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Phase Transitions of Large N Orbifold Gauge Theories

Yasuaki Hikida*

DESY Theory Group, Notkestrasse 85, D-22603 Hamburg, Germany

Abstract

We study the phase structures of $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theories on $\mathbb{R} \times S^3/\mathbb{Z}_k$ with large N. The theory has many vacua labelled by the holonomy matrix along the non-trivial cycle on $S^3/\mathbb{Z}_k$, and for the fermions the periodic and the anti-periodic boundary conditions can be assigned along the cycle. We compute the partition functions of the orbifold theories and observe that phase transitions occur even in the zero 't Hooft coupling limit. With the periodic boundary condition, the vacua of the gauge theory are dual to various arrangements of k NS5-branes. With the anti-periodic boundary condition, transitions between the vacua are dual to localized tachyon condensations. In particular, the mass of a deformed geometry is compared with the Casimir energy for the dual vacuum. We also obtain an index for the supersymmetric orbifold theory.

*E-mail: yasuaki.hikida@desy.de

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1 Introduction

Recently the thermodynamics of large N gauge theories on compact spaces attract much attention. On compact spaces the Gauss constraint restricts physical states into gauge invariant form, and due to this fact the theories are in a confinement phase at low temperature, and undergo a deconfinement transition at a critical temperature. Moreover, the large N gauge theories may have their dual description in terms of string theory on an asymptotic Anti-de Sitter (AdS) space [1]. For example, the partition function for a large N gauge theory on $\mathbb{R} \times S^3$ was computed in [2, 3], and it was shown that the partition function is of order $\mathcal{O}(1)$ at low temperature and of order $\mathcal{O}(N^2)$ above a critical temperature. In the dual gravity theory, the phase transition corresponds to the Hawking-Page transition [4, 5], where the thermal AdS space is dominant at low temperature and the AdS-Schwarzschild black hole is dominant at high temperature.

In this paper, we study the thermodynamics of $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory on $\mathbb{R} \times S^3/\mathbb{Z}_k$.[1] We construct the orbifold theory in the following way. The manifold S^3

[1] The orbifold theory at zero temperature has been studied in [6], see also [7, 8].

has a $U(1)$ symmetry along a cycle, and the orbifold is constructed by dividing $\mathbb{Z}_k$ rotation along the cycle. The background does not include any fixed point, but has a non-trivial cycle due to the orbifold procedure. We can introduce flux along the non-trivial cycle, which gives non-trivial holonomies to the fields. Therefore, the theory admits many vacua labelled by the choice of flux, and this makes the phase diagram richer. Along the non-trivial cycle, we can assign periodic and anti-periodic boundary conditions to fermions along the non-trivial cycle, and this leads to supersymmetric and non-supersymmetric theories at zero temperature. We study the gauge theories perturbatively with respect to the 't Hooft coupling $\lambda = N g_{Ym}^2$.[2] In this paper, we consider the zero 't Hooft coupling limit, where the theories reduce to free field theories. Even in this limit we observe phase transitions due to the compactness of the base manifold.

In this orbifold case the dual gravity description is also available, thus the phase diagram can be extended into the strong coupling region. For the case with the periodic boundary condition, the dual geometry is the orbifold of the thermal AdS space or its deformation by localized massless states in the low temperature phase. In the high temperature phase, the dual geometry is the orbifold of the AdS-Schwarzschild black hole or its deformation. If we perform the T-duality along the non-trivial cycle, then we obtain k NS5-brane configuration, which is parametrized by the positions of k NS5-branes [6]. For the case with the anti-periodic boundary condition, there are localized tachyons at the fixed point of the thermal AdS orbifold in the low temperature phase. The condensation of localized tachyon may resolve the orbifold singularity like [9] and lead to the deformed geometry obtained in [10, 11] called as Eguchi-Hanson soliton. In other words, the gauge theory gives the dual picture of the localized tachyon condensation discussed in [9].[3] At enough high temperature, the dual geometry should be the orbifold of the AdS-Schwarzschild black hole, and there are no localized tachyons in the geometry.

Following the analysis in [2, 3], we obtain the partition functions for the gauge theories in terms of a matrix integral. At low temperature, we find the leading contribution comes from the Casimir energy of the theories on $S^3/\mathbb{Z}_k$. For the case with the periodic boundary condition, the Casimir energy is the same for all the vacua and the same as the mass of the thermal AdS orbifold. For the case with the anti-periodic boundary condition, the Casimir energy is smallest for the vacuum dual to the deformed geometry. Interestingly, the Casimir energy is roughly 4/3 times the mass of the Eguchi-Hanson soliton. In the high temperature limit, the partition function behaves in the same way for all possible holonomies and spin structures. Near the critical temperature, we can perform an analytic computation by using the Gross-Witten ansatz [17] as an approximation, and we can

[2] On the compact space, we have a tunable dimensionless parameter $R\Lambda$, where R is the radius of S^3 and Λ is a cut off scale. If we take $R\Lambda \ll 1$, then the Yang-Mills coupling can be set small even at low energy. Even with this fact we set $R = 1$ for simplicity.

[3] Previous attempts to apply the AdS/CFT correspondence to the localized tachyon condensation have been given in [12, 13, 14, 15, 16].

discuss the dominant contribution to the total partition function.

The organization of this paper is as follows. In the next section, we define the orbifold gauge theories on $\mathbb{R} \times S^3/\mathbb{Z}_k$ with generic holonomy by utilizing the standard orbifold method as in [18]. We compute the partition function for the orbifold gauge theories by following [2, 3]. In section 3 we analyze the partition function and discuss the phase structure. We observe that the dominant contribution comes from the Casimir energy at low temperature, and we compute the Casimir energy for the case with general holonomy. Near the critical temperature we solve the partition function analytically by making use of the analysis in [17]. At very high temperature, we obtain the partition function as an expansion of the temperature T, which does not depend on the choice of holonomy. In section 4 we analyze the large 't Hooft coupling limit in the dual gravity description. We discuss the relations to the arrangement of k NS5-branes for the case with the periodic boundary condition and to the localized tachyon condensation for the case with the anti-periodic boundary condition. Section 5 is devoted to conclusion and discussions. In appendix A we compute the partition function of single scalar particle as an example. In appendix B the index proposed in [19] is computed for the supersymmetric orbifold theory.[4]

2 Orbifold gauge theories

We consider $\mathcal{N} = 4$ super Yang-Mills theories on $\mathbb{R} \times S^3/\mathbb{Z}_k$ with large N $U(N)$ gauge symmetry. The 3-sphere S^3 has $SO(4) \simeq SU(2)_1 \times SU(2)_2$ isometry, and we divide the gauge theories by $4\pi/k$ rotation along the χ-cycle of $U(1)_\chi \subset SU(2)_2$. Since $\pi_1(S^3/\mathbb{Z}_k) = \mathbb{Z}_k$, we can assign a non-trivial holonomy matrix $V = P \exp i \oint A_\chi$ along the non-trivial cycle. Using the $U(N)$ gauge transformation, we can set V as a diagonal matrix $V = \mathrm{diag}(\Omega_1, \cdots, \Omega_N)$. Because of the condition $V^k = 1$, the element should be a k-th root of unity $\Omega_j^k = 1$. Therefore, we can label the vacua by $(n_0, \cdots, n_{k-1})$ with $N = \sum_{I=0}^{k-1} n_I$, where n_I represents the number of j such that $\Omega_j = \omega^I$ ($\omega = \exp(2\pi i/k)$). Then the orbifold theories are defined by projecting the Hilbert space into the orbifold invariant subspace as below. In the following we consider two special vacua. One is the $\mathbb{Z}_k$ symmetric vacuum[5] with $n_I = N/k$ for all I. Due to the $\mathbb{Z}_k$ symmetry the dual geometry can be identified as the standard orbifold. The other vacua do not preserve the $\mathbb{Z}_k$ symmetry, thus the dual geometry should be a deformation of the orbifold. The other important vacuum is with the trivial holonomy $V = 1$ or equivalently $n_0 = N$. In this case the $\mathbb{Z}_k$ symmetry is maximally broken.

[4]Similar computations were done in [20] for quiver gauge theories, which are constructed as orbifolds different from ours.

[5]In this case N is assumed to be $N = k\mathbb{Z}$. However, for large N and finite k, the difference from the general N case should be negligible.

2.1 Partition function

We would like to compute the partition function of gauge invariant operator in the orbifold theories. In general, the counting of gauge invariant operator is an involved task. Fortunately, it was shown in [2, 3] that the partition function of gauge invariant operator can be written in terms of single-particle partition functions as

$$Z(x) = \int [dU] \exp \left[\sum_{\mathcal{R}} \sum_{n=1}^{\infty} \frac{1}{n} z^{\mathcal{R}}(x^n) \chi_{\mathcal{R}}(U^n) \right] . \tag{2.1}$$

At this stage, a gauge group G could be arbitrary, and the sum is taken over all representation $\mathcal{R}$ of the gauge group G. We denote $\chi_{\mathcal{R}}(U)$ as the character for representation $\mathcal{R}$, and $[dU]$ as the Haar measure for the group element U. The partition function of single particle in the representation $\mathcal{R}$ is computed by

$$z^{\mathcal{R}}(x) = \sum_{E} x^E , \tag{2.2}$$

where E denotes the energy eigenvalue. The condition of gauge invariance comes from the integral over U, and the variable U will be identified as the holonomy matrix along the thermal cycle.

We can adopt any gauge group G and representation $\mathcal{R}$ in the formula (2.1), and several interesting examples have their dual gravity description. The most famous one arises from the N D3-brane worldvolume theory, which is dual to superstrings on $AdS_5 \times S^5$. The theory is $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbb{R} \times S^3$ with gauge group $G = U(N)$, where the states are in the adjoint representation. Our interest is on the orbifold gauge theories with holonomy along the non-trivial cycle, where the existence of the holonomy $(n_0, \cdots, n_{k-1})$ breaks the gauge symmetry into $G = \prod_{I=0}^{k-1} U(n_I)$.[6] With respect to the broken gauge group, the states are in the adjoint representation for $U(n_I)$ or in the bi-fundamental representation $(n_I, \bar{n}_J)$ for $U(n_I) \times U(n_J)$.

The spectrum of the orbifold theories can be obtained by projecting the spectrum on S^3 into the orbifold invariant subspace. The spectrum on S^3 can be obtained from the spherical harmonic analysis as in [21]. The theory includes 6 scalers, a gauge field and 4 Majorana fermions. The scalars can be expanded by the scalar spherical harmonics $S_{j,m,\bar{m}}(\Omega)$, where Ω represents the coordinates of S^3. The eigenfunctions of Laplace operator on S^3 are given as

$$\nabla^2 S_{j,m,\bar{m}}(\Omega) = -j(j+2) S_{j,m,\bar{m}}(\Omega) . \tag{2.3}$$

The labels $(m, \bar{m})$ are eigenvalues of J^3 and $\bar{J}^3$ for $SU(2)_1$ and $SU(2)_2$, and they run $-j/2, -j/2+1, \cdots, j/2-1, j/2$. The projection into the orbifold invariant modes is

[6] The diagonal $U(1)$ parts of each $U(n_I)$ may be decoupled from the rest, but the difference can be ignored when n_I are very large. See, however, appendix B for the case of an index.

performed by the projection operator $P = \frac{1}{k}\sum_{I=0}^{k-1}\Gamma^I$, where Γ represents the orbifold action. For a bi-fundamental state $(n_I, \bar{n}_J)$ or an adjoint state with $I = J$, the orbifold action is given by

$$\Gamma = e^{4\pi i \bar{J}_3/k}\omega^{I-J} \ , \tag{2.4}$$

where $\omega = \exp(2\pi i/k)$. The orbifold action consists of two parts. The first part is the phase shift due to the $\mathbb{Z}_k$ rotation along the χ-cycle. The second part is the holonomy for the bi-fundamental state $(n_I, \bar{n}_J)$. We can see from (2.4) that the orbifold invariant modes are restricted to $2\bar{m} = J - I \bmod k$. As a notation we define $0 \le L < k$ subject to $L = J - I \bmod k$. Now we can compute the partition function for the single scalar particle (2.2) as

$$\begin{aligned} z_S^{I,J}(x) &= \frac{(x^{L+1} - x^{L+3} + x^{-L+1} - x^{-L+3})kx^k}{(1-x^2)^2(1-x^k)^2} \\ &+ \frac{(L+1)x^{L+1} - (L-1)x^{L+3} - (L-1)x^{k-L+1} + (L+1)x^{k-L+3}}{(1-x^2)^2(1-x^k)} \ . \end{aligned} \tag{2.5}$$

See appendix A for the detail. We have used the fact that the energy is given by $E = j+1$ for a scaler on S^3 conformally coupled to gravity.

We move to the gauge field, which is expanded by the vector spherical harmonics $V^{\pm}_{j,m,\bar{m}}(\Omega)$.[7] We use the notation such that the vector index is contracted with an auxiliary unit vector $\hat{\xi}_\mu$ as $V^{\pm}_{j,m,\bar{m};\mu}\hat{\xi}^\mu$. The vector spherical harmonics $V^{+}_{j,m,\bar{m}}$ and $V^{-}_{j,m,\bar{m}}$ belong to the representations $(j_1, j_2) = (\frac{j+1}{2}, \frac{j-1}{2})$ and $(\frac{j-1}{2}, \frac{j+1}{2})$, respectively. The eigenvalues of Laplace operator on S^3 are

$$\nabla^2 V^{\pm}_{j,m,\bar{m}}(\Omega) = -(j+1)^2 V^{\pm}_{j,m,\bar{m}}(\Omega) \ . \tag{2.6}$$

The orbifold action to the vector spherical harmonics is the same as in the scalar case (2.4), since the vector index is contracted with an auxiliary unit vector. Therefore, the orbifold projection allows only the modes with $2\bar{m} = J - I \bmod k$ for the bi-fundamental state with $(n_I, \bar{n}_J)$. The partition function is then given by

$$\begin{aligned} z_{V^+}^{I,J}(x) &= \frac{(x^{L+2} - x^{L+4} + x^{-L+2} - x^{-L+4})kx^k}{(1-x^2)^2(1-x^k)^2} \\ &+ \frac{(L+3)x^{L+2} - (L+1)x^{L+4} - (L-3)x^{k-L+2} + (L-1)x^{k-L+4}}{(1-x^2)^2(1-x^k)} \end{aligned} \tag{2.7}$$

[7] A longitudinal mode is expanded by the scalar spherical harmonics as $\vec{\nabla}S$, and we do not consider it.

for $V^+_{j,m,\bar{m}}$ and

$$
\begin{aligned}
z^{I,J}_{V^-}(x) &= \frac{(x^L - x^{L+2} + x^{-L} - x^{-L+2})kx^k}{(1-x^2)^2(1-x^k)^2} \\
&+ \frac{(L-1)x^L - (L-3)x^{L+2} - (L+1)x^{k-L} + (L+3)x^{k-L+2}}{(1-x^2)^2(1-x^k)}
\end{aligned}
\tag{2.8}
$$

for $V^-_{j,m,\bar{m}}$. We have defined $L = J - I \mod k$ ($0 \le L < k$) as before.

Fermions are expanded by the spherical harmonics $F^+_{j,m,\bar{m}}(\Omega)$ and $F^-_{j,m,\bar{m}}(\Omega)$, which belong to $(j_1, j_2) = (\frac{j}{2}, \frac{j-1}{2})$ and $(\frac{j-1}{2}, \frac{j}{2})$. The spinor index is again contracted with an auxiliary spinor χ^α as $F^\pm_{j,m,\bar{m};\alpha}\chi^\alpha$. The eigenvalues of Laplace operator on S^3 are

$$
\nabla^2 F^\pm_{j,m,\bar{m}}(\Omega) = -(j+\tfrac{1}{2})^2 F^\pm_{j,m,\bar{m}}(\Omega) \ . \tag{2.9}
$$

For fermions we can assign two types of boundary conditions along the non-trivial cycle. The orbifold action depends on the boundary condition as

$$
\Gamma = \pm e^{4\pi i \bar{J}_3/k}\omega^{I-J} \ , \tag{2.10}
$$

where $+$ and $-$ means the periodic and the anti-periodic boundary conditions, respectively. For the periodic boundary condition, the orbifold invariant modes are given by those with $2\bar{m} = J - I \mod k$. The anti-periodic boundary condition can be assigned only for even k, and the restriction is shifted by $k/2$ as $2\bar{m} = J - I + k/2 \mod k$. For the periodic boundary condition, the partition function can be computed as

$$
\begin{aligned}
z^{I,J}_{F^+}(x) &= \frac{(x^{L+\frac{3}{2}} - x^{L+\frac{7}{2}} + x^{-L+\frac{3}{2}} - x^{-L+\frac{7}{2}})kx^k}{(1-x^2)^2(1-x^k)^2} \\
&+ \frac{(L+2)x^{L+\frac{3}{2}} - Lx^{L+\frac{7}{2}} - (L-2)x^{k-L+\frac{3}{2}} + Lx^{k-L+\frac{7}{2}}}{(1-x^2)^2(1-x^k)}
\end{aligned}
\tag{2.11}
$$

for $F^+_{j,m,\bar{m}}$ and

$$
\begin{aligned}
z^{I,J}_{F^-}(x) &= \frac{(x^{L+\frac{1}{2}} - x^{L+\frac{5}{2}} + x^{-L+\frac{1}{2}} - x^{-L+\frac{5}{2}})kx^k}{(1-x^2)^2(1-x^k)^2} \\
&+ \frac{Lx^{L+\frac{1}{2}} - (L-2)x^{L+\frac{5}{2}} - Lx^{k-L+\frac{1}{2}} + (L+2)x^{k-L+\frac{5}{2}}}{(1-x^2)^2(1-x^k)}
\end{aligned}
\tag{2.12}
$$

for $F^-_{j,m,\bar{m}}$. For the anti-periodic boundary condition, we should use $L = J - I + k/2 \mod k$ with $0 \le L < k$ instead of $L = J - I \mod k$.

Now we can write up explicitly the partition function (2.1) for $\mathcal{N} = 4$ super Yang-Mills theories on $\mathbb{R} \times S^3/\mathbb{Z}_k$ with holonomy $(n_0, \cdots, n_{k-1})$. The total partition function is given by summing over all the vacua. It is useful to use the formula for the character of bi-fundamental representation as $\chi_{(n_I,\bar{n}_J)}(U) = \mathrm{Tr}\, U_I \, \mathrm{Tr}\, U_J^\dagger$, where the trace is taken in the fundamental representation. The partition function is then given by

$$Z(x) = \int [\prod_I dU_I] \exp\left[\sum_{I,J} \sum_{n=1}^{\infty} \frac{1}{n} z_n^{I,J}(x) \,\mathrm{Tr}(U_I^n)\, \mathrm{Tr}(U_J^{\dagger n})\right] , \tag{2.13}$$

where the single-particle partition function is summarized as

$$z_n^{I,J}(x) = 6 z_S^{I,J}(x^n) + z_{V^+}^{I,J}(x^n) + z_{V^-}^{I,J}(x^n) + (-1)^{n+1} 4[z_{F^+}^{I,J}(x^n) + z_{F^-}^{I,J}(x^n)] \tag{2.14}$$

for the case with the periodic boundary condition and

$$z_n^{I,J}(x) = 6 z_S^{I,J}(x^n) + z_{V^+}^{I,J}(x^n) + z_{V^-}^{I,J}(x^n) + (-1)^{n+1} 4[z_{F^+}^{I,J+\frac{k}{2}}(x^n) + z_{F^-}^{I,J+\frac{k}{2}}(x^n)] \tag{2.15}$$

for the case with the anti-periodic boundary condition.

Finally let us remark on the difference from the D-brane worldvolume theories localized at the fixed point of $\mathbb{C}^n/\Gamma$ with $n = 2, 3$ [18]. Since the orbifold action acts trivially to the worldvolume in those cases, only bi-fundamental matters with $I, I \pm 1$ and adjoint gauge fields (and matters) are left under the orbifold projection. On the other hand, the orbifold action rotates S^3 by $4\pi/k$ in our case, there is a $\bar{m}$ dependent phase in (2.4). Due to this effect, bi-fundamental states with every pairs of I, J (and adjoint states with $I = J$) survive the projection each for matters, gauge field and fermions. The difference would be significant if we compare our case with the duality between superstrings on $AdS_5 \times S^5/\Gamma$ and the gauge theory coming from D3-branes at the fixed point of orbifold action Γ [22].

2.2 Path integral formulation

In the previous subsection, we have obtained the partition function of gauge invariant operator (2.13) in terms of integral over the group manifolds. However, we cannot determine the overall pre-factor in the formulation. In this subsection, we re-derive the partition function in the path integral formulation. In this derivation, we obtain the normalization depending on the Casimir energy of the gauge theories on $\mathbb{R} \times S^3/\mathbb{Z}_k$. The Casimir energy will be important when we consider the phase structure at low temperature. Moreover, we can identify U_I as the holonomy matrix for $U(n_I)$ gauge group along the thermal cycle.

The path integral for the partition function with a finite temperature T may be computed on $S^1 \times S^3/\mathbb{Z}_k$, where S^1 is the thermal cycle with periodicity $\beta = 1/T$. Along the thermal cycle we assign the anti-periodic boundary condition for the fermions.[8] We

[8]Due to this boundary condition, supersymmetry is always broken at a finite temperature even for the theory supersymmetric at zero temperature.

start from fixing the gauge symmetry and then introduce the Faddeev-Popov determinant conjugate to the gauge fixing. We adopt the Coulomb gauge

$$\nabla_a A^a = 0 \tag{2.16}$$

with ∇_a as covariant derivatives along the S^3 direction ($a = 1, 2, 3$). If we do not include a non-trivial holonomy, then there are spatially constant modes of the gauge field. The presence of holonomy $(n_0, \cdots, n_{k-1})$ breaks the gauge group into $\prod_I U(n_I)$ and spatially constant modes are left only for $\prod_I U(n_I)$.[9] The time-dependence of these modes is not fixed by the Coulomb gauge (2.16), and we fix these degrees by

$$\partial_t \alpha^I = 0 \ , \qquad \alpha^I = \frac{1}{\mathrm{Vol}(S^3/\mathbb{Z}_k)} \int d\Omega A_t^I \ , \tag{2.17}$$

where the integration is performed over $S^3/\mathbb{Z}_k$.

First we consider the Faddeev-Popov determinant conjugate to (2.17), which is given by

$$\Delta_{\mathrm{FP}}^I = \det{}'(\partial_t D_t^I) \ , \qquad D_t^I = \partial_t - i[\alpha^I, \ \ ] \ . \tag{2.18}$$

The determinant is taken over the non-zero modes. Diagonalizing the zero modes as $\alpha^I = \mathrm{diag}(\alpha_1^I, \cdots \alpha_{n_I}^I)$, the measure can be written as

$$d\alpha^I = \prod_i d\alpha_i^I \prod_{i,j} |\alpha_i^I - \alpha_j^I| \ , \tag{2.19}$$

where the Van der Monde determinant arises from the integration over the off diagonal elements. Now that the bosonic modes are periodic along the thermal cycle, they can be expanded by the function $\exp(2\pi i n t/\beta)$ with $n \in \mathbb{Z}$.[10] Thus the determinant can be written as

$$\Delta_{\mathrm{FP}}^I = \prod_{i,j} \prod_{n \neq 0} \frac{2\pi i n}{\beta} \left(\frac{2\pi i n}{\beta} - i(\alpha_i^I - \alpha_j^I) \right) \ . \tag{2.20}$$

With the help of the formula $\prod_{n=1}^{\infty}(1 - x^2/n^2) = \sin \pi x/(\pi x)$, we find up to an overall factor

$$[dU_I] = d\alpha^I \Delta_{\mathrm{FP}}^I = \prod_i d\alpha_i^I \prod_{i<j} \sin^2 \left(\frac{\beta(\alpha_i^I - \alpha_j^I)}{2} \right) \ , \tag{2.21}$$

which is the Haar measure of $U_I = \exp(i\beta\alpha^I)$.

[9]The projection under the orbifold action (2.4) removes spatially constant modes with $\bar{m} = 0$ for $I \neq J$ sectors.

[10]For fermionic modes, we should replace n by $n + 1/2$ since we have assigned anti-periodic boundary condition along the thermal cycle.

The Faddeev-Popov determinant conjugate to (2.16) is given by

$$\det \nabla_a D^a = \int [dc d\bar{c}] \exp(-\bar{c}\nabla_a D^a c) \ , \tag{2.22}$$

which should be added to the action of $\mathcal{N}=4$ super Yang-Mills theory. Notice that the ghosts are expanded by the scalar spherical harmonics projected into the orbifold invariant modes. After integrating over the massive modes including the c-ghosts, the partition function is given in terms of integral over the zero modes as

$$Z(T) = \int [\prod_I dU_I] e^{-S(U)} \ . \tag{2.23}$$

Let us first compute the contribution from the gauge field and the c-ghosts. Since the longitudinal modes $\vec{\nabla}S$ and A_t (except the zero modes α^I) are expanded by the scalar spherical harmonics, the contributions to the path integral from the c-ghosts and the longitudinal modes cancel out. Therefore, the contribution reduces to the Gaussian integral over the vector spherical harmonics, which is evaluated as

$$S(U) = \frac{1}{2}\sum_{I,J}\sum_{E}[n_{V^+}^{I,J}(E) + n_{V^-}^{I,J}(E)] \ln\det(-D_t^2 + E^2) \ . \tag{2.24}$$

We denote $n_{V^\pm}^{I,J}(E)$ as the degeneracy of eigenstates with E in the representation $(n_I, \bar{n}_J)$. Following the computation in [3], we find

$$\begin{aligned} S(U) = &\frac{1}{2}\sum_{I,J}\beta n_I n_J \sum_{E}[n_{V^+}^{I,J}(E) + n_{V^-}^{I,J}(E)]E \\ &-\sum_{I,J}\sum_{n=1}^{\infty}\frac{1}{n}[z_{V^+}^{I,J}(e^{-n/T}) + z_{V^-}^{I,J}(e^{-n/T})] \operatorname{Tr}(U_I^n)\operatorname{Tr}(U_J^{\dagger n}) \ . \end{aligned} \tag{2.25}$$

In the same way, we can compute the contributions from scalars and fermions and summarize all the contributions as[11]

$$S(U) = \beta V_0 - \sum_{I,J}\sum_{n=1}^{\infty}\frac{1}{n} z_n^{I,J}(e^{-1/T}) \operatorname{Tr}(U_I^n)\operatorname{Tr}(U_J^{\dagger n}) \ , \tag{2.26}$$

where $z_n^{I,J}$ are the single-particle partition functions (2.14) or (2.15). The first term is the Casimir energy

$$V_0 = \frac{1}{2}\sum_{I,J} n_I n_J \sum_{E}[6n_S^{I,J}(E) + n_{V^+}^{I,J}(E) + n_{V^-}^{I,J}(E) - 4n_{F^+}^{I,J}(E) - 4n_{F^-}^{I,J}(E)]E \ . \tag{2.27}$$

[11] In the expression of (2.23) with (2.26), the normalization has been set by dividing holonomy independent factors. Introducing the holonomy $(n_0, \cdots, n_I)$ along the χ-cycle, the lowest modes in $U(N)/\prod_I U(n_I)$ become space-dependent, and hence they can be fixed by the Coulomb gauge (2.16) instead of (2.17). The normalization of each Faddeev-Popov determinants may depend on the choice of holonomy along the χ-cycle, but the sum of both should not.

Compared with the expression of (2.13), the integral variables U_I are identified with the holonomy matrices $U_I = \exp(i\beta\alpha^I)$ with respect to the gauge group $\prod_I U(n_I)$. In this way we can see that the previous expression only includes the finite temperature contribution. The zero temperature contribution, which comes from the Casimir energy, should be included in the partition function.

3 Phase transitions of the gauge theories

In the previous section, we have obtained the partition function of gauge invariant operator in terms of integral over U_I as (2.23) with (2.26). In this section, we perform the U_I integral in the large N limit[12] and examine the phase structure of the orbifold theories. For large N and fixed k, it is natural to assume that n_I in the label of holonomy $(n_0, \cdots, n_{k-1})$ are very large. In case that some of n_I are very small, then they may be set zero in this limit. We consider two specific vacua in the following. One is the $\mathbb{Z}_k$ symmetric holonomy vacuum with $n_I = N/k$ and the other is the trivial holonomy vacuum with $n_0 = N$. In these cases our assumption is valid. In the next subsection we study the low temperature phase, and in subsection 3.2 we focus on the Casimir energy contribution. In subsection 3.3 we obtain an analytic expression under an approximation near the critical temperature. In subsection 3.4 we take the high temperature limit, where the analysis becomes simpler.

3.1 Critical temperatures

It is convenient to diagonalize the eigenvalues of holonomy matrix U_I as $\exp(i\theta_{I,i})$ with $-\pi \leq \theta_{I,i} < \pi$.[13] For large n_I the discrete elements may be replaced by a continuous parameter θ_I with a density $\rho^I(\theta_I)$. The density has to satisfy $\rho^I(\theta_I) \geq 0$ and the normalization is set as $\int_{-\pi}^{\pi} \rho^I(\theta_I) d\theta_I = 1$. In this approximation the effective action (2.26) becomes

$$S[\rho^I(\theta_I)] = \beta V_0 - \sum_{I,J} n_I n_J \int d\theta_I d\theta'_J \rho^I(\theta_I)\rho^J(\theta'_J) \left[\delta_{I,J} \ln \left|\sin\left(\frac{\theta_I - \theta'_J}{2}\right)\right| + \sum_{n=1}^{\infty} \frac{1}{n} z_n^{I,J}(x) \cos(n(\theta_I - \theta'_J))\right] \tag{3.1}$$

with $x = e^{-1/T}$. The first term in the bracket arises from the change of measure $[dU_I] \to [d\theta_{I,i}]$. In terms of the Fourier transform $\rho_n^I = \int d\theta_I \rho^I(\theta_I) \cos(n\theta_I)$,[14] the effective action

[12] Finite N effects may be examined by following the analysis in [23, 24, 25].

[13] The eigenvalue $\theta_{I,i}$ is related with the zero modes in (2.17) as $\theta_{I,i} = \beta\alpha_i^I$.

[14] We assume that θ_I is distributed symmetrically around $\theta_I = 0$.

(3.1) is given by

$$S[\rho_n^I] = \beta V_0 + \sum_{I,J} n_I n_J \sum_{n=1}^{\infty} \rho_n^I \bar{\rho}_n^J V_n^{I,J}(x) \ , \tag{3.2}$$

where

$$V_n^{I,J}(x) = \frac{1}{n}(\delta_{I,J} - z_n^{I,J}(x)) \ . \tag{3.3}$$

At enough low temperature, the repulsive force coming from the first term of (3.3) dominates, and the uniform distribution $\rho_n^I = 0$ for $n \geq 1$ is the classical solution to the effective action (3.2). Therefore, there are no order $\mathcal{O}(N^2)$ (nor order $\mathcal{O}(N)$) contributions from the effective action except for the Casimir energy term βV_0. An order $\mathcal{O}(1)$ contribution comes from the Gaussian integral as

$$\prod_{n=1}^{\infty} \frac{1}{\det\left(n_I n_J V_n^{I,J}(x)\right)} \ , \tag{3.4}$$

where the determinant is over the labels $0 \leq I, J < k$. As the temperature increased, the second term of (3.3) contributes to the potential, and the determinant would vanish at a critical temperature $x_c = \exp(-1/T_c)$. Above the critical temperature, the distribution becomes non-uniform and the classical contribution is of order $\mathcal{O}(N^2)$.

Let us examine two concrete examples. We start from the trivial holonomy case, where the action (3.2) reads

$$S[\rho_n^0] = \beta V_0 + N^2 \sum_{n=1}^{\infty} |\rho_n^0|^2 \frac{1}{n}(1 - z_n^{0,0}(x)) \ . \tag{3.5}$$

At enough low temperature, the coefficients of $|\rho_n^0|^2$ are positive, and $\rho_n^0 = 0$ for $n \geq 1$ is the saddle point. Now that the coefficients are 1×1 matrices, the determinant (3.4) is simply

$$\prod_{n=1}^{\infty} \frac{1}{1 - z_n^{0,0}(x)} \ . \tag{3.6}$$

We have changed the normalization such that only the Casimir energy term is left at $x = 0$. As the temperature increased, the coefficients of $|\rho_n^0|^2$ become smaller, and at a critical temperature $T = T_c$, a coefficient vanishes. Since the single-particle partition function is a monotonically increasing function of x, the first zero comes from the $n = 1$ part when $1 - z_1^{0,0}(x) = 0$. The critical temperatures x_c and T_c are summarized for small k in Table 1. We should note that for $k = 1$ the critical temperature reduces to the one for $\mathbb{R} \times S^3$ case [2, 3] as $x_c = 7 - 4\sqrt{3} = 0.071797$ or $T_c = 0.379663$.

Another interesting case may be with the $\mathbb{Z}_k$ symmetric holonomy $n_I = N/k$ for all I. In this case the action (3.2) is given by

$$S[\rho_n^I] = \beta V_0 + \frac{N^2}{k^2} \sum_{I,J} \sum_{n=1}^{\infty} \rho_n^I \bar{\rho}_n^J \frac{1}{n}(\delta_{I,J} - z_n^{I,J}(x)) \ . \tag{3.7}$$

k	x_c (periodic)	T_c (periodic)	x_c (anti-periodic)	T_c (anti-periodic)
2	0.095663	0.426090	0.095663	0.426090
4	0.104448	0.442661	0.127999	0.486445
6	0.104684	0.443104	0.139545	0.507777
8	0.104689	0.443113	0.142528	0.513290
10	0.104689	0.443113	0.143136	0.514414

Table 1: *The critical temperatures* $x_c = \exp(-1/T_c)$ *and* T_c *for the* $\mathcal{N} = 4$ *super Yang-Mills theories on* $\mathbb{R} \times S^3/\mathbb{Z}_k$ *with the trivial holonomy* $V = 1$. *We set* $k = 2, 4, 6, 8, 10$. *The periodic and anti-periodic boundary conditions are assigned for the fermions along the* χ*-cycle.*

Notice that the coefficients of ρ_n^I take the form of a circulant matrix since $z_n^{I,J}$ only depends on the difference $I - J$. Using the formula for a circulant determinant (B.11), the determinant (3.4) can be written in a compact form as

$$\prod_{n=1}^{\infty} \prod_{I=0}^{k-1} \frac{1}{1 - \sum_{J=0}^{k-1} \omega^{IJ} z_n^{0,J}(x)} \ . \tag{3.8}$$

As before it is enough to focus on the $n = 1$ factors. If we increase the temperature, then the denominator diverges when $\sum_J z_1^{0,J}(x) = 1$. Among the other factors, this factor gives the divergence with smallest x_c since $z_1^{0,J}(x)$ is positive for all J. Remarkably the critical temperature is the same for all k and for both the spin structures as $x_c = 0.071797$ or $T_c = 0.379663$ as in the $\mathbb{R} \times S^3$ case. Actually this is an expected result since the sum of all sectors with weight one $\sum_J z_1^{0,J}(x)$ is the same as the single-particle partition function for the $\mathbb{R} \times S^3$ case.

3.2 Casimir energies

At low temperature, the determinant (3.4) is of order $\mathcal{O}(1)$ and the contribution from the Casimir energy (2.27) is dominant. The Casimir energy is an important quantity since it is supposed to correspond to the mass of the dual geometry. In order to compute Casimir energy, we have to sum over infinitely many states, and this may lead to a divergent result. Thus we have to choose a regularization, but it is a subtle problem for quantum field theory on a curve background [26]. Fortunately, we will find that the Casimir energies in our case are finite, thus we do not need to worry about this difficult issue.

In order to regularize the infinite sum in the Casimir energy (2.27), we first introduce a cut off factor $e^{-E/\mu}$ as

$$\sum_E (6n_S^{I,J} + n_{V+}^{I,J} + n_{V-}^{I,J} - 4n_{F+}^{I,J} - 4n_{F-}^{I,J}) E e^{-E/\mu} \ , \tag{3.9}$$

and finally take the limit $\epsilon = 1/\mu \to 0$. This regularization may be justified by the fact that no divergent terms are left in the final results. The above regularization can be realized by using the identity ($A = S, V^{\pm}, F^{\pm}$)

$$\sum_E n_A^{I,J}(E) E e^{-\epsilon E} = -\frac{d}{d\epsilon} z_A^{I,J}(e^{-\epsilon}) \ , \tag{3.10}$$

so we need the expansion of single-particle partition functions by ϵ up to $\mathcal{O}(\epsilon)$ as

$$z_S^{I,J}(e^{-\epsilon}) \sim \frac{2}{\epsilon^3 k} - \frac{7\epsilon}{180k} + \frac{\epsilon k}{36} + \frac{\epsilon k^3}{360} - \frac{\epsilon L(k-L)}{6k} - \frac{\epsilon L^2(k-L)^2}{12k} \ , \tag{3.11}$$

$$\begin{aligned} z_{V^\pm}^{I,J}(e^{-\epsilon}) \sim & \frac{2}{\epsilon^3 k} - \frac{1}{\epsilon k} \pm \frac{k}{6} \pm \frac{1}{3k} \mp L \pm \frac{L^2}{k} + \left(\frac{1 \mp 1}{2}\right)\delta_{L,0} + \frac{2\epsilon}{45k} - \frac{5\epsilon k}{36} + \frac{\epsilon k^3}{360} \\ & + \frac{5\epsilon L(k-L)}{6k} - \frac{\epsilon L^2(k-L)^2}{12k} \ , \end{aligned} \tag{3.12}$$

$$\begin{aligned} z_{F^\pm}^{I,J}(e^{-\epsilon}) \sim & \frac{2}{\epsilon^3 k} - \frac{1}{4\epsilon k} \pm \frac{k}{12} \mp \frac{1}{12k} \mp \frac{L}{2} \pm \frac{L^2}{2k} + \frac{83\epsilon}{2880k} - \frac{\epsilon k}{72} + \frac{\epsilon k^3}{360} \\ & + \frac{\epsilon L(k-L)}{12k} - \frac{\epsilon L^2(k-L)^2}{12k} \ . \end{aligned} \tag{3.13}$$

Below we discuss the cases with the periodic and the anti-periodic boundary conditions separately.

For the case with the periodic boundary condition, we have

$$\sum_E (6n_S^{I,J} + n_{V^+}^{I,J} + n_{V^-}^{I,J} - 4n_{F^+}^{I,J} - 4n_{F^-}^{I,J}) E e^{-\epsilon E} = \frac{3}{8k} + \mathcal{O}(\epsilon) \ . \tag{3.14}$$

As mentioned before, the final expression does not depend on the cut off parameter ϵ, which might be due to the large supersymmetry. Moreover, the Casimir energy does not depend on the indices I, J, and this means that the Casimir energy is the same for all choices of holonomy $(n_0, \cdots, n_{k-1})$. This is consistent with the argument in [6] that the vacua with different holonomy are degenerated at zero temperature. Notice that the Casimir energy

$$V_0 = N^2 \frac{3}{16k} \tag{3.15}$$

is precisely the same as the mass of $AdS_5/\mathbb{Z}_k$ as we will discuss below.

For the case with the anti-periodic boundary condition, the above cancellation among I, J dependent terms does not occur in general. With $L = J - I \mod k$ $(0 \le L < k)$ we find

$$\begin{aligned} & \sum_E (6n_S^{I,J} + n_{V^+}^{I,J} + n_{V^-}^{I,J} - 4n_{F^+}^{I,J+\frac{k}{2}} - 4n_{F^-}^{I,J+\frac{k}{2}}) E e^{-\epsilon E} \\ & \qquad = \frac{3}{8k} + \frac{k}{6} - \frac{k^3}{24} + L^2\left[k - \frac{2}{3}\left(2L + \frac{1}{L}\right)\right] + \mathcal{O}(\epsilon) \end{aligned} \tag{3.16}$$

for $0 \leq L \leq \frac{k}{2}$ and L is replaced by $k - L$ for $\frac{k}{2} < L < k$. Notice that the divergent terms proportional to $1/\epsilon^4$ and $1/\epsilon^2$ cancel out even in this case. The Casimir energy depends on the choice of holonomy due to the L-dependence of the zero point energy (3.16). The dominant contribution to the total partition function comes from the vacuum with smallest zero point energy, which is realized with the trivial holonomy. This is because the value inside the bracket in (3.16) is always positive for all $0 \leq L \leq \frac{k}{2}$ if we set $k \geq 4$.[15] The Casimir energy for the trivial holonomy case is given by

$$V_0 = N^2 \left(\frac{3}{16k} + \frac{k}{12} - \frac{k^3}{48} \right) , \tag{3.17}$$

which will be compared with the mass of the dual geometry (4.12). Another interesting case may be with the $\mathbb{Z}_k$ symmetric holonomy. In this case, we sum up every L with the same weight, and hence we except that the cancellation between the sectors with L and $L + k/2$ occurs. This can be confirmed by a direct computation, and the Casimir energy is obtained as (3.15).[16]

Before finishing the arguments on Casimir energy, we would like to make a comment on the validity of regularization adopted here, even though the divergent terms cancel out in the final expressions. Let us write the radius R of S^3 explicitly such as $E = (j+1)/R$ for the scalar case. Then the divergent parts are proportional to $\mu^4 R^3$ and $\mu^2 R$ in each single-particle partition function. For quantum field theory on a curved background, the divergent terms of energy momentum tensor should be absorbed by the renormalization of coefficients in the Einstein-Hilbert action. In our case the divergent terms may be absorbed by the counter terms $a\mu^4 \int \sqrt{g}$ and $b\mu^2 \int \sqrt{g}\mathcal{R}$. See [26] for more detailed discussions.

3.3 Just above the critical temperatures

The eigenvalues distribute uniformly due to the repulsive potential at low temperature, however the eigenvalues get together due to the attractive potential above the critical temperature. In particular, the densities may be gaped and vanish except for $-\theta_{Ic} \leq \theta_I \leq \theta_{Ic}$. The condition that an eigenvalue θ_I does not feel any force is obtained as

$$n_I^2 \int d\theta'_I \rho^I(\theta'_I) \cot\left(\frac{\theta_I - \theta'_I}{2} \right) = 2 \sum_{J=0}^{k-1} n_I n_J \sum_{n=1}^{\infty} z_n^{I,J}(x) \rho_n^J \sin(n\theta_I) \tag{3.18}$$

from the action (3.1). The general solutions subject to the normalization condition $\rho_0^I = 1$ can be obtained by following [27, 3] in principle. However, the analysis is quite complicated generically, so we adopt an approximation by setting $z_n^{I,J}(x) = 0$ for $n > 1$. This

[15] For $k = 2$ the cases with the periodic and anti-periodic boundary conditions lead to the identical result.

[16] The Casimir energy in this case was already computed in [10, 11] by following the general method of [26].

approximation may be justified for small $x \sim x_c$ as in Table 1 by the fact that $z_n^{I,J}(x)$ with $n > 1$ is much smaller than $z_1^{I,J}(x)$. In the following we will explicitly solve these equations for the trivial holonomy case with $V = 1$ and the $\mathbb{Z}_k$ symmetric case with $n_I = N/k$.

Let us begin with the trivial holonomy case, where the condition (3.18) reduces to

$$\int d\theta_0' \rho^0(\theta_0') \cot\left(\frac{\theta_0 - \theta_0'}{2}\right) = 2z_1^{0,0}(x)\rho_1^0 \sin\theta_0 \ . \tag{3.19}$$

This case is almost the same as the $\mathbb{R} \times S^3$ case analyzed in [2, 3]. The solution is given by the form of the Gross-Witten ansatz [17] as

$$\rho^0(\theta_0) = \frac{1}{\pi \sin^2 \frac{\theta_c}{2}} \sqrt{\sin^2\frac{\theta_c}{2} - \sin^2\frac{\theta_0}{2}} \cos\frac{\theta_0}{2} \tag{3.20}$$

for $-\theta_c \leq \theta_0 \leq \theta_c$ and zero for otherwise. The parameter θ_c satisfies

$$\sin^2\frac{\theta_c}{2} = 1 - \sqrt{1 - \frac{1}{z_1^{0,0}(x)}} \ . \tag{3.21}$$

With this solution we can compute the classical action and the free energy $F = -T \ln Z = T\langle S\rangle$ as

$$\frac{F}{N^2} \simeq V_0 - T\left(\frac{1}{2\sin^2\frac{\theta_c}{2}} + \frac{1}{2}\ln\sin^2\frac{\theta_c}{2} - \frac{1}{2}\right) \ . \tag{3.22}$$

Near the critical temperature, it is given as

$$\frac{F}{N^2} \simeq V_0 - \frac{T_H}{4}(T - T_c)\frac{d}{dT}z_1^{0,0}(e^{-1/T})|_{T=T_c} + \mathcal{O}\left((T - T_c)^2\right) \ . \tag{3.23}$$

For the purpose of comparison with the $\mathbb{Z}_k$ symmetric case, we draw plots of the free energies for $k = 4, 6$ in Figure 1. In the Figure we have shifted the zero point energy by $3/(16k)$.

For the $\mathbb{Z}_k$ symmetric case, the condition (3.18) becomes

$$\int d\theta_I' \rho^I(\theta_I') \cot\left(\frac{\theta_I - \theta_I'}{2}\right) = 2\sum_{J=0}^{k-1} z_1^{I,J}(x)\rho_n^J \sin\theta_I \ , \tag{3.24}$$

and the generic solutions are quite complicated. However we only need the solution responsible to the phase transition at the critical temperature x_c satisfying $\sum_J z_1^{0,J}(x_c) = 1$. With the help of the $\mathbb{Z}_k$ symmetry, we assign that the densities of eigenvalue take the same form as $\rho^I(\theta^I) = \rho(\theta^I)$ for all I. Then the solution can be easily found as

$$\rho(\theta_I) = \frac{1}{\pi \sin^2 \frac{\theta_c}{2}} \sqrt{\sin^2\frac{\theta_c}{2} - \sin^2\frac{\theta_I}{2}} \cos\frac{\theta}{2} \tag{3.25}$$

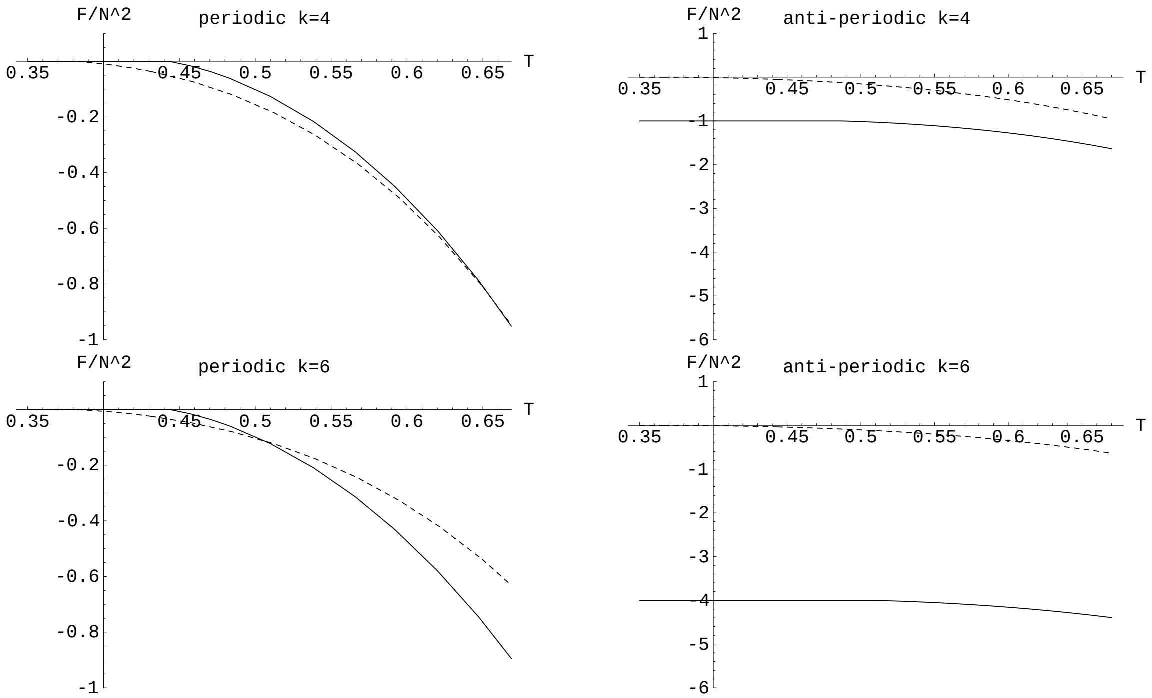

Figure 1: *Free energies $F(T)/N^2$ as functions of T in the cases with the periodic and the anti-periodic boundary conditions and with $k = 4, 6$. The solid lines are for the trivial holonomy case and the dotted lines are for the $\mathbb{Z}_k$ symmetric case.*

for $-\theta_c \leq \theta \leq \theta_c$ and zero for otherwise. In this case θ_c satisfies

$$\sin^2 \frac{\theta_c}{2} = 1 - \sqrt{1 - \frac{1}{\sum_{J=0}^{k-1} z_1^{0,J}(x)}} \, . \tag{3.26}$$

This is valid for x satisfying $\sum_J z_1^{0,J}(x) \geq 1$, and the equality holds for the critical temperature $x = x_c$. The free energy is

$$\frac{F^2}{N^2} \simeq V_0 - \frac{T}{k} \left(\frac{1}{2\sin^2 \frac{\theta_c}{2}} + \frac{1}{2} \ln \sin^2 \frac{\theta_c}{2} - \frac{1}{2} \right) \, , \tag{3.27}$$

and near the critical temperature

$$\frac{F^2}{N^2} \simeq V_0 - \frac{T_H}{4}(T - T_c) \frac{d}{dT} \frac{1}{k} \sum_{J=0}^{k-1} z_1^{0,J}(e^{-1/T})|_{T=T_c} + \mathcal{O}\left((T - T_c)^2\right) \, . \tag{3.28}$$

See Figures 1 for $k = 4, 6$.

Let us compare the free energies for the above two cases. For the case with the periodic boundary condition, the free energy for the $\mathbb{Z}_k$ symmetric case is lower near the critical temperature, since the critical temperature is smaller in this case. Near the critical temperature, the free energy is proportional to $T - T_c$ as in (3.23) and (3.28), and the coefficients in (3.28) behaves like $1/k$.[17] From this reason the free energy for the trivial

[17] This is checked for small k by a numerical computation.

holonomy case is lower at slightly higher temperature for large k as in Figure 1. For the case with the anti-periodic boundary condition, the free energy for the trivial holonomy case is lower than for the $\mathbb{Z}_k$ symmetric case due to the Casimir energy V_0. Since the Casimir energy behaves like k^3, the difference of free energy becomes bigger for larger k.

3.4 High temperature behaviors

At higher temperature the approximation in the previous subsection is not valid any more, and the contribution from $n > 1$ terms in (3.18) should be taken into account. Fortunately, the analysis becomes simpler when we take the high temperature limit $T \gg 1$. This limit is the same as the limit of large radius R of S^3, where we can use the flat space approximation. In this limit, the densities of eigenvalue may be given by the delta function, thus we set $\rho_n^I = 1$ for all n. From (3.3) we find

$$\sum_{n=1}^{\infty} V_n^L = -\zeta(4)T^3\left[\frac{16}{k} + \left(1 - \frac{1}{2^3}\right)\frac{16}{k}\right] - \zeta(2)T\left[-\frac{2}{k} + \left(1 - \frac{1}{2}\right)\left(-\frac{2}{k}\right)\right] + \mathcal{O}\left(\frac{1}{T}\right) , \tag{3.29}$$

which may be read off from the expansion of $z_A^{I,J}(e^{-\epsilon})$ by ϵ. The coefficients of T^3 terms can be given by the degrees of freedom in the flat space limit, where the volume should be divided by k. Since the above expression does not depend on L, the free energy behaves in the same way for all the vacua and for both the spin structures in the high temperature expansion. This is quite natural since the high energy excitations should not depend on the vacuum structure. In summary, the free energy $F = -T \ln Z$ and the expectation value of energy $E = -\frac{\partial}{\partial \beta} \ln Z$ are given by

$$F = -\frac{N^2}{k}\left(\frac{\pi^4}{3}T^4 - \frac{\pi^2}{2}T^2\right) + \mathcal{O}(1) , \qquad E = \frac{N^2}{k}\left(\pi^4 T^4 - \frac{\pi^2}{2}T^2\right) + \mathcal{O}(1) \tag{3.30}$$

for every choice of holonomy and for both the spin structures. This energy was already computed in [28] in a different way.

4 Dual gravity description

In the limit of large 't Hooft coupling the dual gravity description is more appropriate to discuss the phase structure of the gauge theories. In the dual picture the confinement/deconfinement phase transition is described by the Hawking-Page transition [4, 5]. In this section we investigate the Hawking-Page transition in the dual geometries and try to see whether the phase structure continues from the one at zero 't Hooft coupling. In the next subsection we review the Hawking-Page transition between the thermal AdS space and the AdS-Schwarzschild black hole. In subsection 4.2 we move to our orbifold cases. With the periodic boundary condition, the geometries are given by various arrangements of k NS5-branes in a T-dual picture. With the anti-periodic boundary condition, there are localized tachyons at the fixed point of the thermal AdS orbifold. The condensation of the localized tachyon is discussed in subsection 4.3.

4.1 Hawking-Page transition

The boundary gauge theory at finite temperature may be defined on $S^1 \times S^3$ with a thermal cycle, and we have to include all geometries with the same boundary condition to compute a path integral in gravity theory. The geometries are obtained by extending the boundary $S^1 \times S^3$ into the bulk, which can be done in two ways. One of the geometry has the topology of $S^1 \times B^4$, where the 4 dimensional ball B^4 is obtained by filling the inside of S^3. This geometry is the thermal AdS space, where the thermal cycle has a periodicity. The other geometry has the topology $B^2 \times S^3$, which is the AdS-Schwarzschild black hole. We will see that the thermal AdS space is dominant at low temperature and the AdS-Schwarzschild black hole is dominant at high temperature.

The metric of the thermal AdS_5 is given by

$$ds^2 = g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 ds^2_{S^3} \, , \qquad\qquad g(r) = \frac{r^2}{l^2} + 1 \, , \tag{4.1}$$

where $ds^2_{S^3}$ is the metric of S^3 and the Euclidean time is periodic $t \sim t + \beta_l$. The mass of the thermal AdS_5 was computed in [29] as

$$M = \frac{3\pi l^2}{32 G_5} \tag{4.2}$$

by utilizing the boundary stress tensor method. This is precisely the same as the Casimir energy of $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory on $R \times S^3$ given in (3.15) with $k = 1$. The 5 dimensional Newton constant G_5 is written as $G_5 = \pi l^3/(2N)$ in terms of the dual gauge theory. For simplicity we set the AdS radius as $l = 1$.

The metric of the AdS-Schwarzschild black hole is

$$ds^2 = h(r)dt^2 + \frac{dr^2}{h(r)} + r^2 ds^2_{S^3} \, , \qquad\qquad h(r) = r^2 + 1 - \frac{r_0^2}{r^2} \, , \tag{4.3}$$

where the period of the Euclidean time is given by $\beta_h = 2\pi r_+/(2r_+^2 + 1)$. We denote r_+ as the horizon satisfying $h(r_+) = 0$ or equivalently $r_0^2 = r_+^2 + r_+^4$.[18] The Mass of the AdS-Schwarzschild black hole is [29]

$$M = \frac{3\pi r_0^2}{8G_5} + \frac{3\pi}{32G_5} \, , \tag{4.4}$$

[18] There are two solutions $r_\pm$ to this equation, which implies that there are two types of black holes. The bigger and smaller ones r_+, r_- are the radii of horizons of big and small black holes, respectively. We only consider the big black hole since the small black hole has a negative specific heat and hence it is unstable. However the unstable saddle point may be important to understand the phase structure as pointed out in [3, 24, 25].

where the second term is the constant AdS_5 contribution. This mass can be expanded by $T = 1/\beta_h$ at high temperature as [28]

$$M = \frac{3\pi}{8G_5}(\pi^4 T^4 - \pi^2 T^2) + \mathcal{O}(1) \ , \tag{4.5}$$

which is about 3/4 times the energy given in (3.30) with $k = 1$.[19]

Now we can compute the partition function in the gravity description. In the classical approximation, the partition function is obtained from the classical actions for the geometries, which are proportional to the volume as $S = V/(2\pi G_5)$. Since the volume diverges for both cases, we introduce a cut off r_m and examine the difference. The Euclidean time periods are set equal at r_m as $\beta_l \sqrt{g(r_m)} = \beta_h \sqrt{h(r_m)}$. Then the difference of the classical action is computed as [4, 5]

$$\lim_{r_m \to \infty} \frac{V_{BH}(r_m) - V_{TAdS}(r_m)}{2\pi G_5} = \frac{\pi^2(r_+^3 - r_+^5)}{4G_5(2r_+^2 + 1)} \ . \tag{4.6}$$

For small r_+ (low temperature) the above quantity is positive, which means that the thermal AdS space dominates the partition function. The phase transition occurs at $r_+ = 1$ or $T = 3/(2\pi)$, and above the critical temperature the AdS-Schwarzschild black hole dominates.

In the dual gauge theory, the Polyakov loop $\langle \mathrm{Tr}\, U \rangle = \langle \mathrm{Tr}\, P \exp i \oint A_t \rangle$ is an important order parameter. In the confinement phase, the $\mathbb{Z}_N$ symmetry of the theory is unbroken, and hence the Polyakov loop vanishes, which is realized by uniformly distributed eigenvalues θ_i. In the deconfinement phase, since the $\mathbb{Z}_N$ symmetry is broken, the Polyakov loop may have a non-trivial value and the eigenvalues are distributed non-trivially.[20] The Polyakov (Wilson) loop along the path $\mathcal{C}$ may be computed in the gravity side as $\exp(-A)$, where A represents the minimum area of the worldsheet with boundary $\mathcal{C}$ subject to a regularization [31, 32]. For the thermal AdS space, the thermal cycle is not contractible, so the area is infinite and this leads to the vanishing Polyakov loop. For the AdS-Schwarzschild black hole, the thermal cycle shrinks at the horizon, so the area could be finite and hence the Polyakov loop can take a non-zero value. See [5] for more detail.

4.2 Phase transitions of the orbifolds

Due to the $\mathbb{Z}_k$ symmetry the vacuum with the holonomy $n_I = N/k$ for all I is supposed to be dual to the standard orbifold. At low temperature, the dual geometry is given by the orbifold of the thermal AdS space, whose metric is (4.1) with $ds^2_{S^3}$ replaced by

$$ds^2_{\mathrm{orb}} = \frac{1}{4}[(d\chi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2] \ , \tag{4.7}$$

[19] It was argued in [28] that the origin of the 3/4 difference is the same as the one of [30], where the entropy of black 3-branes is compared with the state counting on D3-brane.

[20] Because the eigenvalues collapse in the $\mathbb{Z}_N$ symmetric way, the Polyakov loop may vanish after taking the average. We may use $\langle |\mathrm{Tr}\, U| \rangle$ or $\langle \mathrm{Tr}\, U^2 \rangle$ to avoid this subtlety.

where the variables run $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \chi \leq 4\pi/k$ due to the orbifold identification. This metric of $S^3/\mathbb{Z}_k$ may be useful since we can easily see that the orbifold action acts on the χ-cycle as $\chi \to \chi + 4\pi/k$ and that there is no fixed point on this space. The mass of the thermal AdS orbifold is

$$M = \frac{3\pi}{32kG_5} \ , \tag{4.8}$$

which is $1/k$ times the mass of the thermal AdS space (4.2). This should be compared with the Casimir energy with the $\mathbb{Z}_k$ symmetric holonomy. The Casimir energy is given by (3.15) for both the spin structures and it is precisely the same as (4.8). The geometry of AdS_5 is believed to be stable under α' correction and the stability seems to continue to the standard orbifold.

The thermal AdS orbifold has a fixed point at $r = 0$ and there are closed strings in twisted sectors localized at the fixed point. With the periodic boundary condition for the fermions along the χ-cycle, there are massless states localized at the fixed point, and the different vacua are dual to different excitations of these massless states. The excitation of localized massless states does not change the global geometry, thus all the geometries contribute equally to the partition function. In particular, the mass of the every geometries should be the same as (4.8), which is also the same as the Casimir energy for the every vacua. Therefore, we can say that the phase structure does not change even at the large 't Hooft coupling.[21] For large k, the T-dual picture along the χ-cycle is relevant, where the k NS5-branes are arranged in the dual $\tilde{\chi}$-cycle [33]. In the T-dual picture, the different excitations of massless states correspond to different configurations of k NS5-branes.[22] In particular the vacuum with the trivial holonomy corresponds to the configuration of k coincident NS5-branes. With the anti-periodic boundary condition, there are localized tachyons at the fixed point as in [9]. The different vacua are dual to different condensations of these localized tachyons, which deform the geometry significantly from the thermal AdS orbifold. Since the configurations with tachyon are unstable, the relevant geometry should be the one without tachyon, which will be discussed in the next subsection.

At high temperature, the dual geometry is the orbifold of the AdS-Schwarzschild black hole (4.3) with $ds^2_{S^3}$ replaced by (4.7). The mass of the black hole may be given by the expansion of the temperature as

$$M = \frac{3\pi}{8kG_5}(\pi^4 T^4 - \pi^2 T^2) + \mathcal{O}(1) \ , \tag{4.9}$$

[21]We are not sure whether the phase structure is the same in a middle value of the 't Hooft coupling, but we guess that this is indeed the case.

[22]It was shown in [6] that the dual geometries are also labelled by the integers $(n_0, \cdots, n_{k-1})$ at zero temperature. They discussed how to construct these geometries, where the NS5-branes are replaced by flux and the back-reaction of the flux is taken into account.

which is 3/4 times the energy given in (3.30) as discussed in [28]. Since the geometry does not have any fixed point, there are no light localized modes generically. Due to this fact, the orbifold of the black hole seems to be the most relevant one, even though the every vacua are degenerated at the zero 't Hooft coupling. At relatively small temperature or for large k, there may be nearly massless modes or tachyonic modes near the horizon. With the periodic boundary condition, the T-dual picture is more relevant for large k, where the shift of NS5-branes is given by the nearly massless modes. With the anti-periodic boundary condition, the orbifold of the AdS-Schwarzschild black hole may decay into a resolved AdS orbifold through the tachyon condensation. This type of geometry decay has been discussed in [34, 35]. According to them, a kind of black hole decays into a bubble of noting by a winding tachyon condensation.

4.3 Localized tachyon condensation

If the anti-periodic boundary condition is assigned for the fermions along the χ-cycle, then there are tachyonic modes at the fix point of the thermal AdS orbifold at low temperature. The geometry deformed by a tachyon condensation was proposed by [10, 11] as[23]

$$ds^2 = g(r)dt^2 + \frac{dr^2}{g(r)f(r)} + \frac{r^2}{4}[f(r)(d\chi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2] \ , \tag{4.10}$$

with

$$g(r) = r^2 + 1 \ , \qquad f(r) = 1 - \frac{a^4}{r^4} \ , \qquad a^2 = \left(\frac{k^2}{4} - 1\right) \ . \tag{4.11}$$

They call this geometry as Eguchi-Hanson soliton. We set k even in order to assign the anti-periodic boundary condition for the fermions and also $k > 2$ for $a^2 > 0$. Because of the boundary condition for the fermions, the non-trivial χ-cycle can be pinched off. The boundary metric at $r \to \infty$ is given by (4.7) with $\chi \sim \chi + 4\pi/k$ as supposed to be. The solution is regular everywhere including $r = a$, and there are no tachyonic modes. There is no geometry for $r < a$, and this region may be replaced by the tachyon state or the nothing state in the sense of [35].

The conserved mass of the deformed geometry was computed in [10, 11] as

$$M = \frac{\pi(3 - 4a^2)}{32kG_5} = -\frac{\pi(k^4 - 8k^2 + 4)}{128kG_5} \ , \tag{4.12}$$

where $G_5 = \pi/(2N^2)$ in the gauge theory terminology. Since the non-trivial χ-cycle is pinched off at $r = a$, the Wilson loop along the χ-cycle can take non-trivial value according to the previous discussion on the Polyakov loop. This means that the $\mathbb{Z}_k$ symmetry is broken in this background, and this is consistent with the fact that the dual vacuum has

[23] See also [36].

the trivial holonomy. It is amusing to notice that the mass of the Eguchi-Hanson soliton (4.12) is about 3/4 times the Casimir energy of the gauge theory on the trivial holonomy vacuum (3.17) for large k. This reminds us of the famous 3/4 difference in the context of [30] mentioned above, but their origins are not directly related to each other. It is worth to study this issue furthermore.

At enough high temperature, only the relevant geometry seems to be the orbifold of the AdS-Schwarzschild black hole. Therefore we may observe the phase transition between the Eguchi-Hanson soliton and the orbifold of the AdS-Schwarzschild black hole. As in the previous case the partition function may be obtained from the classical actions for these geometries in the classical approximation. The difference of these actions is given by

$$\lim_{r_m \to \infty} \frac{V_{BH}(r_m) - V_{EH}(r_m)}{2\pi G_5} = \frac{\pi^2 r^+ (r_+^2 - r_+^4 - 2a^2)}{4kG_5(2r_+^2 + 1)} , \tag{4.13}$$

where the region of $r < a$ is removed in the Eguchi-Hanson soliton. The critical temperature is[24]

$$T_c = \frac{2 + \sqrt{1 + 8a^2}}{\sqrt{2\pi^2(1 + \sqrt{1 + 8a^2})}} . \tag{4.14}$$

We can see from (4.13) that the Eguchi-Hanson soliton is dominant at lower temperature and the orbifold of AdS-Schwarzschild black hole is dominant at higher temperature.

5 Conclusion and discussions

We have studied the thermodynamics of $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theories on $\mathbb{R} \times S^3/\mathbb{Z}_k$ with large N. The base manifold $S^3/\mathbb{Z}_k$ has a non-trivial cycle along the χ-direction of (4.7), and a non-trivial holonomy can be assigned along the non-trivial cycle. The theory has multi-vacua associated with the choice of holonomy labelled by k integer numbers $(n_0, \cdots, n_{k-1})$. We can assign the periodic and anti-periodic boundary conditions for the fermions along the non-trivial cycle. On a compact manifold, the Gauss constraint only allows gauge invariant operators, and due to this fact a phase transition occurs even in the zero 't Hooft coupling limit for large N. We have computed the partition function for the large N gauge theories with different holonomies by following the analysis in [2, 3], and examined the phase structure with a special care on the difference between the vacua.

At low temperature, the most relevant contribution to the free energy comes form the Casimir energy. For the case with the periodic boundary condition, the Casimir energy

[24] Notice that the critical temperature of the Hawking-Page transition $T_c = 3/(2\pi)$ is reproduced for $a = 0$.

does not depend on the choice of holonomy, and hence the vacua are degenerated. For the case with the anti-periodic boundary condition, the Casimir energy depends on the choice of holonomy, and the dominant contribution comes from the vacuum with the trivial holonomy $V = 1$. Near the critical temperature, we can obtain an approximate analytic expression of free energy by utilizing the Gross-Witten ansatz [17]. See Figure 1. For enough large k the case with the trivial holonomy seems to dominate for both the spin structures. At high temperature, the free energy is universal as in (3.30), thus the vacua are degenerated for both the spin structures.[25]

In the limit of large 't Hooft coupling, the dual gravity description is more appropriate. In the low temperature phase, the dual geometry is the orbifold of the thermal AdS space or its deformation. The orbifold of the thermal AdS space has the fixed point at $r = 0$, and there are closed string states localized at the fixed point. With the periodic boundary condition, the localized states are massless, and the excitation of these massless states leads to degenerated different geometries. In a T-dual picture, the different excitations correspond to different arrangements of k NS5-branes along the dual $\tilde{\chi}$-cycle. With the anti-periodic boundary condition, the localized states are tachyonic and a condensation of the tachyonic modes leads to the decay of geometry into the regularized geometry (4.10) called as the Eguchi-Hanson soliton [10, 11]. In the viewpoint of the dual gauge theory, the localized tachyon condensation is realized as the transition between different vacua. In particular, we have found that the mass of the Eguchi-Hanson soliton is about 3/4 times the Casimir energy for the dual vacuum with the trivial holonomy. I would like to study the relation between the vacuum transition of the gauge theory and the RG-flow or time-dependent process among the geometries as in [9]. At enough high temperature, the dual geometry is the orbifold of the AdS-Schwarzschild black hole. Since there is no fixed point in this geometry, the orbifold seems to be the most relevant geometry. This implies that the phase structure would vary as the 't Hooft coupling is changed at high temperature.

There are several theories similar to our orbifold gauge theories in the sense that the gauge theory has many vacua and its dual gravity description. One of them is (1+1) dimensional large N gauge theories on a torus [37, 38], where non-trivial holonomy matrices can be assigned along the two cycles. For the case with the periodic boundary condition of the fermions along the spatial cycle, the eigenvalues of spatial holonomy matrix correspond to the positions of N D0-branes along the T-dual spatial cycle [37, 38]. In the high temperature phase the eigenvalues are distributed uniformly, but in the low temperature phase the eigenvalues get together. This is related to the Gregory-Laflamme transition

[25] The masses of lightest twisted string modes are proportional to the horizon radius r_+, and these modes become massless (or tachyonic) in the zero 't Hooft coupling limit since the radius behaves like $r_+ \sim \alpha' \sqrt{\lambda} T$. From this reason we can assign non-trivial expectation values to the lightest modes and the vacua can be degenerated in the gauge theory description.

from a black string wrapped on a spatial cycle into a localized black hole [39]. For the case with the anti-periodic boundary condition, the spatial cycle can shrink, and indeed the thermal AdS_3 is dominant in the low temperature phase. In this phase, the $\mathbb{Z}_N$ symmetry of the holonomy matrix along the spatial cycle has to be broken. In the high temperature phase, the BTZ black hole dominates, and the $\mathbb{Z}_N$ symmetry is preserved. The relation to our cases may be examined for large k limit, where the back-reaction of NS5-branes should be taken into account. I would like to study this relation, for instance, by taking a large N, k limit with keeping a ratio N/k finite.[26]

Other interesting theories are the plane wave matrix model [41] and (2+1) dimensional super Yang-Mills theory on $\mathbb{R} \times S^2$, which are obtained by truncating the $\mathcal{N} = 4$ Yang-Mills theory on $\mathbb{R} \times S^3$ like our orbifold theory. These models share the same symmetry $SU(2|4)$ at zero temperature, and the gravity dual of these models was studied in [6]. The thermodynamics of plane wave matrix model has been studied in [42, 43, 44, 45, 46], and, in particular, the different vacua was compared in [46]. The thermodynamics of large N gauge theory on $\mathbb{R} \times S^2$ should be also interesting. The relation among these models at zero temperature has been discussed in [6, 47, 48, 8], and it is worth while investigating the relation among their thermodynamics.

In this paper, we have taken the zero 't Hooft coupling limit $\lambda = 0$, namely, the free theory limit, thus a next task is to study the effects of non-zero 't Hooft coupling. In the free theory limit, the phase transition is of the first order, but the inclusion of small λ may change the order of phase transition as discussed in [3, 49]. Moreover, we may be able to examine how the phase structure in the zero 't Hooft coupling limit continues to the one in the large 't Hooft coupling limit.[27] For example, the Casimir energy on the trivial vacuum with the anti-periodic boundary condition (3.17) is about 4/3 times the mass of the Eguchi-Hanson soliton (4.12), thus one may wonder what would happen if we include λ correction. In the high temperature phase, we have observed that the relevant geometry is the orbifold of the AdS-Schwarzschild black hole, which is dual to the $\mathbb{Z}_k$ symmetric vacuum. Since the vacua are degenerated in the zero 't Hooft coupling limit, the vacuum structure should depend on the 't Hooft coupling. We would like to study this issue as a future work.

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[26] A similar limit was taken in [40] for the case including N_f fundamental matters with a finite N_f/N, where they observed a third order phase transition just like the Gross-Witten transition [17].

[27] It might be useful to make use of the Penrose limit [41] to examine the intermediate regime. See [50, 51] for the comparison of the Hagedorn temperature.

A The partition function of single scalar particle

As an example we compute here the partition function of single scalar particle. Let us define the following function on S^3 as

$$f(x,y) = \mathrm{Tr}\, x^E y^{2j_2} \ . \tag{A.1}$$

Since a scalar is expanded by the scalar spherical harmonics $S_{j,m,\bar{m}}(\Omega)$, the above function can be easily evaluated as

$$f(x,y) = x + 2x^2(y+1/y) + 3x^3(y^2+1+1/y^2) + \cdots \ . \tag{A.2}$$

The orbifold case is given by restricting the modes into the ones invariant under the orbifold action (2.4).

In the case of the trivial holonomy $V=1$, we have to project the modes into the ones with $2\bar{m} = k\mathbb{Z}$. The projection leads to

$$f(x,y) = x + 3x^3 + \cdots + \sum_{n=1}^{\infty}(y^{nk}+1/y^{nk})[(nk+1)x^{nk+1} + (nk+3)x^{nk+3} + \cdots] \ , \tag{A.3}$$

where only terms proportional to $y^{k\mathbb{Z}}$ are kept. Using a formula

$$Lx^L + (L+2)x^{L+2} + \cdots = \frac{Lx^L - (L-2)x^{L+2}}{(1-x^2)^2} \ , \tag{A.4}$$

the function can be written as

$$f(x,y) = \frac{x+x^3}{(1-x)^2} + \sum_{n=1}^{\infty}(y^{nk}+1/y^{nk})\frac{(nk+1)x^{nk+1} - (nk-1)x^{nk+3}}{(1-x^2)^2} \ . \tag{A.5}$$

With $\sum_{n=1}^{\infty} x^n = x/(1-x)$ and $\sum_{n=1}^{\infty} nx^n = x/(1-x)^2$, we finally find for $y=1$

$$f(x,1) = \frac{(x+x^3)(1-x^{2k}) + 2kx^k(x-x^3)}{(1-x)^2(1-x^k)^2} \ . \tag{A.6}$$

This gives the expression in (2.5) with $I = J$.

The case with non-trivial holonomy can be analyzed in a similar way. For a bi-fundamental scalar $(n_I, \bar{n}_J)$, the orbifold action is given by (2.4). In this case we keep the terms proportional to $y^{k\mathbb{Z}+L}$, and we find

$$\begin{aligned} f(x,y) = &\sum_{n=0}^{\infty} y^{nk+L}[(nk+L+1)x^{nk+L+1} + (nk+L+3)x^{nk+L+3} + \cdots] \\ &+ \sum_{n=1}^{\infty} y^{nk-L}[(nk-L+1)x^{nk-L+1} + (nk-L+3)x^{nk-L+3} + \cdots] \ . \end{aligned} \tag{A.7}$$

Making use of the formula (A.4), we can reduce the above sum into the more simplified form in (2.5).

B An index for the supersymmetric orbifold theory

In this appendix the index proposed in [19] is computed for $\mathcal{N}=4$ $U(N)$ super Yang-Mills theory on $\mathbb{R}\times S^3/\mathbb{Z}_k$ with the periodic boundary condition for the fermions along the χ-cycle.[28] The index in our case is defined as

$$\mathcal{I}=\mathrm{Tr}(-1)^F e^{-\beta\Xi} t^{2(E+j_1)} v^{R_2} w^{R_3}\ , \qquad \Xi=E-2j_1-\frac{3}{2}R_1-R_2-\frac{1}{2}R_3\ . \tag{B.1}$$

Since only states satisfying $\Xi=0$ contribute the index [19], we can set $\beta\to\infty$. As before E is the energy and (j_1,j_2) are spins with respect to $SU(2)_1\times SU(2)_2$ isometry on S^3. The symmetry of the theory is $SU(2|4)$ and R_1,R_2,R_3 are R-charges. With generic holonomy $(n_0,\cdots,n_{k-1})$, the $U(N)$ gauge symmetry is reduced to $\prod_I U(n_I)$, and the states are in adjoint or bi-fundamental representation of the gauge group. As in [19, 20] or in (2.13) the index can be written by an integral of unitary matrices as

$$\mathcal{I}=\int[\prod_I dU_I]\exp\left[\sum_{I,J}\sum_{n=1}^{\infty}\frac{1}{n}f^{I,J}(t^n,v^n,w^n)\,\mathrm{Tr}(U_I^n)\,\mathrm{Tr}(U_J^{\dagger n})\right]\ , \tag{B.2}$$

where the coefficients $f^{I,J}$ are the indices for the single particles with representation $(n_I,\bar{n}_J)$. Notice that the index does not receive any corrections of 't Hooft coupling, since we are counting the modes protected by supersymmetry [19].

Let us compute the indices for single particles. We start from the scalar contribution. The theory includes three complex scalars X,Y,Z with $(R_1,R_2,R_2)=(0,1,0)$, $(1,-1,1)$, $(1,0,-1)$. The scalars are expanded by $S_{j,m,\bar{m}}(\Omega)$, and the condition $\Xi=0$ is satisfied by the modes $S_{j,j/2,\bar{m}}$. The orbifold projection means $2\bar{m}\in L+k\mathbb{Z}$ for the $L=J-I+k\mathbb{Z}$ $(0\le L<k)$ sector, and the contribution can be computed as

$$\frac{t^2(t^{3L}+t^{3(k-L)})(v+1/w+w/v)}{(1-t^6)(1-t^{3k})}\ . \tag{B.3}$$

Since the gauge field does not have any R-charges, the condition $\Xi=0$ is satisfied by the modes $V^+_{j,(j+1)/2,\bar{m}}$ with $2\bar{m}\in L+k\mathbb{Z}$. Thus the contribution from the gauge field is

$$\frac{t^6(t^{3L}+t^{3(k-L)})}{(1-t^6)(1-t^{3k})}\ . \tag{B.4}$$

For the fermions, the condition $\Xi=0$ can be satisfied by the modes $F^+_{j,j/2,\bar{m}}$ with $2\bar{m}\in L+k\mathbb{Z}$ and $(R_1,R_2,R_2)=(1,-1,0)$, $(0,1,-1)$, $(0,0,-1)$. The contribution from these fermions is

$$-\frac{t^4(t^{3L}+t^{3(k-L)})(1/v+w+v/w)}{(1-t^6)(1-t^{3k})}\ . \tag{B.5}$$

[28] See also [52, 20, 53].

The other contribution comes from the modes $F^-_{j,(j-1)/2,\bar{m}}$ with $2\bar{m} \in L+k\mathbb{Z}$ and $(R_1, R_2, R_2) = (1,0,0)$ as

$$\delta_{L,0} - \frac{(t^{3L} + t^{3(k-L)})}{(1-t^6)(1-t^{3k})} \ . \tag{B.6}$$

Interestingly the sum of all contributions can be factorized as

$$f^{I,J}(t,v,w) = \delta_{L,0} - \frac{(t^{3L} + t^{3(k-L)})(1-t^2 v)(1-t^2/w)(1-t^2 w/v)}{(1-t^6)(1-t^{3k})} \ . \tag{B.7}$$

For the indices to converge, we have to assign $t^2 < 1$, $t^2 v < 1$, $t^2/w < 1$, $t^2 w/v < 1$, which implies $\delta_{I,J} - f^{I,J}(t,v,w) > 0$.

In order to perform the integral (B.2), we assume that n_I is very large or zero. Then we can replace the discrete eigenvalues by continuous ones with the densities $\rho^I(\theta_I)$ satisfying $\int d\theta_I \rho^I(\theta_I) = 1$. In this term, the effective action is given by

$$S(\rho^I_n) = \sum_{I,J} n_I n_J \sum_{n=1} \rho^I_n \bar{\rho}^J_n V^{I,J}_n \ , \qquad V^{I,J}_n = \frac{1}{n}(\delta_{I,J} - f^{I,J}(t^n, v^n, w^n)) \ , \tag{B.8}$$

where we denote the Fourier transform of $\rho^I(\theta_I)$ as ρ^I_n. The saddle point of the action is $\rho^I_n = 0$ for $n \neq 0$ as $V^{I,J}_n > 0$, and the index is given by the determinant

$$\mathcal{I} = \prod_n \frac{1}{\det(\sum_{I,J} n_I n_J V^{I,J}_n)} \ . \tag{B.9}$$

The determinant is complicated in general, but it could be written in a simpler form for several cases.

One case is with the trivial holonomy $V = 1$. In this case, we have just 1×1 matrices, so the determinant is simply

$$\mathcal{I} = \prod_n \frac{(1-t^{6n})(1-t^{3kn})}{(1+t^{3kn})(1-t^{2n}v^n)(1-t^{2n}/w^n)(1-t^{2n}w^n/v^n)} \ . \tag{B.10}$$

The N dependent factor is removed by changing the normalization. Another interesting case may be with the $\mathbb{Z}_k$ symmetric holonomy $n_I = N/k$ for all I. In this case, it is useful to utilize a formula for a circulant determinant as in [20]

$$\begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_k \\ f_k & f_1 & f_2 & \cdots & f_{k-1} \\ f_{k-1} & f_k & f_1 & \cdots & f_{k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_2 & f_3 & f_4 & \cdots & f_1 \end{vmatrix} = \prod_{I=0}^{k-1} (f_1 + \omega^I f_2 + \omega^{2I} f_3 + \cdots + \omega^{(k-1)I} f_k) \tag{B.11}$$

with $\omega = \exp(2\pi i/k)$. Using the identity

$$\prod_{I=0}^{k-1} [\sum_{L=0}^{k-1} (t^{3L} + t^{3(k-L)}) \omega^{LI}] = \frac{(1-t^{3k})^k (1-t^6)^k}{(1-t^{3k})^2} \ , \tag{B.12}$$

the index is computed as

$$\mathcal{I} = \prod_n \frac{(1-t^{3kn})^2}{(1-t^{2n}v^n)^k(1-t^{2n}/w^n)^k(1-t^{2n}w^n/v^n)^k} \ . \tag{B.13}$$

We would like to compare these indices with the one from the gravity computation. However there is a subtle problem whether we should remove the contribution from the diagonal $U(1)^{k-1}$ part. In the case of $AdS_5 \times S^5/\mathbb{Z}_k$, the $U(1)^{k-1}$ part of the dual Yang-Mills theory is removed to compare the indices with those from the gravity computation [20]. Suppose that the similar $U(1)$ decoupling should be taken into account even in our case. The contribution from $U(1)$ part is just the same as that of $V=1$ case (B.10), so after the subtraction we have

$$\mathcal{I} = \prod_n \frac{(1-t^{3kn})^2(1+t^{3kn})^{k-1}}{(1-t^{6n})^{k-1}(1-t^{3kn})^{k-1}(1-t^{2n}v^n)(1-t^{2n}/w^n)(1-t^{2n}w^n/v^n)} \ . \tag{B.14}$$

We would like to study the $U(1)$ problem furthermore to examine whether this is indeed the case.

The dual geometry is the orbifold $AdS_5/\mathbb{Z}_k \times S^5$ with the excitation of localized string states at the fixed point, and the different excitations correspond to different vacua of the gauge theory. The index for the supergravity on $AdS_5/\mathbb{Z}_k \times S^5$ can be computed by acting the orbifold projection to the index in the covering space case [19, 20]. The single-particle index can be computed as

$$\begin{aligned} \mathcal{I}'(t,v,w) &= \frac{1}{k}\sum_{I=1}^{k}\left[\frac{t^2 v}{1-t^2 v} + \frac{t^2/w}{1-t^2/w} + \frac{t^2 w/v}{1-t^2 w/v} - \frac{t^3\omega^I}{1-t^3\omega^I} - \frac{t^3/\omega^I}{1-t^3/\omega^I}\right] \\ &= \frac{t^2 v}{1-t^2 v} + \frac{t^2/w}{1-t^2/w} + \frac{t^2 w/v}{1-t^2 w/v} - \frac{2t^{3k}}{1-t^{3k}} \ , \end{aligned} \tag{B.15}$$

and the index including the multi-particle contribution is given as

$$\mathcal{I} = \exp \sum_{n=1}^{\infty} \frac{1}{n}\mathcal{I}'(t^n, v^n, w^n) = \prod_{n=1}^{\infty} \frac{(1-t^{3kn})^2}{(1-t^{2n}v^n)(1-t^{2n}/w^n)(1-t^{2n}w^n/v^n)} \ . \tag{B.16}$$

The index is a slightly different form the both of (B.10) and (B.14), and the difference should be interpreted as the contribution from the localized string states. It is desirable to include these stringy contributions to the index from the AdS side and compare it with the index of the gauge theory on the corresponding vacuum.

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