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Gauge Coupling Unification in Six Dimensions

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Abstract

We compute the one-loop gauge couplings in six-dimensional non-Abelian gauge theories on the T^2/Z_2 orbifold with general GUT breaking boundary conditions. For concreteness, we apply the obtained general formulae to the gauge coupling running in a 6D $SO(10)$ orbifold GUT where the GUT group is broken down to the standard model gauge group up to an extra $U(1)$. We find that the one-loop corrections depend on the parity matrices encoding the orbifold boundary conditions as well as the volume and shape moduli of extra dimensions. When the $U(1)$ is broken by the VEV of bulk singlets, the accompanying extra color triplets also affect the unification of the gauge couplings. In this case, the $B-L$ breaking scale compatible with the gauge coupling unification is sensitive to the change of the compactification scales.

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1 Introduction

Grand Unified Theories(GUTs) [1] have been reconsidered recently in the context of orbifold GUTs [2, 3] where orbifold boundary conditions in extra dimensions are utilized to break down a GUT gauge symmetry to the Standard Model(SM) gauge group and at the same time solve the doublet-triplet splitting problem. In orbifold GUTs, on top of the usual 4D logarithmic running to generate the difference between the SM gauge couplings at low energy [4], the Kaluza-Klein(KK) massive modes of the 4D gauge bosons give rise to additional threshold corrections. In orbifold GUTs, however, there is an ambiguity due to the existence of the non-universal gauge kinetic terms localized at the fixed points [5–7] where the local gauge symmetry is reduced compared to the bulk one. Nonetheless, by making a strong coupling assumption at the cutoff scale [8], the brane-localized gauge couplings may be ignored compared to the bulk gauge coupling, due to the volume suppression of extra dimensions. So, the orbifold GUTs can provide a minimal setup for considering the threshold corrections consistent with a successful gauge coupling unification.

Over the past years, the 5D orbifold GUTs have been much studied as a simplest case, in particular, to compute the resulting effects of the KK massive modes to the gauge coupling unification for one flat [9] or warped [10] extra dimension. For larger GUT groups such as $SO(10)$, however, the 5D case turns out not to be a minimal setup, because a usual Higgs mechanism is required for a further breaking to the SM gauge group or unwanted massless modes of extra components of gauge fields must get massive [11]. In contrast, the 6D case has drawn more attention because it is more economic to obtain the SM gauge group directly from a large GUT group [12,13] and there is more freedom to locate the SM fields for satisfying the experimental requirements such as the flavor structure [14] and the proton lifetime [15]. We may even regard the 6D orbifold GUTs as an effective theory of describing (heterotic) string compactifications as an intermediate GUT [16] with the hope to identify the remnants of string theory within the 6D orbifold field theory.

In this paper, we consider the one-loop effective action for the gauge fields containing zero modes in six-dimensional $\mathcal{N} = 1$ supersymmetric GUTs compactified on a T^2/Z_2 orbifold. This can be regarded as a generalization of the previous findings on orbifolds without gauge symmetry breaking [6]. In the presence of the orbifold boundary conditions that are commuting, we obtain the bulk and brane contributions due to bulk vector and hyper multiplets and identify the necessary counterterms to cancel the divergences appearing in dimensional regularization. A bulk vector multiplet leads to both brane and bulk corrections while a bulk hyper multiplet gives rise only to a bulk correction. The bulk divergences are cancelled by a higher derivative term with universal coefficient whereas the brane divergences are cancelled by brane-localized gauge kinetic terms the coefficients of which depend on the local gauge symmetry at the fixed points. In the case of the cutoff regularization [6], there would be also power-like corrections in the cutoff scale to the gauge couplings, but they don't affect the gauge coupling unification at all. From the obtained effective action, we also derive the general expressions for the running of the effective gauge couplings for zero-mode gauge bosons. In the low energy limit, we consider the running of the gauge couplings, including the non-universal threshold corrections due to KK massive modes[1].

We apply the general formulae for the gauge coupling running in the six-dimensional $SO(10)$ orbifold GUT model proposed in Ref. [12]. This is the minimal setup to break $SO(10)$ down to the SM gauge group up to a $U(1)$ factor only by orbifold boundary conditions without obtaining massless modes from the extra components of gauge bosons. In some realistic $SO(10)$ orbifold GUT models, we discuss about the possibility of having a large volume of extra dimensions compatible with the success of the gauge coupling unification. We assume the breaking scale of the extra $U(1)$ to be lower than the compactification scale in order to ignore the effect of the brane-localized $U(1)$ breaking mass terms.

For the case with isotropic compactification of extra dimensions, we show that the volume dependent term of the KK threshold correction can give a sizable contribution to the differential running of the gauge couplings for the large volume of extra dimensions. In this case, in order for the additional contribution due to extra color triplets to be cancelled by the volume dependent part, the breaking scale of the extra $U(1)$ tends to be close to the compactification scale for the gauge coupling unification. On the other hand, in the case with anisotropic compactification, e.g. in the 5D limit where the bulk gauge group becomes the Pati-Salam $SU(4) \times SU(2)_L \times SU(2)_R$, we show that the shape dependent term of the KK threshold correction can be dominant, giving rise to the 5D power-like threshold corrections with non-universal coefficient in the compactification scales. These power-like corrections in the 5D limit are *calculable*, in contrast to the *uncalculable* power-like corrections in the cutoff scale in the genuine 5D case. Consequently, we show that the allowed contribution of extra color triplets or the breaking scale of the extra $U(1)$ is sensitive to the shape modulus in a phenomenologically successful $SO(10)$ model.

The paper is organized as follows. First we give a brief review on the general boundary conditions for breaking the bulk gauge symmetry on T^2/Z_2. In Section 3, we present the one-loop effective action for gauge bosons in the general 6D orbifold GUTs and derive the running for the effective gauge couplings at low energy. Then, in Section 4, we consider the case with $SO(10)$ bulk group and discuss the gauge coupling unification for some embeddings of the MSSM. Finally the conclusion is drawn. The details on the propagators on GUT orbifolds, the KK summations, the definition of special functions and some $SO(10)$ group theory facts are given in the appendices.

2 Boundary conditions on GUT orbifolds

Before considering particular models, we give a brief sketch for the orbifold breaking of gauge symmetry in a six-dimensional non-Abelian gauge theory with a simple gauge group. Two extra dimensions are compactified on the orbifold T^2/Z_2. For the extra coordinates $z \equiv x^5 + ix^6$, there are double periodicities $z \sim z + 2\pi(R_5 n_5 + iR_6 n_6)$ with radii R_5, R_6 and integer numbers n_5, n_6. Further, when the bulk positions are identified by a Z_2 reflection symmetry as $z \to -z$, there are four fixed points on the orbifold: $z_0 = 0$, $z_1 = \pi R_5$, $z_2 = i\pi R_6$ and $z_3 = \pi R_5 + i\pi R_6$.

[1]For some early works on string theory computation of the one-loop gauge couplings, see Ref. [17, 18]

In order to break the bulk gauge symmetry down to the SM gauge group, let us introduce nontrivial boundary conditions for bulk gauge fields A_M with $M = 0,1,2,3 \equiv \mu$ and $M = 5,6 \equiv m$. The boundary conditions are specified by unitary parity matrices $P_i (i = 0,1,2,3)$ at the fixed points,

$$\begin{aligned} \mathcal{P}_i A_\mu(z)\mathcal{P}_i^{-1} &\equiv P_i A_\mu(-z+z_i)P_i^{-1} = A_\mu(z+z_i), \\ \mathcal{P}_i A_m(z)\mathcal{P}_i^{-1} &\equiv -P_i A_m(-z+z_i)P_i^{-1} = A_m(z+z_i) \end{aligned} \tag{1}$$

where $P_i^2 = 1 (i = 0,1,2,3)$. The above boundary conditions can be rewritten simply in terms of component fields with $A_M = A_M^a T_a$ in the group space as

$$A_\mu^a(-z+z_i) = (Q_i)^a{}_b A_\mu^b(z+z_i), \tag{2}$$

$$A_m^a(-z+z_i) = -(Q_i)^a{}_b A_m^b(z+z_i) \tag{3}$$

where

$$(Q_i)^a{}_b \equiv \mathrm{tr}(T^a P_i T_b P_i). \tag{4}$$

Here the defined matrices $(Q_i)^a{}_b (i = 0,1,2,3)$ fulfill

$$(Q_i)^a{}_{a'}(Q_i)^b{}_{b'}\eta_{ab} = \eta_{a'b'}, \quad f_{abc}(Q_i)^a{}_{a'}(Q_i)^b{}_{b'}(Q_i)^c{}_{c'} = f_{a'b'c'} \tag{5}$$

where η_{ab} is the Killing metric defined by $\mathrm{tr}(T_aT_b) = \eta_{ab}$ on the group space and it is used to raise and low adjoint indices, and f_{abc} are the group structure constants given in the group algebra $[T_a, T_b] = if_{abc}T^c$. Note that $Q_i^2 = 1$ from the Z_2 symmetry and hence Q_i are real symmetric matrices. Eq. (4) and the second property in eq. (5) can be rewritten, respectively, as

$$P_i T^a P_i = (Q_i)^a{}_b T^b, \tag{6}$$

$$Q_i T_G^a Q_i = (Q_i)^a{}_b T_G^b \tag{7}$$

with $(T_G^b)^{ac} = if^{abc}$.

We also discuss on the Wilson lines on a torus in comparison to the local boundary conditions as given above. The boundary conditions along noncontractible loops on a torus are defined by the unitary matrices U_1, U_2 as

$$U_1 A_M(z+2\pi R_5)U_1^{-1} = A_M(z), \tag{8}$$

$$U_2 A_M(z+i2\pi R_6)U_2^{-1} = A_M(z). \tag{9}$$

Since $x^5 + \pi R_5 \to -x^5 + \pi R_5$ is equivalent to $x^5 + \pi R_5 \to -x^5 - \pi R_5 \to -x^5 + \pi R_5$ and similarly for the other coordinate, we obtain the following relations,

$$U_1 = P_1 P_0, \quad U_2 = P_2 P_0. \tag{10}$$

Then, we can see that the consistency conditions for the Wilson lines on orbifolds, $U_1 P_0 U_1 = P_0$ and $U_2 P_0 U_2 = P_0$, are satisfied. Moreover, since $U_2 U_1 P_0 = P_3$ and $[U_1, U_2] = 0$, the parity matrix P_3 can be written as

$$P_3 = P_2 P_0 P_1 = P_1 P_0 P_2. \tag{11}$$

Therefore, the Wilson lines one can consider are not independent of local boundary conditions, and one of the parity actions is not independent.

For simplicity, let us focus on the case with commuting parity matrices, i.e. $[P_i, P_j] = 0$ or $[Q_i, Q_j] = 0$. For these parity actions, the rank of the gauge group is not reduced. In this case, it is convenient to choose the Cartan-Weyl basis such that the orbifold actions become diagonal. In this basis, the generators are organized into Cartan subalgebra generators $H_I, I = 1, \cdots, \text{rank}(G)$, and the remaining generators, $E_\alpha, \alpha = 1, \cdots, (\text{dim(G)} - \text{rank}(G))$, with

$$[H_I, E_\alpha] = \alpha_I E_\alpha, \tag{12}$$

where α_I is the rank$(G)-$dimensional root vector associated with E_α. Then, it is always possible to write the parity matrices as

$$P_i = e^{-2\pi i V_i \cdot H} \tag{13}$$

which defines the rank$(G)-$dimensional twist vector V_i for each fixed point. Thus, the relations (6) become

$$\begin{aligned} P_i H_I P_i &= H_I, & (14)\\ P_i E_\alpha P_i &= e^{-2\pi i \alpha \cdot V_i} E_\alpha. & (15) \end{aligned}$$

In this basis, the matrices Q_i are also diagonal such that $(Q_i)^I{}_J = \delta^I_J$ and $(Q_i)^\alpha{}_\beta = e^{-2\pi i \alpha \cdot V_i} \delta^\alpha_\beta$ and other entries are zero. Here we have that $\alpha \cdot V_i = 0$ or $\frac{1}{2}$ mod $\mathbf{Z}$ for Z_2 actions at the fixed points because $Q_i^2 = 1$. Then, a bulk field takes a combination of parity eigenvalues (p_0, p_1, p_2) with $p_i = +1$ or -1 under three independent Z_2 actions, so it is composed of a subset of basis functions on a torus with radii $2R_5$ and $2R_6$.

3 The effective action on GUT orbifolds

In this section, we present the general formulae for the one-loop effective action in a 6D $\mathcal{N} = 1$ supersymmetric GUT where the bulk gauge symmetry is broken by local boundary conditions at the fixed points on the T^2/Z_2 orbifold as described in the previous section. As a result, we also discuss about the running of the gauge couplings for zero-mode gauge bosons at low energy.

3.1 The one-loop effective action on the T^2/Z_2 orbifold

We consider a 6D $\mathcal{N} = 1$ supersymmetric non-Abelian gauge theory compactified on the orbifold T^2/Z_2. In terms of component fields, a vector multiplet is composed of gauge bosons A_M and (right-handed) symplectic Majorana gauginos λ while a hyper multiplet is composed of two complex hyperscalars $\phi_\pm$ without opposite charges and a (left-handed) hyperino ψ. Since all charged hyperinos have the equal 6D chiralities due to supersymmetry, one is not allowed to write the 6D mass terms for hyper multiplets.

In the process of taking the usual gauge fixing for a non-Abelian gauge theory [6], we also introduce ghost fields c^a. Then, the orbifold boundary conditions for bulk component fields that we are considering are as the following,

$$
\begin{aligned}
A^a_\mu(x,-z+z_i) &= (Q_i)^a{}_b\, A^b_\mu(x,z+z_i), \quad A^a_m(x,-z+z_i) = -(Q_i)^a{}_b\, A^b_m(x,z+z_i), \\
c^a(x,-z+z_i) &= (Q_i)^a{}_b\, c^b(x,z+z_i), \quad \lambda^a(x,-z+z_i) = i\gamma^5\,(Q_i)^a{}_b\,\lambda^b(x,z+z_i), \\
\psi(x,-z+z_i) &= i\gamma^5\,\eta_i P_i\,\psi(x,z+z_i), \\
\phi_+(x,-z+z_i) &= \eta_i P_i\,\phi_+(x,z+z_i), \quad \phi_-(x,-z+z_i) = -\eta_i\phi_-(x,z+z_i)P_i
\end{aligned} \tag{16}
$$

with $i=0,1,2,3$. Here the forms of the parity matrices depend on the representation of a hyper multiplet under the bulk gauge group. Each hyper multiplet can take its own value of η_i as either $+1$ or -1.

Taking into account the group structure of propagators in loops as discussed in the Appendix A and following the similar procedure as in the case with no orbifold breaking of the gauge symmetry in Ref. [6], we obtain the one-loop effective action for the *background* gauge bosons up to quadratic orders as

$$
\begin{aligned}
\Gamma^{(2)}[A_\mu] =& \frac{1}{2g^2}\sum_{\vec{k}}\int\frac{d^4k}{(2\pi)^4}A^a_\mu(-k,-\vec{k})\,A^a_\nu(k,\vec{k})\Big(-(k^2-\vec{k}^2)g^{\mu\nu}+k^\mu k^\nu\Big) \\
&+\frac{i}{2}\sum_{\vec{k},\vec{k}'}\int\frac{d^4k}{(2\pi)^4}A^{b\mu}(-k,-\vec{k}')\,A^\nu_a(k,\vec{k}) \\
&\times\Big\{-\Pi^{\rm G}_{\mu\nu}+4(k^2g_{\mu\nu}-k_\mu k_\nu)\Pi^G_{++}-2\vec{k}\cdot\vec{k}'g_{\mu\nu}(\Pi^G_{+-}+\Pi^G_{-+})-\Pi^{\rm H}_{\mu\nu}\Big\}^a{}_b
\end{aligned} \tag{17}
$$

where $\Pi^{\rm G}_{\mu\nu}=\Pi^g_{\mu\nu,+}+\Pi^g_{\mu\nu,-}+\Pi^\lambda_{\mu\nu}$, $\Pi^{\rm H}_{\mu\nu}=\Pi^h_{\mu\nu,+}+\Pi^h_{\mu\nu,-}+\Pi^\psi_{\mu\nu}$ with

$$
\begin{aligned}
(\Pi^g_{\mu\nu,\pm})^a{}_b =& \sum_{\vec{p},\vec{p}'}\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\Big[\Big\{-(2p+k)_\mu(2p+k)_\nu\tilde{G}_{g,\pm}(p+k,\vec{p}'+\vec{k}',\vec{p}+\vec{k}) \\
&+2ig_{\mu\nu}\delta_{\vec{p}',\vec{p}+\vec{k}-\vec{k}'}\Big\}T_b\tilde{G}_{g,\pm}(p,\vec{p},\vec{p}')T^a\Big],
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
(\Pi^h_{\mu\nu,\pm})^a{}_b =& \sum_{\vec{p},\vec{p}'}\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\Big[\Big\{-(2p+k)_\mu(2p+k)_\nu\tilde{G}_{h,\pm}(p+k,\vec{p}'+\vec{k}',\vec{p}+\vec{k}) \\
&+2ig_{\mu\nu}\delta_{\vec{p}',\vec{p}+\vec{k}-\vec{k}'}\Big\}T_b\tilde{G}_{h,\pm}(p,\vec{p},\vec{p}')T^a\Big],
\end{aligned} \tag{19}
$$

$$
(\Pi^\lambda_{\mu\nu})^a{}_b = \sum_{\vec{p},\vec{p}'}\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\Big[\tilde{D}_\lambda(p,\vec{p},\vec{p}')\gamma_\mu T_b\tilde{D}_\lambda(p+k,\vec{p}'+\vec{k}',\vec{p}+\vec{k})\gamma_\nu T^a\Big], \tag{20}
$$

$$
(\Pi^\psi_{\mu\nu})^a{}_b = \sum_{\vec{p},\vec{p}'}\int\frac{d^4p}{(2\pi)^4}{\rm Tr}\Big[\tilde{D}_\psi(p,\vec{p},\vec{p}')\gamma_\mu T_b\tilde{D}_\psi(p+k,\vec{p}'+\vec{k}',\vec{p}+\vec{k})\gamma_\nu T^a\Big], \tag{21}
$$

and

$$(\Pi^G_{\pm\pm})^a{}_b = \sum_{\vec{p},\vec{p}'}\int \frac{d^4p}{(2\pi)^4}\ \mathrm{Tr}\Big[\tilde{G}_{g,\pm}(p+k,\vec{p}+\vec{k},\vec{p}'+\vec{k}')T_b\tilde{G}_{g,\pm}(p,\vec{p},\vec{p}')T^a\Big], \qquad (22)$$

$$\begin{aligned}(\Pi^G_{\pm\mp})^a{}_b &= \sum_{\vec{p},\vec{p}'}\int \frac{d^4p}{(2\pi)^4}\ \mathrm{Tr}\Big[\tilde{G}_{g,\mp}(p+k,\vec{p}+\vec{k},\vec{p}'+\vec{k}')T_b\tilde{G}_{g,\pm}(p,\vec{p},\vec{p}')T^a\Big] \\ &= (\Pi^G_{\pm,\pm})^a{}_b. \qquad (23)\end{aligned}$$

Here the propagators appearing in the loops are given in the Appendix A. Since we consider the commuting parity matrices, the orbifold actions with respect to the fixed points other that the origin are factorized out of the propagators which would be given for the case with one Z_2 orbifold action only. Then, after identifying various equivalent terms, we get the effective action in a simpler form as a decomposition into bulk and brane parts,

$$\Gamma^{(2)}[A_\mu] = \Gamma_{\rm bulk} + \Gamma_{\rm brane} \qquad (24)$$

with

$$\begin{aligned}\Gamma_{\rm bulk} &= \frac{1}{2}\sum_{\vec{k},\vec{k}'}\int \frac{d^4k}{(2\pi)^4}A^b_\mu(-k,-\vec{k})A_{a\nu}(k,\vec{k}')((k^2-\vec{k}^2)g^{\mu\nu}-k^\mu k^\nu) \\ &\quad\times\Big[-\frac{1}{g^2}\delta^a_b - i(\Pi_G+\Pi_H)^a{}_b(k,\vec{k})\Big]\delta_{\vec{k},\vec{k}'}, \qquad (25) \\ \Gamma_{\rm brane} &= \frac{1}{2}\sum_{\vec{k},\vec{k}'}\int \frac{d^4k}{(2\pi)^4}A^b_\mu(-k,-\vec{k})A_{a\nu}(k,\vec{k}')(k^2g^{\mu\nu}-k^\mu k^\nu)[-4i(\tilde{\Pi}_G)^a{}_b] \qquad (26)\end{aligned}$$

where

$$\begin{aligned}(\Pi_G)^a{}_b(k,\vec{k}) &= \mu^{4-d}\sum_{\vec{p}}\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-\vec{p}^2)[(p+k)^2-(\vec{p}+\vec{k})^2]} \\ &\quad\times\frac{1}{4}\mathrm{tr}_{\rm Adj}\Big[\Big\{1+\cos(2p_5\pi R_5)Q_0Q_1\Big\}\Big\{1+\cos(2p_6\pi R_6)Q_0Q_2\Big\}T^aT_b\Big], \qquad (27) \\ (\Pi_H)^a{}_b(k,\vec{k}) &= -\mu^{4-d}\sum_{\vec{p}}\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-\vec{p}^2)[(p+k)^2-(\vec{p}+\vec{k})^2]} \\ &\quad\times\frac{1}{4}\mathrm{tr}_{\rm R}\Big[\Big\{1+\eta_0\eta_1\cos(2p_5\pi R_5)P_0P_1\Big\}\Big\{1+\eta_0\eta_2\cos(2p_6\pi R_6)P_0P_2\Big\}T^aT_b\Big], \qquad (28) \\ (\tilde{\Pi}_G)^a{}_b(k,\vec{k}',\vec{k}) &= \frac{\mu^{4-d}}{2}\sum_{\vec{p}}\int\frac{d^dp}{(2\pi)^d}\frac{\delta_{-2\vec{p},\vec{k}-\vec{k}'}}{(p^2-\vec{p}^2)[(p+k)^2-(\vec{p}+\vec{k})^2]} \\ &\quad\times\frac{1}{4}\mathrm{tr}_{\rm Adj}\Big[\Big\{1+\cos(2p_5\pi R_5)Q_0Q_1\Big\}\Big\{1+\cos(2p_6\pi R_6)Q_0Q_2\Big\}Q_0T^aT_b\Big]. \qquad (29)\end{aligned}$$

Here μ is the renormalization scale in dimensional regularization with $d=4-\epsilon$, and $\vec{p}=(p_5,p_6)=(\frac{n_5}{2R_5},\frac{n_6}{2R_6})$ with n_5,n_6 being integer, and similarly for $\vec{k}$ and $\vec{k}'$. In simplifying the expressions in the above, for the generators satisfying $P_i T^a = \pm T^a P_i$, we made use of $\cos(2(p_5+k_5)\pi R_5) = \pm\cos(2p_5\pi R_5)$ and $\cos(2(p_6+k_6)\pi R_6) = \pm\cos(2p_6\pi R_6)$. Further, we notice that $\mathrm{tr}_{\mathrm{Adj}}$ is the trace over indices of the adjoint representation and tr_{R} is the trace over indices of the R representation.

From eq. (24), we can see that a vector multiplet gives rise to both bulk and brane-localized corrections while a hyper multiplet only leads to a bulk correction. It has been shown that the absence of the brane-localized corrections due to a hyper multiplet is restricted to the case with even ordered orbifolds [5].

In order to simplify the expression for the bulk contribution (25), we define the quantity

$$\begin{aligned}\Pi^{(\rho_5,\rho_6)} &\equiv \mu^{4-d}\sum_{\vec{p}}\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-\vec{p}^2)[(p+k)^2-(\vec{p}+\vec{k})^2]} \\ &= \frac{i}{(4\pi)^2V}(2\pi\mu)^\epsilon\int_0^1 dx\,\mathcal{J}_0[x(1-x)(k^2+\vec{k}^2), xk_5R_5+\rho_5, xk_6R_6+\rho_6] \quad (30)\end{aligned}$$

where $\vec{p} = (\frac{n_5+\rho_5}{R_5}, \frac{n_6+\rho_6}{R_6})$ with $\rho_5, \rho_6 = 0$ or $\frac{1}{2}$ and n_5, n_6 being integer, $V \equiv (2\pi)^2 R_5 R_6$, and

$$\mathcal{J}_0[c, c_1, c_2] \equiv \sum_{n_1,n_2\in\mathbf{Z}}\int_0^\infty\frac{dt}{t^{1-\epsilon/2}}\,e^{-\pi t[c+a_1(n_1+c_1)^2+a_2(n_2+c_2)^2]} \quad (31)$$

with $a_i = 1/R_{i+4}^2 (i = 1, 2)$. Thus, we can rewrite the bulk contribution due to a vector multiplet as

$$\begin{aligned}(\Pi_G)^a{}_b &= \frac{1}{4}\mathrm{tr}_{\mathrm{Adj}}[T^aT_b]\,(\Pi^{(0,0)}+\Pi^{(0,\frac{1}{2})}+\Pi^{(\frac{1}{2},0)}+\Pi^{(\frac{1}{2},\frac{1}{2})}) \\ &+\frac{1}{4}\mathrm{tr}_{\mathrm{Adj}}[Q_0Q_1T^aT_b]\,(\Pi^{(0,0)}+\Pi^{(0,\frac{1}{2})}-\Pi^{(\frac{1}{2},0)}-\Pi^{(\frac{1}{2},\frac{1}{2})}) \\ &+\frac{1}{4}\mathrm{tr}_{\mathrm{Adj}}[Q_0Q_2T^aT_b]\,(\Pi^{(0,0)}-\Pi^{(0,\frac{1}{2})}+\Pi^{(\frac{1}{2},0)}-\Pi^{(\frac{1}{2},\frac{1}{2})}) \\ &+\frac{1}{4}\mathrm{tr}_{\mathrm{Adj}}[Q_1Q_2T^aT_b]\,(\Pi^{(0,0)}-\Pi^{(0,\frac{1}{2})}-\Pi^{(\frac{1}{2},0)}+\Pi^{(\frac{1}{2},\frac{1}{2})}). \quad (32)\end{aligned}$$

Similarly, we can write the bulk contribution of a hyper multiplet as

$$\begin{aligned}(\Pi_H)^a{}_b &= -\frac{1}{4}\mathrm{tr}_{\mathrm{R}}[T^aT_b]\,(\Pi^{(0,0)}+\Pi^{(0,\frac{1}{2})}+\Pi^{(\frac{1}{2},0)}+\Pi^{(\frac{1}{2},\frac{1}{2})}) \\ &-\frac{1}{4}\eta_0\eta_1\,\mathrm{tr}_{\mathrm{R}}[P_0P_1T^aT_b]\,(\Pi^{(0,0)}+\Pi^{(0,\frac{1}{2})}-\Pi^{(\frac{1}{2},0)}-\Pi^{(\frac{1}{2},\frac{1}{2})}) \\ &-\frac{1}{4}\eta_0\eta_2\mathrm{tr}_{\mathrm{R}}[P_0P_2T^aT_b]\,(\Pi^{(0,0)}-\Pi^{(0,\frac{1}{2})}+\Pi^{(\frac{1}{2},0)}-\Pi^{(\frac{1}{2},\frac{1}{2})}) \\ &-\frac{1}{4}\eta_1\eta_2\mathrm{tr}_{\mathrm{R}}[P_1P_2T^aT_b]\,(\Pi^{(0,0)}-\Pi^{(0,\frac{1}{2})}-\Pi^{(\frac{1}{2},0)}+\Pi^{(\frac{1}{2},\frac{1}{2})}). \quad (33)\end{aligned}$$

Also defining

$$\tilde{\Pi}^{(\rho_5,\rho_6)} \equiv \frac{\mu^{4-d}}{2}\sum_{\vec{p}}\int\frac{d^dp}{(2\pi)^d}\frac{\delta_{-2\vec{p},\vec{k}-\vec{k}'}}{(p^2-\vec{p}^2)[(p+k)^2-(\vec{p}+\vec{k})^2]} \quad (34)$$

where $\vec{p}=(\frac{n_5+\rho_5}{R_5},\frac{n_6+\rho_6}{R_6})$ with $\rho_5,\rho_6=0$ or $\frac{1}{2}$ and n_5,n_6 being integer, we can rewrite the brane contribution as

$$\begin{aligned}(\tilde{\Pi}_G)^a{}_b &= \frac{1}{4}\mathrm{tr}_{\mathrm{Adj}}[Q_0T^aT_b]\,(\tilde{\Pi}^{(0,0)}+\tilde{\Pi}^{(0,\frac{1}{2})}+\tilde{\Pi}^{(\frac{1}{2},0)}+\tilde{\Pi}^{(\frac{1}{2},\frac{1}{2})})\\ &+\frac{1}{4}\mathrm{tr}_{\mathrm{Adj}}[Q_1T^aT_b]\,(\tilde{\Pi}^{(0,0)}+\tilde{\Pi}^{(0,\frac{1}{2})}-\tilde{\Pi}^{(\frac{1}{2},0)}-\tilde{\Pi}^{(\frac{1}{2},\frac{1}{2})})\\ &+\frac{1}{4}\mathrm{tr}_{\mathrm{Adj}}[Q_2T^aT_b]\,(\tilde{\Pi}^{(0,0)}-\tilde{\Pi}^{(0,\frac{1}{2})}+\tilde{\Pi}^{(\frac{1}{2},0)}-\tilde{\Pi}^{(\frac{1}{2},\frac{1}{2})})\\ &+\frac{1}{4}\mathrm{tr}_{\mathrm{Adj}}[Q_3T^aT_b]\,(\tilde{\Pi}^{(0,0)}-\tilde{\Pi}^{(0,\frac{1}{2})}-\tilde{\Pi}^{(\frac{1}{2},0)}+\tilde{\Pi}^{(\frac{1}{2},\frac{1}{2})}).\end{aligned} \quad (35)$$

Thus, we find that the brane-localized contribution at each fixed point corresponds to the brane projection of the bulk quantity by the local parity matrix.

3.2 Counterterms for loop divergences

After the KK summation given in the Appendix B, we can separate the divergent term as

$$\Pi^{(\rho_5,\rho_6)}=\frac{i}{(4\pi)^2V}\frac{\pi}{6}R_5R_6(k^2+\vec{k}^2)\left[\frac{-2}{\epsilon}\right]+\mathcal{O}(\epsilon^0). \quad (36)$$

Thus, the bulk contribution becomes

$$(\Pi_G+\Pi_H)^a{}_b=\frac{1}{4}(\mathrm{tr}_{\mathrm{Adj}}[T^aT_b]-\mathrm{tr}_{\mathrm{R}}[T^aT_b])\frac{i}{(4\pi)^2V}\frac{\pi}{6}R_5R_6(k^2+\vec{k}^2)\left[\frac{-2}{\epsilon}\right]+\mathcal{O}(\epsilon^0). \quad (37)$$

Therefore, we find that the loop corrections generate a divergent higher derivative term the coefficient of which is proportional to the universal $\mathcal{N}=2$ beta function. At the momentum scale higher than the compactification scales, the higher derivative operator becomes important so that the gauge couplings run power-like in momentum scale rather than logarithmically [6]. However, it does not affect the unification of the gauge couplings, even if it is important in determining the value of the unified gauge coupling and the unification scale.

For a given set of ingoing and outgoing momenta of gauge bosons satisfying $\vec{p}=\frac{\vec{k}'-\vec{k}}{2}$, we compute $\tilde{\Pi}^{(\rho_5,\rho_6)}$ as

$$\tilde{\Pi}^{(\rho_5,\rho_6)}=\frac{i}{32\pi^2}\left\{\frac{2}{\epsilon}+\ln(4\pi\mu^2e^{-\gamma_E})-\int_0^1dx\,\ln\left[x(1-x)(k^2+\vec{k}^2)+\left(\frac{\vec{k}'}{2}+(x-\frac{1}{2})\vec{k}\right)^2\right]\right\}. \quad (38)$$

Thus, the brane contribution becomes

$$(\tilde{\Pi}_G)^a{}_b=\frac{1}{4}\frac{i}{32\pi^2}\frac{2}{\epsilon}\sum_{i=0}^{3}\mathrm{tr}_{\mathrm{Adj}}[Q_iT^aT_b]\,\delta^2(z-z_i)+\mathcal{O}(\epsilon^0). \quad (39)$$

Therefore, the appearing divergent term at each fixed point respects the corresponding gauge symmetry which depends on the local orbifold action.

Consequently, in order to subtract the ϵ poles in $(\Pi_G)^a{}_b, (\Pi_H)^a{}_b$ and $(\tilde{\Pi}_G)^a{}_b$ obtained in eqs. (37) and (39), we require the following new counterterms which are not present in the original action,

$$\mathcal{L}_{c.t.} = \int d^2z d^2\theta \left[\frac{1}{2h^2}\mathrm{tr}[W\Box_6 W] + \frac{1}{2}\sum_{i=0}^{3}\left(\frac{1}{g^2_{i,a}}W_a W^a \delta^2(z-z_i)\right)\right] + \mathrm{h.c.} \tag{40}$$

Here h^2 is a dimensionless bulk coupling while $g^2_{i,a}(i=0,1,2,3)$ are dimensionless gauge couplings correspoding to the local gauge groups at the fixed points. Note that the brane-localized gauge coupling can be non-universal so it could affect the predictive power of the orbifold GUTs.

3.3 Limiting cases

For a later use in the running of the zero-mode gauge coupling, let us take the asymptotic limits of the loop corrections. First, in the low momentum limit $k^2 \ll 1/R^2_{5,6}$, with $\vec{k} = \vec{k}' = 0$, by using eq. (B.4), we get the approximate forms for eq. (30),

$$\begin{aligned}
\Pi^{(0,0)} &\approx \frac{i}{(4\pi)^2 V}\bigg\{\frac{\pi}{6}R_5R_6k^2\left[\frac{-2}{\epsilon} - \ln\left[\pi e^{\gamma_E}\mu^2R_5^2|\eta(iu)|^{-4}\right]\right] \\
&\quad - \ln\left[4\pi^2e^{-2}|\eta(iu)|^4R_6^2k^2\right]\bigg\}, & (41)\\
\Pi^{(0,\frac{1}{2})} &\approx \frac{i}{(4\pi)^2 V}\bigg\{\frac{\pi}{6}R_5R_6k^2\left[\frac{-2}{\epsilon} - \ln(\pi e^{\gamma_E}\mu^2R_5^2) - \pi u + 2\sum_{n\geq 1}(-1)^n\ln|1-e^{-2\pi un}|^2\right] \\
&\quad - \ln\left|\frac{\vartheta_1(1/2|iu)}{\eta(iu)}\right|^2\bigg\}, & (42)\\
\Pi^{(\frac{1}{2},0)} &\approx \frac{i}{(4\pi)^2 V}\bigg\{\frac{\pi}{6}R_5R_6k^2\left[\frac{-2}{\epsilon} - \ln(4\pi e^{\gamma_E}\mu^2R_5^2) - 2\sum_{n\geq 1}\ln\left|\frac{1+e^{-\pi un}}{1-e^{-\pi un}}\right|^2\right] \\
&\quad - \ln\left|\frac{\vartheta_1(-iu/2|iu)}{\eta(iu)}e^{-\pi u/4}\right|^2\bigg\}, & (43)\\
\Pi^{(\frac{1}{2},\frac{1}{2})} &\approx \frac{i}{(4\pi)^2 V}\bigg\{\frac{\pi}{6}R_5R_6k^2\left[\frac{-2}{\epsilon} - \ln(4\pi e^{\gamma_E}\mu^2R_5^2) - 2\sum_{n\geq 1}(-1)^n\ln\left|\frac{1+e^{-\pi un}}{1-e^{-\pi un}}\right|^2\right] \\
&\quad - \ln\left|\frac{\vartheta_1(1/2-iu/2|iu)}{\eta(iu)}e^{-\pi u/4}\right|^2\bigg\}. & (44)
\end{aligned}$$

Further, using the fact that $\Pi^{(0,0)} + \Pi^{(0,\frac{1}{2})} + \Pi^{(\frac{1}{2},0)} + \Pi^{(\frac{1}{2},\frac{1}{2})}$ is the same as the KK sum on a torus with each radius double sized and with no Wilson lines, i.e. from the approximate form of the sum,

$$\sum_{\rho_5,\rho_6=0,\frac{1}{2}} \Pi^{(\rho_5,\rho_6)} \approx \frac{i}{(4\pi)^2 V}\Bigg\{\frac{\pi}{6}(4R_5R_6)k^2\left[\frac{-2}{\epsilon} - \ln\left[4\pi e^{\gamma_E}\mu^2 R_5^2|\eta(iu)|^{-4}\right]\right. \\ \left. - \ln\left[16\pi^2 e^{-2}|\eta(iu)|^4 R_6^2 k^2\right]\right]\Bigg\}, \quad (45)$$

we note the useful identity for the theta functions,

$$\Big|\vartheta_1(1/2|iu)\,\vartheta_1(-iu/2|iu)\,\vartheta_1(1/2-iu/2|iu)\Big|^2 = 4|\eta(iu)|^6 e^{\pi u}. \quad (46)$$

In the high momentum limit $k^2 \gg 1/R_{5,6}^2$, with $\vec{k} = \vec{k}' = 0$, eq. (30) becomes, independently of the orbifold actions,

$$\Pi^{(\rho_5,\rho_6)} \approx \frac{i}{(4\pi)^2 V}\Bigg\{\frac{\pi}{6}R_5R_6k^2\left[\frac{-2}{\epsilon} - \ln\frac{\mu^2}{k^2} - \ln\left(4\pi e^{8/3-\gamma_E}\right)\right]\Bigg\}. \quad (47)$$

Therefore, from eqs. (32) and (33), even the finite part of the bulk correction becomes universal at high energy.

3.4 Running of the 4D effective gauge coupling

In this section, we consider the running of the effective gauge coupling which is defined as the coefficient of the kinetic term of a zero-mode gauge boson. It also includes the bulk higher derivative term and the brane kinetic terms.

From the one-loop effective action (24), the zero-mode gauge coupling reads

$$\frac{1}{g^2_{\mathrm{eff},ab}(k^2)} = \frac{1}{g^2_{\mathrm{tree},ab}} - \frac{k^2 V}{h^2_{\mathrm{tree}}}\delta_{ab} + iV(\Pi_G(k,0) + \Pi_H(k,0))^a{}_b + 4i(\tilde{\Pi}_G(k,0,0))^a{}_b \quad (48)$$

with

$$\frac{1}{g^2_{\mathrm{tree},ab}} = \left[\frac{V}{g^2} + \sum_{i=0}^{3}\frac{1}{g^2_{i,a}}\right]\delta_{ab}. \quad (49)$$

When taking the minimal subtraction scheme for divergences (37) and (39) at $k^2 = M_*^2$, where M_* is the 6D fundamental scale, we define the renormalized bulk and brane gauge couplings for $\xi_1\mu^2 = M_*^2$ with $\xi_1 = 4\pi e^{2-\gamma_E}$ at that scale. Then, below the compactification scales ($k^2 \ll 1/R_{5,6}^2$), using eqs. (38), (41)-(44) with (46), we have eq. (48) as

$$\frac{1}{g^2_{\mathrm{eff},ab}(k^2)} = \frac{1}{g^2_{\mathrm{r},a}}\delta_{ab} + \frac{1}{16\pi^2}B_{ab}\ln\frac{M_*^2}{k^2} - \frac{1}{16\pi^2}\sum_{i,j=\pm}B^{ij}_{ab}L_{ij} - \frac{1}{4\pi}\kappa_{ab} \quad (50)$$

where $g_{r,a}$ are the renormalized gauge couplings and the beta functions are

$$\begin{aligned} B_{ab} &= \frac{1}{4}\Big[-\mathrm{tr}_{\mathrm{Adj}}(T^aT_b)+\sum_R \mathrm{tr}_{\mathrm{R}}(T^aT_b)-2\sum_{i=0}^{3}\mathrm{tr}_{\mathrm{Adj}}(Q_iT^aT_b)\Big] \\ &+\frac{1}{4}\sum_{i=1}^{3}\Big[-\mathrm{tr}_{\mathrm{Adj}}(Q_0Q_iT^aT_b)+\sum_R \eta_0^R\eta_i^R\mathrm{tr}_{\mathrm{R}}(P_0P_iT^aT_b)\Big] \end{aligned} \tag{51}$$

and

$$B_{ab}^{++} = \frac{1}{4}\Big[-\mathrm{tr}_{\mathrm{Adj}}(T^aT_b)+\sum_R \mathrm{tr}_{\mathrm{R}}(T^aT_b)\Big], \tag{52}$$

$$B_{ab}^{-+} = \frac{1}{4}\Big[-\mathrm{tr}_{\mathrm{Adj}}(Q_0Q_1T^aT_b)+\sum_R \eta_0^R\eta_1^R\mathrm{tr}_{\mathrm{R}}(P_0P_1T^aT_b)\Big], \tag{53}$$

$$B_{ab}^{+-} = \frac{1}{4}\Big[-\mathrm{tr}_{\mathrm{Adj}}(Q_0Q_2T^aT_b)+\sum_R \eta_0^R\eta_2^R\mathrm{tr}_{\mathrm{R}}(P_0P_2T^aT_b)\Big], \tag{54}$$

$$B_{ab}^{--} = \frac{1}{4}\Big[-\mathrm{tr}_{\mathrm{Adj}}(Q_0Q_3T^aT_b)+\sum_R \eta_0^R\eta_3^R\mathrm{tr}_{\mathrm{R}}(P_0P_3T^aT_b)\Big] \tag{55}$$

with

$$L_{++} = \ln\Big[4e^{-2}|\eta(iu)|^4 uVM_*^2\Big], \tag{56}$$

$$L_{-+} = \ln\Big[\frac{e^{-2}}{4}\Big|\vartheta_1(\frac{1}{2}|iu)\Big|^4 uVM_*^2\Big], \tag{57}$$

$$L_{+-} = \ln\Big[\frac{e^{-2}}{4}\Big|\vartheta_1(-\frac{1}{2}iu|iu)e^{-\pi u/4}\Big|^4 uVM_*^2\Big], \tag{58}$$

$$L_{--} = \ln\Big[\frac{e^{-2}}{4}\Big|\vartheta_1(\frac{1}{2}-\frac{1}{2}iu|iu)e^{-\pi u/4}\Big|^4 uVM_*^2\Big]. \tag{59}$$

Further, κ_{ab} corresponds to the power-like dependence on the momentum scale and it is suppressed by the compactification volume at low energy as in the case without orbifold breaking of the gauge symmetry [6]. B_{ab} are the $\mathcal{N}=1$ beta function coefficients of the logarithmic running due to the massless modes. They are composed of both bulk and brane corrections. On the other hand, B_{ab}^{ij} are the $\mathcal{N}=2$ beta function coefficients for the KK massive modes of the bulk fields. The logarithms L_{ij} have the common volume (V) dependence, but also they are functions of the shape modulus (u), being of different form depending on the parities. Since the KK massive mode correction contains a non-universal part due to the gauge symmetry breaking, they can affect the unification of gauge couplings.

4 Gauge coupling unification in a 6D $SO(10)$ orbifold GUT

We consider a 6D $\mathcal{N}=1$ supersymmetric $SO(10)$ orbifold GUT and compute the gauge coupling running by using the general formulae found in the previous section.

4.1 Orbifold breaking of $SO(10)$

In order to break the bulk $SO(10)$ gauge group down to the SM one, we introduce the parity matrices in eq. (1) or (16) for a fundamental representation [12] as

$$P_0 = I_{10\times 10}, \tag{60}$$
$$P_1 = \mathrm{diag}(-1,-1,-1,1,1)\times\sigma^0, \tag{61}$$
$$P_2 = \mathrm{diag}(1,1,1,1,1)\times\sigma^2, \tag{62}$$

and $P_3=P_1P_2$ from the consistency condition (11). Then, the parity operations P_1, P_2 break $SO(10)$ down to maximal subgroups, the Pati-Salam group $SU(4)\times SU(2)_L\times SU(2)_R$ and the Georgi-Glashow group $SU(5)\times U(1)_X$, respectively. The parity operation P_3 also breaks $SO(10)$ down to the flipped $SU(5)$ but it is not an independent breaking. Thus, the intersection of two maximal subgroups leads to $SU(3)_C\times SU(2)_L\times U(1)_Y\times U(1)_X$ as the remaining gauge group. This can be seen from the gauge bosons with positive parities: $\mathbf{45}$ is decomposed into $(\mathbf{15},\mathbf{1},\mathbf{1})_+ + (\mathbf{6},\mathbf{2},\mathbf{2})_- + (\mathbf{1},\mathbf{3},\mathbf{1})_+ + (\mathbf{1},\mathbf{1},\mathbf{3})_+$ under P_1 (where $\pm$ indicate the parities) and $\mathbf{24}_{\mathbf{0},+} + \mathbf{10}_{-\mathbf{4},-} + \overline{\mathbf{10}}_{\mathbf{4},-} + \mathbf{1}_{\mathbf{0},+}$ under P_2. Then, finally, the extra $U(1)_X$ or $U(1)_{B-L}$ has to be broken further by the VEV of bulk or brane Higgs fields.

For applying the parity action to other representations, from eq. (13), we can rewrite the parity matrices in terms of Cartan-Weyl generators[2] as a special case of eq. (13),

$$P_1 = e^{-2\pi i x_1(-6T_Y+T_X)}, \quad x_1=\frac{1}{2}, \tag{63}$$
$$P_2 = e^{-2\pi i x_2 T_X}, \quad x_2=\frac{1}{8} \tag{64}$$

where T_Y, T_X are the $U(1)_Y$ and $U(1)_X$ generators[3], respectively. We consider a set of hyper multiplets, N_{10} $\mathbf{10}$'s and N_{16} $\mathbf{16}$'s satisfying $N_{10}=2+N_{16}$ for no irreducible anomalies [21, 22]. Both N_{10} and N_{16} have to be even for the absence of localized anomalies unless there are split multiplets at the fixed points [22]. A $\mathbf{10}=(H,G,H^c,G^c)$ is decomposed into $(\mathbf{6},\mathbf{1},\mathbf{1})_- + (\mathbf{1},\mathbf{2},\mathbf{2})_+$ under P_1 and $\mathbf{5}_{-\mathbf{2},-}+\bar{\mathbf{5}}_{\mathbf{2},+}$ under P_2. Then, we get a massless Higgs doublet from H^c of $\mathbf{10}$. On the other hand, a $\mathbf{16}=(Q,L,U,E,D^c,N^c)$ is decomposed into $(\mathbf{4},\mathbf{2},\mathbf{1})_+ + (\bar{\mathbf{4}},\mathbf{1},\mathbf{2})_-$ under P_1 and $\mathbf{10}_{\mathbf{1},-}+\bar{\mathbf{5}}_{-\mathbf{3},+}+\mathbf{1}_{\mathbf{5},+}$ under P_2. Then, we also get a massless lepton doublet from L of $\mathbf{16}$. The $\mathcal{N}=2$ partner of each hyper multiplet has the parity matrices in the group space multiplied by the negative overall parity, due to the discrete choice of the Scherk-Schwarz twist in $SU(2)_R$ space. We note that in eq. (16), $\eta_0=1$ and $\eta_3=\eta_1\eta_2$ for each hyper multiplet.

[2]One has to be careful with multiplying a $U(1)$ phase for the correct parity matrices satisfying $P_i^2=1$.
[3]See the appendix D for details.

4.2 Gauge coupling running at low energy

In order to break the extra $U(1)_X$ gauge symmetry by a usual Higgs mechanism, one can introduce **16** Higgs multiplets in the bulk [13, 14]. In that case, after the orbifolding, on top of SM singlets, one ends up with extra color triplets as zero modes. Since the extra color triplets can get masses of order the $B-L$ breaking scale M_{B-L} at the fixed points, they already start contributing to the running of gauge couplings at that scale. Thus, from the general result (50), we consider the logarithmic running due to zero modes by taking two steps across the $B-L$ breaking scale.

The brane-localized $B-L$ breaking masses for the color triplets can modify their KK massive modes so that there exists an additional contribution to the the gauge coupling running. However, when the $B-L$ breaking scale is below the compactification scale, the new contribution becomes suppressed as M^2_{B-L}/M_c^2 with $M_c \equiv 1/\sqrt{V}$. Thus, henceforth we assume this case to ignore the effect of the brane-localized $B-L$ breaking masses. Then, much below the compactification scale, the running of the 4D effective gauge coupling of the SM gauge group is governed by

$$\begin{aligned}\frac{1}{g^2_{\mathrm{eff},ab}(k^2)} &= \frac{1}{g_u^2}\delta_{ab} + \frac{1}{16\pi^2}B'_{ab}\ln\frac{M^2_{B-L}}{k^2} + \frac{1}{16\pi^2}\tilde{B}_{ab}\ln\frac{M_*^2}{M^2_{B-L}} \\ &\quad -\frac{1}{16\pi^2}\sum_{i,j=\pm} B^{ij}_{ab}L_{ij} + \frac{1}{8\pi^2}\Delta_a\delta_{ab} \end{aligned} \tag{65}$$

where g_u is the universal renormalized gauge coupling[4] and Δ_a are corrections due to renormalized gauge couplings localized at the Pati-Salam and flipped $SU(5)$ fixed points. $B'_{ab} = b_a\delta_{ab}$ are the beta functions of the gauge couplings in the MSSM with $b_a = (33/5, 1, -3)$ given below the $B-L$ breaking scale. Moreover, above the $B-L$ breaking scale, the beta functions for the gauge couplings are given by $\tilde{B}_{ab} = B_{ab} - C_{ab} + \hat{B}_{ab}$ where

$$\begin{aligned}B_{ab} &= \frac{1}{4}\Big[-3\mathrm{tr}_{\mathrm{Adj}}(T^aT_b) + \sum_R \mathrm{tr}_{\mathrm{R}}(T^aT_b)\Big] \\ &\quad +\frac{1}{4}\sum_{i=1}^{3}\Big[-3\mathrm{tr}_{\mathrm{Adj}}(Q_iT^aT_b) + \sum_R \eta_i^R\mathrm{tr}_{\mathrm{R}}(P_iT^aT_b)\Big]\end{aligned} \tag{66}$$

with $Q_3 = Q_1Q_2$, $P_3 = P_1P_2$ and $\eta_3^R = \eta_1^R\eta_2^R$, and $C_{ab} = c_a\delta_{ab}$ is the contribution coming from vector-like massless modes which get tree-level brane masses of order the GUT scale and $\hat{B}_{ab} = \hat{b}_a\delta_{ab}$ comes from the brane-localized fields. Further, the beta functions of the

[4]Although there are also power-like threshold corrections in the cutoff regularization [6, 9], they don't contribute to the differential running of gauge couplings. Nevertheless, the power-like contributions may have the net effect of placing an upper limit on the possible volume of the extra dimensions [23].

KK massive mode corrections are

$$B_{ab}^{++} = \frac{1}{4}\Big[-\mathrm{tr}_{\mathrm{Adj}}(T^aT_b)+\sum_R \mathrm{tr}_{\mathrm{R}}(T^aT_b)\Big], \tag{67}$$

$$B_{ab}^{-+} = \frac{1}{4}\Big[-\mathrm{tr}_{\mathrm{Adj}}(Q_1T^aT_b)+\sum_R \eta_1^R\mathrm{tr}_{\mathrm{R}}(P_1T^aT_b)\Big], \tag{68}$$

$$B_{ab}^{+-} = \frac{1}{4}\Big[-\mathrm{tr}_{\mathrm{Adj}}(Q_2T^aT_b)+\sum_R \eta_2^R\mathrm{tr}_{\mathrm{R}}(P_2T^aT_b)\Big], \tag{69}$$

$$B_{ab}^{--} = \frac{1}{4}\Big[-\mathrm{tr}_{\mathrm{Adj}}(Q_3T^aT_b)+\sum_R \eta_3^R\mathrm{tr}_{\mathrm{R}}(P_3T^aT_b)\Big]. \tag{70}$$

4.2.1 Computation of traces

Now we compute the necessary traces to get the running equations. To this, we define the following invariant quantity including all $SO(10)$ gauge fields,

$$\begin{aligned} B &\equiv B_{ab}F^a_{SO(10)}F^b_{SO(10)} \\ &\equiv B_V + B_M \end{aligned} \tag{71}$$

where

$$B_V \equiv -\frac{3}{4}\Big[\mathrm{tr}_{\mathrm{Adj}}F^2_{SO(10)}+\sum_{i=1}^{3}\mathrm{tr}_{\mathrm{Adj}}(Q_iF^2_{SO(10)})\Big], \tag{72}$$

$$B_M \equiv \sum_R \frac{1}{4}\Big[\mathrm{tr}_R F^2_{SO(10)}+\sum_{i=1}^{3}\eta_i^R\mathrm{tr}_R(P_iF^2_{SO(10)})\Big]. \tag{73}$$

Moreover, similarly we define

$$B^{ij} \equiv B^{ij}_{ab}F^a_{SO(10)}F^b_{SO(10)}. \tag{74}$$

By using the traces for maximal subgroups of $SO(10)$ in the Appendix D, we obtain the following result for the vector multiplet,

$$\begin{aligned} B_V = & -\frac{3}{4}\mathrm{tr}_{\mathrm{Adj}}F^2_{SO(10)}+6\mathrm{tr}_2F^2_{SU(2)_L}+6\mathrm{tr}_2F^2_{SU(2)_R} \\ & -3\mathrm{tr}_5F^2_{SU(5)}+6F^2_{U(1)_X}-3\mathrm{tr}_5F^2_{SU(5)'}+6F^2_{U(1)_{X'}}. \end{aligned} \tag{75}$$

For the hyper multiplets, we also have

$$B_M = B_{10}+B_{16} \tag{76}$$

with

$$B_{10} = \frac{1}{4}N_{10}\mathrm{tr}_{10}F^2_{SO(10)} + \frac{1}{2}\sum_{10}\eta_1^{10}\Big[-\mathrm{tr}_4F^2_{SU(4)} + \mathrm{tr}_2F^2_{SU(2)_L} + \mathrm{tr}_2F^2_{SU(2)_R}\Big], \quad (77)$$

$$B_{16} = \frac{1}{4}N_{16}\mathrm{tr}_{16}F^2_{SO(10)} + \sum_{16}\eta_1^{16}\Big[\mathrm{tr}_2F^2_{SU(2)_L} - \mathrm{tr}_2F^2_{SU(2)_R}\Big] + \frac{1}{4}\sum_{16}\eta_2^{16}\Big[-2\mathrm{tr}_5F^2_{SU(5)} + \frac{3}{2}F^2_{U(1)_X}\Big] + \frac{1}{4}\sum_{16}\eta_1^{16}\eta_2^{16}\Big[-2\mathrm{tr}_5F^2_{SU(5)'} + \frac{3}{2}F^2_{U(1)_{X'}}\Big] \quad (78)$$

Further, we have

$$B^{++} = \frac{1}{4}[-\mathrm{tr}_{\mathrm{Adj}}F^2_{SO(10)} + N_{10}\mathrm{tr}_{10}F^2_{SO(10)} + N_{16}\mathrm{tr}_{16}F^2_{SO(10)}], \quad (79)$$

$$B^{-+} = 2\mathrm{tr}_2F^2_{SU(2)_L} + 2\mathrm{tr}_2F^2_{SU(2)_R} + \frac{1}{2}\sum_{10}\eta_1^{10}\Big[-\mathrm{tr}_4F^2_{SU(4)} + \mathrm{tr}_2F^2_{SU(2)_L} + \mathrm{tr}_2F^2_{SU(2)_R}\Big] + \sum_{16}\eta_1^{16}\Big[\mathrm{tr}_2F^2_{SU(2)_L} - \mathrm{tr}_2F^2_{SU(2)_R}\Big], \quad (80)$$

$$B^{+-} = -\mathrm{tr}_5F^2_{SU(5)} + 2F^2_{U(1)_X} + \frac{1}{4}\sum_{16}\eta_2^{16}\Big[-2\mathrm{tr}_5F^2_{SU(5)} + \frac{3}{2}F^2_{U(1)_X}\Big] \quad (81)$$

$$B^{--} = -\mathrm{tr}_5F^2_{SU(5)'} + 2F^2_{U(1)_{X'}} + \frac{1}{4}\sum_{16}\eta_1^{16}\eta_2^{16}\Big[-2\mathrm{tr}_5F^2_{SU(5)'} + \frac{3}{2}F^2_{U(1)_{X'}}\Big]. \quad (82)$$

Therefore, by reading off the gauge kinetic terms for the SM gauge group in B and B^{ij}, we can find the general expression for the beta function coming from the massless modes in the bulk as $B_{ab} = b_a\delta_{ab}$ with

$$b_a = b_a^V + b_a^{10} + b_a^{16} \quad (83)$$

where

$$b_a^V = (0, -6, -9), \quad (84)$$

$$b_a^{10} = \frac{1}{4}N_{10}(1,1,1) + \frac{1}{4}\sum_{10}\eta_1^{10}(\frac{1}{5}, 1, -1), \quad (85)$$

$$b_a^{16} = \frac{1}{4}(2N_{16} - \sum_{16}\eta_2^{16})(1,1,1) + \frac{1}{4}\sum_{16}\eta_1^{16}(-\frac{6}{5}, 2, 0) + \frac{1}{4}\sum_{16}\eta_1^{16}\eta_2^{16}(\frac{7}{5}, -1, -1), \quad (86)$$

and the beta function for KK massive modes as $B^{ij}_{ab} = b^{ij}_a \delta_{ab}$ with

$$\begin{aligned}
b^{++}_a &= \frac{1}{4}(-8 + N_{10} + 2N_{16})(1,1,1), && (87)\\
b^{-+}_a &= \frac{1}{4}(\frac{12}{5},4,0) + \frac{1}{4}\sum_{10}\eta^{10}_1(\frac{1}{5},1,-1) + \frac{1}{4}\sum_{16}\eta^{16}_1(-\frac{6}{5},2,0), && (88)\\
b^{+-}_a &= \frac{1}{4}(2 + \sum_{16}\eta^{16}_2)(-1,-1,-1), && (89)\\
b^{--}_a &= \frac{1}{4}(\frac{38}{5},-2,-2) + \frac{1}{4}\sum_{16}\eta^{16}_1\eta^{16}_2(\frac{7}{5},-1,-1). && (90)
\end{aligned}$$

Here, in order to get the beta function for $U(1)_Y$, we made use of the relations between $U(1)$ gauge bosons (D.23) in the Appendix D. The corrections due to hyper multiplets in eq. (77) and (78) also contain mixing terms[5] between the $U(1)_Y$ and $U(1)_X$ gauge bosons. After transforming to the canonical gauge kinetic terms, this mixing leads to an overall shift in the $U(1)_X$ charges as well as the coupled renormalization group equations for two $U(1)$ gauge couplings and the $U(1)_X$ charge shift [19]. However, when the extra gauge boson gets a heavy mass for giving the see-saw scale for neutrino masses, the mixing effect is not relevant for the low energy physics while the running of the gauge coupling for a light $U(1)$ gauge boson is not affected by the presence of the mixing term [19].

Consequently, compared to eq. (83), we obtain the relation between beta functions as

$$b_a = (0,-4,-6) + b^{++}_a + b^{-+}_a + b^{+-}_a + b^{--}_a. \quad (91)$$

The first term is only due to the difference between the beta functions of $\mathcal{N}=1$ vector multiplets and $\mathcal{N}=2$ vector multiplets for the SM gauge group. Apart from that, the sum of the $\mathcal{N}=2$ beta functions for the volume dependent part of the KK massive contributions, i.e. $\sum_{ij} b^{ij}_a$, contains only the KK massive modes for the bulk fields containing the zero modes. Therefore, from the beta functions (85), (86), (88) and (90), one can find that the terms proportional to η^R_1 or $\eta^R_1\eta^R_2$, i.e. the orbifold actions associated with Pati-Salam and flipped $SU(5)$ gauge groups generate the non-universal corrections to the gauge couplings.

From the obtained beta functions (83) and (87)-(90), eq. (65) becomes

$$\frac{4\pi}{g^2_{\mathrm{eff},a}(k^2)} = \frac{4\pi}{g^2_u} + \frac{1}{4\pi}\tilde{b}_a \ln\frac{M^2_*}{M^2_{B-L}} + \frac{1}{4\pi}b'_a \ln\frac{M^2_{B-L}}{k^2} - \frac{1}{4\pi}\sum_{i,j=\pm} b^{ij}_a L_{ij} + \frac{\Delta_a}{2\pi}. \quad (92)$$

Here $\tilde{b}_a = b_a - c_a + \hat{b}_a$ is the $\mathcal{N}=1$ beta function above the $B-L$ breaking scale and b^{ij}_a are the beta functions for the KK massive modes.

[5]The gauge kinetic terms localized at Pati-Salam and flipped $SU(5)$ fixed points can also lead to a mixing.

4.2.2 The differential running of the gauge couplings

For a number of hyper multiplets with arbitrary parities, we assume that both vector-like particles (getting brane masses of order the GUT scale) and brane-localized particles fill GUT multiplets, i.e. c_a and $\hat{b}_a$ are universal. Then, we get the general formula for the differential running of gauge couplings as

$$\frac{1}{g_3^2}-\frac{12}{7}\frac{1}{g_2^2}+\frac{5}{7}\frac{1}{g_1^2} = \frac{1}{8\pi^2}\left(\tilde{b}\ln\frac{M_*}{M_{B-L}}-\frac{1}{2}b^{-+}L_{-+}-\frac{1}{2}b^{--}L_{--}\right)+\frac{\tilde{\Delta}}{8\pi^2} \tag{93}$$

where

$$\tilde{b} = \frac{9}{7}-\frac{9}{14}\sum_{10}\eta_1^{10}-\frac{15}{14}\sum_{16}\eta_1^{16}+\frac{3}{7}\sum_{16}\eta_1^{16}\eta_2^{16}, \tag{94}$$

$$b^{-+} = -\frac{9}{7}-\frac{9}{14}\sum_{10}\eta_1^{10}-\frac{15}{14}\sum_{16}\eta_1^{16}, \tag{95}$$

$$b^{--} = \frac{12}{7}+\frac{3}{7}\sum_{16}\eta_1^{16}\eta_2^{16}. \tag{96}$$

Thus, we find a general relation between coefficients as

$$\tilde{b}=\frac{6}{7}+b^{-+}+b^{--}. \tag{97}$$

Then, from eq. (93) with the relation (97), we find the deviation from the 4D SGUT prediction of the QCD coupling at M_Z, i.e. $\Delta\alpha_s\equiv\alpha_s^{KK}-\alpha_s^{SGUT,0}$ as

$$\begin{aligned}\Delta\alpha_s(M_Z) \approx & -\frac{1}{2\pi}\alpha_s^2(M_Z)\Big\{\tilde{b}\ln\frac{M_*}{M_{B-L}}-(\tilde{b}-\frac{6}{7})\ln(M_*\sqrt{V}) \\ & -\frac{1}{2}b^{-+}\ln\Big[\frac{e^{-2}}{4}\Big|\vartheta_1(\frac{1}{2}|iu)\Big|^4 u\Big] \\ & -\frac{1}{2}(\tilde{b}-\frac{6}{7}-b^{-+})\ln\Big[\frac{e^{-2}}{4}\Big|\vartheta_1(\frac{1}{2}-\frac{1}{2}iu|iu)e^{-\pi u/4}\Big|^4 u\Big]+\tilde{\Delta}\Big\}.\end{aligned} \tag{98}$$

The first term corresponds to the contribution due to the extra particles above the $B-L$ scale. The second term is the volume dependent correction due to the KK massive modes while the third part containing the theta functions is the shape dependent correction. The last term $\tilde{\Delta}$ is the effect of the brane-localized gauge couplings.

When $u\sim 1$, the shape dependent term is subdominant compared to the other logarithmic terms. As can be shown explicitly in the specific models, the last term can be also ignored by making a strong coupling assumption at the cutoff scale. Then, the first two logarithms become a dominant contribution. For $\tilde{b}(\tilde{b}-\frac{6}{7})>0$, we can see that the individual logarithm can be large, being compatible with the gauge coupling unification due to a cancellation.

Now we consider the behavior of our result in the 5D limit with $u = R_6/R_5 \gg 1$. In this case, the bulk gauge group becomes the Pati-Salam and there remain only two fixed points with the Pati-Salam group and the SM gauge group enlarged with a $U(1)$ factor, respectively. Some relevant discussion on this limit has been made in Ref. [24], concerning the power-like threshold corrections. Since $|\vartheta_1(z|iu)| \sim 2e^{-\pi u/4}|\sin(\pi z)|$ for $u \gg 1$, the shape dependent terms could give a significant effect on the gauge coupling unification by the non-universal power-like threshold corrections. Thus, in this 5D limit, eq. (98) becomes

$$\begin{aligned}\Delta\alpha_s(M_Z) &\approx -\frac{1}{2\pi}\alpha_s^2(M_Z)\bigg\{\tilde{b}\ln\frac{M_*}{M_{B-L}} - (\tilde{b}-\frac{6}{7})\ln(M_*\sqrt{V}) \\ &\quad -\frac{1}{2}(\tilde{b}-\frac{6}{7})\ln(4e^{-2}u) + \frac{\pi}{2}b^{-+}u + \tilde{\Delta}\bigg\}. \qquad (99)\end{aligned}$$

Therefore, we can interpret that the effective 5D gauge coupling$(1/g_5^2 = 1/(g_4^2 R_6))$ also gets a power-like threshold correction proportional to $u/R_6 \sim 1/R_5$ which is set by the mass scale of heavy gauge bosons belonging to $SO(10)/SU(4)\times SU(2)_L\times SU(2)_R$.

On the other hand, when we take a different 5D limit for $u \ll 1$, the bulk gauge group becomes $SU(5)\times U(1)_X$ and there remain only two fixed points with $SU(5)\times U(1)_X$ and the SM gauge group enlarged with a $U(1)$ factor, respectively. In this case, the appearing power threshold corrections are universal so there is no power threshold correction in eq. (98).

4.3 Gauge coupling unification in some $SO(10)$ orbifold GUT models

Now we are in a position to apply our general formula (98) to particular cases for the unification of the SM gauge couplings. To this purpose, we consider some known $SO(10)$ models of embedding the MSSM into the extra dimensions. In the minimal model(: model I) [13] that contains Higgs fields in the bulk for breaking $U(1)_{B-L}$ and the SM gauge group[6], there are 4 **10**'s with parities (η_1, η_2) such as $H_1 = (+,+)$, $H_2 = (+,-)$, $H_3 = (-,+)$ and $H_4 = (-,-)$, and one pair of **16** and $\overline{\mathbf{16}}$ with parities $\Phi = (-,+)$, $\Phi^c = (-,+)$. Then, the resulting massless modes are two doublet Higgs fields H_1^c and H_2 from H_1 and H_2, and $G_3^c, G_4, (D^c, N^c), (D, N)$ from H_3, H_4, Φ and Φ^c in order. Moreover, each family of quarks and leptons is introduced as a **16** being localized at the fixed point without $SO(10)$ gauge symmetry. After the $B-L$ breaking via the bulk **16**'s with $\langle N\rangle = \langle N^c\rangle \neq 0$, neutrino masses are generated at the fixed points by a usual see-saw mechanism. Moreover, $G_3^c, G_4, (D^c, N^c), (D, N)$ can acquire masses of order the $B-L$ breaking scale by the brane superpotential [13, 14] $W = \lambda NDG_3^c + \lambda' N^c D^c G_4$ for $\langle N\rangle = \langle N^c\rangle \neq 0$. In this case, since $\sum_{10}\eta_1^{10} = 0$, $\sum_{16}\eta_1^{16} = \sum_{16}\eta_1^{16}\eta_2^{16} = -2$, we get the values $\tilde{b} = \frac{18}{7}, b^{-+} = b^{--} = \frac{6}{7}$ in eq. (98). Thus, in the 5D limit, because b^{-+} is nonzero in eq. (99), there exists an effective

[6] In order to cancel the bulk anomalies due to one **45**, we need to add in the bulk two **10**'s. So, it is necessary to have two Higgs doublets of the **10**'s in the bulk unlike in 5D case [3]. Moreover, in order to break the $U(1)_{B-L}$, we need one **16** in the bulk. However, for cancellation of localized and bulk anomalies, one needs one $\overline{\mathbf{16}}$ and two more **10**'s.

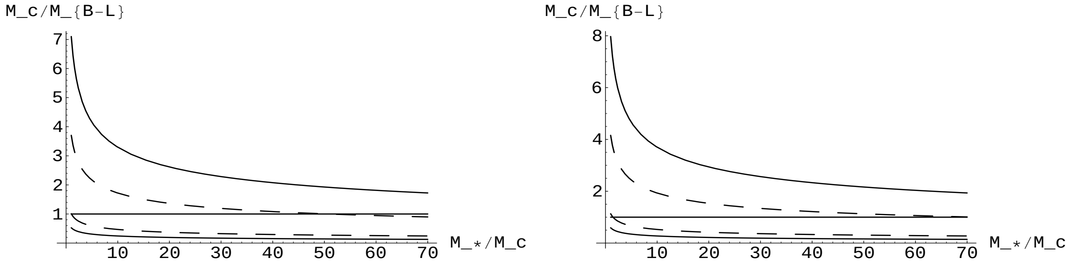

Figure 1: The 1σ and 2σ band of $\Delta\alpha_s$: the model I on the left and the model II on the right for $u = R_6/R_5 \sim 1$. The dashed lines and the continuous lines denote 1σ and 2σ bounds of the experimental data, respectively. Only in the region above the straight line at $M_c/M_{B-L} = 1$, the $B-L$ breaking brane mass terms can be neglected.

5D power-like threshold correction to the QCD coupling so the threshold correction is sensitive to the shape modulus.

We consider another 6D $SO(10)$ GUT model where the realistic flavor structure of the SM was discussed(: model II) [14]. In this case, on top of the minimal model, there are more hyper multiplets: 2 $\mathbf{10}$'s such as $H_5 = (-,+)$ and $H_6 = (-,-)$, and one pair of $\mathbf{16}$ and $\overline{\mathbf{16}}$ with $\phi = (+,+)$ and $\phi^c = (+,+)$. Then, there are additional zero modes G_5^c, G_6, L, L^c from H_5, H_6, ϕ and ϕ^c in order. They are assumed to get brane masses of order the GUT scale. Thus, the running of gauge couplings between the GUT scale and the $B-L$ breaking scale is the same as in the minimal model. In this case, since $\sum_{10}\eta_1^{10} = -2$, $\sum_{16}\eta_1^{16} = \sum_{16}\eta_1^{16}\eta_2^{16} = 0$, we get the values $\tilde{b} = \frac{18}{7}, b^{-+} = 0$ and $b^{--} = \frac{12}{7}$ in eq. (98). Thus, in the 5D limit, because $b^{-+} = 0$ in eq. (99), there is no effective 5D power-like threshold correction to the QCD coupling.

From the data of the electroweak gauge couplings at the scale of the Z mass, one can compare the predicted value of the QCD coupling in a theory to a measure one [25] $\alpha_s^{exp} = 0.1176 \pm 0.0020$. In the 4D supersymmetric GUTs, the prediction without threshold corrections for the QCD coupling is $\alpha_s^{SGUT,0} = 0.130 \pm 0.004$. Thus, in this case, there is a discrepancy from the experimental data as $\delta\alpha_s = \alpha_s^{SGUT,0} - \alpha_s^{exp} = 0.0124 \pm 0.0045$.

First we consider the case with isotropic compactification of the extra dimensions, $u \sim 1$. In both models, since $\tilde{b} = \frac{18}{7}$, we can see from eq. (98) that logarithmic contributions of zero modes and those of KK massive modes appear with opposite signs so that there is a possibility of having the large volume of extra dimensions consistent with perturbativity and gauge coupling unification. Ignoring the unknown brane-localized gauge couplings and the $B-L$ breaking effect, we depict in Fig. 1 the parameter space of (M_c, M_{B-L}) with $u \sim 1$, being compatible with the experimental data. If we take $M_*/M_c \sim 63/\sqrt{C} \sim 22$ for strong coupling assumption[7] at the 6D fundamental scale [8], the correction due

[7]We included the group theory factor $C = 8$ for the $SO(10)$ bulk gauge group in the naive dimensional analysis compared to [8].

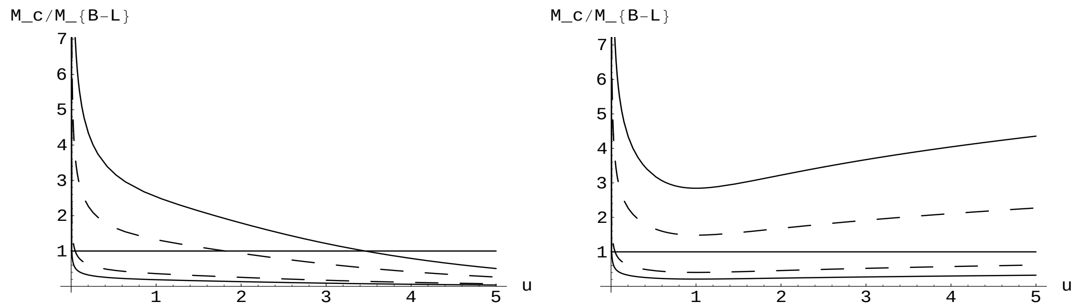

Figure 2: The 1σ and 2σ band of $\Delta\alpha_s$: the model I on the left and the model II on the right for $M_*/M_c \sim 22$. The dashed lines and the continuous lines denote 1σ and 2σ bounds of the experimental data, respectively. Only in the region above the straight line at $M_c/M_{B-L} = 1$, the $B - L$ breaking brane mass terms can be neglected.

to the brane-localized gauge couplings becomes $\tilde{\Delta} = \mathcal{O}(1)$ so it is negligible to the KK threshold corrections which is of order $\ln(M_*/M_c) \sim 3$. In the model I(II), for $M_*/M_c \sim 22$, M_{B-L}/M_c can be as small as 0.23(0.12) at the 2σ level. For $M_{B-L}/M_c \ll 1$, the KK massive modes of the color triplets are modified to $m^2_{n_5,n_6} \approx (n_5/2R_5)^2 + (n_6/2R_6)^2 + cM^2_{B-L}$ where c is of order unity independent of the KK level for $R_5 \neq R_6$ [26]. In this case, the $B-L$ breaking effect to the differential running is estimated as M^2_{B-L}/M^2_c in comparison to $\tilde{\Delta}$ in eq. (98), so it is also suppressed compared to the KK threshold corrections. Apart from the two models, we can consider other possibilities of embedding the matter representations into extra dimensions, like in the field-theory limit of a successful string orbifold compactification [16] where there are two families at the fixed points and one family in the bulk. In view of the general formula (98), however, as far as an extra particle contributes to the running of the gauge couplings above the $B-L$ breaking scale, M_{B-L} tends to be close to M_c for the success of the gauge coupling unification.

Next we consider the shape dependence of the loop corrections. As in the previous case, we make the strong coupling assumption and take the $B-L$ breaking scale to be below the compactification scale. Then, we depict in Fig. 2 the parameter space of (u, M_{B-L}) with $M_*/M_c \sim 22$, being compatible with the experimental data. Therefore, we can see the clear difference between the two models, through the dependence of the compatible $B-L$ breaking scale on the shape modulus. As shown in the limit of anisotropic compactification of the extra dimensions in eq. (99), in the model I, the power-like threshold correction in the effective 5D theory gives a sizable contribution to the differential running of the gauge couplings, thus the $B-L$ breaking scale is more sensitive to the shape modulus.

5 Conclusion

We have considered the one-loop effective action for gauge bosons in six-dimensional orbifold GUT models with a number of hyper multiplets satisfying arbitrary local discrete twists. From the obtained effective action, we encountered the divergences which require the introduction of brane-localized gauge kinetic terms and a bulk higher derivative term. Moreover, we derived the general expressions for the running of the gauge couplings of zero-mode gauge bosons in the low 4D momentum limit. Since the KK massive mode corrections depend on the parity actions, in general they can give a non-universal contribution to the gauge coupling running.

By taking a concrete example such as the $SO(10)$ orbifold GUTs, we estimated the corresponding KK massive mode corrections to the QCD coupling at the scale of the Z mass. The extra $U(1)$ or the $B-L$ symmetry is broken by the VEV of bulk singlets below the unification scale. Then, the extra color triplets, which accompany bulk singlets for a full **16**, also appear as zero modes so that they can lead to an additional logarithmic running of the gauge couplings starting at the $B-L$ breaking scale.

In the case with isotropic compactification of the extra dimensions, there is a partial cancellation between two dominant logarithmic corrections; the volume dependent part of the KK threshold corrections and the threshold corrections due to the extra color triplets. In this case, we argued that the large volume of the extra dimensions can be compatible with the gauge coupling unification and perturbativity. We also considered the case with anisotropic compactification of the extra dimensions for which a 5D orbifold GUT limit can be discussed. In this case, the shape dependent part of the KK threshold corrections corresponds to non-universal power-like corrections in the compactification scales. These power-like corrections are calculable because they are finite in the 6D sense. This situation is in contrast to the genuine 5D orbifold GUTs with non-simple groups where power-like corrections with the cutoff dependence is uncalculable. Further, we showed that for a fixed volome of the extra dimensions compatible with a strong coupling assumption, the $B-L$ breaking scale is sensitive to the change of the shape modulus, in order to maintain the success of the gauge coupling unification.

Acknowledgments

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Appendix A: Propagators on GUT orbifolds

Suppose that the bulk gauge bosons satisfy the boundary conditions,

$$\mathcal{P}_i A_\mu(z)\mathcal{P}_i^{-1} \equiv P_i A_\mu(-z+z_i)P_i^{-1} = A_\mu(z+z_i), \tag{A.1}$$

$$\mathcal{P}_i A_m(z)\mathcal{P}_i^{-1} \equiv -P_i A_m(-z+z_i)P_i^{-1} = A_m(z+z_i) \tag{A.2}$$

where $P_i^2=1(i=0,1,2,3)$ and $[P_i,P_j]=0$. Then, we can write the gauge bosons $\tilde{A}_M$ on orbifolds in terms of gauge fields A_M having $4\pi R_{5,6}$ periodicities along the extra dimensions as

$$\tilde{A}_M(z)=\prod_i\left[\frac{1}{2}(1+\mathcal{P}_i)\right]A_M(z)\prod_j\left[\frac{1}{2}(1+\mathcal{P}_j^{-1})\right]. \tag{A.3}$$

After taking into account the relation (11), the field redefinition becomes in terms of component fields in the group space

$$\begin{aligned}\tilde{A}^a_\mu(z) &= \frac{1}{8}\Big[\delta^a{}_b A^b_\mu(z)+\sum_{i=0}^{2}(Q_i)^a{}_b A^b_\mu(-z+2z_i)+\sum_{i<j\neq 3}(Q_iQ_j)^a{}_b A^b_\mu(z+2z_i-2z_j)\\ &\quad +(Q_0Q_1Q_2)^a{}_b A^b_\mu(-z+2z_0-2z_1+2z_2)\Big],\end{aligned} \tag{A.4}$$

$$\begin{aligned}\tilde{A}^a_m(z) &= \frac{1}{8}\Big[\delta^a{}_b A^b_m(z)-\sum_{i=0}^{2}(Q_i)^a{}_b A^b_m(-z+2z_i)+\sum_{i<j\neq 3}(Q_iQ_j)^a{}_b A^b_m(z+2z_i-2z_j)\\ &\quad -(Q_0Q_1Q_2)^a{}_b A^b_m(-z+2z_0-2z_1+2z_2)\Big]\end{aligned} \tag{A.5}$$

where $(Q_i)^a{}_b\equiv \mathrm{tr}(T^aP_iT_bP_i)$. Thus, the orbifold-compatible functional differentiations for gauge fields are

$$\begin{aligned}(\delta_{12}^{\tilde{A}_{\mu(m)}})^a{}_b &= \frac{1}{8}\Big[\delta^a{}_b\delta^2(z_1-z_2)\pm\sum_{i=0}^{2}(Q_i)^a{}_b\delta^2(z_1+z_2-2z_i)\\ &\quad +\sum_{i<j\neq 3}(Q_iQ_j)^a{}_b\delta^2(z_1-z_2-2z_i+2z_j)\\ &\quad \pm(Q_0Q_1Q_2)^a{}_b\delta^2(z_1+z_2-2z_0+2z_1-2z_2)\Big]\cdot\delta^4(x_1-x_2).\end{aligned} \tag{A.6}$$

Consequently, the propagator of gauge fields in the Feynman gauge is given by

$$\langle\tilde{A}^a_M(z_1)\tilde{A}_{bN}(z_2)\rangle = g_{MN}(\delta_{13}^{\tilde{A}_M})^a{}_c G(z_3-z_4)(\delta_{24}^{\tilde{A}_N})^c{}_b \tag{A.7}$$

where $G(z_3 - z_4)$ is a bulk scalar propagator satisfying $4\pi R_{5,6}$ periodicities. Then, we obtain the propagator of gauge fields in 6D momentum space as

$$\begin{aligned}
\langle \tilde{A}^a_\mu(z_1)\tilde{A}_{b\nu}(z_2)\rangle &\to g_{\mu\nu}\frac{1}{4}(1+Q_0Q_1\cos(2p_5\pi R_5))^a{}_c(1+Q_0Q_2\cos(2p_6\pi R_6))^c{}_d \\
&\times\frac{i}{2}\frac{(\delta_{\vec{p},\vec{p}'}+Q_0\delta_{\vec{p},-\vec{p}'})^d{}_b}{p^2-\vec{p}^2} \equiv g_{\mu\nu}(\tilde{G}_{g,+}(p,\vec{p},\vec{p}'))^a{}_b, \qquad (A.8)\\
\langle \tilde{A}^a_m(z_1)\tilde{A}_{bn}(z_2)\rangle &\to g_{mn}\frac{1}{4}(1+Q_0Q_1\cos(2p_5\pi R_5))^a{}_c(1+Q_0Q_2\cos(2p_6\pi R_6))^c{}_d \\
&\times\frac{i}{2}\frac{(\delta_{\vec{p},\vec{p}'}-Q_0\delta_{\vec{p},-\vec{p}'})^d{}_b}{p^2-\vec{p}^2} \equiv g_{mn}(\tilde{G}_{g,-}(p,\vec{p},\vec{p}'))^a{}_b \qquad (A.9)
\end{aligned}$$

where $\vec{p} = (p_5, p_6) = (n_5/(2R_5), n_6/(2R_6))$ with n_5, n_6 integers. We note that the Wilson line effects encoded into Q_1 and Q_2 are factorized as two matrices in front of the propagator on orbifolds without Wilson lines.

Next let us consider a complex scalar field in the fundamental representation, satisfying orbifold boundary conditions,

$$\mathcal{P}_i\tilde{\phi}(z) \equiv \eta_i P_i\tilde{\phi}(-z+z_i) = \tilde{\phi}(z+z_i) \qquad (A.10)$$

with $\eta_i = +1$ or -1. Similarly, we can write the complex scalar field $\tilde{\phi}$ on orbifolds in terms of a complex scalar field satisfying $4\pi R_{5,6}$ periodicities as

$$\tilde{\phi}(z) = \prod_i\left[\frac{1}{2}(1+\mathcal{P}_i)\right]\phi(z). \qquad (A.11)$$

Then, we can write the above equation in terms of component fields as

$$\begin{aligned}
\tilde{\phi}^a(z) &= \frac{1}{8}\Bigg[\delta^a{}_b\phi(z) + \sum_{i=0}^{2}\eta_i(P_i)^a{}_b\phi(-z+2z_i) + \sum_{i<j\neq 3}\eta_i\eta_j(P_iP_j)^a{}_b\phi^j(z+2z_i-2z_j) \\
&+\eta_0\eta_1\eta_2(P_0P_1P_2)^a{}_b\phi^b(-z+2z_0-2z_1+2z_2)\Bigg]. \qquad (A.12)
\end{aligned}$$

Therefore, the orbifold-compatible functional differentiation for a complex scalar field is

$$\begin{aligned}
(\delta^{\tilde{\phi}}_{12})^a{}_b &= \frac{1}{8}\Bigg[\delta^a{}_b\delta^2(z_1-z_2) + \sum_{i=0}^{2}\eta_i(P_i)^a{}_b\delta^2(z_1+z_2-2z_i) \\
&+\sum_{i<j\neq 3}\eta_i\eta_j(P_iP_j)^a{}_b\delta^2(z_1-z_2-2z_i+2z_j) \\
&+\eta_0\eta_1\eta_2(P_0P_1P_2)^a{}_b\delta^2(z_1+z_2-2z_0+2z_1-2z_2)\Bigg]\cdot\delta^4(x_1-x_2). \qquad (A.13)
\end{aligned}$$

Consequently, the propagator of a complex scalar is given by

$$\langle\tilde{\phi}^a(z_1)\overline{\tilde{\phi}}_b(z_2)\rangle = (\delta^{\tilde{\phi}}_{13})^a{}_cG(z_3-z_4)(\delta^{\tilde{\phi}}_{24})^c{}_b \qquad (A.14)$$

or in 6D momentum space,

$$\begin{aligned}\langle\tilde{\phi}^a(z_1)\tilde{\phi}_b(z_2)\rangle \;&\rightarrow\; \frac{1}{4}(1+\eta_0\eta_1 P_0 P_1\cos(2p_5\pi R_5))^a{}_c(1+\eta_0\eta_2 P_0 P_2\cos(2p_6\pi R_6))^c{}_d \\ &\times\frac{i}{2}\frac{(\delta_{\vec{p},\vec{p}'}+\eta_0 P_0\delta_{\vec{p},-\vec{p}'})^d{}_b}{p^2-\vec{p}^2}\equiv(\tilde{G}_{h,+}(p,\vec{p},\vec{p}'))^a{}_b. \end{aligned} \tag{A.15}$$

For scalar fields of other representations, we only have to replace the parity matrices with the ones for corresponding representations.

Finally let us consider a bulk left-handed fermion in the fundamental representation, satisfying the boundary conditions,

$$\mathcal{P}_i\tilde{\psi}(z)\equiv i\xi_i\gamma^5 P_i\tilde{\psi}(-z+z_i)=\tilde{\psi}(z+z_i) \tag{A.16}$$

with $\xi_i=+1$ or -1. Following the similar procedure, the propagator of a bulk fermion is given in 6D momentum space as

$$\begin{aligned}\langle\tilde{\psi}^a(z_1)\overline{\tilde{\psi}}_b(z_2)\rangle \;&\rightarrow\; \frac{1}{4}(1+\xi_0\xi_1 P_0 P_1\cos(2p_5\pi R_5))^a{}_c(1+\xi_0\xi_2 P_0 P_2\cos(2p_6\pi R_6))^c{}_d \\ &\times\frac{i}{2}\left\{\frac{\delta^d{}_b\delta_{\vec{p},\vec{p}'}}{\not{p}+\gamma_5 p_5+p_6}-\xi_0(P_0)^d{}_b\frac{\delta_{\vec{p},-\vec{p}'}}{\not{p}+\gamma_5 p_5+p_6}i\gamma_5\right\} \\ &\equiv(\tilde{D}_\psi(p,\vec{p},\vec{p}'))^a{}_b. \end{aligned} \tag{A.17}$$

On the other hand, for a bulk right-handed gaugino in the adjoint representation, the propagator in 6D momentum space takes the above form with $\xi_i P_i$ replaced by Q_i and $\not{p}+\gamma_5 p_5+p_6\rightarrow\not{p}+\gamma_5 p_5-p_6$.

Appendix B: KK summations in 6D orbifolds

We consider the following KK summation (with $c\geq 0$, $a_{1,2}>0$, $0\leq c_{1,2}<1$):

$$\begin{aligned}\mathcal{J}_0[c;c_1,c_2] \;&\equiv\; \Gamma[\epsilon/2]\sum_{n_1,n_2\in\mathbf{Z}}\Big[\pi[c+a_1(n_1+c_1)^2+a_2(n_2+c_2)^2]\Big]^{-\epsilon/2} \\ &=\sum_{n_1,n_2\in\mathbf{Z}}\int_0^\infty\frac{dt}{t^{1-\epsilon/2}}\,e^{-\pi t\,[c+a_1(n_1+c_1)^2+a_2(n_2+c_2)^2]}. \end{aligned} \tag{B.1}$$

If $0\leq c/a_1<1$, with notations $\gamma(n_1)\equiv\sqrt{z(n_1)}/\sqrt{a}_2-i\,c_2$; and $z(n_1)\equiv c+a_1(n_1+c_1)^2$, $u\equiv\sqrt{a_1/a_2}$, $s_{\tilde{n}_1}\equiv 2\pi\tilde{n}_1\sqrt{c/a_1}$, $\gamma_E=0.577216...$, we obtain [6] (in the text $a_1=1/R_5^2$, $a_2=1/R_6^2$)

$$\mathcal{J}_0[c;c_1,c_2] = \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon} + \ln\left[4\pi\, a_1\, e^{\gamma_E+\psi(c_1)+\psi(-c_1)}\right]\right] + 2\pi\, u\left[\frac{1}{6} + c_1^2 - \left(c/a_1 + c_1^2\right)^{\frac{1}{2}}\right]$$

$$-\sum_{n_1\in\mathbf{Z}} \ln\left|1 - e^{-2\pi\,\gamma(n_1)}\right|^2 + \sqrt{\pi}\, u \sum_{p\geq 1} \frac{\Gamma[p+1/2]}{(p+1)!}\left[\frac{-c}{a_1}\right]^{p+1}\Big(\zeta[2p+1, 1+c_1] + \zeta[2p+1, 1-c_1]\Big) \quad \text{(B.2)}$$

while if we have $c/a_1 > 1$, then

$$\begin{aligned} \mathcal{J}_0[c;c_1,c_2] &= \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon} + \ln\left[\pi\, c\, e^{\gamma_E - 1}\right]\right] - \sum_{n_1\in\mathbf{Z}} \ln\left|1 - e^{-2\pi\,\gamma(n_1)}\right|^2 \\ &\quad + \frac{4\sqrt{c}}{\sqrt{a_2}} \sum_{\tilde{n}_1>0} \frac{\cos(2\pi\tilde{n}_1\, c_1)}{\tilde{n}_1}\, K_1(s_{\tilde{n}_1}) \end{aligned} \quad \text{(B.3)}$$

The pole structure is the same for both cases; if $c/a_1 > 1$ and except the first square bracket, no power-like terms in c are present (the last one being suppressed due to K_1). Here $\zeta[z,a]$ is the Hurwitz Zeta function and $\psi(x) = d/dx\, \ln\Gamma[x]$ and K_1 is the modified Bessel function.

Finally, we quote here a limiting case for the behaviour of the function $\mathcal{J}_0$

$$\begin{aligned} \mathcal{J}_0[c\ll 1;0,0] &= \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon} + \ln\left[4\pi e^{-\gamma_E} a_1 \left|\eta(i\sqrt{a_1/a_2})\right|^4\right]\right] \\ &\quad - \ln\left[4\pi^2\, |\eta(i\,\sqrt{a_1/a_2})|^4\ a_2^{-1}\right] - \ln c \\ \mathcal{J}_0[c\ll 1;0,1/2] &= \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon} + \ln(4\pi e^{-\gamma_E} a_1) - \pi u + 2\sum_{n\geq 1}(-1)^n \ln|1 - e^{-2\pi u n}|^2\right] \\ &\quad - \ln\left|\frac{\vartheta_1(1/2|iu)}{\eta(iu)}\right|^2 \\ \mathcal{J}_0[c\ll 1;1/2,0] &= \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon} + \ln(\pi e^{-\gamma_E} a_1) - 2\sum_{n\geq 1} \ln\left|\frac{1+e^{-\pi u n}}{1-e^{-\pi u n}}\right|^2\right] \\ &\quad - \ln\left|\frac{\vartheta_1(-iu/2|iu)}{\eta(iu)} e^{-\pi u/4}\right|^2 \\ \mathcal{J}_0[c\ll 1;1/2,1/2] &= \frac{\pi c}{\sqrt{a_1 a_2}}\left[\frac{-2}{\epsilon} + \ln(\pi e^{-\gamma_E} a_1) - 2\sum_{n\geq 1}(-1)^n \ln\left|\frac{1+e^{-\pi u n}}{1-e^{-\pi u n}}\right|^2\right] \\ &\quad - \ln\left|\frac{\vartheta_1(1/2 - iu/2|iu)}{\eta(iu)} e^{-\pi u/4}\right|^2. \end{aligned} \quad \text{(B.4)}$$

Appendix C: Definition of special functions

The Hurwitz Zeta function $\zeta[z,a]$ is defined as

$$\zeta[z,a] = \sum_{n\geq 0}(n+a)^{-z} \tag{C.1}$$

with $\mathrm{Re}\, z > 1$ and $a \neq 0, -1, -2, \cdots$.

The modified Bessel function K_n is defined through

$$\int_0^\infty dx\, x^{\nu-1} e^{-bx^p - ax^{-p}} = \frac{2}{p}\left[\frac{a}{b}\right]^{\frac{\nu}{2p}} K_{\frac{\nu}{p}}(2\sqrt{a\,b}), \quad Re(b),\, Re(a) > 0 \tag{C.2}$$

with

$$K_1[x] = e^{-x}\sqrt{\frac{\pi}{2x}}\left[1 + \frac{3}{8x} - \frac{15}{128x^2} + \mathcal{O}(1/x^3)\right] \tag{C.3}$$

which is strongly suppressed at large argument.

In the text, we used the Dedekind Eta function

$$\begin{aligned} \eta(\tau) &\equiv e^{\pi i\tau/12}\prod_{n\geq 1}(1-e^{2i\pi\tau\, n}), \\ \eta(-1/\tau) &= \sqrt{-i\tau}\,\eta(\tau), \quad \eta(\tau+1) = e^{i\pi/12}\eta(\tau), \end{aligned} \tag{C.4}$$

and the Jacobi Theta function ϑ_1

$$\begin{aligned} \vartheta_1(z|\tau) &\equiv 2q^{1/8}\sin(\pi z)\prod_{n\geq 1}(1-q^n)(1-q^n e^{2i\pi z})(1-q^n e^{-2i\pi z}), \qquad q \equiv e^{2i\pi\tau} \\ &= -i\sum_{n\in\mathbf{Z}}(-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)i\pi z} \end{aligned} \tag{C.5}$$

which has the properties

$$\begin{aligned} \vartheta_1(z|\tau+1) &= e^{i\pi/4}\vartheta_1(z|\tau), \\ \vartheta_1(z+1|\tau) &= -\vartheta_1(z|\tau), \\ \vartheta_1(z+\tau|\tau) &= -e^{-i\pi\tau-2i\pi z}\vartheta_1(z|\tau), \\ \vartheta_1(-z/\tau|-1/\tau) &= e^{i\pi/4}\tau^{1/2}e^{i\pi z^2/\tau}\vartheta_1(z|\tau). \end{aligned} \tag{C.6}$$

Appendix D: Some group theory for $SO(10)$ GUT

- Relations between fundamental and other representations:

  For $SU(N)$ and $SO(2N)$ gauge groups considered in the paper, we have [20]

$$\begin{aligned}
\mathrm{tr}_{\mathrm{Adj}} F^2_{SU(N)} &= 2N\, \mathrm{tr}_N F^2_{SU(N)}, && \text{(D.1)}\\
\mathrm{tr}_{a^{ij}} F^2_{SU(N)} &= (N-2)\, \mathrm{tr}_N F^2_{SU(N)}, && \text{(D.2)}\\
\mathrm{tr}_{a^{ijk}} F^2_{SU(N)} &= \frac{1}{2}(N-2)(N-3)\, \mathrm{tr}_N F^2_{SU(N)}, && \text{(D.3)}\\
\mathrm{tr}_{\mathrm{Adj}} F^2_{SO(2N)} &= 2(N-1)\, \mathrm{tr}_{2N} F^2_{SO(2N)}, && \text{(D.4)}\\
\mathrm{tr}_{2^{N-1}} F^2_{SO(2N)} &= 2^{N-4}\, \mathrm{tr}_{2N} F^2_{SO(2N)} && \text{(D.5)}
\end{aligned}$$

  where the subindex of the trace implies the representation of the group, for instance, $a^{ij}\,(a^{ijk})$ is the second(third) rank totally antisymmetric tensor representation of $SU(N)$. In the text, we take the normalization, $\mathrm{tr}_N(T^aT^b) = \frac{1}{2}\delta^{ab}$ for $SU(N)$ and $\mathrm{tr}_{2N}(T^aT^b) = \delta^{ab}$ for $SO(2N)$.

- Computation of traces:

  By standard representation theory, we can do the decomposition of the quadratic Casimir for an adjoint representation of $SO(10)$: under the Pati-Salam,

$$\begin{aligned}
\mathrm{tr}_{\mathrm{Adj}} F^2_{SO(10)} &= \mathrm{tr}_{\mathrm{Adj}} F^2_{SU(4)} + 4\mathrm{tr}_6 F^2_{SU(4)} + 12\mathrm{tr}_2 F^2_{SU(2)_L} + 12\mathrm{tr}_2 F^2_{SU(2)_R}\\
&\quad + \mathrm{tr}_{\mathrm{Adj}} F^2_{SU(2)_L} + \mathrm{tr}_{\mathrm{Adj}} F^2_{SU(2)_R} \qquad \text{(D.6)}
\end{aligned}$$

  and under the Georgi-Glashow,

$$\mathrm{tr}_{\mathrm{Adj}} F^2_{SO(10)} = \mathrm{tr}_{\mathrm{Adj}} F^2_{SU(5)} + \mathrm{tr}_{10} F^2_{SU(5)} + \mathrm{tr}_{\overline{10}} F^2_{SU(5)} + 8F^2_{U(1)_X}. \qquad \text{(D.7)}$$

  Using the definition $\mathrm{tr}_{\mathrm{Adj}}(Q_i F^2_{SO(10)}) = f_{acd} f_{bef} \eta^{cd} Q_i^{df} F^a_{SO(10)} F^b_{SO(10)}$ and the fact that Q_i is equal to $+1(-1)$ for Z_2-even(odd) modes of gauge fields, we get

$$\begin{aligned}
\mathrm{tr}_{\mathrm{Adj}}(Q_1 F^2_{SO(10)}) &= \mathrm{tr}_{\mathrm{Adj}} F^2_{SU(4)} - 4\mathrm{tr}_6 F^2_{SU(4)} - 12\mathrm{tr}_2 F^2_{SU(2)_L} - 12\mathrm{tr}_2 F^2_{SU(2)_R}\\
&\quad + \mathrm{tr}_{\mathrm{Adj}} F^2_{SU(2)_L} + \mathrm{tr}_{\mathrm{Adj}} F^2_{SU(2)_R}\\
&= -8\mathrm{tr}_2 F^2_{SU(2)_L} - 8\mathrm{tr}_2 F^2_{SU(2)_R} \qquad \text{(D.8)}
\end{aligned}$$

  and

$$\begin{aligned}
\mathrm{tr}_{\mathrm{Adj}}(Q_2 F^2_{SO(10)}) &= \mathrm{tr}_{\mathrm{Adj}} F^2_{SU(5)} - \mathrm{tr}_{10} F^2_{SU(5)} - \mathrm{tr}_{\overline{10}} F^2_{SU(5)} - 8F^2_{U(1)_X}\\
&= 4\mathrm{tr}_5 F^2_{SU(5)} - 8F^2_{U(1)_X} \qquad \text{(D.9)}
\end{aligned}$$

  and likewise

$$\mathrm{tr}_{\mathrm{Adj}}(Q_3 F^2_{SO(10)}) = 4\mathrm{tr}_5 F^2_{SU(5)'} - 8F^2_{U(1)_{X'}}. \qquad \text{(D.10)}$$

  Let us consider a similar decomposition of the index of the other representation of $SO(10)$. First, the index of a fundamental representation of $SO(10)$ is decomposed as

$$\begin{aligned}\mathrm{tr}_{10}F^2_{SO(10)} &= \mathrm{tr}_6F^2_{SU(4)} + 2\mathrm{tr}_2F^2_{SU(2)_L} + 2\mathrm{tr}_2F^2_{SU(2)_R} \\ &= \mathrm{tr}_5F^2_{SU(5)} + \frac{1}{2}F^2_{U(1)_X} + \mathrm{tr}_{\overline{5}}F^2_{SU(5)} + \frac{1}{2}F^2_{U(1)_X}. \end{aligned} \tag{D.11}$$

Then, with the parity matrices in the traces, we get

$$\begin{aligned}\mathrm{tr}_{10}(P_1F^2_{SO(10)}) &= -\mathrm{tr}_6F^2_{SU(4)} + 2\mathrm{tr}_2F^2_{SU(2)_L} + 2\mathrm{tr}_2F^2_{SU(2)_R} \\ &= -2\mathrm{tr}_4F^2_{SU(4)} + 2\mathrm{tr}_2F^2_{SU(2)_L} + 2\mathrm{tr}_2F^2_{SU(2)_R} \end{aligned} \tag{D.12}$$

and

$$\mathrm{tr}_{10}(P_2F^2_{SO(10)}) = -\mathrm{tr}_5F^2_{SU(5)} - \frac{1}{2}F^2_{U(1)_X} + \mathrm{tr}_{\overline{5}}F^2_{SU(5)} + \frac{1}{2}F^2_{U(1)_X} = 0 \tag{D.13}$$

and similarly $\mathrm{tr}_{10}(P_3F^2_{SO(10)}) = 0$. Next, we also do the decomposition of the index of a **16** spinor representation of $SO(10)$ as

$$\begin{aligned}\mathrm{tr}_{16}F^2_{SO(10)} &= 2\mathrm{tr}_4F^2_{SU(4)} + 2\mathrm{tr}_{\overline{4}}F^2_{SU(4)} + 4\mathrm{tr}_2F^2_{SU(2)_L} + 4\mathrm{tr}_2F^2_{SU(2)_R} \\ &= \mathrm{tr}_{10}F^2_{SU(5)} + \mathrm{tr}_{\overline{5}}F^2_{SU(5)} + \frac{1}{40}(10+45+25)F^2_{U(1)_X}. \end{aligned} \tag{D.14}$$

Then, we get the necessary traces for a **16** as

$$\begin{aligned}\mathrm{tr}_{16}(P_1F^2_{SO(10)}) &= 2\mathrm{tr}_4F^2_{SU(4)} - 2\mathrm{tr}_{\overline{4}}F^2_{SU(4)} + 4\mathrm{tr}_2F^2_{SU(2)_L} - 4\mathrm{tr}_2F^2_{SU(2)_R} \\ &= 4\mathrm{tr}_2F^2_{SU(2)_L} - 4\mathrm{tr}_2F^2_{SU(2)_R} \end{aligned} \tag{D.15}$$

and

$$\begin{aligned}\mathrm{tr}_{16}(P_2F^2_{SO(10)}) &= -\mathrm{tr}_{10}F^2_{SU(5)} + \mathrm{tr}_{\overline{5}}F^2_{SU(5)} + \frac{1}{40}(-10+45+25)F^2_{U(1)_X} \\ &= -2\mathrm{tr}_5F^2_{SU(5)} + \frac{3}{2}F^2_{U(1)_X} \end{aligned} \tag{D.16}$$

and likewise

$$\mathrm{tr}_{16}(P_3F^2_{SO(10)}) = -2\mathrm{tr}_5F^2_{SU(5)'} + \frac{3}{2}F^2_{U(1)_{X'}}. \tag{D.17}$$

- Relations between $U(1)$ generators:
  There are three maximal subgroups of $SO(10)$, Georgi-Glashow ($SU(5)\times U(1)_X$) and Pati-Salam ($SU(4)\times SU(2)_L\times SU(2)_R$) and flipped $SU(5)$ ($SU(5)'\times U(1)_{X'}$). For a fundamental representation of $SO(10)$, the $U(1)$ generators are given by

$$\begin{aligned}&Y = \mathrm{diag}(\frac{1}{3},\frac{1}{3},\frac{1}{3},-\frac{1}{2},-\frac{1}{2})\times\sigma^2, \quad X = \mathrm{diag}(2,2,2,2,2)\times\sigma^2, \\ &B-L = \mathrm{diag}(\frac{2}{3},\frac{2}{3},\frac{2}{3},0,0)\times\sigma^2, \quad T_{3R} = \mathrm{diag}(0,0,0,-\frac{1}{2},-\frac{1}{2})\times\sigma^2. \end{aligned} \tag{D.18}$$

Thus, we obtain the relation between $U(1)$ generators appearing in the different subgroups,

$$Y = T_{3R} + \frac{1}{2}(B-L), \quad X = -4T_{3R} + 3(B-L). \tag{D.19}$$

Moreover, by comparing the flipped $SU(5)$ to the Georgi-Glashow as $N \leftrightarrow e^c_L$ and $u^c_L \leftrightarrow d^c_L$ in **16** and $h_1 \leftrightarrow h_2$ in **10**, we get another relation

$$Y = \frac{1}{5}(-Y' + X'), \quad X = \frac{1}{5}(24Y' + X'). \tag{D.20}$$

Using the above relations, we can also find the relations between the $U(1)$ gauge bosons. To this, let us consider the bulk kinetic terms for $U(1)$ gauge bosons A_Y, A_X and a charged field ϕ:

$$\mathcal{L}_{\rm bulk} \supset -\frac{1}{4g_1^2}F_Y^2 - \frac{1}{4g_X^2}F_X^2 + \left|\left[\partial - i(\sqrt{\frac{3}{5}}YA_Y + \frac{1}{\sqrt{40}}XA_X)\right]\phi\right|^2 \tag{D.21}$$

where $g_1 = g_X$ at tree level. By writing

$$\begin{aligned}\sqrt{\frac{3}{5}}YA_Y + \frac{1}{\sqrt{40}}XA_X &= \sqrt{\frac{3}{5}}Y'A_{Y'} + \frac{1}{\sqrt{40}}X'A_{X'} \\ &= T_{3R}A_R + \sqrt{\frac{3}{8}}(B-L)A_{B-L},\end{aligned} \tag{D.22}$$

and using eqs. (D.19) and (D.20), we obtain the relations between the $U(1)$ gauge bosons as

$$\begin{aligned}A_{Y'} &= \frac{1}{5}(-A_Y + 2\sqrt{6}A_X), \quad A_{X'} = \frac{1}{5}(2\sqrt{6}A_Y + A_X), \\ A_R &= \sqrt{\frac{3}{5}}(A_Y - \sqrt{\frac{2}{3}}A_X), \quad A_{B-L} = \sqrt{\frac{3}{5}}(\sqrt{\frac{2}{3}}A_Y + A_X).\end{aligned} \tag{D.23}$$

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