

THE NEXT-TO-LADDER APPROXIMATION FOR LINEAR DYSON-SCHWINGER
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1. INTRODUCTION

Ladder approximations have been one of the most basic attempts to simplify and truncate Dyson–Schwinger equations in field theory in a still meaningful way [1]. From a mathematical viewpoint they simplify the combinatorics of the forest formula considerably, and are solvable by a scaling Ansatz for sufficiently simple kinematics.

Here, we discuss such a scenario, but iterate one- and two-loop skeletons jointly, combining some analytic progress with a thorough discussion of the underlying algebraic properties.

1.1. Purpose of this paper. The main purpose is to sum an infinite series of graphs based on the iteration of two underlying skeleton graphs. We progress in a manner such that our methods can be generalized to any countable number of skeletons. We restrict to linear Dyson Schwinger equations, a case relevant for theories at a fixpoint of the renormalization group. We proceed using one-dimensional Mellin transforms, a privilege of linearity of which we make full use. See [2, 3, 4, 5] for the general approach.

2. THE DYSON–SCHWINGER EQUATION

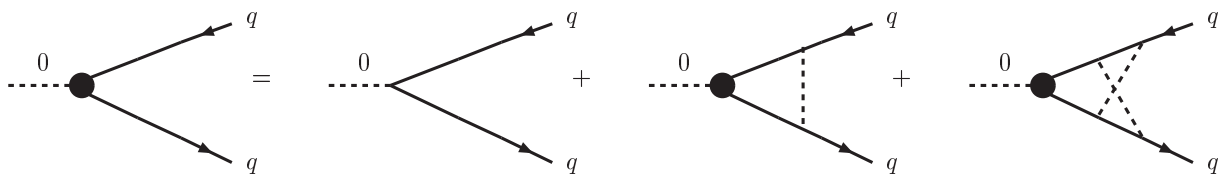
2.1. The integral equation. The equation which we consider is in massless Yukawa theory in four-dimensional Minkowski space $\mathbb{M}$, for pedagogical purposes. We define a renormalized Green function describing the coupling of a scalar particle to a fermion line by

$$(1) \quad \begin{aligned} G_R(a, \ln(-q^2/\mu^2)) &= 1 - a \int_{\mathbb{M}} \frac{d^4 k}{i\pi^2} \left\{ \frac{1}{k\!\!\!/} G_R(a, \ln(-k^2/\mu^2)) \frac{1}{k\!\!\!/} \frac{1}{(k-q)^2} \right\}_- \\ &+ a^2 \int_{\mathbb{M}} \frac{d^4 k}{i\pi^2} \int_{\mathbb{M}} \frac{d^4 l}{i\pi^2} \left\{ \frac{\not{l}(\not{l} + \not{k}) G_R(a, \ln(-(k+l)^2/\mu^2)) (\not{l} + \not{k})(\not{k} + \not{q})}{[(k+l)^2]^2 l^2 (k+q)^2 k^2 (l-q)^2} \right\}_-, \end{aligned}$$

where $\{\}_-$ indicates subtraction at $\mu^2 = -q^2$, so that $G_R(a, 0) = 1$:

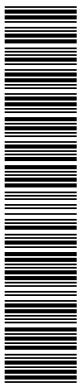
$$(2) \quad \{G_R(a, \ln(-k^2/\mu^2))\}_- = G_R(a, \ln(-k^2/\mu^2)) - G_R(a, \ln(-k^2/-q^2)).$$

The kinematics are such that the fermion has momentum q and the external scalar particle carries zero momentum. The equation can graphically be represented as



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where the blob represents the unknown Green function $G_R(a, \ln(-q^2/\mu^2))$. This linear Dyson–Schwinger equation can be solved by a scaling solution, $L = \ln(-q^2/\mu^2)$,

$$(3) \quad G_R(a, L) = \exp \{-\gamma_G(a)L\}.$$

Indeed, this satisfies the desired normalization and leads to the equation

$$(4) \quad \exp \{-\gamma_G(a)L\} = 1 + (\exp \{-\gamma_G(a)L\} - 1) [aF_1(\gamma_G) + a^2F_2(\gamma_G)],$$

where the two Mellin transforms are the functions determined by

$$(5) \quad F_1 : \rho \rightarrow - \int \frac{d^4k}{i\pi^2} \frac{1}{k^2(k-q)^2} \left(\frac{k^2}{q^2} \right)^{-\rho}$$

and similarly

$$(6) \quad F_2 : \rho \rightarrow \int_{\mathbb{M}} \frac{d^4k}{i\pi^2} \int_{\mathbb{M}} \frac{d^4l}{i\pi^2} \frac{\not{l}(k+l)}{(k+l)^2 l^2 (k+q)^2 k^2 (l-q)^2} \left[\frac{(l+k)^2}{q^2} \right]^{-\rho}.$$

Clearing the factor $[\exp\{-\gamma_G(a)L\} - 1]$ in this equation gives

$$(7) \quad 1 = aF_1(\gamma_G) + a^2F_2(\gamma_G).$$

It remains to determine F_1, F_2 explicitly and solve this implicit equation for γ_G in terms of a . We do so in the next sections but first discuss the perturbative structure behind this solution.

2.2. The algebraic structure. We can identify any graph in this resummation with a word in two letters u, v say, for example:

$$(8) \quad \begin{array}{cccc} \text{---} \triangle \text{---} & \text{---} \triangle \text{---} & \text{---} \triangle \text{---} & \text{---} \triangle \text{---} \\ u & v & uv & vu \end{array}$$

We have renormalized Feynman rules ϕ_R such that

$$(9) \quad \phi_R(u)(L) = -L \lim_{\rho \rightarrow 0} \rho F_1(\rho),$$

and

$$(10) \quad \phi_R(v)(L) = -L \lim_{\rho \rightarrow 0} \rho F_2(\rho).$$

The Green function $G_R(a, L)$ is obtained as the evaluation by ϕ_R of the fixpoint of the combinatorial Dyson Schwinger equation

$$(11) \quad X(a) = \mathbb{I} + aB_+^u(X(a)) + a^2B_+^v(X(a)).$$

We have

$$(12) \quad X(a) = 1 + au + a^2(uu + v) + \dots = \exp_{\Upsilon} [au + a^2v],$$

where Υ is the shuffle

$$(13) \quad H_{\text{lin}} \times H_{\text{lin}} \rightarrow H_{\text{lin}},$$

$$(14) \quad B_+^i(w_1) \Upsilon B_+^j(w_2) = B_+^i(w_1 \Upsilon B_+^j(w_2)) + B_+^j(w_2 \Upsilon B_+^i(w_1)),$$

$\forall i, j \in \{u, v\}$. Note that for example $u \Upsilon u = 2uu$.

The two maps B_+^i are Hochschild one-cocycles, and $X(a)$ is group-like:

$$(15) \quad \Delta X(a) = X(a) \otimes X(a).$$

Correspondingly, decomposing $X(a) = 1 + \sum_{k \geq 1} a^k c_k$, we have

$$(16) \quad \Delta c_k = \sum_{j=0}^k c_j \otimes c_{k-j},$$

with $c_0 = 1$. This is a decorated version of the Hopf algebra of undecorated ladder trees t_n with coproduct $\Delta t_n = \sum_{j=0}^n t_j \otimes t_{n-j}$. Feynman rules become iterated integrals as

$$(17) \quad \phi_R(B_+^i(w))(L) = \int \{ \phi(w) (\ln k^2 / \mu^2) d\mu_i(k) \}_-,$$

where $d\mu_i$ is the obvious integral kernel for $i \in u, v$, cf. (1). Apart from the shuffle product, we have disjoint union as a product which makes the Feynman rules into a character

$$(18) \quad \phi(w_1 \cdot w_2) = \phi(w_1) \phi(w_2).$$

These two commutative products $\Upsilon, \cdot$ allow to express the primitive elements associated with shuffles of letters u, v , see for example [7]:

Theorem 1. *The primitive elements are given by polarization of the primitive elements p_n of the undecorated ladder trees t_n . These are given by $p_n = \frac{1}{n}[S \star Y](t_n)$.*

Here, Y denotes the grading operator, defined by $Y(t_k) = kt_k$ and the star product is defined as usual by $O_1 \star O_2 = \cdot \circ (O_1 \otimes O_2) \circ \Delta$. Polarization of the undecorated primitive elements p_n means that we decorate each vertex of p_n with $u + v$.

The set $\mathcal{P}(u, v)$ of primitive elements is hence spanned by elements p_{i_u, i_v} , where the integeres i_u, i_v count the number of letters u and v in the polarization of $t_{i_u + i_v}$. For example the primitive element corresponding to the undecorated ladder tree t_2 is $p_2 = t_2 - \frac{1}{2}t_1 t_1$. Polarization yields

$$(19) \quad \begin{aligned} p_{2,0} &= \frac{1}{2}(u \Upsilon u - u \cdot u) = uu - \frac{1}{2}u \cdot u, & p_{0,2} &= \frac{1}{2}(v \Upsilon v - v \cdot v) = vv - \frac{1}{2}v \cdot v, \\ p_{1,1} &= u \Upsilon v - u \cdot v = uv + vu - u \cdot v. \end{aligned}$$

3. THE MELLIN TRANSFORMS

The general structure of the Mellin transform can be obtained from quite general considerations. The crucial input comes from powercounting and conformal symmetry.

Theorem 2. *The Mellin transforms above are invariant under the transformation $\rho \rightarrow 1 - \rho$.*

Proof: Explicit computation. We give it here for F_1 . We assume $\Re \rho > 0$ so that F_i is well defined as a function. Then, the conformal inversion $k_\mu \rightarrow k'_\mu = k_\mu / k^2$ gives explicitly

$$(20) \quad - \int \frac{d^4 k'}{i\pi^2} \frac{1}{k'^2 (k' - q)^2} \left(\frac{k'^2}{q^2} \right)^{-1+\rho}$$

for F_1 . F_2 can be treated similarly by conformal inversion in both Minkowski spaces. $\square$

3.1. The Mellin transform of the one-loop kernel. This Mellin transform is readily integrated to deliver

$$(21) \quad F_1(\rho) = -\frac{1}{\rho(1-\rho)},$$

exhibiting the expected conformal symmetry.

3.2. The Mellin transform of the two-loop kernel. Determining this Mellin transform is the core part of this paper. We proceed by making use of the advantage that we remain in four dimensions, and use results of [6]. We are interested in the integral

$$(22) \quad F_2(\rho) = (-q^2)^\rho \int_{\mathbb{M}} \frac{d^4 k}{i\pi^2} \int_{\mathbb{M}} \frac{d^4 l}{i\pi^2} \frac{\not{l}(\not{k} + \not{q})[-(l+k)^2]^{-\rho}}{(k+l)^2 l^2 (k+q)^2 k^2 (l-q)^2}.$$

Integration is over the eight dimensional cartesian product of two Minkowski spaces furnished with a quadratic form

$$(23) \quad a^2 = a_0^2 - a_1^2 - a_2^2 - a_3^2.$$

A simple tensor calculus delivers

$$(24) \quad F_2(\rho) = \frac{1}{2} \{ -2G_4(1, 1 + \rho)G_4(1, 1 + \rho) + I_6(1, 1, \rho, 1, 1, 2 - \rho) + I_6(1, 1, 1 + \rho, 1, 1, 1 - \rho) \},$$

where

$$(25) \quad G_D(a, b) = \frac{\Gamma(a+b-D/2)\Gamma(D/2-a)\Gamma(D/2-b)}{\Gamma(a)\Gamma(b)\Gamma(D-a-b)},$$

so that $G_4(1, 1 + \rho) = \frac{1}{\rho(1-\rho)}$. We use the notation of [6] for I_6 . In this notation, we have $I_6 = \overline{I_6}$. Setting $u \rightarrow 0$ and $v = -\rho$ or $v = 1 - \rho$ we can determine the two I_6 integrals as a limit $u \rightarrow 0$ in Eq.(19)(op.cit.) as

$$(26) \quad I_6(1, 1, 1 - v, 1, 1, 1 + v) = 8 \sum_{n=1}^{\infty} n \zeta_{2n+1} (1 - 2^{-2n}) v^{2n-2},$$

and similarly for $v = 1 - \rho$. We hence find the above DSE in the form

$$(27) \quad 1 = -a \frac{1}{\gamma_G(1 - \gamma_G)} - a^2 \left\{ \frac{1}{\gamma_G^2(1 - \gamma_G)^2} - 4 \sum_{n=1}^{\infty} n \zeta_{2n+1} (1 - 2^{-2n}) [\gamma_G^{2n-2} + (1 - \gamma_G)^{2n-2}] \right\}.$$

4. THE SOLUTION

We can solve for γ_G in the above in two different ways, expressing the solution as an infinite product or via the logarithmic derivative of the Γ function.

4.1. Solution as an infinite product. We have:

$$(28) \quad G_R(a, L) = \exp \left\{ \sum_{p \in \mathcal{P}(u, v)} a^{|p|} \phi_R(p)(L) \right\}.$$

Here, the sum is over all primitives $p \in \mathcal{P}(u, v)$, where $\mathcal{P}$ is the set of primitives assigned to any tree t_n decorated arbitrarily by letters in the alphabet u, v , as described above. The proof is an elementary exercise in the Taylor expansion of the two Mellin transforms. Note that $\phi_R(p)(L)$ is linear in L for primitive p , $\partial_L^2 \phi_R(L) = 0$. We hence find for $\gamma_G(a)$

$$(29) \quad \gamma_G(a) = -\frac{\partial \ln G}{\partial L} \Big|_{L=0} = -\sum_p a^{|p|} \phi_R(p)/L.$$

Convergence of the sum is covered by the implicit function theorem, which provides for γ_G through the two Mellin transforms in the DSE. We hence proceed to the second way of expressing the solution.

4.2. Solution via the ψ -function. We can express the DSE using the logarithmic derivative of the Γ function and we obtain

$$(30) \quad 1 = -a \frac{1}{\gamma_G(1-\gamma_G)} - a^2 \left\{ \frac{1}{\gamma_G^2(1-\gamma_G)^2} + \frac{1}{\gamma_G} [\psi'(1+\gamma_G) - \psi'(1-\gamma_G)] + \frac{1}{1-\gamma_G} [\psi'(2-\gamma_G) - \psi'(\gamma_G)] - \frac{1}{2\gamma_G} \left[\psi' \left(1 + \frac{\gamma_G}{2} \right) - \psi' \left(1 - \frac{\gamma_G}{2} \right) \right] - \frac{1}{2(1-\gamma_G)} \left[\psi' \left(\frac{3-\gamma_G}{2} \right) - \psi' \left(\frac{1+\gamma_G}{2} \right) \right] \right\}.$$

Here

$$(31) \quad \psi'(x) = \frac{d^2}{dx^2} \ln \Gamma(x).$$

Again, the two-loop solution shows explicitly the conformal symmetry $\gamma_G \rightarrow 1-\gamma_G$. Note that the apparent second order poles at $\gamma_G = 0$ and $\gamma_G = 1$ on the rhs are only first order poles upon using standard properties of the logarithmic derivative of the Γ function, as it has to be. This provides an implicit equation for γ_G , which can be solved numerically.

5. CONCLUSIONS

We determined the Mellin transform of the two-loop massless vertex in Yukawa theory. We used it to resum a linear Dyson–Schwinger equation. Following [2, 3, 4], more complete applications to non-linear Dyson–Schwinger equations will be provided elsewhere. Techniques to deal with non-linearity have indeed been developed recently [2, 3, 4], and involve transcendental functions even upon resummation of terms from the first Mellin transform [2]. In the non-linear case one gets indeed results very different from scaling, as has been demonstrated early on in field theory [8]. Finally, we note that the same two-loop Mellin transform also appears in setting up the full DSE in other renormalizable theories [5].

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