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Simple Approach to Renormalize the Cabibbo-Kobayashi-Maskawa Matrix

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Abstract

We present an on-shell scheme to renormalize the Cabibbo-Kobayashi-Maskawa (CKM) matrix. It is based on a novel procedure to separate the external-leg mixing corrections into gauge-independent self-mass and gauge-dependent wave-function renormalization contributions, and to implement the on-shell renormalization of the former with non-diagonal mass counterterm matrices. Diagonalization of the complete mass matrix leads to an explicit CKM counterterm matrix, which automatically satisfies all the following important properties: it is gauge independent, preserves unitarity, and leads to renormalized amplitudes that are non-singular in the limit in which any two fermions become mass degenerate.

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The Cabibbo-Kobayashi-Maskawa (CKM) [1] flavor mixing matrix, which rules the charged-current interactions of the quark mass eigenstates and describes how the heavier ones decay to the lighter ones, is one of the fundamental cornerstones of the Standard Model of elementary particle physics and, in particular, it is the key to our understanding why the weak interactions are not invariant under simultaneous charge-conjugation and parity transformations. In fact, the detailed determination of this matrix is one of the major aims of recent experiments carried out at the B factories [2], as well as the objective of a wide range of theoretical studies [2, 3]. An important theoretical problem associated with the CKM matrix is its renormalization. An early discussion, in the two-generation framework, was given in Ref. [4], focusing mostly on the cancellation of ultraviolet divergences. More recently, there have been a number of interesting papers that address the renormalization of both the divergent and finite contributions at various levels of generality and complexity [5].

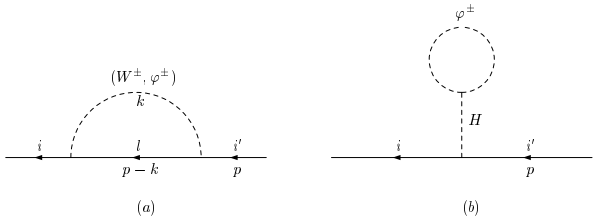

Figure 1: Fermion mixing self-energy diagrams. H and $\phi^\pm$ denote Higgs and charged Goldstone bosons, respectively. Diagram (b) is included to cancel the gauge dependence in the diagonal contribution of diagrams (a).

In this Letter we propose an explicit on-shell framework to renormalize the CKM matrix at the one-loop level, based on a novel procedure to separate the external-leg mixing corrections into gauge-independent "self-mass" (sm) and gauge-dependent "wave-function renormalization" (wfr) contributions, and to implement the on-shell renormalization of the former with non-diagonal mass counterterm matrices. This procedure may be regarded as a simple generalization of Feynman's approach in Quantum Electrodynamics (QED) [6]. We recall that, in QED, the self-energy contribution to an outgoing fermion is given by

$$\Delta\mathcal{M}^{\rm leg} = \overline{u}(p)\Sigma(\not{p})\frac{1}{\not{p}-m}, \tag{1}$$

$$\Sigma(\not{p}) = A + B(\not{p}-m) + \Sigma_{\rm fin}(\not{p}), \tag{2}$$

where $\Sigma(\not{p})$ is the self-energy, A and B are divergent constants, and $\Sigma_{\rm fin}(\not{p})$ is a finite part which is proportional to $(\not{p}-m)^2$ in the vicinity of $\not{p}=m$ and, therefore, vanishes when inserted in Eq. (1). The contribution of A to Eq. (1) exhibits a pole at $\not{p}=m$ and is gauge independent, while that of B is regular at this point, but gauge dependent. They are referred to as sm and wfr contributions, respectively. A is canceled by the

mass counterterm. On the other hand, since the factor $(\not{p} - m)$ cancels the propagator's singularity, in Feynman's approach B is combined with the proper vertex diagrams leading to a gauge-independent result.

In the case of the CKM matrix, one encounters not only diagonal terms as in Eq. (1), but also off-diagonal external-leg contributions generated by the Feynman diagrams of Fig. 1(a). As a consequence, the self-energy corrections to an external leg are of the form

$$\Delta\mathcal{M}^{\rm leg}_{ii'} = \overline{u}_i(p)\Sigma_{ii'}(\not{p})\frac{1}{\not{p} - m_{i'}}, \tag{3}$$

where i denotes the external quark of momentum p and mass m_i, and i' the virtual quark of mass $m_{i'}$.

We evaluate the contributions of Fig. 1 in R_ξ gauge, treating the i and i' quarks on an equal footing. (A detailed account of our analytical work will be presented in a later, longer manuscript [7].) For example, we write

$$\begin{aligned} 2\not{p}a_- &= \not{p}a_- + a_+\not{p} \\ &= (\not{p} - m_i)a_- + a_+(\not{p} - m_{i'}) + m_i a_- + m_{i'}a_+, \end{aligned} \tag{4}$$

where $a_\pm = (1 \pm \gamma_5)/2$ are the chiral projectors. Using this approach, we find that the contributions of Fig. 1 can be classified in four classes: (i) terms with a left factor $(\not{p} - m_i)$; (ii) terms with a right factor $(\not{p} - m_{i'})$; (iii) terms with a left factor $(\not{p} - m_i)$ and a right factor $(\not{p} - m_{i'})$; and (iv) constant terms not involving $\not{p}$. When inserted in Eq. (3), the terms of class (iii) obviously vanish, in analogy with $\Sigma_{\rm fin}(\not{p})$ in Eqs. (1) and (2). The terms of classes (i) and (ii) contain gauge-dependent parts but, when inserted in Eq. (3), they combine to cancel the propagator $(\not{p} - m_{i'})^{-1}$ in both the diagonal $(i = i')$ and off-diagonal $(i \neq i')$ contributions. Thus, they lead to expressions suitable for combination with the proper vertex diagrams. In analogy with B in Eqs. (1) and (2), such expressions are identified as wfr contributions. They satisfy the following important property: all the gauge-dependent and all the divergent wfr contributions to the basic $W \to q_i + \overline{q}_j$ amplitude are independent of i'. Using the unitarity relation $V_{il}V^\dagger_{l'}V_{i'j} = V_{il}\delta_{lj}$ (since the cofactor of this expression depends on m_l, the summation over l is performed later), one then finds that the gauge-dependent and the divergent wfr contributions to the $W \to q_i + \overline{q}_j$ amplitude are independent of CKM matrix elements, except for an overall factor V_{ij}, and depend only on the external-quark masses m_i and m_j. Since the one-loop proper vertex diagrams also only depend on m_i, m_j, and an overall factor V_{ij}, this observation implies that the proof of gauge independence and finiteness of the remaining one-loop corrections to the $W \to q_i + \overline{q}_j$ amplitude is the same as in the unmixed, single-generation case!

In contrast to the contributions of classes (i) and (ii) to Eq. (3), those of class (iv) lead to a multiple of $(\not{p} - m_{i'})^{-1}$ with a cofactor that involves $a_\pm$, but is independent of $\not{p}$. Thus, they are unsuitable to be combined with the proper vertex diagrams and are expected to be separately gauge independent, as we indeed find. In analogy with A in Eqs. (1) and (2), they are identified with sm contributions. Specifically, in the case of an outgoing up-type

quark, the sm contributions from Fig. 1 are given by the gauge-independent expression

$$\begin{aligned}\Delta\mathcal{M}^{\rm sm}_{ii'} = \frac{g^2}{32\pi^2}V_{il}V^\dagger_{li'}\overline{u}_i(p)\Bigg\{& m_i\left(1+\frac{m_i^2}{2m_W^2}\Delta\right)\\ &+\left[m_i a_- + m_{i'}a_+ + \frac{m_i m_{i'}}{2m_W^2}(m_i a_+ + m_{i'}a_-)\right]\\ &\times\left[I\left(m_i^2,m_l\right)-J\left(m_i^2,m_l\right)\right]\\ &-\frac{m_l^2}{2m_W^2}(m_i a_- + m_{i'}a_+)\left[3\Delta + I\left(m_i^2,m_l\right)\right.\\ &\left.+\ J\left(m_i^2,m_l\right)\right]\Bigg\}\frac{1}{\not p - m_{i'}},\end{aligned} \tag{5}$$

where g is the SU(2) gauge coupling, $\Delta = 1/(n-4) + [\gamma_E - \ln(4\pi)]/2 + \ln(m_W/\mu)$, n is the space-time dimension, μ is the 't Hooft mass, γ_E is Euler's constant,

$$\begin{aligned}\{I(p^2,m_l);J(p^2,m_l)\} &= \int_0^1 dx\,\{1;x\}\\ &\times\ln\frac{m_l^2 x + m_W^2(1-x) - p^2x(1-x) - i\varepsilon}{m_W^2},\end{aligned} \tag{6}$$

and m_l are the masses of the virtual down-type quarks in Fig. 1(a). Terms independent of m_l within the curly brackets of Eq. (5) lead to diagonal contributions on account of $V_{il}V^\dagger_{li'} = \delta_{ii'}$. There are other sm contributions involving virtual Z^0, ϕ^0, γ, and H bosons, as well as additional tadpole diagrams, but these are again diagonal expressions of the usual kind.

In order to generate mass counterterms, we proceed as follows. In the weak-eigenstate basis, the bare mass terms are of the form $-\overline{\psi}'^Q_R m_0'^Q \psi'^Q_L$ + h.c., where ψ'^Q_L and ψ'^Q_R are left- and right-handed column spinors involving the three up-type ($Q = U$) and down-type ($Q = D$) quarks, and $m_0'^Q$ are non-diagonal matrices. Writing $m_0'^Q = m'^Q - \delta m'^Q$, where m'^Q and $\delta m'^Q$ are the renormalized and counterterm mass matrices, we consider a biunitary transformation of the quark fields that diagonalizes m'^Q leading to diagonal and real renormalized mass matrices m^Q and to new non-diagonal mass counterterm matrices δm^Q. In the new framework, the mass term is given by

$$\begin{aligned}&-\overline{\psi}\left(m - \delta m^{(-)}a_- - \delta m^{(+)}a_+\right)\psi\\ &= -\overline{\psi}_R\left(m - \delta m^{(-)}\right)\psi_L - \overline{\psi}_L\left(m - \delta m^{(+)}\right)\psi_R,\end{aligned} \tag{7}$$

where m is real, diagonal, and positive, and $\delta m^{(-)}$ and $\delta m^{(+)}$ are arbitrary non-diagonal matrices subject to the hermiticity constraint

$$\delta m^{(+)} = \delta m^{(-)\dagger}. \tag{8}$$

Here we have not exhibited the superscript Q, but it is understood that m and $\delta m^{(\pm)}$ stand for two different sets of matrices involving the up- and down-type quarks. As usual, the

mass counterterms are included in the interaction Lagrangian. Their contribution to the external-leg corrections is given by $-\overline{u}_i(p)\left(\delta m^{(-)}_{ii'}a_- + \delta m^{(+)}_{ii'}a_+\right)/(\not{p} - m_{i'})$. Next we adjust $\delta m^{(\pm)}_{ii'}$ to cancel, as much as possible, the sm contributions given in Eq. (5). The cancellation of the divergent parts is achieved by choosing

$$\left(\delta m^{(-)}_{\rm div}\right)_{ii'} = \frac{g^2 m_i}{64\pi^2 m_W^2}\Delta\left(\delta_{ii'}m_i^2 - 3V_{il}V^\dagger_{li'}m_l^2\right),$$
$$\left(\delta m^{(+)}_{\rm div}\right)_{ii'} = \frac{g^2 m_{i'}}{64\pi^2 m_W^2}\Delta\left(\delta_{ii'}m_i^2 - 3V_{il}V^\dagger_{li'}m_l^2\right), \qquad (9)$$

which satisfies the hermiticity constraint of Eq. (8). Because the functions $I(p^2, m_l)$ and $J(p^2, m_l)$ are evaluated at $p^2 = m_i^2$ in the ii' channel (where i and i' are the external and virtual quarks, respectively) and at $p^2 = m_{i'}^2$ in the $i'i$ channel (where i' and i are the external and virtual quarks, respectively), it is easy to see that it is not possible to cancel all the finite pieces of Eq. (5) in all channels without contradicting Eq. (8). In particular, we note that once the $\delta m^{(\pm)}_{ii'}$ are chosen, the $\delta m^{(\pm)}_{i'i}$ are fixed by Eq. (8). For this reason, we employ the following renormalization prescription: the mass counterterms are chosen to exactly cancel all the contributions to Eq. (5) in the $i' = i$, uc, ut, and ct channels, and all the sm contributions in the $j' = j$, sd, bd, and bs channels in the corresponding down-type-quark expression. (Here j and j' are the incoming and virtual down-type quarks, respectively.) This implies that, after mass renormalization, there are residual sm contributions in the cu, tu, tc, ds, db, and sb channels. However, these residual contributions are finite, gauge independent, and numerically very small. In fact, the fractional corrections they induce in the real parts of V_{ij} reach a maximum value of $\mathcal{O}(4\times 10^{-6})$ for V_{ts}, and they are much smaller in the case of several other CKM matrix elements. Since they are regular in the limits $m_{i'} \to m_i$ or $m_{j'} \to m_j$, they may be regarded as additional finite and gauge-independent contributions to wave-function renormalization that happen to be very small.

We emphasize that with this renormalization prescription the sm corrections are fully canceled in all channels in which the external particle is a u, $\overline{u}$, d, or $\overline{d}$ quark. This is of particular interest since V_{ud}, the parameter associated with $W \to u + \overline{d}$, is by far the most precisely determined CKM matrix element [3].

It is also interesting to note that, since Eq. (9) satisfies Eq. (8), the modified minimal-subtraction ($\overline{\rm MS}$) renormalization, in which only the $1/(n-4) + [\gamma_E - \ln(4\pi)]/2$ terms are subtracted, can be implemented in all non-diagonal channels. More generally, one can consider a renormalization prescription that satisfies the hermiticity condition in all channels by choosing the mass counterterms to cancel the off-diagonal terms in Eq. (5) and the corresponding down-type-quark expression with the functions $I(p^2, m_l)$ and $J(p^2, m_l)$ evaluated at the same fixed p^2 value for all flavors. Since Eq. (5) is explicitly gauge independent, in our formulation there is no restriction in the choice of p^2 other than that it should not generate imaginary parts in the integrals $I(p^2, m_l)$ and $J(p^2, m_l)$. In particular, p^2 can have any value $p^2 \le m_W^2$. Of course, since it is desirable to cancel the sm contributions as much as possible, it is convenient to choose $0 \le p^2 \ll m_W^2$. It

should be pointed out, however, that the $\overline{\rm MS}$ and fixed-p^2 subtraction prescriptions of mass renormalization are not on-shell schemes and lead to residual sm contributions in all off-diagonal channels, which diverge in the limits $m_{i'} \to m_i$ or $m_{j'} \to m_j$.

An alternative formulation, equivalent to the one discussed so far, is obtained by diagonalizing the complete mass matrix $m - \delta m^{(-)} a_- - \delta m^{(+)} a_+$ in Eq. (7). This is achieved by a biunitary transformation

$$\psi_L = U_L \hat{\psi}_L, \qquad \psi_R = U_R \hat{\psi}_R. \tag{10}$$

At the one-loop level, it is sufficient to approximate

$$U_L = 1 + i h_L, \qquad U_R = 1 + i h_R, \tag{11}$$

where h_L and h_R are hermitian matrices of $\mathcal{O}(g^2)$. The diagonalization is implemented by choosing

$$i(h_L)_{ii'} = \frac{m_i \delta m^{(-)}_{ii'} + \delta m^{(+)}_{ii'} m_{i'}}{m_i^2 - m_{i'}^2} \qquad (i \neq i'), \tag{12}$$

while $i(h_R)_{ii'}$ is obtained by exchanging $\delta m^{(-)} \leftrightarrow \delta m^{(+)}$ in Eq. (12). Since the only effect of the diagonal terms of h_L and h_R on the $W q_i \overline{q}_j$ interaction is to introduce phases that can be absorbed in a redefinition of the quark fields, it is convenient to set $(h_L)_{ii} = (h_R)_{ii} = 0$. This analysis is carried out separately to diagonalize the mass matrices of the up- and down-type quarks. Thus, we obtain two pairs of matrices: h_L^U and h_R^U for the up-type quarks and h_L^D and h_R^D for the down-type quarks. Next we consider the effect of this biunitary transformation on the $W q_i \overline{q}_j$ interaction

$$\mathcal{L}_{W q_i \overline{q}_j} = -\frac{g_0}{\sqrt{2}} \overline{\psi}^U_L V \gamma^\lambda \psi^D_L W_\lambda + \text{h.c.}. \tag{13}$$

We readily find that

$$\mathcal{L}_{W q_i \overline{q}_j} = -\frac{g_0}{\sqrt{2}} \overline{\hat{\psi}}^U_L (V - \delta V) \gamma^\lambda \hat{\psi}^D_L W_\lambda + \text{h.c.}, \tag{14}$$

where

$$\delta V = i \left(h_L^U V - V h_L^D \right). \tag{15}$$

It is important to note that $V - \delta V$ satisfies the unitarity condition through $\mathcal{O}(g^2)$:

$$(V - \delta V)^\dagger (V - \delta V) = 1 + \mathcal{O}(g^4). \tag{16}$$

In the $(\hat{\psi}_L, \hat{\psi}_R)$ basis, in which the complete quark mass matrices are diagonal, δV and $V_0 = V - \delta V$ represent the counterterm and bare CKM matrices, respectively. One readily verifies that the term $i h_L^U V$ in δV leads to the same off-diagonal contribution to the $W \to q_i + \overline{q}_j$ amplitude as $\delta m^{U(-)}$ and $\delta m^{U(+)}$ in our previous discussion in the (ψ_L, ψ_R) basis. Similarly, the term $-i V h_L^D$ leads to the same contributions as $\delta m^{D(-)}$ and $\delta m^{D(+)}$. It is important to emphasize that this formulation is consistent with the

unitarity and gauge independence of both the renormalized and bare CKM matrices, V and V_0, respectively.

For completeness, we exhibit the CKM counterterm matrix in component form:

$$\begin{aligned}\delta V_{ij} &= i\left[\left(h_L^U\right)_{ii'} V_{i'j} - V_{ij'}\left(h_L^D\right)_{j'j}\right] \\ &= \frac{m_i^U \delta m_{ii'}^{U(-)} + \delta m_{ii'}^{U(+)} m_{i'}^U}{\left(m_i^U\right)^2 - \left(m_{i'}^U\right)^2} V_{i'j} \\ &\quad - V_{ij'} \frac{m_{j'}^D \delta m_{j'j}^{D(-)} + \delta m_{j'j}^{D(+)} m_j^D}{\left(m_{j'}^D\right)^2 - \left(m_j^D\right)^2},\end{aligned} \tag{17}$$

where it is understood that $i' \neq i$ in the first term on the r.h.s. and $j' \neq j$ in the second, and $\delta m_{ii'}^{U(\pm)}$ and $\delta m_{j'j}^{D(\pm)}$ are the off-diagonal mass counterterms determined by the on-shell renormalization prescriptions proposed in our first formulation. The coefficient of $1/(n-4)$ in Eq. (17) is, of course, common to all renormalization prescriptions for the CKM matrix [5] and also appears in its renormalization group equation [8].

In summary, after introducing a novel procedure to separate the external-leg mixing corrections into gauge-independent sm and gauge-dependent wfr contributions, in analogy with Feynman's treatment in QED, we have implemented their renormalization in two equivalent frameworks. The first one is carried out in a basis in which the renormalized quark matrices are diagonal and the non-diagonal mass counterterm matrices are employed to cancel all the divergent sm contributions, and also their finite parts up to hermiticity constraints. In particular, the sm corrections are fully canceled in the $W \to u + \overline{d}$ amplitude, associated with V_{ud}, the most accurately measured CKM parameter. Residual finite contributions in other channels are very small. We have also pointed out that the proof of gauge independence and finiteness of the remaining one-loop corrections to the $W \to q_i + \overline{q}_j$ amplitude reduces to that in the unmixed, single-generation case. Alternative renormalization prescriptions that are "democratic," in the sense that they do not single out particular off-diagonal channels, were briefly outlined. However, strictly speaking, they are not on-shell schemes and lead to residual sm contributions in all off-diagonal channels, which diverge in the limits $m_{i'} \to m_i$ or $m_{j'} \to m_j$.

The second formulation was obtained by diagonalizing the complete mass matrices, namely the renormalized plus counterterm mass matrices derived in the first approach. In the second framework a CKM counterterm matrix δV was generated which again cancels the divergent and, to the extent allowed by the hermiticity constraints, also the finite parts of the off-diagonal sm contributions. As usual, the diagonal sm contributions are canceled by the mass counterterms, which in this approach are also diagonal. An important feature is that this formulation is consistent with the unitarity and gauge independence of both the renormalized and bare CKM matrices, V and $V_0 = V - \delta V$, respectively.

As is well known, an enduring difficulty, thirty years old, in a satisfactory treatment of the one-loop electroweak corrections to all charged-current processes involving fermions is due to the external off-diagonal self-energy effects depicted in Fig. 1(a). Since the mass renormalization of the usual, diagonal effects must necessarily involve a complete

subtraction of the sm contributions to avoid the propagator's singularity [see Eq. (1)], it is natural to follow the same strategy in the off-diagonal contributions. Thus, an on-shell renormalization procedure to treat all these effects is highly desirable and strongly motivated. Such an objective has been achieved for the first time in this Letter in a way that the following important properties are manifestly satisfied: the CKM counterterm matrix is gauge independent, preserves unitarity, and leads to renormalized amplitudes that are non-singular in the limit in which any two fermions become mass degenerate. Because of the close analogy with QED and the fact that our decomposition procedure is algebraic in nature, it is likely that this approach can be naturally generalized to higher orders.

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