DESY 06-140

DESY 06–140

ISSN 0418 - 9833
August 2006

Path Integral Approach for Quantum Motion on Spaces of Non-constant Curvature According to Koenigs

Christian Grosche

II. Institut für Theoretische Physik

Universität Hamburg, Luruper Chaussee 149

22761 Hamburg, Germany

Contribution to the "XII. International Conference on Symmetry Methods in Physics", July 3–8, Yerevan, Armenia.

Abstract

In this contribution I discuss a path integral approach for the quantum motion on two-dimensional spaces according to Koenigs, for short "Koenigs-Spaces". Their construction is simple: One takes a Hamiltonian from two-dimensional flat space and divides it by a two-dimensional superintegrable potential. These superintegrable potentials are the isotropic singular oscillator, the Holt-potential, and the Coulomb potential. In all cases a non-trivial space of non-constant curvature is generated. We can study free motion and the motion with an additional superintegrable potential. For possible bound-state solutions we find in all three cases an equation of eighth order in the energy E. The special cases of the Darboux spaces are easily recovered by choosing the parameters accordingly.

1 Introduction

In this contribution I discuss the quantum motion on spaces of non-constant curvature according to Koenigs [14], which I will call for short "Koenigs-spaces". The construction of such a space is simple. One takes a two-dimensional flat Hamiltonian, $\mathcal{H}$, including some potential V, and divides $\mathcal{H}$ by a potential $f(x,y)$ $(x,y\in\mathbb{R}^2)$ such that this potential takes on the form of a metric:

$$\mathcal{H}_{\mathrm{Koenigs}}=\frac{\mathcal{H}}{f(x,y)}\quad . \tag{1.1}$$

Such a construction leads to a very rich structure, and attempts to classify such systems are e.g. due to Kalnins et al. [11, 12] and Daskaloyannis and Ypsilantis [2]. Simpler examples of such spaces are the Darboux spaces, where one chooses the potential $f(x,y)$ in such a way that it depends only on one variable [13]. Another choice consists whether one chooses for $f(x,y)$ some arbitrary potential (or some superintegrable potential) and taking into account that the Poisson bracket structure of the observables makes up a reasonable simple algebra [2, 4, 13].

In previous publications we have analyzed the quantum motion on Darboux spaces by means of the path integral [6, 8]. The path integral approach [3, 10] serves as a powerful tool to calculate the propagator, respectively the Green function of the quantum motion in such a space. In the present contribution I apply the path integral technique to three kinds of Koenigs-spaces, where a specific two-dimensional superintegrable potential [7] is chosen. They are the two-dimensional isotropic singular oscillator (Section II), the Holt-potential (section III) and the two-dimensional Coulomb-potential (Section IV). Section V is devoted to a summary and a discussion of the results achieved.

2 Koenigs-Space with Isotropic Singular Oscillator

We start with the first example, where we take for the metric term

$$\begin{aligned}
\mathrm{d}s^2 &= f_I(x,y)(\mathrm{d}x^2+\mathrm{d}y^2)\ , &(2.1)\\
f_I(x,y) &= \alpha(x^2+y^2)+\frac{\beta}{x^2}+\frac{\gamma}{y^2}+\delta\ , &(2.2)
\end{aligned}$$

and $\alpha,\beta,\gamma,\delta$ are constants. The classical Hamiltonian and Lagrangian in $\mathbb{R}^2$ with the isotropic singular oscillator as the superintegrable potential have the form:

$$\begin{aligned}
\mathcal{L} &= \frac{m}{2}\Big((\dot x^2+\dot y^2)-\omega^2(x^2+y^2)\Big)-\frac{\hbar^2}{2m}\bigg(\frac{k_x^2-\frac14}{x^2}+\frac{k_y^2-\frac14}{y^2}\bigg)\ , &(2.3)\\
\mathcal{H} &= \frac{p_x^2+p_y^2}{2m}+\frac{m}{2}\omega^2(x^2+y^2)+\frac{\hbar^2}{2m}\bigg(\frac{k_x^2-\frac14}{x^2}+\frac{k_y^2-\frac14}{y^2}\bigg)\ . &(2.4)
\end{aligned}$$

Counting constants, there are seven independent constants: $\alpha,\beta,\gamma,\delta$, and ω,k_x,k_y. An eighth constant can be added by adding a further constant $\tilde\delta$ into the potential of the

Hamiltonian. It will be omitted in the following. The first Koenigs-space K_{I} is constructed by considering

$$\mathcal{H}_{K_{\mathrm{I}}} = \frac{\mathcal{H}}{f_I(x,y)} \quad , \tag{2.5}$$

hence for the Lagrangian (with potential)

$$\mathcal{L}_{K_{\mathrm{I}}} = \frac{m}{2} f_I(x,y)(\dot{x}^2+\dot{y}^2) - \frac{1}{f_I(x,y)}\left[\frac{m}{2}\omega^2(x^2+y^2) + \frac{\hbar^2}{2m}\left(\frac{k_x^2-\frac{1}{4}}{x^2} + \frac{k_y^2-\frac{1}{4}}{y^2}\right)\right] \quad . \tag{2.6}$$

Setting the potential in square-brackets equal to zero yields the Lagrangian for the free motion in K_{I}. With this information we can set up the path integral in K_{I} including a potential. Because the space is two-dimensional, and the metric is diagonal, the additional quantum potential $\propto \hbar^2$ vanishes. The canonical momentum operators are constructed by

$$p_{x_i} = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial_{x_i}} + \frac{\Gamma_i}{2}\right) \quad , \qquad \Gamma_i = \frac{\partial}{\partial_{x_i}} \ln\sqrt{g} \quad , \tag{2.7}$$

with $x_1 = x, x_2 = y$ and $g = \det(g_{ab})$, (g_{ab}) the metric tensor. For the path integral in the product lattice definition [10] we obtain

$$\begin{aligned}
K^{(K_{\mathrm{I}})}(x'',x',y'',y';T) &= \int\limits_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \int\limits_{y(t')=y'}^{y(t'')=y''} \mathcal{D}y(t) f_I(x,y) \\
&= \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left\{\frac{m}{2} f_I(x,y)(\dot{x}^2+\dot{y}^2)\right.\right. \\
&\qquad \left.\left. -\frac{1}{f_I(x,y)}\left[\frac{m}{2}\omega^2(x^2+y^2) + \frac{\hbar^2}{2m}\left(\frac{k_x^2-\frac{1}{4}}{x^2} + \frac{k_y^2-\frac{1}{4}}{y^2}\right)\right]\right\}\mathrm{d}t\right) \\
G^{(K_{\mathrm{I}})}(x'',x',y'',y';E) &= \frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}s'' K^{(K_{\mathrm{I}})}(x'',x',y'',y';s'')\mathrm{e}^{\mathrm{i}\delta E s''/\hbar} \quad ,
\end{aligned} \tag{2.8}$$

with the time-transformed path integral $K^{(K_{\mathrm{I}})}(s'')$ given by ($\tilde{\omega}^2 = \omega^2 - 2\alpha E/m$)

$$\begin{aligned}
&K^{(K_{\mathrm{I}})}(x'',x',y'',y';s'') \\
&= \int\limits_{x(0)=x'}^{x(s'')=x''} \mathcal{D}x(s) \int\limits_{y(0)=y'}^{y(s'')=y''} \mathcal{D}y(s) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}\Big((\dot{x}^2+\dot{y}^2) - \tilde{\omega}^2(x^2+y^2)\Big)\right.\right. \\
&\qquad\qquad \left.\left. -\frac{\hbar^2}{2m}\left(\frac{k_x^2 - 2m\beta E/\hbar^2 - \frac{1}{4}}{x^2} + \frac{k_y^2 - 2m\gamma E/\hbar^2 - \frac{1}{4}}{y^2}\right)\right]\mathrm{d}s''\right\} \quad .
\end{aligned} \tag{2.9}$$

The path integrals in the variables x and y are both path integrals for the radial harmonic oscillator, however with energy-dependent coefficients. By switching to two-dimensional polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$, the path integral in x,y gives one in r,φ. Furthermore, we get $x^2+y^2 = r^2$, $1/x^2 = 1/r^2\cos^2\varphi$, and $1/y^2 = 1/r^2\sin^2\varphi$. Let us abbreviate $\tilde{k}_x^2 = k_x^2 - 2m\beta E/\hbar^2$, $\tilde{k}_y^2 = k_y^2 - 2m\gamma E/\hbar^2$. In the variable φ we obtain a path integral for the Pöschl–Potential, and in the variable r a radial path integral. The successive path integrations therefore yield

$$K^{(K_\mathrm{I})}(r'',r',\varphi'',\varphi';s'')=\sum_{n_\varphi}\Phi_{n_\varphi}^{(\tilde k_y,\tilde k_x)}(\varphi'')\Phi_{n_\varphi}^{(\tilde k_y,\tilde k_x)}(\varphi')$$
$$\times\frac{m\widetilde\omega\sqrt{r'r''}}{\mathrm{i}\hbar\sin\widetilde\omega s''}\exp\bigg[-\frac{m\widetilde\omega}{2\mathrm{i}\hbar}({r'}^2+{r''}^2)\cot\widetilde\omega s''\bigg]I_\lambda\bigg(\frac{m\widetilde\omega r'r''}{\mathrm{i}\hbar\sin\widetilde\omega s''}\bigg)\quad. \tag{2.10}$$

Here $\lambda=2n_\varphi+\tilde k_x+\tilde k_y+1$, and the $\Phi_{n_\varphi}^{(\tilde k_y,\tilde k_x)}(\varphi)$ are the wave-functions for the Pöschl-Teller potential [1, 10]. $I_\lambda(z)$ is the modified Bessel function [5]. Performing the s''-integration for obtaining the Green function $G(E)$ yields [5, 10]:

$$G^{(K_\mathrm{I})}(r'',r',\varphi'',\varphi';E)=\sum_{n_\varphi}\Phi_{n_\varphi}^{(\tilde k_y,\tilde k_x)}(\varphi'')\Phi_{n_\varphi}^{(\tilde k_y,\tilde k_x)}(\varphi')$$
$$\times\frac{\Gamma[\frac12(1+\lambda-\delta E/\hbar\widetilde\omega)]}{\hbar\widetilde\omega\sqrt{r'r''}\,\Gamma(1+\lambda)}W_{\delta E/2\widetilde\omega,\lambda/2}\bigg(\frac{m\widetilde\omega}{\hbar}r_>^2\bigg)M_{\delta E/2\widetilde\omega,\lambda/2}\bigg(\frac{m\widetilde\omega}{\hbar}r_<^2\bigg)\quad. \tag{2.11}$$

$M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$ are Whittaker-functions [5], and $r_<,r_>$ is the smaller/larger of r',r''. The poles of the Γ-function give the energy-levels of the bound states:

$$\tfrac12(1+\lambda-\delta E/\hbar\widetilde\omega)=-n_r\quad, \tag{2.12}$$

which is equivalent to ($N=n_r+n_\varphi+1=1,2,\ldots$):

$$\delta E\;=\;\hbar\sqrt{\omega^2-\frac{2\alpha}{m}E}\,\left(2N+\sqrt{k_x^2-\frac{2m\beta}{\hbar^2}E}+\sqrt{k_y^2-\frac{2m\gamma}{\hbar^2}E}\right)\quad. \tag{2.13}$$

In general, this quantization condition is an equation of eighth order in E. If we know the bound state energy E_N, we can determine the wavefunctions according to

$$\Psi_N^{(K_\mathrm{I})}(r,\phi)=N_N\Phi_{n_\varphi}^{(\tilde k_y,\tilde k_x)}(\varphi)\Phi_{n_r}^{(RHO,\lambda)}(r)\quad, \tag{2.14}$$

with the normalization constant N_N determined by evaluating the residuum in the Green function (2.11), and the $\Phi_N^{(RHO,\lambda)}(r)$ are the wave-functions of the radial harmonic oscillator [10]. We can recover the flat space limit with $\alpha=\beta=\gamma=0$ with the correct spectrum $E_N=\hbar\omega(N+k_x+k_y)/\delta$.

Note that we also can obtain the quantization condition by explicitly inserting the wave-functions in x and y in (2.9) and performing the s''-integration in (2.8). We do not discuss the continuous spectrum.

3 Koenigs-Space with Holt-Potential

Next we consider for the metric term

$$\mathrm{d}s^2\;=\;f_{II}(x,y)(\mathrm{d}x^2+\mathrm{d}y^2)\quad, \tag{3.1}$$
$$f_{II}(x,y)\;=\;\alpha(x^2+4y^2)+\frac{\beta}{x^2}+\gamma y+\delta \tag{3.2}$$

and $\alpha,\beta,\gamma,\delta$ are constants. The classical Hamiltonian and Lagrangian in $\mathbb{R}^2$ with the Holt-potential as the superintegrable potential have the form:

$$\mathcal{L} = \frac{m}{2}\Big((\dot{x}^2+\dot{y}^2)-\omega^2(x^2+4y^2)\Big)-k_y y-\frac{\hbar^2}{2m}\frac{k_x^2-\frac{1}{4}}{x^2}\ , \tag{3.3}$$

$$\mathcal{H} = \frac{p_x^2+p_y^2}{2m}+\frac{m}{2}\omega^2(x^2+4y^2)+k_y y+\frac{\hbar^2}{2m}\frac{k_x^2-\frac{1}{4}}{x^2}\ . \tag{3.4}$$

Counting constants, there are seven independent constants: $\alpha,\beta,\gamma,\delta$, and ω,k_x,k_y. An eighth constant can be added by adding a further constant $\tilde{\delta}$ into the potential of the Hamiltonian, which is omitted. The second Koenigs-space $K_{\rm II}$ with potential is now constructed by considering

$$\mathcal{H}_{K_{\rm I}}^{(V)}=\frac{\mathcal{H}}{f_{II}(x,y)}\ . \tag{3.5}$$

From the discussion in the Section II it is obvious how to construct the path integral on $K_{\rm II}$. We proceed straightforward to the time-transformed path integral $K^{(K_{\rm II})}(s'')$ which has the form

$$K^{(K_{\rm II})}(x'',x',y'',y';s'')=\int\limits_{x(0)=x'}^{x(s'')=x''}\mathcal{D}x(s)\int\limits_{y(0)=y'}^{y(s'')=y''}\mathcal{D}y(s)$$
$$\times\exp\left\{\frac{\rm i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\Big((\dot{x}^2+\dot{y}^2)-\widetilde{\omega}^2(x^2+4y^2)\Big)-\frac{\hbar^2}{2m}\frac{\tilde{k}_x^2-\frac{1}{4}}{x^2}-(k_y-\gamma E)y\right]{\rm d}s''\right\}. \tag{3.6}$$

Again, $\widetilde{\omega}^2=\omega^2-2\alpha E/m$, $\tilde{k}_x^2=k_x^2-2m\beta E/\hbar^2$. We have in the variable x a singular oscillator, and in the variable y a shifted oscillator with shift $y\to y-(k_y-\gamma E)/(4m\widetilde{\omega}^2)\equiv y-y_E$. However, in comparison to Section II, we cannot separate variables in an analogous way as for $K_{\rm I}$, because the only separating coordinate systems for the Holt-potential are the Cartesian and the parabolic systems, and only in Cartesian coordinates a closed solution is possible. Therefore we must evaluate this path integral by another method. The first possibility consists of writing down the Green functions for the radial singular oscillator $G^{(RHO,\tilde{k}_x)}(E)$ and for the shifted harmonic oscillator $G^{(HO,y_E)}(E)$, respectively. These solutions can be found in e.g. [10]. The final result for the Green function $G^{(K_{\rm II})}(E)$ then has the form

$$G^{(K_{\rm II})}(E)=\frac{\hbar}{2\pi{\rm i}}\int{\rm d}\mathcal{E}G_x^{(RHO,\tilde{k}_x)}(E;x'',x';\mathcal{E})G_y^{(HO,y_E)}\bigg(E;y'',y';-\mathcal{E}-\delta+\frac{(k_y-\gamma E)^2}{8m\widetilde{\omega}^2}\bigg)\ . \tag{3.7}$$

However, this is a very complicated expression, mainly due to the fact that both the Green functions $G^{(RHO)}(E)$ and $G^{(HO,\text{shift})}(E)$ consist of products of Whittaker functions and parabolic cylinder functions, respectively. A better way to analyze the spectral properties is to re-express each kernel in its bound-state wave-functions expansion. Therefore

$$K^{(K_{\rm II})}(x'',x',y'',y';s'')=\sum_{n_x}\Psi_{n_x}^{(RHO,\tilde{k}_x)}(x'')\Psi_{n_x}^{(RHO,\tilde{k}_x)*}(x')\sum_{n_y}\Psi_{n_y}^{(HO,y_E)}(y'')\Psi_{n_y}^{(HO,y_E)*}(y')$$
$$\times{\rm e}^{-{\rm i}s''(k_y-\gamma E)^2/(8m\hbar\widetilde{\omega}^2)}{\rm e}^{-{\rm i}s''\widetilde{\omega}(n_x+\tilde{k}_x+2n_y+3/2)}\ . \tag{3.8}$$

Here, the $\Psi_{n_y}^{(HO,y_E)}(y)$ denote the wave-functions of a shifted harmonic oscillator with shift y_E. Performing the s''-integration similarly as in (2.8) we get the quantization condition $(N = n_x + 2n_y + 3/2)$

$$8m\delta E\Big(\omega^2 - \frac{2\alpha}{m}E\Big) - (k_y - \gamma E)^2 = \hbar\Big(\omega^2 - \frac{2\alpha}{m}E\Big)^{3/2}\left(2N + \sqrt{k_x^2 - \frac{2m\beta}{\hbar^2}E}\right) \quad . \tag{3.9}$$

In general, this is an equation of eighth order in E. The solution in terms of the wave-functions then has the form

$$\Psi_N^{(K_{\rm II})}(x,y) = N_N \Psi_{n_x}^{(RHO,\tilde{k}_x)}(x)\Psi_{n_y}^{(HO,y_E)}(y) \enspace , \tag{3.10}$$

and the normalization constant N_N is determined by the residuum of (3.7) at the energy E_N from (3.9). The correct flat space limit with $\alpha = \beta = \gamma = 0$ is easily recovered with spectrum $E_N = \hbar\omega(N + k_x)/\delta + k_y^2/8m\delta\omega^2$. We do not discuss the continuous spectrum.

4 Koenigs-Space with Coulomb-Potential

For the last example we consider a metric which corresponds to the two-dimensional Coulomb potential $(r^2 = x^2 + y^2)$

$$\begin{aligned} \mathrm{d}s^2 &= f_{III}(x,y)(\mathrm{d}x^2 + \mathrm{d}y^2) \enspace , && (4.1)\\ f_{III}(x,y) &= -\frac{\alpha_1}{r} + \frac{1}{4r^2}\left(\frac{\beta}{\cos^2\frac{\varphi}{2}} + \frac{\gamma}{\sin^2\frac{\varphi}{2}}\right) + \delta && (4.2)\end{aligned}$$

and $\alpha_1, \beta, \gamma, \delta$ are constants. The classical Hamiltonian and Lagrangian in $\mathbb{R}^2$ with the Coulomb potential as the superintegrable potential have the form:

$$\begin{aligned} \mathcal{L} &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{\alpha_2}{r} - \frac{\hbar^2}{8mr^2}\left(\frac{k_1^2 - \frac14}{\cos^2\frac{\varphi}{2}} + \frac{k_2^2 - \frac14}{\sin^2\frac{\varphi}{2}}\right) \enspace , && (4.3)\\ \mathcal{H} &= \frac{p_x^2 + p_y^2}{2m} - \frac{\alpha_2}{r} + \frac{\hbar^2}{8mr^2}\left(\frac{k_1^2 - \frac14}{\cos^2\frac{\varphi}{2}} + \frac{k_2^2 - \frac14}{\sin^2\frac{\varphi}{2}}\right) \enspace . && (4.4)\end{aligned}$$

Counting constants, there are seven independent constants: $\alpha_1, \beta, \gamma, \delta$, and α_2, k_1, k_2. An eight constants can be added by adding a further constant $\tilde{\delta}$ into the potential of the Hamiltonian, which is again omitted. The third Koenigs-space $K_{\rm III}$ is constructed by considering

$$\mathcal{H}_{K_{\rm I}}^{(V)} = \frac{\mathcal{H}}{f_{III}(x,y)} \quad . \tag{4.5}$$

We proceed to the time-transformed path integral $K^{(K_{\rm III})}(s'')$ which has the form

$$\begin{aligned} K^{(K_{\rm III})}(r'', r', \varphi'', \varphi'; s'') &= \int\limits_{r(0)=r'}^{r(s'')=r''} \mathcal{D}r(s) \int\limits_{\varphi(0)=\varphi'}^{\varphi(s'')=\varphi''} \mathcal{D}\varphi(s) r\\ &\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^{s''}\left[\frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{\tilde{\alpha}}{r} - \frac{\hbar^2}{8mr^2}\left(\frac{\tilde{k}_1^2 - \frac14}{\cos^2\frac{\varphi}{2}} + \frac{\tilde{k}_2^2 - \frac14}{\sin^2\frac{\varphi}{2}} - 1\right)\right]\mathrm{d}s''\right\}. \end{aligned} \tag{4.6}$$

Here, $\tilde{k}_1^2 = k_1^2 - 2m\beta E/\hbar^2$, $\tilde{k}_2^2 = k_2^2 - 2m\gamma E/\hbar^2$, $\tilde{\alpha} = \alpha_2 - \alpha_1 E$. As in Section II, it is best to switch to two-dimensional polar coordinates, which is straightforward. We obtain for the Green function in polar coordinates

$$
\begin{aligned}
G^{(K_{\rm III})}(r'',r',\varphi'',\varphi';E) &= \sum_{n_\varphi} \Phi_{n_\varphi}^{(\tilde{k}_2,\tilde{k}_1)}(\tfrac{\varphi''}{2})\Phi_{n_\varphi}^{(\tilde{k}_2,\tilde{k}_1)*}(\tfrac{\varphi'}{2})\\
&\times\frac{1}{\hbar}\sqrt{-\frac{m}{2\delta E}}\,\frac{\Gamma(\frac{1}{2}+\lambda-\kappa)}{\Gamma(2\lambda+1)}W_{\kappa,\lambda}\left(\sqrt{-8m\delta E}\,\frac{r_>}{\hbar}\right)M_{\kappa,\lambda}\left(\sqrt{-8m\delta E}\,\frac{r_>}{\hbar}\right)
\end{aligned}
\tag{4.7}
$$

$\Big(\kappa = (\tilde{\alpha}/\hbar)\sqrt{-m/2\delta E}$, $\lambda = n_\varphi + \tilde{k}_1/2 + \tilde{k}_2/2 + \frac{1}{2}\Big)$. The poles of the Γ-function gives the quantization condition $1/2 + \lambda - \kappa = -n_r$, or more explicitly

$$
1 + n_\varphi + n_r + \frac{1}{2}\sqrt{k_1^2 - \frac{2m\beta}{\hbar^2}E} + \frac{1}{2}\sqrt{k_2^2 - \frac{2m\gamma}{\hbar^2}E} = \frac{\alpha_2 - \alpha_1 E}{\hbar}\sqrt{-\frac{m}{2\delta E}}\ . \tag{4.8}
$$

This is again an equation of eighth order in E. Actually, this quantization condition has the same structure as the quantization condition for the third potential on Darboux Space $D_{\rm II}$, c.f. our recent publication [8]. We consider the special case $k_1 = k_2 = 0$. This gives $(N = 1 + n_\varphi + n_r)$:

$$
N = \frac{\alpha_2 - \alpha_1 E}{\hbar}\sqrt{-\frac{m}{2\delta E}} - \frac{\sqrt{-E}}{2\hbar}\Big(\sqrt{2m\beta} + \sqrt{2m\gamma}\Big)\ . \tag{4.9}
$$

This is a quadratic equation in the energy E with solution

$$
E_\pm = -\frac{B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}\ , \tag{4.10}
$$

$$
\left.\begin{aligned}
&A = m\alpha_1(a_1 - 2) + 2m\delta\left(\sqrt{\beta} + \sqrt{\gamma}\right)^2\ ,\\
&B = 2\delta\hbar^2 N^2 + 2\alpha_2(m - \alpha_1)\ ,\qquad C = m\alpha_2^2\ .
\end{aligned}\right\} \tag{4.11}
$$

We consider the limit $N \to \infty$. In this case, we take the $+$-sign of the square-root expression only, and obtain

$$
E_N \simeq -\frac{m\alpha_2^2}{2\delta\hbar^2 N^2}\ ,\qquad (N \to \infty)\ , \tag{4.12}
$$

showing a Coulomb-behavior of the energy-levels. For the bound-states wave-function we get in the general case $(a = \hbar^2/m\tilde{\alpha})$:

$$
\begin{aligned}
\Psi_N^{(K_{\rm III})}(r,\varphi) &= \frac{N_N}{n_r + \lambda + \frac{1}{2}}\sqrt{\frac{n_r!}{a\Gamma(n_r + 2\lambda + 1)}}\,\Phi_{n_\varphi}^{(\tilde{k}_2,\tilde{k}_1)}(\tfrac{\varphi}{2})\\
&\times\left(\frac{2r}{a(n_r + \lambda + \frac{1}{2})}\right)^\lambda \exp\left[-\frac{r}{a(n_r + \lambda + \frac{1}{2})}\right]L_{n_r}^{(2\lambda)}\left(\frac{2r}{a(n_r + \lambda + \frac{1}{2})}\right)
\end{aligned}
\tag{4.13}
$$

(the $L_n^{(\lambda)}(z)$ are Laguerre polynomials [5]). The wave-functions in r are the well-known Coulomb wave-functions. Note that $\lambda = \lambda(E_N)$. The normalization constant N_N is determined by taking the residuum in the Green function (4.7) for the corresponding energy E_N from (4.8).

We get another special case if we set the potential in $K_{\rm III}$ to zero, i.e., $k_{1,2}=\frac{1}{2}$, $\alpha_2=0$. This yields together with the simplification $\beta=\gamma$

$$N+\sqrt{\frac{1}{4}-\frac{2m\beta E}{\hbar^2}}=-\frac{\alpha_1 E}{\hbar}\sqrt{-\frac{m}{2\delta E}}\ . \tag{4.14}$$

This is a quadratic equation in the energy E with solution

$$E_\pm=-\frac{B}{2A}\pm\frac{B}{2A}\sqrt{1-\frac{4AC}{B^2}}\ , \tag{4.15}$$

$$\left.\begin{aligned}A&=\tfrac{m^2}{\hbar^4}\Big(\tfrac{\alpha_1^2}{2\delta}-4\beta N\Big)^2\ ,\qquad C=(N^2+N)^2-4N^2\ ,\\ B&=\tfrac{2m}{\hbar^2}\bigg[(N^2+N)\Big(\tfrac{\alpha_1^2}{2\delta}-4\beta N\Big)+8\beta\bigg]\ .\end{aligned}\right\} \tag{4.16}$$

We see that even for zero potential, bound states are possible. For $N\to\infty$, the leading term behaves according to $-B/2A\to\hbar^2N/2m\beta$, showing a oscillator-like behavior. We do not discuss the continuous spectrum. This concludes the discussion.

5 Summary and Discussion

In this contribution I have discussed a path integral approach for spaces of non-constant curvature according to Koenigs, which I have for short called "Koenigs-spaces" $K_{\rm I}$, $K_{\rm II}$, and $K_{\rm III}$, respectively. I have found a very rich structure of the spectral properties of the quantum motion on Koenigs-spaces. In the general case with potential, in all three spaces the quantization condition is determined by an equation of eighth order in the energy E. Such an equation cannot be solved explicitly, however special cases can be studied. Indeed in the space $K_{\rm III}$ we have found for such a special case a Coulomb-like spectrum for large quantum numbers. This is very satisfying, because the flat space $\mathbb{R}^2$ is contained as a special case of $K_{\rm III}$. Our systems are also superintegrable, because they admit separation of variables in more that one coordinate system.

Let us note a further feature of these spaces. It is obvious that our solutions remain on a formal level. Neither have we specified an embedding space, nor have we specified boundary conditions on our spaces. Let us consider the space $K_{\rm II}$: We set $\alpha=\beta=\delta=0$ and $\gamma=1$. In this case we obtain a metric which corresponds to the Darboux space $D_{\rm I}$ (modulo change of variables), as discussed in [13]. In $D_{\rm I}$ boundary conditions and the signature of the ambient space is very important, because choosing a positive or a negative signature of the ambient space changes the boundary conditions, and hence the quantization conditions [8].

Furthermore, we can recover the Darboux space $D_{\rm II}$ [6, 8, 13] by setting in our examples in the potential function f all constant to zero except those corresponding to the $1/x^2$-singularity. However, we did not discuss these cases in detail.

In our approach we have chosen examples of superintegrable potentials in two-dimensional space, i.e. the isotropic singular oscillator, the Holt potential and the Coulomb potential, respectively. Other well-known potentials can also be included, for instance the Morse-potential or the (modified) Pöschl–Teller potential. Actually, the incorporation of the Morse-potential leads to the Darboux space $D_{\rm III}$, and the incorporation of the Pöschl–Teller potential to the Darboux space $D_{\rm IV}$ [13]. The quantum motion without potential have been discussed extensively in [6], and with potentials will be discussed in [9]. In these cases, also complicated quantization conditions are found.

In the present contribution I have omitted the discussion of the continuous spectrum. This is on the one hand side due to lack of space, and on the other the specific ambient space has to be taken into account. For instance, in the Darboux space $D_{\rm II}$ we know that the continuous spectrum has the form of $E_p \propto (\hbar^2/2m)p^2 + \text{constant}$. The wave-functions are proportional to K-Bessel functions [6]. However, in Darboux space $D_{\rm I}$ there is no such constant, and the wave-functions have a different form. Furthermore, $D_{\rm II}$ contains as special cases the two-dimensional Euclidean plane and the Hyperbolic plane, respectively. In $K_{\rm II}$ we can find these spaces for a special choice of parameters and the continuous wave-functions are proportional to Whittaker-functions (which reduce to K-Bessel functions and parabolic cylinder functions for specific parameters, respectively). Such a more detailed study will be presented elsewhere.

Acknowledgments

This work was supported by the Heisenberg–Landau program.

The author is grateful to Ernie Kalnins for fruitful and pleasant discussions on super-integrability and separating coordinate systems. I also would like to thank the organizers, in particular G.Pogosyan, of the "XII. International Conference on Symmetry Methods in Physics", July 3–8, Yerevan, Armenia, for the warm hospitality during my stay in Yerevan.

References

[1] Böhm, M., Junker, G.: Path Integration Over Compact and Noncompact Rotation Groups. *J. Math. Phys.* **28** (1987) 1978–1994.
Duru, I.H.: Path Integrals Over SU(2) Manifold and Related Potentials. *Phys. Rev.* **D 30** (1984) 2121–2127.

[2] Daskaloyannis, C., Ypsilantis, K.: Unified Treatment and Classification of Superintegrable Systems with Integrals Quadratic in Momenta on a Two Dimensional Manifold. *J. Math. Phys.* **45** (2006) 042904.

[3] Feynman, R.P., Hibbs, A.: *Quantum Mechanics and Path Integrals.* McGraw Hill, New York, 1965.
Kleinert, H.: *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics.* World Scientific, Singapore, 1990.
Schulman, L.S.: *Techniques and Applications of Path Integration.* John Wiley & Sons, New York, 1981.

[4] Friš, J., Mandrosov, V., Smorodinsky, Ya.A., Uhlir, M., Winternitz, P.: On Higher Symmetries in Quantum Mechanics; *Phys.Lett.* **16** (1965) 354,
Friš, J.; Smorodinskiĭ, Ya.A., Uhlíř, M., Winternitz, P.: Symmetry Groups in Classical and Quantum Mechanics; *Sov.J.Nucl.Phys.* **4** (1967) 444
Winternitz, P., Smorodinskiĭ, Ya.A., Uhlir, M., Fris, I.: Symmetry Groups in Classical and Quantum Mechanics. *Sov. J. Nucl. Phys.* **4** (1967) 444–450.

[5] Gradshteyn, I.S., Ryzhik, I.M.: *Table of Integrals, Series, and Products.* Academic Press, New York, 1980.

[6] Grosche, C.: Path Integration on Darboux Spaces. *Phys. Part. Nucl.* **37** (2006) 368–389.

[7] Grosche, C., Pogosyan, G.S., Sissakian, A.N.: Path Integral Discussion for Smorodinsky-Winternitz Potentials: I. Two- and Three-Dimensional Euclidean Space. *Fortschr. Phys.* **43** (1995) 453–521.

[8] Grosche, C., Pogosyan, G.S., Sissakian, A.N.: Path Integral Approach for Superintegrable Potentials on Spaces of Non-constant Curvature: I. Darboux Spaces D_{I} and D_{II}. *DESY preprint* DESY 06-113, July 2006. *Phys. Part. Nucl.*, to appear.

[9] Grosche, C., Pogosyan, G.S., Sissakian, A.N.: Path Integral Approach for Superintegrable Potentials on Spaces of Non-constant Curvature: II. Darboux Spaces D_{III} and D_{IV}. *DESY preprint* DESY 06-149, August 2006. *Phys. Part. Nucl.*, to appear.

[10] Grosche, C., Steiner, F.: *Handbook of Feynman Path Integrals. Springer Tracts in Modern Physics* **145**. Springer, Berlin, Heidelberg, 1998.

[11] Kalnins, E.G., Kress, J.M., Pogosyan, G., Miller, W.Jr.: Complete Sets of Invariants for Dynamical Systems that Admit a Separation of Variables. *J. Math. Phys.* **43** (2002) 3592–3609.
Infinite-Order Symmetries for Quantum Separable Systems. *Phys. Atom. Nucl.* **68** (2005) 1756-1763.

[12] Kalnins, E.G., Kress, J.M., Miller, W.Jr.: Second Order Superintegrable Systems in Conformally Flat Spaces. I. 2D Classical Structure Theory. *J. Math. Phys.* **46** (2005) 053509.
Second Order Superintegrable Systems in Conformally Flat Spaces. II. The Classical Two-Dimensional Stäckel Transform. *J. Math. Phys.* **46** (2005) 053510.

[13] Kalnins, E.G., Kress, J.M., Miller, W.Jr., Winternitz, P.: Superintegrable Systems in Darboux Spaces. *J. Math. Phys.* **44** (2003) 5811–5848.
Kalnins, E.G., Kress, J.M., Winternitz, P.: Superintegrability in a Two-Dimensional Space of Non-constant Curvature. *J. Math. Phys.* **43** (2002) 970–983.

[14] Koenigs, G.: Sur les géodésiques a intégrales quadratiques. A note appearing in "Lecons sur la théorie générale des surface". Darboux, G., Vol.4, 368–404, Chelsea Publishing, 1972.