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Field Theory and Standard Model

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Abstract

This is a short introduction to the Standard Model and the underlying concepts of quantum field theory.

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Chapter 1

Introduction

In these lectures we shall give a short introduction to the standard model of particle physics with emphasis on the electroweak theory and the Higgs sector, and we shall also attempt to explain the underlying concepts of quantum field theory.

The standard model of particle physics has the following key features:

- As a theory of elementary particles, it incorporates relativity and quantum mechanics, and therefore it is based on quantum field theory.
- Its predictive power rests on the regularisation of divergent quantum corrections and the renormalisation procedure which introduces scale–dependent "running couplings".
- Electromagnetic, weak, strong and also gravitational interactions are all related to local symmetries and described by Abelian and non-Abelian gauge theories.
- The masses of all particles are generated by two mechanisms: confinement and spontaneous symmetry breaking.

In the following chapters we shall explain these points one by one. Finally, instead of a summary, we will briefly recall the history of "The making of the Standard Model"[1].

From the theoretical perspective, the standard model has a simple and elegant structure: It is a chiral gauge theory. Spelling out the details reveals a rich phenomenology which can account for strong and electroweak interactions, confinement and spontaneous symmetry breaking, hadronic and leptonic flavour physics etc. [2, 3]. The study of all these aspects has kept theorists and experimenters busy for three decades. Let us briefly consider these two sides of the standard model before we enter the discussion of the details.

1.1 Theoretical Perspective

The standard model is a theory of fields with spins 0, $\frac{1}{2}$ and 1. The fermions (matter fields) can be arranged in a big vector containing left-handed spinors only:

$$\Psi_L^T = \big(\underbrace{q_{L1}, u_{R1}^C, e_{R1}^C, d_{R1}^C, l_{L1}, (n_{R1}^C)}_{\text{1st family}}, \underbrace{q_{L2}, \ldots,}_{\text{2nd}} \underbrace{\ldots, (n_{R3}^C)}_{\text{3rd}}\big) , \tag{1.1}$$

where the fields are the quarks and leptons, all in threefold family replication. The quarks come in triplets of colour, i.e., they carry an index α, $\alpha = 1, 2, 3$, which we suppressed in the above expression. The left-handed quarks and leptons come in doublets of weak isospin,

$$q_{Li}^{\alpha} = \begin{pmatrix} u_{Li}^{\alpha} \\ d_{Li}^{\alpha} \end{pmatrix} \quad \text{and} \quad l_{Li} = \begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix} ,$$

where i is the family index, $i = 1, 2, 3$. We have included a right-handed neutrino n_R because there is evidence for neutrino masses from neutrino oscillation experiments.

The subscripts L and R denote left- and right-handed fields, respectively, which are eigenstates of the chiral projection operators P_L or P_R. The superscript C indicates the charge conjugate field (the antiparticle). Note that the charge conjugate of a right-handed field is left-handed:

$$P_L\psi_L \equiv \frac{1-\gamma^5}{2}\psi_L = \psi_L , \qquad P_L\psi_R^C = \psi_R^C , \qquad P_L\psi_R = P_L\psi_L^C = 0 , \tag{1.2}$$

$$P_R\psi_R \equiv \frac{1+\gamma^5}{2}\psi_R = \psi_R , \qquad P_R\psi_L^C = \psi_L^C , \qquad P_R\psi_L = P_R\psi_R^C = 0 . \tag{1.3}$$

So all fields in the big column vector of fermions have been chosen left-handed. Altogether there are 48 chiral fermions. The fact that left- and right-handed fermions carry different weak isospin makes the standard model a chiral gauge theory. The threefold replication of quark-lepton families is one of the puzzles whose explanation requires physics beyond the standard model [4].

The spin-1 particles are the gauge bosons associated with the fundamental interactions in the standard model,

$$G_\mu^A,\ A = 1, \ldots, 8 : \qquad \text{the gluons of the strong interactions ,} \tag{1.4}$$

$$W_\mu^I,\ I = 1, 2, 3,\ B_\mu : \qquad \text{the } W \text{ and } B \text{ bosons of the electroweak interactions.} \tag{1.5}$$

These forces are gauge interactions, associated with the symmetry group

$$G_{\mathrm{SM}} = \mathsf{SU}(3)_C \times \mathsf{SU}(2)_W \times \mathsf{U}(1)_Y , \tag{1.6}$$

where the subscripts C, W, and Y denote colour, weak isospin and hypercharge, respectively.

The gauge group acts on the fermions via the covariant derivative D_μ, which is an ordinary partial derivative plus a big matrix A_μ built out of the gauge bosons and the generators of the gauge group:

$$D_\mu \Psi_L = \left(\partial_\mu \mathbb{1} + g A_\mu\right) \Psi_L . \tag{1.7}$$

From the covariant derivative we can also construct the field strength tensor,

$$F_{\mu\nu} = -\frac{1}{g}\left[D_\mu, D_\nu\right] , \tag{1.8}$$

which is a matrix-valued object as well.

The last ingredient of the standard model is the Higgs field Φ, the only spin-0 field in the theory. It is a complex scalar field and a doublet of weak isospin. It couples left- and right-handed fermions together.

Written in terms of these fields, the Lagrangean of the theory is rather simple:

$$\begin{aligned} \text{Ł} = &-\frac{1}{2}\operatorname{tr}\left[F_{\mu\nu}F^{\mu\nu}\right] + \overline{\Psi}_L \imath\gamma^\mu D_\mu \Psi_L + \operatorname{tr}\left[\left(D_\mu\Phi\right)^\dagger D^\mu\Phi\right] \\ &+\mu^2\,\Phi^\dagger\Phi - \frac{1}{2}\lambda\left(\Phi^\dagger\Phi\right)^2 + \left(\frac{1}{2}\Psi_L^T C h \Phi \Psi_L + \text{h.c.}\right) . \end{aligned} \tag{1.9}$$

The matrix C in the last term is the charge conjugation matrix acting on the spinors, h is a matrix of Yukawa couplings. All coupling constants are dimensionless, in particular, there is no mass term for any quark, lepton or vector boson. All masses are generated via the Higgs mechanism which gives a vacuum expectation value to the Higgs field,

$$\langle\Phi\rangle \equiv v = 174\,\text{GeV} . \tag{1.10}$$

The Higgs boson associated with the Higgs mechanism has not yet been found, but its discovery is generally expected at the LHC.

1.2 Phenomenological Aspects

The standard model Lagrangean (1.9) has a rich structure which has led to different areas of research in particle physics:

- The gauge group is composed of three subgroups with different properties:
  - The $\mathsf{SU}(3)$ part leads to quantum chromodynamics, the theory of strong interactions [5]. Here the most important phenomena are asymptotic freedom and confinement: The quarks and gluons appear as free particles only at very short distances, probed in deep-inelastic scattering, but are confined into mesons and baryons at large distances.
  - The $\mathsf{SU}(2) \times \mathsf{U}(1)$ subgroup describes the electroweak sector of the standard model. It gets broken down to the $\mathsf{U}(1)_{\text{em}}$ subgroup of quantum electrodynamics by the Higgs mechanism, leading to massive W and Z bosons which are responsible for charged and neutral current weak interactions, respectively.
- The Yukawa interaction term can be split into different pieces for quarks and leptons:

  $$\frac{1}{2}\Psi_L^T C h \Phi \Psi_L = h_{u\,ij}\bar{u}_{Ri} q_{Lj}\Phi + h_{d\,ij}\bar{d}_{Ri} q_{Lj}\widetilde{\Phi} + h_{e\,ij}\bar{e}_{Ri} l_{Lj}\widetilde{\Phi} + h_{n\,ij}\bar{n}_{Ri} l_{Lj}\Phi \ , \tag{1.11}$$

  where $i, j = 1, 2, 3$ label the families and $\widetilde{\Phi}_a = \epsilon_{ab}\Phi_b^*$. When the Higgs field develops a vacuum expectation value $\langle\Phi\rangle = v$, the Yukawa interactions generate mass terms. The first two terms, mass terms for up-type- and down-type-quarks, respectively, cannot be diagonalised simultaneously, and this misalignment leads to the CKM matrix and flavour physics [6]. Similarly, the last two terms give rise to lepton masses and neutrino mixings [7].

Chapter 2

Quantisation of Fields

In this chapter we will cover some basics of quantum field theory (QFT). For a more in-depth treatment, there are many excellent books on QFT and its application in particle physics, such as [2, 3].

2.1 Why Fields?

2.1.1 Quantisation in Quantum Mechanics

Quantum mechanics is obtained from classical mechanics by a method called quantisation. Consider for example a particle moving in one dimension along a trajectory $q(t)$, with velocity $\dot{q}(t)$ (see Fig. 2.1). Its motion can be calculated in the Lagrangean or the Hamiltonian approach. The Lagrange function $L(q,\dot{q})$ is a function of the position and the velocity of the particle, usually just the kinetic minus the potential energy. The equation of motion is obtained by requiring that the action, the time integral of the Lagrange function, be extremal, or, in other words, that its variation under arbitrary perturbations around the trajectory vanishes:

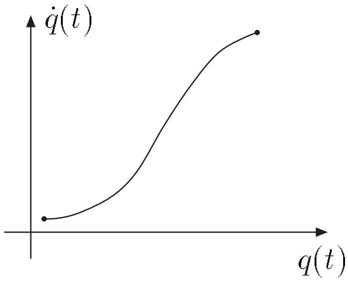


Figure 2.1: *Particle moving in one dimension*

$$\delta S = \delta \int \mathrm{d}t L\left(q(t), \dot{q}(t)\right) = 0 \ . \tag{2.1}$$

The Hamiltonian of the system, which corresponds to the total energy, depends on the coordinate q and its conjugate momentum p rather than $\dot{q}$:

$$H(p,q) = p\dot{q} - L\left(q, \dot{q}\right) \ , \qquad p = \frac{\partial L}{\partial \dot{q}} \, . \tag{2.2}$$

To quantise the system, one replaces the coordinate and the momentum by operators q and p acting on some Hilbert space of states we will specify later. In the Heisenberg picture, the states are time-independent and the operators change with time as

$$q(t) = e^{\imath H t} q(0) e^{-\imath H t} \ . \tag{2.3}$$

Since p and q are now operators, they need not commute, and one postulates the commutation relation

$$[p(0), q(0)] = -\imath\hbar\ , \tag{2.4}$$

where $h = 2\pi\hbar$ is Planck's constant. In the following we shall use units where $\hbar = c = 1$. The commutator (2.4) leads to the uncertainty relation

$$\Delta q \cdot \Delta p \geq \frac{1}{2} . \tag{2.5}$$

Note that on Schrödinger wave functions the operator q is just the coordinate itself and p is $-\imath\partial/\partial q$. In this way the commutation relation (2.4) is satisfied.

As an example example of a quantum mechanical system, consider the harmonic oscillator with the Hamiltonian

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)\ , \tag{2.6}$$

which corresponds to a particle (with mass 1) moving in a quadratic potential with a strength characterised by ω^2. Classically, H is simply the sum of kinetic and potential energy. In the quantum system, we can define new operators as linear combinations of p and q:

$$q = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right)\ , \qquad p = -\imath\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right)\ , \tag{2.7a}$$

$$\text{i.e.}\ ,\quad a = \sqrt{\frac{\omega}{2}}q + \imath\sqrt{\frac{1}{2\omega}}p\ , \qquad a^\dagger = \sqrt{\frac{\omega}{2}}q - \imath\sqrt{\frac{1}{2\omega}}p\ . \tag{2.7b}$$

a and $a^\dagger$ satisfy the commutation relations

$$\left[a, a^\dagger\right] = 1\ . \tag{2.8}$$

In terms of a and $a^\dagger$ the Hamiltonian is given by

$$H = \frac{\omega}{2}\left(aa^\dagger + a^\dagger a\right)\ . \tag{2.9}$$

Since Eqs. (2.7) are linear transformations, the new operators a and $a^\dagger$ enjoy the same time evolution as q and p:

$$a(t) = e^{\imath Ht}a(0)e^{-\imath Ht} = a(0)e^{-\imath\omega t}\ , \tag{2.10}$$

where the last equality follows from the commutator of a with the Hamiltonian,

$$[H, a] = -\omega a\ , \quad \left[H, a^\dagger\right] = \omega a^\dagger\ . \tag{2.11}$$

We can now construct the Hilbert space of states the operators act on. We first notice that the commutators (2.11) imply that a and $a^\dagger$ decrease and increase the energy of a state, respectively. To see this, suppose we have a state $|E\rangle$ with fixed energy, $H|E\rangle = E|E\rangle$. Then

$$Ha|E\rangle = (aH + [H, a])|E\rangle = aE|E\rangle - \omega a|E\rangle = (E - \omega)\,a|E\rangle\ , \tag{2.12}$$

i.e., the energy of a the state $a|E\rangle$ is $(E-\omega)$. In the same way one can show that $Ha^\dagger|E\rangle = (E+\omega)|E\rangle$. From the form of H we can also see that its eigenvalues must be positive. This suggests constructing the space of states starting from a lowest-energy state $|0\rangle$, the vacuum or no-particle state. This state needs to satisfy

$$a|0\rangle = 0\ , \tag{2.13}$$

so its energy is $\omega/2$. States with more "particles", i.e., higher excitations, are obtained by successive application of $a^\dagger$:

$$|n\rangle = \left(a^\dagger\right)^n|0\rangle\ , \quad \text{with} \quad H|n\rangle = \left(n+\frac{1}{2}\right)\omega|n\rangle\ . \tag{2.14}$$

2.1.2 Special Relativity Requires Antiparticles

So far, we have considered nonrelativistic quantum mechanics. A theory of elementary particles, however, has to incorporate special relativity. It is very remarkable that quantum mechanics together with special relativity implies the existence of antiparticles. To see this (following an argument in [8]), consider two system (e.g. atoms) A_1 and A_2 at positions $\vec{x}_1$ and $\vec{x}_2$. Assume that at time t_1 atom A_1 emits an electron and turns into B_1. So the charge of B_1 is one unit higher than that of A_1. At a later time t_2 the electron is absorbed by atom A_2 which turns into B_2 with charge lower by one unit. This is illustrated in Fig. 2.2.

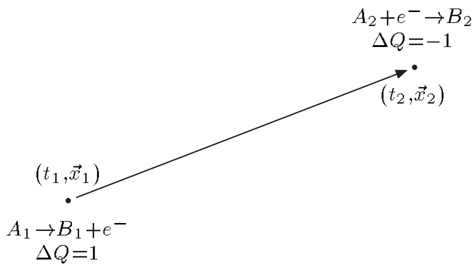


Figure 2.2: *Electron moving from A_1 to A_2*

According to special relativity, we can also watch the system from a frame moving with relative velocity $\vec{v}$. One might now worry whether the process is still causal, i.e., whether the emission still precedes the absorption. In the boosted frame (with primed coordinates), one has

$$t_2' - t_1' = \gamma\left(t_2-t_1\right) + \gamma\vec{v}\left(\vec{x}_2-\vec{x}_1\right)\ , \quad \gamma = \frac{1}{\sqrt{1-\vec{v}^{\,2}}}\,. \tag{2.15}$$

$t_2'-t_1'$ must be positive for the process to remain causal. Since $|\vec{v}|<1$, $t_2'-t_1'$ can only be negative for spacelike distances, i.e., $\left(t_2-t_1\right)^2 - \left(\vec{x}_1-\vec{x}_2\right)^2 < 0$. This, however, would mean that the electron travelled faster than the speed of light, which is not possible according to special relativity. Hence, within classical physics, causality is not violated.

This is where quantum mechanics comes in. The uncertainty relation leads to a "fuzzy" light cone, which gives a non-negligible propagation probability for the electron even for slightly spacelike distances, as long as

$$\left(t_2-t_1\right)^2 - \left(\vec{x}_1-\vec{x}_2\right)^2 \gtrsim -\frac{\hbar^2}{m^2}\,. \tag{2.16}$$

Does this mean causality is violated?

Fortunately, there is a way out: The antiparticle. In the moving frame, one can consider the whole process as emission of a positron at $t = t'_2$, followed by its absorption at a later time $t = t'_1$ (see Fig. 2.3). So we see that quantum mechanics together with special relativity requires the existence of antiparticles for consistency. In addition, particle and antiparticle need to have the same mass.

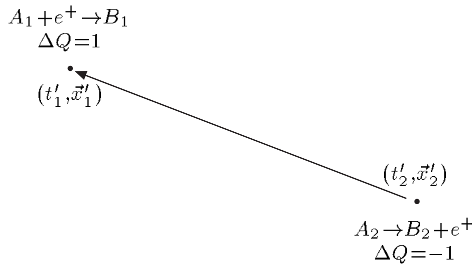


Figure 2.3: *Positron moving from A_2 to A_1*

In a relativistic theory, the uncertainty relation (2.5) also implies that particles cannot be localized below their Compton wavelength

$$\Delta x \geq \frac{\hbar}{mc} . \tag{2.17}$$

For shorter distances the momentum uncertainty $\Delta p > mc$ allows for contributions from multiparticle states, and one can no longer talk about a single particle.

2.2 Multiparticle States and Fields

In the previous section we saw that the combination of quantum mechanics and special relativity has important consequences. First, we need antiparticles, and second, particle number is not well-defined. These properties can be conveniently described by means of fields. A field here is a collection of infinitely many harmonic oscillators, corresponding to different momenta. For each oscillator, we can construct operators and states just as before in the quantum mechanical case. These operators will then be combined into a field operator, the quantum analogue of the classical field. These results can be obtained by applying the method of canonical quantisation to fields.

2.2.1 States, Creation and Annihilation

The starting point is a continuous set of harmonic oscillators, which are labelled by the spatial momentum $\vec{k}$. We want to construct the quantum fields for particles of mass m, so we can combine each momentum $\vec{k}$ with the associated energy $\omega_k = k^0 = \sqrt{\vec{k}^2 + m^2}$ to form the momentum 4-vector k. This 4-vector satisfies $k^2 \equiv k^\mu k_\mu = m^2$. For each k we define creation and annihilation operators, both for particles (a, $a^\dagger$) and antiparticles (b, $b^\dagger$), and construct the space of states just as we did for the harmonic oscillator in the previous section.

For the states we again postulate the vacuum state, which is annihilated by both particle and antiparticle annihilation operators. Each creation operator $a^\dagger(k)$ ($b^\dagger(k)$) creates a (anti)particle with momentum k, so the space of states is:

$$\begin{aligned}
\text{vacuum: } & |0\rangle \ , \quad a(k)|0\rangle = b(k)|0\rangle = 0 \\
\text{one-particle states: } & a^\dagger(k)|0\rangle \ , \ b^\dagger(k)|0\rangle \\
\text{two-particle states: } & a^\dagger(k_1)a^\dagger(k_2)|0\rangle \ , \ a^\dagger(k_1)b^\dagger(k_2)|0\rangle \ , \ b^\dagger(k_1)b^\dagger(k_2)|0\rangle \\
& \vdots
\end{aligned}$$

Like in the harmonic oscillator case, we also have to postulate the commutation relations of these operators, and we choose them in a similar way: operators with different momenta correspond to different harmonic oscillators and hence they commute. Furthermore, particle and antiparticle operators should commute with each other. Hence, there are only two non-vanishing commutators ("canonical commutation relations"):

$$\left[a(k), a^\dagger(k')\right] = \left[b(k), b^\dagger(k')\right] = (2\pi)^3\, 2\omega_k\, \delta^3\!\left(\vec{k} - \vec{k}'\right) , \tag{2.18}$$

which are the counterparts of relation (2.8). The expression on the right-hand side is the Lorentz-invariant way to say that only operators with the same momentum do not commute (the $(2\pi)^3$ is just convention).

Since we now have a continuous label for the creation and annihilation operators, we need a Lorentz-invariant way to sum over operators with different momentum. The four components of k are not independent, but satisfy $k^2 \equiv k_\mu k^\mu = m^2$, and we also require positive energy, that is $k^0 = \omega_k > 0$. Taking these things into account, one is led to the integration measure

$$\begin{aligned}
\int \overline{\mathrm{d}k} &\equiv \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, 2\pi\, \delta\left(k^2 - m^2\right)\, \Theta\left(k^0\right) \\
&= \int \frac{\mathrm{d}^4 k}{(2\pi)^3} \delta\left(\left(k^0 - \omega_k\right)\left(k^0 + \omega_k\right)\right)\, \Theta\left(k^0\right) \\
&= \int \frac{\mathrm{d}^4 k}{(2\pi)^3} \frac{1}{2\omega_k} \left(\delta\left(k^0 - \omega_k\right) + \delta\left(k^0 + \omega_k\right)\right)\, \Theta\left(k^0\right) \\
&= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{2\omega_k} .
\end{aligned} \tag{2.19}$$

The numerical factors are chosen such that they match those in Eq. (2.18) for the commutator of $a(k)$ and $a^\dagger(k)$.

2.2.2 Charge and Momentum

Now we have the necessary tools to construct operators which express some properties of fields and states. The first one is the operator of 4-momentum, i.e., of spatial momentum and energy. Its construction is obvious, since we interpret $a^\dagger(k)$ as a creation operator for a state with 4-momentum k. That means we just have to count the number of particles with each momentum and sum the contributions:

$$P^\mu = \int \overline{\mathrm{d}k}\, k^\mu \left(a^\dagger(k) a(k) + b^\dagger(k) b(k)\right) . \tag{2.20}$$

This gives the correct commutation relations:

$$\left[P^\mu, a^\dagger(k)\right] = k^\mu a^\dagger(k) , \qquad \left[P^\mu, b^\dagger(k)\right] = k^\mu b^\dagger(k) , \tag{2.21a}$$

$$\left[P^\mu, a(k)\right] = -k^\mu a(k) , \qquad \left[P^\mu, b(k)\right] = -k^\mu b(k) . \tag{2.21b}$$

Another important operator is the charge. Since particles and antiparticles have opposite charges, the net charge of a state is proportional to the number of particles minus the number of antiparticles:

$$Q = \int \overline{\mathrm{d}k} \left(a^\dagger(k)a(k) - b^\dagger(k)b(k) \right) , \tag{2.22}$$

and one easily verifies

$$\left[Q, a^\dagger(k)\right] = a^\dagger(k) \,, \qquad \left[Q, b^\dagger(k)\right] = -b^\dagger(k) \,. \tag{2.23}$$

We now have confirmed our intuition that $a^\dagger(k)$ $\left(b^\dagger(k)\right)$ creates a particle with 4-momentum k and charge +1 (-1). Both momentum and charge are conserved: The time derivative of an operator is equal to the commutator of the operator with the Hamiltonian, which is the 0-component of P^μ. This obviously commutes with the momentum operator, but also with the charge:

$$\imath \frac{\mathrm{d}}{\mathrm{d}t} Q = [Q, H] = 0 \,. \tag{2.24}$$

So far, this construction applied to the case of a complex field. For the special case of neutral particles, one has $a = b$ and $Q = 0$, i.e., the field is real.

2.2.3 Field Operator

We are now ready to introduce field operators, which can be thought of as Fourier transform of creation and annihilation operators:

$$\phi(x) = \int \overline{\mathrm{d}k} \left(e^{-\imath kx} a(k) + e^{\imath kx} b^\dagger(k) \right) . \tag{2.25}$$

A spacetime translation is generated by the 4-momentum in the following way:

$$e^{\imath yP} \phi(x) e^{-\imath yP} = \phi(x+y). \tag{2.26}$$

This transformation can be derived from the transformation of the a's:

$$\begin{aligned} e^{\imath yP} a^\dagger(k) e^{-\imath yP} &= a^\dagger(k) + \imath y_\mu \left[P^\mu, a^\dagger(k)\right] + \mathcal{O}\left(y^2\right) &(2.27)\\ &= \left(1 + \imath yk + \cdots\right) a^\dagger(k) &(2.28)\\ &= e^{\imath yk} a^\dagger(k) \,. &(2.29) \end{aligned}$$

The commutator with the charge operator is

$$[Q, \phi(x)] = -\phi(x) \,, \qquad \left[Q, \phi^\dagger\right] = \phi^\dagger \,. \tag{2.30}$$

The field operator obeys the (free) field equation,

$$\left(\Box + m^2\right) \phi(x) = \int \overline{\mathrm{d}k} \left(-k^2 + m^2\right) \left(e^{-\imath kx} a(k) + e^{\imath kx} b^\dagger(k) \right) = 0, \tag{2.31}$$

where $\Box = \partial^2/\partial t^2 - \vec{\nabla}^2$ is the d'Alambert operator.

2.2.4 Propagator

Now we can tackle the problem of causal propagation that led us to introduce antiparticles. We consider the causal propagation of a charged particle between $x_1^\mu = (t_1, \vec{x}_1)$ and $x_2^\mu = (t_2, \vec{x}_2)$, see Fig. (2.4). The field operator creates a state with charge ± 1 "at position $(t, \vec{x})$",

$$Q\, \phi(t, \vec{x})|0\rangle = -\phi(t, \vec{x})|0\rangle \ , \tag{2.32}$$

$$Q\, \phi^\dagger(t, \vec{x})|0\rangle = \phi^\dagger(t, \vec{x})|0\rangle \ . \tag{2.33}$$

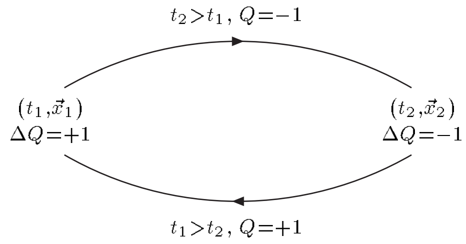


Figure 2.4: *Propagation of a particle or an antiparticle, depending on the temporal order.*

Depending on the temporal order of x_1 and x_2, we interpret the propagation of charge either as a particle going from x_1 to x_2 or an antiparticle going the other way. Formally, this is expressed as the time-ordered product (using the Θ-function, $\Theta(\tau) = 1$ for $\tau > 0$ and $\Theta(\tau) = 0$ for $\tau < 0$):

$$\mathrm{T}\, \phi(x_2)\phi^\dagger(x_1) = \Theta(t_2 - t_1)\phi(x_2)\phi^\dagger(x_1) + \Theta(t_1 - t_2)\phi^\dagger(x_1)\phi(x_2)\,. \tag{2.34}$$

The vacuum expectation value of this expression is the Feynman propagator:

$$\begin{aligned} i\Delta_{\mathrm{F}}(x_2 - x_1) &= \left\langle 0 \,\middle|\, \mathrm{T}\, \phi(x_2)\phi^\dagger(x_1) \middle|\, 0 \right\rangle \\ &= i \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{e^{ik(x_2 - x_1)}}{k^2 - m^2 + i\varepsilon} \ , \end{aligned} \tag{2.35}$$

where we used the Θ-function representation

$$\Theta(\tau) = -\frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \mathrm{d}\omega \frac{e^{-i\omega\tau}}{\omega + i\epsilon}\,. \tag{2.36}$$

This Feynman propagator is a Green function for the field equation,

$$\left(\Box + m^2\right) \Delta_{\mathrm{F}}(x_2 - x_1) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\left(-p^2 + m^2\right)}{p^2 - m^2 + i\varepsilon} e^{-ip(x_2 - x_1)} = -\delta^4\left(x_2 - x_1\right) \,. \tag{2.37}$$

It is causal, i.e. it propagates particles into the future and antiparticles into the past.

2.3 Canonical Quantisation

All the results from the previous section can be derived in a more rigorous manner by using the method of canonical quantisation which provides the step from classical to quantum mechanics. We now start from classical field theory, where the field at point $\vec{x}$ corresponds to the position q in classical mechanics, and we again have to construct the conjugate momentum variables and impose commutation relations among them.

Let us consider the Lagrange density for a complex scalar field ϕ. Like the Lagrangean in classical mechanics, the free Lagrange density is just the kinetic minus the potential energy density,

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi\,. \tag{2.38}$$

The Lagrangean has a $\mathsf{U}(1)$-symmetry, i.e., under the transformation of the field

$$\phi \to \phi' = e^{\imath\alpha}\phi\,, \quad \alpha = \text{const.}\,, \tag{2.39}$$

it stays invariant. From Noether's theorem, there is a conserved current j_μ associated with this symmetry,

$$j^\mu = \imath\phi^\dagger \overleftrightarrow{\partial}^\mu \phi = \imath\left(\phi^\dagger\partial^\mu\phi - \partial^\mu\phi^\dagger\phi\right)\,, \qquad \partial_\mu j^\mu = 0\,. \tag{2.40}$$

The space integral of the time component of this current is conserved in time:

$$Q = \int \mathrm{d}^3x\, \imath\phi^\dagger\overleftrightarrow{\partial}^0\phi\,, \quad \partial_0 Q = 0\,. \tag{2.41}$$

The time derivative vanishes because we can interchange derivation and integration and then replace $\partial_0 j^0$ by $\partial_i j^i$ since $\partial_\mu j^\mu = \partial_0 j^0 + \partial_i j^i = 0$. So we are left with an integral of a total derivative which we can transform into a surface integral via Gauss' theorem. Since we always assume that all fields vanish at spatial infinity, the surface term vanishes.

Now we need to construct the "momentum" $\pi(x)$ conjugate to the field ϕ. Like in classical mechanics, it is given by the derivative of the Lagrangean with respect to the time derivative of the field,

$$\pi(x) = \frac{\partial \mathrm{L}}{\partial\dot{\phi}(x)} = \dot{\phi}^\dagger(x)\,, \qquad \pi^\dagger(x) = \frac{\partial \mathrm{L}}{\partial\dot{\phi}^\dagger(x)} = \dot{\phi}\,. \tag{2.42}$$

At this point, we again replace the classical fields by operators which act on some Hilbert space of states and which obey certain commutation relations. The commutation relations we have to impose are analogous to Eq. (2.4). The only non-vanishing commutators are the ones between field and conjugate momentum, at different spatial points but at equal times,

$$[\pi(t,\vec{x}), \phi(t,\vec{x}')] = \left[\pi^\dagger(t,\vec{x}), \phi^\dagger(t,\vec{x}')\right] = -\imath\delta^3\left(\vec{x}-\vec{x}'\right)\,, \tag{2.43}$$

all other commutators vanish.

These relations are satisfied by the field operator defined in Eq. (2.25) via the (anti)particle creation and annihilation operators. Its field equation can be derived from the Lagrangean,

$$\partial_\mu \frac{\partial \mathrm{L}}{\partial(\partial_\mu\phi)} - \frac{\partial \mathrm{L}}{\partial\phi} = \left(\Box + m^2\right)\phi^\dagger = 0\,. \tag{2.44}$$

From the Lagrangean and the momentum, we can also construct the Hamiltonian density,

$$\mathscr{H} = \pi\dot{\phi} + \pi^\dagger\dot{\phi}^\dagger - \mathrm{L} = \pi^\dagger\pi + \left(\vec{\nabla}\phi^\dagger\right)\left(\vec{\nabla}\phi\right) + m^2\phi^\dagger\phi\,. \tag{2.45}$$

Note that canonical quantisation yields Lorentz invariant results, although it requires the choice of a particular time direction.

2.4 Fermions

Fermions are what makes calculations unpleasant.

In the previous section we considered a scalar field, which describes particles with spin 0. In the standard model, there is just one fundamental scalar field, the Higgs field, which still remains to be discovered. There are other bosonic fields, gauge fields which carry spin 1 (photons, $W^\pm$, Z^0 and the gluons). Those are described by vector fields which will be discussed in Chapter 3. Furthermore, there are the matter fields, fermions with spin $\frac{1}{2}$, the quarks and leptons.

To describe fermionic particles, we need to introduce new quantities, spinor fields. These are four-component objects (but not vectors!) ψ, which are defined via a set of γ-matrices. These four-by-four matrices are labelled by a vector index and act on spinor indices. They fulfill the anticommutation relations (the Clifford or Dirac algebra),

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}\,, \tag{2.46}$$

with the metric $g_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. The numerical form of the γ-matrices is not fixed, rather, one can choose among different possible representations. A common representation is the so-called chiral or Weyl representation, which is constructed from the Pauli matrices:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}\,, \qquad\qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\,. \tag{2.47}$$

This representation is particularly useful when one considers spinors of given chiralities. However, for other purposes, other representations are more convenient. Various rules and identities related to γ-matrices are collected in Appendix A.

The Lagrangean for a free fermion contains, just as for a scalar, the kinetic term and the mass:

$$\mathcal{L} = \overline{\psi}\imath\partial\!\!\!/\psi - m\overline{\psi}\psi\,. \tag{2.48}$$

The kinetic term contains only a first-order derivative, the operator $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$. The adjoint spinor $\overline{\psi}$ is defined as $\overline{\psi} \equiv \psi^\dagger\gamma^0$. (The first guess $\psi^\dagger\psi$ is not Lorentz invariant.) To derive the field equation, one has to treat ψ and $\overline{\psi}$ as independent variables. The Euler-Lagrange equation for $\overline{\psi}$ is the familiar Dirac equation:

$$0 = \frac{\partial\mathcal{L}}{\partial\overline{\psi}} = \left(\imath\partial\!\!\!/ - m\right)\psi\,, \tag{2.49}$$

since $\mathcal{L}$ does not depend on derivatives of $\overline{\psi}$.[1]

The Lagrangean again has a $\mathsf{U}(1)$-symmetry, the multiplication of ψ by a constant phase,

$$\psi \to \psi' = e^{\imath\alpha}\psi\,, \qquad \overline{\psi} \to \overline{\psi}' = e^{-\imath\alpha}\overline{\psi}\,, \tag{2.50}$$

which leads to a conserved current and, correspondingly, to a conserved charge,

$$j^\mu = \overline{\psi}\gamma^\mu\psi\,, \quad \partial^\mu j_\mu = 0\,, \quad Q = \int \mathrm{d}^3x\,\overline{\psi}\gamma^0\psi\,. \tag{2.51}$$

[1]Of course one can shift the derivative from ψ to $\overline{\psi}$ via integration by parts. This slightly modifies the computation, but the result is still the same.

2.4.1 Canonical Quantisation of Fermions

Quantisation proceeds along similar lines as in the scalar case. One first defines the momentum π_α conjugate to the field ψ_α ($\alpha = 1, \ldots, 4$),

$$\pi_\alpha = \frac{\partial \mathrm{L}}{\partial \dot{\psi}_\alpha} = \imath \left(\overline{\psi}\gamma^0\right)_\alpha = \imath \psi^\dagger_\alpha \,. \tag{2.52}$$

Instead of imposing commutation relations, however, for fermions one has to impose anticommutation relations. This is a manifestation of the Pauli exclusion principle which can be derived from the spin-statistics theorem. The relations are again postulated at equal times ("canonical anticommutation relations"):

$$\{\pi_\alpha(t,\vec{x}), \psi_\beta(t,\vec{x}\,')\} = -\imath \delta_{\alpha\beta}\delta^3\left(\vec{x} - \vec{x}\,'\right) \,, \tag{2.53a}$$

$$\{\pi_\alpha(t,\vec{x}), \pi_\beta(t,\vec{x}\,')\} = \{\psi_\alpha(t,\vec{x}), \psi_\beta(t,\vec{x}\,')\} = 0 \,. \tag{2.53b}$$

In order to obtain creation and annihilation operators, we again expand the field operator in terms of plane waves. Because of the four-component nature of the field, now a spinor $u(p)$ occurs, where p is the momentum four-vector of the plane wave:

$$\left(\imath \partial\!\!\!/ - m\right) u(p) e^{-\imath p x} = 0 \,, \tag{2.54}$$

which implies

$$\left(p\!\!\!/ - m\right) u(p) = 0 \,. \tag{2.55}$$

This is an eigenvalue equation for the 4×4-matrix $p_\mu \gamma^\mu$, which has two solutions for $p^2 = m^2$ and $p^0 > 0$. They are denoted $u^{(1,2)}(p)$ and represent positive energy particles. Taking a positive sign in the exponential in Eq. (2.54), which is equivalent to considering $p^0 < 0$, we obtain two more solutions, $v^{(1,2)}(p)$ that can be interpreted as antiparticles. The form of these solutions depends on the representation of the γ-matrices. For the Weyl representation they are given in the appendix.

The eigenspinors determined from the equations ($\imath = 1, 2$),

$$\left(p\!\!\!/ - m\right) u^{(i)}(p) = 0 \,, \qquad \left(p\!\!\!/ + m\right) v^{(i)}(p) = 0 \,, \tag{2.56}$$

obey the identities:

$$\overline{u}^{(i)}(p) u^{(j)}(p) = -\overline{v}^{(i)}(p) v^{(j)}(p) = 2m\delta_{ij} \,, \tag{2.57}$$

$$\sum_i u^{(i)}_\alpha(p)\overline{u}^{(i)}_\beta(p) = \left(p\!\!\!/ + m\right)_{\alpha\beta} \,, \qquad \sum_i v^{(i)}_\alpha(p)\overline{v}^{(i)}_\beta(p) = \left(p\!\!\!/ - m\right)_{\alpha\beta} \,. \tag{2.58}$$

These are the ingredients we need to define creation and annihilation operators in terms of the spinor field $\psi(x)$ and its conjugate $\overline{\psi}(x)$:

$$\psi(x) = \int \overline{\mathrm{d}p} \sum_i \left(b_i(p) u^{(i)}(p) e^{-\imath p x} + d_i^\dagger(p) v^{(i)}(p) e^{\imath p x}\right) \,, \tag{2.59a}$$

$$\overline{\psi}(x) = \int \overline{\mathrm{d}p} \sum_i \left(b_i^\dagger(p) \overline{u}^{(i)}(p) e^{\imath p x} + d_i(p) \overline{v}^{(i)}(p) e^{-\imath p x}\right) \,. \tag{2.59b}$$

Here, as before,

$$\overline{\mathrm{d}p} = \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{2E_p}\,, \quad E_p = \sqrt{\vec{p}^2 + m^2}\,. \tag{2.60}$$

Inverting Eq. (2.59a) one obtains

$$b_i(p) = \int \mathrm{d}^3 x\, \overline{u}^{(i)}(p) e^{\imath px} \gamma^0 \psi(x)\,, \tag{2.61}$$

and similar equations for the other operators.

The creation and annihilation operators inherit the anticommutator algebra from the field operators,

$$\left\{ b_i(\vec{p}), b_j^\dagger(\vec{p}\,') \right\} = \left\{ d_i(\vec{p}), d_j^\dagger(\vec{p}\,') \right\} = (2\pi)^3 2E_p \delta^3 \left(\vec{p} - \vec{p}\,'\right)\,, \tag{2.62a}$$

$$\left\{ b_i(\vec{p}), d_j(\vec{p}\,') \right\} = \text{all other anticommutators} = 0\,. \tag{2.62b}$$

The momentum and charge operators are again constructed from the creation and annihilation operators by "counting" the number of particles in each state and summing over all states,

$$P^\mu = \int \overline{\mathrm{d}k}\, k^\mu \left(b^\dagger(k) b(k) + d^\dagger(k) d(k) \right)\,, \tag{2.63}$$

$$Q = \int \overline{\mathrm{d}k} \left(b^\dagger(k) b(k) - d^\dagger(k) d(k) \right)\,. \tag{2.64}$$

These operators have the correct algebraic relations, which involve commutators, since P^μ and Q are bosonic operators (not changing the number of fermions in a given state):

$$\left[P^\mu, b_i^\dagger(p) \right] = p^\mu b_i^\dagger(p)\,, \qquad \left[P^\mu, d_i^\dagger(p) \right] = p^\mu d_i^\dagger(p)\,, \tag{2.65}$$

$$\left[Q, b_i^\dagger(p) \right] = b_i^\dagger(p)\,, \qquad \left[Q, d_i^\dagger(p) \right] = -d_i^\dagger(p)\,. \tag{2.66}$$

An operator we did not encounter in the scalar case is the spin operator $\vec{\Sigma}$. It has three components, corresponding to the three components of an angular momentum vector[2]. Only one combination of these components is, however, measurable. This is specified by a choice of quantisation axis, i.e., a spatial unit vector $\vec{s}$. The operator that measures the spin of a particle is given by the scalar product $\vec{s} \cdot \vec{\Sigma}$. Creation operators for particles with definite spin satisfy the commutation relations

$$\left[\vec{s} \cdot \vec{\Sigma}, d_\pm^\dagger(p) \right] = \mp \frac{1}{2} d_\pm^\dagger(p)\,, \qquad \left[\vec{s} \cdot \vec{\Sigma}, b_\pm^\dagger(p) \right] = \pm \frac{1}{2} b_\pm^\dagger(p)\,. \tag{2.67}$$

In summary, all these commutation relations tell us how to interpret the operators $d_\pm^\dagger(p)$ ($b_\pm^\dagger(p)$): They create spin-$\frac{1}{2}$ fermions with four-momentum p^μ, charge $+1$ (-1)

[2]Actually, Σ is constructed as a commutator of γ-matrices and as such has six independent components. But three of these correspond to Lorentz boosts which mix time and spatial directions. $\vec{\Sigma}$ is the spin operator in the rest frame.

and spin orientation $\pm\frac{1}{2}$ ($\mp\frac{1}{2}$) relative to the chosen axis $\vec{s}$. Their conjugates $d_\pm(p)$ and $b_\pm(p)$ annihilate those particles.

This immediately leads to the construction of the Fock space of fermions: We again start from a vacuum state $|0\rangle$, which is annihilated by the annihilation operators, and construct particle states by successive application of creation operators:

$$\begin{aligned} \text{vacuum: } & |0\rangle\ , \quad b_i(p)|0\rangle = d_i(p)|0\rangle = 0 \\ \text{one-particle states: } & b_i^\dagger(p)|0\rangle\ ,\ d_i^\dagger(p)|0\rangle \\ \text{two-particle states: } & b_i^\dagger(p_1)d_j^\dagger(p_2)|0\rangle\ ,\ldots \\ & \vdots \end{aligned}$$

At this point we can verify that the Pauli principle is indeed satisfied, due to the choice of anticommutation relations in Eq. (2.53). For a state of two fermions with identical quantum numbers, we would get

$$\underbrace{b_i^\dagger(p)\, b_i^\dagger(p)}_{\text{anticommuting}}\,|X\rangle = -b_i^\dagger(p)\, b_i^\dagger(p)|X\rangle = 0\,, \tag{2.68}$$

where $|X\rangle$ is an arbitrary state. Had we quantised the theory with commutation relations instead, the fermions would have the wrong (i.e., Bose) statistics.

The final expression we need for the further discussion is the propagator. By the same reasoning as in the scalar case, it is obtained as the time-ordered product of two field operators. The Feynman propagator S_{F} for fermions, which is now a matrix-valued object, is given by

$$\begin{aligned} \imath S_{\mathrm{F}}(x_1-x_2)_{\alpha\beta} &= \left\langle 0 \,\middle|\, \mathrm{T}\,\psi_\alpha(x_1)\overline{\psi}_\beta(x_2) \,\middle|\, 0 \right\rangle \\ &= \imath\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\frac{\left(\not{p}+m\right)_{\alpha\beta}}{p^2-m^2+\imath\varepsilon}e^{-\imath p(x_1-x_2)}\,. \end{aligned} \tag{2.69}$$

This completes our discussion on the quantisation of free scalar and spinor fields.

2.5 Interactions

So far, we have considered free particles and their propagation. A theory of elementary particles obviously needs interactions. Unfortunately, they are much more difficult to handle, and little is known rigorously (except in two dimensions). Hence, we have to look for approximations.

By far the most important approximation method is perturbation theory where one treats the interaction as a small effect, a perturbation, to the free theory. The interaction strength is quantified by a numerical parameter, the coupling constant, and one expresses physical quantities as power series in this parameter. This approach has been very successful and has led to many celebrated results, like the precise prediction of the anomalous magnetic moment of the electron, despite the fact that important conceptual problems still remain to be resolved.

2.5.1 ϕ^4 Theory

Let us consider the simplest example of an interacting theory, involving only one real scalar field with a quartic self-interaction (a cubic term would look even simpler, but then the theory would not have a ground state since the energy would not be bounded from below):

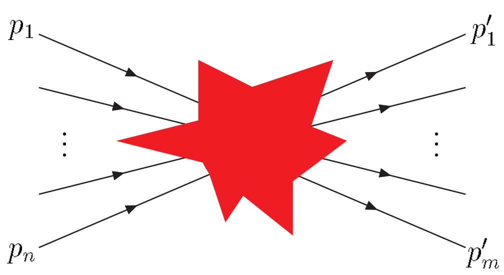


Figure 2.5: *Scattering of n incoming particles, producing m outgoing ones with momenta $p_1, \ldots, p_n$ and $p'_1, \ldots, p'_m$, respectively.*

$$\begin{aligned} \mathrm{L} &= \mathrm{L}_0 + \mathrm{L}_\mathrm{I} \\ &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \,. \end{aligned} \tag{2.70}$$

L_0 is the free Lagrangean, containing kinetic and mass term, while L_I is the interaction term, whose strength is given by the dimensionless coupling constant λ.

In perturbation theory we can calculate various physical quantities, in particular scattering cross sections for processes like the one in Fig. 2.5: n particles with momenta p_i interact, resulting in m particles with momenta p'_j. Since the interaction is localised in a region of spacetime, particles are free at infinite past and future. In other words, we have free asymptotic states

$$|p_1, \ldots, p_n\,, \mathrm{in}\rangle \text{ at } , t = -\infty \quad \text{and} \quad |p'_1, \ldots, p'_m\,, \mathrm{out}\rangle \text{ at } t = +\infty\,. \tag{2.71}$$

The transition amplitude for the scattering process is determined by the scalar product of incoming and outgoing states, which defines a unitary matrix, the so-called S-matrix (S for scattering),

$$\langle p'_1, \ldots, p'_m\,, \mathrm{out}\,|\,p_1, \ldots, p_n\,, \mathrm{in}\rangle = \langle p'_1, \ldots, p'_m\,|S|\,p_1, \ldots, p_n\rangle\ . \tag{2.72}$$

Detailed techniques have been developed to obtain a perturbative expansion for the S-matrix from the definition (2.72). The basis are Wick's theorem and the LSZ-formalism. One starts from a generalisation of the propagator, the time-ordered product of k fields,

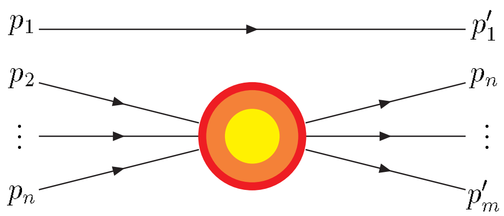


Figure 2.6: *A disconnected diagram: One particle does not participate in the interaction.*

$$\begin{aligned} &\tau(x_1, \ldots, x_k) \\ &\qquad = \langle 0\,|\,\mathrm{T}\,\phi(x_1), \ldots \phi(x_k)|\,0\rangle\ . \end{aligned} \tag{2.73}$$

First, disconnected pieces involving non-interacting particles have to be subtracted (see Fig. 2.6), and the blob in Fig. 2.5 decomposes into a smaller blob and straight lines just passing from the left to the right side. From the Fourier transform

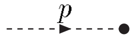
$$\tau(x'_1, \ldots, x'_m, x_1, \ldots, x_n) \xrightarrow{\mathrm{F.T.}} \tilde{\tau}(p'_1, \ldots, p'_m, p_1, \ldots, p_n) \tag{2.74}$$

one then obtains the amplitude for the scattering process

$$\langle p'_1, \ldots, p'_m\,|S|\,p_1, \ldots, p_n\rangle = (2\pi)^4\,\delta^4\left(\sum_{\mathrm{out}} p'_i - \sum_{\mathrm{in}} p_i\right) i\mathcal{M}\,, \tag{2.75}$$

where the matrix element $\mathcal{M}$ contains all the dynamics of the interaction. Due to the translational invariance of the theory, the total momentum is conserved. The matrix element can be calculated perturbatively up to the desired order in the coupling λ via a set of Feynman rules. To calculate the cross section for a particular process, one first draws all possible Feynman diagrams with a given number of vertices and then translates them into an analytic expression using the Feynman rules.

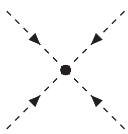
For the ϕ^4 theory, the Feynman diagrams are all composed out of three building blocks: External lines corresponding to incoming or outgoing particles, propagators and 4-vertices. The Feynman rules read:

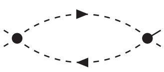
i. 

1 External lines: For each external line, multiply by 1 (i.e., external lines don't contribute to the matrix element in this theory). However, one needs to keep track of the momentum of each particle entering or leaving the interaction. The momentum direction is indicated by the arrow.

ii. 

$\dfrac{\imath}{p^2 - m^2 + \imath\varepsilon}$ Propagators between vertices are free propagators corresponding to the momentum of the particle. Note that particles of internal lines need not be on-shell, i.e., $p^2 = m^2$ need not hold!

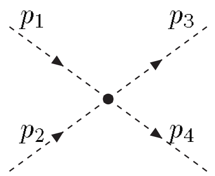
iii. 
$-\imath\lambda$ Vertices yield a factor of the coupling constant. In this theory, there is only one species of particles, and the interaction term does not contain derivatives, so there is only one vertex, and it does not depend on the momenta.

iv. 
$\displaystyle\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$ The momenta of internal loops are not fixed by the incoming momenta. For each undetermined loop momentum p, one integrates over all values of p.

As an example, let us calculate the matrix element for the $2 \to 2$ scattering process to second order in λ. The relevant diagrams are collected in Fig. (2.7). The first-order diagram simply contributes a factor of $-\imath\lambda$, while the second-order diagrams involve an integration:

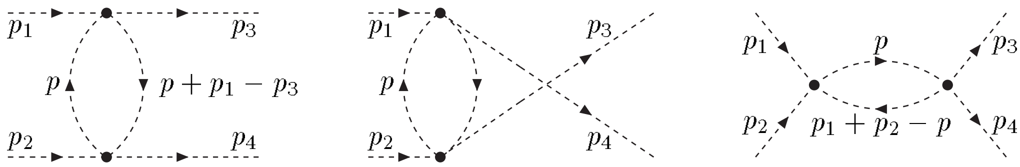
$$\begin{aligned}
\imath\mathcal{M} = -\imath\lambda &+ \frac{1}{2}(-\imath\lambda)^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\imath}{p^2 - m^2} \frac{\imath}{(p + p_1 - p_3)^2 - m^2} \\
&+ \frac{1}{2}(-\imath\lambda)^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\imath}{p^2 - m^2} \frac{\imath}{(p + p_1 - p_4)^2 - m^2} \\
&+ \frac{1}{2}(-\imath\lambda)^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\imath}{p^2 - m^2} \frac{\imath}{(p_1 + p_2 - p)^2 - m^2} + \mathcal{O}\left(\lambda^3\right) .
\end{aligned} \tag{2.76}$$

The factors of $\frac{1}{2}$ are symmetry factors which arise if a diagram is invariant under interchange of internal lines. The expression for $\mathcal{M}$ has a serious problem: The integrals do



(a) Tree graph

Figure 2.7: *Feynman graphs for $2 \to 2$ scattering in ϕ^4 theory to second order. The one-loop graphs all are invariant under the interchange of the internal lines and hence get a symmetry factor of $\frac{1}{2}$.*



(a) One-loop graphs

not converge. This can be seen by counting the powers of the integration variable p. For p much larger that incoming momenta and the mass, the integrand behaves like p^{-4}. That means that the integral depends logarithmically on the upper integration limit,

$$\int^{\Lambda} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2} \frac{1}{(p + p_1 - p_3)^2 - m^2} \xrightarrow{p \gg p_i, m} \int^{\Lambda} \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{-1}{p^4} \propto \ln \Lambda \,. \tag{2.77}$$

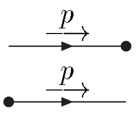
Divergent loop diagrams are ubiquitous in quantum field theory. They can be cured by regularisation, i.e., making the integrals finite by introducing some cutoff parameter, and renormalisation, where this additional parameter is removed in the end, yielding finite results for observables. This will be discussed in more detail in the chapter on quantum corrections.

2.5.2 Fermions

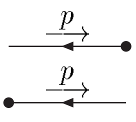
We can augment the theory by adding a fermionic field ψ, with a Lagrangean including an interaction with the scalar ϕ,

$$\mathrm{L}_\psi = \underbrace{\overline{\psi}\left(\imath \partial\!\!\!/ - m\right)\psi}_{\text{free Lagrangean}} - \underbrace{g\overline{\psi}\phi\psi}_{\text{interaction}} \,. \tag{2.78}$$

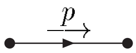
There are additional Feynman rules for fermions. The lines carry two arrows, one for the momentum as for the scalars and one for the fermion number flow, which basically distinguishes particles and antiparticles. The additional rules are:

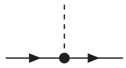
i. 

$u(p)$ $\overline{u}(p)$ Incoming or outgoing particles get a factor of $u(p)$ or $\overline{u}(p)$, respectively.

ii. 

$\overline{v}(p)$ $v(p)$ Incoming or outgoing antiparticles get a factor of $\overline{v}(p)$ or $v(p)$, respectively.

iii.  $\dfrac{\imath\left(\not{p}+m\right)}{p^2-m^2+\imath\varepsilon}$ Free propagator for fermion with momentum p.

iv.  $-\imath g$ The fermion-fermion-scalar vertex yields a factor of the coupling constant. Again, there is no momentum dependence.

Chapter 3

Gauge Theories

In addition to spin-0 and spin-$\frac{1}{2}$ particles, the standard model contains spin-1 particles. They are the quanta of vector fields which can describe strong and electroweak interactions. The corresponding theories come with a local ("gauge") symmetry and are called gauge theories.

3.1 Global Symmetries v Gauge Symmetries

Consider a complex scalar field with the Lagrangean

$$\mathrm{L} = \partial_\mu \phi^\dagger \partial^\mu - V\left(\phi^\dagger \phi\right) , \tag{3.1}$$

which is a generalisation of the one considered in Eq. (2.38). This theory has a $\mathsf{U}(1)$ symmetry under which $\phi \to \phi' = \exp\{\imath\alpha\}\, \phi$ with constant parameter α. Usually it is sufficient to consider the variation of the fields and the Lagrangean under infinitesimal transformations,

$$\delta\phi = \phi' - \phi = \imath\alpha\phi\,, \qquad \delta\phi^\dagger = -\imath\alpha\phi^\dagger\,, \tag{3.2}$$

where terms $\mathcal{O}\left(\alpha^2\right)$ have been neglected. To derive the Noether current, Eq. (2.40), we compute the variation of the Lagrangean under such a transformation:

$$\begin{aligned}
\delta\mathrm{L} &= \frac{\partial \mathrm{L}}{\partial \phi}\delta\phi + \frac{\partial \mathrm{L}}{\partial \phi^\dagger}\delta\phi^\dagger + \frac{\partial \mathrm{L}}{\partial\left(\partial_\mu \phi\right)} \underbrace{\delta\left(\partial_\mu \phi\right)}_{=\partial_\mu \delta\phi} + \frac{\partial \mathrm{L}}{\partial\left(\partial_\mu \phi^\dagger\right)}\delta\left(\partial_\mu \phi^\dagger\right) \\
&= \underbrace{\left(\frac{\partial \mathrm{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathrm{L}}{\partial\left(\partial_\mu \phi\right)}\right)}_{=0 \text{ by equation of motion}} \delta\phi + \underbrace{\left(\frac{\partial \mathrm{L}}{\partial \phi^\dagger} - \partial_\mu \frac{\partial \mathrm{L}}{\partial\left(\partial_\mu \phi^\dagger\right)}\right)}_{=0} \delta\phi^\dagger \\
&\quad + \partial_\mu \left(\frac{\partial \mathrm{L}}{\partial\left(\partial_\mu \phi\right)}\delta\phi + \frac{\partial \mathrm{L}}{\partial\left(\partial_\mu \phi^\dagger\right)}\delta\phi^\dagger\right) \\
&= \alpha\partial_\mu\left(\imath\partial^\mu\phi^\dagger\phi - \imath\phi^\dagger\partial^\mu\phi\right) \\
&= -\alpha\partial_\mu j^\mu\,.
\end{aligned} \tag{3.3}$$

Since the Lagrangean is invariant, $\delta\mathrm{L} = 0$, we obtain a conserved current for solutions of the equations of motion,

$$\partial_\mu j^\mu = 0\,. \tag{3.4}$$

From the first to the second line we have used that

$$\frac{\partial \mathrm{L}}{\partial\left(\partial_\mu \phi\right)} \partial_\mu \delta\phi = \partial_\mu \left(\frac{\partial \mathrm{L}}{\partial\left(\partial_\mu \phi\right)} \delta\phi\right) - \left(\partial_\mu \frac{\partial \mathrm{L}}{\partial\left(\partial_\mu \phi\right)}\right) \delta\phi \tag{3.5}$$

by the Leibniz rule.

The above procedure can be generalised to more complicated Lagrangeans and symmetries. The derivation does not depend on the precise form of L, and up to the second line of (3.3), it is independent of the form of $\delta\phi$. As a general result, a symmetry of the Lagrangean always implies a conserved current, which in turn gives a conserved quantity (often referred to as charge, but it can be angular momentum or energy as well).

What is the meaning of such a symmetry? Loosely speaking, it states that "physics does not change" under such a transformation. This, however, does not mean that the solutions to the equations of motion derived from this Lagrangean are invariant under such a transformation. Indeed, generically they are not, and only $\phi \equiv 0$ is invariant.

As an example, consider the Mexican hat potential,

$$V(\phi^\dagger\phi) = -\mu^2 \phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2 . \tag{3.6}$$

This potential has a ring of minima, namely all fields for which $|\phi|^2 = \mu^2/(2\lambda)$. This means that any constant ϕ with this modulus is a solution to the equation of motion,

$$\Box\phi + \frac{\partial V}{\partial\phi}(\phi, \phi^\dagger) = \Box\phi - \phi^\dagger\left(\mu^2 - 2\lambda\phi^\dagger\phi\right) = 0 . \tag{3.7}$$

These solutions are not invariant under $\mathsf{U}(1)$ phase rotations. On the other hand, it is obvious that any solution to the equations of motion will be mapped into another solution under such a transformation.

This situation is analogous to the Kepler problem: A planet moving around a stationary (very massive) star. The setup is invariant under spatial rotations around the star, i.e., the symmetries form the group $\mathsf{SO}(3)$. This group is three-dimensional (meaning that any rotation can be built from three independent rotations, e.g. around the three axes of a Cartesian coordinate system). Thus there are three conserved charges which correspond to the three components of angular momentum. The solutions of this problem – the planet's orbits – are ellipses in a plane, so they are not at all invariant under spatial rotations, not even under rotations in the plane of motion. Rotated solutions, however, are again solutions.

In particle physics, most experiments are scattering experiments at colliders. For those, the statement that "physics does not change" translates into "transformed initial states lead to transformed final states": If one applies the transformation to the initial state and performs the experiment, the result will be the same as if one had done the experiment with the untransformed state and transformed the result.

There is a subtle, but important, difference between this and another type of symmetry, gauge symmetry. A gauge transformation is also a transformation which leaves the Lagrangean invariant, but it does relate identical states which describe exactly the same physics.

This might be familiar from electrodynamics. One formulation uses electric and magnetic fields $\vec{E}$ and $\vec{B}$, together with charge and current densities ρ and $\vec{j}$. These fields and sources are related by Maxwell's equations:

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0\,, \qquad \vec{\nabla} \cdot \vec{B} = 0\,, \tag{3.8a}$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j}\,, \qquad \vec{\nabla} \cdot \vec{E} = \rho\,. \tag{3.8b}$$

The first two of these can be identically solved by introducing the potentials ϕ and $\vec{A}$, which yield $\vec{E}$ and $\vec{B}$ via

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\,, \qquad \vec{B} = \vec{\nabla} \times \vec{A}\,. \tag{3.9}$$

So we have reduced the six components of $\vec{E}$ and $\vec{B}$ down to the four independent ones ϕ and $\vec{A}$. However, the correspondence between the physical fields and the potentials is not unique. If some potentials ϕ and $\vec{A}$ lead to certain $\vec{E}$ and $\vec{B}$ fields, the transformed potentials

$$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda\,, \qquad \phi' = \phi - \frac{\partial \Lambda}{\partial t}\,, \tag{3.10}$$

where Λ is a scalar field, give the same electric and magnetic fields.

This transformation (3.9) is called gauge transformation. It is a symmetry of the theory, but it is different from the global symmetries we considered before. First, it is a local transformation, i.e., the transformation parameter Λ varies in space and time. Second, it relates physically indistinguishable field configurations, since solutions of the equations of motion for electric and magnetic fields are invariant. It is important to note that this gauge transformation is inhomogeneous, i.e., the variation is not multiplicative, but can generate non-vanishing potentials from zero. Potentials that are related to $\phi = 0$ and $\vec{A} = 0$ by a gauge transformation are called pure gauge.

Phrased differently, we have expressed the physical fields $\vec{E}$ and $\vec{B}$ in terms of the potentials ϕ and $\vec{A}$. These potentials still contain too many degrees of freedom for the physical fields $\vec{E}$ and $\vec{B}$, since different potentials can lead to the same $\vec{E}$ and $\vec{B}$ fields. So the description in terms of potentials is redundant, and the gauge transformation (3.10) quantifies just this redundancy. Physical states and observables have to be invariant under gauge transformations.

3.2 Abelian Gauge Theories

The easiest way to come up with a gauge symmetry is to start from a global symmetry and promote it to a gauge one, that is, demand invariance of the Lagrangean under local transformations (where the transformation parameter is a function of spacetime). To see this, recall the Lagrangean with the global $\mathsf{U}(1)$ symmetry from the preceding section,

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi^\dagger \phi)\,,$$

and the transformation

$$\phi \to \phi' = e^{\imath\alpha}\phi\,, \quad \delta\phi = \phi' - \phi = \imath\alpha\phi\,.$$

If we now allow spacetime dependent parameters $\alpha(x)$, the Lagrangean is no longer invariant. The potential part still is, but the kinetic term picks up derivatives of $\alpha(x)$, so the variation of the Lagrangean is

$$\delta \mathrm{L} = \imath \partial_\mu \alpha \left(\partial^\mu \phi^\dagger \phi - \phi^\dagger \partial^\mu \phi \right) = -\partial_\mu \alpha \, j^\mu \,, \tag{3.11}$$

the derivative of α times the Noether current of the global symmetry derived before.

The way to restore invariance of the Lagrangean is to add another field, the gauge field, with a gauge transformation just like the electromagnetic potentials in the previous section, combined into a four-vector $A^\mu = (\phi, \vec{A})$:

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \,. \tag{3.12}$$

The factor $\frac{1}{e}$ is included for later convenience. We can now combine the inhomogeneous transformation of A_μ with the inhomogeneous transformation of the derivative in a covariant derivative D_μ:

$$D_\mu \phi = \left(\partial_\mu + \imath e A_\mu \right) \phi \,. \tag{3.13}$$

This is called covariant derivative because the differentiated object $D_\mu \phi$ transforms in the same way as the original field,

$$\begin{aligned} D_\mu \phi \longrightarrow \left(D_\mu \phi \right)' &= \left(\partial_\mu + \imath e A'_\mu \right) \phi' \\ &= \partial_\mu \left(e^{\imath \alpha(x)} \phi \right) + \imath e \left(A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \right) e^{\imath \alpha(x)} \phi \\ &= e^{\imath \alpha(x)} D_\mu \phi \,. \end{aligned} \tag{3.14}$$

So we can construct an invariant Lagrangean from the field and its covariant derivative:

$$\mathrm{L} = \left(D_\mu \phi \right)^\dagger \left(D^\mu \phi \right) - V\left(\phi^\dagger \phi \right) \,. \tag{3.15}$$

So far this is a theory of a complex scalar with $\mathsf{U}(1)$ gauge invariance. The gauge field A_μ, however, is not a dynamical field, i.e., there is no kinetic term for it. This kinetic term should be gauge invariant and contain derivatives up to second order. In order to find such a kinetic term, we first construct the field strength tensor from the commutator of two covariant derivatives:

$$\begin{aligned} F_{\mu\nu} &= -\frac{\imath}{e} \left[D_\mu, D_\nu \right] = -\frac{\imath}{e} \left[\left(\partial_\mu + \imath e A_\mu \right), \left(\partial_\nu + \imath e A_\nu \right) \right] \\ &= -\frac{\imath}{e} \left(\left[\partial_\mu, \partial_\nu \right] + \left[\partial_\nu, \imath e A_\nu \right] + \left[\imath e A_\mu, \partial_\nu \right] - e^2 \left[A_\mu, A_\nu \right] \right) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \,. \end{aligned} \tag{3.16}$$

To check that this is a sensible object to construct, we can redecompose A_μ into the scalar and vector potential ϕ and $\vec{A}$ and spell out the field strength tensor in electric and magnetic fields,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_2 & 0 \end{pmatrix} \,. \tag{3.17}$$

This shows that the field strength is gauge invariant, as $\vec{E}$ and $\vec{B}$ are. Of course, this can also be shown by straightforward calculation,

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = -\frac{1}{e}\left(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu\right) \alpha(x) = 0\,, \tag{3.18}$$

so it is just the antisymmetry in μ and ν that ensures gauge invariance.

The desired kinetic term is now just the square of the field strength tensor,

$$\mathcal{L}_{\text{gaugekin}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\,, \tag{3.19a}$$

or, in terms of $\vec{E}$ and $\vec{B}$ fields,

$$\mathcal{L} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) \tag{3.19b}$$

The coupling to scalar fields via the covariant derivative can also be applied to fermions. To couple a fermion ψ to the gauge field, one simply imposes the gauge transformation

$$\psi \to \psi' = e^{\imath\alpha} \psi\,. \tag{3.20}$$

In the Lagrangean, one again replaces the ordinary derivative with the covariant one. The Lagrangean for a fermion coupled to a $\mathsf{U}(1)$ gauge field is quantum electrodynamics (QED), if we call the fields electron and photon:

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}\left(\imath \not{D} - m\right) \psi\,. \tag{3.21}$$

Finally, let us note that for a $\mathsf{U}(1)$ gauge theory, different fields may have different charges under the gauge group (as e.g. quarks and leptons indeed do). For fields with charge q (in units of elementary charge), we have to replace the gauge transformations and consequently the covariant derivative as follows:

$$\psi_q \to \psi_q' = e^{\imath q \alpha} \psi_q\,, \qquad D_\mu^{(q)} \psi_q = \left(\partial_\mu + \imath q e A_\mu\right) \psi_q\,. \tag{3.22}$$

What have we done so far? We started from a Lagrangean, Eq. (3.1) with a global $\mathsf{U}(1)$ symmetry (3.2). We imposed invariance under local transformations, so we had to introduce a new field, the gauge field A_μ. This field transformed inhomogeneously under gauge transformations, just in a way to make a covariant derivative. This covariant derivative was the object that coupled the gauge field to the other fields of the theory. To make this into a dynamical theory, we added a kinetic term for the gauge field, using the field strength tensor. Alternatively, we could have started with the gauge field and tried to couple it to other fields, and we would have been led to the transformation properties (3.2). This is all we need to construct the Lagrangean for QED. For QCD and the electroweak theory, however, we need a richer structure: non-Abelian gauge theories.

3.3 Non-Abelian Gauge Theories

To construct non-Abelian theories in the same way as before, we first have to discuss non-Abelian groups, i.e., groups whose elements do not commute. We will focus on the groups $\mathsf{SU}(n)$, since they are most relevant for the standard model. $\mathsf{SU}(n)$ is the group of $n \times n$ complex unitary matrices with determinant 1. To see how many degrees of freedom there are, we have to count: A $n \times n$ complex matrix U has n^2 complex entries, equivalent to $2n^2$ real ones. The unitarity constraint, $U^\dagger U = \mathbb{1}$, is a matrix equation, but not all component equations are independent. Actually, $U^\dagger U$ is Hermitean, $\left(U^\dagger U\right)^\dagger = U^\dagger U$, so the diagonal entries are real and the lower triangle is the complex conjugate of the upper one. Thus, there are $n + 2 \cdot \frac{1}{2} n(n-1)$ real constraints. Finally, by taking the determinant of the unitarity constraint, $\det\left(U^\dagger U\right) = |\det U|^2 = 1$. Hence, restricting to $\det U = 1$ eliminates one more real degree of freedom. All in all, we have $2n^2 - n - 2 \cdot \frac{1}{2} n(n-1) - 1 = n^2 - 1$ real degrees of freedom in the elements of $\mathsf{SU}(n)$.

This means that any $U \in \mathsf{SU}(n)$ can be specified by $n^2 - 1$ real parameters α_a. The group elements are usually written in terms of these parameters and $n^2 - 1$ matrices T^a, the generators of the group, as an exponential

$$U = \exp\left\{\imath \alpha_a T^a\right\} = \mathbb{1} + \imath \alpha_a T^a + \mathcal{O}\left(\alpha^2\right) , \tag{3.23}$$

and one often considers only infinitesimal parameters.

The generators are usually chosen as Hermitean matrices[1]. The product of group elements translates into commutation relations for the generators,

$$\left[T^a, T^b\right] = \imath f^{abc} T^c , \tag{3.24}$$

with the antisymmetric structure constants f^{abc}, which of course also depend on the choice of generators.

In the standard model, the relevant groups are $\mathsf{SU}(2)$ for the electroweak theory and $\mathsf{SU}(3)$ for QCD. $\mathsf{SU}(2)$ has three parameters. The generators are usually chosen to be the Pauli matrices, $T^a = \frac{1}{2}\sigma^a$, whose commutation relations are $\left[\sigma^a, \sigma^b\right] = \imath \varepsilon^{abc} \sigma^c$. The common generators of $\mathsf{SU}(3)$ are the eight Gell-Mann matrices, $T^a = \frac{1}{2}\lambda^a$.

To construct a model with a global $\mathsf{SU}(n)$ symmetry, we consider not a single field, but an n-component vector Φ_i, $i = 1, \ldots, n$ (called a multiplet of $\mathsf{SU}(n)$), on which the matrices of $\mathsf{SU}(n)$ act by multiplication :

$$\Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_n \end{pmatrix} \longrightarrow \Phi' = U\Phi , \quad \Phi^\dagger = \left(\Phi_1^\dagger, \cdots, \Phi_n^\dagger\right) \longrightarrow \left(\Phi^\dagger\right)' = \Phi^\dagger U^\dagger . \tag{3.25}$$

Now we see why we want unitary matrices U: A product $\Phi^\dagger \Phi$ is invariant under such a transformation. This means that we can generalise the Lagrangean (3.1) in a straightforward way to include a non-Abelian symmetry:

$$\mathrm{L} = (\partial_\mu \Phi)^\dagger (\partial^\mu \Phi) - V\left(\Phi^\dagger \Phi\right) . \tag{3.26}$$

[1] Actually, the generators live in the Lie algebra of the group, and so one can choose any basis one likes, Hermitean or not.

If we allow for local transformations $U = U(x)$, we immediately encounter the same problem as before: The derivative term is not invariant, because the derivatives act on the matrix U as well,

$$\partial_\mu \Phi \to \partial_\mu \Phi' = \partial_\mu (U\Phi) = U\partial_\mu \Phi + (\partial_\mu U)\,\Phi\,. \tag{3.27}$$

To save the day, we again need to introduce a covariant derivative consisting of a partial derivative plus a gauge field. This time, however, the vector field needs to be matrix-valued, i.e., $A_\mu = A_\mu^a T^a$, where T^a are the generators of the group. We clearly need one vector field per generator, as each generator represents an independent transformation in the group.

The transformation law of A_μ is chosen such that the covariant derivative is covariant,

$$\begin{aligned}
(D_\mu \Phi)' &= [(\partial_\mu + \imath g A_\mu)\,\Phi]' \\
&= \left(\partial_\mu + \imath g A'_\mu\right)(U\Phi) \\
&= U\left(\partial_\mu + U^{-1}(\partial_\mu U) + \imath g U^{-1} A'_\mu U\right)\Phi \\
&\stackrel{!}{=} U D_\mu \Phi\,.
\end{aligned} \tag{3.28}$$

This requirement fixes the transformation of A_μ to be

$$A'_\mu = U A_\mu U^{-1} - \frac{1}{g} U\partial_\mu U^{-1}\,. \tag{3.29}$$

In the Abelian case this reduces to the known transformation law, Eq. (3.12).

For infinitesimal parameters $\alpha = \alpha^a T^a$, the matrix $U = \exp\{\imath\alpha\} = 1 + \imath\alpha$, and Eq. (3.29) becomes

$$A'_\mu = A_\mu - \frac{1}{g}\partial_\mu \alpha + \imath\,[\alpha, A_\mu]\ , \tag{3.30}$$

or for each component

$$A_\mu^{a\prime} = A_\mu^a - \frac{1}{g}\partial_\mu \alpha^a - f^{abc}\alpha^b A_\mu^c\,. \tag{3.31}$$

Sometimes it is convenient to write down the covariant derivative in component form:

$$(D_\mu \Phi)_i = \left(\partial_\mu \delta_{ij} + \imath g T_{ij}^a A_\mu^a\right)\Phi_j\,. \tag{3.32}$$

Next we need a kinetic term, which again involves the field strength, the commutator of covariant derivatives:

$$\begin{aligned}
F_{\mu\nu} &= -\frac{1}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu + \imath g\left[A_\mu, a_\nu\right] = F_{\mu\nu}^a T^a\,, \\
F_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c\,.
\end{aligned} \tag{3.33}$$

Now we see that the field strength is more that just the derivative: There is a quadratic term in the potentials. This leads to a self-interaction of gauge fields, like in QCD, where the gluons interact with each other. This is the basic reason for confinement, unlike in QED, where the photon is not charged.

Furthermore, when we calculate the transformation of the field strength, we find that it is not invariant, but transforms as

$$F_{\mu\nu} \to F'_{\mu\nu} = U F_{\mu\nu} U^{-1}\,, \tag{3.34}$$

i.e., it is covariant. There is an easy way to produce an invariant quantity out of this: the trace. Since $\mathrm{tr}\, AB = \mathrm{tr}\, BA$, the Lagrangean

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} \tag{3.35}$$

is indeed invariant, as $\mathrm{tr}\,(UF^2U^{-1}) = \mathrm{tr}\,(U^{-1}UF^2) = \mathrm{tr}\, F^2$. In the second step we have used a normalisation convention,

$$\mathrm{tr}\left(T^aT^b\right) = \frac{1}{2}\delta^{ab}\,, \tag{3.36}$$

and every generator is necessarily traceless. The factor $\frac{1}{2}$ is arbitrary and could be chosen differently, with compensating changes in the coefficient of the kinetic term.

By choosing the gauge group $\mathsf{SU}(3)$ and coupling the gauge field to fermions, the quarks, we can write down the Lagrangean of quantum chromodynamics (QCD):

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu} + \overline{q}\left(\imath \not{D} - m\right)q\,, \tag{3.37}$$

where $a = 1, \ldots, 8$ counts the gluons and q is a three-component (i.e. three-colour) quark.

3.4 Quantisation

So far we have only discussed classical gauge theories. If we want to quantise the theory and find the Feynman rules for diagrams involving gauge fields, we have a problem: We have to make sure we do not count field configurations of A_μ which are pure gauge, nor that we count separately fields which differ only by a gauge transformation, since those are meant to be physically identical. On the more technical side, the naïve Green function for the free equation of motion does not exist. In the Abelian case, the equation is

$$\partial_\mu F^{\mu\nu} = \Box A^\nu - \partial^\nu\partial_\mu A^\mu = \left(\Box g^{\mu\nu} - \partial^\nu\partial^\mu\right)A_\mu = 0\,. \tag{3.38}$$

The Green function should be the inverse of the differential operator in brackets, but the operator is not invertible. Indeed, it annihilates every pure gauge mode, as it should,

$$\left(\Box g^{\mu\nu} - \partial^\nu\partial^\mu\right)\partial_\mu\Lambda = 0\,, \tag{3.39}$$

so it has zero eigenvalues. Hence, the propagator must be defined in a more clever way.

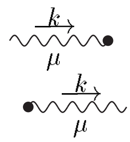
One way out would be to fix the gauge, i.e., simply demand a certain gauge condition like $\vec{\nabla}\cdot\vec{A} = 0$ (Coulomb gauge) or $n_\mu A^\mu = 0$ with a fixed 4-vector (axial gauge). It turns out, however, that the loss of Lorentz invariance causes many problems in calculations.

A better way makes use of Faddeev–Popov ghosts. In this approach, we add two terms to the Lagrangean, the gauge-fixing term and the ghost term. The gauge-fixing term is not gauge invariant, but rather represents a certain gauge condition which can be chosen freely. The fact that it is not gauge invariant means that now the propagator is well-defined, but the price to pay is that it propagates too many degrees of freedom, namely gauge modes. This is compensated by the propagation of ghosts, strange fields which are scalars but anticommute and do not show up as physical states but only as internal lines in loop calculations. It turns out that gauge invariance is not lost but rather traded for a different symmetry, BRST-symmetry, which ensures that we get physically sensible results.

For external states, we have to restrict to physical states, of which there are two for massless bosons. They are labelled by two polarisation vectors $\epsilon_\mu^\pm$ which are transverse, i.e., orthogonal to the momentum four-vector and the spatial momentum, $k_\mu \epsilon^\mu = \vec{k}\vec{\epsilon} = 0$.

The form of the gauge fixing and ghost terms depends on the gauge condition we want to take. A common (class of) gauge is the covariant gauge which depends on a parameter ξ, which becomes Feynman gauge (Landau/Lorenz gauge) for $\xi = 1$ ($\xi = 0$)

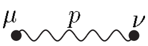
We now list the Feynman rules for a non-Abelian gauge theory (QCD) coupled to fermions (quarks) and ghosts. The fermionic external states and propagators are listed in Section 2.5.2.

i. 

$\epsilon_\mu(k)$

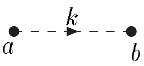
$\epsilon^*_\mu(k)$

For each external line one has a polarisation vector.

ii. 

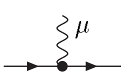
$$\frac{-i\delta^{ab}}{k^2 + i\varepsilon} \times \left(g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2} \right)$$

The propagator for gauge bosons contains the parameter ξ.

iii. 

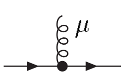
$$\frac{-i\delta^{ab}}{k^2 + i\varepsilon}$$

The propagator for ghosts is the one of scalar particles. There are no external ghost states.

iv. 

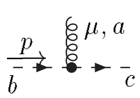
$ie\gamma^\mu$

In QED, there is just one vertex between photon and electron.

v. 

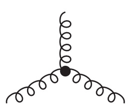
$i\frac{g}{2}\gamma^\mu \lambda^a$

In QCD, the basic quark-quark-gluon vertex involves the Gell-Mann matrices.

vi. 

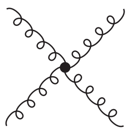
$-gf^{abc}p^\mu$

The ghosts couple to the gauge field.

vii. 

$gf^{abc}k_\mu$ + permutations

Three-gluon self-interaction.

viii.  $-\frac{1}{4}g^2 f^{abc} f^{ade} g^{\mu\nu} g^{\rho\sigma}$ +permutations Four-gluon self-interaction.

Chapter 4

Quantum Corrections

Now that we have the Feynman rules, we are ready to calculate quantum corrections [3, 5, 9]. As a first example we will consider the anomalous magnetic moment of the electron at one-loop order. This was historically, and still is today, one of the most important tests of quantum field theory. The calculation is still quite simple because the one-loop expression is finite. In most cases, however, one encounters divergent loop integrals. In the following sections we will study these divergences and show how to remove them by renormalisation. Finally, as an application, we will discuss the running of coupling constants and asymptotic freedom.

4.1 Anomalous Magnetic Moment

The magnetic moment of the electron determines its energy in a magnetic field,

$$H_{\text{mag}} = -\vec{\mu} \cdot \vec{B} \,. \tag{4.1}$$

For a particle with spin $\vec{s}$, the magnetic moment is aligned in the direction of $\vec{s}$, and for a classical spinning particle of mass m and charge e, its magnitude would be the Bohr magneton, $e/(2m)$. In the quantum theory, the magnetic moment is different, which is expressed by the Landé factor g_e,

$$\vec{\mu}_e = g_e \frac{e}{2m} \vec{s} \,. \tag{4.2}$$

We now want to calculate g_e in QED. To lowest order, this just means solving the Dirac equation in an external electromagnetic field $A^\mu = (\phi, \vec{A})$,

$$\left(\imath \not{D} - m\right) \psi = \left[\gamma^\mu \left(\imath \partial_\mu - e A_\mu\right) - m\right] \psi = 0 \,. \tag{4.3}$$

For a bound nonrelativistic electron a stationary solution takes the form

$$\psi(x) = \begin{pmatrix} \varphi(\vec{x}) \\ \chi(\vec{x}) \end{pmatrix} e^{-\imath E t} \,, \quad \text{with} \quad \frac{E-m}{m} \ll 1 \,. \tag{4.4}$$

It is convenient to use the following representation of the Dirac matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \,, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \,. \tag{4.5}$$

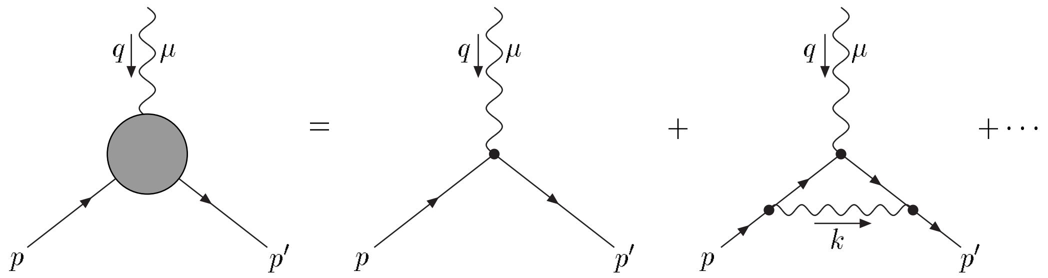


Figure 4.1: *Tree level and one-loop diagram for the magnetic moment.*

One then obtains the two coupled equations

$$[(E - e\phi) - m]\,\varphi - \left(-\imath\vec{\nabla} - e\vec{A}\right)\cdot\vec{\sigma}\chi = 0\,, \tag{4.6a}$$

$$[\underbrace{-(E - e\phi) - m}_{\approx -2m}]\,\chi + \left(-\imath\vec{\nabla} - e\vec{A}\right)\cdot\vec{\sigma}\varphi = 0\,. \tag{4.6b}$$

The coefficient of χ in the second equation is approximately independent of ϕ, so we can solve the equation to determine χ in terms of φ,

$$\chi = \frac{1}{m}\left(-\imath\vec{\nabla} - e\vec{A}\right)\cdot\vec{\sigma}\varphi\,. \tag{4.7}$$

Inserting this into (4.6a), we get the Pauli equation,

$$\left[\frac{1}{2m}\left(-\imath\vec{\nabla} - e\vec{A}\right)^2 + e\phi - \frac{e}{2m}\vec{B}\cdot\vec{\sigma}\right]\varphi = (E - m)\,\varphi\,. \tag{4.8}$$

This is a Schrödinger-like equation which implies (since $\vec{s} = \frac{1}{2}\vec{\sigma}$),

$$H_{\mathrm{mag}} = -2\frac{e}{2m}\vec{s}\vec{B}\,. \tag{4.9}$$

Hence, the Landé factor of the electron is $g_e = 2$.

In QED, the magnetic moment is modified by quantum corrections. The magnetic moment is the spin-dependent coupling of the electron to a photon in the limit of vanishing photon momentum. Diagrammatically, it is contained in the blob on the left side of Fig. 4.1, which denotes the complete electron-photon coupling. The tree-level diagram is the fundamental electron-photon coupling. There are several one-loop corrections to this diagram, but only the so-called vertex correction, where an internal photon connects the two electron lines, gives a contribution to the magnetic moment. All other one-loop diagrams concern only external legs, such as an electron-positron-bubble on the incoming photon, and will be removed by wave-function renormalisation.

The expression for the tree-level diagram is

$$\imath\overline{u}(p')e\gamma^\mu u(p)\,. \tag{4.10}$$

Note that the photon becomes on-shell only for $q \to 0$, so no polarisation vector is included. The matrix element of the electromagnetic current can be decomposed via the

Gordon identity into convection and spin currents,

$$\overline{u}(p')\gamma^\mu u(p) = \overline{u}(p')\left(\frac{(p+p')^\mu}{2m} + \frac{\imath}{2m}\sigma^{\mu\nu}(p'-p)_\nu\right)u(p)\,. \tag{4.11}$$

Here the first term can be viewed as the net flow of charged particles, the second one is the spin current. Only this one is relevant for the magnetic moment, since it gives the spin-dependent coupling of the electron.

In order to isolate the magnetic moment from the loop diagram, we first note that the corresponding expression will contain the same external states, so it can be written as

$$\imath\overline{u}(p')e\Gamma^\mu(p,q)u(p)\,, \quad q = p' - p\,, \tag{4.12}$$

where $\Gamma^\mu(p,q)$ is the correction to the vertex due to the exchange of the photon. We can now decompose Γ^μ into different parts according to index structure and extract the term $\propto\sigma^{\mu\nu}$. Using the Feynman rules, we find for Γ^μ in Feynman gauge ($\xi = 1$),

$$\begin{aligned} \imath e\Gamma^\mu(p,q) = (-\imath e)^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{-\imath g_{\rho\sigma}}{k^2+\imath\varepsilon}\gamma^\rho \frac{\imath\left(\not{p}'-\not{k}+m\right)}{(p'-k)^2-m^2+\imath\varepsilon}\gamma^\mu \\ \times\frac{\imath\left(\not{p}-\not{k}+m\right)}{(p-k)^2-m^2+\imath\varepsilon}\gamma^\sigma\,. \end{aligned} \tag{4.13}$$

This integral is logarithmically divergent, as can be seen by power counting, since the leading term is $\propto k^2$ in the numerator and $\propto k^6$ in the denominator.

On the other hand, the part $\propto\sigma^{\mu\nu}q_\mu$ is finite and can be extracted via some tricks:

- Consider first the denominator of the integral (4.13). It is the product of three terms of the form $(\text{momentum})^2 - m^2$, which can be transformed into a sum at the expense of further integrations over the so-called Feynman parameters x_1 and x_2,

$$\frac{1}{A_1A_2A_3} = 2\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1}\mathrm{d}x_2 \frac{1}{\left[A_1x_1 + A_2x_2 + A_3(1-x_1-x_2)\right]^3}\,. \tag{4.14}$$

- After introducing the Feynman parameters, the next trick is to shift the integration momentum $k \to k'$, where

$$A_1x_1 + A_2x_2 + A_3(1-x_1-x_2) = \underbrace{(k - x_1p' - x_2p)}_{k'}{}^2 - (x_1p' + x_2p)^2 + \imath\varepsilon\,. \tag{4.15}$$

  Note that one must be careful when manipulating divergent integrals. In principle, one should first regularise them and then perform the shifts on the regularised integrals, but in this case, there is no problem.

- For the numerator, the important part is the Dirac algebra of γ-matrices. A standard calculation gives (see appendix)

$$\begin{aligned} &\gamma^\nu\left(\not{p}'-\not{k}+m\right)\gamma^\mu\left(\not{p}-\not{k}+m\right)\gamma_\nu \\ &\quad = -2m^2\gamma_\mu - 4\imath m\sigma^{\mu\nu}(p'-p)_\nu - 2\not{p}\gamma_\mu\not{p}' + \mathcal{O}(k) + \mathcal{O}(k^2)\,. \end{aligned} \tag{4.16}$$

Here we have used again the Gordon formula to trade $(p+p')_\nu$ for $\sigma_{\nu\rho}q^\rho$, which only is allowed if the expression is sandwiched between on-shell spinors $\overline{u}(p')$ and $u(p)$.

- Now the numerator is split into pieces independent of k, linear and quadratic in k. The linear term can be dropped under the integral. The quadratic piece leads to a divergent contribution which we will discuss later. The integral over the k-independent part in the limit $q^\mu \to 0$ yields

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{\left[k^2-(x_1+x_2)^2 m^2 + \imath\varepsilon\right]^3} = -\frac{\imath}{32\pi^2} \frac{1}{(x_1+x_2)^2 m^2} . \tag{4.17}$$

Now all that is left are the parameter integrals over x_1 and x_2.

Finally, one obtains the result, usually expressed in terms of the fine structure constant $\alpha = e^2/(4\pi)$,

$$\imath e \overline{u}(p')\Gamma^\mu u(p) = +\imath e \overline{u}(p') \left(\frac{\alpha}{2\pi} \frac{\imath}{2m} \sigma^{\mu\nu} q_\nu + \cdots \right) u(p) , \tag{4.18}$$

where the dots represent contributions which are not $\propto \sigma^{\mu\nu}q_\nu$.

Comparison with the Gordon decomposition (4.11) gives the one-loop correction to the Landé factor,

$$g = 2\left(1+\frac{\alpha}{2\pi}\right) . \tag{4.19}$$

This correction was first calculated by Schwinger in 1948. It is often expressed as the anomalous magnetic moment a_e,

$$a_e = \frac{g-2}{2} . \tag{4.20}$$

Today, a_e is known up to three loops analytically and to four loops numerically [10]. The agreement of theory and experiment is impressive:

$$\begin{aligned} a_e^{\text{exp}} &= (1159652185.9 \pm 3.8) \cdot 10^{-12} , \\ a_e^{\text{th}} &= (1159652175.9 \pm 8.5) \cdot 10^{-12} . \end{aligned} \tag{4.21}$$

This is one of the cornerstones of our confidence in quantum field theory.

4.2 Divergences

The anomalous magnetic moment we calculated in the last section was tedious work, but at least the result was finite. Most other expressions, however, have divergent momentum integrals. One such example is the vertex function Γ^μ we already considered. It has contributions which are logarithmically divergent. We can isolate these by setting $q=0$, which yields

$$\Gamma^\mu(p,0) = -2\imath e^2 \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \int \frac{\mathrm{d}^4k}{(2\pi)^2} \frac{\gamma^\nu \not{k} \gamma_\mu \not{k} \gamma_\nu}{\left[k^2-(x_1+x_2)^2 m^2 + \imath\varepsilon\right]^3} . \tag{4.22}$$

This expression is treated in two steps:

- First we make the integral finite in a step called regularisation. In this step, we have to introduce a new parameter of mass dimension 1. An obvious choice would be a cutoff Λ which serves as an upper bound for the momentum integration. One might even argue that there should be a cutoff at a scale where quantum gravity becomes important, although a regularisation parameter has generally no direct physical meaning.

- The second step is renormalisation. The divergences are absorbed into the parameters of the theory. The key idea is that the "bare" parameters which appear in the Lagrangean are not physical, so they can be divergent. Their divergences are chosen such as to cancel the divergences coming from the divergent integrals.

- Finally, the regulator is removed. Since all divergences have been absorbed into the parameters of the theory, the results remain finite for infinite regulator. Of course, one has to make sure the results do not depend on the regularisation method.

The cutoff regularisation, while conceptually simple, is not a convenient method, as it breaks Lorentz and gauge invariance. Symmetries, however, are very important for all calculations, so a good regularisation scheme should preserve as many symmetries as possible. We will restrict ourselves to dimensional regularisation, which is the most common scheme used nowadays.

4.2.1 Dimensional Regularisation

The key idea is to define the theory not in four, but in $d = 4 - \epsilon$ dimensions [9]. If ϵ is not an integer, the integrals do converge. Non-integer dimensionality might seem weird, but in the end we will take the limit of $\epsilon \to 0$ and return to four dimensions. This procedure is well-defined and just an intermediate step in the calculation.

Let us consider some technical issues. In d dimensions, the Lorentz indices "range from 0 to d", in the sense that

$$g^{\mu\nu} g_{\nu\mu} = d \,, \tag{4.23}$$

and there are d γ-matrices obeying the usual algebra,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1} \,, \quad \operatorname{tr}(\mathbb{1}) = 4 \,. \tag{4.24}$$

γ-matrix contractions are also modified due to the change in the trace of $g_{\mu\nu}$, such as

$$\gamma^\mu\gamma^\nu\gamma_\mu = -(2-\epsilon)\gamma^\nu \,, \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - \epsilon\gamma^\nu\gamma^\rho \,. \tag{4.25}$$

The tensor structure of diagrams can be simplified as follows. If a momentum integral over k contains a factor of $k_\mu k_\nu$, this must be proportional to $g_{\mu\nu}k^2$, since it is of second order in k and symmetric in $(\mu\nu)$. The only symmetric tensor we have is the metric (as long as the remaining integrand depends only on the square of k and the squares of the external momenta p_i), and the coefficient can be obtained by contracting with $g^{\mu\nu}$ to yield

$$\int \frac{\mathrm{d}^4 k}{(2\pi)^4} k_\mu k_\nu f\left(k^2, p_i^2\right) = \frac{1}{d}\, g_{\mu\nu} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} k^2 f\left(k^2, p_i^2\right) \,. \tag{4.26}$$

The measure of an integral changes from d^4k to d^dk. Since k is a dimensionful quantity[1] (of mass dimension 1), we need to compensate the change in dimensionality by a factor of μ^ϵ, where μ is an arbitrary parameter of mass dimension 1. The mass dimensions of fields and parameters also change. They can be derived from the condition that the action, which is the d-dimensional integral over the Lagrangean, be dimensionless. Schematically (i.e., without all numerical factors), a Lagrangean of gauge fields, scalars and fermions reads

$$\begin{aligned} \mathrm{L} = & \left(\partial_\mu A_\nu\right)^2 + e\partial_\mu A^\mu A_\nu A^\nu + e^2\left(A_\mu A^\mu\right)^2 \\ & + \left(\partial_\mu \phi\right)^2 + \overline{\psi}\left(\imath \partial\!\!\!/ - m\right)\psi + e\overline{\psi} A\!\!\!/ \psi + m^2\phi^2 + \cdots \ . \end{aligned} \tag{4.27}$$

The condition of dimensionless action, $[S]=0$, translates into $[\mathrm{L}]=d$, since $\left[\mathrm{d}^dx\right]=-d$. Derivatives have mass dimension 1, and so do masses. That implies for the dimensions of the fields (and the limit as $d\to 4$),

$$[A_\mu] = \frac{d-2}{2} \to 1 \ , \qquad [\phi] = \frac{d-2}{2} \to 1 \ , \tag{4.28}$$

$$[\psi] = \frac{d-1}{2} \to \frac{3}{2} \ , \qquad [e] = 2 - \frac{d}{2} \to 0 \ . \tag{4.29}$$

How do we evaluate a d-dimensional integral? One first transforms to Euclidean space replacing k^0 by $\imath k_4$, so that the Lorentzian measure d^dk becomes d^dk_E. In Euclidean space, one can easily convert to spherical coordinates and perform the integral over the angular variables, which gives the "area" of the d-dimensional "unit sphere",

$$\int \frac{\mathrm{d}^dk_\mathrm{E}}{(2\pi)^d} f\left(k^2\right) = \underbrace{\int \frac{\mathrm{d}\Omega_d}{(2\pi)^d}}_{\frac{1}{2^{d-1}\pi^{d/2}}\frac{1}{\Gamma(d/2)}} \int_0^\infty \mathrm{d}k_\mathrm{E}\, k_\mathrm{E}^{d-1} f\left(k^2\right) \ . \tag{4.30}$$

The remaining integral can then be evaluated, again often using Γ-functions. The result is finite for $d\neq 4$, but as we let $d\to 4$, the original divergence appears again in the form of $\Gamma(2-d/2)$. The Γ-function has poles at negative integers and at zero, so the integral exists for noninteger dimension. In the limit $d\to 4$, or equivalently, $\epsilon\to 0$, one has

$$\Gamma\left(2-\frac{d}{2}\right) = \Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma_\mathrm{E} + \mathcal{O}(\epsilon) \ , \tag{4.31}$$

with the Euler constant $\gamma_\mathrm{E} \simeq 0.58$.

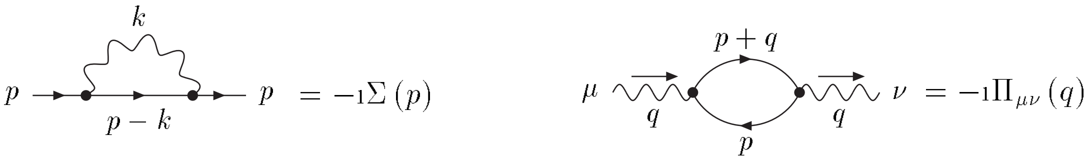
As an example, consider the logarithmically divergent integral (cf. (4.22))

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{1}{\left(k^2+C\right)^2} \ , \tag{4.32}$$

where $C = \left(x_1+x_2\right)^2 m^2$. In d Euclidean dimensions, this becomes

$$\mu^\epsilon \int \frac{\mathrm{d}^4k_\mathrm{E}}{(2\pi)^4} \frac{1}{\left(k_\mathrm{E}^2+C\right)^2} = \frac{\mu^\epsilon \Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}\,\Gamma(2)} \frac{1}{C^{2-d/2}} = \frac{1}{8\pi^2}\frac{1}{\epsilon} + \cdots \tag{4.33}$$

[1] In our units where $\hbar = c = 1$, the only dimension is mass, so everything can be expressed in powers of GeV. The basic quantities have [mass] = [energy] = [momentum] = 1 and [length] = [time] = −1, so $[\mathrm{d}x^\mu] = -1$ and $[\partial_\mu] = 1$.



(a) The electron self-energy

(b) The vacuum polarisation

Figure 4.2: *One-loop corrections to the propagators of electron and photon.*

For the original expression (4.22) we thus obtain

$$\Gamma^{\mu}(p,0) = \frac{\alpha}{2\pi}\frac{1}{\epsilon}\gamma^{\mu} + \mathcal{O}(1)\;. \tag{4.34}$$

What have we achieved? In four dimensions, the result is still divergent. However, the situation is better than before: We have separated the divergent part from the finite one and can take care of the divergence before taking the limit $\epsilon \to 0$. This is done in the procedure of renormalisation.

There are more divergent one-loop graphs where we can achieve the same: the electron self-energy Σ in Fig. 4.2(a) (linearly divergent) and the photon self-energy or vacuum polarisation $\Pi_{\mu\nu}$ in Fig. 4.2(b) (quadratically divergent). The self-energy graph has two divergent terms,

$$\Sigma(p) = \frac{3\alpha}{2\pi}\frac{1}{\epsilon}m - \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\not{p} - m\right) + \mathcal{O}(1)\,, \tag{4.35}$$

which contribute to the mass renormalisation and the wave function renormalisation, respectively. The vacuum polarisation seems more complicated since it is a second rank tensor. However, the tensor structure is fixed by gauge invariance which requires

$$q^{\mu}\Pi_{\mu\nu}(q) = 0\;. \tag{4.36}$$

Therefore, because of Lorentz invariance,

$$\Pi_{\mu\nu}(q) = \left(g_{\mu\nu}q^2 - q_{\mu}q_{\nu}\right)\Pi\left(q^2\right)\;. \tag{4.37}$$

The remaining scalar quantity $\Pi(q^2)$ has the divergent part

$$\Pi\left(q^2\right) = \frac{2\alpha}{3\pi}\frac{1}{\epsilon} + \mathcal{O}(1)\,. \tag{4.38}$$

4.2.2 Renormalisation

So far we have isolated the divergences, but they are still there. How do we get rid of them? The crucial insight is that the parameters of the Lagrangean, the "bare" parameters, are not observable. Rather, the sum of bare parameters and loop-induced corrections are physical. Hence, divergencies of bare parameters can cancel against divergent loop corrections, leaving physical observables finite.

To make this more explicit, let us express, as an example, the QED Lagrangean in terms of bare fields A_0^μ and ψ_0 and bare parameters m_0 and e_0,

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_{0\,\nu} - \partial_\nu A_{0\,\mu}\right)\left(\partial^\mu A^{0\,\nu} - \partial^\nu A^{0\,\mu}\right) + \overline{\psi}_0\left(\gamma^\mu\left(\imath\partial_\mu - e_0 A_{0\,\mu}\right) - m_0\right)\psi_0 \,. \tag{4.39}$$

The "renormalised fields" A_μ and ψ and the "renormalised parameters" e and m are then obtained from the bare ones by multiplicative rescaling,

$$A_{0\,\mu} = \sqrt{Z_3}A_\mu \,, \qquad \psi_0 = \sqrt{Z_2}\psi \,, \tag{4.40}$$

$$m_0 = \frac{Z_m}{Z_2}m \,, \qquad e_0 = \frac{Z_1}{Z_2\sqrt{Z_3}}\mu^{2-d/2}e \,. \tag{4.41}$$

Note that coupling and electron mass now depend on the mass parameter μ,

$$e = e(\mu) \,, \quad m = m(\mu) \,. \tag{4.42}$$

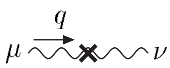
In terms of the renormalized fields and parameters the Lagrangean (4.39) reads

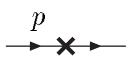
$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) + \overline{\psi}\left(\gamma^\mu\left(\imath\partial_\mu - eA_\mu\right) - m\right)\psi + \Delta\mathcal{L} \,, \tag{4.43}$$

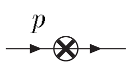
where $\Delta\mathcal{L}$ contains the divergent counterterms,

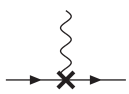
$$\begin{aligned}\Delta\mathcal{L} = &-\left(Z_3 - 1\right)\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(Z_2 - 1\right)\overline{\psi}\imath\partial\!\!\!/\psi \\ &-\left(Z_m - 1\right)m\overline{\psi}\psi - \left(Z_1 - 1\right)e\overline{\psi}A\!\!\!/\psi \,.\end{aligned} \tag{4.44}$$

The counterterms have the same structure as the original Lagrangean and lead to new vertices in the Feynman rules:

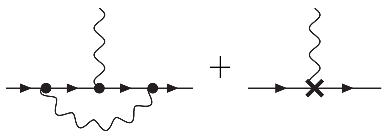
i.  $-\imath\left(Z_3 - 1\right) \times \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right)$ Photon wave function counterterm (countertems are generically denoted by ✕). It has the same tensor structure as the vacuum polarisation.

ii.  $-\imath\left(Z_2 - 1\right)p\!\!\!/$ Electron wave function counterterm.

iii.  $-\imath\left(Z_m - 1\right)m$ Electron mass counterterm.

iv.  $-\imath e\left(Z_1 - 1\right)\gamma^\mu$ Vertex counterterm.

The renormalisation constants Z_i are determined by requiring that the counterterms cancel the divergences. They can be determined as power series in α. The lowest order counterterms are $\mathcal{O}(\alpha)$ and have to be added to the one-loop diagrams. Calculating e.g. the $\mathcal{O}(\alpha)$ correction to the electron-photon vertex, one has

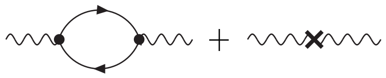


$$= -\imath e\gamma^\mu \left(\frac{\alpha}{2\pi}\frac{1}{\epsilon} + (Z_1 - 1) + \mathcal{O}(1) \right) . \tag{4.45}$$

Demanding that the whole expression be finite determines the divergent part of Z_1,

$$Z_1 = 1 - \frac{\alpha}{2\pi}\frac{1}{\epsilon} + \mathcal{O}(1) . \tag{4.46}$$

Similarly, the $\mathcal{O}(\alpha)$ vacuum polarisation now has two contributions,



$$= -\imath \left(g_{\mu\nu}q^2 - q_\mu q_\nu\right) \left(\frac{2\alpha}{3\pi}\frac{1}{\epsilon} + (Z_3 - 1) + \mathcal{O}(1) \right) , \tag{4.47}$$

which yields

$$Z_3 = 1 - \frac{2\alpha}{3\pi}\frac{1}{\epsilon} + \mathcal{O}(1) . \tag{4.48}$$

The other constants Z_2 and Z_m are fixed analogously. A Ward identity, which follows from gauge invariance, yields the important relation $Z_1 = Z_2$. The finite parts of the renormalisation constants are still undetermined. There are different ways to fix them, corresponding to different renormalisation schemes. All schemes give the same results for physical quantities, but differ at intermediate steps.

Having absorbed the divergences into the renormalised parameters and fields, we can safely take the limit $\epsilon \to 0$. The theory now yields well-defined relations between physical observables. Divergencies can be removed to all orders in the loop expansion for renormalisable theories [3, 9]. Quantum electrodynamics and the standard model belong to this class. The proof is highly non-trivial and has been a major achievement in quantum field theory!

4.2.3 Running Coupling in QED

Contrary to the bare coupling e_0, the renormalised coupling $e(\mu)$ depends on the renormalisation scale μ (cf. (4.41)),

$$e_0 = \frac{Z_1}{Z_2\sqrt{Z_3}}\mu^{-2+d/2}e(\mu) = e(\mu)\mu^{-\epsilon/2}Z_3^{-\frac{1}{2}} ,$$

where we have used the Ward identity $Z_1 = Z_2$. It is very remarkable that the scale dependence is determined by the divergencies. To see this, expand Eq. (4.41) in ϵ and $e(\mu)$,

$$\begin{aligned} e_0 &= e(\mu)\left(1 - \frac{\epsilon}{2}\ln\mu + \cdots\right)\left(1 + \frac{1}{\epsilon}\frac{\alpha}{3\pi} + \cdots\right) \\ &= e(\mu)\left(\frac{1}{\epsilon}\frac{e^2(\mu)}{12\pi^2} + 1 - \frac{e^2(\mu)}{24\pi^2}\ln\mu + \mathcal{O}\left(\epsilon, e^4(\mu)\right)\right) , \end{aligned} \tag{4.49}$$

where we have used $\alpha = e^2/(4\pi)$. Since the bare mass e_0 does not depend on μ, differentiation with respect to μ yields

$$0 = \mu \frac{\partial}{\partial \mu} e_0 = \mu \frac{\partial}{\partial \mu} e - \frac{e^3}{24\pi^2} + \mathcal{O}\left(e^5\right) \ , \tag{4.50}$$

and therefore

$$\mu \frac{\partial}{\partial \mu} e = \frac{e^3}{24\pi^2} + \mathcal{O}\left(e^5\right) \equiv \beta(e) \, . \tag{4.51}$$

This equation is known as the renormalisation group equation, and the function on the right hand side of Eq. (4.51) is the so-called the β function ,

$$\beta(e) = \frac{b_0}{(4\pi)^2} e^3 + \mathcal{O}\left(e^5\right) \ , \quad \text{with} \quad b_0 = \frac{2}{3} \, . \tag{4.52}$$

The differential equation (4.51) can easily be integrated. Using a given value of e at a scale μ_1, the coupling α at another scale μ is given by

$$\alpha(\mu) = \frac{\alpha\left(\mu_1\right)}{1 - \alpha\left(\mu_1\right) \frac{b_0}{(2\pi)} \ln \frac{\mu}{\mu_1}} \, . \tag{4.53}$$

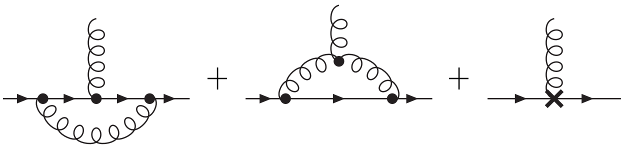
Since $b_0 > 0$, the coupling increases with μ until it approaches the so-called Landau pole where the denominator vanishes and perturbation theory breaks down.

What is the meaning of a scale dependent coupling? This becomes clear when one calculates physical quantities, such as a scattering amplitude at some momentum transfer q^2. In the perturbative expansion one then finds terms $\propto e^2(\mu)\log(q^2/\mu^2)$. Such terms make the expansion unreliable unless one chooses $\mu^2 \sim q^2$. Hence, $e^2(q^2)$ represents the effective interaction strength at a momentum (or energy) scale q^2 or, alternatively, at a distance of $r \sim 1/q$.

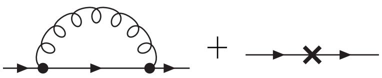
The positive β function in QED implies that the effective coupling strength decreases at large distances. Qualitatively, this can be understood as the effect of "vacuum polarisation": Electron-positron pairs etc. from the vacuum screen any bare charge at distances larger than the corresponding Compton wavelength. Quantitatively, one finds that the value $\alpha(0) = \frac{1}{137}$, measured in Thompson scattering, increases to $\alpha(M_Z^2) = \frac{1}{127}$, the value conveniently used in electroweak precision tests.

4.2.4 Running Coupling in QCD

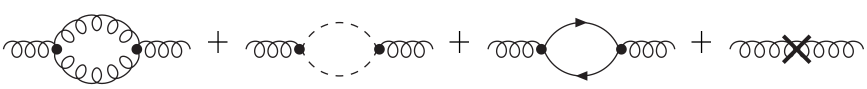
Everything we did so far for QED can be extended to non-Abelian gauge theories, in particular to QCD [5]. It is, however, much more complicated, since there are more diagrams to calculate, and we will not be able to discuss this in detail. The additional diagrams contain gluon self-interactions and ghosts, and they lead to similar divergences, which again are absorbed by renormalisation constants. Schematically, these are



$$\rightsquigarrow Z_1 \,, \tag{4.54}$$



$$\rightsquigarrow Z_2 \,, \tag{4.55}$$



$$\rightsquigarrow Z_3 \,. \tag{4.56}$$

The renormalised coupling can again be defined as in QED, Eq. (4.41),

$$g_0 = \frac{Z_1}{Z_2\sqrt{Z_3}}\mu^{-2+d/2}\, g \,. \tag{4.57}$$

The coefficients of the $1/\epsilon$-divergences depend on the gauge group and on the number of different fermions. For a $\mathsf{SU}(N_c)$ gauge group with N_f flavours of fermions, one obtains the β function for the gauge coupling g,

$$\mu\frac{\partial}{\partial\mu}g = \frac{b_0}{(4\pi)^2}g^3 + \mathcal{O}\left(g^5\right) \,, \quad b_0 = -\left(\frac{11}{3}N_c - \frac{4}{3}N_f\right) \,. \tag{4.58}$$

Note that for $N_f < 11N_c/4$ the coefficient is negative! Hence, the coupling decreases at high momentum transfers or short distances. The calculation of this coefficient earned the Nobel Prize in 2004 for Gross, Politzer and Wilczek. The decrease of the coupling at short distances is the famous phenomenon of asymptotic freedom. As a consequence, one can treat in deep-inelastic scattering quarks inside the proton as quasi-free particles, which is the basis of the parton model.

The coupling at a scale μ can again be expressed in terms of the coupling at a reference scale μ_1,

$$\alpha(\mu) = \frac{\alpha\left(\mu_1\right)}{1 + \alpha\left(\mu_1\right)\frac{|b_0|}{(2\pi)}\ln\frac{\mu}{\mu_1}} \,. \tag{4.59}$$

The analogue of the Landau pole now occurs at small μ or large distances. For QCD with$N_c = 3$ and $N_f = 6$, the pole is at the "QCD scale" $\Lambda_{\mathrm{QCD}} \simeq 300$ MeV. At the QCD scale gluons and quarks are strongly coupled and colour is confined [5]. Correspondingly, the inverse of Λ_{QCD} gives roughly the size of hadrons, $r_{\mathrm{had}} \sim \Lambda_{\mathrm{QCD}}^{-1} \sim 0.7$ fm.

Chapter 5

Electroweak Theory

So far we have studied QED, the simplest gauge theory, and QCD, the prime example of a non-Abelian gauge theory. But there also are the weak interactions, which seem rather different. They are short-ranged, which requires massive messenger particles, seemingly inconsistent with gauge invariance. Furthermore, weak interactions come in two types, charged and neutral current-current interactions, which couple quarks and leptons differently. Charged current interactions, mediated by the $W^{\pm}$ bosons, only involve left-handed fermions and readily change flavour, as in the strange quark decay $s \to ue^-\overline{\nu}_e$. Neutral current interactions, on the other hand, couple both left- and right-handed fermions, and flavour-changing neutral currents are strongly suppressed.

Despite these differences from QED and QCD, weak interactions also turn out to be described by a non-Abelian gauge theory. Yet the electroweak theory is different because of two reasons: It is a chiral gauge theory, and the gauge symmetry is spontaneously broken.

5.1 Quantum Numbers

In a chiral gauge theory, the building blocks are massless left- and right-handed fermions,

$$\psi_L = \frac{1}{2}\left(1-\gamma^5\right)\psi_L\,, \qquad \psi_R = \frac{1}{2}\left(1+\gamma^5\right)\psi_R\,, \tag{5.1}$$

with different gauge quantum numbers. For one generation of standard model particles, we will have seven chiral spinors: Two each for up- and down-type quark and charged lepton, and just one for the neutrino which we will treat as massless in this chapter, i.e., we omit the right-handed one. The electroweak gauge group is a product of two groups, $\mathsf{G}_{EW} = \mathsf{SU}(2)_W \times \mathsf{U}(1)_Y$. Here the subscript W stands for "weak isospin", which is the quantum number associated with the $\mathsf{SU}(2)_W$ factor, and the $\mathsf{U}(1)$ charge is the hypercharge Y.

The assignment of quantum numbers, which corresponds to the grouping into representations of the gauge group, is obtained as follows: The non-Abelian group $\mathsf{SU}(2)_W$ has a chargeless one-dimensional singlet ($\mathbf{1}$) representation and charged multidimensional representations, starting with the two-dimensional doublet ($\mathbf{2}$) representation[1].

[1] Here we use "representation" as meaning "irreducible representation". Of course we can build reducible representations of any dimension.

We are not allowed to mix quarks and leptons, since weak interactions do not change colour, nor left- and right-handed fields, which would violate Lorentz symmetry. The $\mathsf{U}(1)_Y$ factor is Abelian, so it only has one-dimensional representations. This means we can assign different hypercharges we to the various singlets and doublets of $\mathsf{SU}(2)_W$.

Furthermore, we know that charged currents connect up- with down-type quarks and charged leptons with neutrinos, and that the $W^\pm$ bosons couple only to left-handed fermions. This suggests to form doublets from u_L and d_L, and from e_L and ν_L, and to keep the right-handed fields as singlets. So we obtain the $\mathsf{SU}(2)_W$ multiplets

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} , \qquad u_R , \qquad d_R , \qquad l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} , \qquad e_R , \tag{5.2}$$

with the hypercharges (which we will justify later)

$$\begin{array}{rccccc} \text{field:} & q_L & u_R & d_R & l_L & e_R \\ \text{hypercharge:} & \frac{1}{6} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{2} & -1 \end{array} . \tag{5.3}$$

With these representations, we can write down the covariant derivatives. The $\mathsf{SU}(2)_W$ has three generators, which we choose to be the Pauli matrices, and therefore three gauge fields W_μ^I, $I = 1, 2, 3$. The $\mathsf{U}(1)_Y$ gauge field is B_μ, and the coupling constants are g and g', respectively. The covariant derivatives acting on the left-handed fields are

$$D_\mu \psi_L = \left(\partial_\mu + \imath g W_\mu + \imath g' Y B_\mu\right) \psi_L , \quad \text{where } W_\mu = \tfrac{1}{2}\sigma^I W_\mu^I , \tag{5.4}$$

while the right-handed fields are singlets under $\mathsf{SU}(2)_W$, and hence do not couple to the W bosons,

$$D_\mu \psi_R = \left(\partial_\mu + \imath g' Y B_\mu\right) \psi_R . \tag{5.5}$$

From the explicit form of the Pauli matrices,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma^2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} , \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{5.6}$$

we see that W_μ^1 and W_μ^2 mix up- and down-type quarks, while W_μ^3 does not, like the $\mathsf{U}(1)$ boson B_μ.

It is often convenient to split the Lagrangean into the free (kinetic) part and the interaction Lagrangean, which takes the form (current)·(vector field). In the electroweak theory, one has

$$\mathrm{L}_{\text{int}} = -g J_{W,\mu}^I W^{I\,\mu} - g' J_{Y,\mu} B^\mu , \tag{5.7}$$

with the currents

$$J_{W,\mu}^I = \overline{q}_L \gamma_\mu \tfrac{1}{2}\sigma^I q_L + \overline{l}_L \gamma_\mu \tfrac{1}{2}\sigma^I l_L , \tag{5.8}$$

$$J_{Y,\mu} = \frac{1}{6}\overline{q}_L \gamma_\mu q_L - \frac{1}{2}\overline{l}_L \gamma_\mu l_L + \frac{2}{3}\overline{u}_R \gamma_\mu u_R - \frac{1}{3}\overline{d}_R \gamma_\mu d_R - \overline{e}_R \gamma_\mu e_R . \tag{5.9}$$

These currents have to be conserved, $\partial_\mu J^\mu = 0$, to allow a consistent coupling to gauge bosons.

5.1.1 Anomalies

Before considering the Higgs mechanism which will lead to the identification of the physical $W^\pm$, Z and γ bosons of the standard model, let us briefly discuss anomalies. We will see that the choice of hypercharges in (5.3) is severely constrained by the consistency of the theory.

Suppose we have a classical field theory with a certain symmetry and associated conserved current. After quantising the theory, the resulting quantum field theory might not have that symmetry anymore, which means the current is no longer conserved. This is called an anomaly. Anomalies are not a problem for global symmetries, where the quantised theory just lacks that particular symmetry. For gauge symmetries, however, the currents have to be conserved, otherwise the theory is inconsistent.

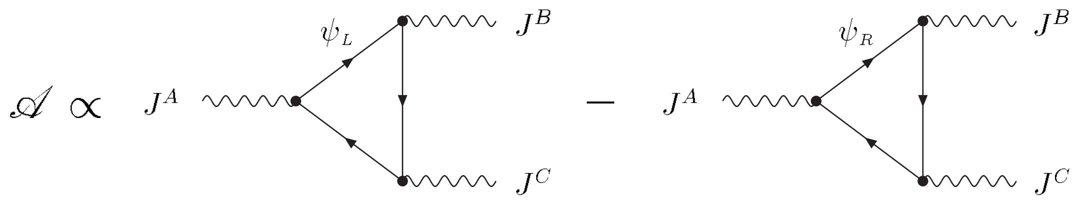


Figure 5.1: *The gauge anomaly is given by triangle diagrams with chiral fermions in the loop.*

Anomalies are caused by certain one-loop diagrams, the so-called triangle diagrams (see Fig 5.1). The left- and right-handed fermions contribute with different sign, so if they have the same quantum numbers, the anomaly vanishes. This is the case in QED and QCD, which thus are automatically anomaly free. In general, for currents J^A, J^B and J^C, the anomaly $\mathscr{A}$ is the difference of the traces of the generators T^A, T^B and T^C in the left- and right-handed sectors,

$$\mathscr{A} = \mathrm{tr}\left[\left\{T^A, T^B\right\} T^C\right]_L - \mathrm{tr}\left[\left\{T^A, T^B\right\} T^C\right]_R \stackrel{!}{=} 0\,. \qquad (5.10)$$

Here the trace is taken over all fermions. For the electroweak theory, in principle there are four combinations of currents, containing three, two, one or no $\mathsf{SU}(2)_W$ current. However, the trace of any odd number of σ^I matrices vanishes, so we only have to check the $\mathsf{SU}(2)_W^2\mathsf{U}(1)_Y$ and $\mathsf{U}(1)_Y^3$ anomalies.

The $\mathsf{SU}(2)_W$ generators are $\frac{1}{2}\sigma^I$, whose anticommutator is $\left\{\frac{1}{2}\sigma^I, \frac{1}{2}\sigma^J\right\} = \frac{1}{2}\delta^{IJ}$. Furthermore, only the left-handed fields contribute, since the right-handed ones are $\mathsf{SU}(2)_W$ singlets. Hence the $\mathsf{SU}(2)_W^2\mathsf{U}(1)_Y$ anomaly is

$$\mathscr{A} = \mathrm{tr}\left[\left\{\frac{1}{2}\sigma^I, \frac{1}{2}\sigma^J\right\} Y\right]_L = \frac{1}{2}\delta^{IJ}\ \mathrm{tr}\left[Y\right]_L = \frac{1}{2}\delta^{IJ}\left(\underbrace{3}_{N_c}\cdot\frac{1}{6} - \frac{1}{2}\right) = 0\,. \qquad (5.11)$$

We see that it only vanishes if quarks come in three colours!

The $\mathsf{U}(1)_Y^3$ anomaly also vanishes:

$$\begin{aligned}\mathscr{A} &= \mathrm{tr}\left[\{Y,Y\}\,Y\right]_L - \mathrm{tr}\left[\{Y,Y\}\,Y\right]_R = 2\Big(\mathrm{tr}\left[Y^3\right]_L - \mathrm{tr}\left[Y^3\right]_R\Big)\\ &= 2\left(3\cdot 2\left(\frac{1}{6}\right)^3 + 2\left(-\frac{1}{2}\right)^3 - 3\left(\frac{2}{3}\right)^3 - 3\left(-\frac{1}{3}\right)^3 - (-1)^3\right)\\ &= 0\;.\end{aligned} \tag{5.12}$$

This vanishing of the anomaly is again related to the number of colours. It does not vanish in either the left- or right-handed sector, nor in the quark and lepton sector individually. Hence, the vanishing of anomalies provides a deep connection between quarks and leptons in the standard model, which is a hint to grand unified theories where anomaly cancellation is often automatic.

Anomaly cancellation is not restricted to the electroweak gauge currents, but applies to the strong force and gravity as well: Mixed $\mathsf{SU}(3)_C$-$\mathsf{U}(1)_Y$ anomalies vanish by the same argument as above: Only the $\mathsf{SU}(3)_C^2\mathsf{U}(1)_Y$ triangle contributes, but it is $\mathrm{tr}\,[Y]_L - \mathrm{tr}\,[Y]_R = 0$. The same is true for the last possible anomaly, the gravitational one, where two non-Abelian gauge currents are replaced by the energy-momentum tensor $T_{\mu\nu}$.

Hence, the standard model is anomaly free, as it should be. For this, all particles of one generation with their strange hypercharges have to conspire to cancel the different anomalies. A "standard model" without quarks, for instance, would not be a consistent theory, nor a "standard model" with four colours of quarks. Note that a right-handed neutrino, suggested by neutrino masses, does not pose any problem, since it is a complete singlet, without any charge, and thus it does not contribute to any gauge anomaly.

5.2 Higgs Mechanism

The electroweak model discussed so far bears little resemblance to the physics of weak interactions. The gauge bosons W_μ^I and B_μ are massless, implying long-range forces, because a mass term $m^2 W_\mu W^\mu$ would violate gauge invariance. Furthermore, the fermions are massless as well, again because of gauge invariance: A mass term mixes left- and right-handed fermions,

$$m\overline{\psi}\psi = m\left(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\right)\,, \tag{5.13}$$

and since these have different gauge quantum numbers, such a term is not gauge invariant. The way out is the celebrated Higgs mechanism: Spontaneous symmetry breaking generates masses for the gauge bosons and fermions without destroying gauge invariance. A simpler version of this effect is what happens in superconductors: The condensate of Cooper pairs induces an effective mass for the photon, so that electromagnetic interactions become short-ranged, leading to the Meissner–Ochsenfeld effect where external magnetic fields are expelled from the superconductor, levitating it.

The key ingredient for the Higgs mechanism is a complex scalar field Φ, which is a doublet under $\mathsf{SU}(2)_W$ with hypercharge $-\frac{1}{2}$, which has four real degrees of freedom. The crucial feature of the Higgs field is its potential ,which is of the Mexican hat form:

$$\mathrm{L} = \left(D_\mu\Phi\right)^\dagger\left(D^\mu\Phi\right) - V\left(\Phi^\dagger\Phi\right)\,, \tag{5.14}$$

with

$$
\begin{aligned}
D_\mu \Phi &= \left(\partial_\mu + \imath g W_\mu - \tfrac{1}{2} g' B_\mu\right) \Phi \ , \\
V\left(\Phi^\dagger \Phi\right) &= -\mu^2\, \Phi^\dagger \Phi + \frac{1}{2}\lambda \left(\Phi^\dagger \Phi\right)^2 \ , \quad \mu^2 > 0 \ .
\end{aligned}
\tag{5.15}
$$

This potential has a minimum away from the origin, at $\Phi^\dagger\Phi = v^2 \equiv \mu^2/\lambda$. In the vacuum, the Higgs field settles in this minimum. At first sight, the minimisation of the potential only fixes the modulus $\Phi^\dagger\Phi$, i.e., one of the four degrees of freedom. The other three, however, can be eliminated by a gauge transformation, and we can choose the following form of Φ, which is often referred to as unitary gauge:

$$
\Phi = \begin{pmatrix} v + \frac{1}{\sqrt{2}} H(x) \\ 0 \end{pmatrix} \ , \quad H = H^* \tag{5.16}
$$

Here we have eliminated the upper component and the imaginary part of the lower one. We have also shifted the lower component to the vacuum value, so that the dynamical field $H(x)$ vanishes in the vacuum.

In unitary gauge, the Higgs Lagrangean (5.14) becomes

$$
\begin{aligned}
\mathrm{L} = {} & \frac{\lambda}{2} v^4 \\
& + \frac{1}{2}\partial_\mu H\, \partial^\mu H - \lambda v^2 H^2 + \frac{\lambda}{\sqrt{2}} v H^3 + \frac{\lambda}{8} H^4 \\
& + \frac{1}{4}\left(v + \frac{1}{\sqrt{2}} H\right)^2 \left(W^1_\mu, W^2_\mu, W^3_\mu, B_\mu\right) \begin{pmatrix} g^2 & 0 & & \\ 0 & g^2 & & \\ & & g^2 & g g' \\ & & g g' & g'^2 \end{pmatrix} \begin{pmatrix} W^{1\,\mu} \\ W^{2\,\mu} \\ W^{3\,\mu} \\ B^\mu \end{pmatrix} .
\end{aligned}
\tag{5.17}
$$

The first line could be interpreted as vacuum energy density, i.e., a cosmological constant. However, such an interpretation is on shaky grounds in quantum field theory, so we will ignore this term[2]. The second line describes a real scalar field H of mass $m_H^2 = 2\lambda v^2$ with cubic and quartic self-interactions. The most important line, however, is the last one: It contains mass terms for the vector bosons! A closer look at the mass matrix reveals that it only is of rank three, so it has one zero eigenvalue, and the three remaining ones are g^2, g^2, and $(g^2 + g'^2)$. In other words, it describes one massless particle, two of equal nonzero mass and one which is even heavier, i.e., we have identified the physical γ, $W^\pm$ and Z bosons.

The massless eigenstate of the mass matrix, i.e., the photon, is the linear combination $A_\mu = -\sin\theta_\mathrm{W} W^3_\mu + \cos\theta_\mathrm{W} B_\mu$, the orthogonal combination is the Z boson, $Z_\mu = \cos\theta_\mathrm{W} W^3_\mu + \sin\theta_\mathrm{W} B_\mu$. Here we have introduced the Weinberg angle θ_W, which is defined by

$$
\sin\theta_\mathrm{W} = \frac{g'}{\sqrt{g^2 + g'^2}} \ , \quad \cos\theta_\mathrm{W} = \frac{g}{\sqrt{g^2 + g'^2}} \ . \tag{5.18}
$$

[2] Generally, nothing prevents us from adding an arbitrary constant to the Lagrangean, obtaining any desired "vacuum energy". For example, the Higgs potential is often written as $\left(\Phi^\dagger\Phi - v^2\right)^2$, so that its expectation value vanishes in the vacuum. These potentials just differ by the a shift $\sim v^4$, and are indistinguishable within QFT.

To summarise, the theory contains the following mass eigenstates:

- Two charged vector bosons $W^{\pm}$ with mass $M_W^2 = \frac{1}{2} g^2 v^2$,
- two neutral vector bosons with masses $M_Z = \frac{1}{2}\left(g^2 + g'^2\right) v^2 = M_W^2 \cos^{-2}\theta_{\mathrm{W}}$ and $M_\gamma = 0$,
- and one neutral Higgs boson with mass $m_H^2 = 2\lambda v^2$.

The Higgs mechanism and the diagonalisation of the vector boson mass matrix allow us to rewrite the interaction Lagrangean (5.7), which was given in terms of the old fields W_μ^I and B_μ and their currents (5.8) and (5.9), in terms of the physical field. The associated currents are separated into a charged current (for $W_\mu^{\pm}$) and neutral currents (for A_μ and Z_μ):

$$\mathrm{L}_{\mathrm{CC}} = -\frac{g}{\sqrt{2}} \sum_{i=1,2,3} \left(\bar{u}_{Li}\gamma^\mu d_{Li} + \bar{\nu}_{Li}\gamma^\mu e_{Li}\right) W_\mu^+ + \text{h.c.}\,, \tag{5.19}$$

$$\begin{aligned} \mathrm{L}_{\mathrm{NC}} &= -g J_\mu^3 W^{3\,\mu} - g' J_{Y\,\mu} B^\mu \\ &= -e J_{\mathrm{em}\,\mu} A^\mu - \frac{e}{\sin 2\theta_{\mathrm{W}}} J_{Z\,\mu} Z^\mu\,, \end{aligned} \tag{5.20}$$

with the electromagnetic and Z currents

$$J_{\mathrm{em}\,\mu} = \sum_{\substack{i=u,d,c,\\ s,t,b,e,\mu,\tau}} \bar{\psi}_i \gamma_\mu Q_i \psi_i\,, \quad \text{with the electric charge} \quad Q_i = T_i^3 + Y_i\,, \tag{5.21}$$

$$J_{Z\,\mu} = \sum_{\substack{i=u,d,c,s,t,b\\ e,\mu,\tau,\nu_e,\nu_\mu,\nu_\tau}} \bar{\psi}_i \gamma_\mu \left(v_i - a_i\gamma^5\right) \psi_i\,. \tag{5.22}$$

Here the fermions ψ_i are the sum of left- and right-handed fields,

$$\psi_i = \psi_{Li} + \psi_{Ri}\,. \tag{5.23}$$

The coupling to the photon, the electric charge Q, is given by the sum of the third component of weak isospin T^3 ($\pm\frac{1}{2}$ for doublets, zero for singlets) and the hypercharge Y. This reproduces the known electric charges of quarks and leptons, which justifies the hypercharge assignments in (5.3). The coupling constant e is related to the original couplings and the weak mixing angle:

$$e = g \sin\theta_{\mathrm{W}} = g' \cos\theta_{\mathrm{W}}\,. \tag{5.24}$$

The photon couples only vector-like, i.e., it does not distinguish between different chiralities. The Z boson, on the other hand, couples to the vector- and axial-vector currents of different fermions ψ_i (i.e., their left-and right-handed components) with different strengths. They are given by the respective couplings v_i and a_i, which are universal for all families. In particular, the Z couples in the same way to all leptons, a fact known as lepton universality.

The Higgs mechanism described above is also called spontaneous symmetry breaking. This term, however, is somewhat misleading: Gauge symmetries are never broken, but only hidden. The Lagrangean (5.17) only has a manifest $\mathsf{U}(1)$ symmetry associated with the massless vector field, so it seems we have lost three gauge symmetries. This, however, is just a consequence of choosing the unitary gauge. The Higgs mechanism can also be described in a manifestly gauge invariant way, and all currents remain conserved.

The "spontaneous breaking of gauge invariance" reshuffles the degrees of freedom of the theory: Before symmetry breaking, we have the complex Higgs doublet (four real degrees of freedom) and four massless vector fields with two degrees of freedom each, so twelve in total. After symmetry breaking (and going to unitary gauge), three Higgs degrees of freedom are gone (one remaining), but they have resurfaced as extra components of three massive vector fields[3] (nine), and one vector field stays massless (another two). So there still are twelve degrees of freedom.

5.3 Fermion Masses and Mixings

The Higgs mechanism generates masses not only for the gauge bosons, but also for the fermions. As already emphasized, direct mass terms are not allowed in the standard model. There are, however, allowed Yukawa couplings of the Higgs doublet to two fermions. They come in three classes, couplings to quark doublets and either up- or down-type quark singlets, and to lepton doublet and charged lepton singlets. Each term is parametrised by a 3×3-matrix in generation space,

$$\mathrm{L}_{\mathrm{Y}} = (h_u)_{ij}\, \overline{q}_{L\,i} u_{R\,j} \Phi + (h_d)_{ij}\, \overline{q}_{L\,i} d_{R\,j} \widetilde{\Phi} + (h_e)_{ij}\, \overline{l}_{L\,i} e_{R\,j} \widetilde{\Phi} + \text{h.c.}\,, \tag{5.25}$$

where $\widetilde{\Phi}$ is given by $\widetilde{\Phi}_a = \epsilon_{ab} \Phi^*_b$.

These Yukawa couplings effectively turn into mass terms once the electroweak symmetry is spontaneously broken: A vacuum expectation value $\langle \Phi_X \rangle = v$ inserted in the Lagrangean (5.25) yields

$$\mathrm{L}_m = (m_u)_{ij}\, \overline{u}_{L\,i} u_{R\,j} + (m_d)_{ij}\, \overline{d}_{L\,i} d_{R\,j} + (m_e)_{ij}\, \overline{e}_{L\,i} e_{R\,j} + \text{h.c.}\,. \tag{5.26}$$

Here the mass matrices are $m_u = h_u v$ etc., and u_L, d_L and e_L denote the respective components of the quark and lepton doublets q_L and l_L.

The mass matrices thus obtained are in general not diagonal in the basis where the charged current is diagonal. They can be diagonalised by bi-unitary transformations,

$$V^{(u)\dagger} m_u \widetilde{V}^{(u)} = \operatorname{diag}(m_u, m_c, m_t)\ , \tag{5.27a}$$

$$V^{(d)\dagger} m_d \widetilde{V}^{(d)} = \operatorname{diag}(m_d, m_s, m_b)\ , \tag{5.27b}$$

$$V^{(e)\dagger} m_e \widetilde{V}^{(e)} = \operatorname{diag}(m_e, m_\mu, m_\tau)\ , \tag{5.27c}$$

with unitary matrices V,

$$V^{(u)\dagger} V^{(u)} = \mathbb{1}\,, \quad \text{etc.}$$

[3]Remember that a massless vector only has two (transverse) degrees of freedom, while a massive one has a third, longitudinal, mode.

This amounts to a change of basis from the weak eigenstates (indices $i, j, \ldots$) to mass eigenstates (with indices $\alpha, \beta, \ldots$):

$$u_{Li} = V^{(u)}_{i\alpha} u_{L\alpha}\,, \qquad d_{Li} = V^{(d)}_{i\alpha} d_{L,\alpha}\,, \qquad u_{Ri} = \widetilde{V}^{(u)}_{i\alpha} u_{R\alpha}\,, \qquad d_{Ri} = \widetilde{V}^{(d)}_{i\alpha} d_{R\alpha}\,. \tag{5.28}$$

The up- and down-type matrices $V^{(u)}$ and $V^{(d)}$ are not identical, which has an important consequence: The charged current couplings are now no longer diagonal, but rather

$$\mathrm{L}_{\mathrm{CC}} = -\frac{g}{\sqrt{2}} V_{\alpha\beta} \overline{u}_{L\alpha} \gamma^\mu d_{L\beta} W^+_\mu + \mathrm{h.c.}\,, \tag{5.29}$$

with the CKM matrix

$$V_{\alpha\beta} = {V^{(u)}}^\dagger_{\alpha i} {V^{(d)}}_{i\beta}\,, \tag{5.30}$$

which carries the information about flavour mixing in charged current interactions. Because of the unitarity of the transformations, there is no flavour mixing in the neutral current.

We saw that the Higgs mechanism generates fermion masses since direct mass terms are not allowed due to gauge invariance. There is one possible exception: a right-handed neutrino, which one may add to the standard model to have also neutrino masses. It is a singlet of the standard model gauge group and can therefore have a Majorana mass term which involves the charge conjugate fermion

$$\psi^C = C\overline{\psi}^T\,, \tag{5.31}$$

where $C = \imath\gamma^2\gamma^0$ is the charge conjugation matrix. As the name suggests, the charge conjugate spinor has charges opposite to the original one. It also has opposite chirality, $P_L \psi^C_R = \psi_R$. Thus we can produce a mass term $\overline{\psi^C}\psi$ (remember that a mass term always requires both chiralities), which only is gauge invariant for singlet fields.

So a right-handed neutrino ν_R can have the usual Higgs coupling and a Majorana mass term,

$$\mathrm{L}_{\nu,\mathrm{mass}} = h_{\nu\, ij} \overline{l}_{L\,i} \nu_{R\,j} \Phi + \frac{1}{2} M_{ij} \overline{\nu}_{R\,i} \nu_{R\,j} + \mathrm{h.c.}\,, \tag{5.32}$$

where i, j again are family indices.

The Higgs vacuum expectation value v turns the coupling matrix h_ν into the Dirac mass matrix $m_D = h_\nu v$. The eigenvalues of the Majorana mass matrix M can be much larger than the Dirac masses, and a diagonalisation of the (ν_L, ν_R) system leads to three light modes ν_i with the mass matrix

$$m_\nu = -m_D M^{-1} m_D^T\,. \tag{5.33}$$

Large Majorana masses naturally appear in grand unified theories. For $M \sim 10^{15}$ GeV, and $m_D \sim m_t \sim 100$ GeV for the largest Dirac mass, one finds $m_\nu \sim 10^{-2}$ eV, which is consistent with results from neutrino oscillation experiments. This "seesaw mechanism", which explains the smallness of neutrino masses masses as a consequence of large Majorana mass terms, successfully relates neutrino physics to grand unified theories.

5.4 Predictions

The electroweak theory contains four parameters, the two gauge couplings and the two parameters of the Higgs potential: g, g', μ^2 and λ. They can be traded for four other parameters, which are more easily measured: The fine-structure constant α, the Fermi constant G_F and the Z boson mass M_Z, which are known to great accuracy, and the Higgs mass m_H which is not yet known.

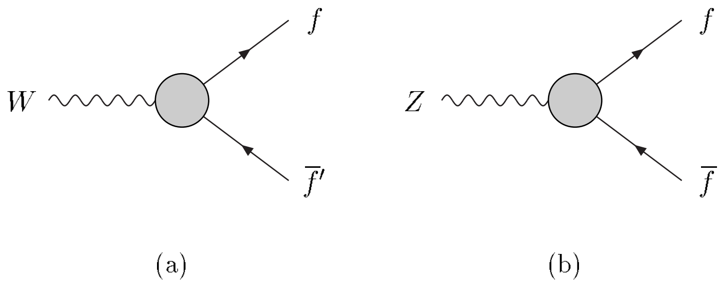


Figure 5.2: *Decays of the W and Z bosons into two fermions. In W decays, the fermion and antifermion can have different flavour. The grey blobs indicate higher order corrections which must be included to match the experimental precision.*

At LEP, W and Z bosons were produced in huge numbers. There are many observables related to their production and decay (Fig. 5.2). These include:

- The W mass M_W and the decay widths Γ_W and Γ_Z.
- Ratios of partial decay widths, for example, the ratio of the partial Z width into bottom quarks to that into all hadrons,

$$R_b = \frac{1}{\Gamma(Z \to \text{hadrons})}\Gamma\big(Z \to b\bar{b}\big) \ . \tag{5.34}$$

- Forward-backward asymmetries: In $e^+e^- \to Z/\gamma \to f\bar{f}$ reactions, the direction of the outgoing fermion is correlated with the incoming electron. This is quantified by the asymmetries $A^f_{\rm fb}$,

$$A^f_{\rm fb} = \frac{\sigma^f_{\rm f} - \sigma^f_{\rm b}}{\sigma^f_{\rm f} + \sigma^f_{\rm b}}\,, \quad \text{for } f = \mu, \tau, b, c\,, \tag{5.35}$$

where $\sigma^f_{\rm f}$ is the cross section for an outgoing fermion in the forward direction, i.e., $\theta \in [0, \pi/2]$ in Fig. 5.4, while $\sigma^f_{\rm b}$ is the cross section for backward scattering.

Also important are double, left-right and forward-backward asymmetries,

$$A^{\rm fb}_{\rm LR} = \frac{\sigma^f_{\rm Lf} - \sigma^f_{\rm Lb} - \sigma^f_{\rm Rf} + \sigma^f_{\rm Rb}}{\sigma^f_{\rm Lf} + \sigma^f_{\rm Lb} + \sigma^f_{\rm Rf} + \sigma^f_{\rm Rb}} \equiv \frac{3}{4}A_f \ . \tag{5.36}$$

The reason for these asymmetries is the presence of the axial couplings a_i in the Z boson current (5.22), which lead to different cross sections for the processes $Z \to f_L\overline{f}_R$ and $Z \to f_R\overline{f}_L$. Thus, one can deduce the a_i and v_i couplings for

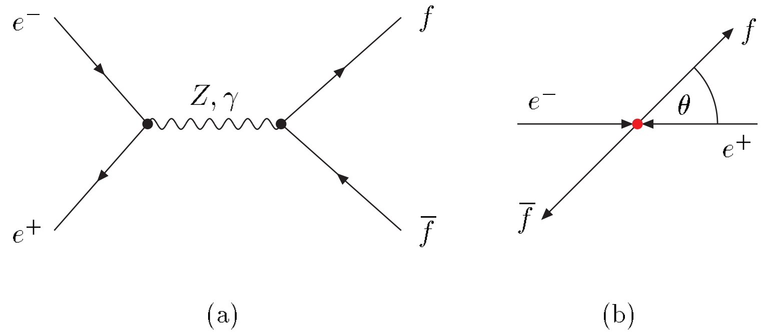


Figure 5.3: *The forward-backward asymmetry $A_{\rm fb}$: In the process $e^+e^- \to Z/\gamma \to f\overline{f}$, there is a correlation between the directions of the outgoing fermion and the incoming electron. This asymmetry has been measured for several types of final state fermions, mostly at LEP with center of mass energy $\sqrt{s} = M_Z$.*

fermions from the forward-backward asymmetries, and finally the weak mixing angle, on which the vector- and axial-vector couplings of the Z boson depend,

$$\sin^2\theta_{\rm eff}^{\rm lept} = \frac{1}{4}\left(1 - \frac{v_l}{a_l}\right) . \tag{5.37}$$

- Electroweak measurements by now are very precise, and require the inclusion of W boson loops in theoretical calculations, so that they test the non-Abelian nature of the electroweak theory. The theoretical predictions critically depend on the the electromagnetic coupling at the electroweak scale, $\alpha(m_Z)$, which differs from the low energy value $\alpha(0)$ in particular by hadronic corrections, $\Delta\alpha_{\rm had}(m_Z)$.

  An important observable is the ρ parameter, defined by

$$\rho = \frac{M_W^2}{M_Z^2\cos^2\theta_{\rm W}} . \tag{5.38}$$

  At tree level, $\rho = 1$. Loop corrections to the masses of the gauge bosons, and therefore to ρ, due to quark or Higgs boson loops as in Fig. 5.4, are an important prediction of the electroweak theory.

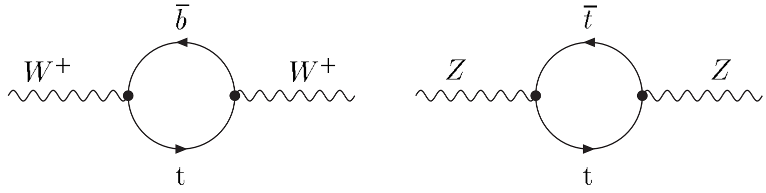
  The tree level value $\rho = 1$ is protected by an approximate $\mathsf{SU}(2)$ symmetry, called custodial symmetry, which is only broken by the $\mathsf{U}(1)_{\mathsf{Y}}$ gauge interaction and by Yukawa couplings. Thus the corrections depend on the fermion masses, and are dominated by the top quark, as in Fig 5.4(a). The leading correction is

$$\Delta\rho^{(t)} = \frac{3G_F m_t^2}{8\pi^2\sqrt{2}} \propto \frac{m_t^2}{M_W^2} . \tag{5.39}$$

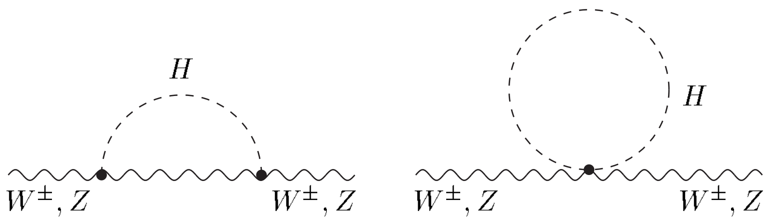
  This led to the correct prediction of the top mass from electroweak precision data before the top quark was discovered at the TeVatron.

  The correction due to the Higgs boson diagrams in Fig. 5.4(b) again depends on the Higgs mass, but this time the effect is only logarithmic:

$$\Delta\rho^{(H)} = -C\ln\frac{m_H^2}{M_W^2} . \tag{5.40}$$

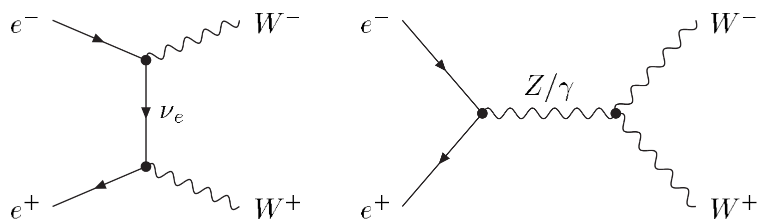


(a) Heavy quark corrections



(b) Higgs corrections

Figure 5.4: *Radiative corrections to the masses of the W and Z bosons, which depend on the masses of the particles in the loop. Diagrams with gauge boson self-interactions have been omitted.*



(a) (b)

Figure 5.5: *The process $e^+e^- \to W^+W^-$. The diagrams of panel (b) contain triple gauge boson vertices, γWW and ZWW.*

From this relation, one can obtain a prediction for the mass of the Higgs boson. Clearly, the accuracy of this prediction strongly depends on the experimental error on the top mass, which affects ρ quadratically.

However, the Higgs mass (weakly) influences many other quantities, and from precision measurements one can obtain a fit for the Higgs mass. This is shown in the famous blue-band plot, Fig. 6.3.

- A characteristic prediction of any non-Abelian gauge theory is the self-interaction of the gauge bosons. In the electroweak theory, this can be seen in the process $e^+e^- \to W^+W^-$.

  The tree-level diagrams are given in Fig. 5.5, and Fig. 5.6(a) shows the measured cross section from LEP, compared with theoretical predictions. Clearly, the full calculation including all diagrams agrees well with data, while the omission of the γWW and ZWW vertices leads to large discrepancies. For the process $e^+e^- \to ZZ$, on the other hand, there is no triple gauge boson (ZZZ or γZZ) vertex, so at tree level one only has the t-channel diagram which is similar to the diagram in Fig. 5.5(a), but with an electron instead of the neutrino. The agreement between theory and data is evident from Fig. 5.6(b).

5.4.1 Fermi Theory

The exchange of a W boson with momentum q in a Feynman diagram contributes a factor of $(M_W^2 - q^2)^{-2}$ to the amplitude. For low-energy processes like muon decay (see Fig. 5.7), the momentum transfer is much smaller than the mass of the W boson. Hence, to good approximation one can ignore q^2 and replace the propagator by M_W^{-2}. This amounts to introducing an effective four-fermion vertex (see Fig. 5.8),

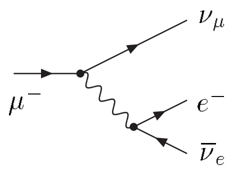


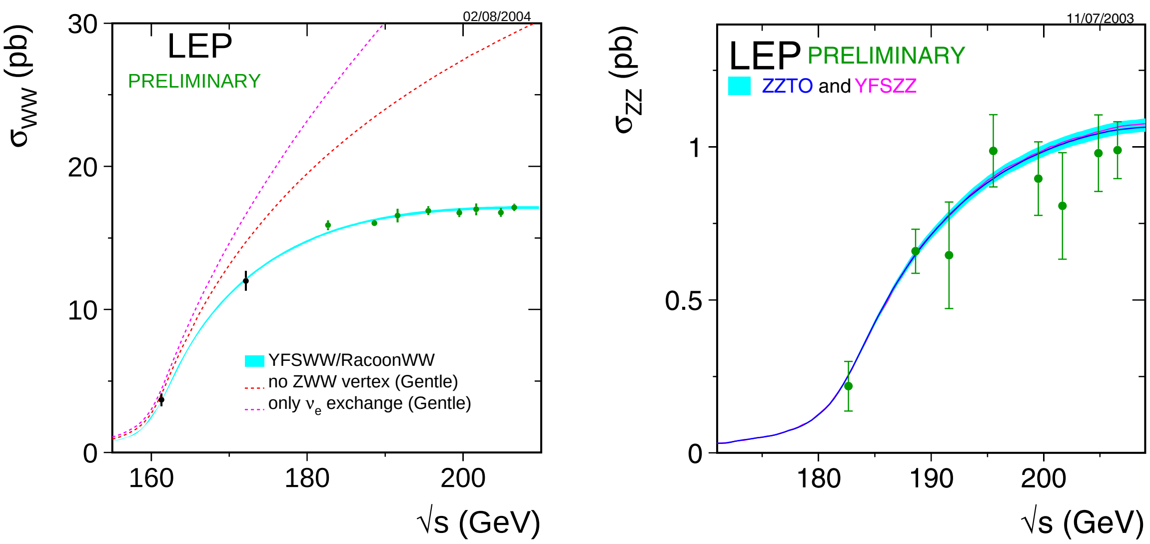
Figure 5.7: *μ decay*

$$\mathrm{L}_{\mathrm{CC}}^{\mathrm{eff}} = -\frac{G_{\mathrm{F}}}{\sqrt{2}} J_{\mathrm{CC}}^{\mu} J_{\mathrm{CC}\mu}^{\dagger} \,, \tag{5.41}$$

where G_{F} is Fermi's constant,

$$G_{\mathrm{F}} = \frac{g^2}{4\sqrt{2}\, M_W^2} = \frac{1}{2\sqrt{2}\, v^2} \,, \tag{5.42}$$

which is inversely proportional to the Higgs vacuum expectation value v^2. A four-fermion theory for the weak interactions was first introduced by Fermi in 1934. Since it is not



(a) W boson pair production cross section at LEP2. Predictions which ignore the WWW vertex deviate substantially.

(b) For Z pair production, there is no triple Z vertex, which agrees well with the experimental result.

Figure 5.6: *Gauge boson pair production cross sections at LEP2 energies. From [11].*

renormalisable, it cannot be considered a fundamental theory. However, one can use it as an effective theory at energies small compared to the W mass. This is sufficient for many applications in flavour physics, where the energy scale is set by the masses of leptons, kaons and B mesons.

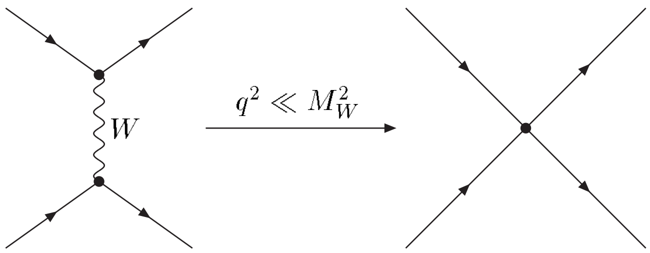


Figure 5.8: *W boson exchange can be described in terms of the Fermi theory, an effective theory for momentum transfers small compared to the W mass. The W propagator is replaced by a four-fermion vertex $\propto G_F$.*

5.5 Summary

The electroweak theory is a chiral gauge theory with gauge group $\mathsf{SU}(2)_W \times \mathsf{U}(1)_Y$. This symmetry is spontaneously broken down to $\mathsf{U}(1)_{\mathrm{em}}$ by the Higgs mechanism which generates the gauge boson and Higgs masses, and also all fermion masses, since direct mass terms are forbidden by gauge invariance.

The electroweak theory is extremely well tested experimentally, to the level of 0.1%, which probes loop effects of the non-Abelian gauge theory. The results of a global electroweak fit are shown in Fig. 5.9. There is one deviation of almost 3σ, all other quantities agree within less than 2σ.

This impressive agreement is only possible due to two properties of the electroweak interactions: They can be tested in lepton-lepton collisions, which allow for very precise measurements, and they can be reliably calculated in perturbation theory. QCD, on the other hand, requires hadronic processes which are experimentally known with less accuracy and also theoretically subject to larger uncertainties.

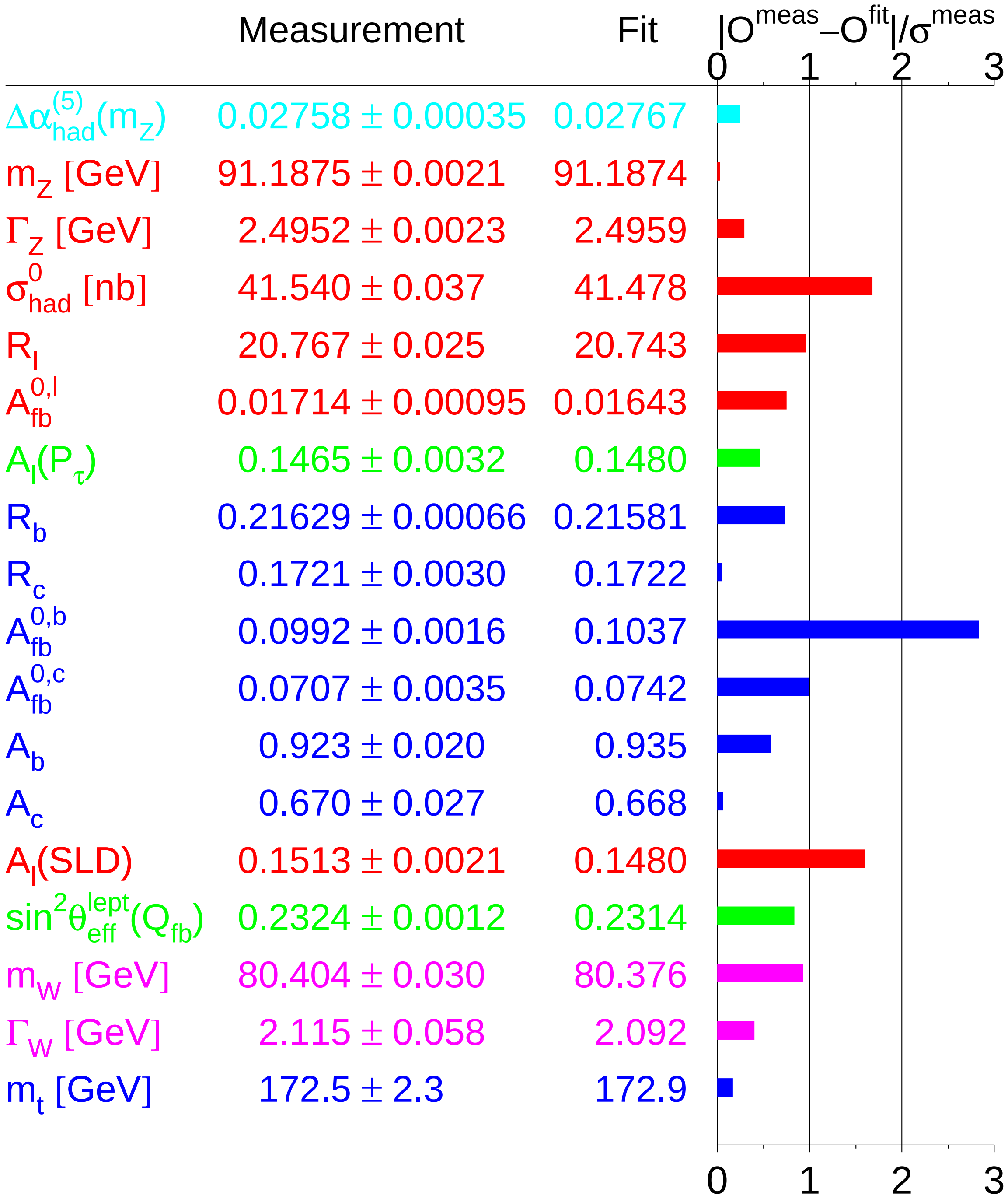


Figure 5.9: *Results of a global fit to electroweak precision data. The right column shows the deviation of the fit from measured values in units of the standard deviation. From [11].*

Chapter 6

The Higgs Profile

The only missing building block of the standard model is the Higgs boson. Spontaneously broken electroweak symmetry, however, is a cornerstone of the standard model, and so the discovery of the Higgs boson and the detailed study of its interactions is a topic of prime importance for the LHC and also the ILC.

The investigation of the Higgs sector can be expected to to give important insight also on physics beyond the standard model. Since the Higgs is a scalar particle, its mass is subject to quadratically divergent quantum corrections, and an enormous "fine-tuning" of the tree-level mass term is needed to keep the Higgs light (this is usually referred to as the "naturalness problem" of the Higgs sector). Such considerations have motivated various extensions of the standard model:

- Supersymmetry retains an elementary scalar Higgs (and actually adds four more), while radiative corrections with opposite signs from bosons and fermions cancel.
- Technicolour theories model the Higgs as a composite particle of size $1/\Lambda_{\mathrm{TC}}$, where $\Lambda_{\mathrm{TC}} \sim 1$ TeV is the confinement scale of a new non-Abelian gauge interaction. These theories generically have problems with electroweak precision tests and the generation of fermion masses.
- A related idea regards the Higgs as a pseudo-Goldstone boson of some approximate global symmetry spontaneously broken at an energy scale above the electroweak scale. The Higgs mass is then related to the explicit breaking of this symmetry.
- In theories with large extra dimensions new degrees of freedom occur, and the Higgs field can be identified, for instance, as the fifth component of a five-dimensional vector field.

All such ideas can be tested at the LHC and the ILC, since the unitarity of WW scattering implies that the standard model Higgs and/or other effects related to electroweak symmetry breaking become manifest at energies below ~ 1 TeV.

6.1 Higgs Couplings and Decay

Suppose a resonance is found at the LHC with a mass above 114 GeV and zero charge. How can one establish that it indeed is the Higgs?

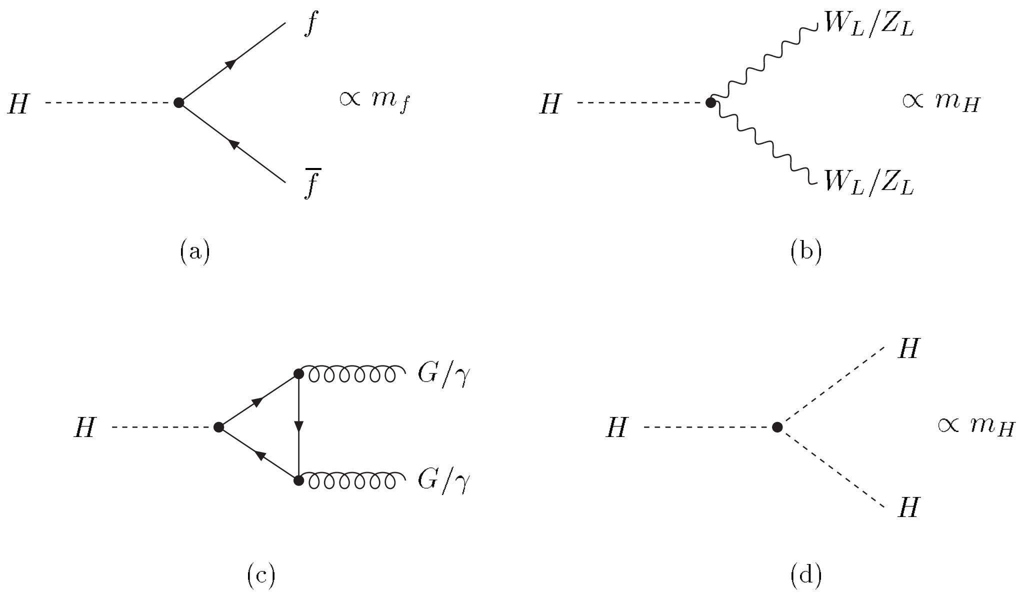


Figure 6.1: *Higgs boson decays. Tree-level couplings are proportional to masses, but there also are loop-induced decays into massless particles. The cubic Higgs self-coupling can be probed at the ILC and possibly at the LHC.*

The Higgs boson can be distinguished from other scalar particles as they occur, for instance, in supersymmetric theories, by its special couplings to standard model particles. All couplings are proportional to the mass of the particle, since it is generated by the Higgs mechanism. Hence, the Higgs decays dominantly into the heaviest particles kinematically allowed, which are $t\bar{t}$ or, for a light Higgs, $b\bar{b}$ and $\tau\bar{\tau}$ pairs. It also has a strong coupling $\propto m_H$ to the longitudinal component of W and Z bosons. The tree-level diagrams are given in Figs. 6.1(a) and 6.1(b). In addition, there are important loop-induced couplings to massless gluons and photons (see Fig. (6.1(c)).

The tree level decay widths in the approximation $m_H \gg m_f, M_W$ are given by

$$\Gamma(H \to f\bar{f}) = \frac{G_{\mathrm{F}} m_H m_f^2}{4\pi\sqrt{2}} N_c \, , \tag{6.1a}$$

$$\Gamma(H \to Z_L Z_L) = \frac{1}{2}\Gamma(H \to W_L W_L) = \frac{G_{\mathrm{F}} m_H^3}{32\pi\sqrt{2}} \, . \tag{6.1b}$$

The branching fractions of the Higgs into different decay products strongly depend on the Higgs mass, as shown in Fig. 6.2. For a heavy Higgs, with $m_H > 2M_W$, the decay into a pair of W bosons dominates. At the threshold the width increases by two orders of magnitude, and it almost equals the Higgs mass at $m_H \sim 1$ TeV where the Higgs dynamics becomes nonperturbative. For a light Higgs with a mass just above the present experimental limit, $m_H > 114$ GeV, the decay into two photons might be the best possible detection channel given the large QCD background for the decay into two gluons at the LHC. It is clearly an experimental challenge to establish the mass dependence of the Higgs couplings, so the true discovery of the Higgs is likely to take several years of LHC data!

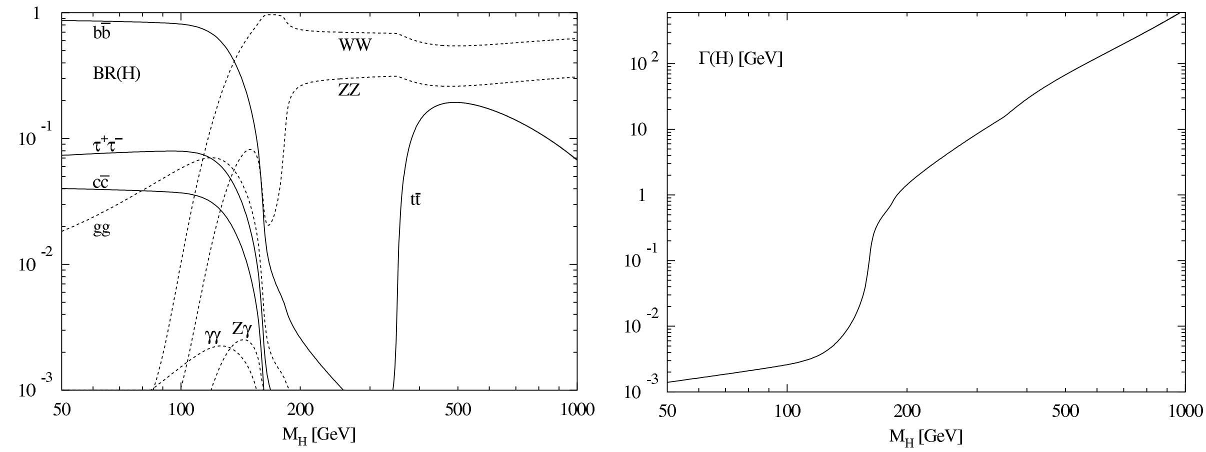


Figure 6.2: *Left: Higgs branching ratios as function of the Higgs mass. Right: Higgs decay width as function of the Higgs mass. It increases by two orders of magnitude at the WW threshold. From [12].*

6.2 Higgs Mass Bounds

We now turn to the issue of the Higgs mass. Within the standard model, $m_H^2 = 2\lambda v^2$ is a free parameter which cannot be predicted. There are, however, theoretical consistency arguments which yield stringent upper and lower bounds on the Higgs mass.

Before we present these argument, we first recall the experimental bounds:

- The Higgs has not been seen at LEP. This gives a lower bound on the mass, $m_H > 114$ GeV.

- The Higgs contributes to radiative corrections, in particular for the ρ parameter. Hence, precision measurements yield indirect constraints on the Higgs mass. The result of a global fit is shown in the blue-band plot, Fig. 6.3. The current 95% confidence level upper bound is $m_H < 185$ GeV, an impressive result! One should keep in mind, however, that the loop corrections used to determine the Higgs mass strongly depend on the top mass as well. A shift of a few GeV in the top mass, well within the current uncertainties, can shift the Higgs mass best fit by several tens of GeV, as can be seen by comparing he plots in Fig. 6.3.

Theoretical bounds on the Higgs mass arise, even in the standard model, from two consistency requirements: (Non-)Triviality and vacuum stability. In the minimal supersymmetric standard model (MSSM), on the other hand, the Higgs self-coupling is given by the gauge couplings, which implies the upper bound $m_H \lesssim 135$ GeV.

The mass bounds in the standard model arise from the scale dependence of couplings, as explained in Chapter 4. Most relevant are the quartic Higgs self-coupling λ and the top quark Yukawa coupling h_t which gives the top mass via $m_t = h_t v$. Other Yukawa couplings are much smaller and can be ignored. The renormalisation group equations

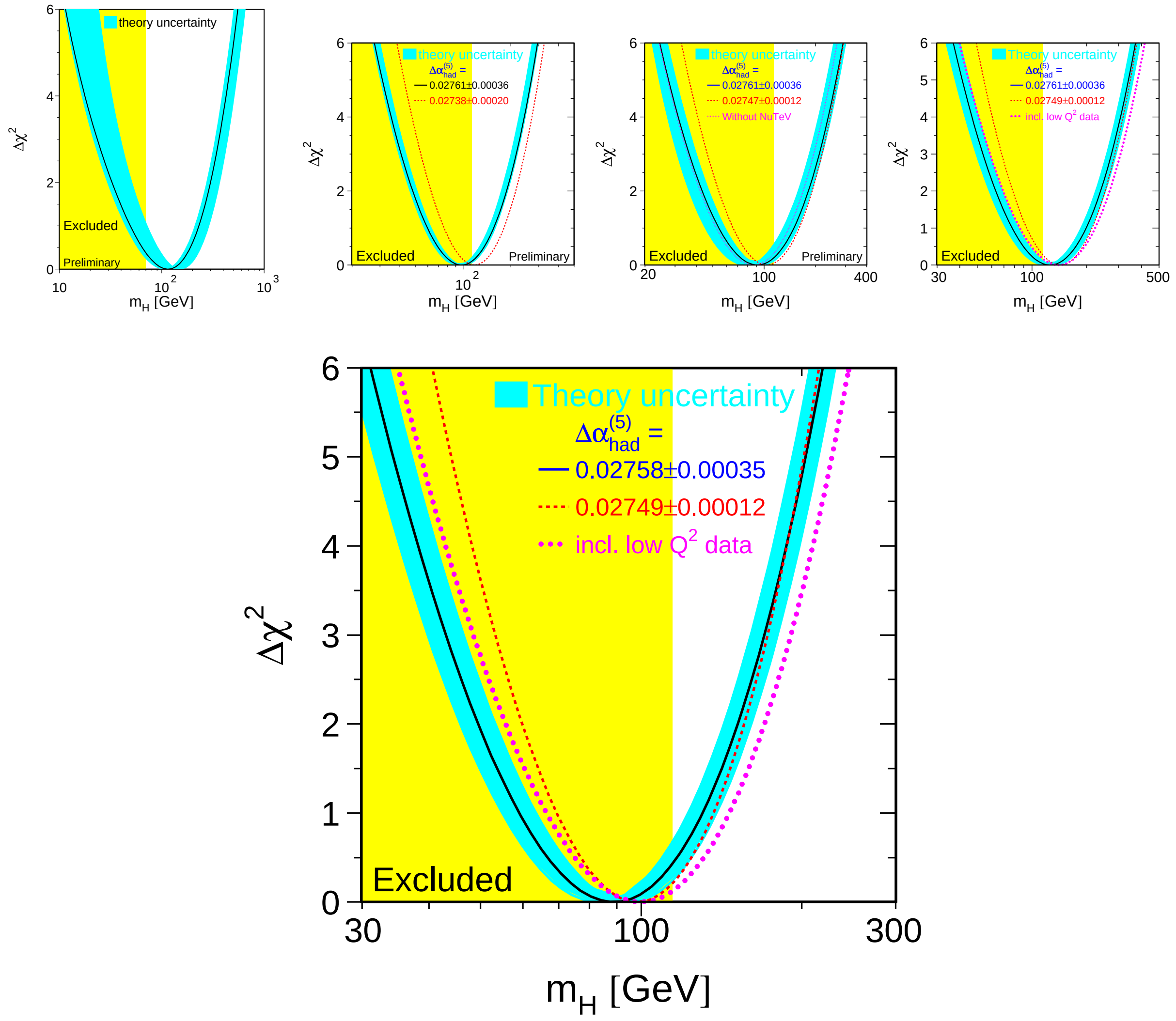


Figure 6.3: *The blue-band plot showing the constraints on the Higgs mass from precision measurements. The small plots show the same plot from winter conferences of different years: 1997, 2001, 2003 and 2005 (left to right). The big plot dates from winter 2006. The best fit and the width of the parabola vary, most notable due to shifts in the top mass and its uncertainty. From [11].*

for the couplings $\lambda(\mu)$ and $h_t(\mu)$ are

$$\mu \frac{\partial}{\partial \mu} \lambda(\mu) = \beta_\lambda(\lambda, h_t) = \frac{1}{(4\pi)^2} \left(12\lambda^2 - 12h_t^4 + \dots\right) \;, \tag{6.2a}$$

$$\mu \frac{\partial}{\partial \mu} h_t(\mu) = \beta_\lambda(\lambda, h_t) = \frac{h_t}{(4\pi)^2} \left(\frac{9}{2} h_t^2 - 8g_s^2 + \dots\right) \;. \tag{6.2b}$$

These equations imply that h_t decreases with increasing μ whereas the behaviour of $\lambda(\mu)$ depends on the initial condition $\lambda(v)$, i.e., on the Higgs mass.

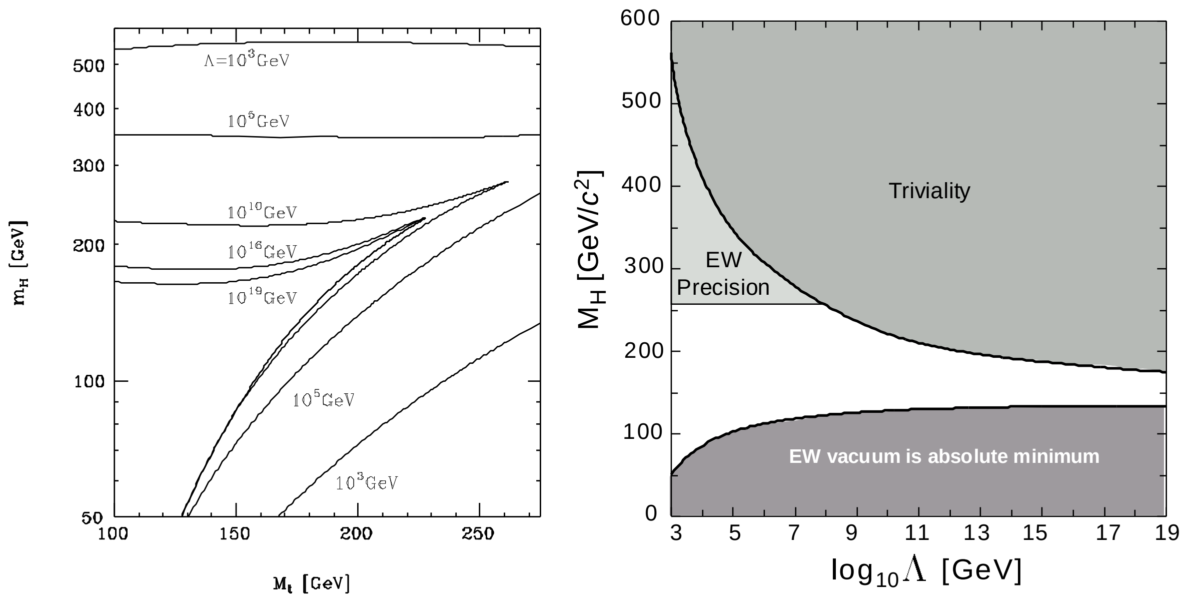
For the standard model to be a consistent theory from the electroweak scale v up to some high-energy cutoff Λ, one needs to satisfy the following two conditions in the range $v < \mu < \Lambda$:

- The triviality bound: $\lambda(\mu) < \infty$. If λ would hit the Landau pole at some scale

$\mu_{\rm L} < \Lambda$, a finite value $\lambda(\mu_{\rm L})$ would require $\lambda(v) = 0$, i.e., the theory would be "trivial".

- The vacuum stability bound: $\lambda(\mu) > 0$. If λ would become negative, the Higgs potential would not be bounded from below anymore, and the electroweak vacuum would no longer be the ground state of the theory.

These two requirements define allowed regions in the m_H-m_t–plane as function of the cutoff Λ (see Fig. (6.4a)). For a given top mass, this translates into an upper and lower bound on the Higgs mass. For increasing Λ, the allowed region shrinks, and for the known top quark mass and $\Lambda \sim \Lambda_{\rm GUT} \sim 10^{16}$ GeV, the Higgs mass is constrained to lie in a narrow region, 130 GeV $< m_H <$ 180 GeV (see Fig. (6.4b)).



(a) (b)

Figure 6.4: *Bounds on the Higgs and top mass from triviality and vacuum stability. Panel (a) shows the combined bounds for different values of Λ (from [13]). Panel (b) gives the bounds on the Higgs mass for the known top mass (from [14]).*

The impressively narrow band of allowed Higgs masses, which one obtains from the triviality and vacuum stability bounds, assumes that the standard model is valid up to $\Lambda_{\rm GUT}$, the scale of grand unification. This might seem a bold extrapolation, given the fact that our present experimental knowledge ends at the electroweak scale, $\sim 10^2$ GeV. There are, however, two indications for such a "desert" between the electroweak scale and the GUT scale: First, the gauge couplings empirically unify at the GUT scale, especially in the supersymmetric standard model, if there are no new particles between $\sim 10^2$ GeV and $\Lambda_{\rm GUT}$; second, via the seesaw mechanism, the evidence for small neutrino masses is also consistent with an extrapolation to $\Lambda_{\rm GUT}$ without new physics at intermediate scales.

Chapter 7

History and Outlook

Finally, instead of a summary, we shall briefly recall the history of "The making of the Standard Model" following a review by S. Weinberg [1]. It is very instructive to look at this process as the interplay of some "good ideas" and some "misunderstandings" which often prevented progress for many years.

1. A "good idea" was the quark model, proposed in 1964 independently by Gell-Mann and Zweig. The hypothesis that hadrons are made out of three quarks and antiquarks allowed one to understand their quantum numbers and mass spectrum in terms of an approximate $\mathsf{SU}(3)$ flavour symmetry, the "eightfold way". Furthermore, the deep-inelastic scattering experiments at SLAC in 1968 could be interpreted as elastic scattering of electrons off point-like partons inside the proton, and it was natural to identify these partons with quarks.

   But were quarks real or just some mathematical entities? Many physicists did not believe in quarks since no particles with third integer charges were found despite many experimental searches.

2. Another "good idea" was the invention of non-Abelian gauge theories by Yang and Mills in 1954. The local symmetry was the isospin group $\mathsf{SU}(2)$, and one hoped to obtain in this way a theory of strong interactions with the ρ-mesons as gauge bosons. Only several years later, after the $V-A$-structure of the weak interactions had been identified, Bludman, Glashow, Salam and Ward and others developed theories of the weak interactions with intermediate vector bosons.

   But all physical applications of non-Abelian gauge theories seemed to require massive vector bosons because no massless ones had been found, neither in strong nor weak interactions. Such mass terms had to be inserted by hand, breaking explicitly the local gauge symmetry and thereby destroying the rationale for introducing non-Abelian local symmetries in the first place. Furthermore, it was realized that non-Abelian gauge theories with mass terms would be non-renormalisable, plagued by the same divergencies as the four-fermion theory of weak interactions.

3. A further "good idea" was spontaneous symmetry breaking: There can be symmetries of the Lagrangean that are not symmetries of the vacuum. According to the Goldstone theorem there must be a massless spinless particle for every spontaneously broken global symmetry. On the other hand, there is no experimental

evidence for any massless scalar with strong or weak interactions. In 1964 Higgs and Englert and Brout found a way to circumvent Goldstone's theorem: The theorem does not apply if the symmetry is a gauge symmetry as in electrodynamics or the non-Abelian Yang–Mills theory. Then the Goldstone boson becomes the helicity-zero part of the gauge boson, which thereby acquires a mass.

But again, these new developments were applied to broken symmetries in strong interactions, and in 1967 Weinberg still considered the chiral $\mathsf{SU}(2)_{\mathrm{L}} \times \mathsf{SU}(2)_{\mathrm{R}}$ symmetry of strong interactions to be a gauge theory with the ρ and $a1$ mesons as gauge bosons. In the same year, however, he then applied the idea of spontaneous symmetry breaking to the weak interactions of the leptons of the first family, (ν_L, e_L) and e_R (he did not believe in quarks!). This led to the gauge group $\mathsf{SU}(2) \times \mathsf{U}(1)$, massive W and Z bosons, a massless photon and the Higgs boson!

The next steps on the way to the Standard Model are well known: The proof by 't Hooft and Veltman that non-Abelian gauge theories are renormalisable and the discovery of asymptotic freedom by Gross and Wilczek and Politzer. Finally, it was realised that the infrared properties of non-Abelian gauge theories lead to the confinement of quarks and massless gluons, and the generation of hadron masses. So, by 1973 "The making of the Standard Model" was completed!

Since 1973 many important experiments have confirmed that the Standard Model is indeed the correct theory of elementary particles:

- 1973: discovery of neutral currents
- 1979: discovery of the gluon
- 1983: discovery of the W and Z bosons
- 1975 - 2000: discovery of the third family, τ, b, t and ν_τ
- During the past decade impressive quantitative tests have been performed of the electroweak theory at LEP, SLC and Tevatron, and of QCD at LEP, HERA and Tevatron.

Today, there are also a number of "good ideas" on the market, which lead beyond the Standard Model. These include grand unification, dynamical symmetry breaking, supersymmetry and string theory. Very likely, there are again some "misunderstandings" among theorists, but we can soon hope for clarifications from the results of the LHC.

We would like to thank the participants of the school for stimulating questions and the organisers for arranging an enjoyable and fruitful meeting in Kitzbühel.

Appendix A

Vectors, Spinors and γ-Algebra

A.1 Metric Conventions

Our spacetime metric is mostly minus,

$$g_{\mu\nu} = \mathrm{diag}(+,-,-,-) \,, \tag{A.1}$$

so timelike vectors v^μ have positive norm $v_\mu v^\mu > 0$. The coordinate four-vector is $x^\mu = (t, \vec{x})$ (with upper index), and derivatives with respect to x^μ are denoted by

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) . \tag{A.2}$$

Greek indices $\mu, \nu, \rho, \ldots$ run from 0 to 3, purely spatial vectors are indicated by an vector arrow.

A.2 γ-Matrices

In four dimensions, the γ-matrices are defined by their anticommutators,

$$\{\gamma_\mu, \gamma_\nu\} = 2 g_{\mu\nu} \mathbb{1} \,, \quad \mu = 0, \ldots, 3 \tag{A.3}$$

In addition, $\gamma_0 = \gamma_0^\dagger$ is Hermitean while the $\gamma_i = -\gamma_i^\dagger$ are anti-Hermitean, and all γ^μ are traceless. The matrix form of the γ-matrices is not fixed by the algebra, and there are several common representations, like the Dirac and Weyl representations, Eqs. (4.5) and (2.47), respectively. However, the following identities hold regardless of the representation.

The product of all γ-matrices is

$$\gamma^5 = \imath \gamma^0 \gamma^1 \gamma^2 \gamma^3 \tag{A.4}$$

which is Hermitean, squares to one and anticommutes with all γ-matrices,

$$\left\{\gamma^5, \gamma^\mu\right\} = 0 \,. \tag{A.5}$$

The chiral projectors $P_{L/R}$ are defined as

$$P_{L/R} = \frac{1}{2}\left(1 \pm \gamma^5\right) , \quad P_L P_R = P_R P_L = 0 , \quad P_{L/R}^2 = P_{L/R} . \tag{A.6}$$

To evaluate Feynman diagrams like for the anomalous magnetic moment, one often needs to contract several γ-matrices such as

$$\gamma^\mu \gamma_\mu = 4 \tag{A.7a}$$
$$\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu \tag{A.7b}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4 g^{\nu\rho} \tag{A.7c}$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu \quad \text{etc.} \tag{A.7d}$$

For a vector v^μ we sometimes use the slash $\not{v} = \gamma^\mu v_\mu$

A.3 Dirac, Weyl and Majorana Spinors

The solutions of the Dirac equation in momentum space are fixed by the equations

$$(\not{p} - m)\, u^{(i)}(p) = 0 \qquad\qquad (\not{p} + m)\, v^{(i)}(p) = 0 . \tag{A.8}$$

Here it is convenient to choose the Weyl representation (2.47) of the Dirac matrices,

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} , \quad \Rightarrow \quad \gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & \\ 0 & \mathbb{1}_2 \end{pmatrix} .$$

In this basis, the spinors $u(p)$ and $v(p)$ are given by

$$u^s(p) = \begin{pmatrix} \sqrt{E\mathbb{1}_2 + \vec{p}\cdot\vec{\sigma}}\; \xi^s \\ \sqrt{E\mathbb{1}_2 - \vec{p}\cdot\vec{\sigma}}\; \xi^s \end{pmatrix} , \quad v^s(p) = \begin{pmatrix} \sqrt{E\mathbb{1}_2 + \vec{p}\cdot\vec{\sigma}}\; \eta^s \\ -\sqrt{E\mathbb{1}_2 - \vec{p}\cdot\vec{\sigma}}\; \eta^s \end{pmatrix} . \tag{A.9}$$

Here ξ and η are two-component unit spinors. Choosing the momentum along the z-axis and e.g. $\xi = (1,0)^T$, the positive-energy spinor becomes

$$u^+ = \begin{pmatrix} \sqrt{E + p_z} \\ 0 \\ \sqrt{E - p_z} \\ 0 \end{pmatrix} , \tag{A.10}$$

which has spin $+\frac{1}{2}$ along the z-axis. For $\xi = (0,1)^T$, the spin is reversed, and similar for η and the negative energy spinors.

The spinors considered so far are called Dirac spinors: They are restricted only by the Dirac equation and have four degrees of freedom (particle and antiparticle, spin up and spin down). There are two restricted classes of spinors, Weyl and Majorana spinors, which only have two degrees of freedom.

Weyl or chiral spinors are subject to the constraint

$$P_L \psi_L = \psi_L \quad \text{or} \quad P_R \psi_R = \psi_R \tag{A.11}$$

and correspond to purely left- or right-handed fermions. In the language of u's and v's, chiral spinors correspond to sums $u \pm \gamma^5 v$.Chiral spinors can have a kinetic term, but no usual mass term, since

$$\overline{(\psi_L)} = \overline{P_L \psi_L} = (P_L \psi_L)^\dagger \gamma^0 = \psi_L^\dagger P_L \gamma^0 = \overline{\psi}_L P_R \tag{A.12}$$

and hence

$$\overline{\psi_L}\, \psi_L = \overline{\psi}_L \underbrace{P_R P_L}_{=0} \psi_L = 0\,. \tag{A.13}$$

However, there is the possibility of a Majorana mass term via the charge conjugate spinor ψ^C:

$$\psi^C = C \overline{\psi}^T \quad \text{with the charge conjugation matrix} \quad C = \imath \gamma^0 \gamma^2\,. \tag{A.14}$$

ψ^C is of opposite chirality to ψ, so it can be used to build a bilinear $\overline{\psi}^C \psi$ for a mass term. However, this term violates all symmetries under which ψ is charged, so it is only acceptable for complete singlets, like right-handed neutrinos.

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