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Analytic Solutions for Tachyon Condensation with General Projectors

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Abstract

The tachyon vacuum solution of Schnabl is based on the wedge states, which close under the star product and interpolate between the identity state and the sliver projector. We use reparameterizations to solve the long-standing problem of finding an analogous family of states for arbitrary projectors and to construct analytic solutions based on them. The solutions simplify for special projectors and allow explicit calculations in the level expansion. We test the solutions in detail for a one-parameter family of special projectors that includes the sliver and the butterfly. Reparameterizations further allow a one-parameter deformation of the solution for a given projector, and in a certain limit the solution takes the form of an operator insertion on the projector. We discuss implications of our work for vacuum string field theory.

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1 Introduction

The first analytic solution of open string field theory (OSFT) [1], corresponding to condensation of the open string tachyon, was recently constructed by Schnabl [2] and further studied in

[3, 4, 5, 6, 7, 8]. The starting point of [2] is a clever gauge-fixing condition, which makes the infinite system of equations of motion amenable to a recursive analysis. Schnabl's gauge choice for the open string field Ψ is

$$\mathcal{B}_0 \Psi = 0\,, \tag{1.1}$$

where $\mathcal{B}_0$ is the antighost zero mode in the conformal frame $z = f_{\mathcal{S}}(\xi) = \frac{2}{\pi}\arctan\xi$ of the sliver[1]:

$$\begin{aligned} \mathcal{B}_0 \equiv \oint \frac{dz}{2\pi i}\, z\, b(z) &= \oint \frac{d\xi}{2\pi i}\, \frac{f_{\mathcal{S}}(\xi)}{f'_{\mathcal{S}}(\xi)}\, b(\xi) \\ &= \oint \frac{d\xi}{2\pi i}\,(1+\xi^2)\, \arctan\xi\; b(\xi) = b_0 + \frac{2}{3}\, b_2 - \frac{2}{15}\, b_4 + \ldots\,. \end{aligned} \tag{1.2}$$

The sliver state W_∞ is a nontrivial projector of the open string star algebra, *i.e.*, a string field different from the identity that squares to itself [9, 10, 11, 12]. The wedge states W_α with $\alpha \geq 0$ are a family of states which interpolate between the sliver W_∞ and the identity $W_0 \equiv \mathcal{I}$, and they obey the abelian relation:

$$W_\alpha * W_\beta = W_{\alpha+\beta}\,. \tag{1.3}$$

Schnabl's solution is constructed in terms of a state ψ_α, with $\alpha \geq 0$, which is the wedge state $W_{\alpha+1}$ with suitable operator insertions. One defines the derivative state

$$\psi'_\alpha \equiv \frac{d\psi_\alpha}{d\alpha}\,, \quad \alpha \geq 0, \tag{1.4}$$

and then Schnabl's solution can be written as follows:[2]

$$\Psi = \lim_{N\to\infty} \Big[-\psi_N + \sum_{n=0}^{N} \psi'_n \Big]\,. \tag{1.5}$$

A simple description of the states ψ_α was presented in [3] using the CFT formulation of OSFT [13].

While the sliver was historically the first example of a projector, it was soon realized that infinitely many projectors exist [14]. Let us restrict attention to the subset of string fields known as surface states. A surface state is specified by a local coordinate map $z = f(\xi)$ from the canonical half-disk $\mathbb{D}^+ \equiv \{\xi \mid \Im\xi \geq 0\,, |\xi| \leq 1\}$ to a region in the upper-half plane (UHP) $\mathbb{H} \equiv \{z \mid \Im z \geq 0\}$. The surface state $|f\rangle$ is defined by its inner product

$$\langle\, \phi, f\,\rangle = \langle\, f\circ\phi(0)\,\rangle_{\mathbb{H}} \tag{1.6}$$

[1] For convenience, we have rescaled the traditional conformal frame of the sliver by a factor of $2/\pi$. This does not change the sliver state because of the $SL(2,R)$ invariance of the vacuum, nor does it affect the definition of $\mathcal{B}_0$.

[2] We use the conventions of [3], and the solution differs from that in [2] by an overall sign. See the beginning of section 2 of [3] for more details.

with any state ϕ in the Fock space. The condition that $|f\rangle$ is a projector is $f(i) = \infty$ [14], namely, the local coordinate curve goes to the boundary of $\mathbb{H}$ at the open string midpoint $\xi = i$. (Throughout this paper, we will restrict our considerations to "single-split" projectors, *i.e.*, surface states whose coordinate curve goes to infinity *only* at the open string midpoint.) The associated open string functional $\Psi_f[X(\sigma)]$ is *split*, namely, it is the product of a functional of the left half of the string times a functional of the right half of the string, $\Psi_f[X] = \Psi_f^L(X_L)\Psi_f^R(X_R)$. In the half-string formalism of OSFT [15, 16], where string fields are regarded as operators acting on the space of half-string functionals, surface state projectors are interpreted as rank-one projectors [14]. From this viewpoint, all surface state projectors should be equivalent. This is the intuition provided by finite dimensional vector spaces, where all rank-one projectors are related by similarity transformations.

These observations raise the natural question of whether Schnabl's solution, based on the sliver projector, can be generalized to solutions based on a generic surface state projector. In this paper we find that this is indeed the case. We also find, however, that the solution technically simplifies for the subclass of *special* projectors [5], which includes the sliver as its canonical representative. While we give a geometric description of the solution associated with a general projector, with the technology currently available we are able to evaluate its explicit Fock space expansion only when the projector is special.

It is useful at this point to recall some facts about special projectors [5]. The crucial algebraic property of a special projector is that the zero mode $\mathcal{L}_0$ of the energy-momentum tensor in the frame of the projector[3],

$$\mathcal{L}_0 \equiv \oint \frac{dz}{2\pi i}\, zT(z) = \oint \frac{d\xi}{2\pi i}\frac{f(\xi)}{f'(\xi)}T(\xi)\,, \tag{1.7}$$

and its BPZ conjugate $\mathcal{L}_0^\star$ obey

$$[\mathcal{L}_0, \mathcal{L}_0^\star] = s(\mathcal{L}_0 + \mathcal{L}_0^\star)\,, \quad s > 0\,. \tag{1.8}$$

The sliver is a special projector with $s = 1$ and the butterfly is a special projector with $s = 2$. The sliver and the butterfly fit into an infinite "hypergeometric" collection of special projectors — one projector $P_\infty^{(s)}$ for each real $s \geq 1$ — which was briefly described in [5]. We believe that the hypergeometric collection contains all the single-split special projectors. For special projectors, we shall use the notations

$$L \equiv \frac{\mathcal{L}_0}{s}\,, \quad L^\star \equiv \frac{\mathcal{L}_0^\star}{s}\,. \tag{1.9}$$

[3]The definition of a special projector further requires the conformal frame $f(\xi)$ to obey certain regularity conditions [5] which guarantee that the operator $L^+ = \frac{1}{s}(\mathcal{L}_0 + \mathcal{L}_0^\star)$ has a non-anomalous left/right decomposition.

In terms of L and $L^\star$ the algebra (1.8) takes the canonical form

$$[L, L^\star] = L + L^\star\,. \tag{1.10}$$

For any special projector P_∞ a family of states P_α with $\alpha \geq 0$ analogous to the wedge state family of the sliver is described by the following simple expression:

$$P_\alpha = e^{-\frac{\alpha}{2}L^+}\mathcal{I}\,, \quad L^+ \equiv L + L^*\,. \tag{1.11}$$

The states in the family interpolate between the projector P_∞ and the identity $P_0 \equiv \mathcal{I}$, and they obey the relation:

$$P_\alpha * P_\beta = P_{\alpha+\beta}\,. \tag{1.12}$$

We are now in the position to outline our strategy. Our starting point is the fact that *all* single-split, twist-invariant projectors can be related to one another by a reparameterization of the open string coordinate. Reparameterizations are generated infinitesimally by the star-algebra derivations $K_n = L_n - (-)^n L_{-n}$ and are familiar gauge symmetries of OSFT [17, 18]. Given a generic projector P_∞, there exists a finite reparameterization that relates it to the sliver, formally implemented by an operator e^H, with H a linear combination of K_n's:

$$P_\infty = e^H\, W_\infty\,, \quad H = \sum_{n=1}^{\infty} a_n K_n\,. \tag{1.13}$$

Acting with e^H on the solution Ψ_{W_∞} associated with the sliver W_∞, we find the solution Ψ_{P_∞} associated with P_∞:

$$\Psi_{P_\infty} \equiv e^H \Psi_{W_\infty}\,. \tag{1.14}$$

By construction, Ψ_{P_∞} is gauge equivalent to Ψ_{W_∞}. The idea of using reparameterizations as a solution-generating technique was already noted in [3]. The solution Ψ_{P_∞} will take the form (1.5), with the replacement of all the elements associated with the sliver by the corresponding elements associated with P_∞. In particular, we can define an abelian family of states interpolating between the identity and a generic projector P_∞ simply by taking $P_\alpha \equiv e^H W_\alpha$. If we write

$$W_\alpha = e^{-\frac{\alpha}{2}L^+_{\mathcal{S}}}\,\mathcal{I}\,, \tag{1.15}$$

where the subscript $\mathcal{S}$ in $L^+_{\mathcal{S}}$ denotes that it is an operator related to the sliver, we have

$$P_\alpha \equiv e^H\, W_\alpha = e^H\, e^{-\frac{\alpha}{2}L^+_{\mathcal{S}}}\, e^{-H}\, e^H\,\mathcal{I} \equiv e^{-\frac{\alpha}{2}L^+}\,\mathcal{I}\,. \tag{1.16}$$

Note that the identity is annihilated by H and we have *defined*

$$L^+ \equiv e^H\, L^+_{\mathcal{S}}\, e^{-H}\,. \tag{1.17}$$

Similarly we take

$$L \equiv e^{H} L_{\mathcal{S}} e^{-H}, \quad L^{\star} \equiv e^{H} L^{\star}_{\mathcal{S}} e^{-H}. \tag{1.18}$$

The operators L and $L^{\star}$ are BPZ conjugates of each other since $H^{\star} = -H$, and they obey the canonical algebra (1.10). If the projector P_{∞} is special, the definition (1.18) turns out to coincide with (1.9), but for general projectors the operator L is not proportional to $\mathcal{L}_0$.

It is in practice prohibitively difficult to determine the operator H. The construction, while motivated by the above considerations, must be realized differently. The main result of this paper is to give a geometric description of the reparameterization procedure and a concrete implementation using the CFT language of OSFT. In particular we provide a geometric description for the family of interpolating states P_{α} associated with an arbitrary projector that makes the abelian relation (1.12) obvious.

The description simplifies further for the case of a special projector. It should be emphasized that the geometrical construction of the family of states has been a long-standing question — there have been several attempts for the butterfly. In this paper we find out that the answer is quite simple if one uses the conformal frame of the projector itself.

It is remarkable that projectors play a central role in the construction of the analytic tachyon solution. Projectors have been intensively studied in the context of vacuum string field theory (VSFT) [19, 20]. In its simplest incarnation, VSFT is the conjecture that the OSFT action expanded around the tachyon vacuum has a kinetic operator $\mathcal{Q}$ of the form [20]:

$$\mathcal{Q} = \frac{c(i) - c(-i)}{2i}. \tag{1.19}$$

Taking a matter/ghost factorized ansatz for classical solutions, $\Psi = \Psi_g \otimes \Psi_m$, the VSFT equations of motion reduce to projector equations for the matter part Ψ_m. VSFT correctly describes the classical dynamics of D-branes [11, 12, 21], but it is somewhat singular. For example, the overall constant in front of the VSFT action must be taken to be formally infinite. It is believed that VSFT arises from OSFT, expanded around the tachyon vacuum, by a singular field redefinition. Moreover, the operator (1.19) is expected to be the leading term of a more complicated kinetic operator that involves the matter energy-momentum tensor as well, as discussed in more detail in [22]. One specific example of such a field redefinition given in [20] was the reparameterization that maps wedge states to one another, which in a singular limit formally maps all wedge states to the sliver. Interestingly, this reparameterization emerges naturally in the context of this paper. Indeed, it turns out that for each projector P_{∞} there is a reparameterization that leaves the projector invariant but maps the states in the interpolating family to one another. It takes P_{α} to $P_{e^{2\beta}\alpha}$, where β is an arbitrary real number. If we implement this reparameterization on the sliver-based solution and take the large β limit, all wedge states approach the sliver and the solution takes the form of an operator insertion on the sliver. A closely related approach in constructing a solution in a series expansion was proposed some time ago in [23] and investigated further in [24]. It would be interesting to find a systematic way to derive the kinetic operator of VSFT starting from a suitably reparameterized version of the tachyon vacuum solution.

We begin in section 2 with a general introduction to reparameterizations. After reviewing basic definitions and algebraic properties, we explain why any two regular twist-invariant surface states can be related by a reparameterization. The geometrical reason is simple. A surface state can be defined by what we call the *reduced* surface: it is the surface $\mathbb{H}$ for the inner product in (1.6) minus the local coordinate patch. In this picture the open string is a parameterized boundary curve created by removing the patch. The two string endpoints and the string midpoint define three special points on the boundary of the reduced surface. Given two surface states, the Riemann mapping theorem ensures that there is a conformal map between the reduced surfaces that maps the two endpoints and the midpoint of one string into those of the other. This map defines a relationship between the parameterizations of the two open strings; this is the induced reparameterization. When the surface state is a projector, the reduced surface is split in two at the point where the open string midpoint reaches the boundary of the full surface. When we map the reduced surfaces of two projectors to each other, each of the split surfaces of one reduced surface is mapped to a split surface of the other reduced surface. Since each split surface has only two special points (a string midpoint and a string endpoint), the conformal map has a one-parameter ambiguity.[4]

In section 3 we use the above insights to give the geometric construction of the abelian family P_α associated with a generic projector. In fact, once we choose a map R that relates the sliver to the chosen projector, the surface states P_α are obtained from the wedge states by a reparameterization naturally induced by R. This construction represents the surface states P_α using the conformal frame of the projector: the local coordinate patch is that of the projector but the surface only covers part of the UHP. The geometric description of the surface states P_α simplifies in this conformal frame — a fact that was missed in the earlier attempts to describe them. In §3.2 we specialize to special projectors, for which we find remarkable simplification. The reparameterization map that relates the sliver to the special projector in the hypergeometric collection with the parameter s can be chosen to be simply $R(z) = z^s$, where z is the coordinate in the UHP. For any fixed s, the regions of the UHP needed to represent states P_α with different

[4] The maps of the two split surfaces are related by a symmetry constraint, so there are no two independent parameters.

values of α are related to one another by rescaling. This is related to the fact that for special projectors the operator L defined in (1.18) is proportional to $\mathcal{L}_0$, which is the dilation operator in the conformal frame of the projector.

In section 4 we begin by discussing the algebraic framework of the tachyon vacuum solution. We then present our main result, the CFT construction of the solution using reparameterizations. We also present a detailed analysis of various operator insertions in the CFT description and derive useful formulas. In section 5 we use the operator formalism to derive an expression for the solutions associated with special projectors. The solution is written as a sequence of normal-ordered operators acting on the vacuum and can be readily expanded in level. Our expression has two parameters, $s \in [1, \infty)$ labeling the special projectors and $\beta \in (-\infty, \infty)$ labeling the reparameterizations of the solution that leave the projector invariant.

In section 6 we give the level expansion of the solutions for special projectors up to level four. We first set $\beta = 0$ and examine the dependence of the energy on s to level zero, two, and four. We find that as the level is increased the energy density approaches the expected value that cancels the D-brane tension. The solutions constructed by our method can be written in terms of even-moded total Virasoro operators and even-moded antighost operators in addition to the modes of c ghost. This structure imposes additional constraints, and thus the solutions belong to a resticted sector of the universal subspace of the CFT. We then examine the most accurate expression for the solution in the Siegel gauge computed in [25] and find evidence that it does not belong to the restricted universal subspace at level four. We thus conclude that the solution in the Siegel gauge cannot be obtained by our construction. In §6.4 we examine the solution for a fixed value of s and in the limit as β becomes large. The leading term in the solution takes the form of an insertion of the c ghost in P_β multiplied by $e^{2\beta/s}$ and by a finite, calculable coefficient. We offer some concluding remarks in section 7.

2 Reparameterizations

In this section we describe some general facts about reparameterizations. The first three subsections are for a review of well-known material. In §2.1 we define the notion of midpoint-preserving reparameterization φ of the open string coordinate, $t \to t' = \varphi(t)$, with $t = e^{i\sigma}$. Corresponding to φ there is an operator U_φ acting on the space of string fields that obeys a number of algebraic properties, as explained in §2.2. The transformation $\Psi \to U_\varphi \Psi$ is a gauge transformation of OSFT with a vanishing inhomogeneous term, as we review in §2.3. Finally, in §2.4 we explain the key idea: any two regular twist-invariant surface states can be related to one another by a unique reparameterization. For surface states that correspond to single-split projectors, an interesting and useful ambiguity arises.

In the rest of the paper we shall use these facts to find solutions of OSFT corresponding to a general projector, starting from Schnabl's solution corresponding to the sliver. By construction, all these solutions will be gauge equivalent.

2.1 Definitions

Let us start by recalling the definition of midpoint-preserving reparameterizations (henceforth, simply reparameterizations) [17]. A reparameterization of the open string coordinate is a map $\sigma \to \sigma' = \rho(\sigma)$ (with $\sigma\,,\sigma' \in [0\,,\,\pi]$) that obeys

$$\rho(\pi - \sigma) = \pi - \rho(\sigma)\,. \tag{2.1}$$

Note that this is a much stronger condition on ρ than just fixing the midpoint $\sigma = \pi/2$: it implies that points at equal parameter distance from the midpoint remain at equal parameter distance after the map. We will use the coordinate $t \equiv \exp(i\sigma)$ defined on the unit semicircle in the upper half plane. It follows from (2.1) that a map $t \to t' = \varphi(t)$ (with $|t| = |t'| = 1$, $\Re t \geq 0\,, \Re t' \geq 0$) is a reparameterization if

$$\varphi\left(-\frac{1}{t}\right) = -\frac{1}{\varphi(t)}\,. \tag{2.2}$$

For an infinitesimal reparameterization we write the general ansatz

$$\varphi(t) = t + \epsilon\, v(t) + O(\epsilon^2)\,, \tag{2.3}$$

where ϵ is an infinitesimal real parameter and $v(t)$ is a complex vector. We deduce from (2.2) that the vector field $v(t)$ must be BPZ odd:

$$v\left(-\frac{1}{t}\right) = \frac{1}{t^2}\, v(t)\,. \tag{2.4}$$

Hence $v(t)$ is a linear combination of the BPZ odd vector fields v_{K_n} corresponding to the familiar derivations $K_n = L_n - (-1)^n L_{-n}$:

$$v(t) = \sum_{n=1}^{\infty} a_n v_{K_n} = \sum_{n=1}^{\infty} a_n \,\left(t^{n+1} - (-1)^n t^{-n+1}\right)\,. \tag{2.5}$$

By definition, reparameterizations preserve the unit norm of $|t|$. Using (2.3) this condition gives

$$t\, v(t)^* + t^* v(t) = 0\,, \tag{2.6}$$

which implies that the coefficients a_n in (2.5) satisfy

$$a_n = (-)^n a_n^* \,. \tag{2.7}$$

We see that a_n must be real for n even and imaginary for n odd.

A finite reparameterization $\varphi(t)$ can be obtained by exponentiation of a vector $v(t)$ of the form (2.5):

$$\exp(v(t)\partial_t)\; t = \varphi(t) \,. \tag{2.8}$$

Indeed, the condition (2.4) implies that $\varphi(t)$ satisfies (2.2). Moreover, (2.6) implies that $\varphi(t)$ has unit norm. In general $\varphi(t)$ is defined only on the unit semicircle with $|t| = 1$ and cannot be extended to a holomorphic function inside the local coordinate half-disk $\mathbb{D}^+$. If v is a finite linear combination of v_{K_n} vectors, $\varphi(t)$ can be extended to a finite annulus in the upper-half plane $\mathbb{H}$ containing the unit semicircle.

2.2 The operator U_φ

We now consider the operator U_φ that implements a finite reparameterization. The operator is defined to act on any operator $\mathcal{O}(t)$ in the CFT as

$$U_\varphi\, \mathcal{O}(t) U_\varphi^{-1} = \varphi \circ \mathcal{O}(t) \,. \tag{2.9}$$

This is the same relation one has for operators that realize the conformal maps used for surface states, the difference being that here the action on $\mathcal{O}$ is only defined for $|t| = 1$ and typically does not extend to the origin. We write[5]

$$U_\varphi = e^{-H} \,, \quad H = \sum_{n=1}^{\infty} a_n K_n \,, \quad a_n = (-1)^n a_n^* \,. \tag{2.10}$$

We can verify that the reality condition on the coefficients a_n guarantees that U_φ preserves the reality condition of the string field. In OSFT the string field Ψ obeys the reality condition:

$$\Psi = \mathrm{hc}^{-1} \circ \mathrm{bpz}(\Psi) \,. \tag{2.11}$$

BPZ conjugation (bpz) and hermitian conjugation (hc) act on Virasoro generators as follows:

$$\mathrm{bpz}(L_n) = (-1)^n L_{-n} \,, \qquad \mathrm{hc}(L_n) = L_{-n} \,. \tag{2.12}$$

[5]We use the symbol H rather than K since we reserve the latter for the operator introduced in [5]: $K \equiv \widetilde{L}^+ \equiv L_R^+ - L_L^+$.

For any operator $\mathcal{O}$ we let $\mathcal{O}^\star$ denote its BPZ conjugate. Recalling that BPZ conjugation is a linear transformation while hermitian conjugation is an *anti*-linear transformation, we easily check that reparameterizations preserve the reality of the string field:

$$\begin{aligned}\mathrm{hc}^{-1}\circ \mathrm{bpz}\,(U_\varphi|\Psi\rangle) &= \mathrm{hc}^{-1}\left(\langle \mathrm{bpz}(\Psi)|e^{\sum_n a_n K_n}\right)\\ &= e^{-\sum_n (-1)^n a_n^* K_n}|\mathrm{hc}^{-1}\circ \mathrm{bpz}(\Psi)\rangle\\ &= e^{\sum_n a_n K_n}|\Psi\rangle = U_\varphi|\Psi\rangle\,.\end{aligned} \tag{2.13}$$

The operator U_φ obeys the following formal properties:

$$U_\varphi^\star \;=\; U_\varphi^{-1}\,, \tag{2.14}$$

$$[Q_B, U_\varphi] \;=\; 0\,, \tag{2.15}$$

$$U_\varphi\mathcal{I} \;=\; U_\varphi^\star\mathcal{I} = \mathcal{I}\,, \tag{2.16}$$

$$U_\varphi\Psi_1 * U_\varphi\Psi_2 \;=\; U_\varphi(\Psi_1 * \Psi_2)\,, \qquad \forall\, \Psi_1\,,\Psi_2\,. \tag{2.17}$$

These identities are the exponentiated version of the following familiar properties of $H = \sum_{n=1}^\infty a_n K_n$:

$$H^\star \;=\; -H\,, \tag{2.18}$$

$$[Q_B, H] \;=\; 0\,, \tag{2.19}$$

$$H\mathcal{I} \;=\; 0\,, \tag{2.20}$$

$$H\Psi_1 * \Psi_2 + \Psi_1 * H\Psi_2 \;=\; H(\Psi_1 * \Psi_2)\,, \qquad \forall\, \Psi_1\,,\Psi_2\,. \tag{2.21}$$

The properties (2.14)–(2.17) can also be understood from the viewpoint of OSFT without reference to the operator H. For example, since points at equal parameter distance from the midpoint remain at equal parameter distance after reparameterizations, (2.17) follows at once from the picture of the star product as gluing of half open string functionals. Similarly, (2.16) follows, at least formally, from the understanding of the identity string field as the functional that identifies the left and the the right halves of the open string. In [5] it was found that the property (2.20) may fail to hold for certain singular BPZ odd operators H. The finite reparameterizations that we explicitly consider in this paper appear to be perfectly smooth, and we believe that they obey all the formal properties (2.14)–(2.17). Following the discussion of [5], we note that a regular H should admit a left/right decomposition $H = H_L + H_R$ that is non-anomalous:

$$[\,H_L\,,H_R] = 0\,, \quad H_L(A*B) = (H_LA)*B\,, \quad H_R(A*B) = A*(H_RB) \tag{2.22}$$

for general string fields A and B.

2.3 Reparameterizations as gauge symmetries

Reparameterizations are well-known gauge symmetries of OSFT. (See, for example, [18] for an early general discussion.) Infinitesimal gauge transformations take the familiar form

$$\delta_\Lambda \Psi = Q_B\Lambda + \Psi * \Lambda - \Lambda * \Psi\,, \tag{2.23}$$

where, in the classical theory, Ψ carries ghost number one and the gauge parameter Λ carries ghost number zero. Choose now $\Lambda = H_R\mathcal{I} = -H_L\mathcal{I}$. The inhomogeneous term in (2.23) vanishes since $[Q_B, H_R] = 0$ and $Q_B\mathcal{I} = 0$. Using (2.22) we have

$$\delta_{H_R\mathcal{I}}\Psi = \Psi * (H_R\mathcal{I}) + (H_L\mathcal{I}) * \Psi = H_R(\Psi * \mathcal{I}) + H_L(\mathcal{I} * \Psi) = (H_R + H_L)\Psi = H\Psi\,. \tag{2.24}$$

This shows that the infinitesimal reparameterization generated by H can be viewed as an infinitesimal gauge transformation with gauge parameter $H_R\mathcal{I}$. Exponentiating this relation, we claim that

$$U_\varphi\Psi \equiv e^H\Psi = \mathcal{U}_\varphi^{-1} * \Psi * \mathcal{U}_\varphi\,, \tag{2.25}$$

where the string fields $\mathcal{U}_\varphi$ and $\mathcal{U}_\varphi^{-1}$ are defined by

$$\begin{aligned}\mathcal{U}_\varphi &\equiv \exp_*(H_R\mathcal{I}) \equiv \mathcal{I} + H_R\mathcal{I} + \frac{1}{2}H_R\mathcal{I} * H_R\mathcal{I} + \ldots \frac{1}{n!}(H_R\mathcal{I})^n + \ldots\,, &(2.26)\\ \mathcal{U}_\varphi^{-1} &\equiv \exp_*(-H_R\mathcal{I}) \equiv \mathcal{I} - H_R\mathcal{I} + \frac{1}{2}H_R\mathcal{I} * H_R\mathcal{I} + \ldots \frac{(-1)^n}{n!}(H_R\mathcal{I})^n + \ldots\,, &(2.27)\end{aligned}$$

and they obey

$$\mathcal{U}_\varphi^{-1} * \mathcal{U}_\varphi = \mathcal{U}_\varphi * \mathcal{U}_\varphi^{-1} = \mathcal{I}\,. \tag{2.28}$$

It is straighforward to check that for arbitrary string field A,

$$\exp_*(H_L\mathcal{I}) * A = e^{H_L}A\,, \qquad \text{and} \qquad A * \exp_*(H_R\mathcal{I}) = e^{H_R}A\,. \tag{2.29}$$

These identities, together with $[H_L, H_R] = 0$, can be used to show that the equality in (2.25) holds. The right-hand side of (2.25) has the structure of a finite gauge-transformation in OSFT:

$$\Psi \;\to\; \mathcal{U}_\varphi^{-1} * \Psi * \mathcal{U}_\varphi + \mathcal{U}_\varphi^{-1} * Q_B\mathcal{U}_\varphi\,, \tag{2.30}$$

where the inhomogeneous term $\mathcal{U}_\varphi^{-1} * Q_B\mathcal{U}_\varphi$ is identically zero.

Since reparameterizations are gauge symmetries, it is clear that they map a classical solution of OSFT to other gauge-equivalent classical solutions. If Ψ is a solution then $U_\varphi\Psi$ is also a solution, as is verified using the formal properties (2.15) and (2.17):

$$Q_B\Psi + \Psi * \Psi = 0 \;\longrightarrow\; U_\varphi(Q_B\Psi + \Psi * \Psi) = 0 \;\longrightarrow\; Q_BU_\varphi\Psi + U_\varphi\Psi * U_\varphi\Psi = 0\,. \tag{2.31}$$

It is also clear that Ψ and U_φ have the same vacuum energy. Indeed, using (2.14) and (2.15),

$$\langle U_\varphi \Psi, Q_B U_\varphi \Psi\rangle = \langle \mathrm{bpz}(U_\varphi \Psi)|\, Q_B U_\varphi |\Psi\rangle = \langle \mathrm{bpz}(\Psi)|\, U_\varphi^{-1} U_\varphi Q_B |\Psi\rangle = \langle \Psi, Q_B \Psi\rangle\,. \tag{2.32}$$

Furthermore, from (2.14) and (2.17),

$$\langle U_\varphi \Psi, U_\varphi \Psi * U_\varphi \Psi\rangle \equiv \langle \mathrm{bpz}(U_\varphi \Psi)| U_\varphi \Psi * U_\varphi \Psi\rangle = \langle \mathrm{bpz}(\Psi)| U_\varphi^{-1} U_\varphi |\Psi * \Psi\rangle = \langle \Psi, \Psi * \Psi\rangle\,. \tag{2.33}$$

The two equations (2.32) and (2.33) guarantee that if the equations of motion for Ψ are obeyed when contracted with Ψ itself, the same is true for $U_\varphi \Psi$.

2.4 Reparameterizations of surface states

We now explain how reparameterizations can be used to relate surface states. Consider a twist-invariant surface states $|f\rangle$, specified as usual by a local coordinate map $z = f(\xi)$ from the canonical half-disk $\mathbb{D}^+$ to a region in the upper half plane $\mathbb{H}$. (Both $\mathbb{D}^+$ and $\mathbb{H}$ are defined above (1.6).) We denote by $\mathcal{V}^{(f)}$ the *reduced* surface corresponding to the surface state $|f\rangle$. The reduced surface is defined as the complement of the local coordinate half-disk in $\mathbb{H}$:

$$\mathcal{V}^{(f)} \equiv \mathbb{H}/f(\mathbb{D}^+)\,. \tag{2.34}$$

The reduced surface $\mathcal{V}^{(f)}$ has two types of boundary. The first type is the boundary where open string boundary conditions apply; it is the part of the boundary of $\mathbb{H}$ which belongs to $\mathcal{V}^{(f)}$. The second type is provided by the coordinate curve C_f which represents the open string:

$$\mathcal{C}_f \equiv \{f(t) \in \mathbb{H},\ |t| = 1\,, \Im(t) \geq 0\}\,. \tag{2.35}$$

Let us assume for the time being that the local coordinate curve does not go to infinity anywhere. Then $\mathcal{V}^{(f)}$ has the topology of a disk. The twist invariance $f(-\xi) = -f(\xi)$, together with the standard conjugation symmetry $(f(\xi))^* = f(\xi^*)$, implies that $f(\xi) = -(f(-\xi^*))^*$ so $\mathcal{V}^{(f)}$ is invariant under a reflection about the imaginary z axis. We now claim that given any two such surface states $|f\rangle$ and $|g\rangle$, there exists a reparameterization φ (depending of course on f and g) that relates them:

$$|g\rangle = U_\varphi |f\rangle\,. \tag{2.36}$$

This is shown as follows. By the Riemann mapping theorem, there exists a holomorphic map $z' = \widehat{R}(z)$ relating the reduced surfaces $\mathcal{V}^{(f)}$ and $\mathcal{V}^{(g)}$:

$$\mathcal{V}^{(g)} = \widehat{R}(\mathcal{V}^{(f)})\,. \tag{2.37}$$

We construct the map using the symmetry of the problem: first we uniquely map the region to the right of the imaginary axis of $\mathcal{V}^{(f)}$ to that of $\mathcal{V}^{(g)}$ by requiring that $f(1)$, $f(i)$, and infinity are mapped to $g(1)$, $g(i)$, and infinity, respectively. We then extend the map to the left of the imaginary line using Schwarz's reflection principle, which applied here gives $\widehat{R}(z) = -(\widehat{R}(-z^*))^*$. The map $\widehat{R}$ so constructed takes the local coordinate curve C_f to the local coordinate curve C_g (defined by (2.35) with f replaced by g):

$$C_g = \widehat{R}(C_f)\,. \tag{2.38}$$

A reparameterization $t' = \varphi(t)$ of the two coordinate curves is defined implicitly by the relation

$$\widehat{R}(f(t)) \equiv g(\varphi(t))\,. \tag{2.39}$$

It follows from the above construction that φ is a reparameterization. Indeed one readily verifies that

$$\begin{aligned} \widehat{R}\Big(f\Big(-\frac{1}{t}\Big)\Big) &= \widehat{R}(f(-t^*)) = \widehat{R}(-(f(t))^*) = -(\widehat{R}(f(t)))^* \\ &= -(g(\varphi(t)))^* = g(-(\varphi(t))^*) = g\Big(-\frac{1}{\varphi(t)}\Big)\,, \end{aligned} \tag{2.40}$$

which establishes that (2.2) holds.

We now give a formal argument that explains why (2.36) holds. The surface state $\langle f|$ is defined by its overlap with a generic state $|\Psi\rangle$. Without loss of generality, we can restrict to states $|\Psi\rangle = |X_b\rangle$ which are eigenstates of the position operator[6] $\hat{X}(t)$,

$$\hat{X}(t)|X_b\rangle = X_b(t)|X_b\rangle\,. \tag{2.41}$$

The overlap $\langle f|X_b\rangle$ is computed by the path-integral over $\mathcal{V}^{(f)}$, where we impose open string boundary conditions on the portion of the boundary with $\Im z = 0$ and the boundary conditions $X(f(t)) = X_b(t)$ on the coordinate curve $\mathcal{C}_f$. Schematically,

$$\langle f|X_b\rangle = \int_{z\in\mathcal{V}^{(f)}} [dX(z)]\, e^{-S_{BCFT}[X]} \quad \text{with } \; X(f(t)) \equiv X_b(t) \text{ on } \; \mathcal{C}_f\,. \tag{2.42}$$

Applying the reparameterization $z \to z' = \widehat{R}(z)$, we see that $\langle f|X_b\rangle$ is equivalently computed by the path-integral over $\mathcal{V}^{(g)}$, provided we keep track of how the boundary conditions are mapped,

$$\langle f|X_b\rangle = \int_{z'\in\mathcal{V}^{(g)}} [dX(z')]\, e^{-S_{BCFT}[X]} \quad \text{with } \; X(g(t')) \equiv X_b(\varphi^{-1}(t')) \text{ on } \; \mathcal{C}_g\,. \tag{2.43}$$

[6]For notational simplicity, we are focussing on the matter part of the CFT.

The path-integral in (2.43) can now be interpreted as computing the overlap of the surface state $\langle g|$ with the position eigenstate $|X_b \circ \varphi^{-1}\rangle$. Thus

$$\langle f|X_b\rangle = \langle g|X_b \circ \varphi^{-1}\rangle \,. \tag{2.44}$$

To proceed, we note that the reparameterization U_φ that gives

$$U_\varphi \hat{X}(t) U_\varphi^{-1} = \hat{X}(\varphi(t))\,, \tag{2.45}$$

will also give

$$U_\varphi |X_b\rangle = |X_b \circ \varphi^{-1}\rangle \,. \tag{2.46}$$

Indeed

$$\hat{X}(t)\, U_\varphi |X_b\rangle = U_\varphi U_\varphi^{-1}\, \hat{X}(t)\, U_\varphi |X_b\rangle = U_\varphi \hat{X}(\varphi^{-1}(t))\, |X_b\rangle = X_b(\varphi^{-1}(t))\, U_\varphi |X_b\rangle \,, \tag{2.47}$$

confirming that $U_\varphi|X_b\rangle$ is the $\hat{X}(t)$ eigenstate of eigenvalue $X_b \circ \varphi^{-1}(t)$, as stated in (2.46). Back in (2.44), we see that

$$\langle f|X_b\rangle = \langle g|\, U_\varphi |X_b\rangle \qquad \forall\ |X_b\rangle \,, \tag{2.48}$$

which implies

$$\langle g| = \langle f|\, U_\varphi^{-1} = \langle f|\, U_\varphi^\star \,. \tag{2.49}$$

This is the BPZ conjugate of the claimed relation (2.36).

So far we have assumed that the coordinate curves C_f and C_g do not reach infinity. It is vital for us to consider projectors, for which the coordinate curve *does* reach infinity at the open string midpoint: $f(i) = \infty$. If we assume that the midpoint is the only point for which $f(t)$ is infinite then the reduced surface $\mathcal{V}^{(f)}$ splits into two disks $\mathcal{V}^{(f)-}$ and $\mathcal{V}^{(f)+}$, with $\Re z < 0$ and $\Re z > 0$, respectively, joined at the point at infinity. The claim (2.36) still holds in this case: any two such twist-invariant projectors $|f\rangle$ and $|g\rangle$ can be related by a reparameterization φ. We define the map $\widehat{R}$ for $\mathcal{V}^{(f)+}$ and, as before, we extend it to $\mathcal{V}^{(f)-}$. Again, the map $\widehat{R} : \mathcal{V}^{(f)+} \to \mathcal{V}^{(g)+}$, is guaranteed to exist by the Riemann mapping theorem, but this time it is not unique. While before $f(i)$ and ∞ provided two different points whose maps could be constrained, now they are the same one. We partially fix the $SL(2,R)$ symmetry by requiring that $f(1)$ and $f(i) = \infty$ are mapped to $g(1)$ and $g(i) = \infty$, respectively. There is one degree of freedom left unfixed, so there exists a one parameter family of analytic maps from $\mathcal{V}^{(f)+}$ to $\mathcal{V}^{(g)+}$. This redundancy will play an important role in the following.

Finally, we note that we can never hope to relate regular surface states to projectors using reparameterizations, since the topologies of the reduced surfaces $\mathcal{V}^{(f)}$ are different in the two classes.

3 Abelian families for general projectors

The basic building block of Schnabl's solution is the state ψ_α, which is constructed from the wedge state $W_{\alpha+1}$ by adding suitable operator insertions. In this section we generalize the wedge states W_α, associated with the sliver W_∞, to states P_α associated with a generic twist-invariant projector P_∞. In the next section we shall deal with the operator insertions and construct the analog of the state ψ_α for a generic projector.

As we have explained in §2.4, given a projector P_∞, there exists a reparameterization φ that relates it to the sliver:

$$W_\infty = U_\varphi P_\infty \,. \tag{3.1}$$

(There is in fact a one-parameter family of such reparameterizations. For now we simply choose one of them.) We define P_α by

$$P_\alpha \equiv U_\varphi^{-1} W_\alpha = U_{\varphi^{-1}} W_\alpha \,. \tag{3.2}$$

It follows from (2.16) that $P_0 = \mathcal{I}$ and from (2.17) that the states P_α obey the same abelian relation as W_α:

$$P_\alpha * P_\beta = U_{\varphi^{-1}} W_\alpha * U_{\varphi^{-1}} W_\alpha = U_{\varphi^{-1}}(W_\alpha * W_\beta) = U_{\varphi^{-1}}(W_{\alpha+\beta}) = P_{\alpha+\beta} \,. \tag{3.3}$$

In §3.1 we give a geometric construction of P_α by determining the shape of the associated one-punctured disk $\mathcal{P}_\alpha$ in the presentation where the local coordinate patch is that of the projector P_∞. In §3.2 we focus on special projectors, for which the construction simplifies considerably and the reparameterization to the sliver can be given in closed form. For a special projector the corresponding abelian family obeys a remarkable geometric property: the surfaces $\mathcal{P}_\alpha$ with different values of α are related to one another by overall conformal scaling.

3.1 Abelian families by reparameterizations

Given a single-split, twist-invariant projector $|f\rangle$, we wish to find a reparameterization that relates it to the sliver. In the notations of §2.4, we write the sliver as $|W_\infty\rangle \equiv |g\rangle$ with $z' = g(\xi) = \frac{2}{\pi}\arctan(\xi)$ and look for a one-parameter family of conformal maps $\widehat{R}_\beta : \mathcal{V}^{(f)} \to \mathcal{V}^{(g)}$. From now on we shall drop the superscript in $\mathcal{V}^{(f)} \to \mathcal{V}$, and we rename the sliver's coordinate $z' \to z_{\mathcal{S}}$ and the sliver's region $\mathcal{V}^{(g)} \to \mathcal{U}$.

To describe the conformal maps $\widehat{R}_\beta(z)$ we need to define a set of curves and regions in the conformal plane. We denote by C_0^+ and C_0^- the right and left parts, respectively, of the coordinate curve C_0 of the projector $|f\rangle$. It is convenient to extend C_0^+ and C_0^- by complex

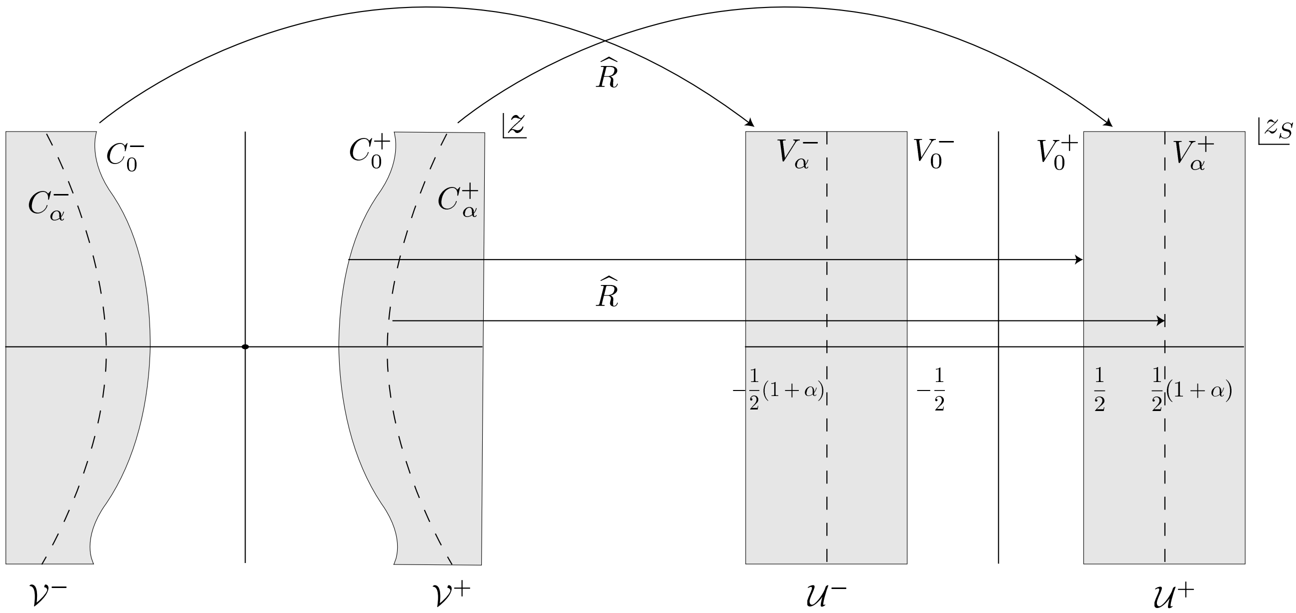

Figure 1: Left: Coordinate curves $C_0^\pm$ of the projector and (shaded) regions $\mathcal{V}^\pm$ to the left and right of the coordinate disk. Right: Coordinate curves $V_0^\pm$ for the sliver and (shaded) regions $\mathcal{U}^\pm$ to the left and right of the coordinate disk. The map $\widehat{R}$ relates the reduced surfaces of the two projectors. It takes $\mathcal{V}^\pm$ to $\mathcal{U}^\pm$ and defines the reparameterization that relates the two projectors.

conjugation to curves on the full plane, making the extended curves invariant under complex conjugation. For twist invariance of the projectors, the curve C_0^- is determined by C_0^+: $z \in C_0^-$ if $-z \in C_0^+$. The curve C_0^- is the mirror image of C_0^+ across the imaginary axis. (See Figure 1.)

Let $\mathcal{V}^+$ denote the region of the z-plane to the right of C_0^+ and let $\mathcal{V}^-$ denote the region of the z-plane to the left of C_0^-. Since the coordinate curves reach the point at infinity, both $\mathcal{V}^+$ and $\mathcal{V}^-$ are conformally equivalent to the UHP, with the role of the real axis in the UHP played by the curves $C_0^\pm$. The union of $\mathcal{V}^+$ and $\mathcal{V}^-$ is $\mathcal{V}$, the surface of the projector minus its coordinate disk. Let us define analogous regions $\mathcal{U}^\pm$ for the sliver as follows:

$$\mathcal{U}^+ = \Big\{ z_{\mathcal{S}} \Big| \, \Re(z_{\mathcal{S}}) \geq \frac{1}{2} \Big\} , \quad \mathcal{U}^- = \Big\{ z_{\mathcal{S}} \Big| \, \Re(z_{\mathcal{S}}) \leq -\frac{1}{2} \Big\} . \tag{3.4}$$

It is also useful to define vertical lines $V_\alpha^\pm$ in the sliver frame:

$$V_\alpha^\pm = \Big\{ \; z_{\mathcal{S}} \; \Big| \; \Re(z_{\mathcal{S}}) = \pm\frac{1}{2}(1+\alpha) \Big\} . \tag{3.5}$$

The boundaries of $\mathcal{U}^\pm$ are $V^\pm$. Both $\mathcal{U}^+$ and $\mathcal{U}^-$ are conformally equivalent to the UHP, with the role of the real axis in the UHP played by the lines V_0^+ and V_0^-. (See Figure 1.)

We are interested in the map

$$R : \mathcal{V}^+ \; \to \; \mathcal{U}^+ , \qquad z_{\mathcal{S}} = R(z) . \tag{3.6}$$

The map must exist since both regions are conformal to the UHP. Of course, the map will take the boundary C_0^+ to the boundary V_0^+. We impose two additional conditions:

1. The intersection of C_0^+ with the real axis is mapped to $z_{\mathcal{S}} = 1/2$.

2. The point at infinity on C_0^+ is mapped to the point at infinity on V_0^+.

The map R commutes with the operation of complex conjugation: $R(z^*) = (R(z))^*$. Thus the portion of the real axis contained in $\mathcal{V}^+$ is mapped to the portion of the real axis contained in $\mathcal{U}^+$. We can then define the map $\widehat{R}$ that maps the whole $\mathcal{V}$ to the whole $\mathcal{U}$ as follows:

$$\widehat{R}(z) = \begin{cases} R(z) & \text{if } z \in \mathcal{V}_0^+ \,, \\ -R(-z) & \text{if } z \in \mathcal{V}_0^- \,. \end{cases} \tag{3.7}$$

It is easy to check that $\widehat{R}$ is an odd function:

$$\widehat{R}(-z) = -\widehat{R}(z)\,. \tag{3.8}$$

The map $\widehat{R}$ describes a reparameterization between the projector and the sliver. Indeed, letting $f(\xi)$ denote the coordinate function of the projector and $f_{\mathcal{S}}(\xi_{\mathcal{S}})$ denote the coordinate function of the sliver, we have the relation $\xi_{\mathcal{S}} = f^{-1} \circ \widehat{R} \circ f(\xi)$. As befits a reparameterization, it satisfies the condition in (2.2).

As we have already remarked, the reparameterization $\widehat{R}(z)$ is not unique: we only specified two out of the three conditions needed to determine a map $\mathbb{H} \to \mathbb{H}$ uniquely. The remaining ambiguity is that of post-composition with the self maps of $\mathcal{U}^+$ that leave the points $z_{\mathcal{S}} = 1/2$ and $z_{\mathcal{S}} = \infty$ invariant. Given a function $R_0(z)$ that realizes the map in (3.6) with the conditions listed above, we can generate a one-parameter family $R_\beta(z)$ of maps that satisfy the same conditions as follows:

$$R_\beta(z) \equiv e^{-2\beta}\left(R_0(z) - \frac{1}{2}\right) + \frac{1}{2}\,, \tag{3.9}$$

with $-\infty < \beta < \infty$ an arbitrary real constant. It is clear that the map is a scaling about $z = 1/2$ with scale factor $e^{-2\beta}$. With R_β replacing R in (3.7) we obtain a family $\widehat{R}_\beta$ of reparameterizations. We will later use this ambiguity to produce, for any fixed projector, a family of solutions parameterized by β.

Let us continue our analysis, assuming that a choice of $\widehat{R}$ has been made for the projector under consideration. Since the function $\widehat{R}(z)$ is invertible we can define the curves $C_\alpha^\pm$ as the image under the inverse function $\widehat{R}^{-1}$ of the vertical lines $V_\alpha^\pm$:

$$C_\alpha^\pm \equiv \widehat{R}^{-1}(V_\alpha^\pm)\,. \tag{3.10}$$

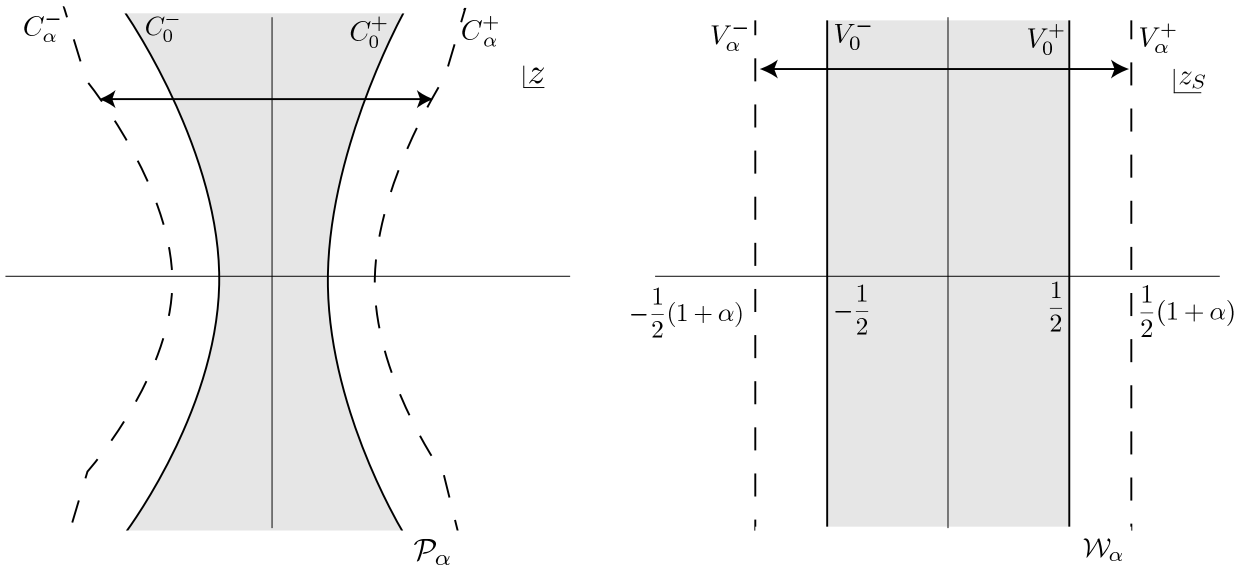

Figure 2: Left: The surface $\mathcal{P}_\alpha$ with its coordinate disk shaded. Right: The wedge surface $\mathcal{W}_\alpha$ with its coordinate disk shaded.

It follows from $\widehat{R}(C_\alpha^\pm) = V_\alpha^\pm$ that

$$\Re(\widehat{R}(z)) = \frac{1}{2}(1+\alpha)\,, \qquad z \in C_\alpha^+\,. \tag{3.11}$$

The various lines $V_\alpha^\pm$ and $C_\alpha^\pm$ are shown in Fig 1.

We now proceed to the key step in the construction: we introduce a family P_α of states associated with the projector that is related by a reparameterization to the wedge states. Consider first the surface $\mathcal{W}_\alpha$ for the wedge state W_α given by

$$\text{Wedge state surface } \mathcal{W}_\alpha : \quad -\frac{1}{2}(1+\alpha) \leq \Re(z_{\mathcal{S}}) \leq \frac{1}{2}(1+\alpha)\,. \tag{3.12}$$

This surface is shown on the right side of Figure 2. We write

$$\mathcal{W}_\alpha = (V_\alpha^-\,, V_\alpha^+)\,, \tag{3.13}$$

where (C, C') denotes the region between the curves C and C'. The coordinate disk for $\mathcal{W}_\alpha$ is (V_0^-, V_0^+). Using $z_{\mathcal{S}}^\pm$ for coordinates on $V_\alpha^\pm$, the identification for the surface is described as follows:

$$z_{\mathcal{S}}^+ - z_{\mathcal{S}}^- = 1+\alpha\,. \tag{3.14}$$

Now define the surface

$$\mathcal{P}_\alpha \equiv (C_\alpha^-, C_\alpha^+) = \big(\, \widehat{R}^{-1}(V_\alpha^-)\,,\, \widehat{R}^{-1}(V_\alpha^+)\,\big), \tag{3.15}$$

with the identification inherited from that of the vertical lines in (3.14). The surface $\mathcal{P}_\alpha$ is shown on the left side of Figure 2. The coordinate disk in $\mathcal{P}_\alpha$ is the region (C_0^-, C_0^+), or $\mathcal{P}_0$ without the identification. It follows that the complement of the coordinate disk in $\mathcal{W}_\alpha$ is mapped by $\widehat{R}^{-1}$ to the complement of the coordinate disk in $\mathcal{P}_\alpha$. We have thus related the states P_α and W_α by a reparameterization. Using $z^\pm$ for coordinates on $C_\alpha^\pm$, the identification (3.14) becomes

$$\widehat{R}(z^+) - \widehat{R}(z^-) = 1 + \alpha\,. \tag{3.16}$$

Using (3.7) this gives

$$R(z^+) + R(-z^-) = 1 + \alpha\,. \tag{3.17}$$

A few comments are in order. Since $\mathcal{P}_0$ is the coordinate disk of the projector with its boundaries identified, this is simply another surface for the identity state. Moreover, the limit of $\mathcal{P}_\alpha$ as $\alpha \to \infty$ is expected to be the surface for the projector itself. In fact, the curves to be identified are going to infinity, and the identification becomes immaterial because infinity is a single point in the UHP. We thus obtain the UHP with the coordinate patch of the projector — this is the surface for the projector.

In order to describe star products of wedge states it is convenient to use an alternative presentation of the region (3.12). We use the transition function (3.14) to move the region (V_α^-, V_0^-) to the right of V_α^+. Since the image of $z_{\mathcal{S}}^- = -1/2$ is $z_{\mathcal{S}}^+ = (1+2\alpha)/2$, we have

$$\mathcal{W}_\alpha = (V_0^-, V_{2\alpha}^+)\,, \tag{3.18}$$

with the identification in (3.14) still operational. (See Figure 3.) Similarly, the surface $\mathcal{P}_\alpha$ can also be represented as

$$\mathcal{P}_\alpha = (C_0^-, C_{2\alpha}^+)\,, \tag{3.19}$$

with the identification in (3.17) still operational. (See Figure 3.)

The gluing for the star product of wedge states is performed simply by translation with a real parameter in the sliver frame. Using the representation (3.18), the two vertical lines to be glued are always in $\mathcal{U}^+$. This induces the identification between two C^+ curves in $\mathcal{V}^+$ for the star product of the states P_α. If the curve C_α^+ described with a coordinate $z_<$ is to be glued to $C_{\alpha+\gamma}^+$ with a coordinate $z_>$, then $z_<$ and $z_>$ are related by

$$R(z_>) - R(z_<) = \frac{\gamma}{2}\,. \tag{3.20}$$

The right-hand side is the real translation parameter that relates the curves $R(C_\alpha^+)$ and $R(C_{\alpha+\gamma}^+)$.

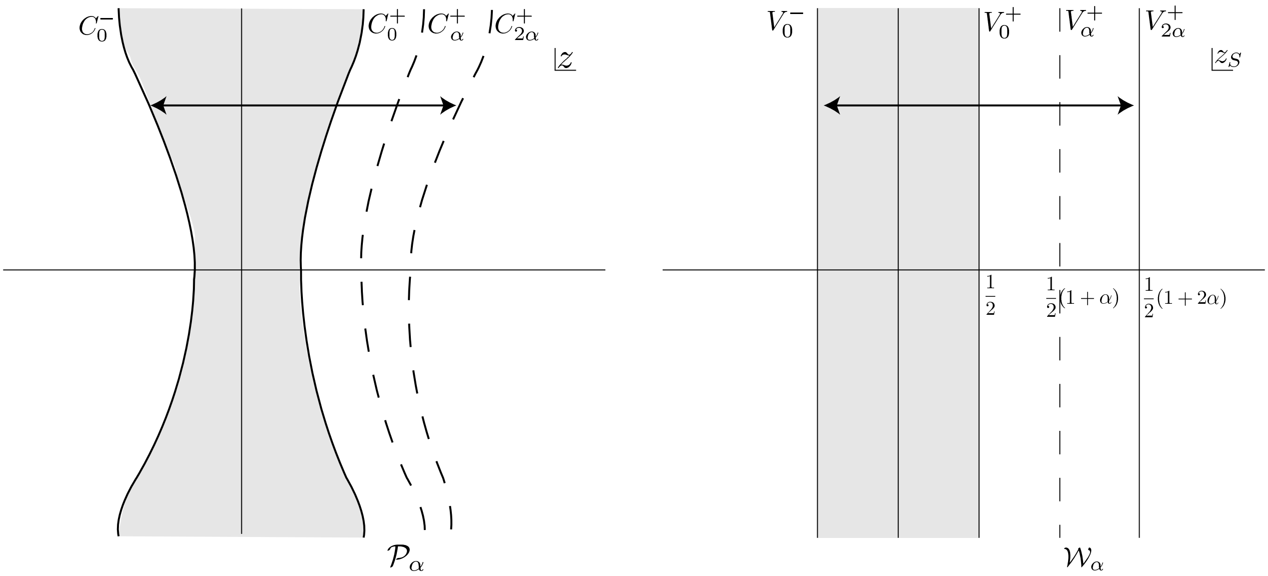

Figure 3: Left: The surface $\mathcal{P}_\alpha$ presented as the region between C_0^- and $C_{2\alpha}^+$. Right: The wedge surface $\mathcal{W}_\alpha$ presented as the region between V_0^- and $V_{2\alpha}^+$.

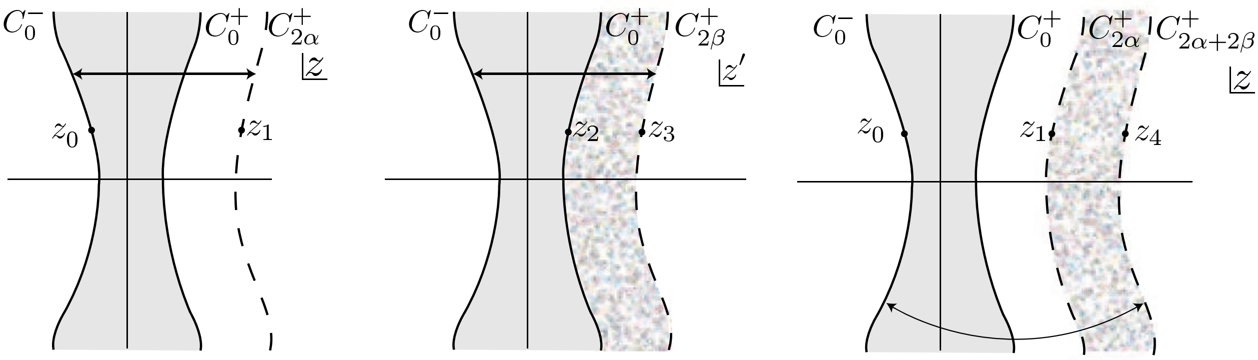

Figure 4: Left: $\mathcal{P}_\alpha$ presented as the region between C_0^- and $C_{2\alpha}^+$. Middle: $\mathcal{P}_\beta$ presented as the region between C_0^- and $C_{2\beta}^+$. Right: The surface $\mathcal{P}_{\alpha+\beta}$ obtained by gluing the complement of the coordinate disk in $\mathcal{P}_\beta$ to $\mathcal{P}_\alpha$.

We now demonstrate the abelian relation $P_\alpha * P_\beta = P_{\alpha+\beta}$ geometrically. We present $\mathcal{P}_\alpha$ as the region $(C_0^-, C_{2\alpha}^+)$ and $\mathcal{P}_\beta$ as the region $(C_0^-, C_{2\beta}^+)$, as shown in Figure 4. The surface for $P_\alpha * P_\beta$ is obtained by mapping the region $(C_0^+, C_{2\beta}^+)$ in $\mathcal{P}_\beta$ to the immediate right of $C_{2\alpha}^+ \in \mathcal{P}_\alpha$ and by gluing together $C_{2\alpha}^+ \in \mathcal{P}_\alpha$ and $C_0^+ \in \mathcal{P}_\beta$. Using coordinates $z \in \mathcal{P}_\alpha$ and $z' \in \mathcal{P}_\beta$, the gluing identification that follows from (3.20) is

$$R(z) - R(z') = \alpha\,. \tag{3.21}$$

When $z' \in C_{2\beta}^+$, we have

$$\Re(R(z)) = \Re(R(z')) + \alpha = \frac{1}{2}(1 + 2\beta) + \alpha = \frac{1}{2}(1 + 2(\alpha + \beta))\,, \tag{3.22}$$

where we made use of (3.11). It thus follows that, after gluing, the image of $C_{2\beta}^+$ in the z-plane is the curve $C_{2\alpha+2\beta}^+$. The composite surface is the region $(C_0^-, C_{2\alpha+2\beta}^+)$ shown on the right side of Figure 4. To fully confirm that this is simply $\mathcal{P}_{\alpha+\beta}$ we must examine the identification between C_0^- and $C_{2(\alpha+\beta)}^+$. Let $z_0 \in C_0^-$ and $z_1 \in C_{2\alpha}^+$ denote two points identified in $\mathcal{P}_\alpha$ (see Figure 4):

$$R(z_1) + R(-z_0) = 1 + \alpha\,. \tag{3.23}$$

Let $z_2 \in C_0^+ \in \mathcal{P}_\beta$ denote the point identified with z_1 by the following relation:

$$R(z_1) - R(z_2) = \alpha\,. \tag{3.24}$$

Let $z_3 \in C_{2\beta}^+ \in \mathcal{P}_\beta$ be the point associated with z_2 on account of having the same imaginary value after mapping by R:

$$R(z_3) - R(z_2) = \beta\,. \tag{3.25}$$

Finally, let $z_4 \in C_{2(\alpha+\beta)}^+$ in the z-plane denote the point glued to z_3:

$$R(z_4) - R(z_3) = \alpha\,. \tag{3.26}$$

The relation between z_4 and z_0 is the identification derived from the gluing procedure. To find this relation we note that the last three equations imply that $R(z_1) = R(z_4) - \beta$. Together with (3.23) we obtain $R(z_4) + R(-z_0) = 1 + \alpha + \beta$, which is the expected gluing relation on $\mathcal{P}_{\alpha+\beta}$. This completes the verification that $P_\alpha * P_\beta = P_{\alpha+\beta}$.

3.2 Abelian families for special projectors

For single-split special projectors, the maps $R(z)$ that relate them to the sliver are explicitly given by

$$R(z) = z^s\,, \tag{3.27}$$

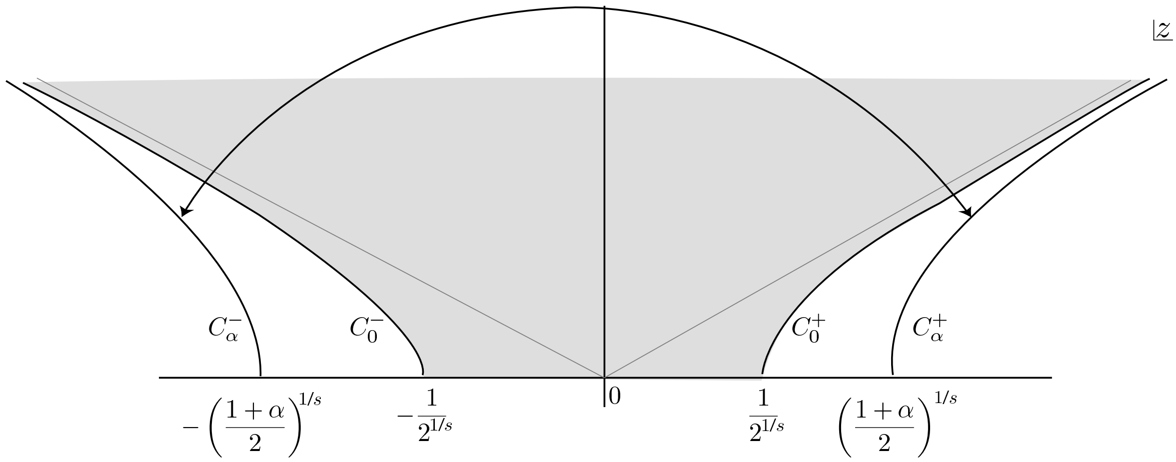

Figure 5: The surface $\mathcal{P}_\alpha$ for an arbitrary special projector with parameter s. The curves C_α^- and C_α^+ are identified via the relation $(z^+)^s + (-z^-)^s = 1 + \alpha$. The local coordinate patch is the region between C_0^- and C_0^+.

where s is the parameter appearing in the algebra $[\mathcal{L}_0, \mathcal{L}_0^\star] = s(\mathcal{L}_0 + \mathcal{L}_0^\star)$ of the special projector. We will explain (3.27) in §5.1. The full map from the complement of the coordinate disk in the projector to the complement of the coordinate disk of the sliver given by (3.7) is

$$\widehat{R}(z) = \begin{cases} z^s & \text{if } z \in \mathcal{V}^+ , \\ -(-z)^s & \text{if } z \in \mathcal{V}^- . \end{cases} \tag{3.28}$$

It follows from (3.27) that the coordinate curve C_0^+ is the s-th root of the sliver line V_0^+. Similarly C_α^+ is the s-th root of V_α^+. The surface $\mathcal{P}_\alpha$ associated with a special projector with parameter s is shown in Figure 5.

Another key feature of special projectors is that we can write the map from $\mathcal{P}_\alpha$ to $\mathbb{H}$ in terms of the map $z = f(\xi)$ that defines the projector. Recall that $f(\xi)$ maps the upper-half disk of ξ to the region (C_0^-, C_0^+) — this is $\mathcal{P}_0$ without the identification The map $f(\xi)$ is known explicitly for special projectors, as we shall review in §5.1.

The first step in constructing the map from $\mathcal{P}_\alpha$ to $\mathbb{H}$ is relating the curves C_α^+ to the curve C_0^+. From the relation (3.11) we have

$$\Re(z^s) = \frac{1}{2}(1+\alpha) \quad \text{for} \quad z \in C_\alpha^+ \quad \text{and} \quad \Re(z^s) = \frac{1}{2} \quad \text{for} \quad z \in C_0^+ . \tag{3.29}$$

It follows that C_α^+ is obtained from C_0^+ by a constant scaling! Indeed,

$$z' \in C_\alpha^+ , \; z \in C_0^+ \;\to\; z' = (1+\alpha)^{1/s} \, z . \tag{3.30}$$

Since it appears frequently later, we define the scaling function $I_{\alpha,s}$ as follows:

$$I_{\alpha,s}(z) \equiv (1+\alpha)^{1/s} z \,. \tag{3.31}$$

Because of the reflection symmetry about the imaginary axis, C_α^- is obtained from C_0^- by the same constant scaling. The identification for $\mathcal{P}_\alpha$ is also properly transformed by the scaling. Indeed, using (3.17) we have

$$(z^{+\prime})^s + (-z^{-\prime})^s = 1+\alpha \;\;\text{for } \mathcal{P}_\alpha \quad \text{and} \quad (z^{+})^s + (-z^{-})^s = 1 \;\;\text{for } \mathcal{P}_0\,, \tag{3.32}$$

and the scaling $z^{\pm\prime} = (1+\alpha)^{1/s}\, z^\pm$ relates the identifications. We thus have a full mapping of the surfaces:

$$\boxed{\mathcal{P}_\alpha = I_{\alpha,s}(\,\mathcal{P}_0\,) \;\;\text{for special projectors}\,.} \tag{3.33}$$

For a general projector, this map is difficult to obtain and does not follow directly from the knowledge of $R(z)$ and $f(\xi)$.

We now claim that the map from $\mathcal{P}_\alpha$ to $\mathbb{H}$ is given by the following function h_α:

$$h_\alpha = f_I \circ f^{-1} \circ I_{\alpha,s}^{-1}\,. \tag{3.34}$$

The function $I_{\alpha,s}^{-1}$ scales $\mathcal{P}_\alpha$ down to $\mathcal{P}_0$, with the identification applied to the boundary of $\mathcal{P}_0$. The function f^{-1} then maps $\mathcal{P}_0$ to the upper-half disk with the inherited identification. Finally, the function f_I is defined by

$$f_I(\xi) = \frac{\xi}{1-\xi^2}\,. \tag{3.35}$$

This is the function that defines the identity state: it maps the upper-half disk of ξ, with the left and right parts of the semicircle boundary identified via $\xi \sim -1/\xi$, to $\mathbb{H}$. It is then clear that h_α maps $\mathcal{P}_\alpha$ to $\mathbb{H}$.

The surface state P_α corresponding to the surface $\mathcal{P}_\alpha$ is defined by

$$\langle\, \phi, P_\alpha \,\rangle \equiv \langle\, f \circ \phi(0)\,\rangle_{\mathcal{P}_\alpha} = \langle\, f_\alpha \circ \phi(0)\,\rangle_{\mathbb{H}} \tag{3.36}$$

for any state ϕ in the Fock space. The correlation function on $\mathcal{P}_\alpha$ in the projector frame has been mapped to that on the UHP on the right-hand side, where f_α is given by

$$f_\alpha = h_\alpha \circ f = f_I \circ f^{-1} \circ I_{\alpha,s}^{-1} \circ f\,. \tag{3.37}$$

This is the expression obtained in [5]. (See (3.35) of [5].) In that work, however, the presentation of $\mathcal{P}_\alpha$ using the conformal frame of the projector was not given, and a geometric proof of the relation $P_\alpha * P_\beta = P_{\alpha+\beta}$ was not provided. The above results will be useful later in our calculations on the tachyon vacuum solutions. For a general projector, the calculation of f_α is complicated because the map from $\mathcal{P}_\alpha$ to $\mathcal{P}_0$ is nontrivial.

We conclude this section with an example. Aside from the sliver, the simplest and most familiar projector is the butterfly state. The butterfly is a special projector with $s = 2$. Recall that the conformal frame of the butterfly is defined by

$$z = f(\xi) = \frac{\xi}{\sqrt{1+\xi^2}}\,. \tag{3.38}$$

Let us see that the butterfly is related to the sliver through the reparameterization induced by

$$R(z) = z^2\,. \tag{3.39}$$

The full map (3.7) between the complements of the coordinate disks is then given by

$$z_{\mathcal{S}} = \widehat{R}(z) = \begin{cases} z^2 & \text{if } z \in \mathcal{V}^+\,, \\ -z^2 & \text{if } z \in \mathcal{V}^-\,. \end{cases} \tag{3.40}$$

Since the butterfly is a special projector with $s = 2$, the square of the coordinate curve must be a straight line or a set of straight lines [5]. Points on the coordinate curve are $f(\xi)$ for $\xi = e^{i\theta}$, so we have

$$z^2 = (f(e^{i\theta}))^2 = \frac{e^{2i\theta}}{1+e^{2i\theta}} = \frac{e^{i\theta}}{2\cos\theta} = \frac{1}{2} + \frac{i}{2}\tan\theta\,. \tag{3.41}$$

The points here span a vertical line with real part equal to $1/2$. For $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, we obtain the full vertical line so we indeed find that $\widehat{R}$ maps $C_0^+ \to V_0^+$. For $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, (3.41) shows that z^2 also spans the full vertical line with its real part equal to $1/2$. With the minus sign in the second case of (3.40), we find that $\widehat{R}$ maps $C_0^- \to V_0^-$.

If we write $z = x + i\,y$, with x and y real, it follows from the real part of (3.41) that the butterfly coordinate curve is part of the hyperbola given by

$$\Re\,(z^2) = x^2 - y^2 = \frac{1}{2}\,. \tag{3.42}$$

In fact, the full coordinate curve is the part of the hyperbola that lies on $\mathbb{H}$.

Consider now the surface $\mathcal{P}_0$, namely, the region in $\mathbb{H}$ in between C_0^- and C_0^+. Let $z_+ \in C_0^+$ and $z_- \in C_0^-$. How do we write the identification of C_0^- and C_0^+ as an analytic relation between z_- and z_+? From (3.17) we have

$$z_+^2 + z_-^2 = 1\,. \tag{3.43}$$

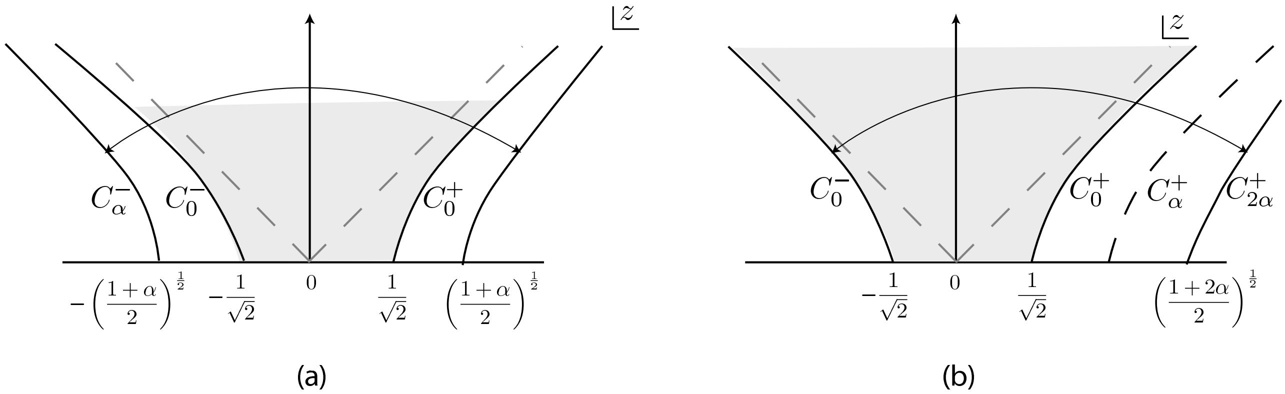

Figure 6: (a) The surface $\mathcal{P}_\alpha$ in the butterfly family. The curves C_α^- and C_α^+ are identified. The coordinate patch is that of the butterfly itself. (b) The same surface, with the complement of the coordinate patch placed completely to the right of the patch. The curves C_0^- and $C_{2\alpha}^+$ are identified.

This correctly identifies $z_- = -1/\sqrt{2}$ with $z_+ = 1/\sqrt{2}$. We can confirm (3.43) by recalling that the identification is induced by that of ξ and $-1/\xi$. Therefore the point $z_- = f(\xi_-)$ is identified with $z_+ = f(\xi_+)$ when $\xi_+ = -1/\xi_-$. This gives

$$z_-^2 = \frac{\xi_-^2}{1+\xi_-^2} = \frac{1}{1+\xi_+^2} = 1 - z_+^2 \tag{3.44}$$

in agreement with (3.43).

The surface $\mathcal{P}_\alpha$ associated with the butterfly projector is obtained by a dilation $z \to (1+\alpha)^{1/2} z$ of $\mathcal{P}_0$, as we have seen in (3.33). Under this dilation the bounding curves C_0^+ and C_0^- in (3.42) become the curves C_α^+ and C_α^- whose points satisfy

$$z \in C_\alpha^\pm \quad \to \quad \Re(z^2) = \frac{1}{2}(1+\alpha)\,. \tag{3.45}$$

Their identification is obtained from (3.43) by the dilation:

$$z_+^2 + z_-^2 = 1+\alpha\,. \tag{3.46}$$

The surface $\mathcal{P}_\alpha$ is the region between C_α^- and C_α^+. The coordinate disk can be viewed as $\mathcal{P}_0$, without identifications, inside $\mathcal{P}_\alpha$. The surface $\mathcal{P}_\alpha$ is shown in Figure 6(a).

We can use the identification (3.46) to move the region (C_α^-, C_0^-) to the right of C_α^+. Since points $z_- \in C_0^-$ satisfy $\Re(z_-^2) = 1/2$, (3.46) shows that under the identification they become

$$\Re(z_+^2) = \frac{1}{2}(1+2\alpha) \quad \to \quad z_+ \in C_{2\alpha}^+\,, \tag{3.47}$$

where we have used (3.45). The surface $\mathcal{P}_\alpha$ can therefore be described as the region between C_0^- and $C_{2\alpha}^+$, with these two curves identified via (3.46). This presentation is shown in Figure 6(b).

4 Solutions from reparameterizations

In this section we construct the tachyon vacuum solution associated with a general twist-invariant projector. We begin §4.1 with a review of the algebraic structure of Schnabl's solution. We then give a formal construction of the solution associated with a general projector using reparameterizations. In §4.2 we present the CFT description of the states ψ_α and ψ'_α for a general projector. In the last subsection we analyze the various operator insertions in more detail and geometrically confirm that they obey the expected algebraic properties.

4.1 Review of the algebraic construction

Schnabl's solution Ψ consists of two pieces and is defined by a limit:

$$\Psi = \lim_{N\to\infty}\Big[-\psi_N + \sum_{n=0}^{N}\psi'_n\Big]\,. \tag{4.1}$$

The "phantom piece" ψ_N does not contribute to inner products with states in the Fock space in the limit. Namely,

$$\lim_{N\to\infty}\langle\,\phi,\psi_N\,\rangle = 0 \tag{4.2}$$

for any state ϕ in the Fock space. On the other hand, the piece involving the sum of ψ'_n is the limit $\lambda\to 1$ of a state Ψ_λ,

$$\Psi_\lambda \equiv \sum_{n=0}^{\infty}\lambda^{n+1}\,\psi'_n\,, \tag{4.3}$$

which formally satisfies the equation of motion for all λ,

$$Q_B\Psi_\lambda + \Psi_\lambda * \Psi_\lambda = 0\,. \tag{4.4}$$

The state Ψ_λ can be formally written as a pure-gauge configuration [3] and is considered to be gauge-equivalent to $\Psi = 0$ for $|\lambda| < 1$. The equation (4.4) for any λ is equivalent to the following relations for ψ'_n with integer n:

$$\begin{aligned} Q_B\psi'_0 &= 0\,, && (4.5)\\ Q_B\psi'_n &= -\sum_{m=0}^{n-1}\psi'_m * \psi'_{n-m-1}\,, \qquad n>0\,. && (4.6)\end{aligned}$$

There is a simple algebraic construction of the states ψ'_n, which we now review. It helps to use the abstract notation of [5] even though for the time being all the operators are meant to be those associated with the sliver. The left and right parts of the operator $L^+ = L + L^\star$ are denoted by L_L^+ and L_R^+, respectively, and $L^+ = L_L^+ + L_R^+$. The operator $K \equiv \widetilde{L}^+$ is defined by $K = \widetilde{L}^+ = L_R^+ - L_L^+$. For the sliver, its explicit form derived in [2] is

$$K = \widetilde{L}^+ = \frac{\pi}{2} K_1 = \frac{\pi}{2} \left(L_1 + L_{-1} \right) . \tag{4.7}$$

The antighost operators B, $B^\star$, $B^+ = B + B^\star$, $\widetilde{B}^+ = B_R^+ - B_L^+$ are similarly defined by replacing $T(z) \to b(z)$ or $L_n \to b_n$. Thus for the sliver,

$$\widetilde{B}^+ = \frac{\pi}{2} \left(b_1 + b_{-1} \right) . \tag{4.8}$$

In this language, we can write

$$\psi_0 = C|P_1\rangle , \tag{4.9}$$

$$\psi_n = -C|P_1\rangle * |P_{n-1}\rangle * B_L^+ C|P_1\rangle , \quad n > 0 , \tag{4.10}$$

as well as

$$\psi_0' = -Q_B B_L^+ C|P_1\rangle \tag{4.11}$$

$$\psi_n' = C|P_1\rangle * |P_{n-1}\rangle * B_L^+ L_L^+ C|P_1\rangle , \quad n > 0 , \tag{4.12}$$

where the operator C is

$$C \equiv \frac{2}{\pi} c_1 . \tag{4.13}$$

Again, at this stage all objects are defined in the sliver frame. In particular, $|P_\alpha\rangle$ is the wedge state $|W_\alpha\rangle$ and $|P_1\rangle$ is just the $SL(2,R)$-invariant vacuum $|0\rangle$.

It was algebraically shown in [3] that the string fields ψ_n' defined by (4.11) and (4.12) satisfy (4.5) and (4.6). In the proof, one uses the abelian algebra $P_\alpha * P_\beta = P_{\alpha+\beta}$, standard properties of the BRST operator (Q_B is a nilpotent derivation of the star algebra and annihilates the vacuum state), as well as the following identities:

$$\widetilde{B}^+|P_1\rangle = (B_R^+ - B_L^+)\,|P_1\rangle = 0 , \tag{4.14}$$

$$\widetilde{B}^+ C|P_1\rangle = (B_R^+ - B_L^+)\, C|P_1\rangle = |P_1\rangle , \tag{4.15}$$

$$(B_R^+ \phi_1) * \phi_2 = (-1)^{\phi_1} \phi_1 * (B_L^+ \phi_2) . \tag{4.16}$$

The first two equations (4.14) and (4.15) are immediately checked using $|P_1\rangle = |0\rangle$ and the expansions (4.8) and (4.13). The identity (4.14) can also be understood as a special case of the familiar conservation laws obeyed by wedge states,

$$\begin{aligned} \widetilde{L}^+|P_\alpha\rangle &= (L_R^+ - L_L^+)|P_\alpha\rangle = 0 , \\ \widetilde{B}^+|P_\alpha\rangle &= (B_R^+ - B_L^+)|P_\alpha\rangle = 0 . \end{aligned} \tag{4.17}$$

The last identity (4.16) is obtained by observing that for any derivation $D = D_L + D_R$ one has

$$(D_R\phi_1) * \phi_2 = \,-\,(-1)^{\phi_1 \cdot D}\phi_1 * (D_L\phi_2)\,. \tag{4.18}$$

For $D = \widetilde{B}^+$ we find (4.16), while for $D = K$ we obtain

$$(L^+_R\phi_1) * \phi_2 = \phi_1 * (L^+_L\phi_2)\,. \tag{4.19}$$

Let us confirm that ψ'_n as defined in (4.12) is indeed the derivative with respect to n of the state ψ_n in (4.10). Since $|P_\alpha\rangle = e^{-\frac{\alpha}{2}L^+}|\mathcal{I}\rangle$, we have

$$\frac{d}{d\alpha}|P_\alpha\rangle = -\frac{1}{2}\,L^+|P_\alpha\rangle = \,-\,L^+_R\,|P_\alpha\rangle\,, \tag{4.20}$$

where we have used (4.17). With the help of (4.19) we find that

$$\frac{d}{dn}\psi_n = C|P_1\rangle * L^+_R|P_{n-1}\rangle * B^+_L C|P_1\rangle = C|P_1\rangle * |P_{n-1}\rangle * L^+_L B^+_L C|P_1\rangle\,, \tag{4.21}$$

as claimed. Note that L^+_L and B^+_L commute because $L^+_L = \{Q_B, B^+_L\}$ and $(B^+_L)^2 = 0$.

One can also show that the solution satisfies the gauge condition $B\Psi = 0$. The algebraic properties that guarantee this fact are

$$\begin{aligned} \{B\,, C\} = \{B^\star\,, C\} = \;0\,, \\ L\,C\,|P_1\rangle = \,-\,C\,|P_1\rangle\;, \end{aligned} \tag{4.22}$$

which follow immediately from the mode expansions on B, L, and C in the sliver frame. To show that (4.22) imply $B\psi_n = B\psi'_n = 0$, the following identities are useful. Writing $B = \frac{1}{2}(B^- + B^+_L + B^+_R)$, one can prove that

$$B(\psi_1 * \psi_2) = B\psi_1 * \psi_2 + (-1)^{\psi_1}\psi_1 * (B - B^+_L)\psi_2\,. \tag{4.23}$$

For a larger number of factors we have

$$B(\psi_1 * \psi_2 * \ldots \psi_n) = (B\psi_1) * \ldots * \psi_n + \sum_{m=2}^{n}(-)^{\sum_{k=1}^{m-1}\psi_k}\;\psi_1 * \ldots * (B - B^+_L)\psi_m * \ldots * \psi_n\,. \tag{4.24}$$

One can actually make manifest the fact that ψ'_n is annihilated by B in the following way:

$$\psi'_n = \frac{1}{n}B\Big(C\,|P_1\rangle * |P_{n-1}\rangle * (L^+_L + \tfrac{1}{n})C\,|P_1\rangle\Big)\,. \tag{4.25}$$

We have seen in the previous section that a generic single-split projector P_∞ can be related to the sliver W_∞ by a reparameterization φ as $P_\infty = U^{-1}_\varphi W_\infty$. This allowed us to construct the abelian family P_α from the wedge states by the same transformation $P_\alpha \equiv U_\varphi^{-1} W_\alpha$. We now proceed to define operators associated with P_∞ by similarity transformations of the corresponding operators associated with the sliver. From now on we use the subscript $\mathcal{S}$ to denote objects in the sliver frame, and objects without the subscript are those in the frame of P_∞. We have

$$\begin{aligned}
C &\equiv U_\varphi^{-1} C_{\mathcal{S}} \, U_\varphi \,, &(4.26)\\
L &\equiv U_\varphi^{-1} L_{\mathcal{S}} \, U_\varphi \,, &(4.27)\\
L^\star &\equiv U_\varphi^{-1} L_{\mathcal{S}}^\star \, U_\varphi \,, &(4.28)\\
L^\pm &\equiv U_\varphi^{-1} L_{\mathcal{S}}^\pm \, U_\varphi = L \pm L^\star \,, &(4.29)\\
L_L^+ &\equiv U_\varphi^{-1} (L_L^+)_{\mathcal{S}} \, U_\varphi \,, &(4.30)\\
L_R^+ &\equiv U_\varphi^{-1} (L_R^+)_{\mathcal{S}} \, U_\varphi \,, &(4.31)
\end{aligned}$$

and analogous expressions for the antighost operators B, $B^\star$, $B^\pm$, B_R^+, B_L^+. Because of the formal property (2.14), $L^\star$ in (4.28) *is* the BPZ conjugate of L in (4.27), so our notation is consistent. It is also consistent to use L_L^+ and L_R^+ in (4.30) and (4.31) since reparameterizations preserve the left/right decomposition of operators. As we will see explicitly in §4.3, the operators L_L^+ and L_R^+ are, respectively, the left and right parts of the operator L^+ defined in (4.29). It is also obvious that all the algebraic properties (4.14), (4.15), (4.16), (4.17), and (4.22) are obeyed by the operators in the frame of P_∞.

The states ψ_n associated with P_∞ are given by

$$\psi_n \equiv U_\varphi^{-1} \, \psi_{n\,\mathcal{S}} = -C|P_1\rangle * |P_{n-1}\rangle * B_L^+ C|P_1\rangle \,, \tag{4.32}$$

and ψ_n' associated with P_∞ are similarly obtained. Finally, the solution Ψ associated with P_∞ is obtained from the sliver's solution $\Psi_{\mathcal{S}}$ as

$$\Psi = U_\varphi^{-1} \Psi_{\mathcal{S}} \,. \tag{4.33}$$

Clearly, it takes the same form (4.1), with the understanding that the states ψ_n' and ψ_N are now those in the frame of P_∞.

4.2 Solutions in the CFT formulation

We now translate the above formal construction into a geometric description. In the CFT formulation, the state $\psi_{n\,\mathcal{S}}$ in the sliver frame is defined by

$$\langle\, \phi, \psi_{n\,\mathcal{S}} \,\rangle = \Big\langle\, f_{\mathcal{S}} \circ \phi(0) \;\, c(1) \int_{-V_\alpha^+} \frac{dz}{2\pi i}\, b(z) \;\, c(n+1) \,\Big\rangle_{\mathcal{W}_{n+1}} , \tag{4.34}$$

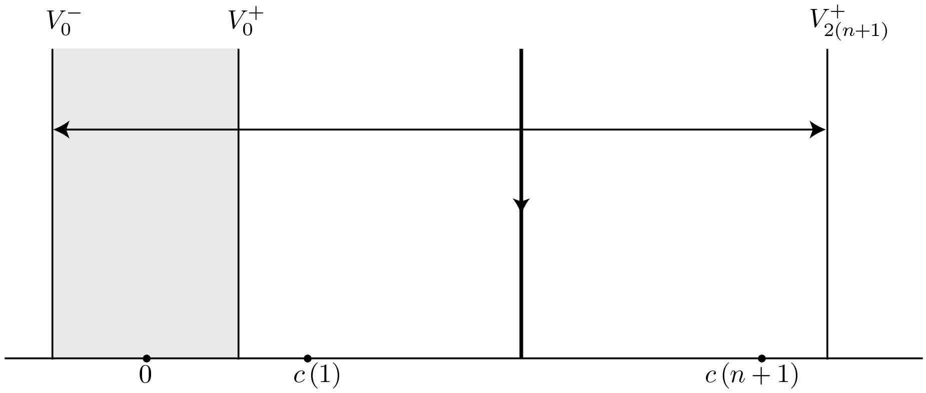

Figure 7: A diagram of the correlator on $\mathcal{W}_{n+1}$ used in (4.34) to describe the solution in the sliver frame. Shown are ghost insertions at $z_{\mathcal{S}} = 1$ and $z_{\mathcal{S}} = n+1$. The vertical line in between these insertions represents the antighost line integral.

for any state ϕ in the Fock space, where $1 < \alpha < 2\,n+1$. A pictorial representation of the correlator is given in Figure 7. The contour V_α^+ is oriented in the direction of increasing imaginary $z_{\mathcal{S}}$, and by $-V_\alpha^+$ we denote the same contour with opposite orientation. The expression (4.34) is the direct geometric translation of the algebraic expression (4.10), as explained in detail in [3]. Recall the change in the normalization of $f_{\mathcal{S}}$.

Let us apply the reparameterization U_φ^{-1} to the state $\psi_{n\,\mathcal{S}}$. Geometrically, this amounts to mapping the region $(V_0^+, V_{2(n+1)}^+)$, including the operator insertions, by the conformal transformation R^{-1} used to construct the state $|P_{n+1}\rangle$ from the wedge state $|W_{n+1}\rangle$. It is straightforward to calculate the transformations of the operator insertions in (4.34). We find that the state ψ_n associated with a general projector is given by

$$\langle\, \phi, \psi_n \,\rangle = \langle\, f \circ \phi(0)\; \mathcal{C}(1)\, \mathcal{B}\, \mathcal{C}(2\,n+1)\,\rangle_{\mathcal{P}_{n+1}} \tag{4.35}$$

for any state ϕ in the Fock space, where

$$\mathcal{C}(\alpha) \equiv R'\left(R^{-1}\left(\frac{1+\alpha}{2}\right)\right)\, c\left(R^{-1}\left(\frac{1+\alpha}{2}\right)\right)\,, \qquad \mathcal{B} \equiv \int \frac{dz}{2\pi i}\,\frac{b(z)}{R'(z)}\,. \tag{4.36}$$

The contour of the integral for $\mathcal{B}$ can be taken to be $-C_\alpha^+$ with $1 < \alpha < 2\,n+1$. (The orientation of the contour C_α^+, inherited from the orientation of V_α^+, is directed towards increasing imaginary z). In general, when $\mathcal{B}$ is located between two operators, the contour of the integral must run between the two operators. Note that $\mathcal{C}(\alpha)$ is nothing but the operator $c(z_{\mathcal{S}})$, with $z_{\mathcal{S}} = \frac{1}{2}(1+\alpha)$, expressed in the frame $z = R^{-1}(z_{\mathcal{S}})$. The argument α of $\mathcal{C}$ denotes the label of the line C_α^+ that contains the insertion. The surface and insertions for the correlator indicated in (4.35) are shown in Figure 8.

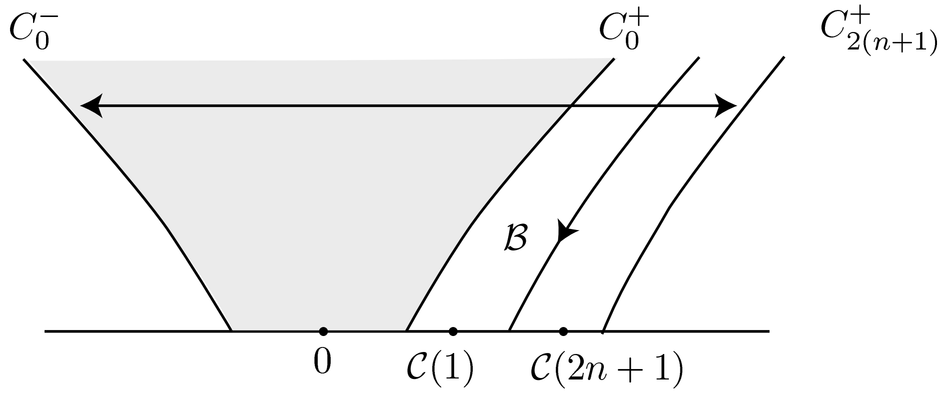

Figure 8: The surface and insertions relevant to the correlator (4.35) used to define ψ_n. The surface $\mathcal{P}_{n+1}$ includes two ghost insertions $\mathcal{C}$ and an antighost line integral $\mathcal{B}$.

This definition of ψ_n is valid for $n > 0$, and ψ_0 can be defined by the limit $n \to 0$:

$$\psi_0 \equiv \lim_{n\to 0} \psi_n \,. \tag{4.37}$$

Let us calculate ψ_0 explicitly. The anticommutation relation of $\mathcal{B}$ and $\mathcal{C}$ is given by

$$\{\, \mathcal{B}\,,\, \mathcal{C}(\alpha)\,\} = \mathcal{B}\,\mathcal{C}(\alpha) + \mathcal{C}(\alpha)\,\mathcal{B} = 1\,. \tag{4.38}$$

Note that the contour for $\mathcal{B}$ in the term $\mathcal{B}\,\mathcal{C}(\alpha)$ should be $-C_\beta^+$ with $\beta < \alpha$, and the contour for $\mathcal{B}$ in the term $\mathcal{C}(\alpha)\,\mathcal{B}$ should be $-C_\beta^+$ with $\beta > \alpha$. Using this anticommutation relation, the inner product $\langle\, \phi, \psi_n \,\rangle$ in the limit $n \to 0$ is given by

$$\lim_{n\to 0} \langle\, \phi, \psi_n \,\rangle = \langle\, f \circ \phi(0)\, \mathcal{C}(1)\,\rangle_{\mathcal{P}_1} \,. \tag{4.39}$$

This gives the CFT description of the state $\psi_0 = C|P_1\rangle$ in (4.9) for a general projector. It coincides with the state obtained by reparameterization from the sliver's ψ_0.

Another useful expression for the inner product $\langle\, \phi, \psi_n \,\rangle$ is

$$\langle\, \phi, \psi_n \,\rangle = \, - R'(R^{-1}(1))^2 \left\langle\, c(-R^{-1}(1))\ f \circ \phi(0)\ c(R^{-1}(1)) \int_{C_\alpha^+} \frac{dz}{2\pi i}\, \frac{b(z)}{\widehat{R}'(z)} \,\right\rangle_{\mathcal{P}_{n+1}} , \tag{4.40}$$

where $\alpha > 1$, and we have mapped the operator $\mathcal{C}(2\,n+1)$ to $\widehat{R}'(\widehat{R}^{-1}(-1))\, c(\widehat{R}^{-1}(-1)) = R'(R^{-1}(1))\, c(-R^{-1}(1))$ using the identification (3.17) for the surface $\mathcal{P}_{n+1}$. Note that ϕ must be Grassmann even in order for the inner product to be nonvanishing. We will use (4.40) in the next section.

Let us now consider ψ'_n. Taking a derivative of $\psi_{n\,\mathcal{S}}$ with respect to n is equivalent to an insertion of the operator

$$\int_{-V_\alpha^+} \frac{dz}{2\pi i}\, T(z) \tag{4.41}$$

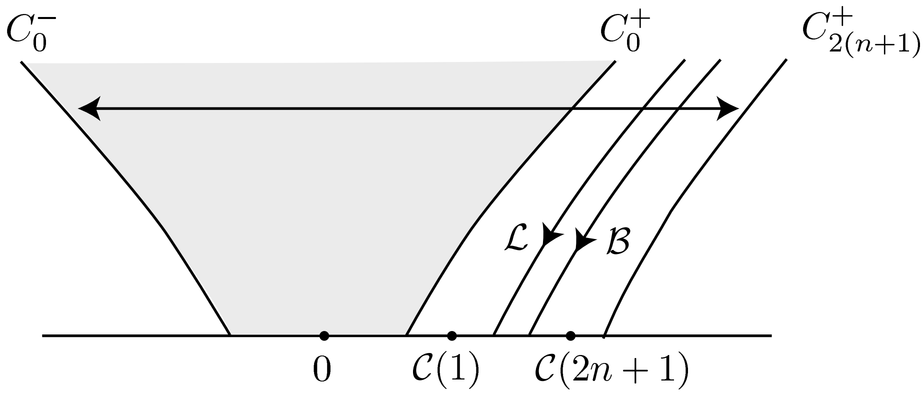

Figure 9: The surface and insertions relevant to the correlator (4.43) used to define ψ'_n. The surface $\mathcal{P}_{n+1}$ includes two ghost insertions $\mathcal{C}$, an antighost line integral $\mathcal{B}$, and a stress-tensor line integral $\mathcal{L}$.

in (4.34), with $1 < \alpha < 2\,n+1$. See [3] for more details. Since the operator is transformed by R^{-1} to

$$\mathcal{L} \equiv \int \frac{dz}{2\pi i}\,\frac{T(z)}{R'(z)}\,, \tag{4.42}$$

the geometric translation of (4.12) for a general projector is

$$\langle\, \phi, \psi'_n \,\rangle = \langle\, f \circ \phi(0)\ \mathcal{C}(1)\, \mathcal{L}\, \mathcal{B}\, \mathcal{C}(2\,n+1)\,\rangle_{\mathcal{P}_{n+1}}\,, \tag{4.43}$$

where the contour of the integral for $\mathcal{L}$ can be taken to be $-C^+_\alpha$ with $1 < \alpha < 2\,n+1$. The surface and insertions for this correlator are shown in Figure 9.

Note that $\mathcal{B}$ and $\mathcal{L}$ commute. In general, when $\mathcal{L}$ is located between two operators, the contour of the integral must run between the two operators. The definition (4.43) is valid for n in the range $n > 0$. As in the case of ψ_0, the state ψ'_0 can be defined by the limit $n \to 0$:

$$\psi'_0 = \lim_{n\to 0} \psi'_n\,. \tag{4.44}$$

Using the anticommutation relation (4.38), the inner product $\langle\, \phi, \psi'_n \,\rangle$ can be written as

$$\begin{aligned} \langle\, \phi, \psi'_n \,\rangle &= \langle\, f \circ \phi(0)\ \mathcal{C}(1)\, \mathcal{B}\, \mathcal{L}\, \mathcal{C}(2\,n+1)\,\rangle_{\mathcal{P}_{n+1}} \\ &= \langle\, f \circ \phi(0)\ \mathcal{L}\, \mathcal{C}(2\,n+1)\,\rangle_{\mathcal{P}_{n+1}} - \langle\, f \circ \phi(0)\ \mathcal{B}\, \mathcal{C}(1)\, \mathcal{L}\, \mathcal{C}(2\,n+1)\,\rangle_{\mathcal{P}_{n+1}}\,. \end{aligned} \tag{4.45}$$

It is trivial to take the limit $n \to 0$ for the first term. The limit of the second term can be calculated using the formula

$$\lim_{\epsilon\to 0} \mathcal{C}(\alpha)\, \mathcal{L}\, \mathcal{C}(\alpha+\epsilon) = \lim_{\epsilon\to 0} \mathcal{C}(\alpha)\, [\, \mathcal{L}, \mathcal{C}(\alpha+\epsilon)\,] = Q_B \cdot \mathcal{C}(\alpha)\,, \tag{4.46}$$

where $Q_B \cdot \mathcal{O}$ is the BRST transformation of $\mathcal{O}$. The inner product $\langle\, \phi, \psi_0' \,\rangle$ is thus

$$\langle\, \phi, \psi_0' \,\rangle = \langle\, f \circ \phi(0)\ \mathcal{L}\, \mathcal{C}(1) \,\rangle_{\mathcal{P}_1} - \langle\, f \circ \phi(0)\ \mathcal{B}\ Q_B \cdot \mathcal{C}(1) \,\rangle_{\mathcal{P}_1} \,. \tag{4.47}$$

This gives the geometric translation of the state $\psi_0' = -L_L^+ C|P_1\rangle + B_L^+ Q_B C|P_1\rangle = -Q_B B_L^+ C|P_1\rangle$ in (4.11) for a general projector, as we will explain further in the next subsection. The state coincides with the state obtained by reparameterization from the sliver's ψ_0'.

4.3 Operator insertions in the geometric language

The expressions of ψ_n and ψ_n' in (4.35), (4.40), and (4.43) are the central results of this section. While the solution constructed from these states are guaranteed to satisfy the equation of motion because it is related to Schnabl's solution by a reparameterization, it is also possible to confirm this directly without referring to the reparameterization. In this subsection we offer a more detailed analysis of how various operator insertions are presented in the CFT formulation. It is then straightforward to confirm that the equation of motion is satisfied using the formulas in this subsection. The techniques developed in this subsection will be useful in handling operator insertions in the conformal frame of a general projector.

Let us begin with the operator L. It is, by definition, obtained from $L_{\mathcal{S}}$ by the reparameterization φ, where φ is implicitly defined by the relation $\widehat{R}(f(t)) = f_{\mathcal{S}}(\varphi(t))$ in (2.39). The function $f_{\mathcal{S}}(t)$ becomes $f_{\mathcal{S}}(\varphi(t)) = \widehat{R}(f(t))$, and thus L in the general projector frame $z = f(\xi)$ is given by $L_{\mathcal{S}}$ in the sliver frame $z_{\mathcal{S}} = f_{\mathcal{S}}(\xi_{\mathcal{S}})$ by the conformal transformation $z = \widehat{R}^{-1}(z_{\mathcal{S}})$:

$$\begin{aligned} L \equiv U_\varphi^{-1} L_{\mathcal{S}}\, U_\varphi = U_\varphi^{-1} \left(\int_{V_0^+ - V_0^-} \frac{dz_{\mathcal{S}}}{2\pi i}\, z_{\mathcal{S}}\, T(z_{\mathcal{S}}) \right) U_\varphi = \int_{C_0^+ - C_0^-} \frac{dz}{2\pi i}\, \frac{\widehat{R}(z)}{\widehat{R}'(z)}\, T(z) \\ = \int_{C_0^+} \frac{dz}{2\pi i}\, \frac{R(z)}{R'(z)}\, T(z) + \int_{C_0^-} \frac{dz}{2\pi i}\, \frac{R(-z)}{R'(-z)}\, T(z) \,. \end{aligned} \tag{4.48}$$

In obtaining the second line we made use of (3.7). For special projectors, $R(z) = z^s$ and the expression for L simplifies to

$$L = \frac{1}{s} \oint \frac{dz}{2\pi i}\, z\, T(z) = \frac{\mathcal{L}_0}{s} \,. \tag{4.49}$$

The operator $\mathcal{L}_0$ is the Virasoro zero mode in the frame of the projector. This is the definition of L given in [5]. If the projector is not special, (4.49) does not hold. Generically the expansion of L in ordinary Virasoro operators L_n contains terms with negative n.

The inner product $\langle\, L\,\phi, P_\alpha \,\rangle$ for any state ϕ in the Fock space is given by

$$\begin{aligned}\langle\, L\,\phi, P_\alpha \,\rangle &= \left\langle \int_{C_0^+ - C_0^-} \frac{dz}{2\pi i}\, \frac{\widehat{R}(z)}{\widehat{R}'(z)}\, T(z)\, f\circ\phi(0) \right\rangle_{\mathcal{P}_\alpha} \\ &= \left\langle \int_{C_0^+} \frac{dz}{2\pi i}\, \frac{R(z)}{R'(z)}\, T(z)\, f\circ\phi(0) \right\rangle_{\mathcal{P}_\alpha} + \left\langle \int_{C_0^-} \frac{dz}{2\pi i}\, \frac{R(-z)}{R'(-z)}\, T(z)\, f\circ\phi(0) \right\rangle_{\mathcal{P}_\alpha} .\end{aligned} \tag{4.50}$$

This provides the CFT representation of the state $L^\star\,|P_\alpha\rangle$ because $\langle\, L\,\phi, P_\alpha \,\rangle = \langle\, \phi, L^\star\, P_\alpha \,\rangle$.

Next, we wish to derive a representation of $L\,|P_\alpha\rangle$. To this end, we need an expression for $\langle\, L^\star\,\phi, P_\alpha \,\rangle$. While it is possible to construct $L^\star$ from $L^\star_{\mathcal{S}}$ by the reparameterization φ as in (4.48), it is instructive to understand BPZ conjugation directly on the surface $\mathcal{P}_\alpha$. BPZ conjugation is, by definition, performed by the map $I(\xi) = -1/\xi$ in the ξ coordinate. For an operator in the z-plane, BPZ conjugation requires mapping the operator to the ξ coordinate, performing the conjugation, and mapping the resulting operator back to the z coordinate. The full conformal transformation is then

$$z' = I_f(z) = f\circ I\circ f^{-1}(z)\,, \quad I(\xi) = -1/\xi\,. \tag{4.51}$$

This relation between z' and z is nothing but the identification between z_+ and z_- for $\mathcal{P}_0$, namely,

$$R(z_+) + R(-z_-) = 1\,. \tag{4.52}$$

Let us apply this geometric understanding of BPZ conjugation to the operator L. The map I_f transforms the two integrals in (4.48) as follows:

$$\begin{aligned}\int_{C_0^+} \frac{dz_+}{2\pi i}\, \frac{R(z_+)}{R'(z_+)}\, T(z_+) \quad &\to \; -\int_{C_0^-} \frac{dz_-}{2\pi i}\, \frac{R(-z_-)}{R'(-z_-)}\, T(z_-) + \int_{C_0^-} \frac{dz_-}{2\pi i}\, \frac{T(z_-)}{R'(-z_-)}\,, \\ \int_{C_0^-} \frac{dz_-}{2\pi i}\, \frac{R(-z_-)}{R'(-z_-)}\, T(z_-) \quad &\to \; -\int_{C_0^+} \frac{dz_+}{2\pi i}\, \frac{R(z_+)}{R'(z_+)}\, T(z_+) + \int_{C_0^+} \frac{dz_+}{2\pi i}\, \frac{T(z_+)}{R'(z_+)}\,.\end{aligned} \tag{4.53}$$

Thus the inner product $\langle\, L^\star\,\phi, P_\alpha \,\rangle$ is given by

$$\langle\, L^\star\,\phi, P_\alpha \,\rangle = \; -\left\langle \int_{C_0^+ - C_0^-} \frac{dz}{2\pi i}\, \frac{\widehat{R}(z)}{\widehat{R}'(z)}\, T(z)\, f\circ\phi(0) \right\rangle_{\mathcal{P}_\alpha} + \left\langle \int_{C_0^+ + C_0^-} \frac{dz}{2\pi i}\, \frac{T(z)}{\widehat{R}'(z)}\, f\circ\phi(0) \right\rangle_{\mathcal{P}_\alpha} . \tag{4.54}$$

Recalling (4.50), we can write

$$\langle\, L^\star\,\phi, P_\alpha \,\rangle = \; -\,\langle\, L\,\phi, P_\alpha \,\rangle + \left\langle \int_{C_0^+ + C_0^-} \frac{dz}{2\pi i}\, \frac{T(z)}{\widehat{R}'(z)}\, f\circ\phi(0) \right\rangle_{\mathcal{P}_\alpha} . \tag{4.55}$$

It immediately follows that

$$\langle\, L^{+}\phi, P_\alpha\,\rangle = \langle\, (\, L + L^\star\,)\,\phi, P_\alpha\,\rangle = \Big\langle \int_{C_0^+ + C_0^-} \frac{dz}{2\pi i}\,\frac{T(z)}{\widehat{R}'(z)}\, f\circ\phi(0)\,\Big\rangle_{\mathcal{P}_\alpha}, \tag{4.56}$$

and thus the operator L^+ is

$$L^{+} = \int_{C_0^+} \frac{dz}{2\pi i}\,\frac{T(z)}{R'(z)} + \int_{C_0^-} \frac{dz}{2\pi i}\,\frac{T(z)}{R'(-z)}\,. \tag{4.57}$$

From these expressions, we easily confirm the algebra $[\,L, L^\star\,] = L + L^\star$,

$$\begin{aligned}
[\,L, L^\star\,] = [\,L, L + L^\star\,] &= \int_{C_0^+ + C_0^-} \frac{dw}{2\pi i}\,\frac{1}{\widehat{R}'(w)} \oint \frac{dz}{2\pi i}\,\frac{\widehat{R}(z)}{\widehat{R}'(z)}\,T(z)\,T(w) \\
&= \int_{C_0^+ + C_0^-} \frac{dw}{2\pi i}\,\frac{T(w)}{\widehat{R}'(w)} = L + L^\star\,,
\end{aligned} \tag{4.58}$$

where the contour of the integral of z encircles w counterclockwise, and we have neglected surface terms of the form $\widehat{R}(w)\,T(w)/\widehat{R}'(w)^2$ for integration by parts with respect to w. Whether or not the surface terms vanish should be checked for a given $\widehat{R}(z)$ by evaluating them in a coordinate where the midpoint of the open string is located at a finite point.

We now consider the operators L_L^+ and L_R^+. Since C_0^- and C_0^+ are respectively the left and right parts of the coordinate curve, the expression in (4.56) splits as follows:

$$\begin{aligned}
\langle\, L_R^+\,\phi, P_\alpha\,\rangle &= \Big\langle \int_{C_0^+} \frac{dz}{2\pi i}\,\frac{T(z)}{R'(z)}\, f\circ\phi(0)\,\Big\rangle_{\mathcal{P}_\alpha}, \\
\langle\, L_L^+\,\phi, P_\alpha\,\rangle &= \Big\langle \int_{C_0^-} \frac{dz}{2\pi i}\,\frac{T(z)}{R'(-z)}\, f\circ\phi(0)\,\Big\rangle_{\mathcal{P}_\alpha}.
\end{aligned} \tag{4.59}$$

The BPZ conjugation map I_f acts as

$$I_f:\quad \int_{C_0^+} \frac{dz_+}{2\pi i}\,\frac{T(z_+)}{R'(z_+)} \to \int_{C_0^-} \frac{dz_-}{2\pi i}\,\frac{T(z_-)}{R'(-z_-)}\,, \tag{4.60}$$

so we see

$$(L_R^+)^\star = L_L^+\,. \tag{4.61}$$

Since BPZ conjugation is an involution, we also have $(L_L^+)^\star = L_R^+$.

Using the presentation of $\mathcal{P}_\alpha$ as the region between C_0^- and $C_{2\alpha}^+$ and recalling that these curves are identified by (3.17), we can rewrite $\langle\, L_L^+\,\phi, P_\alpha\,\rangle$ in (4.59) as

$$\langle\, L_L^+\,\phi, P_\alpha\,\rangle = \Big\langle \int_{C_{2\alpha}^+} \frac{dz}{2\pi i}\,\frac{T(z)}{R'(z)}\, f\circ\phi(0)\,\Big\rangle_{\mathcal{P}_\alpha}. \tag{4.62}$$

Since $(L_R^+)^\star = L_L^+$ and $(L_L^+)^\star = L_R^+$, the inner products $\langle\, \phi, L_R^+ P_\alpha \,\rangle$ and $\langle\, \phi, L_L^+ P_\alpha \,\rangle$ are given by

$$\begin{aligned}\langle\, \phi, L_R^+ P_\alpha \,\rangle &= \left\langle\, f \circ \phi(0) \int_{C_{2\alpha}^+} \frac{dz}{2\pi i} \frac{T(z)}{R'(z)} \,\right\rangle_{\mathcal{P}_\alpha},\\ \langle\, \phi, L_L^+ P_\alpha \,\rangle &= \left\langle\, f \circ \phi(0) \int_{C_0^+} \frac{dz}{2\pi i} \frac{T(z)}{R'(z)} \,\right\rangle_{\mathcal{P}_\alpha}.\end{aligned} \tag{4.63}$$

We see that the states $L_R^+|P_\alpha\rangle$ and $L_L^+|P_\alpha\rangle$ are both represented as the region between C_0^+ and $C_{2\alpha}^+$ with the same operator inserted on different locations: it is on the right edge for $L_R^+|P_\alpha\rangle$ and on the left edge for $L_L^+|P_\alpha\rangle$. Since there are no operator insertions in the region between C_0^+ and $C_{2\alpha}^+$, the contour $C_{2\alpha}^+$ can be deformed to C_0^+, and we confirm that the states are the same:

$$L_R^+|P_\alpha\rangle = L_L^+\,|P_\alpha\rangle\,. \tag{4.64}$$

Let us next consider the star multiplication of states with insertions of L_R^+ or L_L^+. We take $P_\alpha * (L_L^+ P_\beta)$ as an example, but the generalization to other cases is straightforward. The operator L_L^+ of $L_L^+ P_\beta$ is represented by an integral over C_0^+ on $\mathcal{P}_\beta$ in (4.63). For the gluing of the star product we need the identification of curves in two different coordinate systems. A curve C_q^+ in the $z_<$ coordinate is mapped to $C_{q+\gamma}^+$ in the $z_>$ coordinate when $z_<$ and $z_>$ are related by

$$R(z_>) = R(z_<) + \frac{\gamma}{2}\,. \tag{4.65}$$

Under this identification the operator insertion in (4.63) takes the same form in the two coordinates:

$$\int_{C_q^+} \frac{dz_<}{2\pi i} \frac{T(z_<)}{R'(z_<)} = \int_{C_{q+\gamma}^+} \frac{dz_>}{2\pi i} \frac{T(z_>)}{R'(z_>)}\,. \tag{4.66}$$

The operator integrated over C_0^+ on $\mathcal{P}_\beta$ is thus mapped to the same operator integrated over $C_{2\alpha}^+$ on the surface $\mathcal{P}_{\alpha+\beta} = (C_0^-, C_{2\alpha+2\beta}^+)$ for the star product $P_\alpha * (L_L^+ P_\beta)$. It follows from the first equation in (4.63) that the star product can also be interpreted as $(L_R^+ P_\alpha) * P_\beta$. We have thus shown that

$$(L_R^+ P_\alpha) * P_\beta = P_\alpha * (L_L^+ P_\beta)\,. \tag{4.67}$$

The antighost field $b(z)$ transforms in the same way as the energy-momentum tensor $T(z)$. Therefore the formulas we have derived for the energy-momentum tensor based on its transformation properties also apply to the antighost. The equations in (4.59), for example, become

$$\begin{aligned}
\langle\, B_R^+\,\phi, P_\alpha\,\rangle &= \left\langle\, \int_{C_0^+} \frac{dz}{2\pi i}\, \frac{b(z)}{R'(z)}\, f\circ\phi(0)\,\right\rangle_{\mathcal{P}_\alpha}\,,\\
\langle\, B_L^+\,\phi, P_\alpha\,\rangle &= \left\langle\, \int_{C_0^-} \frac{dz}{2\pi i}\, \frac{b(z)}{R'(-z)}\, f\circ\phi(0)\,\right\rangle_{\mathcal{P}_\alpha} = \left\langle\, \int_{C_{2\alpha}^+} \frac{dz}{2\pi i}\, \frac{b(z)}{R'(z)}\, f\circ\phi(0)\,\right\rangle_{\mathcal{P}_\alpha}\,,
\end{aligned} \tag{4.68}$$

and the equations in (4.63) become

$$\begin{aligned}
\langle\, \phi, B_R^+ P_\alpha\,\rangle &= \left\langle\, f\circ\phi(0)\, \int_{C_{2\alpha}^+} \frac{dz}{2\pi i}\, \frac{b(z)}{R'(z)}\,\right\rangle_{\mathcal{P}_\alpha}\,,\\
\langle\, \phi, B_L^+ P_\alpha\,\rangle &= \left\langle\, f\circ\phi(0)\, \int_{C_0^+} \frac{dz}{2\pi i}\, \frac{b(z)}{R'(z)}\,\right\rangle_{\mathcal{P}_\alpha}\,.
\end{aligned} \tag{4.69}$$

We also have the analogs of (4.61), (4.64), and (4.67)

$$(B_R^+)^\star = B_L^+\,, \qquad (B_L^+)^\star = B_R^+\,, \tag{4.70}$$

$$\widetilde{B}^+|P_\alpha\rangle = (B_R^+ - B_L^+)\,|P_\alpha\rangle = 0\,, \tag{4.71}$$

$$(B_R^+ P_\alpha) * P_\beta = P_\alpha * (B_L^+ P_\beta)\,. \tag{4.72}$$

Finally let us examine the operator C. As we have discussed in the calculation of ψ_0 in §4.2, the state $C|P_1\,\rangle$ for a general projector is given by

$$\langle\, \phi, CP_1\,\rangle = \langle\, f\circ\phi(0)\,\mathcal{C}(1)\,\rangle_{\mathcal{P}_1} = R'(R^{-1}(1))\,\langle\, f\circ\phi\; c(R^{-1}(1))\,\rangle_{\mathcal{P}_1} \tag{4.73}$$

for any state ϕ in the Fock space, where $\mathcal{P}_1$ is represented by the region between C_0^- and C_2^+. Let us confirm that $C|P_1\rangle$ satisfies the relation (4.15). We need to show that $\langle\, \phi, \widetilde{B}^+ CP_1\,\rangle = \langle\, \phi, P_1\,\rangle$ for any state ϕ in the Fock space. Since ϕ must be Grassmann odd in order to have a nonvanishing inner product, there is an extra minus sign in taking the BPZ conjugate of $\widetilde{B}^+$, and we have

$$\langle\, \phi, \widetilde{B}^+ CP_1\,\rangle = \langle\, \phi, (B_R^+ - B_L^+)\,CP_1\,\rangle = -\langle\, (B_R^+ - B_L^+)^\star\,\phi, CP_1\,\rangle = \langle\, (B_R^+ - B_L^+)\,\phi, CP_1\,\rangle\,. \tag{4.74}$$

The relevant correlation function can be written using (4.68), and it can be evaluated as follows:

$$\begin{aligned}
\langle\, (B_R^+ - B_L^+)\,\phi, CP_1\,\rangle &= R'(R^{-1}(1))\left\langle\, \int_{C_0^+ - C_2^+} \frac{dz}{2\pi i}\, \frac{b(z)}{R'(z)}\, f\circ\phi(0)\, c(R^{-1}(1))\,\right\rangle_{\mathcal{P}_1}\\
&= R'(R^{-1}(1))\left\langle\, f\circ\phi(0)\,\left[\oint \frac{dz}{2\pi i}\, \frac{b(z)}{R'(z)}\, c(R^{-1}(1))\right]\right\rangle_{\mathcal{P}_1} = \langle\, f\circ\phi(0)\,\rangle_{\mathcal{P}_1}\,,
\end{aligned} \tag{4.75}$$

where the contour of the integral in the last line encircles $z = R^{-1}(1)$ counterclockwise. This concludes the confirmation that $\widetilde{B}^+ C|P_1\rangle = |P_1\rangle$.

5 Operator construction of the solution

In this section we give an explicit operator construction of the solution Ψ for the most general single-split special projector for arbitrary value of the reparameterization parameter β introduced in (3.9). We begin in §5.1 with a discussion of single-split special projectors. They form a "hypergeometric collection," indexed by a parameter $s \geq 1$. Then in §5.2 we derive an operator expression for the state ψ_n, the key ingredient of Ψ in (1.5). The result, given in (5.50), takes the form of normal-ordered operators acting on the $SL(2,R)$-invariant vacuum. It holds for any projector in the hypergeometric collection.

5.1 The hypergeometric collection

In a previous paper [5], a family of special projectors with a parameter $s \geq 1$ was introduced. It was demanded that the vector field $v_{\mathcal{L}_{-s}}$ associated with the Virasoro operator $\mathcal{L}_{-s}$ in the frame $z = \tilde{f}(\xi)$ take the form:[7]

$$v_{\mathcal{L}_{-s}}(\tilde{f}) \equiv \frac{s}{(\tilde{f}^s)'} = \frac{(1+\xi^2)^s}{\xi^{s-1}}\,, \tag{5.1}$$

or, equivalently,

$$\frac{d\tilde{f}^s}{d\xi} = \frac{s\xi^{s-1}}{(1+\xi^2)^s}\,. \tag{5.2}$$

By integrating this differential equation, $\tilde{f}(\xi)$ was found to be

$$\tilde{f}(\xi) = \xi\left({}_2F_1\left[\frac{s}{2}, s\,; 1+\frac{s}{2}\,; -\xi^2\right]\right)^{1/s}\,. \tag{5.3}$$

It turns out that for even s the operator $\mathcal{L}_{-s}$ is proportional to L^+ while for each odd s it is proportional to $K = L_R^+ - L_L^+$. More precisely, we found that

$$q(s)\,\mathcal{L}_{-s} = \begin{cases} L^+ & \text{for } s \text{ even}\,, \\ K & \text{for } s \text{ odd}\,, \end{cases} \qquad \text{with} \quad q(s) = \frac{\Gamma(s/2+1)\Gamma(s/2)}{\Gamma(s+1)}\,. \tag{5.4}$$

It will be convenient to fix the normalization of $\tilde{f}(\xi)$ by introducing a rescaled $f(\xi)$ with $f(\xi=1) = 2^{-1/s}$. To implement this, we simply take

$$f(\xi) = 2^{-\frac{1}{s}}\,\frac{\tilde{f}(\xi)}{\tilde{f}(1)} = 2^{-\frac{1}{s}}\,\xi\left(\frac{{}_2F_1\left[\frac{s}{2}, s\,; 1+\frac{s}{2}\,; -\xi^2\right]}{{}_2F_1\left[\frac{s}{2}, s\,; 1+\frac{s}{2}\,; -1\right]}\right)^{1/s}\,. \tag{5.5}$$

[7] We reserve the use of f for the map with a different normalization. The map $\tilde{f}(\xi)$ here corresponds to $f(\xi)$ of [5].

Noting that

$$
{}_2F_1\left[\frac{s}{2}, s\,; 1+\frac{s}{2}\,; -1\right] \equiv \alpha(s) = \frac{2^{-s}\sqrt{\pi}\,\Gamma[1+\frac{s}{2}]}{\Gamma[\frac{1}{2}+\frac{s}{2}]}\,, \tag{5.6}
$$

a short computation shows that with the new normalization

$$
\frac{1}{s}\,\mathcal{L}_{-s} = \begin{cases} L^{+} & \text{for } s \text{ even}\,, \\ K & \text{for } s \text{ odd}\,. \end{cases} \tag{5.7}
$$

This means that in the z-coordinate of the projector we have

$$
\frac{1}{s}\,\frac{1}{z^{s-1}} = \begin{cases} v^{+}(z) & \text{for } s \text{ even}\,, \\ \epsilon(t(z))\,v^{+}(z) & \text{for } s \text{ odd}\,, \end{cases} \qquad \text{with} \quad z = f(t)\,,\ t = e^{i\theta}\,. \tag{5.8}
$$

In here we have used the step function $\epsilon(t)$ defined in [5], eqn. (2.37). By definition, the vector v corresponding to $L = \mathcal{L}_0/s$ is

$$
v = \frac{1}{s}\,z\,. \tag{5.9}
$$

It now follows from $v + v^{\star} = v^{+}$ that

$$
v^{\star}(z) = \begin{cases} \frac{1}{s}\cdot\frac{1-z^s}{z^{s-1}} & \text{for } s \text{ even}\,, \\ \frac{1}{s}\cdot\frac{\epsilon-z^s}{z^{s-1}} & \text{for } s \text{ odd}\,. \end{cases} \tag{5.10}
$$

The hypergeometric conformal frames are projectors for all real $s \geq 1$: $f(i) = \infty$. Moreover the midpoint $\xi = i$ is the only singular point, so the projectors are single-split. These properties and the precise shape of the coordinate curve can be deduced from the differential equation (5.2). A little algebra gives

$$
\frac{dF(\theta)}{d\theta} = \frac{i\,s}{2^{s+1}\tilde{f}(1)^s}\,\frac{1}{(\cos\theta)^s}\,, \qquad F(\theta) \equiv (f(e^{i\theta}))^s\,. \tag{5.11}
$$

By twist symmetry it is sufficient to consider the part of the curve with $0 \leq \theta \leq \pi/2$. The differential equation (5.11) must be supplemented with the initial condition $F(0) = f(1)^s = 1/2$. Since the right-hand side of (5.11) is purely imaginary we see at once that $\Re(F(\theta)) = 1/2$ for $0 \leq \theta < \pi/2$. It follows also that for $s \geq 1$, $\Im(F(\theta))$ is a monotonically increasing function in the interval $0 \leq \theta < \pi/2$ with $\lim_{\theta\to\pi/2^-}\Im(F(\theta)) = +\infty$. We recognize $\mathcal{F}_0 = \{F(\theta)\,|0 \leq \theta < \pi/2\}$ as the vertical line $V_0^{+} = \{z_{\mathcal{S}}\,|\,\Re(z_{\mathcal{S}}) = 1/2)\}$, the positive part of the sliver's coordinate curve. We conclude that the reparameterization mapping the hypergeometric projector with $s > 1$ to the sliver is simply

$$
z \to z_{\mathcal{S}} = R(z) = z^s\,, \quad \Re z > 0\,, \tag{5.12}
$$

a fundamental fact that we had so far claimed without proof.

It seems to us plausible that the hypergeometric collection contains all the single-split special projectors. It was shown in [5] (section 7.2) that for a conformal frame to be special the function $z_{\mathcal{S}} = F(\theta)$, $0 \le \theta \le \pi/2$, needs to be piece-wise linear in the $z_{\mathcal{S}}$-plane. On the other hand we also saw in [5] (section 7.3) that corners in $\mathcal{F}_0$ seem to lead to operators K that fail to kill the identity, thus violating one of the conditions required to have a special projector. If corners are not allowed anywhere, the intersection of $\mathcal{F}_0$ with the real line must be orthogonal and then $\mathcal{F}_0 = V_0^+$, up to a real scaling constant. This would imply that all single-split projectors are in the hypergeometric collection.

For integer s the hypergeometric function can be expressed in terms of elementary functions. For the first few integer values one finds

$$\begin{aligned}
s=1: &\quad f(\xi) = \frac{2}{\pi}\arctan\xi\,,\\
s=2: &\quad f(\xi) = \frac{\xi}{\sqrt{1+\xi^2}}\,,\\
s=3: &\quad f(\xi) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{3}}\Big(\arctan\xi - \frac{\xi(1-\xi^2)}{(1+\xi^2)^2}\Big)^{1/3}\,,\\
s=4: &\quad f(\xi) = x\Big(\frac{3+x^2}{(1+x^2)^3}\Big)^{1/4}\,,\\
s=5: &\quad f(\xi) = \Big(\frac{2}{\pi}\Big)^{\frac{1}{5}}\Big(\arctan\xi - \frac{\xi(1-\xi^2)(3+14\xi^2+3\xi^4)}{3(1+\xi^2)^4}\Big)^{1/5}\,,\\
s=6: &\quad f(\xi) = x\Big(\frac{10+5x^2+x^4}{(1+x^2)^5}\Big)^{1/6}\,.
\end{aligned} \tag{5.13}$$

For $s=1$ we recover the sliver frame with a scaling. For $s=2$ we recover the butterfly. For $s=3$ we recover the projector in (7.56) of [5]. For $s=4$ we have the projector with $a=4/3$ in (6.3) of [5].

For arbitrary s, a series expansion gives $L = \mathcal{L}_0/s$ with a simple analytic form:

$$\begin{aligned}
\mathcal{L}_0 &= L_0 + 2\sum_{k=1}^{\infty}\frac{s!!}{(s-2k)!!}\frac{s!!}{(s+2k)!!}L_{2k}\\
&= L_0 + \frac{2s}{2+s}L_2 + \frac{2s(s-2)}{(2+s)(4+s)}L_4 + \frac{2s(s-2)(s-4)}{(2+s)(4+s)(6+s)}L_6 + \ldots
\end{aligned} \tag{5.14}$$

For even s the operator L contains a finite number of terms and therefore so does L^+. This is consistent with (5.7), since according to (5.1) $\mathcal{L}_{-s}$ involves a finite number of operators for any integer s.

5.2 The solution in operator form

To obtain the operator representation of the solution we will begin with equation (4.40). For notational clarity it is useful to introduce the definition

$$r \equiv R^{-1}(1)\,, \quad \text{or} \quad R(r) = 1\,. \tag{5.15}$$

Moreover, letting $n \to n-2$, we have that (4.40) gives

$$\langle\, \phi, \psi_{n-2}\,\rangle = \;-\,(R'(r))^2 \left\langle\, c(-r)\;\, f\circ\phi(0)\;\, c(r) \int_{C_\gamma^+} \frac{dz}{2\pi i}\,\frac{b(z)}{R'(z)}\,\right\rangle_{\mathcal{P}_{n-1}}, \tag{5.16}$$

with $1 < \gamma \leq n-1$. The surface $\mathcal{P}_{n-1}$ in this correlator is defined by the reparameterization function R. Our goal is to obtain a formula for the state ψ_{n-2} as a string of operators acting on the vacuum. The operators must be normal ordered so that evaluation in the level expansion is possible.

In order to incorporate the reparameterizations that act within the family of surface states associated with a projector we take R to be β-dependent as in (3.9),

$$R_\beta(z) = e^{-2\beta}\Big(R_0(z) - \frac{1}{2}\Big) + \frac{1}{2}\,, \tag{5.17}$$

where R_0 is the "original" function and R_β the function obtained by reparameterization. For generic projectors, the state ψ_{n-2} can be evaluated explicitly only if certain conformal maps are known. For the case of special projectors in the hypergeometric collection, full and explicit evaluation is possible. Our result is an operator formula for ψ_{n-2} that depends on the parameter s of the special projector and the parameter β in (5.17).

5.2.1 Reparameterizations within a family

Let us begin with some preparatory results concerning the relations between operators and surfaces defined by R_β and those defined by R_0. Using (5.17) one can readily verify that

$$\frac{R_\beta(z)}{R'_\beta(z)} = \frac{R_0(z)}{R'_0(z)} + \frac{1}{2}\Big(e^{2\beta} - 1\Big)\frac{1}{R'_0(z)}\,. \tag{5.18}$$

Letting L, L^* denote operators defined by R and $\bar{L}, \bar{L}^\star$ denote operators defined by R_0, equation (4.48) gives

$$\begin{aligned} L &= \int_{C_0^+} \frac{dz}{2\pi i}\,\frac{R(z)}{R'(z)}\,T(z) + \int_{C_0^-} \frac{dz}{2\pi i}\,\frac{R(-z)}{R'(-z)}\,T(z)\,, \\ &= \bar{L} + \frac{1}{2}\Big(e^{2\beta} - 1\Big)\Big[\, \int_{C_0^+} \frac{dz}{2\pi i}\,\frac{T(z)}{R'_0(z)} + \int_{C_0^-} \frac{dz}{2\pi i}\,\frac{T(z)}{R'_0(-z)}\,\Big]\,. \end{aligned} \tag{5.19}$$

We have therefore obtained

$$L = \bar{L} + \frac{1}{2}\big(e^{2\beta} - 1\big)(\bar{L} + \bar{L}^\star)\,. \tag{5.20}$$

Analogous relations hold for the operators associated with the antighost field $b(z)$.

It is interesting to examine L for some special values of β. As $\beta = 0$, we get $L = \bar{L}$. As β becomes arbitrarily large and positive L becomes proportional to $\bar{L}^+$:

$$L \;\to\; \frac{1}{2}e^{2\beta}(\bar{L} + \bar{L}^\star)\,, \quad \text{as} \quad \beta \to \infty\,. \tag{5.21}$$

As β becomes arbitrarily large and negative L approaches $\bar{L}^-$:

$$L \;\to\; \frac{1}{2}(\bar{L} - \bar{L}^\star)\,, \quad \text{as} \quad \beta \to -\infty\,. \tag{5.22}$$

The transition from R_0 to R can be viewed as a reparameterization, as discussed around equation (3.9). Indeed, a short calculation gives

$$L = e^{\beta(\bar{L}-\bar{L}^\star)}\,\bar{L}\,e^{-\beta(\bar{L}-\bar{L}^\star)}\,, \tag{5.23}$$

showing that $\bar{L} - \bar{L}^\star$ generates the reparameterization that maps the R_0-based operators to the R-based operators.

Let us compare surfaces defined by R_β and surfaces defined by R_0. Since R maps C_α^+ to V_α^+, we find

$$z \in C_\alpha^+ \quad \to \quad \Re(R_\beta(z)) = \frac{1}{2}(1+\alpha)\,. \tag{5.24}$$

For such z we also have

$$\Re(R_0(z)) = e^{2\beta}\Big(\frac{1}{2}(1+\alpha) - \frac{1}{2}\Big) + \frac{1}{2} = \frac{1}{2}(1 + e^{2\beta}\alpha)\,. \tag{5.25}$$

Since we are focusing on a single curve in the projector we conclude that

$$C_\alpha^+ = \bar{C}^+_{e^{2\beta}\alpha}\,, \tag{5.26}$$

where the bar indicates a curve defined by R_0. We thus have the identification of surfaces

$$\mathcal{P}_\alpha = \overline{\mathcal{P}}_{e^{2\beta}\alpha}\,, \tag{5.27}$$

where the overline indicates a surface defined by R_0. Note that the surface $\mathcal{P}_0$ coincides with $\overline{\mathcal{P}}_0$. This means that the function $z = f(\xi)$ that defines the projector does not depend on β.

The last ingredient we consider is the antighost insertion in (5.16). We wish to rewrite it in terms of a closed contour integral that involves R_0. We begin by noting the equality

$$\int_{C_\gamma^+} \frac{dz}{2\pi i}\frac{b(z)}{R'(z)} = e^{2\beta}\int_{C_{n-1}^+} \frac{dz}{2\pi i}\frac{b(z)}{R_0'(z)}\,, \tag{5.28}$$

which follows from (5.17) and contour deformation. To rewrite the right-hand side in terms of an integral over a closed contour we recall that on the surface $\mathcal{P}_{n-1}$ the identification of points on C^+_{n-1} and C^-_{n-1} is given by (3.17):

$$R_\beta(z^+) + R_\beta(-z^-) = n\,, \tag{5.29}$$

In terms of R_0 the identification reads

$$R_0(z^+) + R_0(-z^-) = 1 + (n-1)e^{2\beta}. \tag{5.30}$$

We now consider the integral

$$\int_{C^+_{n-1}-C^-_{n-1}} \frac{dz}{2\pi i}\,\frac{\widehat{R}_0(z)}{\widehat{R}_0'(z)}\,b(z) = \int_{C^+_{n-1}} \frac{dz^+}{2\pi i}\,\frac{R_0(z^+)}{R_0'(z^+)}\,b(z^+) + \int_{C^-_{n-1}} \frac{dz^-}{2\pi i}\,\frac{R_0(-z^-)}{R_0'(-z^-)}\,b(z^-)\,. \tag{5.31}$$

Using (5.30) and its differential form $R_0'(z^+)dz^+ - R_0'(-z^-)dz^- = 0$, we can write the second integral above as an integral over C^+_{n-1}. We then find a cancellation and we are left with

$$\int_{C^+_{n-1}-C^-_{n-1}} \frac{dz}{2\pi i}\,\frac{\widehat{R}_0(z)}{\widehat{R}_0'(z)}\,b(z) = \big(1+(n-1)e^{2\beta}\big)\int_{C^+_{n-1}} \frac{dz}{2\pi i}\,\frac{b(z)}{R_0'(z)}\,, \tag{5.32}$$

or, equivalently,

$$\int_{C^+_{n-1}} \frac{dz}{2\pi i}\frac{b(z)}{R_0'(z)} = \frac{1}{1+(n-1)e^{2\beta}}\int_{C^+_{n-1}-C^-_{n-1}} \frac{dz}{2\pi i}\,\frac{\widehat{R}_0(z)}{\widehat{R}_0'(z)}\,b(z)\,. \tag{5.33}$$

Back in (5.28) and using again contour deformation, we find

$$\int_{C^+_\gamma} \frac{dz}{2\pi i}\frac{b(z)}{R_\beta'(z)} = \frac{e^{2\beta}}{1+(n-1)e^{2\beta}}\int_{C^+_\gamma-C^-_\gamma} \frac{dz}{2\pi i}\,\frac{\widehat{R}_0(z)}{\widehat{R}_0'(z)}\,b(z)\,, \quad \text{on} \quad \mathcal{P}_{n-1}\,. \tag{5.34}$$

This is our desired result.

5.2.2 Operator formula

We are now in a position to derive an operator result beginning with (5.16). As a first step we use (5.34) to obtain

$$\langle\,\phi,\psi_{n-2}\,\rangle = -\,\frac{R_0'(r)^2\,e^{-2\beta}}{1+(n-1)e^{2\beta}}\Big\langle \int_{C^+_\gamma-C^-_\gamma} \frac{dz}{2\pi i}\,\frac{\widehat{R}_0(z)}{\widehat{R}_0'(z)}\,b(z)\;c(-r)\;f\circ\phi(0)\;c(r)\,\Big\rangle_{\overline{\mathcal{P}}_{e^{2\beta}(n-1)}}\,. \tag{5.35}$$

Note that we have expressed the surface in terms of the function R_0. Moving the antighost insertion contours inwards we pick up contributions from each of the ghost insertions and we remain with an antighost insertion that effectively surrounds the insertion of the test state ϕ:

$$\begin{aligned}\langle\,\phi,\psi_{n-2}\,\rangle =\ & \frac{R_0'(r)R_0(r)e^{-2\beta}}{1+(n-1)e^{2\beta}}\Big(\langle\, c(-r)\, f\circ\phi\,\rangle_{\overline{\mathcal{P}}_{e^{2\beta}(n-1)}}+\langle\, f\circ\phi\; c(r)\,\rangle_{\overline{\mathcal{P}}_{e^{2\beta}(n-1)}}\Big)\\ &+\ \frac{R_0'(r)^2\, e^{-2\beta}}{1+(n-1)e^{2\beta}}\Big\langle c(-r)\Big[\int_{C_\gamma^+-C_\gamma^-}\frac{dz}{2\pi i}\,\frac{\widehat{R}_0(z)}{\widehat{R}_0'(z)}\,b(z)\;f\circ\phi(0)\Big]\,c(r)\Big\rangle_{\overline{\mathcal{P}}_{e^{2\beta}(n-1)}}.\end{aligned} \tag{5.36}$$

Here $0\leq\gamma<1$. This is the most simplified expression we have obtained for ψ_{n-2} when the projector is completely general.

Let us now assume that we have a special projector with parameter s. We thus take

$$R_0(z)=z^s\quad\to\quad\frac{R_0(z)}{R_0'(z)}=\frac{1}{s}\,z\,, \tag{5.37}$$

which implies that

$$\int_{C_\gamma^+-C_\gamma^-}\frac{dz}{2\pi i}\,\frac{\widehat{R}_0(z)}{\widehat{R}_0'(z)}\,b(z)=\frac{1}{s}\int_{C_\gamma^+-C_\gamma^-}\frac{dz}{2\pi i}\,z\,b(z)=\frac{1}{s}\oint\frac{dz}{2\pi i}\,z\,b(z)\,. \tag{5.38}$$

Notice the great simplification: all that is left of the antighost insertion is a holomorphic integral encircling the origin. We also define

$$a_n\equiv\Big(1+(n-1)e^{2\beta}\Big)^{-1/s}\,, \tag{5.39}$$

and confirm that

$$R_0(r)=r^s=\frac{1}{2}\big(1+e^{2\beta}\big)\,. \tag{5.40}$$

Using the above relations (5.36) can be written as

$$\begin{aligned}\langle\,\phi,\psi_{n-2}\,\rangle =\ & sr^{2s-1}(a_n)^s e^{-2\beta}\Big(\langle\, c(-r)\, f\circ\phi\,\rangle_{\overline{\mathcal{P}}_{e^{2\beta}(n-1)}}+\langle\, f\circ\phi\; c(r)\,\rangle_{\overline{\mathcal{P}}_{e^{2\beta}(n-1)}}\Big)\\ &+\ s\,r^{2s-2}(a_n)^s\,e^{-2\beta}\Big\langle c(-r)\Big[\oint\frac{dz}{2\pi i}\,z\,b(z)\;f\circ\phi(0)\Big]\,c(r)\Big\rangle_{\overline{\mathcal{P}}_{e^{2\beta}(n-1)}}.\end{aligned} \tag{5.41}$$

To map the correlators to the upper-half plane we first scale $\overline{\mathcal{P}}_{e^{2\beta}(n-1)}$ down to $\mathcal{P}_0$. This requires the scaling map

$$z'=a_n z\,, \tag{5.42}$$

with a_n defined in (5.39). We let $\tilde{f} \equiv a_n \circ f$ and perform the scaling, finding

$$\begin{aligned}\langle\, \phi, \psi_{n-2} \,\rangle = sr^{2s-1}(a_n)^{s-1}e^{-2\beta}\Big(\langle\, c(-a_n r)\, \tilde{f}\circ\phi \,\rangle_{\mathcal{P}_0} + \langle\, \tilde{f}\circ\phi\; c(a_n r)\,\rangle_{\mathcal{P}_0}\Big) \\ +\; s\, r^{2s-2}(a_n)^{s-2}\, e^{-2\beta}\Big\langle c(-a_n r)\Big[\oint \frac{dz}{2\pi i}\, z\, b(z)\; \tilde{f}\circ\phi(0)\Big]\, c(a_n r)\Big\rangle_{\mathcal{P}_0}.\end{aligned} \tag{5.43}$$

The map

$$g \equiv f_I \circ f^{-1} \tag{5.44}$$

takes $\mathcal{P}_0$ to the upper half plane $\mathbb{H}$. Letting

$$f_{n-1} \equiv g \circ \tilde{f} = f_I \circ f^{-1} \circ a_n \circ f\,, \tag{5.45}$$

we map the correlators by g and find, noting that g is an odd function,

$$\begin{aligned}\langle\, \phi, \psi_{n-2} \,\rangle = sr^{s}(a_n r)^{s-1}\frac{e^{-2\beta}}{g'(a_n r)}\Big(\langle\, c(-g(a_n r))\, f_{n-1}\circ\phi \,\rangle_{\mathbb{H}} + \langle\, f_{n-1}\circ\phi\; c(g(a_n r))\,\rangle_{\mathbb{H}}\Big) \\ +\; sr^{s}(a_n r)^{s-2}\,\frac{e^{-2\beta}}{(g'(a_n r))^2}\Big\langle c(-g(a_n r))\big[\,\widehat{B}\; f_{n-1}\circ\phi(0)\big]\, c(g(a_n r))\Big\rangle_{\mathbb{H}}.\end{aligned} \tag{5.46}$$

Here all correlators are now on the upper half plane $\mathbb{H}$ and

$$\widehat{B} \equiv \oint \frac{dz}{2\pi i}\frac{g^{-1}(z)}{(g^{-1})'(z)}\, b(z)\,. \tag{5.47}$$

Note that the $\widehat{B}$ insertion is β independent and n independent.

Since the operator $I \circ f_{n-1} \circ \phi(0)$ corresponds to $\langle\phi|\, U^{\star}_{f_{n-1}}$ in the state-operator correspondence, it is convenient to perform a final map by $I(z) = -1/z$. Noting that the test state ϕ must be Grassman even, the result is

$$\begin{aligned}\langle\, \phi, \psi_{n-2} \,\rangle = \gamma\, s(a_n r)^{s-1}\frac{g(a_n r)^2}{g'(a_n r)}\,\Bigg[\,\Big\langle I\circ f_{n-1}\circ\phi\; c\Big(\frac{1}{g(a_n r)}\Big)\Big\rangle_{\mathbb{H}} + \Big\langle I\circ f_{n-1}\circ\phi\; c\Big(-\frac{1}{g(a_n r)}\Big)\Big\rangle_{\mathbb{H}} \\ +\; \frac{g(a_n r)^2}{a_n r g'(a_n r)}\Big\langle I\circ f_{n-1}\circ\phi(0)\, \widehat{B}^{\star}\; c\Big(-\frac{1}{g(a_n r)}\Big)\, c\Big(\frac{1}{g(a_n r)}\Big)\Big\rangle_{\mathbb{H}}\Bigg],\end{aligned} \tag{5.48}$$

where we defined

$$\gamma \equiv e^{-2\beta} r^{s} = \frac{1}{2}(1 + e^{-2\beta})\,. \tag{5.49}$$

We can now read out the operator expression for ψ_{n-2}:

$$\boxed{\begin{aligned}\psi_{n-2} = \;& \gamma\, s(a_n r)^{s-1}\,\frac{g(a_n r)^2}{g'(a_n r)}\,U^\star_{f_{n-1}}\Big[\,c\Big(-\frac{1}{g(a_n r)}\Big)+c\Big(\frac{1}{g(a_n r)}\Big)\\ &\qquad\qquad + \frac{g(a_n r)^2}{a_n r g'(a_n r)}\,\widehat{B}^\star\, c\Big(-\frac{1}{g(a_n r)}\Big)\,c\Big(\frac{1}{g(a_n r)}\Big)\Big]|0\rangle\,.\end{aligned}} \tag{5.50}$$

Equation (5.50) is the expected result: a formula for the state ψ_{n-2} in which operators act on the $SL(2,R)$-invariant vacuum. The state depends on both s and β. Moreover, as we will see in the following section, we can readily find a level expansion of the solution. We recall that the "phantom" piece ψ_N of the solution in (1.5) does not contribute in the level expansion, so we have

$$\langle\, \phi, \Psi\,\rangle = \sum_{n=2}^{\infty}\langle\, \phi, \psi'_{n-2}\,\rangle \tag{5.51}$$

for any state ϕ in the Fock space.

6 Level and other expansions

In this section we will expand and analyze the operator form (5.50) of the solution. We set up the level expansion computation for arbitrary s and β in §6.1. We proceed up to level four, but give the ingredients necessary to carry the computations to arbitrary order, if so desired.

In §6.2 we consider the special case $\beta = 0$ and compute the vacuum expectation values of fields up to level four for arbitrary values of s. This allows us to compute the level zero, two, and four vacuum energies as functions of s. For $s \geq 1$ we find numerical evidence consistent with convergence of the vacuum energy to the expected value of minus the D-brane tension.

Recall that for $s < 1$ the special frames are not projectors. The string field Ψ which provides a solution for $s \geq 1$ is therefore not expected to provide a solution for $s < 1$. Indeed, for $s < 1$ we find numerical evidence consistent with the energy failing to converge to the expected value.

In §6.3 we show that the tachyon vacuum solution in the Siegel gauge cannot be obtained in the present framework. The framework imposes constraints on expectation values that we show are not satisfied in the most accurate version of the Siegel gauge solution known to date.

Finally, in §6.4 we consider the limit $\beta \to \infty$ of the solution. This limit is of some interest because the surface states used to build the solution approach the surface state of the projector. For large β the solution provides an analytic expression closely related to the alternative level expansion scheme introduced in [23] and explored further in [24]. In this scheme, the string field solution is written in terms of operators of increasing level inserted at the midpoint of a regulated projector. Our solution is given in terms of exponentials of β and has a leading divergent term as well as terms that vanish as $\beta \to \infty$.

6.1 Level expansion preliminaries

We now set up the level expansion of the solution (5.50). We begin by level expanding the operators $U^\star_{f_{n-1}}$ and $\widehat{B}^*$. We then write out the level four string field and compute the expectation values of the various components. The results are given in terms of infinite sums that we evaluate numerically.

The operator $U^\star_{f_{n-1}}$ is defined by the function $f_{n-1}(\xi)$ introduced in (5.45):

$$f_{n-1} = f_I \circ f^{-1} \circ a_n \circ f \,. \tag{6.1}$$

It is most convenient to obtain a factorized form in which

$$U_{f_{n-1}} = e^{\bar{t}_0 L_0}\, e^{\bar{t}_2 L_2}\, e^{\bar{t}_4 L_4}\, e^{\bar{t}_6 L_6} \cdots \,, \tag{6.2}$$

with calculable coefficients $\bar{t}_n$. The bpz dual is immediately written

$$U^\star_{f_{n-1}} = \cdots e^{\bar{t}_6 L_{-6}}\, e^{\bar{t}_4 L_{-4}}\, e^{\bar{t}_2 L_{-2}}\, e^{\bar{t}_0 L_0} \,. \tag{6.3}$$

Given an arbitrary function $f(\xi)$ that defines a surface state and has an expansion

$$f(\xi) = \xi + f_2 \xi^3 + f_4 \xi^5 + f_6 \xi^7 + f_8 \xi^9 + \cdots \,, \tag{6.4}$$

the first few $\bar{t}_n$ coefficients are obtained following the steps indicated in appendix A of [2]. We find that they are given by

$$\begin{aligned}
\bar{t}_2 &= f_2 \,, \\
\bar{t}_4 &= f_4 - \frac{3}{2} f_2^2 \,, \\
\bar{t}_6 &= f_6 - 3 f_2 f_4 + 2 f_2^3 \,, \\
\bar{t}_8 &= f_8 - 3 f_2 f_6 - \frac{5}{2} f_4^2 + 9 f_2^2 f_4 - \frac{19}{4} f_2^4 \,.
\end{aligned} \tag{6.5}$$

Using this result and the power series expansion of f_{n-1} we can readily calculate the coefficients $\bar{t}_n$ needed to obtain $U_{f_{n-1}}$ to level four:

$$e^{\bar{t}_0} = a_n \,, \quad \bar{t}_2 = \frac{-s + 2a_n^2(1+s)}{2+s} \,, \quad \bar{t}_4 = -\frac{(s-2)s + 8a_n^4(1+s)}{2(2+s)(4+s)} \,. \tag{6.6}$$

With these we get

$$U^{\star}_{f_{n-1}} = \cdots\, e^{\bar{t}_4 L_{-4}}\, e^{\bar{t}_2 L_{-2}}\, (a_n)^{L_0}\,. \tag{6.7}$$

The expansion of $\widehat{B}^{\star}$ is easier to obtain. Recalling (5.47) and the relation $g = f_I \circ f^{-1}$ we find

$$\widehat{B} = \sum_{n=0}^{\infty} \beta_n\, b_n = b_0 + \frac{4(1+s)}{2+s}\, b_2 - \frac{16(1+s)}{(2+s)(4+s)} b_4 + \cdots \quad . \tag{6.8}$$

Note that both the Virasoro operators and the antighost operators in the above expansions are even moded.

The level expansion of the string is obtained by the action on the vacuum of arbitrary ghost oscillators, even moded Virasoro operators, and even moded antighost oscillators. The string field up to level four is thus given by

$$\begin{aligned}
\Psi_4 = \; - \Big(& t\, c_1\, |0\rangle \\
& + u\, c_{-1}\, |0\rangle + v\, L_{-2} c_1\, |0\rangle + w\, b_{-2} c_0 c_1\, |0\rangle \\
& + A\, L_{-4} c_1\, |0\rangle + B L_{-2} L_{-2} c_1\, |0\rangle + C\, c_{-3}\, |0\rangle + E\, b_{-2} c_{-2} c_1\, |0\rangle + F\, L_{-2} c_{-1}\, |0\rangle \\
& + w_2\, b_{-2} c_{-1} c_0\, |0\rangle + w_3\, b_{-4} c_0 c_1\, |0\rangle + w_4\, L_{-2} b_{-2} c_0 c_1\, |0\rangle \Big)\,.
\end{aligned} \tag{6.9}$$

The first line contains the level-zero tachyon, the second line contains the three level-two fields, and the last two lines contain the eight level-four fields. In this expansion the Virasoro operators include matter and ghost contributions and have zero central charge.

To describe the solution, assume a general expansion in a basis of Fock space states

$$\Psi = \sum_i \phi^{(i)}\, |\mathcal{O}_i\rangle \;. \tag{6.10}$$

Up to level four, the states $|\mathcal{O}_i\rangle$ and the expansion coefficients $\phi^{(i)}$ are those in (6.9). Our goal is to compute those expansion coefficients, since they are the expectation values of the component fields. Assume now that ψ_{n-2}, given in (5.50), is also expanded in the same basis:

$$\psi_{n-2} = \sum_i \phi^{(i)}_n\, |\mathcal{O}_i\rangle \;. \tag{6.11}$$

Using (5.51) we have

$$\Psi = \sum_{n=2}^{\infty} \psi'_{n-2} = \sum_i \sum_{n=2}^{\infty} (\partial_n \phi^{(i)}_n)\, |\mathcal{O}_i\rangle \;. \tag{6.12}$$

Comparing with (6.10) we find that the vevs are given by

$$\phi^{(i)} = \sum_{n=2}^{\infty} \partial_n\, \phi^{(i)}_n\,. \tag{6.13}$$

We can now expand the solution (5.50) to level four. Since the combination $a_n r$ appears repeatedly both by itself and as the argument of g we introduce the notation

$$\tilde{a} \equiv a_n r\,, \quad g \equiv g(\tilde{a})\,. \tag{6.14}$$

Using the expansion (6.7) of $U^\star_{f_{n-1}}$ and the expansion (6.8) of $\widehat{B}$, together with (6.11), we find that the expansion of (5.50) yields

$$\begin{aligned}
&t_n = 2\gamma r s\,\tilde{a}^{s-2}\,\frac{g^2}{g'}\Big(1-\frac{g}{\tilde{a}g'}\Big), \quad u_n = \frac{\gamma}{r}\frac{\tilde{a}^2}{g^2}\,t_n\,, \quad v_n = \gamma r\bar{t}_2\,t_n\,, \quad w_n = -2\frac{\gamma}{r}s\beta_2\,\tilde{a}^{s-1}\,\frac{g^3}{g'^2}\,,\\
&A_n = \gamma r\bar{t}_4\,t_n\,, \quad B_n = \frac{1}{2}\,\gamma r\bar{t}_2^2\,t_n\,, \quad C_n = \frac{\gamma}{r^3}\frac{\tilde{a}^4}{g^4}\,t_n\,, \quad E_n = -2\frac{\gamma}{r^3}s\,\tilde{a}^{s+1}\,\beta_2\,\frac{g}{g'^2}\,,\\
&F_n = \frac{\gamma}{r}\frac{\tilde{a}^2}{g^2}\,\bar{t}_2\,t_n\,, \quad (w_2)_n = -\frac{\gamma}{r^3}E_n\,, \quad (w_3)_n = -2\frac{\gamma}{r^3}s\,\tilde{a}^{s+1}\,\beta_4\frac{g^3}{g'^2}\,,\\
&(w_4)_n = -2\frac{\gamma}{r}s\,\tilde{a}^{s-1}\,\bar{t}_2\beta_2\,\frac{g^3}{g'^2}\,.
\end{aligned} \tag{6.15}$$

The powers of r here arise from the factor $(a_n)^{L_0} = (\tilde{a}/r)^{L_0}$ in $U^\star_{f_{n-1}}$ — see (6.7). In the above formulae all appearances of a_n are in the combination $\tilde{a}$. Note, however, that the coefficients $\bar{t}_2$ and $\bar{t}_4$ have a_n dependence. Following (6.13), the expectation value of A, for example, would be given by

$$A = \sum_{n=2}^{\infty} \partial_n A_n\,. \tag{6.16}$$

For arbitrary β and s, the derivatives with respect to n of the component fields in (6.15) give long and complicated expressions. Therefore, we do not attempt any further simplification of the string field.

6.2 Level Expansion for $\beta = 0$

In this subsection we set $\beta = 0$ and explore the solution for various values of s. We calculate explicitly the expectation values of level four fields and use them evaluate the approximate energy of the solution. We find numerical evidence consistent with the energy converging to the expected value of -1 (in units of the D-brane tension) for $s \geq 1$. For $s < 1$ we can still use (5.50) to calculate a string field but given that the $s < 1$ surface states are not projectors, we have no reason to believe that the constructed field is a solution. Indeed, a level computation of the energy in those cases suggests that it does not converge to minus one.

For $\beta = 0$ we have $r = 1$ and the solution in (5.50) reduces to

$$\begin{aligned}\psi_{n-2} = \ & s(a_n)^{s-1}\frac{g(a_n)^2}{g'(a_n)}\,U^\star_{f_{n-1}}\Big[c\Big(-\frac{1}{g(a_n)}\Big)+c\Big(\frac{1}{g(a_n)}\Big) \\ & \qquad\qquad\qquad + \frac{g(a_n)^2}{a_n g'(a_n)}\,\widehat{B}^\star\,c\Big(-\frac{1}{g(a_n)}\Big)\,c\Big(\frac{1}{g(a_n)}\Big)\Big]|0\rangle\,.\end{aligned} \tag{6.17}$$

This time we write

$$a \equiv a_n = n^{-1/s}\,, \quad g \equiv g(a)\,, \quad g' \equiv g'(a)\,, \tag{6.18}$$

and the results in (6.15) simplify to

$$\begin{aligned} & t_n = 2s\,a^{s-2}\,\frac{g^2}{g'}\Big(1-\frac{g}{ag'}\Big)\,, \quad u_n = \frac{a^2}{g^2}\,t_n\,, \quad v_n = \bar{t}_2\,t_n\,, \quad w_n = -2s\beta_2\,a^{s-1}\frac{g^3}{g'^2}\,, \\ & A_n = \bar{t}_4\,t_n\,, \quad B_n = \frac{1}{2}\bar{t}_2^2\,t_n\,, \quad C_n = \frac{a^4}{g^4}\,t_n\,, \quad E_n = -2s\,a^{s+1}\,\beta_2\,\frac{g}{g'^2}\,, \\ & F_n = \frac{a^2}{g^2}\,\bar{t}_2\,t_n\,, \quad (w_2)_n = -E_n\,, \quad (w_3)_n = -2s\,a^{s+1}\,\beta_4\frac{g^3}{g'^2}\,, \quad (w_4)_n = -2s\,a^{s-1}\,\bar{t}_2\beta_2\,\frac{g^3}{g'^2}\,. \end{aligned} \tag{6.19}$$

These formulae, together with (6.13) allow the evaluation of the level four expectation values. As in [2], no simple closed form seems possible and the computation must be done numerically.

The level four string field in (6.9) can be rewritten using matter Virasoro operators. Expanding the Virasoro operators in (6.9) into matter and ghost parts one obtains the string field

$$\begin{aligned}\Psi_4 = \ -\Big(\, & t'\,c_1\,|0\rangle \\ & + u'\,c_{-1}\,|0\rangle + v'\,L^m_{-2}c_1\,|0\rangle + w'\,b_{-2}c_0c_1\,|0\rangle \\ & + A'\,L^m_{-4}c_1\,|0\rangle + B'L^m_{-2}L^m_{-2}c_1\,|0\rangle + C'\,c_{-3}\,|0\rangle + D'\,b_{-3}c_{-1}c_1\,|0\rangle + E'\,b_{-2}c_{-2}c_1\,|0\rangle \\ & + F'\,L^m_{-2}c_{-1}\,|0\rangle + w_2'\,b_{-2}c_{-1}c_0\,|0\rangle + w_3'\,b_{-4}c_0c_1\,|0\rangle + w_4'\,L^m_{-2}b_{-2}c_0c_1\,|0\rangle\Big)\,,\end{aligned} \tag{6.20}$$

	$s = 0.6$	$s = 0.8$	$s = 1$	$s = 1.2$	$s = 1.4$	$s = 2.0$	$s = 3.0$
$t(s)$	0.52860	0.53755	0.55347	0.57278	0.59361	0.65779	0.75882
$u(s)$	0.02935	0.03881	0.04367	0.04600	0.04694	0.04634	0.04268
$v(s)$	0.04541	0.09289	0.13765	0.17939	0.21840	0.32231	0.46548
$w(s)$	0.09945	0.11908	0.13108	0.13860	0.14330	0.14857	0.14617

Table 1: The expectation values of all fields up to level two calculated using the exact analytic expressions as a function of the parameter s.

where the primed fields are given by

$$\begin{aligned} t' &= t & u' &= u + 3v \\ v' &= v & w' &= w - 2v \\ A' &= A & B' &= B \\ C' &= C + 7A + 15B + 5F & D' &= -5A + 3B + F \\ E' &= E - 6A - 8B + 4w_4 & F' &= F + 6B \\ w_2' &= w_2 + 12B + 2F - 3w_4 & w_3' &= w_3 - 4A \\ w_4' &= w_4 - 4B & & \end{aligned} \tag{6.21}$$

Note that in (6.20) we had to introduce a field D' to multiply the state $b_{-3}c_{-1}c_1\,|0\rangle$. Since $\widehat{B}$ only has even-moded oscillators, that state arises from (5.50) only after expanding the total Virasoro operators in $U^\star_{f_{n-1}}$ into matter and ghost parts. Note also that the state $L^m_{-3}c_0|0\rangle$ does not arise in the expansion. The expansion in ghost and matter parts cannot generate odd-moded Virasoro operators, only odd-moded antighost operators.

We can now consider some numerical work. For $s = 1$ we find the expectation value $t = 0.553466$, $u = 0.0436719$, $v = 0.137646$, and $w = 0.131082$. These imply $t' = 0.553466$, $u' = 0.45661$, $v' = 0.137646$, and $w' = -0.14421$ in complete agreement with [2]. We have also checked that the expectation values of the level four fields for $s = 1$ agree with those in [2]. For $s = 2$ we find

$$t = 0.65779\,, \quad u = 0.04634\,, \quad v = 0.32231\,, \quad w = 0.14857\,. \tag{6.22}$$

Vacuum expectation values for these and other values of s are listed in Table 1.

	$s = 0.6$	$s = 0.8$	$s = 1$	$s = 1.2$	$s = 1.4$	$s = 2.0$	$s = 3.0$
$E_0(s)$	-0.91925	-0.95064	-1.00777	-1.07934	-1.15927	-1.42348	-1.8943
$E_2(s)$	-0.91495	-0.96663	-1.00782	-1.02736	-1.02271	-0.87438	-0.2896
$E_4(s)$	-0.91389	-0.97221	-1.0045	-1.00843	-0.99591	-0.98916	-1.4827

Table 2: The energy calculated at levels zero, two, and four, for several values of the parameter s.

The energy, normalized to minus one, can be computed using the vevs of the fields and the kinetic terms in the string field theory. To level zero, two, and four we get

$$\begin{aligned}
E_0 &= \frac{2\pi^2}{3}\Big(-\frac{1}{2}t^2\Big)\,,\\
E_2 &= E_0 + \frac{2\pi^2}{3}\Big(-\frac{1}{2}u^2 + 3u(v-w) + 2(v-w)^2\Big)\,,\\
E_4 &= E_2 + \frac{2\pi^2}{3}\Big(\ 4A^2 + 24AB + 5AC - 6AE + 18AF - 8Aw_3 - 24Aw_4\\
&\qquad - 3BC + 8BE - 24Bw_2 - 24Bw_3 + CF - Cw_2 - 5Cw_3 + 3Cw_4\\
&\qquad - \frac{3}{2}E^2 + 6EF + 3Ew_2 + 6Ew_3 - 8Ew_4 - \frac{13}{2}F^2 - 5Fw_2 - 18Fw_3\\
&\qquad - 2w_2^2 + 24w_2w_4 + 4w_3^2 + 24w_3w_4\Big)\,.
\end{aligned} \tag{6.23}$$

In Figure 10 we plot energies as a function of $s \in [0.6, 2.0]$. There are three curves: the level-zero energy $E_0(s)$, the level-two energy $E_2(s)$, and the level-four energy $E_4(s)$. At each level the energy was computed using the exact numerical values for all the fields. For $s \geq 1$ the various curves are consistent with an energy that approaches the correct value. For $s < 1$ the plot suggests that the energy will not approach the correct value. Some particular values are also tabulated in Table 2. Note how efficient the convergence is for $s = 2$, while for $s = 0.6$ it appears that the energy will not move much beyond the value -0.91.

6.3 No Siegel gauge in the family

The solution for the tachyon vacuum in the Siegel gauge is a state in the universal subspace of the total CFT: the ghost number one subspace spanned by all states built on the vacuum by acting with finite numbers of ghost and antighost oscillators as well as finite number of matter Virasoro operators. Apart from an $SU(1,1)$ symmetry that relates certain expectation values no additional relations are known.

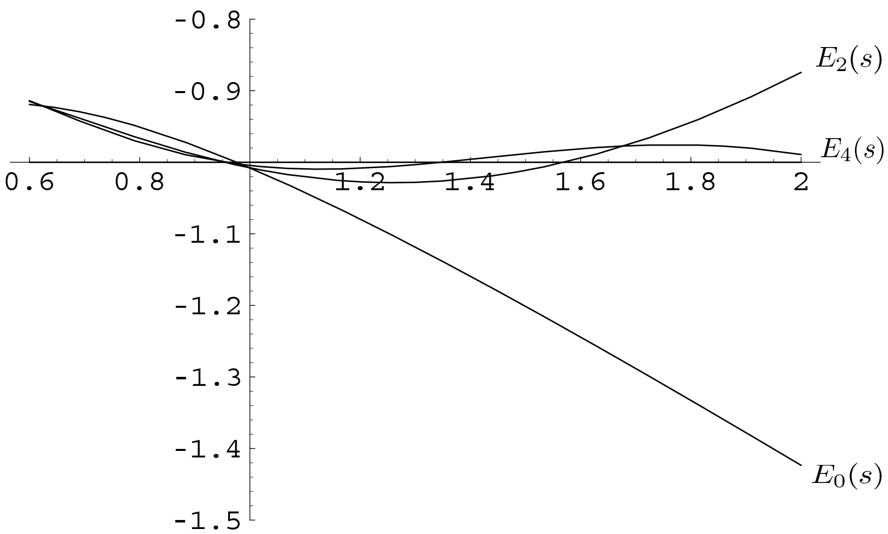

Figure 10: Plot of the energies $E_0(s), E_2(s)$, and $E_4(s)$ computed at levels zero, two, and four, respectively. The exact value is -1.

It is clear from the form of ψ_n that the solution Ψ belongs to a *constrained* universal space where states are built acting on the vacuum with arbitrary ghost oscillators, *even*-moded antighost oscillators, and *even*-moded total Virasoro operators. Before imposing any gauge condition, the level four universal subspace contains 10 states, while the level four constrained space has only 8 states.

As we show now, at level four the Siegel gauge expectation values must satisfy an additional relation if it is to lie on the constrained universal space. This condition is not satisfied.

In the Siegel gauge we can use the expansion (6.20) of the string field. The question is whether the values of the primed fields in the Siegel gauge are consistent with expectation values for the unprimed fields. Can we solve for the unprimed fields using (6.21)? There is a constraint, however. We readily find that

$$D' = -5A' - 3B' + F' \,. \tag{6.24}$$

This is a constraint that must be satisfied by the Siegel gauge solution, if it is to have the structural form required by the general s solution. From [25] we have

$$\begin{aligned} A' &= -0.005049 \,, \\ B' &= -0.000681 \,, \\ F' &= 0.001234 \,. \end{aligned} \tag{6.25}$$

This together with (6.24) predicts $D' = 0.028522$. The value from [25], however, is $D' = 0.01976$, in clear disagreement. We conclude that we cannot reach the Siegel gauge solution for any value of the parameter s.

6.4 Projector expansion

In [23] a variant of level expansion was proposed in which the string field solution is written in terms of operators of increasing level inserted at the midpoint of a regulated projector surface state. The original discussion used the butterfly state but this was extended to large classes of projectors in [24]. In this section we show how to obtain a possibly related expansion using the β parameter in the limit of large β.

In the solution (5.50) and in its level expansion we noted the repeated appearance of $a_n r = \tilde{a}$, which is given by

$$\tilde{a} = a_n r = \Big[\frac{1}{2} \cdot \frac{1+e^{2\beta}}{1+(n-1)e^{2\beta}}\Big]^{1/s} . \tag{6.26}$$

For $\beta \to \infty$ we get a finite limit

$$\lim_{\beta\to\infty} \tilde{a} = \Big(\frac{1}{2n-2}\Big)^{1/s} \equiv \bar{a} . \tag{6.27}$$

We also note that for large β

$$r \simeq 2^{-1/s} e^{2\beta/s} , \quad a_n \simeq 2^{1/s} \bar{a}\, e^{-2\beta/s} . \tag{6.28}$$

Let us separate the factor $U_f^\star$ from $U_{f_{n-1}}^\star$. We recall (6.1), which implies that

$$U_{f_{n-1}} = U_{f_I\circ f^{-1}}\, (a_n)^{L_0}\, U_f . \tag{6.29}$$

It follows that

$$U_{f_{n-1}}^\star = U_f^\star \, \Big[(a_n)^{L_0}\, U_{f_I\circ f^{-1}}^\star\, (a_n)^{-L_0}\Big]\, (a_n)^{L_0} . \tag{6.30}$$

Since f_{n-1} is independent of the overall scale of f, we can assume that $f(z) \sim z + \ldots$ in evaluating $U_{f_I\circ f^{-1}}$. We can then write an expansion without an L_0 term:

$$U_{f_I\circ f^{-1}}^\star = \cdots e^{\bar{d}_6 L_{-6}}\, e^{\bar{d}_4 L_{-4}}\, e^{\bar{d}_2 L_{-2}} . \tag{6.31}$$

Here the $\bar{d}_n$ are calculable coefficiencts that are independent of β. We then have

$$U_{f_{n-1}}^\star = U_f^\star \Big(\cdots e^{\bar{d}_6 a_n^6 L_{-6}}\, e^{\bar{d}_4 a_n^4 L_{-4}}\, e^{\bar{d}_2 a_n^2 L_{-2}}\Big)\, (a_n)^{L_0} . \tag{6.32}$$

The string field will be an expansion in powers of $e^{2\beta/s}$. The leading term in the expansion of the string field will occur when $U_{f_{n-1}}^\star$ acts on the tachyon state, the state with $L_0 = -1$. In this case, to leading order in $e^{2\beta/s}$, the above factor in parenthesis is equal to one, and we have

$$U_{f_{n-1}}^\star c_1|0\rangle \simeq U_f^\star c_1|0\rangle \cdot \frac{1}{\bar{a}}\, 2^{-1/s}\, e^{2\beta/s} . \tag{6.33}$$

It now follows from (5.50) that

$$\psi_{n-2} \simeq \frac{1}{2}\, s\, \bar{a}^{s-1}\, \frac{g(\bar{a})^2}{g'(\bar{a})}\, \frac{1}{\bar{a}}\, 2^{-1/s}\, e^{2\beta/s}\, U_f^\star c_1|0\rangle \cdot 2\Big(1 - \frac{g(\bar{a})}{\bar{a}g'(\bar{a})}\Big)\,, \tag{6.34}$$

or, equivalently,

$$\psi_{n-2} \simeq U_f^\star c_1|0\rangle \cdot\; 2^{-1-\frac{1}{s}} s\, e^{2\beta/s}\, 2\bar{a}^{s-2}\, \frac{g(\bar{a})^2}{g'(\bar{a})}\, \Big(1 - \frac{g(\bar{a})}{\bar{a}g'(\bar{a})}\Big)\,. \tag{6.35}$$

This means that to leading order in the expansion the string field is given by

$$|\Psi\rangle = U_f^\star\, c_1|0\rangle \;\cdot\; 2^{-1-\frac{1}{s}}\, e^{2\beta/s}\, 2\sum_{n=2}^{\infty} \partial_n \Big(\bar{a}^{s-2}\, \frac{g(\bar{a})^2}{g'(\bar{a})}\, \Big(1 - \frac{g(\bar{a})}{\bar{a}g'(\bar{a})}\Big)\Big)\,. \tag{6.36}$$

This is the general result, valid for all arbitrary $s \geq 1$. Note that this term diverges paramaterically with β. For the case of the sliver, the string field becomes

$$|\Psi\rangle = U_f^\star\, c_1|0\rangle \;\cdot \frac{1}{4}e^{2\beta} \;\cdot 2\sum_{n=2}^{\infty} \partial_n \Big(\frac{g^2(\bar{a})}{\bar{a}g'(\bar{a})}\Big(1 - \frac{g(\bar{a})}{\bar{a}g'(\bar{a})}\Big)\Big)\,, \qquad s = 1\,, \tag{6.37}$$

with $g(z) = \frac{1}{2}\tan(\pi z)$. Recalling the definition of $\bar{a}$ in (6.27) one can easily evaluate the above expression numerically. The result is

$$|\Psi\rangle = U_f^\star\, c_1|0\rangle \;\cdot \frac{1}{4}e^{2\beta} \;\cdot (0.39545107)\,. \tag{6.38}$$

We will not attempt the calculation of the subleading terms in the solution. In the work of [23] the leading term of the solution is a divergent coefficient that multiplies a ghost insertion on a regulated projector. The regulation parameter and the divergent coefficient are related, and this helps produce finite energy. While the expansion of the solution around the sliver in this subsection is well defined in calculating coefficients in front of states in the Fock space, it is not well defined in calculating the energy of the solution. It would be interesting to find a more systematic way to expand the solution for large β, in particular, in the context of VSFT.

7 Concluding Remarks

We find it tantalizing that projectors play a significant role in the construction of solutions of OSFT. Projectors are essentially the solutions of vacuum string field theory (VSFT), so this fact should help relate OSFT to VSFT and, with some luck, to obtain a regular form of VSFT. In addition to finding new solutions of OSFT, the development of VSFT may pave the way for further progress in this field.

The role of projectors was somewhat hidden in the tachyon vacuum solution of Schnabl [2]. The $\mathcal{L}_0$, $\mathcal{L}_0^\star$ structure associated with the geometry of the wedge states seemed to be the central and necessary ingredient. In [5] it was found that the $\mathcal{L}_0$, $\mathcal{L}_0^\star$ structure is not unique to the wedge states. Including other conditions required by solvability, one is led to special projectors.

In this work we have used reparameterizations to show that *any* twist-invariant, single-split projector furnishes a solution. It is not required to have a special projector, but the form of the solution simplifies considerably for that case. This is a satisfying conclusion: each single-split projector furnishes a solution in a different gauge, and all single-split projectors are allowed.

Our methods using reparameterizations do not immediately apply to multiple-split projectors, *i.e.*, conformal frames where the coordinate curve goes to infinity at other points besides the string midpoint. These projectors are not related by regular reparameterizations to the sliver. Examples of multiple-split special projectors were given in [5]. It is not difficult to construct formal solutions for a certain class of multiple-split special projectors by inserting operators analogous to those in section 4, but it is not obvious if the calculation of their energies is well defined.

While the idea of using reparameterizations is certainly not new, it was generally felt that concrete computations would be difficult since the operators that perform reparameterizations are extremely difficult to construct. We found a way to implement the necessary reparameterizations without constructing the operators.

One particularly interesting by-product is the construction of an abelian algebra of states for any projector. The surface states interpolate between the identity and the projector. For the sliver this is the familiar algebra of wedge states. We believe, although we have not proven, that the wedge states are the unique states that interpolate between the identity and the sliver and star-multiply among themselves. If this is the case, the possibility of reparameterizations implies that the interpolating family must be a canonical unique object for any projector. In this sense there is no preferred projector and our use of the sliver is recognized to be just a technical tool.

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