A canonical tree-of-tangles theorem for submodular separation systems

Christian Elbracht Jakob Kneip

7th September 2020

We show that every structurally submodular separation system admits a canonical tree set which distinguishes its tangles.

The concept of *tangles* has its origins in Robertson's and Seymours graph minor project [6], where tangles were introduced as a unifying framework with which to describe and study highly cohesive substructures in graphs. A central theorem in their work is a *tree-of-tangles theorem*, which roughly says that the tangles of the graph give rise to a tree-shaped decomposition of that graph with each tangle in a different part.

Since their inception the theory of tangles has seen a number of advancements: it has been discovered [3, 5] that the notion of tangles can be formulated more abstractly and does not require an underlying graph structure, making it applicable to a wider field of combinatorial structures. A *separation system* in this abstract set-up is an axiomatisation of separations of well-known structures such as graphs or matroids: a set whose elements we call *separations* that is equipped with a poset structure in which all the properties important for tangle theory such as 'nested', 'orientation', or 'consistency' can be expressed [5].

This higher level of abstraction, and of stripping away the superfluous information about the underlying graph, have facilitated a number of cleaner proofs and stronger results. A recent result [3] of Diestel, Hundertmark, and Lemanczyk extends Robertsons's and Seymours tangle-tree theorem to tangles outside graph theory by finding a *tree set*, a set of pairwise nested separations, and in addition to this achieves a significant strengthening: the tree set found in [3] can be built *canonically.* The latter means that the construction of the tree set can be carried out using exclusively invariants of the given combinatorial structure. Having a canonical way of constructing the tree set is desirable, for instance, for reproducibility of results when implementing an algorithm for this construction: the canonicity guarantees that the algorithm will construct the same tree set regardless of how the separation system and tangles to be distinguished are presented to it as input.

Establishing a canonical tree-of-tangles theorem has been a long-standing goal in tangle theory since the original proof in [6] relied on a technique that is unable to produce canonical results. A first breakthrough towards this goal was achieved in [1], which managed to establish such a canonical theorem for tangles in graphs. With a similar overall strategy [3] could then extend this canonical result to arbitrary separation systems and the most general class of tangles called *profiles*.

A central ingredient in the proof of [3]'s result is an *order function* on the separations considered, similar to the order $|A \cap B|$ of a separation (A, B) of a graph that was already used in [6] and [1]. In this setting one then considers the separation system $\vec{S}_k$ of all separations of order less than k and studies its tangles. In analogy to the function $(A, B) \mapsto |A \cap B|$ from graphs this order function is usually assumed to be submodular. This submodularity of the order function has a structural effect on the separation system $\vec{S}_k$: for any two separations in $\vec{S}_k$ at least one of their pairwise join and meet (which are separations given by opposite 'corners' of the given pair of separations) again lies in $\vec{S}_k$.

Later Diestel, Erde, and Weißauer [2] showed that the latter structural condition by itself is already sufficiently strong for proving tree-of-tangles theorems: tangle theory can be meaningfully studied without the hitherto usual assumption of a submodular order function, further widening its applicability. If a separation system has this structural property but not necessarily a submodular order function then it is *structurally submodular* or simply *submodular* if the context is clear. The tree-of-tangles theorem established in [2] then reads as follows:

Theorem 1 ([2, Theorem 6])**.** *Let $\vec{S}$ be a structurally submodular separation system and $\mathcal{P}$ a set of profiles of S. Then $\vec{S}$ contains a tree set N that distinguishes $\mathcal{P}$.*

The tree set N *distinguishes* $\mathcal{P}$ if each pair of profiles in $\mathcal{P}$ differs on some separation in N. For formal definitions we refer the reader to [5].

This Theorem 1 is even more widely applicable than the result of [3], but has one major downside: it does not yield canonicity since the proof in [2] chooses certain separations arbitrarily.

In this note we present a proof which makes use of invariants of $\vec{S}$ and $\mathcal{P}$ only, and thereby establishes the following canonical version of Theorem 1:

Theorem 2. *Let $\vec{S}$ be a structurally submodular separation system and $\mathcal{P}$ a set of profiles of S. Then there is a nested set $N = N(\vec{S}, \mathcal{P}) \subseteq S$ which distinguishes $\mathcal{P}$. This $N(\vec{S}, \mathcal{P})$ can be chosen canonically: if $\varphi\colon \vec{S} \to \vec{S'}$ is an isomorphism of separation systems and $\mathcal{P}' := \{\varphi(P) \mid P \in \mathcal{P}\}$ then $\varphi(N(\vec{S}, \mathcal{P})) = N(\vec{S'}, \mathcal{P}')$.*

Note that in our formulation of canonicity in Theorem 2 we do not require that the map φ preserves pairwise joins and meets. If φ does preserve them then its image $\vec{S'}$ is also structurally submodular and $\mathcal{P}'$ a set of profiles of $\vec{S'}$. If φ does not preserve pairwise joins and meets but only the partial order on $\vec{S}$ then it may happen that $\vec{S'}$ is no longer a structurally submodular separation system or that the orientations of $\vec{S'}$ contained in $\mathcal{P}'$ are no longer profiles. However even in those cases since φ preserves nestedness we can nevertheless follow the construction of $N(\vec{S}, \mathcal{P})$ to obtain a nested set $N(\vec{S'}, \mathcal{P}')$ which distinguishes all orientations in $\mathcal{P}'$, and we will have $\varphi(N(\vec{S}, \mathcal{P})) = N(\vec{S'}, \mathcal{P}')$ as asserted in Theorem 2. In other words: the construction of $N(\vec{S}, \mathcal{P})$ uses only invariants of $\vec{S}$

and $\mathcal{P}$ and does not depend on the way in which $\vec{S}$ is embedded in some lattice structure of separations. If there is an embedding of $\vec{S}$ into a lattice of separations that makes $\vec{S}$ structurally submodular and $\mathcal{P}$ a set of profiles of S, then the construction of $N(\vec{S}, \mathcal{P})$ succeeds not only inside this lattice structure, but also for any other embedding of $\vec{S}$.

For a full introduction to tangle theory and its terminology and notation we refer the reader to [5]. The remainder of this note is dedicated to the proof of Theorem 2.

A common tool in proving tree-of-tangles theorems is the so called fish lemma:

Lemma 3 ([5, Lemma 3.2])**.** *Let $r, s \in S$ be two crossing separations. Every separation t that is nested with both r and s is also nested with all four corner separations of r and s.*

A *universe of separations* is a separation system whose poset is a lattice with pairwise join and meet operations $\vee$ and $\wedge$. For the rest of this note let $\vec{S}$ be a structurally submodular separation system inside some universe $\vec{U}$ of separations and $\mathcal{P}$ a set of profiles of S.

We need the following additional terminology. A separation $\vec{s} \in \vec{S}$ is *exclusive (for $\mathcal{P}$)* if it lies in exactly one profile in $\mathcal{P}$. If $P \in \mathcal{P}$ is the profile containing an exclusive separation $\vec{s}$ then we might also say that $\vec{s}$ is *P-exclusive (for $\mathcal{P}$)*. Observe that if $\vec{r}$ is P-exclusive for $\mathcal{P}$, then so is every $\vec{s} \in P$ with $\vec{r} \le \vec{s}$.

For each $P \in \mathcal{P}$ let M_P consist of the maximal elements of the set of all P-exclusive separations. Equivalently, M_P is the set of all maximal elements of P that are exclusive for $\mathcal{P}$.

Our strategy for proving Theorem 2 will be to canonically pick nested P-exclusive representatives of all profiles $P \in \mathcal{P}$ that contain exclusive separations, then discard from $\mathcal{P}$ and $\vec{S}$ all those profiles P for whom we selected a representative and all those separations not nested with these representatives, respectively. Iterating this procedure will yield the canonical nested set.

In order for this strategy to work we must ensure that the sets M_P are not all empty. Our first lemma addresses this:

Lemma 4. *If $\mathcal{P}$ is non-empty, then some M_P is non-empty.*

The existence of exclusive separations and thus Lemma 4 is actually an immediate consequence of Theorem 1: if $N \subseteq S$ is a nested set which distinguishes $\mathcal{P}$, and each element of N distinguishes some pair of profiles in $\mathcal{P}$, then any maximal element of $\vec{N}$ is exclusive for $\mathcal{P}$. In other words, the separations labelling the incoming edges of leaves of the tree associated with N are exclusive.

To avoid the proof of Theorem 2 relying on its non-canonical version, let us give an independent proof of Lemma 4.

Proof of Lemma 4. If $\mathcal{P}$ consists of only one profile the assertion is trivial. For $|\mathcal{P}| \ge 2$ we show the following stronger claim by induction on $|\mathcal{P}|$:

If $|\mathcal{P}| \ge 2$ there is for each $P \in \mathcal{P}$ a separation that is exclusive but not P-exclusive for $\mathcal{P}$.

For the base case $|\mathcal{P}| = 2$ observe that any separation distinguishing the two profiles in $\mathcal{P}$ has two exclusive orientations, one in each profile.

Suppose now that $|\mathcal{P}| > 2$ and that the claim holds for all non-singleton proper subsets of $\mathcal{P}$. Let $P \in \mathcal{P}$ be the given fixed profile and set $\mathcal{P}' := \mathcal{P} \setminus \{P\}$. By the induction hypothesis applied to $\mathcal{P}'$ and an arbitrary profile there is an exclusive separation $\vec{r}$ for $\mathcal{P}'$, contained in some $Q \in \mathcal{P}'$. Applying the induction hypothesis again to $\mathcal{P}'$ and Q yields another separation $\vec{s}$ that is exclusive for $\mathcal{P}'$ and lies in some $Q' \in \mathcal{P}'$ with $Q \neq Q'$.

If either of $\vec{r}$ and $\vec{s}$ is also exclusive for $\mathcal{P}$ then we are done. So suppose not, that is, suppose we have $\vec{r}, \vec{s} \in P$. Then $r \neq s$, and hence $\vec{r}$ and $\vec{s}$ must be incomparable by the consistency of Q and Q'. If $\vec{r} \leq \overleftarrow{s}$ then $\overleftarrow{s}$ is Q-exclusive for $\mathcal{P}$. Thus we may assume that r and s cross.

By submodularity of S one of $\vec{r} \vee \overleftarrow{s}$ and $\overleftarrow{r} \vee \vec{s}$ lies in $\vec{S}$; by symmetry we may assume that $(\vec{r} \vee \overleftarrow{s}) \in \vec{S}$. Since $\vec{s}$ is Q'-exclusive we have $\overleftarrow{s} \in Q$ and hence $(\vec{r} \vee \overleftarrow{s}) \in Q$ by the profile property. From $(\vec{r} \vee \overleftarrow{s}) \geq \vec{r}$ we infer that $(\vec{r} \vee \vec{s})$ is Q-exclusive for $\mathcal{P}'$. Moreover we cannot have $(\vec{r} \vee \overleftarrow{s}) \in P$: it would be inconsistent with $\vec{s} \in P$ as r and s cross.

Therefore $\vec{r} \vee \overleftarrow{s}$ is exclusive but not P-exclusive for $\mathcal{P}$. □

We remark that the stronger assertion used for the induction hypothesis in this proof, too, can be established immediately using Theorem 1: for $|\mathcal{P}| \geq 2$ the tree associated with the nested set $N \subseteq S$ distinguishing $\mathcal{P}$ has at least two leaves, and hence some leaf for which the separation labelling its incoming edge does not lie in the fixed profile P. (See [4] for the precise relationship between tree sets and trees.)

Returning to the proof of Theorem 2, let us find a way to canonically pick representatives of those $P \in \mathcal{P}$ with non-empty M_P in such a way that these representatives are nested with each other. For the 'canonically'-part of this we will make use of the fact that the sets M_P themselves are invariants of $\mathcal{P}$ and $\vec{S}$. For the nestedness we start by showing that separations from different M_P's cannot cross at all:

Lemma 5. *For $P \neq P'$ all $\vec{r} \in M_P$ and $\vec{s} \in M_{P'}$ are pairwise nested.*

Proof. Suppose some $\vec{r} \in M_P$ and $\vec{s} \in M_{P'}$ cross. By submodularity one of $\vec{r} \vee \overleftarrow{s}$ and $\overleftarrow{r} \vee \vec{s}$ lies in $\vec{S}$; by symmetry we may suppose that $(\vec{r} \vee \overleftarrow{s}) \in \vec{S}$. Then P, too, contains this separation since $\overleftarrow{s} \in P$. But $(\vec{r} \vee \overleftarrow{s})$ is also P-exclusive and strictly larger than $\vec{r}$, a contradiction. □

It is possible, however, that the set M_P itself is not nested. In fact the elements of M_P all cross each other, unless $\mathcal{P} = \{P\}$: any $\vec{r}$ and $\vec{s}$ in M_P that are nested must point towards each other by maximality. But every other profile in $\mathcal{P}$ contains both $\overleftarrow{r}$ and $\overleftarrow{s}$ and would then be inconsistent. If we want to represent a $P \in \mathcal{P}$ with non-empty M_P by an element of M_P, we are therefore limited to picking at most one element of M_P. However there is no canonical way of singling out an element of M_P to be the representative of P; we must therefore find another way of choosing an invariant P-exclusive separation, using M_P only as a starting point.

Clearly each non-empty M_P has an infimum in the ambient lattice $\vec{U}$. If this infimum of M_P lies in $\vec{S}$ then it is also the infimum of M_P as measured in the poset $\vec{S}$, and hence an invariant of $\vec{S}$ and $\mathcal{P}$. Therefore the infimum of an M_P is a canonical choice for a representative of P, provided that this infimum happens to lie in $\vec{S}$ and be P-exclusive. Our next lemma shows that this is indeed always the case:

Lemma 6. *Let $P \in \mathcal{P}$ with $M_P \neq \emptyset$ and $\mathcal{P} \neq \{P\}$ be given. Then M_P has an infimum $\vec{s}_P$ in the poset $\vec{S}$, and $\vec{s}_P$ is P-exclusive for $\mathcal{P}$. Moreover if some $t \in S$ is nested with M_P then t is also nested with s_P.*

Proof. Fix an enumeration $M_P = \{\vec{r}_1, \dots, \vec{r}_n\}$ and some $t \in S$ that is nested with M_P. For $i = 1, \dots, n$ let $\vec{s}_i := \vec{r}_1 \wedge \dots \wedge \vec{r}_i$, where these infima are taken in $\vec{U}$. We show by induction on i that $\vec{s}_i$ lies in $\vec{S}$, is P-exclusive for $\mathcal{P}$, and is nested with t; this yields the claim for $i = n$.

The case $i = 1$ is trivially true, so suppose that $i > 1$ and that $\vec{s}_{i-1} = (\vec{r}_1 \wedge \dots \wedge \vec{r}_{i-1})$ is already known to lie in $\vec{S}$ and be P-exclusive and nested with t.

If $s_{i-1} = r_i$ there is nothing to show, so suppose that $s_{i-1} \neq r_i$. Let us first treat the case that $\vec{r}_i$ and $\vec{s}_{i-1}$ are nested. Clearly the two cannot point away from each other since P is consistent. If $\vec{r}_i$ and $\vec{s}_{i-1}$ are comparable then $\vec{s}_i = (\vec{r}_i \wedge \vec{s}_{i-1})$ equals one of the two and hence is as claimed. Finally, if $\vec{r}_i$ and $\vec{s}_{i-1}$ point towards each other, we obtain a contradiction: for then their inverses point away from each other, making every profile in $\mathcal{P}$ other than P inconsistent. Thus if $\vec{r}_i$ and $\vec{s}_{i-1}$ are nested the induction hypothesis holds for $\vec{s}_i$.

Let us now consider the case that $\vec{r}_i$ and $\vec{s}_{i-1}$ cross. Then $\vec{r}_i \vee \vec{s}_{i-1}$ cannot lie in $\vec{S}$ since it would be P-exclusive and strictly larger than $\vec{r}_i \in M_P$. Therefore $(\vec{r} \wedge \vec{s}_{i-1}) \in \vec{S}$, that is, $\vec{s}_i \in \vec{S}$. By consistency we have that $\vec{s}_i \in P$. Every profile in $\mathcal{P}$ other than P contains $\overleftarrow{r}_i$ as well as $\overleftarrow{s}_{i-1}$ and hence $\overleftarrow{s}_{i-1}$ by the profile property, which shows that $\vec{s}_i$ is P-exclusive. Finally, by Lemma 3, $\vec{s}_i$ is also nested with t. ☐

It remains to show that after picking as a representative for each $P \in \mathcal{P}$ with exclusive separations the infimum of M_P, the set of separations in $\vec{S}$ that are nested with all these representatives is still rich enough to distinguish all profiles in $\mathcal{P}$ for which we have not yet picked a representative.

For this let $\vec{S'} \subseteq \vec{S}$ be the system of all those separations that are nested with all M_P, and let $\mathcal{P}' \subseteq \mathcal{P}$ be the set of those profiles Q that have empty M_Q. Our next lemma says that if we restrict ourselves to $\vec{S'}$, we can still distinguish $\mathcal{P}'$:

Lemma 7. *The separation system $\vec{S'}$ is submodular and distinguishes $\mathcal{P}'$.*

Proof. The fact that $\vec{S'}$ is submodular is immediate from Lemma 3. For the latter, let Q and Q' be distinct profiles in $\mathcal{P}'$; we shall show that some $\vec{s'} \in \vec{S'}$ distinguishes them. For this choose a separation $s \in S$ which distinguishes Q and Q' and which is nested with M_P for as many $P \in \mathcal{P}$ as possible. If s is nested with all M_P we are done; otherwise there is some $P \in \mathcal{P}$ for which s crosses something in M_P.

So suppose that there is a $P \in \mathcal{P}$ for which s is not nested with M_P. Among all $\vec{s'} \in \vec{S}$ which distinguish Q and Q' and which are nested with each $M_{P'}$ with which s is nested, pick a minimal $\vec{s'}$ with $\vec{s'} \in P$. We claim that this $\vec{s'}$ is nested with M_P, contradicting the choice of s.

To see this, suppose that $\vec{s'}$ crosses some $\vec{r} \in M_P$. Then $\vec{r} \vee \vec{s'}$ cannot lie in $\vec{S}$ since that would be a strictly larger P-exclusive separation than $\vec{r}$. Hence $(\vec{r} \wedge \vec{s'}) \in \vec{S}$. By $P \notin \{Q, Q'\}$ we have that both Q and Q' contain $\overleftarrow{r}$, and hence this corner separation distinguishes Q and Q' as well. However, by Lemma 3 and Lemma 5, this $\vec{r} \wedge \vec{s'}$ would be nested with each $M_{P'}$ with which s was nested, while being strictly smaller than $\vec{s'}$, a contradiction. □

If M_P is non-empty let us write $\vec{s}_P$ for its infimum in $\vec{S}$ as in Lemma 6. We are now ready to prove Theorem 2 by induction.

Proof of Theorem 2. We proceed by induction on $|\mathcal{P}|$. If $|\mathcal{P}| \leq 1$ there is nothing to show, so suppose that $|\mathcal{P}| > 1$ and that the assertion holds for all proper subsets of $\mathcal{P}$.

Recall that $\vec{S'} \subseteq \vec{S}$ consists of all separations in $\vec{S}$ that are nested with all sets M_P and that $\mathcal{P}' \subseteq \mathcal{P}$ is the set of all $P \in \mathcal{P}$ with empty M_P. Clearly both $\vec{S'}$ and $\mathcal{P}'$ are invariants of $\vec{S}$ and $\mathcal{P}$ since the sets M_P themselves are invariants. For each non-empty M_P let $\vec{s}_P$ be its infimum in $\vec{S}$ as described in Lemma 6. Then

$$N_1 := \{ s_P \mid P \in \mathcal{P} \smallsetminus \mathcal{P}' \}$$

is clearly a canonical set. From Lemma 6 we further know that N_1 distinguishes all profiles in $\mathcal{P}' \smallsetminus \mathcal{P}$ from each other and from each profile in $\mathcal{P}'$.

By Lemma 5 every element of M_P is nested with every element of $M_{P'}$ for all $P \neq P'$. Applying the 'moreover'-part of Lemma 6 twice thus implies that s_P is nested with every element of $M_{P'}$ and subsequently with $s_{P'}$. Therefore N_1 is a nested set. Likewise every separation in $\vec{S'}$ is nested with N_1.

Let us apply the induction hypothesis to $\mathcal{P}'$ in $\vec{S'}$, as made possible by the Lemmas 4 and 7, yielding a canonical nested set $N_2 \subseteq S'$ which distinguishes $\mathcal{P}'$. Since $\vec{S'}$ and $\mathcal{P}'$ themselves are invariants of $\vec{S}$ and $\mathcal{P}$ we have that the union $N_1 \cup N_2$ is the desired canonical nested set. □

References

[1] J. Carmesin, R. Diestel, M. Hamann, and F. Hundertmark, *Canonical tree-decompositions of finite graphs I. Existence and algorithms*, J. Combin. Theory Ser. B **116** (2016), 1–24, doi:10.1016/j.jctb.2014.04.001.

[2] R. Diestel, J. Erde, and D. Weißauer, *Structural submodularity and tangles in abstract separation systems*, J. Combin. Theory (Series A) **167C** (2019), 155–180, doi:10.1016/j.jcta.2019.05.001.

[3] R. Diestel, F. Hundertmark, and S. Lemanczyk, *Profiles of separations: in graphs, matroids, and beyond*, Combinatorica **39** (2019), no. 1, 37–75, doi:10.1007/s00493-017-3595-y.

[4] R. Diestel, *Tree sets*, Order **35** (2018), 171–192, doi:10.1007/s11083-017-9425-4.

[5] R. Diestel, *Abstract separation systems*, Order **35** (2018), 157–170, doi:10.1007/s11083-017-9424-5.

[6] N. Robertson and P. D. Seymour, *Graph minors. X. Obstructions to tree-decomposition*, J. Combin. Theory (Series B) **52** (1991), 153–190, doi:10.1016/0095-8956(91)90061-N.

Christian Elbracht, `christian.elbracht@uni-hamburg.de`
Jakob Kneip, `jakob.kneip@uni-hamburg.de`
Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany