

DESY 04-002

An Analytical Expression for the Non-Singlet  
Structure Functions  
at Small  $x$  in the Double Logarithmic  
Approximation

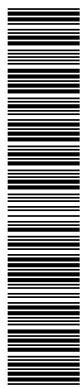
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January 14, 2004

**Abstract**

A simple analytic expression for the non-singlet structure function  $f_{NS}$  is given. The expression is derived from the result of Ref. [1] obtained by low  $x$  resummation of the quark ladder diagrams in the double logarithmic approximation of perturbative QCD.

arXiv:hep-ph/0401030 v2 14 Jan 2004



The small  $x$  behavior of the non-singlet structure functions in DIS are of crucial importance for the determination of the quark densities. As an example, a verification of the Gottfried sum rules requires an extrapolation of the experimental data to region of low  $x$  [2] which can be done using non-singlet structure functions.

The small  $x$  behavior of the non-singlet structure functions  $f_{NS}$  was studied in Ref. [1]. Using the method of infrared evolution equation (IREE) the pQCD double logarithmic contribution to  $f_{NS}$  was calculated. The result was shown to differ dramatically from the one predicted by the low  $x$  approximation of the conventional DGLAP equation. The non-singlet structure function  $f_{NS}$  found in Ref. [1] was presented in the form of a complex  $\omega$  integral, the inverse Mellin transform of a given partial wave. Though one can use this result to make general estimates and read off the leading asymptotics, it is not really “user friendly” and hard for numerical implementations.

Recently a detailed study of the quark ladder double logarithmic resummations was carried out in Ref. [3]. Several methods were analyzed and compared. In particular, a comparison of a direct diagrammatic resummation using Bethe-Salpeter equation with IREE leads to a new insight in the structure of the latter. A particular outcome of this comparison is a clear understanding of the way the  $\omega$  integrations should be performed.

In the present paper we apply the results of analysis of Ref. [3] to the results of Ref. [1] and derive a closed form expression for  $f_{NS}$ . Fortunately, as observed in Ref. [3], a unique choice of a closed  $\omega$  integration path allows the integration to be performed giving rise to a resulting expression in terms of modified Bessel functions  $I_\nu$ . This expression is most easy to handle in any further applications.

The structure function  $f_{NS}$  is proportional to the imaginary part of the forward photon scattering amplitude projected onto the flavor non-singlet state. In the double logarithmic approximation of pQCD this amounts for summing up of diagrams built of two  $t$ -channel quarks and  $s$ -channel gluons forming a ladder. The quark ladders can be resummed using the IREE in partial wave representation.

We define the partial wave through the ansatz:

$$M(Q^2, s) = \int \frac{d\omega}{2\pi i} \left(\frac{s}{\mu^2}\right)^\omega \tilde{F}(\omega, Q^2/\mu^2). \quad (0.2)$$

The function  $\tilde{F}(\omega, Q^2/\mu^2)$  is a partial wave representation of the  $s$ -channel contribution to the quark-photon elastic amplitude  $M$ . The imaginary part is obtained from the  $u$ -crossed amplitude which at high energies approximately equals  $M(Q^2, -s)$ . The scale  $\mu$  is introduced as an auxiliary infrared cutoff parameter, the minimal quark transverse momentum. For a more detailed description of the method of IREE we refer the reader to the original work [4] as well as to some applications in Ref. [1, 5, 6].

The IREE for  $\tilde{F}$  is obtained by differentiating (0.2) with respect to

$\ln \mu^2$ . The equation reads

$$\omega \tilde{F} + \frac{\partial \tilde{F}}{\partial y} = \frac{1}{8\pi^2} f_0(\omega) \tilde{F}(\omega, y) \quad (0.3)$$

with  $y \equiv \ln(Q^2/\tilde{\mu}^2)$ , The function  $f_0$  was introduced in Ref. [4]

$$f_0(\omega) = 4\pi^2 (\omega - \sqrt{\omega^2 - \omega_0^2}); \quad \omega_0^2 = 2\alpha_s C_F/\pi \quad (0.4)$$

The function  $\tilde{F}(\omega, y)$  solving the IREE (0.3) has the form [4, 1]:

$$\tilde{F}(\omega, y) = \frac{C_0}{g^2 C_F} f_0(\omega) e^{-(\omega - f_0/8\pi^2)y} \quad \text{for } y \geq 0, \quad (0.5)$$

The coefficient  $C_0$  contains information about coupling of the photon to the ladder,  $C_0 = -4\pi\alpha_{em}\epsilon(A) \cdot \epsilon(A')$ , with  $\epsilon$  being the photon polarization vectors. In the Born approximation the above choice of  $C_0$  corresponds to a scattering off a free quark with initial condition proportional to  $\delta(1-x)$ .

The  $s$ -channel part of the elastic amplitude  $M$  reads

$$M(Q^2, s, \tilde{\mu}) = \frac{C_0}{g^2 C_F} \int \frac{d\omega}{2\pi i} \left(\frac{s}{Q^2}\right)^\omega f_0(\omega) \exp[y f_0(\omega)/8\pi^2]. \quad (0.6)$$

The non-singlet structure function  $f_{NS}$  is related to  $M$  via

$$-\pi C_0 f_{NS} = e_q^2 \text{Im} M \simeq -\pi e_q^2 \frac{\partial M}{\partial \ln s} \quad (0.7)$$

and this results in the following expression

$$f_{NS} = \frac{e_q^2}{g^2 C_F} \int \frac{d\omega}{2\pi i} \left(\frac{s}{Q^2}\right)^\omega \omega f_0(\omega) \exp[y f_0(\omega)/8\pi^2]. \quad (0.8)$$

The results above are essentially copied from Ref. [1]. What is our new observation is that we can proceed further in performing the  $\omega$  integrations in Eqs. (0.6) and (0.8). The procedure is very similar to the one described in the Appendix B of Ref. [3].

As was emphasized in Ref. [3] the  $\omega$ -integration path  $C_{cut}$  goes around the square root branch cut from  $-\omega_0$  to  $\omega_0$ .

$$M = \frac{C_0}{g^2 C_F} 4\pi^2 \int_{C_{cut}} \frac{d\omega}{2\pi i} \left(\frac{s}{Q^2}\right)^\omega (\omega - \sqrt{\omega^2 - \omega_0^2}) \left(\frac{Q^2}{\mu^2}\right)^{(\omega - \sqrt{\omega^2 - \omega_0^2})/2} \quad (0.9)$$

Denote by  $\xi = \ln(s/Q^2)$  and  $\eta = \ln(s/\mu^2)$ . Let us introduce a new complex variable

$$z = (\omega - \sqrt{\omega^2 - \omega_0^2})/\omega_0. \quad (0.10)$$

The  $\omega$  integral then turns into a contour integral in the  $z$ -plane, and the integration path encircles the origin  $z = 0$ . We obtain:

$$M = 2\pi^2 \omega_0^2 \frac{C_0}{g^2 C_F} \left[ \int_{0+} \frac{dz}{2\pi i} \left(\frac{1}{z} - z\right) e^{\eta \omega_0 z/2 + \xi \omega_0/2 z} \right] = \quad (0.11)$$

$$C_0 \left( I_0(\omega_0 \sqrt{\xi \eta}) - \frac{\xi}{\eta} I_2(\omega_0 \sqrt{\xi \eta}) \right).$$

The expression (0.11) is very similar to the one found in Ref. [3] for the non-singlet structure function of photon.

Finally, the expression for  $f_{NS}$  is given by

$$f_{NS} = \frac{e_q^2 \pi^2 \omega_0^3}{g^2 C_F} \left[ \int_{0^+} \frac{dz}{2\pi i} \left( \frac{1}{z^2} - z^2 \right) e^{\eta \omega_0 z/2 + \xi \omega_0/2z} \right] = \quad (0.12)$$

$$\frac{e_q^2 \omega_0}{2} \left( \frac{\sqrt{\eta}}{\sqrt{\xi}} I_1(\omega_0 \sqrt{\xi \eta}) - \frac{\xi^{3/2}}{\eta^{3/2}} I_3(\omega_0 \sqrt{\xi \eta}) \right).$$

Eq. (0.12) is the main result of this paper. In a somewhat different context a very similar expression can be also found in Ref. [7]. The expression (0.12) is correct for positive  $\xi$  and  $\eta$ . At  $x = 1$  ( $\xi = 0$ ), the  $\omega$  integral is divergent, which corresponds to  $\delta(1-x)$  initial condition.

An additional comment is in order. Strictly speaking the result obtained above should be called a structure function of a free quark rather than of a proton. This is because a particular choice of the initial conditions was implemented. We believe, however, that in the Regge limit considered above factorization holds implying the expression (0.12) is a good approximation for a proton structure modulo an overall normalization constant. An alternative approach to the problem based on the NLO DGLAP equation and a low scale convolution can be found in Ref. [8].

In the Regge limit  $\ln s/Q^2 = \ln 1/x \gg \ln Q^2/\mu^2 \gg 1$  the leading asymptotics of the Bessel functions coincide and cancel.

$$f_{NS} \sim \left( \frac{1}{x} \right)^{\omega_0} \frac{e_q^2}{\ln^{3/2}(1/x)} \left( \frac{Q^2}{\mu^2} \right)^{\omega_0/2} \quad (0.13)$$

Up to the pre-exponential factor this asymptotics was correctly found in Ref. [1]. Exploring another limit  $\ln Q^2/\mu^2 \gg \ln 1/x \gg 1$  we reproduce the low  $x$  approximation of the DGLAP equation

$$f_{NS} \sim \frac{e_q^2}{\ln(1/x)} e^{\omega_0 \sqrt{\ln(1/x) \ln Q^2/\mu^2}} \quad (0.14)$$

Note that only the first term ( $I_1$ ) contributes to this asymptotics. Eq. (0.12) gives an analytic expression for the non-singlet structure function  $f_{NS}$  and it provides a smooth interpolation between two high energy limits. It is worth a comment, however, that the low  $x$  approximation of the DGLAP equation is reproduced with a constant coupling constant only. An extensive work on inclusion of the running coupling effects has been done in Ref. [9].

## Acknowledgments

The author is very grateful to Jochen Bartels for reading of the manuscript and for the fruitful collaboration in Ref. [3] which lead to the observation reported in this paper.

The author would like to thank Andrei Kataev for a very informative and stimulating discussion.

The valuable comments from Johannes Bluemlein, Boris Ermolaev and Yura Kovchegov are greatly acknowledged.

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