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T-duality with H-flux: non-commutativity, T-folds and $G \times G$ structure

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Abstract

Various approaches to T-duality with NSNS three-form flux are reconciled. Non-commutative torus fibrations are shown to be the open-string version of T-folds. The non-geometric T-dual of a three-torus with uniform flux is embedded into a generalized complex six-torus, and the non-geometry is probed by D0-branes regarded as generalized complex submanifolds. The non-commutativity scale, which is present in these compactifications, is given by a holomorphic Poisson bivector that also encodes the variation of the dimension of the world-volume of D-branes under monodromy. This bivector is shown to exist in $SU(3) \times SU(3)$ structure compactifications, which have been proposed as mirrors to NSNS-flux backgrounds. The two $SU(3)$-invariant spinors are generically not parallel, thereby giving rise to a non-trivial Poisson bivector. Furthermore we show that for non-geometric T-duals, the Poisson bivector may not be decomposable into the tensor product of vectors.

Contents

1 Introduction

Compactifications with H-flux are known to give rise to topology changes and even to non-geometric situations when T-duality is performed along directions which have non-trivial support of the NSNS H-flux [1, 2, 3, 4, 5, 6, 7, 8]. Non-geometry occurs for example in the very simple situation of a three-torus endowed with an H-flux proportional to its volume form. Consider namely the three-torus as a trivial T^2-fibration over a circle. Upon T-duality along the fibre, the metric picks up a factor that makes it shrink under monodromy around the base circle. The monodromy around the base is a non-trivial element of the $O(2, 2; \mathbb{Z})$ group acting on the two-torus. This prevents a three-dimensional global Riemannian description from existing. Further T-dualizing along the base leads to more pathological situations, where points do not exist even in a local coordinate patch, and the fibres are conjectured to become non-associative [9, 10]. We will restrict ourselves to the case of two T-dualities, and assume that local coordinate patches do exist. Progress in the description of non-associative T-duals was achieved in the recent paper [11], which also contains observations on the open-string metric and non-commutativity for two T-dualities that have some overlap with ours.

Essentially three conjectures have been put forward for the description of the T-dual of a torus with H-flux:

(I) Field of non-commutative tori: Mathai and Rosenberg proposed that T-dualizing along a two-torus with non-zero H-flux yields a fibration by (or more precisely: field of) *non-commutative* tori. In particular, this fibration is encoded in a closed one-form, which is obtained by integrating the NSNS flux along the fibre directions [7, 12, 13].

(II) T-folds: these are spaces where T-dualities can act as transition functions between local patches [8]. The T-dualized directions are doubled, and T-duality transformations may patch the doubled fibres together. A sigma model with a T-fold as its target space was proposed, and its boundary conditions were studied in [14, 15, 16, 17, 18].

(III) $G \times G$ structure compactifications: $SU(3) \times SU(3)$ structure manifolds are characterized in terms of a pair of pure spinors, constructed as bilinear combinations of a pair $SU(3)$-invariant spinors of Cliff(6). In case the $SU(3)$-invariant spinors are not parallel to each other, their linear independence is encoded by a non-vanishing one-form, and the discrepancy between left- and right-moving complex structures is a potential source of non-geometry and/or non-commutativity. Moreover, [19, 59] suggest the relevance of $SU(3) \times SU(3)$ structures for mirrors of NSNS flux compactifications.

These directions of research have developed somewhat independently from each other, and it is natural to ask if they are compatible. It is also natural to expect that techniques from generalized complex geometry à la Hitchin and Gualtieri [20, 21] should bring some insights into the problem for at least two reasons:

firstly, generalized complex (GC) spaces have been related to non-commutativity in two instances: a non-commutativity scale is induced by the $(0, 2)$ component of a B-field [22], and the master equation of the generalized B-model [23] admits deformations by holomorphic Poisson bivectors into a Poisson sigma model, which is known to induce star-products in the algebra of observables [24];

secondly, the doubling of the torus fibres in T-folds reminds one of the sum of tangent and cotangent spaces considered in generalized complex geometry. But GC spaces have more structure than T-folds, indeed, in [8, 17] T-folds were pointed out to be a real version of GC spaces. Moreover, elements of $O(2, 2; \mathbb{Z})$ called B-transforms and β-transforms act on maximally isotropic subspaces as symmetries of the inner product.

We shall therefore use as a main technical tool the geometry of pure spinors, that are in one-to-one correspondence with generalized complex branes, and building blocks for $SU(3) \times SU(3)$ structure compactifications.

Our conjectures, which we will justify in the case of tori with H-flux, are:

- (I) vs. (II): The proposal (I) by Mathai and Rosenberg claims that the T-dual to a T^3 compactification with H-flux along two of the T-dualized directions yields a non-commutative torus fibration. This is reconciled with Hull's T-fold proposal by showing that the metric seen by the open strings on a T-fold is precisely the one on the non-commutative torus fibration. Thus, the proposal (I) is the open-string version of (II). This connection is discussed from various independent angles in sections 2, 3 and 4.

- (II) vs. (III): when both approaches are applicable as for the T^6 with H-flux, they yield the same T-dual or mirror geometry.

- (III) vs. (I): We show that for a generic $SU(3) \times SU(3)$ structure compactification, where the two $SU(3)$-invariant spinors are not aligned, there exists a Poisson bivector which parametrizes non-commutative deformations. The non-commutativity is however again only relevant for the open-string sector. This relation is discussed in section 5. As for the mirror of a six-torus with H-flux, we observe that the Poisson bivector can in fact not be decomposed in terms of vectors, which seems to indicate that not all the possible non-commutativity scales are inherited from $SU(3) \times SU(3)$ structures.

2 T-folds and non-commutative tori

In this section we shall mainly be concerned with the connection between non-commutativity and T-folds. We shall study this in the case of the simplest non-trivial example, which already illustrates the main point: the Mathai–Rosenberg non-commutative torus-fibrations are the open-string version of T-folds. This observation will then be discussed from the generalized geometry point of view in the next section.

The simplest example that exhibits all the key features is the T^3-compactification with k units of NSNS three-form flux $H \in H^3(T^3, \mathbb{Z})$. We shall generally refer to NSNS-flux supported on a torus bundle E with base B and fibre F of the type $H \in H^n(F) \otimes H^{3-n}(B)$ as an n-legged H-flux. Thus, the one-legged case is known to have a purely geometric T-dual. Our main focus is on the two-legged case, which will be shown to have a non-geometric T-dual.

In order to understand the T-dual along two fibre directions, we consider the three-torus as a T^2-bundle over S^1 (parametrized by x) and dualize along the fibre directions parametrized by y and z. The metric and B-field can be chosen as

$$ds^2 = dx^2 + dy^2 + dz^2 \,, \qquad B = kx\, dy \wedge dz \,. \tag{2.1}$$

Due to the B-field the monodromy $\mathcal{M}_k$ around the S^1 is non-trivial and reads

$$\mathcal{M}_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -k & 1 & 0 \\ k & 0 & 0 & 1 \end{pmatrix} , \tag{2.2}$$

in a basis adapted to the coordinates $(y, z, \tilde{y}, \tilde{z})$, where $\tilde{y}$ and $\tilde{z}$ are T-dual to y and z. Naively applying the standard Buscher rules along the fibres yields the T-dual background

$$ds^2 = dx^2 + \frac{1}{1+k^2x^2}(dy^2 + dz^2) \,, \qquad B = \frac{kx}{1+k^2x^2}\, dy \wedge dz \,, \tag{2.3}$$

and the monodromy obtained after action of the T-duality matrix

$$g_{yz} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

along the fibres is

$$\mathcal{W}_k = g_{yz}^{-1} \mathcal{M}_k g_{yz} = \begin{pmatrix} 1 & 0 & 0 & -k \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{2.4}$$

As this is a non-trivial element (which is not merely a B-field shift or an element in the geometrically acting $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$) of the T-duality group $O(2,2;\mathbb{Z})$, the resulting space is an example of a T-fold as defined by Hull in [8].

The alternative proposal by Mathai and Rosenberg [7, 12, 13] claims that the T-dual is a field C of non-commutative tori[1], $A_\theta \to C \to S^1$, where the non-commutativity scale θ depends on the base-coordinate x as

$$\theta = kx \, . \tag{2.5}$$

This proposal arose from a K-theoretical point of view by showing that the H-twisted K-theory of T^3, $K_H(T^3)$, is the same as the algebraic K-theory of the algebra associated to the field of non-commutative tori

$$K_H(T^3) = K(C) \, . \tag{2.6}$$

It is furthermore supported by the fact that it consistently generalizes the case of geometric fluxes and the T-duality action defined in this fashion is, thanks to Morita equivalence, of order two. In this approach, the action of T-duality is realized in terms of taking the crossed-product algebra [12].

We propose that both pictures are in fact valid, and are describing different aspects of the same T-dual compactification. More precisely, we shall argue that the proposal (I) is the open-string version of the T-fold proposal (II). Starting from the T-fold compactification (2.3), there is an associated open-string metric G and Theta-tensor Θ introduced and studied in [25, 26, 27, 28], which are related to the closed-string metric g and B-field B by (setting $2\pi\alpha' = 1$)

$$G^{ij} = (g+B)^{-1}_{(ij)} \, , \qquad \Theta^{ij} = (g+B)^{-1}_{[ij]} \, . \tag{2.7}$$

These are the metric and spacetime non-commutativity parameter, which the open-strings see. For the background in (2.3) we obtain

$$ds^2 = dx^2 + d\tilde{y}^2 + d\tilde{z}^2 \, , \qquad \Theta = kx \partial\tilde{y} \wedge \partial\tilde{z} \, . \tag{2.8}$$

This is precisely the non-commutative torus fibration which was proposed as the T-dual spacetime in [7]. Similar backgrounds with a varying, meaning space-dependent, non-commutativity parameter have been discussed before in [29, 30].

How do we interpret this connection? The key point is to realize that the K-theory analysis depends on the open-string data (or open-string algebra). As advocated by Witten in [31], the K-theory for H-flux backgrounds has a formulation in terms of the algebraic K-theory of a (non-)commutative algebra [32], which on the other hand can be interpreted as the open-string algebra [33, 31]. This algebra is non-commutative when $H \neq 0$. Thus in order to prove the conjectured correspondence, it remains to show that the algebra C is precisely the algebra of open-string field theory in this background.

[1] The precise definition is in terms of the direct integral of non-commutative torus algebras $C = \int_{\theta \in S^1} A_\theta d\theta$, with non-commutativity parameter θ varying along the base S^1.

On more general grounds one is then led to propose the following relation: consider a principal T^2-bundle $E \to M$ with H-flux such that $H_2 \neq 0$, where $H_2 \in H^2(T^2) \otimes H^1(M)$ ("two-legged case"). Then the T-dual along the fibre-directions is given by a T-fold. The associated open-string metric and Θ-tensor can be computed from (2.7) and the resulting space will generically be non-commutative, with an associated non-commutative algebra, A. The conjecture is then, that A is precisely the algebra proposed by Mathai and Rosenberg as the T-dual, *i.e.*, it is obtained as a crossed product algebra $A = C(E, H) \rtimes \mathbb{R}^2$, where $C(E, H)$ is the C^*-algebra of the T^2-bundle E with H-flux and the crossed product is taken with respect to the $\mathbb{R}^2$-action, which is induced from the T^2-action on the bundle, with the K-theory of the two algebras agreeing.

3 Probing non-geometry by generalized complex branes

In this section the same conclusion is reached as in the last section by embedding the discussion into the setup of generalized complex geometry. It is shown that the T-dual of the background with H-flux is given by a β-transformed background. Again, this is observed in the open-string sector, and we show this by probing the T-fold geometry with generalized complex D-branes.

3.1 Generalized complex structures, B-transforms and β-transforms

Let us recall a few definitions from generalized complex (GC) geometry [21]. Given an n-dimensional manifold M, a generalized almost complex structure on M is defined as an almost complex structure on the sum of tangent and cotangent bundles $TM \oplus T^*M$. For example, such a structure can be induced by an ordinary complex structure J on M

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix} , \tag{3.1}$$

in which case it will sometimes be termed a diagonal GC structure, or by a symplectic form ω on M

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} , \tag{3.2}$$

where the matrices are written in a base adapted to the direct sum. Hybrid examples, other than these two extreme ones, are classified by a generalized Darboux theorem [21], saying that any GC space is locally the sum of a complex space and a symplectic space. For the existence of hybrid GC structures with no underlying complex or symplectic structure, and their relevance for $\mathcal{N} = 1$ supersymmetric compactifications in string theory see [34, 35]. For the present discussion where the (non-)geometry is probed by D0-branes, we shall restrict ourselves to GC structures of the form $\mathcal{J}_J$, thus generalizing the B-model.

Here we would like to relax the requirement that the space on which the GC structure acts be globally of the form $TM \oplus T^*M$, and we only assume that it is made of patches that look like the sum of local tangent and cotangent spaces. The definitions are therefore to be understood in the neighborhood of some point p (which we assume to be still well-defined), that is on $T_pM \oplus T_p^*M$.

The sum $T_pM \oplus T_p^*M$ is naturally endowed with an inner product of signature (n, n),

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\iota_X \eta + \iota_Y \xi)\,, \tag{3.3}$$

whose matrix in the same basis as above reads

$$\mathcal{G} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

The inner product is conserved by an action of the group $O(n, n)$ whose generic element decomposes into a block-diagonal part (encoding an orthogonal transformation of the tangent space and the induced orthogonal transformation of the cotangent space), and off-diagonal blocks that can be exponentiated into B-transforms

$$\exp B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \tag{3.5}$$

$$B : X + \xi \mapsto X + \xi + \iota_X B\,, \tag{3.6}$$

and β-transforms

$$\exp \beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \tag{3.7}$$

$$\beta : X + \xi \mapsto X + \iota_\xi \beta + \xi\,, \tag{3.8}$$

where B and β are antisymmetric blocks identified with a two-form $B_{\mu\nu}$ and a bivector $\beta^{\mu\nu}$.

A B-transform acts by conjugation on generalized almost complex structures, thus mapping the two generalized almost complex structures $\mathcal{J}_J$ and $\mathcal{J}_\omega$ to the structures

$$\mathcal{J}_J(B) = \begin{pmatrix} J & 0 \\ BJ + J^t B & -J^t \end{pmatrix} \tag{3.9}$$

and

$$\mathcal{J}_\omega(B) = \begin{pmatrix} \omega^{-1} B & -\omega^{-1} \\ \omega + B\omega^{-1} B & -B\omega^{-1} \end{pmatrix}, \tag{3.10}$$

which we will encounter in section 5.

3.2 D-branes as generalized complex submanifolds

Let H be a closed three-form. A generalized submanifold is defined in [21] as a submanifold N endowed with a two-form B such that $H|_N = dB$. The generalized tangent bundle τ_N^B of this generalized submanifold is defined as the B-transform of the sum of the tangent bundle TN and conormal bundle (or annihilator) $\mathrm{Ann}\, TN$, namely:

$$\tau_N^B = \{X + \xi \in TN \oplus T^*M|_N,\ \xi|_N = \iota_X B\}\,, \tag{3.11}$$

so that $\tau_N^0 = TN \oplus \mathrm{Ann}\, TN$. A generalized tangent bundle is a maximally isotropic subspace (*i.e.*, it is isotropic with respect to $\mathcal{G}$ and it has the maximal possible dimension for an isotropic space in ambient signature (n, n), namely n.) Moreover, all the maximally isotropic subspaces are of this form, for some submanifold N and two-form B. This is the origin of the one-to-one correspondence between generalized submanifolds and pure spinors, which will be used in subsection 3.4.

Given a GC structure $\mathcal{J}$, a generalized complex brane was defined in [21] to be a generalized submanifold whose generalized tangent bundle is stable under the action of $\mathcal{J}$. In the case of $\mathcal{J} = \mathcal{J}_J$, the compatibility condition gives rise to the B-branes, as expected due to the localization properties of the B-model on complex parameters [36]. The submanifold N namely has to be a complex submanifold, and F has to be of type $(1,1)$ with respect to J

$$\begin{aligned} &J(TN) \subset TN \\ &J^*(\iota_X F) + \iota_{JX} F = 0\,. \end{aligned} \tag{3.12}$$

In the other extreme case of $\mathcal{J} = \mathcal{J}_\omega$, it yields all possible types of A-branes, including the non-Lagrangian ones [37, 38]. These are two tests of the idea that D-branes in generalized geometries are generalized submanifolds. This idea has passed further tests: calibrating forms and pure spinors encoding stability conditions [39, 40] for topological branes are correctly exchanged by mirror symmetry [41, 42, 43, 44, 45, 46], and the study of morphisms between generalized tangent bundles [47] generalizes the K-theoretic description of D-branes by taking winding numbers into account in the resolution of vortex equations of the Yang–Mills–Higgs model [48, 49, 36]. Although all the generalized tangent bundles are n-dimensional, a generalized submanifold associated to a p-dimensional submanifold N will be sometimes referred to as a generalized Dp-brane, and p will be called the ordinary dimension of the brane.

It is important for the description of D-branes in generalized geometries to note that the projection of a subspace on the tangent space is unchanged under a B-transform. A B-transform just switches on an Abelian field strength with magnitude B along the brane. However, a β-transform shifts the dimension of the projection of the brane on the tangent space (the ordinary dimension of the brane) by the rank of β. Let us review the linear-algebraic case where the ambient space is $V \oplus V^*$ for some vector space V. A β-transform of a maximally isotropic subspace $\mathrm{Ann}\, F \oplus F$, where F is a subspace of V^*, reads as a graph over F, in the notations of [21]

$$L(F, \beta) = \{X + \xi \in V \oplus F,\ X|_F = \iota_\xi \beta\}\,. \tag{3.13}$$

The intersection of this space and V is just the annihilator of F, because it is trivially embedded in $V \oplus F$ as

$$L(F, \beta) \cap V = \{X + 0 \in V \oplus F,\ X|_F = 0\} = \mathrm{Ann}\, F \ \ (= L(F, 0))\ . \tag{3.14}$$

The vector part of any element of $L(F, \beta)$ therefore decomposes into an element of $\mathrm{Ann}\, F$ and an element of the image of $\beta : V^* \to V$, and the decomposition is unique because the graph condition $X|_F = \iota_\xi \beta$ implies that the intersection between $\mathrm{Ann}\, F$ and the image of β is zero-dimensional. Let $\pi_V : V \oplus V^* \to V$ denote the projection onto V. We have therefore argued that

$$\pi_V L(F,\beta) = \mathrm{Im}\beta \oplus (L(F,\beta) \cap V) \ , \tag{3.15}$$

and therefore

$$\dim(\pi_V L(F,\beta)) = \dim(L(F,\beta)\cap V) + \mathrm{rk}\beta = \dim \mathrm{Ann}\, F + \mathrm{rk}\beta = \dim(\pi_V L(F,0)) + \mathrm{rk}\beta. \tag{3.16}$$

3.3 T-duality maps B-transforms to β-transforms

As we have just motivated the idea that Abelian D-branes may be identified with GC submanifolds, and since β-transforms can change the ordinary dimension of such submanifolds, it is natural to look for the connection between β-transforms and monodromies on T-folds, in the picture (II) of non-geometry. D-branes wrapped on T-folds can come back to themselves with a different dimension after monodromy. We are going to describe how T-dualities map B-transforms to β-transforms, together with the corresponding effects on D-branes.

3.3.1 Geometric three-torus with H-flux and B-transforms

Consider again the flat three-torus with uniform H-flux, with the same coordinates as above. Consider two D2-branes wrapping fibres over two points of the base, one at $x = 0$ and one at generic x. Going from the first to the second involves a B-transform by the two-form

$$B(x) = kx dy \wedge dz \ . \tag{3.17}$$

Going from $x = 0$ to generic x namely switches a two-form along the brane. The boundary conditions for open strings ending on a D2-brane wrapping a torus over the point x (with embedding coordinates $X, Y, Z(\sigma,\tau)$ and the obvious notation) read

$$\begin{aligned} \partial_\sigma Y + kx\partial_\tau Z &= 0, \\ \partial_\sigma Z - kx\partial_\tau Y &= 0. \end{aligned} \tag{3.18}$$

The matrix of the B-transform in a basis adapted to the coordinates (y, z) and the dual coordinates $(\tilde{y}, \tilde{z})$ reads

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -kx & 1 & 0 \\ kx & 0 & 0 & 1 \end{pmatrix} \tag{3.19}$$

3.3.2 Geometric T-dual with a connection

It is instructive to perform first the T-duality along the y direction. The D2-branes wrapping the two fibres in question become D1-branes, and parametrizing the base by an angle θ with $kx = \tan\theta$, we observe that the D1-branes are rotated with respect to each other within the fibre. This reflects the fact that they now live on a torus with a connection

$$\begin{aligned} \partial_\sigma(-\sin\theta Z + \cos\theta Y) &= 0, \\ \partial_\tau(\cos\theta Z + \sin\theta Y) &= 0. \end{aligned} \tag{3.20}$$

In the same basis as before, T-duality is encoded by the matrix

$$g_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.21}$$

and the B-transform is therefore replaced by one with matrix

$$g' = g_y^{-1} g g_y = \begin{pmatrix} 1 & kx & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -kx & 1 \end{pmatrix}. \tag{3.22}$$

Let us describe these D1-branes in terms of maximally isotropic[2] subspaces. Start at $x = 0$ with a D1-brane wrapping the y circle. The corresponding pure spinor is the sum of the tangent and conormal bundles of the y circle, with coordinates

$$S^1 \oplus \mathrm{Ann} S^1 = \{y, z = 0, \xi^1 = 0, \xi^2\}. \tag{3.23}$$

Acting on it with g' yields the coordinates $(y, kxy, -kx\xi^2, \xi^2)$, which means that there are Dirichlet conditions along the one-dimensional subspace of the two-torus at $x = l$ with equation:

$$\tan\theta Y - Z = 0. \tag{3.24}$$

This is consistent with the fact that there is now a connection on the torus, and taking x to be equal to 1 (the period of the coordinate along the base) and requiring the D1-brane to come back to itself does indeed give rise to the identification of the twisted torus

$$(x, y, z) \sim (x + 1, y, z + ky), \tag{3.25}$$

as it should [3].

3.3.3 Non-geometric T-dual space and β-transforms

Let us perform one more T-duality, along the z direction, and get to the non-geometric space. The matrix acting on the T^2-fibre, in going from $x = 0$ to generic x, in a basis adapted to the real coordinates $(y, z, \tilde{y}, \tilde{z})$ is obtained from g through conjugation by the T-duality matrix

$$g_{yz} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

[2] Isotropic is understood with respect to the inner product on the sum of the two-torus and the dual two-torus; we do not specify the embedding into T^6 yet; the coordinate on base only plays the role of a parameter as it is not acted on by the T-dualities we consider.

It therefore reads

$$g'' = g_{yz}^{-1} g g_{yz} = \begin{pmatrix} 1 & 0 & 0 & -kx \\ 0 & 1 & kx & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.26}$$

which we recognize as a β-transform by the bivector field

$$\beta(x) = kx\partial_y \wedge \partial_z \,. \tag{3.27}$$

Since β-transforms affect the vector part of maximally isotropic subspaces, there is no way of twisting the torus to bring the D0-brane back to itself after a monodromy around the base. Moreover, β-transforms are also associated to open paths on the base, showing that attaching an open string to two D0-branes sitting over different points of the base is impossible, unless T-dualities are allowed to patch the coordinate charts together. As open strings can wind around the base before attaching themselves to the second brane, they are sensitive to the global effect of non-geometry, even if the two points on the base can be put in one single coordinate patch for the purposes of local differential geometry. It is crucial for such a global effect that the base be non-simply-connected.

To sum up, T-dualities therefore relate D-branes located in different fibres. Hence they are needed as changes of charts, as predicted by the proposal (II). Moreover, the transformations of the corresponding pure spinors are dictated by a bivector field $\beta^{\mu\nu}(x) = kx\partial_y \wedge \partial_z$ depending on the coordinate along the base in the same way as the tensor θ of the proposal (I).

3.4 Generalized D0-branes on the non-geometric T-dual

As points might be disturbed by global effects in non-geometric spaces, we would like to probe non-geometry by generalized D0-branes. Of course, in order to be able to use techniques from generalized geometry for describing T-duals of the three-torus with H-flux, we first have to embed the three-torus into a six-torus.

Let us consider a generalized B-model, and pick a complex structure of the form $\mathcal{J}_J$, with J an ordinary complex structure on the six-torus. We still have a choice for the complex structure J: we can either consider the T^2-fibre as an elliptic curve in this complex structure (which would make B a tensor of type $(1,1)$ and a valid field strength for a D2-brane of type B wrapping the elliptic curve), or pick a complex structure in which y and z are components of different complex coordinates. This way B would have a nonzero component of type $(0,2)$ and the dual torus with coordinates $\tilde{y}$ and $\tilde{z}$ could not support a D2-brane of the B-model. Let us choose the second option in order to single out the role of the $(0,2)$ components and their possible influence on non-commutativity.

The way we embed the three-torus into a six-torus is therefore the following: the T^2-fibre coordinates y and z are real parts of complex coordinates $y + iy'$ and $z + iz'$, where y' and z' are coordinates along additional circles, and the base is combined with a third additional circle with coordinate x' into an elliptic curve. In the sequel we shall denote the local complex coordinates we have just described by

$$z^1 = x + ix' , \qquad z^2 = y + iy' , \qquad z^3 = z + iz' . \tag{3.28}$$

This way B is not of type $(1,1)$ and will therefore contribute non-commutative deformations as argued in [22]. Moreover, the x-dependence means that Morita equivalence cannot be used to gauge non-commutativity away, since the B-field will assume non-rational values. But for the time being, we are interested in the effect of the $(0,2)$ and $(2,0)$ components of the B-field in terms of T-duality transformations, as an illustration of (II). The connection with non-commutativity using the language of (I) and (III) will be made in sections 4 and 5.

A few comments about the choice of GC structure are in order: we restrict ourselves to diagonal GC structures, thus generalizing the B-model. We shall see in section 4 that deformations of the generalized B-model are indeed sufficient to explain the connection between non-geometry and non-commutativity, but the reason why it is *a priori* sufficient to consider a diagonal GC structure is that only such structures allow generalized D0-branes, which are the point-like objects with which one would like to probe non-geometry. Of course it is well-known that D-branes corresponding to GC structures of the form $\mathcal{J}_\omega$ do not include D0-branes, moreover the generalized Darboux theorem implies that hybrid GC structures locally have some A-type boundary conditions that forbid D0-branes.

Let us consider generalized D0-branes for the GC structure we have just described, and the way they transform under monodromy. They are not affected by B-transforms because the graph condition of definition (3.11) is empty. On the other hand, their ordinary dimension is raised upon a β-transform by an amount equal to the rank of β, as was explained above.

In order to work out the effect of the monodromy on D0-branes, we are going to use the description of generalized tangent bundles by pure spinors. The mapping between isotropic spaces and spinors is made manifest by the action of sections of $TM \oplus T^*M$ on $\Lambda^\bullet M$, which carries a representation of Clifford(n,n):

$$(X + \xi).\phi = \iota_X \phi + \xi \wedge \phi, \tag{3.29}$$

$$(X + \xi).((X + \xi).\phi) = \langle X + \xi, X + \xi \rangle \phi. \tag{3.30}$$

Given a spinor, one can associate to it its null space in $V \oplus V^*$. Maximally isotropic subspaces are therefore in one-to-one correspondence with pure spinors.

In the case of a generalized D0-brane, the pure spinor to be considered is the holomorphic three-form

$$\Omega := dz^1 \wedge dz^2 \wedge dz^3 , \tag{3.31}$$

in a local patch where z^1, z^2, z^3 are complex coordinates associated to the complex structure we have described on the six-torus. The annihilator is locally of the form

$$TM^{(0,1)} \oplus T^*M^{(1,0)} = \mathrm{Vect}\left(\frac{\partial}{\partial \bar{z}^1}, \frac{\partial}{\partial \bar{z}^3}, \frac{\partial}{\partial \bar{z}^3}\right) \oplus \mathrm{Vect}\left(dz^1, dz^2, dz^3\right) . \tag{3.32}$$

Let us write the components of β in a way adapted to the local complex coordinates, so that $\beta^{\mu\nu}$ is the $(-2,0)$ part of β, and $\beta^{\bar{\mu}\bar{\nu}}$ and $\beta^{\mu\bar{\nu}}$ do not appear in the β-transform because they act on components of the annihilator that are zero (and stay so, because the one-form part is not transformed by β). The transformation rules are therefore

$$\xi^{\bar{\mu}}\bar{\partial}_\mu + \xi_\mu dz^\mu \longrightarrow (\xi^{\bar{\mu}})'\bar{\partial}_\mu + (\xi^\mu)'\partial_\mu + \xi_\mu dz^\mu, \tag{3.33}$$

where

$$\begin{aligned} (\xi^{\bar{\mu}})' &= \beta^{\bar{\mu}\nu}\xi_\nu + \xi^{\bar{\mu}}\,, \\ (\xi^\mu)' &= \beta^{\mu\nu}\xi_\nu\,. \end{aligned} \tag{3.34}$$

The vector space spanned by the vectors $(\xi^{\bar{\mu}})'\partial_{\bar{\mu}}$ is still the whole subspace $TM^{(0,1)}$, whereas the projection of the annihilator of Ω on $TM^{(1,0)}$ is made two-dimensional by the monodromy, since $\mathrm{rk}\,\beta^{\mu\nu} = \mathrm{rk}(dz^2 \wedge dz^3) = 2$.

As expected, the dimension of the projection on the tangent space is shifted by the rank of the $(0,-2)$ part of the bivector field β. This establishes that there is no zero-dimensional global section of the vector part. This phenomenon was observed in the context of non-commutative deformations in [22], where it was called the uncertainty principle for topological D-branes (commutators of equations of complex submanifolds cannot vanish, and this prevents D-branes from wrapping maximal-codimension submanifolds). In the present case, the change of type[3] of a pure spinor under monodromy is equivalent to the lack of a global splitting between momenta and winding numbers. The space obtained by T-duality from the generalized complex T^6 with H-flux can therefore not be globally of the form $L \oplus L^*$, with L a maximally isotropic subspace. This is the absence of global polarization that appeared in the real case for T-folds, and it is encoded by the $(0,-2)$ part of the bivector field β.

4 Non-commutativity from Lagrangian deformations in the BV procedure

In the previous section our consideration of the T-dual of a complex six-torus with H-flux has shown the necessity of T-folds for the description of the global topology, when no global polarization exists, which reproduces the criterion of [8] for the objects (II). What about proposal (I) and non-commutativity along the dual T^2-fibres? The connection comes from the construction of topological string theory on GC spaces in [23], where it was explained that a tensor β of type $(0,-2)$ can deform the B-model with generalized complex target space, inducing star-products on the fibre. In order to make contact with [12] we are going to show how the β-transform induces this very deformation of the generalized B-model on the T-dual of the GC six-torus. For a categorical viewpoint on the Fourier–Mukai equivalence between deformations of complex tori (either in a non-commutative direction parametrized by a holomorphic Poisson structure or in a B-field direction), see [50].

[3] A pure spinor can be written in a unique way as $\theta^1 \wedge \cdots \wedge \theta^n \wedge e^F$, where $\theta^1, \ldots, \theta^n$ are complex one-forms and F is a complex two-form; the integer n is called the *type* of the pure spinor.

4.1 β-transforms and the generalized B-model

The Batalin–Vilkovisky (BV) formalism requires a nilpotent operator Q and an odd differential operator of second order Δ. They act on the graded space of fields and induce an odd symplectic structure, for which the master action S is a Hamiltonian function. The odd Laplacian Δ induces an antibracket via the formula

$$(F, G) = (-1)^{|F|}\Delta(F \wedge G) - \Delta F \wedge G - (-1)^{|F|} F \wedge \Delta G\,. \tag{4.1}$$

The condition $Q^2 = 0$ then induces the master equation

$$(S, S) = 0\,. \tag{4.2}$$

From now on, as is required by the BV procedure, we shall give fermionic statistics to the vector and form coordinates, or in other words reverse the parity on the fibres of the tangent and cotangent bundles. When computing the action of a sigma model, one has to pull back vector and form fields on the world-sheet Σ, which induces a change of statistics on the tangent bundle of the world-sheet, which is now denoted $\hat{\Sigma}$. As far as the B-model is concerned, the graded space of fields is (in a local coordinate patch) the space of observables of the B-model. The operators Q and Δ are the anti-holomorphic and holomorphic differentials,

$$\begin{aligned} Q &= \bar{\partial} = d\xi^{\bar{\mu}} \frac{\partial}{\partial \bar{\xi}^{\bar{\mu}}} \\ \Delta &= \partial = dz^{\mu} \frac{\partial}{\partial \xi^{\mu}} = \frac{\partial}{\partial \xi_{\mu}} \frac{\partial}{\partial \xi^{\mu}}\,, \end{aligned} \tag{4.3}$$

where in re-expressing ∂ as a second-order differential operator, use has been made of the observation that one-forms may act on the de Rham complex as derivatives with respect to vector coordinates [51]. This way the coordinates ξ_μ and z^μ are canonically conjugate to each other, and one has to add antifields to be paired with $\bar{z}^\mu$ and $\xi^{\bar{\mu}}$ (because Δ is degenerate on the subspace they span), called $\bar{z}^*_\mu$ and $\xi^*_{\bar{\mu}}$. As shown in [23], the master action for the generalized B-model then reads

$$S = \int_{\hat{\Sigma}} \left(\xi_\mu dz^\mu + \xi^{\bar{\mu}} z^*_{\bar{\mu}} \right)\,. \tag{4.4}$$

The allowed deformations of the generalized B-model involve holomorphic bivector fields. A Lagrangian submanifold L of the space of fields has indeed to be chosen to compute the gauge-fixed partition-function

$$Z_L := \int_L DX e^{-S[X]}\,, \tag{4.5}$$

and the invariance of this path integral under change of the gauge-fixing condition is equivalent to its invariance under the Lagrangian deformations of L. A variation in the gauge-fixing condition amounts to a Lagrangian deformation of the Lagrangian submanifold L, namely one where the momenta are derived from a density

$$\delta p^i = \frac{\partial \Xi}{\partial x^i}\,. \tag{4.6}$$

Starting with a Lagrangian submanifold with equation given by the vanishing of all momenta $p^i = 0$, invariance of Z_L under Lagrangian deformations is expressed by the following chain of equalities

$$\delta_L Z = \int_L \delta p^i \frac{\delta}{\delta p^i} e^{-S(x_i)} = \int_L \frac{\partial \Xi}{\partial x^i}\frac{\delta}{\delta p^i} e^{-S(x_i)} = -\int_L \Xi \frac{\partial}{\partial p_i}\frac{\partial}{\partial x^i}\left(e^{-S(x_i)}\right) = 0\,, \tag{4.7}$$

which implies the quantum master equation

$$\Delta(e^{-S[X]}) = 0\,. \tag{4.8}$$

Expanding in powers of a deformation of the master action gives rise to the Maurer–Cartan equation. In the case of the generalized B-model, splitting into tensors of different types shows [23] that the deformation of the generalized B-model by a holomorphic bivector field is allowed (moreover, the sum of tangent and cotangent spaces is one of the geometries recently addressed by Ikeda in the deformation theory of BV structures [52]).

In the present context, the lack of global polarization induces deformations of the BV structure when going from one patch of coordinates to antother. It is instructive to see how derivatives are affected by the monodromies described in (3.33). Let us work out the deformation of the antibracket adapted to the isotropic subspace $TM^{(0,1)} \oplus T^*M^{(1,0)}$ we started with in the previous section

$$(F,G) = F\frac{\partial}{\partial \xi^\mu}\frac{\partial}{\partial \xi_\mu}G - F\frac{\partial}{\partial \xi_\mu}\frac{\partial}{\partial \xi^\mu}G\,. \tag{4.9}$$

The change of coordinates induced by a β-transform

$$\begin{aligned} \xi'^\mu &= \beta^{\mu\nu}\xi_\nu + \xi^\mu\,,\\ \xi'^\mu &= \beta^{\mu\nu}\xi_\nu + \xi^\mu\,, \end{aligned} \tag{4.10}$$

induces the following changes in derivatives on the cotangent space

$$\begin{aligned} \frac{\partial}{\partial \xi^\mu} &= \frac{\partial}{\partial \xi'_\nu}\frac{\partial \xi'_\nu}{\partial \xi^\mu} + \frac{\partial}{\partial \xi'^\nu}\frac{\partial \xi'^\nu}{\partial \xi^\mu} = \frac{\partial}{\partial \xi'^\mu}\,,\\ \frac{\partial}{\partial \xi_\mu} &= \frac{\partial}{\partial \xi'_\nu}\frac{\partial \xi'_\nu}{\partial \xi_\mu} + \frac{\partial}{\partial \xi'^\nu}\frac{\partial \xi'^\nu}{\partial \xi_\mu} = \frac{\partial}{\partial \xi'_\mu} + \beta^{\mu\nu}\frac{\partial}{\partial \xi'^\nu}\,, \end{aligned} \tag{4.11}$$

and the antibracket now includes pairs of derivatives with respect to the vector coordinates, so that the monodromy shifts the antibracket by a Poisson bracket:

$$(F,G)' = (F,G) + 2F\frac{\partial}{\partial \xi'^\mu}\beta^{\mu\nu}\frac{\partial}{\partial \xi'^\nu}G\,. \tag{4.12}$$

We have therefore shown that the data of the BV structure do change from patch to patch in the non-geometric T-dual of the GC six-torus. We are going to work directly on the master action, since the β-transform is of the Lagrangian type, so that the T-duality that has been seen to bind together the coordinate patches, is also deforming the master action.

4.2 From the generalized B-model to the Poisson sigma model

We have seen in the previous section that in a special case an obstruction to the existence of a global generalized complex form $L \oplus L^*$ for the T-duals is encoded by a holomorphic bivector field $\beta^{\mu\nu}$. The link with [7] is provided by the choice of an isotropic submanifold involved in the BV gauge-fixing procedure. A global such choice is impossible as soon as the β-transform is non-trivial, and this leads to a deformation of the generalized B-model by the holomorphic bivector field β. The resulting model is precisely the Poisson sigma model that appears as the β-deformation of the topological $\mathcal{J}$-model constructed by Pestun [23]. Star-products emerge from the Poisson structure by deformation quantization [53, 54]. Of course this is no accident. Relevance of the Kontsevich formula in non-commutative gauge theory along D-branes appeared for example in [27, 55, 56].

Consider the master action that is obtained from the BV procedure for the B-model with a generalized complex manifold as a target space [23], *i.e.*, a target space endowed with a GC structure of the diagonal form $\mathcal{J}_J$. We therefore start with the master action on the patch with complex coordinates $(z^\mu, \bar{z}^\mu)$

$$S = \int_{\hat{\Sigma}} \left(\xi_\mu dz^\mu + \xi^{\bar{\mu}} z^*_{\bar{\mu}} \right). \tag{4.13}$$

As for the $\xi^{\bar{\mu}}$, they span the bundle $TM^{(1,0)}$ that is not modified by the monodromy, and the $z^*_{\bar{\mu}}$ are conjugate to the antiholomorphic base coordinates that are untouched by the monodromy. The second term is decoupled in the initial patch, and will stay so under monodromy.

But the first term, as it is endowed with holomorphic indices, is affected by the monodromy. This is due to the fact that the circle base cannot be covered by a single patch. Let us choose a construction of the spinor bundle where the differential forms act by differentiation with respect to the dual coordinates. This corresponds to choosing the pure spinor Ω as the vacuum, and vector fields as creation operators, as explained by Witten in [51]. In a local patch we therefore identify dz^μ with $\partial/\partial\xi_\mu$, so that the relevant term in the master action transforms as follows

$$\left(\xi_\mu \frac{\partial}{\partial \xi_\mu}\right)' = \xi_\mu \frac{\partial}{\partial \xi_\mu} + \xi_\mu \beta^{\mu\nu} \frac{\partial}{\partial \xi^\nu}, \tag{4.14}$$

and the result, in a representation where differential forms act by multiplication, as

$$S' = \int_{\hat{\Sigma}} \left(\xi_\mu dz^\mu + \beta^{\mu\nu} \xi_\mu \xi_\nu \right), \tag{4.15}$$

which of the form $S + \delta_\beta S$, with $\delta_\beta S$ induced by the holomorphic bivector β. Of course we can rewrite the expression for S' in terms of coordinates, vectors and forms with conventional statistics, both on the world-sheet Σ and the target space, by taking multiplications to be wedge products.

Since the non-zero components of the bivector field β are along the directions y and z, and they depend only of the coordinate x, the parameter β is a Poisson bivector field

$$\beta^{\mu\nu} \partial_\nu \beta^{\rho\sigma} + \beta^{\rho\nu} \partial_\nu \beta^{\rho\mu} + \beta^{\sigma\nu} \partial_\nu \beta^{\mu\rho} = 0. \tag{4.16}$$

The resulting model with action S' is the Poisson sigma model studied by Cattaneo and Felder in [24]. We are therefore left with T-dual fibres forming a field of non-commutative tori over a circle. A subspace of the T-dual T^6 is therefore non-commutative, with the non-commutativity scale predicted by (I).

For a given topology of the world-sheet, we find a deformation of the product of observables into a star-product $*_\beta$ as in [24]:

$$f *_{\beta(a_1)} g(a) = \int_{X_\infty = a} DX f(X(0))g(X(1))e^{i(S+\delta_\beta S)[X]}\,, \tag{4.17}$$

where $0, 1, \infty$ are the coordinates of the points of insertion of observables on the boundary of the world-sheet, and a_1 is the component of the coordinates of the point a along the direction x. The continuous dependence on a_1 comes from the definition (3.27) for the bivector β. The fact that non-commutativity shows up in the boundary correlators is the sign that the open-string sector is crucial for the equivalence between (I) and (II).

5 $SU(3) \times SU(3)$ structure and non-commutativity

It is argued in [19] that the mirror of a Calabi–Yau compactification with magnetic H-flux [2] possesses an $SU(3) \times SU(3)$ structure. $SU(3) \times SU(3)$ structure compactifications are described in terms of pure spinors, made from bilinears of $SU(3)$-invariant spinors of Cliff(6). In case those invariant spinors are not parallel and the type of the associated pure spinors is not globally defined, the resulting compactification is conjectured to be non-geometric. In case the two pure spinors still have a globally constant type, the compactification has global $SU(2)$ structure and is still geometric, an example of which is $T^2 \times K3$. The proposal in [19] should in particular be consistent with the non-commutative T-dual conjecture, when both setups are applicable.

In this section we show that precisely in the case when the two $SU(3)$-invariant spinors are not parallel, there is a non-trivial Poisson bivector, which yields a non-commutative deformation of the open-string background. If the pure spinors have a uniform expression, the Poisson bivector is constant and thus one can dispose of the non-commutativity by Morita equivalence. This of course is not possible in the case when the Poisson bivector is dependent on the remaining coordinates.

5.1 $SU(3) \times SU(3)$ structure manifolds

Consider Type II compactifications on six-manifolds with $SU(3) \times SU(3)$ structure [20, 21, 5, 57, 58, 59] (for a detailed introdution and references see [60]). As such they are characterized by a pair of no-where vanishing $SU(3)$-invariant spinors $\eta^{1,2}$, which arise in the decomposition of the two $SO(9,1)$ spinors $\epsilon^{1,2}$ of Type II under $SO(3,1) \times SO(6)$. If $\eta^1 = \eta^2$ the structure group is reduced to $SU(3)$, which in particular incorporates the case of standard Calabi–Yau compactifications. If on the other hand the spinors are not parallel to each other throughout the manifold, one speaks of an $SU(2)$ structure. The latter is characterized by a non-vanishing vector field. Defining

$$\eta^2_+ = c\eta^1_+ + (v + iw)\eta^1_-\,, \qquad c \in \mathbb{C}, \tag{5.1}$$

the vector field in question is

$$\nu_m := \eta_+^{1\,\dagger}\gamma_m\eta_-^2 = v_m - iw_m\,. \tag{5.2}$$

The spinors $\eta^{1,2}$ can also be combined to construct a pair of $SU(3,3)$ bi-spinors

$$\Phi_\pm = \eta_+^1 \otimes \eta_\pm^{2\,\dagger}\,, \tag{5.3}$$

which are pure (*i.e.*, they are annihilated by half of the Γ-matrices). Moreover, via the standard Clifford map, they are in one-to-one correspondence with (formal sums of) differential forms. On an $SU(3)$ structure manifold the pure spinors $\Phi_\pm$ correspond to the $(3,0)$ form Ω and the (exponential of the) $(1,1)$ form J. Generically however, there are two independent two- and three-forms

$$\begin{aligned} J_\pm &= j \pm v \wedge w \\ \Omega_\pm &= \omega \wedge (v \pm iw)\,. \end{aligned} \tag{5.4}$$

Here j and ω parametrize the local $SU(2)$ structure in the transverse directions to v and w. For the present purposes it is instructive to note that in the case of non-geometric spaces the notion of transversality holds only locally, since it can be spoiled by a β-transform.

The associated two pure spinors are

$$\Phi_+ = \frac{1}{8}(\bar{c}e^{-ij} - i\omega) \wedge e^{-iv\wedge w}\,, \qquad \Phi_- = -\frac{1}{8}(e^{-ij} + ic\omega) \wedge (v + iw)\,. \tag{5.5}$$

Raising an index on the two-forms $J_\pm$ we obtain two complex structures, $I_\pm$, which on the other hand define a generalized complex structure $\mathcal{I}$ as will be discussed in the next section. Thus, one important point to notice is that $SU(3)\times SU(3)$ structure implies generically that there are two independent complex structures, which arise from the two $SU(3)$-invariant spinors $\eta^{1,2}$.

5.2 Non-commutative deformations

To begin with, let us review some known facts about non-commutativity: In [22] Kapustin presents a criterion when a compactification allows for non-commutative deformations. Consider first the case of closed B. Then for a Calabi–Yau manifold with metric G and B-field and two (not necessarily equal) complex structures $I_\pm$ compatible with the Levi–Civita connection, one may define the generalized complex structure

$$\mathcal{I} = \begin{pmatrix} \frac{1}{2}(I_+ + I_-) + \delta P\, B & -\delta P \\ \delta J + B\delta P\, B + \frac{1}{2}B(I_+ + I_-) + \frac{1}{2}(I_+ + I_-)^t B & -\frac{1}{2}(I_+ + I_-)^t - B\delta P \end{pmatrix}, \tag{5.6}$$

where the bivector part is defined as

$$\delta P = \frac{1}{2}(J_+^{-1} - J_-^{-1})\,, \tag{5.7}$$

with $J_\pm = GI_\pm$. Furthermore $\delta J = 1/2(J_+ - J_-)$. The complex structure is

$$I = \frac{1}{2}(I_+ + I_-) + \delta P\, B\,, \tag{5.8}$$

whereas the Poisson bivector, which parametrizes the non-commutative deformations, is given by

$$\theta = -\frac{1}{2} I \delta P \,. \tag{5.9}$$

In particular, the non-commutativity is non-trivial only if the two complex structures are unequal $I_+ \neq I_-$, since otherwise $\delta P = 0$. In [40] this was generalized to the case of $H = dB \neq 0$. The only difference from the above equations is that the complex structures $I_\pm$ now have to be covariantly constant with respect to the connection with torsion $H_{ij}{}^k$. Note however that as in the T-fold case, non-commutativity arises only at the level of D-branes, *i.e.*, the closed-string background remains commutative, albeit not necessarily geometric, as the left- and right-moving modes on the world-sheet differ. In particular, a generic $SU(3) \times SU(3)$ structure compactification yields a pair of distinct complex structures, and thus two distinct realizations of the $\mathcal{N} = 2$ super-conformal algebra for left- and right-movers, respectively. In this sense, the world-sheet theory is very much alike the situation for asymmetric orbifolds.

In particular, we can then determine δP in the case of $SU(3) \times SU(3)$ structure compactifications

$$\delta P = -\frac{1}{2}((GI_+)^{-1} - (GI_-)^{-1}) \,, \tag{5.10}$$

which has again non-vanishing Poisson bivector $\theta = -1/2 I \delta P$ if the two complex structures differ.

In case of the two spinors being never parallel, which corresponds to $v + iw \neq 0$, *i.e.*, we have an $SU(2)$ structure at least locally, the corresponding θ is non-zero. So this is indeed the case, when there are non-trivial non-commutative deformations. In fact we can write the Poisson bivector entirely in terms of the one-form ν that characterizes the $SU(2)$ structure

$$\delta P = w \wedge v = \frac{1}{2i} \nu \wedge \bar{\nu} \,, \tag{5.11}$$

which means that the non-commutativity is governed only by the vectors in (5.2).

If the $SU(2)$ structure is global, such as for $K3$-compactifications, the resulting non-commutative deformations are constant and thus of minor interest to the present discussion. The interesting cases arise, when the above description is only local. Then the Poisson bivector is not constant and one cannot get rid of it by Morita equivalence. Thus, the lack of global definition for the type of the pure spinors (because of the change of dimension between two different base-points that was illustrated above) makes it unlikely that the linear independence between two $SU(3)$ spinors can be described by a globally-defined vector field. We may obtain a Poisson bivector δP, but it need not be of the form $v \wedge w$. We shall present an explicit example in the next section.

5.3 Torus with H-flux and mirror symmetry

An illustrative example is T^6 in the complex coordinates (3.28) with H-flux $H \in H^3(T^6, \mathbb{Z})$, which allows to use the language of $SU(3) \times SU(3)$ structures. We consider a triple T-duality along x', y, z, which in this context corresponds to considering the mirror. The three-form is $\Omega = dz^1 \wedge dz^2 \wedge dz^3$ and the $(1,1)$-form is $J = \sum_i dz^i \wedge d\bar{z}^i$. In particular, the T^3 that the T-duality acts upon is a special Lagrangian cycle. Note that this is different from the T-duality transformations encountered in the previous sections for T^3 with H-flux, however, as in that case, we consider only two-legged H-flux, *i.e.*, T-duality acts in only two directions supporting the H-flux.

The setup for the square torus is simple enough, and the generalized complex structure $\mathcal{I}$ is diagonal. However switching on the H-flux yields

$$\mathcal{I}(B) = \begin{pmatrix} I & 0 \\ BI + I^t B & -I^t \end{pmatrix}, \tag{5.12}$$

The T-dual complex structure was determined in [22] to be

$$\mathcal{I}'(B) = \begin{pmatrix} -I^t & BI + I^t B \\ 0 & I \end{pmatrix}, \tag{5.13}$$

so that the Poisson bivector is read off to be $\delta P = BI + I^t B$, which is not necessarily vanishing, as expected. Moreover, if we insist that the β-transformed D0-brane be a generalized D0-brane with respect to $\mathcal{I}'$ after monodromy, we obtain the constraint $\beta = \beta^{(0,2)}$, in terms of the decomposition with respect to I. The deformation parameter is once more seen to be a (0,-2) tensor.

Let us connect this non block-diagonal GC structure to the language of maximally isotropic subspaces we used to probe the non-geometry by D0-branes. Consider again a graph of some bivector field β over some subspace F of the cotangent space

$$L(F,\beta) = \{X + \xi \in V \oplus F,\ X|_F = \iota_\xi \beta\}, \tag{5.14}$$

and require stability of this graph under the action of $\mathcal{I}'$. We are led to the following equation, that must hold for every element ξ in F

$$-(I^t)^\mu_\nu(\beta^{\nu\rho}\xi_\rho) + (\delta P)^{\mu\nu}\xi_\nu = \beta^{\mu\nu} I^\rho_\nu \xi_\rho\,, \tag{5.15}$$

Eliminating the coordinates ξ_ρ we observe that the $(0,-2)$ part of β must equal the Poisson bivector field

$$\beta I + I^t \beta = \delta P\,. \tag{5.16}$$

We therefore see that the non-diagonal block of the GC structure in the T-dual picture is precisely the parameter of the β-transform that is undergone by any D0-brane. Whenever δP is non-zero, the dimension of the projection of a D-brane onto the tangent space is non-zero. Therefore δP induces non-geometry in the sense that point-like D-branes cannot be put on a GC space with non-zero bivector block.

As stated in theorem (5.4) of [61] for mirrors of complex tori, the two generalized complex structures $\mathcal{J}_J(B)$ and $\mathcal{J}_\omega(B)$ shown in formulae (3.9) and (3.10) are exchanged by mirror symmetry. Indeed, if $g_{x'yz}$ is the element of $O(6,6)$ encoding T-duality in the x', y, z directions, the mirror exchange

$$(\mathcal{I}(B))' = g_{x'yz}\mathcal{J}(B) g^{-1}_{x'yz}\,, \tag{5.17}$$

holds.

As we already argued, the image of $\mathcal{J}_J$ by T-duality is not block-diagonal anymore, and we may read off the Poisson bivector δP as

$$\delta P = \frac{1}{2} kx \left(\partial_{y'} \wedge \partial_z + \partial_y \wedge \partial_{z'} \right) \,, \tag{5.18}$$

which is not decomposable as the tensor product of two vectors. Hence this is a case of generalized type of T-fold. In this example of a non-geometric T-dual, the non-commutativity scale depends on the base coordinate x in a way that prevents to gauge it away by Morita equivalence, and furthermore it does not come from a globally defined vector field that encodes the linear independence between two $SU(3)$-invariant spinors. The lack of a global polarization in T-folds (II) can therefore be traced in the formalism of (III) as the non-decomposability of the Poisson bivector field δP.

So far our consideration of the T-dual of a complex torus with H-flux has shown the necessity of T-folds for the description of the global topology, when no global choice of type for a pure spinor exists, which reproduces the criterion of [8]. What about non-commutativity along the dual T^2-fibres? The connection comes from the construction of topological string theory on GC spaces and sigma models with bistructures in [62, 63, 23, 64]. It was explained that a tensor β of type $(0,-2)$ can deform the B-model with generalized complex target space, yielding star-products induced by the Poisson bivector β. In order to make contact with [12] it would be interesting to see how the β-transform induces this very same deformation of the generalized B-model.

The results in this section should also be derivable from the action of T-duality on spinors as advocated by Hassan [65, 66, 67]. In the geometric case, in particular for the the Lunin–Maldacena background [68], the analysis was performed in [69] and it is easily observed from their results that δP in this case is of the form $v \wedge w$, and non-commutativity does not occur between coordinates but as relative phases between ordered products of fields, corresponding to global $U(1)$ symmetries.

6 Conclusion

T-duality in the presence of NSNS fluxes provides the first stepping stone to understanding generalized versions of mirror symmetry à la Strominger–Yau–Zaslow (SYZ) [70]. The present paper discusses this issue, thereby merging various existing proposals for the T-dual. If the NSNS H-flux is supported only on one T-dualized direction the dual is again geometric and consensus has been reached on the T-dual geometry throughout the literature. Controversy starts when two T-dualized directions are spanned by the H-flux. Our present investigations concern the case of two directions, applied to tori and torus fibrations. We have shown that the proposal (I) of Mathai and Rosenberg, claiming the dual to be a field of non-commutative tori, can be viewed as the open-string version of Hull's T-fold proposal (II). Secondly, we have shown that generalized geometries provide an alternative setup for studying the T-dual or mirror, and can be reconciled with the non-commutativity proposal by explicit construction of a Poisson bivector, which depends crucially on the background H-flux. As argued in [22, 71], this bivector parametrizes non-commutative deformations of the open strings. We found that in the case of two-legged H-flux, the bivector field is in fact not uniform, but varies along the base of the torus-fibration.

On more general grounds it would be interesting to understand the precise conditions for the dual space to be non-geometric. The key ingredient for the deformation by the bivector field is of course the multiple-connectedness of the base of the fibration. The present discussion could be extended to complicated fibrations over a multiply-connected base with H-flux. On the other hand, T-dualizing along a two-torus carrying a non-zero B-field and fibered over a contractible space, as in the sequence of β-deformations in [68], can lead to a geometric T-dual (orbifolds with torsion in that instance). This is consistent with the fact that any loop on the base can be shrunk and included in a local coordinate patch (thus removing the β-transformed term in the formula (4.15) in the simply-connected cases), and also with the fact that non-commutativity of the dual two-torus in the proposal of [12] is measured by classes in the first integral cohomology group of the base (thus setting the star-product to the ordinary product in the simply-connected cases).

Clearly it would be very interesting to extend the present discussions to more general setups, in particular, a generalized SYZ construction would be the most natural next problem to be addressed. The argument of SYZ for the existence of T^3-fibrations of Calabi–Yau manifolds rested on the fact that a D0-brane had only its position as a modulus. In the case of T-folds, the modulus β is a modulus of Lagrangian deformations and prevents D0-branes from existing, just as non-commutativity does.

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