-order starting from V_i. It follows from $E(P) \subseteq \mathsf{span}_{M_{1-i}}(F)$ that for every $k < n$ the vertices v_k and v_{k+1} are either in the same connected component of F or both of them is reachable from V_{1-i} in F. Thus by induction v_0 is reachable from V_{1-i} in F but then the path witnessing this is in $\mathcal{P}$. □

The core of our topological variant is the following unpublished result:

Theorem 4.1 (N. Bowler and J. Carmesin)**.** *For every compact graph-like space Γ the edge sets of the pseudo-circles in Γ define a cofinitary matroid on $E(\Gamma)$ in a means of its circuits.*

[4]the circuits are the edge sets of the graph theoretic cycles

Proof. Let Γ be fixed. We show that axioms $(\mathcal{C}1)$-$(\mathcal{C}3)$ and (cF) hold for $\mathcal{C} := \{E(C) : C \text{ is a pseudo-circle in } \Gamma\}$. A pseudo-circle C must have at least one edge by definition thus $E(C) \neq \varnothing$. Suppose for a contradiction that $E(C) \subsetneq E(D)$ for some pseudo-circles. Then for $e \in E(D) \setminus E(C)$ and $f \in E(C)$ the space $D - e - f$ is still connected which contradicts the definition of the pseudo-circle.

We proceed with the strong circuit elimination axiom. Let C be a pseudo-circle with $e \in E(C)$. Suppose that $X \subseteq E(C) - e$ and there is a family $\{C_x : x \in X\}$ of pseudo-circles such that $E(C_x) \cap X = \{x\}$ and $e \notin E(C_x)$ for $x \in X$. We need to find a pseudo-circle D with

$$e \in E(D) \subseteq \left[E(C) \cup \bigcup_{x \in X} E(C_x)\right] \setminus X =: F.$$

Let us denote the end-vertices of e by u and w. We may assume $u \neq w$ since otherwise loop e is suitable for D. By Fact 2.2 it is sufficient to find an arc A between u and w with $E(A) \subseteq F$. To do so it is enough to show that u and w are in the same connected component of the graph-like subspace $\Gamma(F)$ (see Theorem 2.1). Suppose for a contradiction that it is not the case. Then there is an empty topological cut in $\Gamma(F)$ separating u and w, i.e., there is a bipartition $V(\Gamma) = U \cup W$ with $U \ni u$ and $W \ni w$ where U and W are open in $V(\Gamma)$ and for each $f \in F$, the end-vertices of f are either both in U or both in W. The pseudo-arc $C - e$ (see Fact 2.2) between u and w must have an edge f between U and W. Since $f \notin F$, we must have $f \in X$. But then $\Gamma(F)$ contains the pseudo-arc $C_f - f$ that joins the end-vertices of f, thus some $g \in E(C_f - f) \subseteq F$ goes between U and W which is a contradiction.

Finally, we check (cF). Let $e \in E$ and let $\mathcal{F}$ be a nested family of subsets of E such that for every $X \in \mathcal{F}$ there is a pseudo-circle C in Γ with $e \in E(C) \subseteq X$. Suppose for a contradiction that the intersection Y of the elements of $\mathcal{F}$ does not contain such a pseudo-circle. Let u and w be the end-vertices of e. Note that e cannot be a loop so $u \neq w$. Then the graph-like subspace $H := \Gamma(Y - e)$ does not contain any pseudo-arcs between u and w by Fact 2.2. Since H is a compact graph-like space, Theorem 2.1 ensures that H admits a bipartition $H = U \cup W$ into open sets separating u and w. We can lift it up to obtain disjoint open sets $U' \supseteq U$ and $W' \supseteq W$ in Γ (for example if $f \in E(\Gamma) \setminus E(H)$ with say $\iota_f^\Gamma(0) = v \in U$, then we add $\{f\} \times (0, \frac{1}{2})$ to U). The open sets $\{f\} \times (0,1)$ for $f \in E(\Gamma) \setminus E(H)$ together with U' and W' form an open cover of Γ. Since there exists a finite subcover, there is a finite $F \subseteq E(\Gamma) \setminus E(H)$ such that $\Gamma = [F \times (0,1)] \cup U' \cup W'$. Then each $f \in E(\Gamma)$ with one end-vertex in U' and the other in W' must be in F. Since $\mathcal{F}$ is nested, there is an $X \in \mathcal{F}$ such that $X \cap F \subseteq \{e\}$. Therefore in the graph-like subspace $\Gamma(X - e)$ there is no pseudo-arc between u and w. But it implies by Fact 2.2 that there is no pseudo-circle in $\Gamma(X)$ through e which is a contradiction. $\square$

Let us state an important consequence of Fact 2.2.

Corollary 4.2. *Assume that graph-like space Γ induces a matroid M and let $F \subseteq E(\Gamma)$. Then for each $e \in \mathsf{span}_M(F)$ there is a pseudo-arc in $\Gamma(F)$ between the end-vertices of e.*

Now we are ready to prove the compact graph-like space version of Pym's theorem which we repeat here for convenience.

Theorem. *Assume that $\Gamma = (V, E)$ is a compact graph-like space, $V_0, V_1 \subseteq V$ are closed sets and $\mathcal{A}_i$ are V_0V_1-pseudo-arc systems in Γ for $i \in \{0,1\}$. Then there exists a V_0V_1-pseudo-arc system $\mathcal{A}$ with $V(\mathcal{A}) \cap V_i \supseteq V(\mathcal{A}_i) \cap V_i$ for $i \in \{0,1\}$.*

Proof. Since Γ/V_i is a compact graph-like space, Theorem 4.1 ensures that it induces a cofinitary matroid M_i for $i \in \{0,1\}$. Then Theorem 1.4 gives an $F \in M_0 \cap M_1$ with $E(\mathcal{A}_{1-i}) \subseteq \mathsf{span}_{M_i}(F)$ for $i \in \{0,1\}$. It is easy to check using Theorem 2.1 that each connected component of $\Gamma(F)$ is a tree-like space meeting V_i at most once for $i \in \{0,1\}$. Each connected component of $\Gamma(F)$ which meets both V_i contains a unique V_0V_1-pseudo-arc by Fact 2.3. We define $\mathcal{A}$ to be the set of these pseudo-arcs and show that it is as desired.

Let $i \in \{0,1\}$ and $A \in \mathcal{A}_i$ be arbitrary where $V(A) \cap V_i = \{v_i\}$. It is enough to show that v_i and some $v \in V_{1-i}$ are in the same component of $\Gamma(F)$ (see Fact 2.3). Suppose for a contradiction that it is not the case. Since $\Gamma(F)$ is compact and Hausdorff, the Šura-Bura lemma[5] (see for example in [14]) guarantees that the connected component containing an $x \in \Gamma(F)$ can be obtained as the intersection of the clopen subsets of $\Gamma(F)$ containing x. Thus for every $v \in V_{1-i}$ there is a $\Gamma(F)$-clopen U_v containing v but not v_i. Combining this with the compactness of V_{1-i}, we can find a $\Gamma(F)$-open bipartition $U_0 \cup U_1 = \Gamma(F)$ with $U_i \ni v_i$ and $U_{1-i} \supseteq V_{1-i}$. Since A joins v_i and V_{1-i}, there must be some $e \in E(A)$ having one end-vertex u_0 in U_0 and the other u_1 in U_1. Let v_{1-i} be the vertex representing the equivalence class of V_{1-i} in $\Gamma(F)/V_{1-i}$. On the one hand, $U_i' := U_i$ and $U_{1-i}' := U_{1-i} \setminus V_{1-i} + v_{1-i}$ is an open bipartition of $\Gamma(F)/V_{1-i}$ with $U_i' \ni u_i$, $U_{1-i}' \ni u_{1-i}$. On the other hand, we guaranteed that $e \in \mathsf{span}_{M_{1-i}}(F)$ and hence by Corollary 4.2 the vertices u_0 and u_1 are in the same connected component of $\Gamma(F)/V_{1-i}$ which is a contradiction. $\square$

One cannot omit the assumption that the sets V_i are closed in Theorem 1.6. Indeed, let us consider the graph-like tree T that we obtain as the end compactification of the graph on Figure 1 (where the newly added vertex is u_ω). We define $V_0 := \{u_i : i \leq \omega\}$ and $V_1 := \{w_i : i < \omega\}$. Let $\mathcal{A}_0$ consists of the unique pseudo-arc (actually arc, i.e., homeomorphic with $[0,1]$) between u_ω and w_0 and let $\mathcal{A}_1$ consists of the unique (vertical) arcs joining u_i and w_i for $i < \omega$. Any non-trivial pseudo-arc A with one end u_ω goes through the unique neighbour of w_i for infinitely many $i < \omega$. It means that those w_i cannot be connected to V_0 with an arc disjoint from A. Thus there is not even a V_0V_1-arc system $\mathcal{A}$ with $V(\mathcal{A}) \cap V_0 \ni u_\omega$ and $V(\mathcal{A}) \supseteq V_1$.

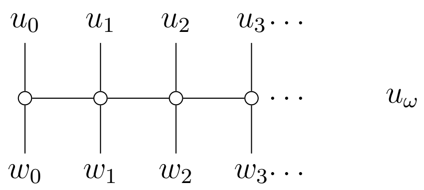

FIGURE 1. Topological Pym may fail if the sets V_i are not closed.

If $V_0 \cap V_1 \neq \varnothing$ in Theorem 1.6, then the pseudo-arcs in the systems A_i meeting $V_0 \cap V_1$ are necessarily trivial. One might prefer a sufficient condition in this special case of

[5]Components and quasi-components coincide in compact Hausdorff spaces.

Theorem 1.6 where the sets V_i are disjoint. Let us point out that in this case "being closed" is an unnecessarily strong restriction for the sets V_i. One can show that it would imply that the pseudo-arc systems $\mathcal{A}_i$ must be finite. A weaker sufficient condition that we get instead directly from Theorem 1.6 is the following: there are closed sets $K_0, K_1 \subseteq V(\Gamma)$ with $K_i \setminus K_{1-i} = V_i$ for $i \in \{0, 1\}$.

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