The Neutralino Sector in the U(1)-Extended Supersymmetric Standard Model

S.Y. Choi[1,2], H.E. Haber[3], J. Kalinowski[4] and P.M. Zerwas[2,3]

[1] *Department of Physics and RIPC, Chonbuk National University, Jeonju 561-756, Korea*

[2] *Deutsches Elektronen-Synchrotron DESY, D-22603 Hamburg, Germany*

[3] *SCIPP, University of California, Santa Cruz, CA 95064, USA*

[4] *Institute of Theor. Physics, Warsaw University, 00681 Warsaw, Poland*

Abstract

Motivated by grand unified theories and string theories we analyze the general structure of the neutralino sector in the USSM, an extension of the Minimal Supersymmetric Standard Model that involves a broken extra U(1) gauge symmetry. This supersymmetric U(1)-extended model includes an Abelian gauge superfield and a Higgs singlet superfield in addition to the standard gauge and Higgs superfields of the MSSM. The interactions between the MSSM fields and the new fields are in general weak and the mixing is small, so that the coupling of the two subsystems can be treated perturbatively. As a result, the mass spectrum and mixing matrix in the neutralino sector can be analyzed analytically and the structure of this 6-state system is under good theoretical control. We describe the decay modes of the new states and the impact of this extension on decays of the original MSSM neutralinos, including radiative transitions in cross-over zones. Production channels in cascade decays at the LHC and pair production at e^+e^- colliders are also discussed.

arXiv:hep-ph/0612218v1 18 Dec 2006

1   Introduction

Adding an extra $U(1)_X$ broken gauge symmetry to the gauge symmetries of the Standard Model is well motivated by grand unified theories [1]. The corresponding supersymmetric extension that generalizes the minimal supersymmetric Standard Model (MSSM) often appears as the low energy effective theory of superstring theories [2]. This $U(1)_X$ extended supersymmetric gauge theory shall henceforth be denoted as the USSM.

The Higgs sector associated with the broken $U(1)_X$ gauge symmetry provides an elegant solution to the μ problem in supersymmetric theories [3, 4]. An effective μ parameter is generated by the vacuum expectation value of the new singlet Higgs field S, which breaks the $U(1)_X$ gauge symmetry. This is the same mechanism employed by the next-to-minimal supersymmetric standard model (NMSSM) [5]. However, the USSM possesses an additional advantage by avoiding the extra discrete symmetries of the NMSSM that, in the canonical version, result in the existence of domain-walls that are incompatible with the observed energy density of the universe. Moreover, the upper bound on the mass of the lightest Higgs boson of the MSSM is relaxed in the USSM due to contributions from the new singlet-doublet Higgs interactions and the $U(1)_X$ D-terms [6]. Various scenarios of this type have been discussed in the literature, see e.g. Refs. [7,8], in which the $U(1)_X$ gauge symmetry is embedded in the grand unification group E_6 (or one of its rank-five subgroups).

Including the extra symmetry, the gauge group is extended to $G = SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_X$ with the couplings g_3, g_2, g_Y, g_X, respectively. The matter particle content in the supersymmetric theory includes, potentially among others, the left-handed chiral superfields $\hat{L}_i$, $\hat{E}_i^c$; $\hat{Q}_i$, $\hat{U}_i^c$, $\hat{D}_i^c$, where the subscript i denotes the generation index, and the Higgs superfields $\hat{H}_d$, $\hat{H}_u$, $\hat{S}$. The usual MSSM Yukawa terms $\hat{W}_Y$ of the MSSM superpotential (i.e. without the μ term) are augmented by an additional term that couples the iso-singlet to the two iso-doublet Higgs fields:

$$\hat{W} = \hat{W}_Y + \lambda \hat{S} \left(\hat{H}_u \hat{H}_d \right). \tag{1.1}$$

The coupling λ is dimensionless. Gauge invariance of the superpotential $\hat{W}$ under $U(1)_X$ requires the $U(1)_X$ charges to satisfy $Q_{H_d} + Q_{H_u} + Q_S = 0$ and corresponding relations between the $U(1)_X$ charges of Higgs and matter fields. [In the following, we use $Q_1 = Q_{H_d}$ and $Q_2 = Q_{H_u}$ for notational convenience.] The effective μ parameter is generated by the vacuum expectation value $\langle S \rangle$ of the scalar S-field.

Compared with the MSSM, the USSM Higgs sector is extended by a single scalar state. The neutralino sector includes an additional pair of higgsino and gaugino states, while the chargino sector remains unaltered. The complexity of phenomena increases dramatically by this extension but the structure remains transparent if the original and the new degrees of freedom are coupled weakly as naturally demanded [see below].

The supersymmetric particle spectrum of the USSM has received limited attention so far in the

literature [3, 8–11]. In this report we attempt a systematic analytical analysis of the neutralino system, based on the well-motivated assumption of weak coupling between the original MSSM and the new additional gaugino/higgsino subsystem. In contrast to the MSSM where exact solutions of the mass spectrum and mixing parameters can be constructed mathematically in closed form (see e.g. Ref. [12]), this is not possible anymore for the supersymmetric $U(1)_X$ model in which the eigenvalue equation for the masses is a 6th order polynomial equation. However, analogously to the NMSSM [13], if the mass scales of the supersymmetric particles are set by higgsino and gaugino parameters of the order the supersymmetry (SUSY)-breaking scale, $M_{SUSY} \sim \mathcal{O}(10^3 \text{ GeV})$, while the interaction between the new singlet and the MSSM fields is of the order of the electroweak scale, $v \sim \mathcal{O}(10^2 \text{ GeV})$, then the perturbative expansion of the solution in v/M_{SUSY} provides an excellent approximation to the mass spectrum and yields a good understanding of the main features of the mixing matrix.

Once the masses and mixings are determined, the couplings of the neutralinos to the electroweak gauge bosons and to the scalar/fermionic matter particles are fixed. Decay widths and production rates can subsequently be predicted for squark cascades at the LHC [14] and pair production in e^+e^- collisions at linear colliders [15]. Of particular interest are the radiative transitions between neutralinos in cross-over zones, where the masses of two neutralinos are nearly degenerate.

The report is organized as follows. In Sect. 2 we first describe the general basis of the neutralino sector in the USSM. Subsequently, for the naturally expected weak coupling between the MSSM and the new subsystem, the properties of the new higgsino and gaugino are derived in Sect. 3. It is shown to what extent the properties of the standard neutralinos are modified. The spectrum and the mixings are determined analytically in a weak-coupling perturbative expansion. The neutralino masses are determined to second-order, whereas the mixing matrix elements are determined to first-order in the weak coupling. The accuracy of the perturbative results will become apparent by comparing the analytic approximations with the numerical solutions, thereby demonstrating that a satisfactory understanding of the system can be achieved. As an illustration we will study the limit in which the gaugino mass parameters are significantly larger than the higgsino mass parameters, where both sets of parameters are assumed to be much larger than the electroweak scale. A general description of the neutralino couplings and decay widths is given in Sect. 4, including photon transitions. We also discuss production cross sections in e^+e^- collisions and cascade decay chains of squarks at the LHC that involve neutralinos. Section 5 summarizes and concludes this report. Technical details of the analytical diagonalization procedures for the 6×6 neutralino mass matrix for non-degenerate and degenerate levels are given in three appendices.

2 The USSM Neutralino Sector

2.1 Supersymmetric kinetic mixing

In a theory with two U(1) gauge symmetries, the two sectors can mix, consistently with all gauge symmetries, through the coupling of the kinetic parts of the two gauge bosons [16]. In the basis in which the couplings between matter and gauge fields have the canonical minimal-interaction form, the pure gauge part of the Lagrangian for the $U(1)_Y \times U(1)_X$ theory can be written

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} Y^{\mu\nu} Y_{\mu\nu} - \frac{1}{4} X^{\mu\nu} X_{\mu\nu} - \frac{\sin\chi}{2} Y^{\mu\nu} X_{\mu\nu}, \qquad (2.1)$$

where the parameter $\sin\chi$ is introduced to characterize the gauge kinetic mixing [17]. This Lagrangian generalizes to

$$\mathcal{L}_{\text{gauge}} = \frac{1}{32} \int d^2\theta \left\{ \hat{W}_Y \hat{W}_Y + \hat{W}_X \hat{W}_X + 2\sin\chi \, \hat{W}_Y \hat{W}_X \right\}, \qquad (2.2)$$

in a supersymmetric theory, where $\hat{W}_Y$ and $\hat{W}_X$ are the chiral superfields associated with the two gauge symmetries.[1]

The gauge/gaugino part of the Lagrangian can be converted back to the canonical form by the following GL(2,$\mathbb{R}$) transformation of the superfields [16, 17, 19]:

$$\begin{pmatrix} \hat{W}_Y \\ \hat{W}_X \end{pmatrix} = \begin{pmatrix} 1 & -\tan\chi \\ 0 & 1/\cos\chi \end{pmatrix} \begin{pmatrix} \hat{W}_B \\ \hat{W}_{B'} \end{pmatrix}, \qquad (2.3)$$

which acts on the gauge boson and gaugino components of the chiral superfields in the same form. The transformation alters the $U(1)_Y \times U(1)_X$ part of the covariant derivative to

$$D_\mu = \partial_\mu + i g_Y Y B_\mu + i \left(-g_Y Y \tan\chi + \frac{g_X}{\cos\chi} Q_X \right) B'_\mu \qquad (2.4)$$

$$= \partial_\mu + i g_Y Y B_\mu + i g_X Q'_X B'_\mu. \qquad (2.5)$$

The choice of the kinetic mixing matrix in the form given by Eq. (2.3) is motivated by the fact that the hypercharge sector of the Standard Model is left unaltered by this transformation, and the new effects are separated in the X sector (see, e.g., Ref. [20] for an alternative choice). Consequently, the effective $U(1)_X$ charge is shifted from its original value Q_X to

$$Q'_X = \frac{Q_X}{\cos\chi} - \frac{g_Y}{g_X} Y \tan\chi. \qquad (2.6)$$

Specifically, the $U(1)_X$ charge of any field is shifted by an amount proportional to their hypercharge Y and the mixing parameter $\sin\chi$. Thus, as a result of the kinetic mixing, new interactions among the

---

[1]The normalization of the superfield $\hat{W} = \overline{D}^2 D \hat{V}$ follows the conventions of Ref. [18], where $\hat{V}$ is the corresponding vector superfield.

gauge bosons and matter fields are generated even for matter fields with zero $U(1)_X$ charge originally.

In grand unification theories the two $U(1)$ groups are orthogonal at the unification scale but small mixing [16] can be induced through loop effects when the theory evolves down to the electroweak scale. In string theories, kinetic mixing can be induced at the tree level [19]; however, such mixing effects must remain small in order to guarantee the general agreement between SM analyses and precision data in a natural way [21].

2.2   The USSM neutralino mass matrix

The Lagrangian of the neutralino system follows from the superpotential in Eq. (1.1), complemented by the gaugino $SU(2)_L$, $U(1)_Y$ and $U(1)_X$ mass terms of the soft-supersymmetry breaking electroweak Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{mass}}^{\mathrm{gaugino}} &= -\frac{1}{2}M_2\widetilde{W}^a\widetilde{W}^a - \frac{1}{2}M_Y\tilde{Y}\tilde{Y} - \frac{1}{2}M_X\tilde{X}\tilde{X} - M_{YX}\tilde{Y}\tilde{X} + \mathrm{h.c.} \\
&= -\frac{1}{2}M_2\widetilde{W}^a\widetilde{W}^a - \frac{1}{2}M_1\tilde{B}\tilde{B} - \frac{1}{2}M_1'\tilde{B}'\tilde{B}' - M_K\tilde{B}\tilde{B}' + \mathrm{h.c.}\,,
\end{aligned} \tag{2.7}
$$

where the $\widetilde{W}^a$ $(a = 1, 2, 3)$, $\tilde{Y}$ and $\tilde{X}$ are the (two-component) $SU(2)_L$, $U(1)_Y$ and $U(1)_X$ gaugino fields, and

$$
M_1 \equiv M_Y\,, \qquad M_1' \equiv \frac{M_X}{\cos^2\chi} - \frac{2\sin\chi}{\cos^2\chi}M_{YX} + M_Y\tan^2\chi\,, \qquad M_K \equiv \frac{M_{YX}}{\cos\chi} - M_Y\tan\chi\,. \tag{2.8}
$$

In parallel to the gauge kinetic mixing discussed in Sect. 2.1, the Abelian gaugino mixing mass parameter M_{YX} is assumed small compared with the mass scales of the gaugino and higgsino fields.

After breaking the electroweak and $U(1)_X$ symmetries spontaneously due to non-zero vacuum expectation values of the iso-doublet and the iso-singlet Higgs fields,

$$
\langle H_u\rangle = \frac{\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}\,, \qquad \langle H_d\rangle = \frac{\cos\beta}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix}\,, \qquad \langle S\rangle = \frac{1}{\sqrt{2}}v_s\,, \tag{2.9}
$$

the doublet higgsino mass and the doublet higgsino-singlet higgsino mixing parameters,

$$
\mu \equiv \lambda\frac{v_s}{\sqrt{2}} \quad \text{and} \quad \mu_\lambda \equiv \lambda\frac{v}{\sqrt{2}}\,, \tag{2.10}
$$

are generated. The USSM neutral gaugino-higgsino mass matrix can be written in the following block matrix form[2],

$$
\mathcal{M}_6 = \begin{pmatrix} \mathcal{M}_4 & X \\ X^T & \mathcal{M}_2 \end{pmatrix}\,, \tag{2.11}
$$

---

[2]Although our initial exploratory analysis is carried out at tree-level, loop corrections can easily be included following the procedures of Ref. [22].

where $\mathcal{M}_4$ is the neutral gaugino-higgsino mass matrix of the MSSM, $\mathcal{M}_2$ corresponds to the new sector containing the singlet higgsino (singlino) and the new U(1)-gaugino $\tilde{B}'$ that is orthogonal to the bino $\tilde{B}$, and X describes the coupling of the two sectors via the neutralino mass matrix. More explicitly, in a basis of two-component spinor fields $\xi \equiv (\tilde{B}, \tilde{W}^3, \tilde{H}_d^0, \tilde{H}_u^0, \tilde{S}, \tilde{B}')^T$, the full neutralino mass matrix is given by [10]:

$$
\mathcal{M}_6 = \left(
\begin{array}{cccc|cc}
M_1 & 0 & -m_Z\,c_\beta\,s_W & m_Z\,s_\beta\,s_W & 0 & M_K \\
0 & M_2 & m_Z\,c_\beta\,c_W & -m_Z\,s_\beta\,c_W & 0 & 0 \\
-m_Z\,c_\beta\,s_W & m_Z\,c_\beta\,c_W & 0 & -\mu & -\mu_\lambda\,s_\beta & Q'_1 m_v\,c_\beta \\
m_Z\,s_\beta\,s_W & -m_Z\,s_\beta\,c_W & -\mu & 0 & -\mu_\lambda\,c_\beta & Q'_2 m_v\,s_\beta \\
\hline
0 & 0 & -\mu_\lambda\,s_\beta & -\mu_\lambda\,c_\beta & 0 & Q'_S m_s \\
M_K & 0 & Q'_1 m_v\,c_\beta & Q'_2 m_v\,s_\beta & Q'_S m_s & M'_1
\end{array}
\right) , \qquad (2.12)
$$

where the various gaugino mass parameters M_1, M_2, M'_1 and M_K have been defined in Eqs. (2.7) and (2.8). Notice the absence of a diagonal mass parameter of the new singlino in contrast to the NMSSM where the cubic self-interaction generates this singlet mass term [13]. Two additional mass mixing parameters,

$$
m_v \equiv g_X v \quad \text{and} \quad m_s \equiv g_X v_s \,, \qquad (2.13)
$$

are generated after gauge symmetry breaking and the effective charges Q'_1, Q'_2 and Q'_S are defined by

$$
Q'_1 \equiv \frac{Q_1}{\cos\chi} + \frac{1}{2}\frac{g_Y}{g_X}\tan\chi, \quad Q'_2 \equiv \frac{Q_2}{\cos\chi} - \frac{1}{2}\frac{g_Y}{g_X}\tan\chi, \quad Q'_S \equiv \frac{Q_S}{\cos\chi} \,, \qquad (2.14)
$$

in terms of the Q_i defined below Eq. (1.1). As usual, $\tan\beta \equiv v_2/v_1$ is the ratio of the vacuum expectation values of the two neutral SU(2) Higgs doublet fields, $s_\beta \equiv \sin\beta$, $c_\beta \equiv \cos\beta$, and s_W, c_W are the sine and cosine of the electroweak mixing angle θ_W.

In general, the neutralino mass matrix $\mathcal{M}_6$ is a complex symmetric matrix. To diagonalize this matrix, we introduce a unitary matrix N^6 such that

$$
\tilde{\chi}_k^0 = N_{k\ell}^6 \,(\tilde{B}, \tilde{W}^3, \tilde{H}_d, \tilde{H}_u, \tilde{S}, \tilde{B}')_\ell \,, \qquad (2.15)
$$

where the physical neutralino states are ordered by some convention. A typical choice, motivated by experimental analyses, is the ordering of $\tilde{\chi}_k^0$ [$k = 1, .., 6$] according to ascending mass values. As an intermediate step, we shall often refer to an auxiliary convention, in which the ordering of states $\tilde{\chi}_{k'}^0$, denoted by primed subscripts, follows the order of the original $(\tilde{B}, \tilde{W}^3, \tilde{H}_d^0, \tilde{H}_u^0, \tilde{S}, \tilde{B}')$ basis.

Given the neutralino mass matrix $\mathcal{M}_6$, the physical neutralino masses m_k^{ph}, which are real non-negative numbers, and the neutralino mixing matrix elements $N_{k\ell}^6$ can be calculated. The mass term in the Lagrangian is given by:

$$
-\mathcal{L}_{\text{mass}} = \tfrac{1}{2}\,\xi^T \mathcal{M}_6\,\xi + \text{h.c.} = \tfrac{1}{2}\sum_{k=1}^{6} m_k^{ph}\,\tilde{\chi}_k^0 \tilde{\chi}_k^0 + \text{h.c.} \,, \qquad (2.16)
$$

5

The transformation of the two-component fields generates the diagonalized mass matrix for the physical neutralino states,

$$(N^6)^* \, \mathcal{M}_6 \, (N^6)^\dagger = \mathrm{diag}(m_1^{ph}, \, m_2^{ph}, \, \ldots, \, m_6^{ph}), \qquad m_k^{ph} \geq 0 \,. \qquad (2.17)$$

Mathematically, this transformation is the Takagi diagonalization [23–27] of a general complex symmetric matrix; see Appendix A for further details. Physically, the unitary matrix N^6 determines the couplings of the mass-eigenstates $\tilde{\chi}_k^0$ to other particles.

If $\mathcal{M}_6$ is complex, then CP is violated in the neutralino sector of the theory if no diagonal matrix of phases P exists such that $P^T \mathcal{M}_6 P$ is real. If P exists, then the neutralino interaction-eigenstates can be rephased to produce a real neutralino mass matrix, and the neutralino sector is CP-conserving.[3] If $\mathcal{M}_6$ is real, then the Takagi diagonalization of Eq. (2.17) still applies but can easily be carried out in two steps. First the real symmetric matrix $\mathcal{M}_6$ can be diagonalized by an orthogonal matrix V^6:

$$V^6 \, \mathcal{M}_6 \, (V^6)^T = \mathrm{diag}(m_1, \, m_2, \, \ldots, \, m_6) \,, \qquad (2.18)$$

where the eigenvalues m_k are real but not necessarily positive. The Takagi diagonalization of $\mathcal{M}_6$, which yields real non-negative diagonal mass elements, can then be achieved in a second step by taking $m_k^{ph} = |m_k|$ and defining the unitary matrix N^6 in Eq. (2.17) by $N^6 = (P^6 V^6)^*$, where P^6 is a diagonal phase matrix with elements $P_{k\ell}^6 = \varepsilon_k^{1/2} \delta_{k\ell}$. Here, $\varepsilon_k \equiv m_k/m_k^{ph} = \pm 1$ is the sign of m_k, which is also proportional to the CP-quantum number [28] of the neutralino $\tilde{\chi}_k^0$. More precisely, the *relative* CP-quantum numbers of $\tilde{\chi}_k^0$ and $\tilde{\chi}_\ell^0$, which is the physical quantity of interest, is given by $\varepsilon_k \varepsilon_\ell$.

Although the ordering of states $\tilde{\chi}_k^0$ in ascending mass values is convenient, it is often useful to adopt an intermediate auxiliary convention. Note that the neutralino mass matrix is easily diagonalized in the limit of $M_K = v = 0$ (i.e., before the coupling of the MSSM with the new gaugino/singlino block is introduced). In this limit, $\mathcal{M}_6$ is real after rephasing the neutralino interaction-eigenstates (if necessary). That is, without loss of generality, we can choose M_1, M_1', M_2 and μ to be real in this limit, in which case Eq. (2.18) yields the following mass eigenvalues: $m_{k'} = \{M_1, M_2, \mu, -\mu, m_{5'}, m_{6'}\}$, where $m_{5',6'} = \frac{1}{2} M_1' \left[1 \mp \sqrt{1 + (2Q_S' m_s/M_1')^2} \right]$ (with $m_{5'} < m_{6'}$). Away from this limit, the mass-eigenstates $\tilde{\chi}_{k'}^0$ will be defined such that their masses are continuously connected to the masses of the corresponding states in the $M_K = v = 0$ limit. This defines an alternative ordering of the states $\tilde{\chi}_{k'}^0$ which will be indicated with primed subscripts.

We shall present a set of techniques for computing analytic approximations of the physical neutralino masses, $m_{k'}^{ph}$ and the corresponding neutralino mixing matrix elements $N_{k'\ell'}^6$. As previously indicated,

---

[3]In this context, the neutralino sector refers to the neutralino kinetic energy and mass terms, plus terms that couple the neutralinos to the gauge bosons. In this restricted sector, the neutralinos would be states of definite CP quantum number. Of course, it is possible to introduce CP-violating interactions through the neutralino couplings to other particles, e.g. matter particles of the USSM. In this case, radiative corrections could transmit these effects into the neutralino mass matrix.

the primed subscripts denote that these quantities refer to the physical neutralino states $\tilde{\chi}^0_{k'}$, whose ordering is specified above. Of course, at the end of the computation, one can convert to an ascending mass ordering convention by an appropriate relabeling of the states, masses and mixing matrix elements.

3 Small Mixing Scenarios

3.1 General analysis

It is well known that the MSSM neutralino mass matrix $\mathcal{M}_4$ can be diagonalized analytically (see, e.g., Ref. [13]). In contrast, the diagonalization of the new USSM 6×6 neutralino mass matrix $\mathcal{M}_6$ cannot be performed analytically in closed form. However, the case of physical interest is one in which both the couplings of the MSSM higgsino doublets to the singlet higgsino and to the $\mathrm{U}(1)_X$ gaugino, and the coupling of the $\mathrm{U}(1)_Y$ and $\mathrm{U}(1)_X$ gaugino singlets are weak, i.e. the elements of the 4×2 submatrix X in Eq. (2.11) are small. Then, an approximate analytical solution can be found following the procedure given in Appendix B.

As an initial step, the 4×4 MSSM submatrix $\mathcal{M}_4$ and the new 2×2 singlino-$\mathrm{U}(1)_X$ gaugino submatrix $\mathcal{M}_2$ are separately diagonalized:

$$\overline{\mathcal{M}}^D_4 = N^{4*}\mathcal{M}_4 N^{4\dagger} = \mathrm{diag}(\overline{m}_{1'}, \overline{m}_{2'}, \overline{m}_{3'}, \overline{m}_{4'}),\tag{3.1}$$

$$\overline{\mathcal{M}}^D_2 = N^{2*}\mathcal{M}_2 N^{2\dagger} = \mathrm{diag}(\overline{m}_{5'}, \overline{m}_{6'}),\tag{3.2}$$

where the $\overline{m}_{k'}$ are real and non-negative. Here we use primed subscripts to indicate that the neutralino states are continuously connected to the corresponding states in the $M_K = v = 0$ limit, as discussed at the end of Sect. 2. The above procedure results in a partial Takagi diagonalization of the full neutralino mass matrix, $\mathcal{M}_6$:

$$\overline{\mathcal{M}}_6 \equiv \begin{pmatrix} N^{4*} & \mathbb{0} \\ \mathbb{0}^T & N^{2*} \end{pmatrix} \begin{pmatrix} \mathcal{M}_4 & X \\ X^T & \mathcal{M}_2 \end{pmatrix} \begin{pmatrix} N^{4\dagger} & \mathbb{0} \\ \mathbb{0}^T & N^{2\dagger} \end{pmatrix} = \begin{pmatrix} \overline{\mathcal{M}}^D_4 & N^{4*}X N^{2\dagger} \\ N^{2*}X^T N^{4\dagger} & \overline{\mathcal{M}}^D_2 \end{pmatrix}.\tag{3.3}$$

where $\mathbb{0}$ is a 4×2 matrix of zeros. The upper left and lower right blocks of $\overline{\mathcal{M}}_6$ are diagonal with real non-negative entries, but the upper right and lower left off-diagonal blocks are non-zero.

Performing a block-diagonalization of $\overline{\mathcal{M}}_6$ will remove the non-zero off-diagonal blocks while leaving the diagonal blocks approximately diagonal up to second order, due to the weak coupling of the two subsystems. That is,

$$\mathcal{M}^D_6 = \overline{N}^{6*}_B \overline{\mathcal{M}}_6 \overline{N}^{6\dagger}_B = \mathrm{diag}(m^{ph}_{1'}, m^{ph}_{2'}, m^{ph}_{3'}, m^{ph}_{4'}, m^{ph}_{5'}, m^{ph}_{6'}),\tag{3.4}$$

where

$$\overline{N}^6_B \simeq \begin{pmatrix} \mathbb{1}_{4\times 4} - \frac{1}{2}\Omega\Omega^\dagger & \Omega \\ -\Omega^\dagger & \mathbb{1}_{2\times 2} - \frac{1}{2}\Omega^\dagger\Omega \end{pmatrix} \times \mathrm{diag}(e^{-i\phi_{1'}}, \dots, e^{-i\phi_{6'}}).\tag{3.5}$$

7

A detailed derivation will be presented in Appendix B. The elements of the 4×2 mixing matrix Ω are given by:

$$\text{Re}\,\Omega_{i'j'} \equiv \frac{\text{Re}\,(N^{4*}XN^{2\dagger})_{i'j'}}{\overline{m}_{i'} - \overline{m}_{j'}}, \qquad\qquad \text{Im}\,\Omega_{i'j'} \equiv \frac{\text{Im}\,(N^{4*}XN^{2\dagger})_{i'j'}}{\overline{m}_{i'} + \overline{m}_{j'}}, \qquad (3.6)$$

with $i' = 1', \ldots, 4'$ and $j' = 5', 6'$. After the block-diagonalization, the upper left 4×4 and the lower right 2×2 blocks need not be re-diagonalized up to second order in the small mixing X between the blocks, but the corresponding eigenvalues are shifted $\overline{m}_{k'} \to m_{k'}^{ph}$ to second order in the small mixing. The physical neutralino masses $m_{k'}^{ph}$ are given by:

$$m_{i'}^{ph} \simeq \overline{m}_{i'} + \sum_{j'=5}^{6} \left\{ \frac{[\,\text{Re}\,(N^{4*}XN^{2\dagger})_{i'j'}]^2}{\overline{m}_{i'} - \overline{m}_{j'}} + \frac{[\,\text{Im}\,(N^{4*}XN^{2\dagger})_{i'j'}]^2}{\overline{m}_{i'} + \overline{m}_{j'}} \right\}, \qquad [i' = 1', \ldots, 4'], \quad (3.7)$$

$$m_{j'}^{ph} \simeq \overline{m}_{j'} - \sum_{i'=1}^{4} \left\{ \frac{[\,\text{Re}\,(N^{4*}XN^{2\dagger})_{i'j'}]^2}{\overline{m}_{i'} - \overline{m}_{j'}} - \frac{[\,\text{Im}\,(N^{4*}XN^{2\dagger})_{i'j'}]^2}{\overline{m}_{i'} + \overline{m}_{j'}} \right\}, \qquad [j' = 5', 6'], \qquad (3.8)$$

The diagonal matrix of phases is chosen such that the $m_{k'}^{ph}$ are real and non-negative, with the phases $\phi_{k'}$ given by:

$$\phi_{i'} \simeq - \sum_{j'=5}^{6} \frac{\overline{m}_{j'}}{\overline{m}_{i'}(\overline{m}_{i'}^2 - \overline{m}_{j'}^2)} \,\text{Re}\,(N^{4*}XN^{2\dagger})_{i'j'} \,\text{Im}\,(N^{4*}XN^{2\dagger})_{i'j'}, \qquad [i' = 1', \ldots, 4'], \quad (3.9)$$

$$\phi_{j'} \simeq \sum_{i'=1}^{4} \frac{\overline{m}_{i'}}{\overline{m}_{j'}(\overline{m}_{i'}^2 - \overline{m}_{j'}^2)} \,\text{Re}\,(N^{4*}XN^{2\dagger})_{i'j'} \,\text{Im}\,(N^{4*}XN^{2\dagger})_{i'j'}, \qquad [j' = 5', 6']. \qquad (3.10)$$

The (perturbative) Takagi diagonalization of the neutralino mass matrix $\mathcal{M}_6$ has now been achieved, with the (real and non-negative) neutralino masses given by Eqs. (3.7) and (3.8), and the neutralino mixing matrix given by:

$$N^6 = \overline{N}_B^6 \begin{pmatrix} N^4 & \mathbb{O} \\ \mathbb{O}^{\mathbf{T}} & N^2 \end{pmatrix}. \qquad (3.11)$$

The validity of the perturbative expansion relies on the assumption that[4]

$$\left| \frac{\text{Re}\,(N^{4*}XN^{2\dagger})_{i'j'}}{\overline{m}_{i'} - \overline{m}_{j'}} \right| \ll 1, \qquad (3.12)$$

for all choices of $i' = 1', \ldots, 4'$ and $j' = 5', 6'$. That is, only degeneracies between the 4×4 block $\overline{\mathcal{M}}_4^D$ and the 2×2 block $\overline{\mathcal{M}}_2^D$ are potentially problematic. In particular, in the so-called cross-over zones in which the masses $\overline{m}_{i'} \simeq \overline{m}_{j'}$ exhibit a near degeneracy and the corresponding residue $\text{Re}\,(N^{4*}XN^{2\dagger})_{i'j'} \neq 0$, mixing effects are enhanced and the analytical formalism in Appendix C must be applied.

---

[4]Since the $\overline{m}_{k'}$ are non-negative, and by definition of order M_{SUSY}, the conditions $|\,\text{Im}\,(N^{4*}XN^{2\dagger})_{i'j'}/(\overline{m}_{i'} + \overline{m}_{j'})| \ll 1$ are automatically satisfied.

3.2 The case of a real neutralino mass matrix

We shall present numerical case studies under the assumption that the parameters of the neutralino mass matrix are real. The general analysis then simplifies, since a real symmetric mass matrix can always be diagonalized by a similarity transformation, $V\mathcal{M}V^T$, where V is real and orthogonal. Since some of the mass eigenvalues of a real symmetric matrix may be negative, we complete the Takagi diagonalization, $N^*\mathcal{M}N^\dagger$, by introducing a suitable diagonal matrix of phases P and identifying the unitary neutralino mixing matrix by $N = (PV)^*$, as indicated below Eq. (2.18). In this case, the (perturbative) neutralino mass matrix diagonalization can be performed using the three-step procedure of Ref. [13]:

[1] *Diagonalization of the submatrices $\mathcal{M}_4$ and $\mathcal{M}_2$*
In the first step, we diagonalize the (real symmetric) MSSM matrix $\mathcal{M}_4$:

$$\widetilde{\mathcal{M}}_4^D = V^4 \mathcal{M}_4 (V^4)^T = \mathrm{diag}(\tilde{m}_{1'}, \tilde{m}_{2'}, \tilde{m}_{3'}, \tilde{m}_{4'}) \,. \tag{3.13}$$

The mass eigenvalues, which are real but need not be non-negative, are denoted by $\tilde{m}_{i'}$ for $i' = 1', .., 4'$. The orthogonal diagonalization matrix V^4 is given explicitly in Ref. [12] for the most general choice of gaugino and higgsino mass parameters. Simple analytic forms for the neutralino mass and mixing matrix elements can be found in limits where either the gaugino parameters are much larger than the higgsino parameter or vice versa [29].

The exact analytic diagonalization of the new 2×2 submatrix $\mathcal{M}_2$ singlet higgsino-U$(1)_X$ gaugino submatrix $\mathcal{M}_2$ is straightforward. The matrix:

$$\mathcal{M}_2 = \begin{pmatrix} 0 & Q'_S m_s \\ Q'_S m_s & M'_1 \end{pmatrix} \tag{3.14}$$

is diagonalized by an orthogonal rotation V^2 as

$$\widetilde{\mathcal{M}}_2^D = V^2 \mathcal{M} (V^2)^T = \mathrm{diag}\left(\tilde{m}_{5'}, \tilde{m}_{6'}\right) \,. \tag{3.15}$$

The eigenvalues $\tilde{m}_{5',6'}$ are given by

$$\tilde{m}_{5',6'} = \frac{M'_1}{2} \left(1 \mp \sqrt{1 + (2Q'_S m_s / M'_1)^2} \right) \,. \tag{3.16}$$

The orthogonal diagonalization matrix V^2 is given by:

$$V^2 = \begin{pmatrix} \cos\theta_s & -\sin\theta_s \\ \sin\theta_s & \cos\theta_s \end{pmatrix} \,, \tag{3.17}$$

where the angle θ_s satisfies the relations:

$$\cos\theta_s = \frac{\left(\sqrt{1+x^2}+1\right)^{1/2}}{\sqrt{2}\,(1+x^2)^{1/4}} \qquad \text{and} \qquad \sin\theta_s = \mathrm{sign}(x)\frac{\left(\sqrt{1+x^2}-1\right)^{1/2}}{\sqrt{2}\,(1+x^2)^{1/4}} \,, \tag{3.18}$$

9

with $x \equiv 2Q'_S m_s / M'_1$.

Two limits are of particular interest:

(i) If $m_s \gg |M'_1|$, then the masses and the mixing parameters are approximately given by

$$\tilde{m}_{5'} \simeq -|Q'_S| m_s , \qquad \tilde{m}_{6'} \simeq |Q'_S| m_s , \quad \text{and} \quad \sin\theta_s \simeq \text{sign}(x)/\sqrt{2} , \qquad (3.19)$$

corresponding to maximal mixing due to the large off-diagonal entries in the mass matrix $\mathcal{M}_2$.

(ii) In the opposite limit, $|M'_1| \gg m_s$, and the mass eigenvalues and mixing angle are approximately given by

$$\tilde{m}_{5'} \simeq -Q'^2_S m_s^2 / M'_1 , \quad \tilde{m}_{6'} \simeq M'_1 + Q'^2_S m_s^2 / M'_1 , \quad \text{and} \quad \sin\theta_s \simeq Q'_S m_s / M'_1 . \qquad (3.20)$$

This is a typical see-saw type mixing phenomenon. The heavy 6th state is a $U(1)_X$ gaugino-dominated state, whereas the 5th neutralino state is a singlet-higgsino dominated state.

[2] *Block-diagonalization of $\mathcal{M}_6$*

We can now perform a block-diagonalization of $\mathcal{M}_6$:

$$V^6 \mathcal{M}^6 (V^6)^T = \widetilde{V}^6_B \begin{pmatrix} \mathcal{M}'^D_4 & V^4 X V^{2T} \\ V^2 X^T V^{4T} & \mathcal{M}'^D_2 \end{pmatrix} \widetilde{V}^{6T}_B = \text{diag}(m_{1'}, \dots, m_{4'}, m_{5'}, m_{6'}), \quad (3.21)$$

where

$$\widetilde{V}^6_B \simeq \begin{pmatrix} \mathbb{1}_{4\times 4} - \frac{1}{2}\Omega\Omega^T & \Omega \\ -\Omega^T & \mathbb{1}_{2\times 2} - \frac{1}{2}\Omega^T\Omega \end{pmatrix} , \qquad (3.22)$$

and the elements of the real matrix Ω are given by [cf. Eq. (3.6)]:

$$\Omega_{i'j'} \equiv \frac{(V^4 X V^{2T})_{i'j'}}{\tilde{m}_{i'} - \tilde{m}_{j'}} , \qquad (3.23)$$

with $i' = 1', .., 4'$ and $j' = 5', 6'$. That is, the orthogonal matrix V^6 is conveniently split into the matrices V^4 and V^2 that diagonalize the 4×4 and 2×2 submatrices $\mathcal{M}_4$ and $\mathcal{M}_2$ respectively, and into the matrix $\widetilde{V}^6_B$ that performs the subsequent block-diagonalization [13]:

$$V^6 \simeq \widetilde{V}^6_B \begin{pmatrix} V^4 & \mathbb{O} \\ \mathbb{O}^T & V^2 \end{pmatrix} . \qquad (3.24)$$

After the block-diagonalization, the mass eigenvalues are shifted to second order in the perturbation X. The shifts are given by [cf. Eq. (3.7) and (3.8)]:

$$m_{i'} = \tilde{m}_{i'} + \sum_{j'=5'}^{6'} \frac{[(V^4 X V^{2T})_{i'j'}]^2}{\tilde{m}_{i'} - \tilde{m}_{j'}} , \qquad [i' = 1', .., 4'] , \qquad (3.25)$$

$$m_{j'} = \tilde{m}_{j'} - \sum_{i'=1'}^{4'} \frac{[(V^4 X V^{2T})_{i'j'}]^2}{\tilde{m}_{i'} - \tilde{m}_{j'}} , \qquad [j' = 5', 6'] . \qquad (3.26)$$

As expected, the eigenvalues fulfill the trace formula

$$\sum_{k'=1'}^{6'} m_{k'} = M_1 + M_2 + M_1',$$

(3.27)

which is independent of the higgsino mass and the mixing parameters.

The perturbative results obtained above are valid if $|(V^4 X V^{2\,T})_{i'j'}/(\tilde{m}_{i'} - \tilde{m}_{j'})| \ll 1$ for all possible choices of i' and j'. In the regime of near-degeneracy, $\tilde{m}_{i'} \simeq \tilde{m}_{j'}$, the perturbation theory breaks down, and the analytic approach of Appendix C must be employed. Note that $\tilde{m}_{i'} = -\tilde{m}_{j'}$ is *not* a case of mass-eigenvalue degeneracy, so that the perturbative results obtained above should be reliable. This may seem to be in conflict with results of the previous subsection, since the latter corresponds to the *degenerate* case of $\overline{m}_{i'} = \overline{m}_{j'}$, where we identify the *positive* masses $\overline{m}_{k'} = |\tilde{m}_{k'}|$ in the notation of Sect. 3.1. However, a more careful analysis reveals that the condition given by Eq. (3.12) does *not* apply, since in the case of opposite sign mass eigenvalues, the residue $\mathrm{Re}\,(N^{4\,*} X N^{2\,\dagger})_{i'j'} = \mathrm{Re}\,(iV^4 X V^{2\,T})_{i'j'} = 0.$[5]

The higgsino doublet-singlet and the higgsino doublet-U$(1)_X$ gaugino mixings generate additional singlino and U$(1)_X$ gaugino components in the wave functions of the original MSSM neutralinos $\tilde{\chi}_{i'}^0$ $[\,i' = 1',..,4'\,]$ of the size

$$V_{i'j'}^6 \approx \sum_{k'=5'}^{6'} \Omega_{i'k'} V_{k'j'}^2 \quad [i' = 1',..,4'; j' = 5',6']$$

(3.28)

which is linear in the mixing parameter to first approximation as expected for off-diagonal elements. Reciprocally, the MSSM gaugino/higgsino components and the singlino and U$(1)_X$ gaugino components in the wave functions of $\tilde{\chi}_{5'}^0$ and $\tilde{\chi}_{6'}^0$ are reduced to

$$V_{j'i'}^6 \approx -\sum_{l'=1'}^{4'} \Omega_{l'j'} V_{l'i'}^4 \qquad [i' = 1',..,4';\, j' = 5',6']$$

$$V_{j'k'}^6 \approx V_{j'k'}^2 - \frac{1}{2}\left(\Omega^T \Omega V^2\right)_{j'k'} \qquad [j',k' = 5',6']$$

(3.29)

with $V_{j'k'}^6$ differing from $V_{j'k'}^2$ only to second order in the mixing, as expected for diagonal elements.

[3] *Ensuring that the physical neutralino masses are non-negative*

The diagonalization of a real symmetric matrix by an orthogonal similarity transformation produces a diagonal matrix with real but not necessarily non-negative elements. Hence, some of the eigenvalues $m_{k'}$ will typically be negative. Defining the unitary matrix $N^6 = (P^6 V^6)^*$, where P^6 is a diagonal matrix whose $k'k'$ element is 1 [i] if $m_{k'}$ is non-negative [negative], the Takagi diagonalization of the neutralino mass matrix is achieved with non-negative neutralino masses. In particular, the unitary neutralino mixing matrix $N^{6\,*}$ appears (instead of the real orthogonal matrix V^6) in the corresponding Feynman rules involving the neutralino mass-eigenstates.

---

[5]As in step **[3]** below, we identify $N^M = (P^M V^M)^*$ for $M = 2$ and 4, respectively, and $(P^4)_{i'i'}^{-1}(P^2)_{j'j'}^{-1} = -i$ for opposite sign mass eigenvalues $\tilde{m}_{i'}$ and $\tilde{m}_{j'}$.

3.3 Large gaugino mass parameters

To illustrate the previous general discussion we shall first give a detailed parametric analysis in the limit in which all gaugino masses are much larger than the higgsino masses, and both sets much larger than the electroweak and the kinetic mixing scales, i.e. $M_1, M_2, M_1' \gg \mu, v_s \gg v, M_K$. All neutralino mass matrix parameters will be taken real.

[1] Starting again with the diagonalization of the MSSM submatrix $\mathcal{M}_4$, the diagonalization matrix V^4 defined in Eq. (3.24) can be parameterized up to second order according to standard MSSM procedure (see, e.g., Ref. [12]), as

$$V^4 \simeq \begin{pmatrix} V_G & \mathbb{0} \\ \mathbb{0}^T & V_H \end{pmatrix} \begin{pmatrix} \mathbb{1}_{2\times2} & V_x \\ -V_x^T & \mathbb{1}_{2\times2} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{2\times2} & \mathbb{0} \\ \mathbb{0}^T & R_{\pi/4} \end{pmatrix}. \tag{3.30}$$

The effect of the 2×2 rotation $R_{\pi/4} \equiv (\mathbb{1} - i\tau_2)/\sqrt{2}$ [where $\vec{\tau} \equiv (\tau_1, \tau_2, \tau_3)$ are the 2×2 Pauli matrices] is to shift the $\{34\}$ off-diagonal elements $[-\mu, -\mu]$ onto the diagonal axis $[\mu, -\mu]$. The matrix, V_x,

$$V_x = \begin{pmatrix} -c_+ \, s_W \, m_Z/M_1 & -c_- \, s_W \, m_Z/M_1 \\ c_+ \, c_W \, m_Z/M_2 & c_- \, c_W \, m_Z/M_2 \end{pmatrix}, \tag{3.31}$$

with the abbreviations $c_\pm \equiv (c_\beta \pm s_\beta)/\sqrt{2}$, removes the mixing between the blocks of the two gaugino and the two higgsino states. V_G and V_H rescale the gaugino and higgsino blocks themselves:

$$V_G \approx \mathbb{1}_{2\times2} - \frac{1}{2}\begin{pmatrix} s_W^2 \, m_Z^2/M_1^2 & 0 \\ 0 & c_W^2 \, m_Z^2/M_2^2 \end{pmatrix},$$

$$V_H \approx \mathbb{1}_{2\times2} - \frac{1}{2}\begin{pmatrix} (1+s_{2\beta})\, M_{12}''^2 m_Z^2/2M_1^2 M_2^2 & 0 \\ 0 & (1-s_{2\beta})\, M_{12}''^2 m_Z^2/2M_1^2 M_2^2 \end{pmatrix}, \tag{3.32}$$

with $M_{12}''^2 \equiv M_1^2 c_W^2 + M_2^2 s_W^2$. V_G and V_H relate to a diagonal form of the gaugino-higgsino mass matrix for large $M_{1,2}$ and μ. Their off-diagonal matrix elements are of second order and can be omitted consistently as they would only affect the eigenvalues at fourth order.

The 2×2 diagonalization matrix defined in Eq. (3.24) can be parameterized up to second order as

$$V^2 \approx \begin{pmatrix} 1 - Q_S'^2 m_s^2/2M_1'^2 & -Q_S' m_s/M_1' \\ Q_S' m_s/M_1' & 1 - Q_S'^2 m_s^2/2M_1'^2 \end{pmatrix}. \tag{3.33}$$

The 2×2 matrix V^2 generates a diagonal form of the singlino-$\mathrm{U}(1)_X$ gaugino mass matrix for $M_1' \gg m_s$.

After these steps are performed, the 4×4 and 2×2 mass submatrices are diagonal and the complete

symmetric mass matrix $\mathcal{M}_6$ takes the intermediate form

$$
\begin{pmatrix} V^4 & \mathbb{O} \\ \mathbb{O}^T & V^2 \end{pmatrix} \mathcal{M}_6 \begin{pmatrix} V^{4T} & \mathbb{O} \\ \mathbb{O}^T & V^{2T} \end{pmatrix} \simeq \left(\begin{array}{cccc|cc} \tilde{m}_{1'} & & & & 0 & M_K \\ & \tilde{m}_{2'} & & & 0 & 0 \\ & & \tilde{m}_{3'} & & +\mu_\lambda c_- & Q'_- m_v \\ & & & \tilde{m}_{4'} & -\mu_\lambda c_+ & Q'_+ m_v \\ \hline 0 & 0 & +\mu_\lambda c_- & -\mu_\lambda c_+ & \tilde{m}_{5'} & \\ M_K & 0 & Q'_- m_v & Q'_+ m_v & & \tilde{m}_{6'} \end{array} \right), \quad (3.34)
$$

where, in obvious notation, zero elements of the diagonal blocks are suppressed for easier reading, and $Q'_\pm \equiv (Q'_1 c_\beta \pm Q'_2 s_\beta)/\sqrt{2}$. The diagonal elements $\tilde{m}_{k'}$ are given by

$$
\begin{aligned}
\tilde{m}_{1'} &= M_1 + \frac{m_Z^2}{M_1} s_W^2\,, & \tilde{m}_{3'} &= \mu - \frac{m_Z^2 M_{12}}{M_1 M_2} c_+^2\,, & \tilde{m}_{5'} &= \mu_\kappa\,, \\
\tilde{m}_{2'} &= M_2 + \frac{m_Z^2}{M_2} c_W^2\,, & \tilde{m}_{4'} &= -\mu - \frac{m_Z^2 M_{12}}{M_1 M_2} c_-^2\,, & \tilde{m}_{6'} &= M_1' - \mu_\kappa\,,
\end{aligned} \quad (3.35)
$$

where $c_\pm$ is defined below Eq. (3.31) and

$$
M_{12} \equiv M_1 c_W^2 + M_2 s_W^2\,, \qquad \mu_\kappa \equiv -Q_S'^2 m_s^2 / M_1'\,. \quad (3.36)
$$

The parameter μ_κ can be identified with the NMSSM-type singlino mass parameter [13]. Note that $\tilde{m}_{5'} = \mu_\kappa$ is small compared to all the other neutralino masses in the limit of large gaugino mass parameters considered in this subsection.

[2] The block-diagonalization of the 6-dimensional intermediate matrix [Eq. (3.34)] can be performed by choosing the proper form of Ω in V^6. In the limit of large gaugino mass parameters and small singlino mass $\mu_\kappa \ll \mu \ll M_1, M_2, M_1'$, the 4×2 mixing matrix Ω is reduced to the simple expression

$$
\Omega \approx \begin{pmatrix} 0 & M_K/(M_1 - M_1') \\ 0 & 0 \\ \mu_\lambda c_-/\mu & -Q'_- m_v/M_1' \\ \mu_\lambda c_+/\mu & -Q'_+ m_v/M_1' \end{pmatrix}. \quad (3.37)
$$

As a result of the block diagonalization of Eq. (3.34), the mass eigenvalues are shifted according to Eq. (3.25). The resulting mass eigenvalues to the desired order are given by:

$$
\begin{aligned}
m_{1'} &\approx M_1 + \frac{m_Z^2}{M_1} s_W^2 + \frac{M_K^2}{M_1 - M_1'}\,, & m_{4'} &\approx -\mu - \frac{m_Z^2 M_{12}}{M_1 M_2} c_-^2 - \frac{\mu_\lambda^2 c_+^2}{\mu} + \frac{Q_+'^2 m_v^2}{M_1'}\,, \\
m_{2'} &\approx M_2 + \frac{m_Z^2}{M_2} c_W^2\,, & m_{5'} &\approx \mu_\kappa + \frac{\mu_\lambda^2}{\mu} s_{2\beta}\,, \\
m_{3'} &\approx \mu - \frac{m_Z^2 M_{12}}{M_1 M_2} c_+^2 + \frac{\mu_\lambda^2 c_-^2}{\mu} + \frac{Q_+'^2 m_v^2}{M_1'}\,, & m_{6'} &\approx M_1' - \mu_\kappa + \frac{m_v^2(Q_+'^2 + Q_-'^2)}{M_1'} - \frac{M_K^2}{M_1 - M_1'}\,.
\end{aligned} \quad (3.38)
$$

Note that the sum rule given by Eq. (3.27) is satisfied.

As expected, while the large SU(2) gaugino mass $m_{2'}$ is not affected by the singlino and the U(1)$_X$ gaugino, the MSSM U(1) mass $m_{1'}$ is affected by the U(1) kinetic mixing. All the higgsino states are modified by the interactions between the MSSM and the new subsystem. The value of $m_{5'}$ is raised by the interaction with the MSSM higgsinos, but remains small nevertheless.

The mixing in the wave-functions is described by the components of Ω alone since the 4×4 matrix V^4 and the 2×2 matrix V^2 deviate from unity only to second order in the small parameters of the order of the SUSY scales $[i = 1', .., 4']$:

$$
\begin{aligned}
&V^6_{i'5'} \approx \frac{\mu_\lambda}{\mu} \left(0,\, 0,\, c_-,\, c_+\right)_{i'}, &\qquad &V^6_{5'5'} \approx 1 - \frac{Q_S'^2 m_s^2}{2 M_1'^2} - \frac{\mu_\lambda^2}{2\mu^2}, \\[2mm]
&V^6_{5'i'} \approx -\frac{\mu_\lambda}{\mu} \left(0,\, 0,\, c_\beta,\, s_\beta\right)_{i'}, &\qquad &V^6_{5'6'} \approx -\frac{Q_S' m_s}{M_1'}, \\[2mm]
&V^6_{i'6'} \approx \left(\frac{M_K}{M_1 - M_1'},\, 0,\, -\frac{Q_-' m_v}{M_1'},\, -\frac{Q_+' m_v}{M_1'} \right)_{i'}, &\qquad &V^6_{6'5'} \approx \frac{Q_S' m_s}{M_1'}, \\[2mm]
&V^6_{6'i'} \approx \left(\frac{-M_K}{M_1 - M_1'},\, 0,\, \frac{Q_1' m_v c_\beta}{M_1'},\, \frac{Q_2' m_v s_\beta}{M_1'} \right)_{i'}, &\qquad &V^6_{6'6'} \approx 1 - \frac{Q_S'^2 m_s^2}{2 M_1'^2} - \frac{m_v^2 (Q_+'^2 + Q_-'^2)}{2 M_1'^2} - \frac{M_K^2}{2(M_1 - M_1')^2}.
\end{aligned}
$$
(3.39)

The non-trivial mixing between two U(1) gaugino states, elements $\{1'6'\}$ and $\{6'1'\}$, is generated by the non-zero Abelian gauge kinetic and mass mixing with non-zero M_K. The analysis above fails when $M_1 \approx M_1'$; this region of near degeneracy can be handled analytically using the results of Appendix C.

In Eqs. (3.38) and (3.39), perturbative corrections up to second order have been included for the masses and diagonal mixing matrix elements, whereas only the first order corrections have been given for the off-diagonal mixing matrix elements. This follows the usual procedure of stationary perturbation theory in quantum mechanics, which associates second-order corrections to the eigenvalues with the first-order corrections to the wave function. Consequently, the zeros that appear in some of the matrix elements of $V^6_{k'\ell'}$, should be interpreted as approximate. For example, $V^6_{2'6'}$ and $V^6_{6'2'}$ are expected to receive higher order perturbative corrections and hence be shifted away from zero. Nevertheless, the fact that the magnitude of these matrix elements are so suppressed will have some dramatic consequences for the behavior of the $\tilde{\chi}^0_{2'}$ and $\tilde{\chi}^0_{6'}$ masses in regions of near-degeneracy.

[3] The final step is to identify $N^6 = (P^6 V^6)^*$, where P^6 is a diagonal matrix whose $k'k'$ element is 1 (i) if $m_{k'}$ is non-negative (negative). The physical masses m_k^{ph} are given by the absolute values of the m_k given above. The neutralino states can then be reordered in ascending (non-negative) mass if desired.

The results of this subsection are easily generalized for the case of M_1, M_2, M_1', μ, $v_s \gg v$, M_K. As long as the MSSM gaugino and higgsino parameters, $M_{1,2}$ and μ remain significantly larger than the electroweak scale v, the couplings between the MSSM and the new fields, generated by X, remain weak

and the diagonalization of the mass matrix can still be performed analytically. However, instead of the approximate values $\tilde{m}_{5',6'}$ in Eqs. (3.34) and (3.35) the exact solutions (3.16) must be used, and for V^2 the general rotation matrix (3.17) must be inserted. The approximation ceases to be valid at isolated points where $X/(\tilde{m}_{i'} - \tilde{m}_{j'})$ is no longer a small perturbation, due to the degeneracy of mass eigenvalues $\tilde{m}_{i'} \approx \tilde{m}_{j'}$. In these cross-over zones the analysis described in Appendix C must be applied.

3.4   An illustrative example

To illustrate the properties of the two new neutralinos and the impact of the coupling of the two subsystems on the original MSSM neutralinos, we study, numerically and analytically, the evolution of the neutralino mass spectrum and representative examples for the mixing of the particles from a very light new $\text{U}(1)_X$ gaugino across typical MSSM mass scales up to very high scales. Gauge kinetic mixing has only a small impact on the spectrum and it will therefore be neglected in the illustrative example. Throughout the evolution, including all intermediate regions, the coupling between the new states and the MSSM states remains weak, apart from regions of mass degeneracy. The evolution affects primarily the spectrum of the two new neutralino states. In the initial limit, M_1' small, two medium-heavy degenerate states, $m_{5',6'} \sim O(v_s)$, are realized in the spectrum. At the end of the chain, M_1' large, the spectrum is of a see-saw type, including one heavy and one nearly zero-mass state.

As an illustrative example, we take $M_2 = 1.5$ TeV, $m_s = 1.2$ TeV, $\mu = 0.3$ TeV and $M_K = 0$, and we assume the gaugino unification relation $M_1 = (5/3) \tan^2 \theta_W M_2 \approx 0.5 M_2$. Also, for the numerical analysis in this paper, we set $\tan \beta = 5$. We adopt the N-model charge assignments [8],

$$ Q_1 = -\frac{3}{2\sqrt{10}}, \qquad Q_2 = -\frac{2}{2\sqrt{10}}, \qquad Q_S = \frac{5}{2\sqrt{10}}. \tag{3.40} $$

For definiteness we fix the gauge coupling at $g_X \simeq 0.46$, evolved from its E_6 unification value of $\sqrt{5/3}\, g_Y$ down to the electroweak scale; however the results are not very sensitive to this assumption. We could also choose to fix M_X at its gaugino unification value under the assumption that all gaugino masses unify at the grand unification scale. This would correspond to a value of $M_1' \approx M_X = M_1 = 750$ GeV (neglecting kinetic mixing effects). However, to illustrate the structure of the system in various scenarios, we shall be slightly more general by allowing M_1' to vary over a large range of values ($0 \leq M_1' \leq 5$ TeV).

To be specific, we choose the evolution with M_1'

$$ \text{from:} \quad M_1' \quad \ll \quad v \quad \ll \quad \mu \quad \ll \quad M_1, M_2, v_s $$
$$ \text{to:} \qquad\qquad\quad v \quad \ll \quad \mu \quad \ll \quad M_1, M_2, v_s \quad \ll \quad M_1'. $$

The evolution of the six (positive) neutralino masses[6] and the values of two typical V^6 mixing elements,

---

[6]The eigenvalues $4'$ and $5'$ of the mass matrix [Eq. (3.34)] are negative, while all the other eigenvalues are positive. Level crossing will therefore occur only between $2'$-$6'$ and $4'$-$5'$ when M_1' is increased. The physical neutralino masses are given by the absolute values of the corresponding mass eigenvalues.
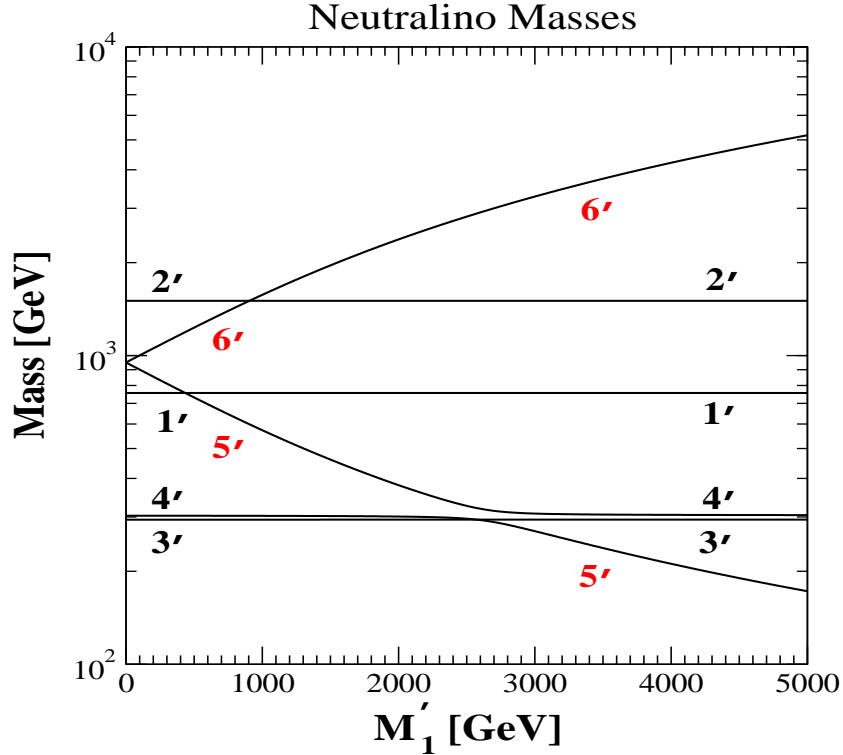
Figure 1: *The evolution of the six neutralino masses when varying the $U(1)_X$ gaugino mass parameter M_1'. The values used for the parameters are given in the text. The numbers with primes characterize the nature of the neutralino states connected with the ordering of the states when evolving from $M_1' = 0$. Note that the $2'$ and $6'$ curves and the $4'$ and $5'$ curves, respectively, do not actually touch. This can be seen more clearly in Fig. 9, where these near intersection regions are expanded. The $1'$ and $5'$ curves, corresponding to opposite-sign mass eigenvalues, intersect for small M_1' but affect each other only weakly.*

$\{5'4'\}$ and $\{5'6'\}$, are shown in Figs. 1 and 2. The neutralino state mixings are exemplified by the V^6 matrix elements $\{5'4'\}$ and $\{5'6'\}$ as representative for gaugino and higgsino mixings of the MSSM and the new states, as well as the mixing among the new gaugino and singlino states themselves.

When the new $U(1)_X$ gaugino mass parameter M_1' is varied from small to very large values, the pattern of neutralino masses evolves in an interesting way, as shown in Fig. 1. For small M_1' the set of parameters chosen in the previous paragraph, leads to a heavy SU(2) MSSM gaugino $\tilde{\chi}_{2'}^0$. It is followed by the two new states, mixed maximally in the $U(1)_X$ gaugino and singlino sector, $\tilde{\chi}_{5'}^0$ and $\tilde{\chi}_{6'}^0$. The fourth heaviest state is the U(1) MSSM gaugino $\tilde{\chi}_{1'}^0$. The lightest states are the two MSSM higgsinos $\tilde{\chi}_{4'}^0$ and $\tilde{\chi}_{3'}^0$. If M_1' is shifted to higher values, the mass eigenvalues in the new sector move apart, generating strong cross-over patterns whenever a mass from the new block comes close to one of the MSSM masses. This is realized at small $M_1' \approx \tilde{m}_{2'} - Q_S'^2 m_s^2 / \tilde{m}_{2'} \approx 0.91$ TeV for the neutralino $\tilde{\chi}_{6'}^0$ in the new block and the SU(2) MSSM neutralino $\tilde{\chi}_{2'}^0$; later between the new-block state $\tilde{\chi}_{5'}^0$ and the MSSM higgsino $\tilde{\chi}_{4'}^0$ for
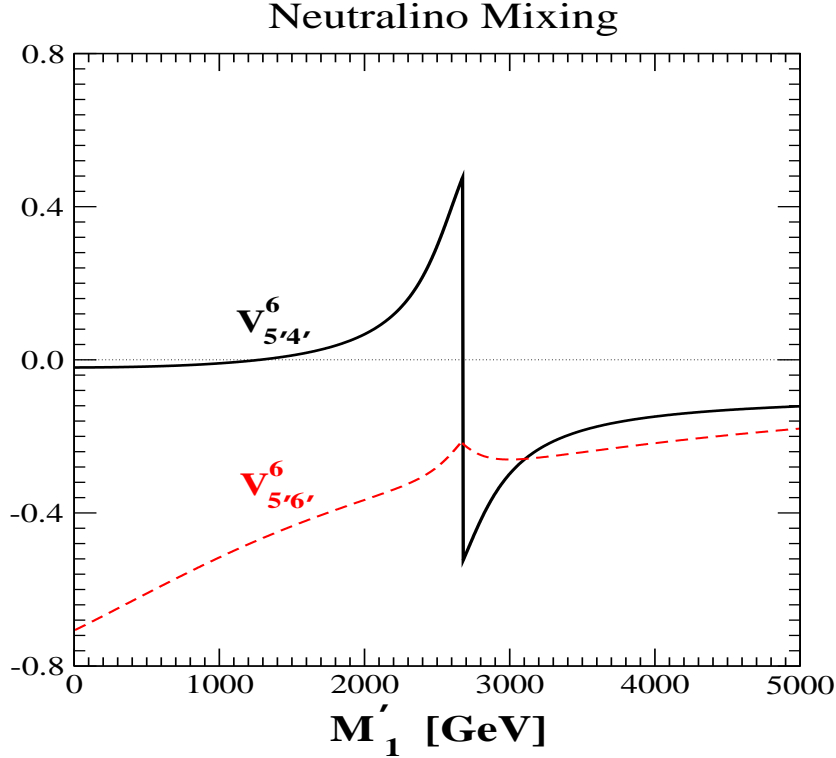
Figure 2: *The evolution of two representative mixing matrix elements when the $U(1)_X$ gaugino mass parameter M_1' is varied from small to large values. Large variations of the $5'4'$ parameter occur in the cross-over zone near $M_1' = 2.6$ TeV.*

$M_1' \approx \tilde{m}_{4'} - Q_S'^2 m_s^2 / \tilde{m}_{4'} \approx 2.68$ TeV. For very large M_1', $\tilde{\chi}_{5'}^0$ approaches the singlino state with a small mass value $|\tilde{m}_{5'}| \sim Q_S'^2 m_s^2 / M_1'$, and $\tilde{\chi}_{6'}^0$ the pure $U(1)_X$ gaugino state with a very large mass $\tilde{m}_{6'} \sim M_1'$.

Outside the cross-over regions the approximate analytical mass spectra nearly coincide with the exact (numerically computed) solutions for the eigenvalues as demonstrated in Table 1 for three M_1' values.

The mixing pattern is more directly reflected in the elements of the rotation matrix V^6, as shown in Fig. 2. For zero kinetic mixing, $\tilde{\chi}_{5'}^0$ and $\tilde{\chi}_{6'}^0$ do not overlap with the U(1) MSSM gaugino, since $V_{1'5'}^6$, $V_{1'6'}^6 \approx 0$. Their overlap with the MSSM higgsinos, $V_{5'4'}^6$, is small except in the cross-over zone. The mixing $V_{5'6'}^6$ between the new $U(1)_X$ gaugino and singlino states is reduced from maximal mixing $-1/\sqrt{2}$ for vanishing $U(1)_X$ gaugino mass parameter M_1' to nearly zero mixing at asymptotically large M_1'. The moderate change in the $5'$-$4'$ cross-over zone is a reflection of the $5'4'$ variations by unitarity of the neutralino mixing matrix.

Table 1: *Comparison between the exact and approximate neutralino masses $m_{\tilde{\chi}_i^0}$ [in GeV] for three values of M_1'. The values of the other parameters are defined in the text.*

$\tilde{\chi}_i^0$	$M_1' = 400$ GeV			$M_1' = 2000$ GeV			$M_1' = 4000$ GeV		
m [GeV]	Exact	Appr.	$\Delta m/m$	Exact	Appr.	$\Delta m/m$	Exact	Appr.	$\Delta m/m$
1	294.0	295.8	0.6%	294.1	295.9	0.6%	211.6	211.4	-0.1%
2	302.7	303.2	0.1%	301.0	301.4	0.1%	294.2	296.0	0.6%
3	756.5	755.6	-0.1%	380.3	380.3	0.0%	304.7	305.3	0.2%
4	770.1	770.1	0.0%	756.5	755.6	-0.1%	756.5	755.6	-0.1%
5	1170.6	1170.6	0.0%	1504.8	1504.3	0.0%	1504.8	1504.3	0.0%
6	1504.8	1504.3	0.0%	2379.0	2379.0	0.0%	4213.9	4213.9	0.0%

4   Neutralino Production and Decays

Neutralino production rates in various channels and decay properties in various modes are affected by the mixing of the neutralino states and by the mass and kinetic mixings of the gauge bosons associated with the broken $U(1)_X$ and $SU(2){\times}U(1)_Y$ gauge symmetries.

The Z and Z' bosons can mix through kinetic coupling, as analyzed before, and mass mixing induced by the exchange of the Higgs fields, for example, charged under both $U(1)'$s. The resulting Z and Z' mixing is described by the mass-squared matrix

$$M_{ZZ'}^2 = \begin{pmatrix} m_Z^2 & \Delta_Z^2 \\ \Delta_Z^2 & m_{Z'}^2 \end{pmatrix}, \tag{4.1}$$

where the matrix elements are given by

$$
\begin{aligned}
m_Z^2 &= \frac{1}{4}g_Z^2 v^2 \,, \\
m_{Z'}^2 &= g_X^2 v^2 \left(Q_1'^2 c_\beta^2 + Q_2'^2 s_\beta^2 \right) + g_X^2 v_S^2 Q_S'^2 \,, \\
\Delta_Z^2 &= \frac{1}{2}g_Z g_X v^2 \left(Q_1' c_\beta^2 - Q_2' s_\beta^2 \right) \,,
\end{aligned}
\tag{4.2}
$$

and where $g_Z^2 \equiv g_2^2 + g_Y^2$. The eigenvalues of $M_{ZZ'}^2$ and the Z and Z' mixing angle follow from

$$
\begin{aligned}
m_{Z_1, Z_2}^2 &= \frac{1}{2}\left(m_Z^2 + m_{Z'}^2 \mp \sqrt{(m_Z^2 - m_{Z'}^2)^2 + 4\Delta_Z^4} \right) \,, \\
\tan 2\theta_{ZZ'} &= -2\Delta_Z^2/(m_{Z'}^2 - m_Z^2) \,.
\end{aligned}
\tag{4.3}
$$

The phenomenological constraints typically require this mixing angle to be less than a few times 10^{-3} [21], although values as much as ten times larger may be possible in some models with a light Z'

and reduced couplings [30].

For the neutralino production processes in e^+e^- annihilation it is sufficient to consider the neutralino-neutralino-$Z_{1,2}$ vertices

$$\langle \tilde{\chi}^0_{iL}|Z_1|\tilde{\chi}^0_{jL}\rangle = -g_Z \mathcal{Z}_{ij}\cos\theta_{ZZ'} - g_X \mathcal{Z}'_{ij}\sin\theta_{ZZ'}\,,$$
$$\langle \tilde{\chi}^0_{iL}|Z_2|\tilde{\chi}^0_{jL}\rangle = +g_Z \mathcal{Z}_{ij}\sin\theta_{ZZ'} - g_X \mathcal{Z}'_{ij}\cos\theta_{ZZ'}\,, \tag{4.4}$$

with $i,j = 1,..,6$ and $g_Z = g_2/c_W$; $L \to R$ can be switched by substituting $\mathcal{Z}_{ij} \to -\mathcal{Z}^*_{ij}$ and $\mathcal{Z}'_{ij} \to -\mathcal{Z}'^*_{ij}$. Explicitly, the couplings $\mathcal{Z}_{ij}$ and $\mathcal{Z}'_{ij}$ are given in terms of the USSM neutralino mixing matrix N by[7]

$$\mathcal{Z}_{ij} = \frac{1}{2}\left(N_{i3}N^*_{j3} - N_{i4}N^*_{j4}\right)\,,$$
$$\mathcal{Z}'_{ij} = Q'_1 N_{i3}N^*_{j3} + Q'_2 N_{i4}N^*_{j4} + Q'_S N_{i5}N^*_{j5}\,. \tag{4.5}$$

Sfermion t/u-channel exchanges require the fermion-sfermion-neutralino vertices (with the fermion masses neglected):

$$\langle \tilde{\chi}^0_{iR}|\tilde{f}_L|f_L\rangle = -\sqrt{2}\left[g_2(I^f_3 N^*_{i2} + (e_f - I^f_3)N^*_{i1}t_W) + g_X Q'_{f_L}N^*_{i6}\right]\,,$$
$$\langle \tilde{\chi}^0_{iL}|\tilde{f}_R|f_R\rangle = +\sqrt{2}\left[g_2\,e_f\,t_W N_{i1} + g_X\,Q'_{f_R}N_{i6}\right]\,. \tag{4.6}$$

In Eq. (4.6) the coupling to the higgsino component, which is proportional to the fermion mass, has been neglected. These would have to be included if one were to study, e.g., the neutralino interaction with the top quark and squark.

For completeness, we also provide the fermion-fermion-$Z_{1,2}$ vertices:

$$\langle f_L|Z_1|f_L\rangle = -g_Z(I^f_3 - e_f s^2_W)\cos\theta_{ZZ'} - g_X Q'_{f_L}\sin\theta_{ZZ'}\,,$$
$$\langle f_L|Z_2|f_L\rangle = +g_Z(I^f_3 - e_f s^2_W)\sin\theta_{ZZ'} - g_X Q'_{f_L}\cos\theta_{ZZ'}\,. \tag{4.7}$$

When switching from $L \to R$ in Eq. (4.7), the corresponding SU(2)×U(1) and U(1)$_X$ charges must be changed accordingly. $I^f_3 \equiv I^f_{3L}$ is the SU(2) isospin component (note that $I^f_{3R} = 0$), e_f is the electric charge of the fermion f and $Q'_{f_{L,R}}$ are the effective U(1)$_X$ charges of the left/right-handed fermions.

The neutralino production and decay properties in the USSM model with the additional gaugino and singlino states depend crucially on their masses with respect to the MSSM neutralino masses. If they are much heavier than the other states, they will rarely be produced and so are practically unobservable. In contrast, if the singlino is lighter than the other states, a singlino-dominated state will be the lightest supersymmetric particle (LSP) into which the other neutralino states will decay, possibly through cascades.

In the following subsections, we present a brief description of the general formalism of neutralino production and the subsequent cascade decays of the neutralinos. Once charges and mixing matrices

---

[7]For simplicity of notation, the USSM neutralino mixing matrix N^6 will be denoted by N in this section.

are generalized to the present $U(1)_X$ theory, the phenomenological infrastructure for cross sections and decay widths can be copied from the MSSM.

4.1 Singlino Production in e^+e^- Annihilation

The production processes of a neutralino pair in e^+e^- annihilation,[8]

$$e^+e^- \to \tilde{\chi}_i^0 \tilde{\chi}_j^0 \quad [i,j=1\text{--}6]\,, \tag{4.8}$$

are generated by s-channel Z_1 and Z_2 exchanges, and t- and u-channel $\tilde{e}_{L,R}$ exchanges. The transition amplitudes,

$$T\left(e^+e^- \to \tilde{\chi}_i^0 \tilde{\chi}_j^0\right) = \frac{e^2}{s}\, Q_{\alpha\beta}\left[\bar{v}(e^+)\gamma_\mu P_\alpha u(e^-)\right]\left[\bar{u}(\tilde{\chi}_i^0)\gamma^\mu P_\beta v(\tilde{\chi}_j^0)\right]\,, \tag{4.9}$$

are built up by products of chiral neutralino currents and chiral fermion currents, coupled by bilinear "charges" Q_{LL}, Q_{LR} $etc.$ The four generalized bilinear charges correspond to independent helicity amplitudes, describing the neutralino production processes for polarized electrons/positrons [12]. They can be parameterized by the fermion and neutralino currents and the propagators of the exchanged (s)particles as follows:

$$Q_{LL} = +\frac{D_{Z_1}}{s_W^2 c_W^2} F_{1L} \mathcal{Z}_{1ij} + \frac{D_{Z_2}}{s_W^2 c_W^2} F_{2L} \mathcal{Z}_{2ij} - \frac{D_{uL}}{s_W^2} L_i L_j^*\,,$$

$$Q_{LR} = -\frac{D_{Z_1}}{s_W^2 c_W^2} F_{1L} \mathcal{Z}_{1ij}^* - \frac{D_{Z_2}}{s_W^2 c_W^2} F_{2L} \mathcal{Z}_{2ij}^* + \frac{D_{tL}}{s_W^2} L_i^* L_j\,,$$

$$Q_{RL} = +\frac{D_{Z_1}}{s_W^2 c_W^2} F_{1R} \mathcal{Z}_{1ij} + \frac{D_{Z_2}}{s_W^2 c_W^2} F_{2R} \mathcal{Z}_{2ij} + \frac{D_{tR}}{s_W^2} R_i R_j^*\,,$$

$$Q_{RR} = -\frac{D_{Z_1}}{s_W^2 c_W^2} F_{1R} \mathcal{Z}_{1ij}^* - \frac{D_{Z_2}}{s_W^2 c_W^2} F_{2R} \mathcal{Z}_{2ij}^* - \frac{D_{uR}}{s_W^2} R_i^* R_j\,. \tag{4.10}$$

The first two terms in each bilinear charge are generated by Z_1 and Z_2 exchanges and the third term by selectron exchange; $D_{Z_{1,2}}$, $D_{tL,R}$ and $D_{uL,R}$ denote the scaled s-channel $Z_{1,2}$ propagators and the t- and u-channel left/right-type selectron propagators

$$D_{Z_{1,2}} = \frac{s}{s - m_{Z_{1,2}}^2 + i m_{Z_{1,2}} \Gamma_{Z_{1,2}}} \quad \text{and} \quad D_{(t,u)L,R} = \frac{s}{(t,u) - m_{\tilde{f}_{L,R}}^2}\,, \tag{4.11}$$

with $s=(p_{e^-}+p_{e^+})^2$, $t=(p_{e^-}-p_{\tilde{\chi}_i^0})^2$ and $u=(p_{e^-}-p_{\tilde{\chi}_j^0})^2$ denoting the Mandelstam variables for neutralino pair production in e^+e^- collisions. The couplings $F_{iL,R}$ of the gauge bosons Z_i ($i=1,2$) to a fermion pair are given by

$$F_{1L} = +\left(I_3^f - e_f s_W^2\right) c_{ZZ'} + \frac{g_X}{g_Z} Q'_{f_L} s_{ZZ'}; \qquad F_{1R} = -e_f s_W^2\, c_{ZZ'} + \frac{g_X}{g_Z} Q'_{f_R} s_{ZZ'}\,,$$

$$F_{2L} = -\left(I_3^f - e_f s_W^2\right) s_{ZZ'} + \frac{g_X}{g_Z} Q'_{f_L} c_{ZZ'}; \qquad F_{2R} = +e_f s_W^2\, s_{ZZ'} + \frac{g_X}{g_Z} Q'_{f_R} c_{ZZ'}\,, \tag{4.12}$$

---

[8]Recall that the numbering $\tilde{\chi}_i^0$ [$i=1,\ldots,6$] of the neutralinos [without primed subscripts] refers to ascending mass ordering.

where $s_{ZZ'} \equiv \sin\theta_{ZZ'}$, $c_{ZZ'} \equiv \cos\theta_{ZZ'}$, $I_3^f = -1/2$ and $e_f = -1$ for the electron charges. Finally, the matrices $\mathcal{Z}_{1,2ij}$ and the vectors L_i and R_i are defined by $(t_W = \tan\theta_W)$

$$\mathcal{Z}_{1ij} = +\frac{1}{2}\left(N_{i3}N_{j3}^* - N_{i4}N_{j4}^*\right)c_{ZZ'} + \frac{g_X}{g_Z}\left(Q_1'N_{i3}N_{j3}^* + Q_2'N_{i4}N_{j4}^* + Q_S'N_{i5}N_{j5}^*\right)s_{ZZ'},$$

$$\mathcal{Z}_{2ij} = -\frac{1}{2}\left(N_{i3}N_{j3}^* - N_{i4}N_{j4}^*\right)s_{ZZ'} + \frac{g_X}{g_Z}\left(Q_1'N_{i3}N_{j3}^* + Q_2'N_{i4}N_{j4}^* + Q_S'N_{i5}N_{j5}^*\right)c_{ZZ'},$$

$$L_i = +I_3^f N_{i2} + (e_f - I_3^f)t_W N_{i1} + \frac{g_X}{g_2}Q_{f_L}'N_{i6},$$

$$R_i = -e_f t_W N_{i1} + \frac{g_X}{g_2}Q_{f_R}'N_{i6}. \tag{4.13}$$

The e^+e^- annihilation cross sections follow from the squares of the relevant couplings,

$$\sigma\left[e^+e^- \to \tilde{\chi}_i^0\tilde{\chi}_j^0\right] = \mathcal{S}_{ij}\frac{\pi\alpha^2}{2s}\lambda_{\text{PS}}^{1/2}\int_{-1}^{1}\left\{\left[1 - (\mu_i^2 - \mu_j^2)^2 + \lambda_{\text{PS}}\cos^2\Theta\right]\mathcal{Q}_1\right.$$

$$\left. + 4\mu_i\mu_j\mathcal{Q}_2 + 2\lambda_{\text{PS}}^{1/2}\mathcal{Q}_3\cos\Theta\right\}d\cos\Theta, \tag{4.14}$$

where $\mathcal{S}_{ij} = (1+\delta_{ij})^{-1}$ is a statistical factor which is equal to 1 for $i \neq j$ and $1/2$ for $i = j$; $\mu_i = m_{\tilde{\chi}_i^0}/\sqrt{s}$, Θ is the polar angle of the produced neutralinos; and $\lambda_{\text{PS}} \equiv \lambda_{\text{PS}}(1, \mu_i^2, \mu_j^2)$ denotes the familiar 2-body phase space function. The quartic charges $\mathcal{Q}_i$ $(i = 1, 2, 3)$ are given by products of bilinear charges:

$$\mathcal{Q}_1 = \frac{1}{4}\left[|Q_{RR}|^2 + |Q_{LL}|^2 + |Q_{RL}|^2 + |Q_{LR}|^2\right],$$

$$\mathcal{Q}_2 = \frac{1}{2}\text{Re}\left[Q_{RR}Q_{RL}^* + Q_{LL}Q_{LR}^*\right],$$

$$\mathcal{Q}_3 = \frac{1}{4}\left[|Q_{RR}|^2 + |Q_{LL}|^2 - |Q_{RL}|^2 - |Q_{LR}|^2\right]. \tag{4.15}$$

The integration over the polar angle Θ can easily be performed analytically.

The production cross sections for the three pairings of the two lightest neutralinos, $\{11\}, \{12\}$ and $\{22\}$, are illustrated in Fig. 3 as functions of M_1'. For the parameter set defined in Sect. 3.4, the corresponding Z' mass is $M_{Z_2} = 949$ GeV and the ZZ' mixing angle is $\theta_{ZZ'} = 3.3 \times 10^{-3}$; these parameters are compatible with existing limits [21, 30]. The center-of-mass energy of the e^+e^- collider is set to 800 GeV. Of course, if Z_2 is in the reach of the collider, running on the Z_2 resonance would be the most natural way to explore all the facets of the new particle sector in an optimal way.

For small values of M_1' the cross section $\sigma\{\tilde{\chi}_1^0\tilde{\chi}_2^0\}$ is of similar size as the MSSM prediction for the mixed higgsino pairs, $\tilde{\chi}_{3'}^0\tilde{\chi}_{4'}^0$ (cf. Fig. 1), modified only by the Z' contribution. However, at and beyond the cross-over points with the new singlino type neutralino $\tilde{\chi}_{5'}^0$, dramatic changes set in for pairs involving the lightest neutralino. Since the couplings of the mixed pair, $\tilde{\chi}_{5'}^0\tilde{\chi}_{3'}^0$, are suppressed to both the Z and Z' vector bosons, the cross section $\sigma\{\tilde{\chi}_1^0\tilde{\chi}_2^0\}$ drops significantly. In contrast, the rising
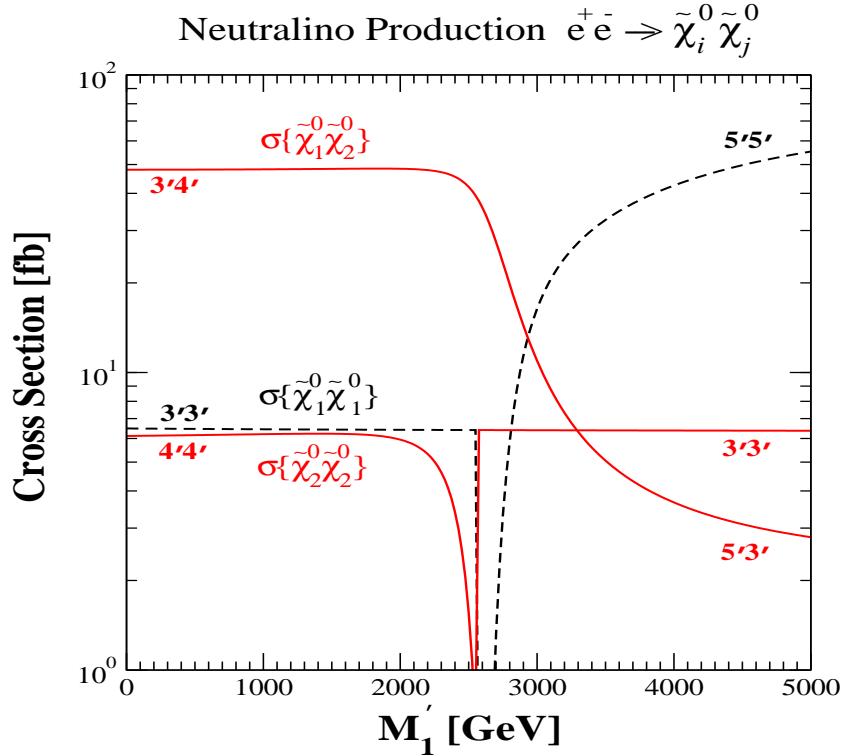
Figure 3: *The production cross sections for $\tilde{\chi}_1^0\tilde{\chi}_1^0$, $\tilde{\chi}_1^0\tilde{\chi}_2^0$ and $\tilde{\chi}_2^0\tilde{\chi}_2^0$ neutralino pairs in e^+e^- collisions when varying the $U(1)_X$ gaugino mass parameter M_1'. The R and L selectron masses are chosen as $m_{\tilde{e}_{R,L}} = 701$ GeV in this example.*

$\tilde{\chi}_{5'}^0\tilde{\chi}_{5'}^0$ coupling to Z' increases the cross section of the diagonal pair $\sigma\{\tilde{\chi}_1^0\tilde{\chi}_1^0\}$ with rising M_1'. [The cross section for the diagonal pair $\sigma\{\tilde{\chi}_2^0\tilde{\chi}_2^0\}$ does not change as the MSSM higgsino character is modified only transiently in the 5'-4' cross-over zone.]

The presence of the extra gauge boson Z_2 with a mass of ~ 1 TeV alters the neutralino-pair production cross sections $\sigma\{\tilde{\chi}_1^0\tilde{\chi}_1^0\}$ and $\sigma\{\tilde{\chi}_2^0\tilde{\chi}_2^0\}$ in the USSM significantly compared with the MSSM, as demonstrated in Table 2. The production of light neutralino pairs, diagonal pairs in particular, are greatly enhanced although the light neutralino masses are nearly identical in the two models.

Table 2: *Comparison of production cross sections between the MSSM and the USSM. The value for M_1' is set to zero in the USSM. For other values of M_1' see Fig. 3.*

| Cross Section [fb]	$\sigma\{\tilde{\chi}_1^0\tilde{\chi}_1^0\}$	$\sigma\{\tilde{\chi}_1^0\tilde{\chi}_2^0\}$	$\sigma\{\tilde{\chi}_2^0\tilde{\chi}_2^0\}$
USSM	6.5	48.0	6.1
MSSM	1.7×10^{-3}	67.1	8.5×10^{-3}

4.2 Neutralino cascade decays and sfermion decays

If kinematically allowed, the two-body decays of neutralinos into a neutralino and an electroweak gauge bosons $Z_{1,2}$ are among the dominant channels. The widths of the decays, $\tilde{\chi}_i^0 \to \tilde{\chi}_j^0 Z_k$ ($k = 1, 2$), are given by

$$\Gamma[\tilde{\chi}_i^0 \to \tilde{\chi}_j^0 Z_k] = \frac{g_Z^2 \lambda_{\rm PS}^{1/2}}{16\pi m_{\tilde{\chi}_i^0}} \left\{ |\mathcal{Z}_{kij}^2| \left[\frac{(m_{\tilde{\chi}_i^0}^2 - m_{\tilde{\chi}_j^0}^2)^2}{m_{Z_k}^2} + m_{\tilde{\chi}_i^0}^2 + m_{\tilde{\chi}_j^0}^2 - 2m_{Z_k}^2 \right] + 6 m_{\tilde{\chi}_i^0} m_{\tilde{\chi}_j^0} \, {\rm Re}\left(\mathcal{Z}_{kij}^2\right) \right\} \,, \quad (4.16)$$

where $\lambda_{\rm PS} \equiv \lambda_{\rm PS}(1, m_{\tilde{\chi}_j^0}^2/m_{\tilde{\chi}_i^0}^2, m_{Z_k}^2/m_{\tilde{\chi}_i^0}^2)$, with $\mathcal{Z}_{1ij}$ and $\mathcal{Z}_{2ij}$ defined in Eq. (4.13).

Two examples, $\tilde{\chi}_i^0 \to \tilde{\chi}_1^0 Z_1$ for $i = 6, 3$, illustrate the evolution of the widths with M_1' in Fig. 4. The neutralinos $\tilde{\chi}_6^0$ and $\tilde{\chi}_1^0$ are identified with the MSSM SU(2) gaugino and the lighter of the MSSM higgsinos for small M_1', and with the U(1)$_X$ gaugino and the singlino for large M_1', respectively [cf. Fig. 1]. Even after $\tilde{\chi}_6^0$ crosses to the U(1)$_X$ gaugino at the 2'-6' cross-over zone, the width increases due to an increasing phase space factor (due to the increasing mass difference) and the fact that $\tilde{\chi}_6^0$ has a significant singlino component. However, once the state $\tilde{\chi}_1^0$ becomes singlino-dominated above the 5'-4' cross-over zone, the width of the decay $\tilde{\chi}_6^0 \to \tilde{\chi}_1^0 Z_1$ drops dramatically as the mixing between the
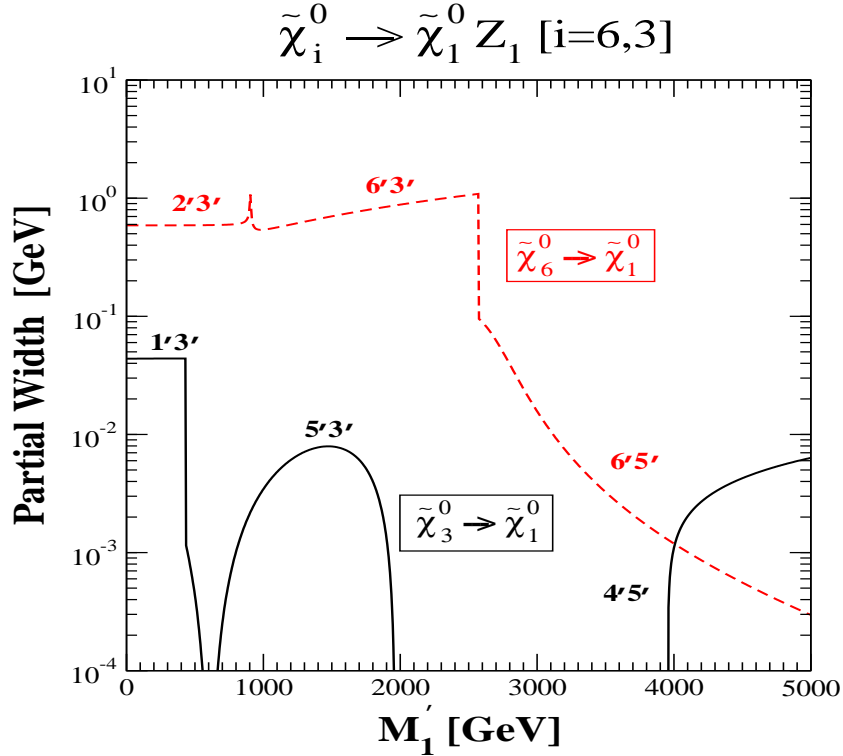
Figure 4: *The evolution of the neutralino decays $\tilde{\chi}_i^0 \to \tilde{\chi}_1^0 Z_1$ [i = 6, 3] when varying the U(1)$_X$ gaugino mass parameter M_1'. The notation follows the previous figures.*

U(1)$_X$ gaugino and the singlino state is strongly suppressed for large M_1'. The state $\tilde{\chi}_3^0$ is the MSSM U(1) gaugino for small M_1', the singlino-dominated state for moderate M_1' and the heavier of the MSSM higgsinos for large M_1'. As the $\tilde{\chi}_3^0$ mass drops, even only slightly, the two-body decay $\tilde{\chi}_3^0 \to \tilde{\chi}_1^0 Z_1$ is kinematically forbidden for moderate M_1'. However, the mode is kinematically allowed again and its magnitude increases when the mass of the singlino-dominated $\tilde{\chi}_1^0$ decreases sufficiently.

Similarly, the two-body decays of the charginos into a neutralino and the $W^\pm$ gauge boson are expected to be among the dominant channels if kinematically allowed. The widths of the decays, $\tilde{\chi}_i^\pm \to \tilde{\chi}_j^0 W^\pm$, are given by

$$\Gamma[\tilde{\chi}_i^\pm \to \tilde{\chi}_j^0 W^\pm] = \frac{g_2^2 \lambda_{\mathrm{PS}}^{1/2}}{16\pi m_{\tilde{\chi}_i^\pm}} \left\{ \frac{|\mathcal{W}_{Lij}|^2 + |\mathcal{W}_{Rij}|^2}{2} \left[\frac{(m_{\tilde{\chi}_i^\pm}^2 - m_{\tilde{\chi}_j^0}^2)}{m_W^2} + m_{\tilde{\chi}_i^\pm}^2 + m_{\tilde{\chi}_j^0}^2 - 2m_W^2 \right] \right.$$
$$\left. - 6\, m_{\tilde{\chi}_i^\pm} m_{\tilde{\chi}_j^0}\, \mathrm{Re}\left(\mathcal{W}_{Lij}\mathcal{W}_{Rij}^*\right) \right\}, \tag{4.17}$$

where $\lambda_{\mathrm{PS}} \equiv \lambda_{\mathrm{PS}}(1, m_{\tilde{\chi}_j^0}^2/m_{\tilde{\chi}_i^\pm}^2, m_W^2/m_{\tilde{\chi}_i^\pm}^2)$ and the $\mathcal{W}_{L,R}$ are defined as

$$\mathcal{W}_{Lij} = U_{Li1}^* N_{j2} + \frac{1}{\sqrt{2}} U_{Li2}^* N_{j3}, \qquad \mathcal{W}_{Rij} = U_{Ri1}^* N_{j2}^* - \frac{1}{\sqrt{2}} U_{Ri2}^* N_{j4}^*. \tag{4.18}$$

The unitary matrices U_L and U_R diagonalize the chargino mass matrix via the singular value decomposition [24] $U_R \mathcal{M}_C U_L^\dagger = \mathrm{diag}\left\{ m_{\tilde{\chi}_1^\pm}, m_{\tilde{\chi}_2^\pm} \right\}$. Explicit formulae for the chargino masses and mixing matrices can be found in Refs. [31, 32].

At the LHC, sfermion decays, $\tilde{f} \to f \tilde{\chi}_i^0$ can produce complex cascades, as heavier neutralinos are often produced in the initial decay and subsequently decay through a number of steps before the lightest neutralino (which is presumably the LSP) is produced to end the chain. Thus, cascade decays are of great experimental interest at the LHC. The width of the sfermion 2-body decay into a fermion and a neutralino follows from

$$\Gamma[\tilde{f} \to f \tilde{\chi}_i^0] = \frac{g_2^2 \lambda_{\mathrm{PS}}^{1/2}}{16\pi m_{\tilde{f}}} |g_{\tilde{f}i}|^2 \left(m_{\tilde{f}}^2 - m_{\tilde{\chi}_i^0}^2 - m_f^2 \right), \tag{4.19}$$

where $\lambda_{\mathrm{PS}} \equiv \lambda_{\mathrm{PS}}(1, m_f^2/m_{\tilde{f}}^2, m_{\tilde{\chi}_i^0}^2/m_{\tilde{f}}^2)$, the couplings $g_{\tilde{f}_L i} = L_i$ and $g_{\tilde{f}_R i} = R_i$ are defined in terms of the neutralino mixing matrix N and the appropriate fermion charges in Eq. (4.13).

The rates for the reverse decays, neutralino decays to sfermions plus fermions, $\tilde{\chi}_i^0 \to \tilde{f}\bar{f}, \bar{\tilde{f}}f$ are given by the corresponding partial widths[9]

$$\Gamma[\tilde{\chi}_i^0 \to \tilde{f}f] = \frac{g_2^2 \lambda_{\mathrm{PS}}^{1/2} N_C^f}{32\pi\, m_{\tilde{\chi}_i^0}} |g_{\tilde{f}i}|^2 \left(m_{\tilde{\chi}_i^0}^2 + m_f^2 - m_{\tilde{f}}^2 \right), \tag{4.20}$$

---

[9]As the decay rates into $\tilde{f}\bar{f}$ and $\bar{\tilde{f}}f$ are the same, we shall henceforth denote *either* of the final states by $\tilde{f}f$.
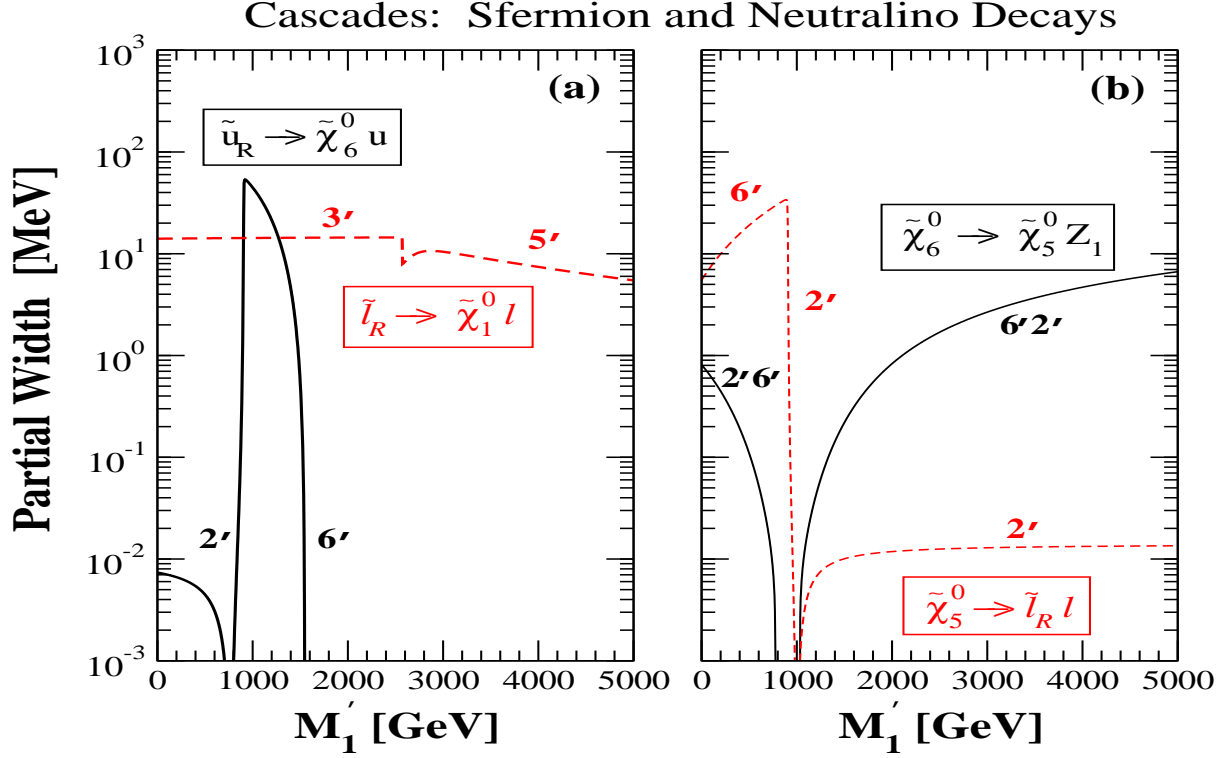
Figure 5: *The evolution of (a) the sfermion decays, $\tilde{u}_R \to \tilde{\chi}_6^0 u$ and $\tilde{\ell}_R \to \tilde{\chi}_1^0 \ell$, and (b) the neutralino decays, $\tilde{\chi}_6^0 \to \tilde{\chi}_5^0 Z$ and $\tilde{\chi}_5^0 \to \tilde{\ell}_R \ell$, when varying the $U(1)_X$ gaugino mass parameter M_1'. The R-type slepton and R-type u-squark masses are $m_{\tilde{\ell}_R} = 701$ GeV and $m_{\tilde{u}_R} = 2000$ GeV, respectively.*

with the same couplings as before, $\lambda_{\mathrm{PS}} \equiv \lambda_{\mathrm{PS}}(1, m_{\tilde{f}}^2/m_{\tilde{\chi}_i^0}^2, m_f^2/m_{\tilde{\chi}_i^0}^2)$ and the color factor $N_C^f = 1, 3$ for leptons and quarks, respectively. [Analogous expressions hold for chargino decays.]

Supersymmetric particles will be analyzed at the LHC primarily in cascade decays of some initially produced squark or gluino. In the $U(1)_X$ extended model, the cascade chains may be extended compared with the MSSM by an additional step due to the presence of two new neutralino states, for example,

$$\tilde{u}_R \to u\tilde{\chi}_6^0 \to u[Z_1\tilde{\chi}_5^0] \to uZ_1[\ell\tilde{\ell}_R] \to uZ_1\ell[\ell\tilde{\chi}_1^0].$$

At each step in the decay chain, we have placed the decay products from the previous step within brackets. The end result of the cascade above is the final state $uZ_1\ell\ell\tilde{\chi}_1^0$ with visible particles/jet u, $Z_1 \simeq Z$, and two ℓ's.

For the parameter set introduced earlier, the partial widths involved in the cascade steps are shown for evolving M_1' in Fig. 5. The sfermion decays $\tilde{u}_R \to \tilde{\chi}_6^0 u$ and $\tilde{\ell}_R \to \tilde{\chi}_1^0 \ell$ are shown in the left panel and the neutralino decays $\tilde{\chi}_6^0 \to \tilde{\chi}_5^0 Z_1$ and $\tilde{\chi}_5^0 \to \tilde{\ell}_R \ell$ are shown in the right panel. The first step $\tilde{u}_R \to \tilde{\chi}_6^0 u$ corresponds to the decay of the R-type u-squark to $\tilde{\chi}_6^0$, which coincides with the MSSM SU(2) gaugino for

Table 3: *The comparison of decay widths between the USSM and the MSSM. The state $\tilde{\chi}_i^0$ in the table denotes the second heaviest neutralino, i.e. $\tilde{\chi}_5^0$ in the USSM and $\tilde{\chi}_3^0$ in the MSSM. The value of M_1' is set to zero in the USSM. For other values of M_1' see Fig. 5.*

Decay Width [MeV]	$\Gamma[\tilde{u}_R \to \tilde{\chi}_i^0 u]$	$\Gamma[\tilde{\chi}_i^0 \to \tilde{\ell}_R \ell]$	$\Gamma[\tilde{\ell}_R \to \tilde{\chi}_1^0 \ell]$
USSM	130.0	5.5	14.1
MSSM	3294.6	18.9	15.0

small M_1' and, after the $2'$-$6'$ cross-over zone, with the $U(1)_X$ gaugino. The width increases dramatically before the decay is forbidden kinematically for M_1' larger than 1.5 TeV. The second step $\tilde{\chi}_6^0 \to \tilde{\chi}_5^0 Z_1$ in this cascade chain corresponds to the decay of the MSSM $SU(2)$ gaugino for small M_1', changing to the $U(1)_X$ gaugino decay thereafter. The dependence of this two-body decay mode on M_1' is mainly of kinematic nature; the decay is not allowed for M_1' between ~ 0.8 TeV and ~ 1.0 TeV. Just beyond the $2'$-$6'$ cross-over zone, it increases very sharply and keeps increasing moderately with M_1' thereafter. The pattern for the third decay $\tilde{\chi}_5^0 \to \tilde{\ell}_R \ell$ is mainly determined by the size of the $U(1)_Y$ and $U(1)_X$ gauge components of the state $\tilde{\chi}_5^0$. Before the $2'$-$6'$ cross-over zone the state is a $U(1)_X$ gaugino so that the width is large. But the width is strongly suppressed for moderate and large M_1' for which $\tilde{\chi}_5^0$ is a MSSM $SU(2)$ gaugino with very small mixing with the two MSSM $U(1)$ and $U(1)_X$ gauginos. The width for the final decay $\tilde{\ell}_R \to \tilde{\chi}_1^0 \ell$ remains moderate as the $U(1)$ components of the $\tilde{\chi}_1^0$ state are small. Beyond the cross-over zone, the width decreases with the suppressed $U(1)_X$ component.

Conventional chains like $\tilde{q} \to q\tilde{\chi}_i^0 \to q[\ell\tilde{\ell}] \to q\ell[\ell\tilde{\chi}_1^0]$ may also be observed in the $U(1)_X$ extended model. However, the partial widths in the USSM can be very different from the MSSM. As an example, we consider the cascade chains, in which the intermediate neutralino state $\tilde{\chi}_i^0$ is the second heaviest neutralino, i.e. $\tilde{\chi}_5^0$ in the USSM and $\tilde{\chi}_3^0$ in the MSSM. As demonstrated in Table 3, the width for the decay of $\tilde{u}_R$ to the second heaviest neutralino in the USSM is much smaller than in the MSSM.

These cascade chains should only be taken as representative theoretical examples. A systematic phenomenological survey needs significantly more detailed analyses.

4.3 Decays to Higgs bosons

The USSM Higgs sector includes two Higgs doublets H_u and H_d as well as the SM singlet field S [8, 33–35]. Their interactions are determined by the gauge interactions and the superpotential in Eq. (1.1). Including soft SUSY breaking terms and radiative corrections, the resulting effective Higgs potential consists of four parts:

$$V_H = V_F + V_D + V_{\text{soft}} + \Delta V, \tag{4.21}$$

where the F, D and soft-breaking terms V_F, V_D and V_{soft} are given by

$$
V_F = |\lambda|^2 |H_u \cdot H_d|^2 + \lambda^2 |S|^2 (|H_u|^2 + |H_d|^2) \,,
$$

$$
V_D = \frac{g_Z^2}{8} \left(|H_d|^2 - |H_u|^2 \right)^2 + \frac{g_2^2}{2} \left(|H_u|^2 |H_d|^2 - |H_u \cdot H_d|^2 \right) + \frac{g_X^2}{2} \left(Q_1' |H_d|^2 + Q_2' |H_u|^2 + Q_S' |S|^2 \right)^2 \,,
$$

$$
V_{\text{soft}} = m_1^2 |H_d|^2 + m_2^2 |H_u|^2 + m_S^2 |S|^2 + \left(\lambda A_\lambda S\, H_u \cdot H_d + \text{h.c} \right) \,, \tag{4.22}
$$

with $H_u \cdot H_d \equiv H_u^+ H_d^- - H_u^0 H_d^0$. The structure of the F term V_F is the same as in the NMSSM without the self-interaction of the singlet field. However the D term V_D contains a new ingredient: the terms proportional to g_X^2 are D-term contributions due to the extra $\text{U}(1)_X$ which are not present in the MSSM or NMSSM. The soft SUSY breaking terms are collected in V_{soft}. The tree-level Higgs potential is CP-conserving [34]. That is, one can rephase the Higgs fields to absorb the phases of the potentially complex coefficient λA_λ. Thus, without loss of generality, we will assume that these parameters are real.

The term ΔV in Eq. (4.21) represents the radiative corrections to the Higgs effective potential [36]. The dominant contributions at one-loop are generated by top quark and scalar top quark (stop) loops due to the large Yukawa couplings; these terms are the same as in the MSSM. All the other model-dependent contributions do not contribute significantly at one-loop order [33]. Therefore, we will ignore these subdominant model-dependent radiative corrections in the following analysis.

The set of soft SUSY breaking parameters in the tree-level Higgs potential includes the soft masses m_1^2, m_2^2 and m_S^2 and the trilinear coupling A_λ. Radiative corrections are affected by many other soft SUSY breaking parameters that generate masses of scalar tops and their mixings: the SU(2) and U(1) soft SUSY breaking scalar top masses m_Q, m_U, the stop trilinear parameter A_t, the supersymmetric mass scale M_{SUSY} and, spuriously, the renormalization scale Q. To simplify the analysis of the Higgs spectrum it is useful to express the soft masses m_1^2, m_2^2, m_S^2 in terms of $v_s, v, \tan\beta$ and the other parameters. The tree-level Higgs masses and couplings depend on four variables only: $\lambda, v_s, \tan\beta$ and A_λ. In the numerical analysis, we take 1 TeV for the new parameters, $m_Q, m_U, A_t, Q, M_{\text{SUSY}}$ and A_λ.

Decays involving Higgs bosons can be quite different for different Higgs boson mass spectra. We first decompose the neutral Higgs states into real and imaginary parts as follows:

$$
H_d^0 = \frac{1}{\sqrt{2}} \left(v \cos\beta + h \cos\beta - H \sin\beta + iA\, \sin\beta \sin\varphi \right) \,,
$$

$$
H_u^0 = \frac{1}{\sqrt{2}} \left(v \sin\beta + h \sin\beta + H \cos\beta + iA\, \cos\beta \sin\varphi \right) \,,
$$

$$
S = \frac{1}{\sqrt{2}} \left(v_s + N + iA\, \cos\varphi \right) \,, \tag{4.23}
$$

where the CP-odd mixing angle φ is determined by $\tan\varphi = 2v_s / v \sin 2\beta$ and all the Goldstone states are removed by adopting the unitary gauge. Subsequently the CP-even states (h, H, N) are rotated onto

the mass eigenstates H_i ($i = 1, 2, 3$), labeled in order of ascending mass, by applying the orthogonal rotation matrix O^H:

$$(H_1, H_2, H_3)_k = (h, H, N)_a \, O^H_{ak} \,, \tag{4.24}$$

The resulting Higgs mass spectrum consists of three CP-even scalars, one CP-odd scalar, and two charged Higgs bosons.

Generally, the width of a 2-body neutralino or chargino $\tilde{\chi}_i$ decay to a neutralino or chargino $\tilde{\chi}_j$ and a Higgs boson ϕ_k ($H_{1,2,3}$ or A) is given by

$$\Gamma[\tilde{\chi}_i \to \tilde{\chi}_j \phi_k] = \frac{g_2^2 \lambda_{\mathrm{PS}}^{1/2}}{32 \pi m_{\tilde{\chi}_i}} \left\{ \left(m_{\tilde{\chi}_i}^2 + m_{\tilde{\chi}_j}^2 - m_{\phi_k}^2 \right) \left(|C_{ijk}^L|^2 + |C_{ijk}^R|^2 \right) + 4 m_{\tilde{\chi}_i} m_{\tilde{\chi}_j} \, \mathrm{Re} \left(C_{ijk}^L C_{ijk}^{R*} \right) \right\} \,, \tag{4.25}$$

where $\lambda_{\mathrm{PS}} \equiv \lambda_{\mathrm{PS}}(1, m_{\tilde{\chi}_j}^2/m_{\tilde{\chi}_i}^2, m_{\phi_k}^2/m_{\tilde{\chi}_i}^2)$ and the left/right couplings $C_{ijk}^{L/R}$ must be specified in each individual case.

(i) For the decay of a neutralino $\tilde{\chi}_i^0$ to a neutralino $\tilde{\chi}_j^0$ and a scalar Higgs boson H_k, $\tilde{\chi}_i^0 \to \tilde{\chi}_j^0 H_k$, the couplings are given by,

$$\begin{aligned}
C_{ijk}^R(\tilde{\chi}_i^0 \to \tilde{\chi}_j^0 H_k) = &-\frac{1}{2} \left[(N_{i2} - N_{i1} t_W)(N_{j3} c_\beta - N_{j4} s_\beta) - \sqrt{2} \frac{\lambda}{g_2} \left(N_{i3} s_\beta + N_{i4} c_\beta \right) N_{j5} \right. \\
&\left. + 2 \frac{g_X}{g_2} N_{i6} \left(Q_1' N_{j3} c_\beta + Q_2' s_\beta N_{j4} \right) \right] O_{1k}^H \\
&+ \frac{1}{2} \left[(N_{i2} - N_{i1} t_W)(N_{j3} s_\beta + N_{j4} c_\beta) + \sqrt{2} \frac{\lambda}{g_2} \left(N_{i3} c_\beta - N_{i4} s_\beta \right) N_{j5} \right. \\
&\left. + 2 \frac{g_X}{g_2} N_{i6} \left(Q_1' N_{j3} s_\beta - Q_2' c_\beta N_{j4} \right) \right] O_{2k}^H \\
&+ \frac{1}{2} \left[\sqrt{2} \frac{\lambda}{g_2} N_{i3} N_{j4} - 2 \frac{g_X}{g_2} Q_S' N_{i6} N_{j6} \right] O_{3k}^H \quad + \quad (i \leftrightarrow j), \tag{4.26}
\end{aligned}$$

$$C_{ijk}^L(\tilde{\chi}_i^0 \to \tilde{\chi}_j^0 H_k) = C_{ijk}^{R*}(\tilde{\chi}_i^0 \to \tilde{\chi}_j^0 H_k) \,. \tag{4.27}$$

While the first term in each of the two square brackets of Eq. (4.26) are reminiscent of the MSSM couplings $\tilde{\chi}_i^0 \tilde{\chi}_j^0 h$ and $\tilde{\chi}_i^0 \tilde{\chi}_j^0 H$ respectively, the other terms are genuinely new in origin, arising from the extra interaction terms in the USSM superpotential and the extra U(1)$_X$ gauge interactions.

The partial widths for the kinematically allowed decays $\tilde{\chi}_{4,5}^0 \to \tilde{\chi}_1^0 H_1$ are shown in the left panel of Fig. 6 as a function of M_1'. In the areas in which $\tilde{\chi}_{4,5}^0$ and $\tilde{\chi}_1^0$ nearly coincide with the MSSM neutralinos, the partial widths do not depend on M_1'.

(ii) Similarly, a 2-body neutralino decay to a neutralino and a CP-odd Higgs boson, $\tilde{\chi}_i^0 \to \tilde{\chi}_j^0 A$,

Neutralino Decays to Higgs Bosons
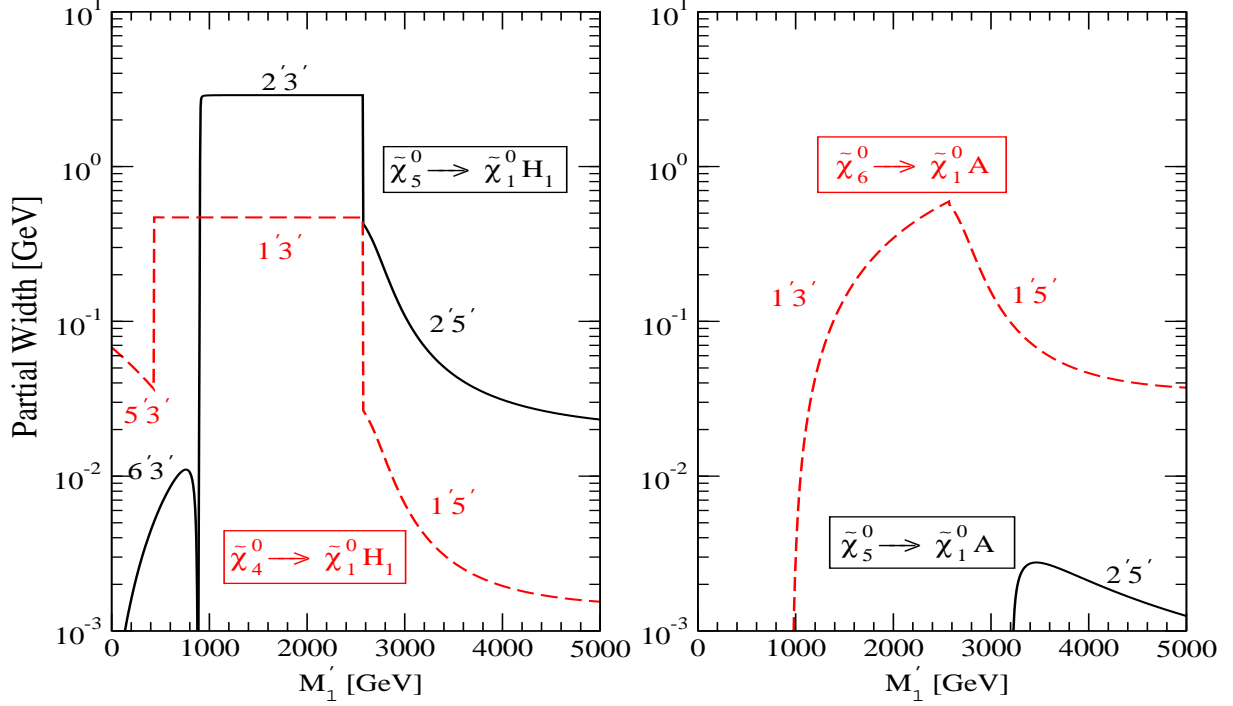
Figure 6: *The decay widths for $\tilde{\chi}^0_{5,4} \to \tilde{\chi}^0_1 H_1$ (left) and $\tilde{\chi}^0_{5,6} \to \tilde{\chi}^0_1 A_1$ (right) for the parameter set given in the text. For the purposes of example, the Higgs mass parameter M_A is set to 1.25 TeV.*

follows Eq. (4.25) with the left/right couplings given by

$$
\begin{aligned}
C^R_{ij}(\tilde{\chi}^0_i \to \tilde{\chi}^0_j A) =& -\frac{i}{2}\left[(N_{i2} - N_{i1}t_W)(N_{j3}s_\beta - N_{j4}c_\beta) + \sqrt{2}\frac{\lambda}{g_2}(N_{i3}c_\beta + N_{i4}s_\beta)N_{j5}\right. \\
&\left. + 2\frac{g_X}{g_2}N_{i6}\left(Q'_1 N_{j3}s_\beta + Q'_2 N_{j4}c_\beta\right)\right]\sin\varphi \\
&- \frac{i}{2}\left[\sqrt{2}\frac{\lambda}{g_2}N_{i3}N_{j4} + 2\frac{g_X}{g_2}Q'_S N_{i6}N_{j5}\right]\cos\varphi \quad + \quad (i \leftrightarrow j),
\end{aligned}
\tag{4.28}
$$

$$
C^L_{ij}(\tilde{\chi}^0_i \to \tilde{\chi}^0_j A) = C^{R\,*}_{ij}(\tilde{\chi}^0_i \to \tilde{\chi}^0_j A).
\tag{4.29}
$$

Again, only the first term in the square brackets is similar to the MSSM $\tilde{\chi}^0_i\tilde{\chi}^0_j A$ coupling.

The widths for the kinematically allowed decays $\tilde{\chi}^0_{5,6} \to \tilde{\chi}^0_1 A$ are shown in the right panel of Fig. 6 as a function of M'_1 in the benchmark scenario. In contrast to the scalar case, only a few decays are kinematically allowed since the CP-odd scalar A is heavy.

(iii) For completeness, we describe the decays of charginos to a neutralino and charged Higgs boson $\tilde{\chi}^-_i \to \tilde{\chi}^0_j H^-$ ($i = 1, 2; j = 1, 2, \ldots, 6$). These follow a similar pattern, but with the last index of the

coupling removed:

$$C_{ij}^L(\tilde{\chi}_i^- \rightarrow \tilde{\chi}_j^0 H^-) = -s_\beta \left[N_{j3}^* U_{Li1}^* - \frac{1}{\sqrt{2}} \left(N_{j2}^* + N_{j1}^* t_W \right) U_{Li2}^* \right] - \sqrt{2} \frac{g_X}{g_2} Q_1' N_{j6}^* U_{Li2}^* - \frac{\lambda}{g_2} c_\beta N_{j5}^* U_{Li2}^* \,, \quad (4.30)$$

$$C_{ij}^R(\tilde{\chi}_i^- \rightarrow \tilde{\chi}_j^0 H^-) = -c_\beta \left[N_{j4} U_{Ri1}^* + \frac{1}{\sqrt{2}} \left(N_{j2} + N_{j1} t_W \right) U_{Ri2}^* \right] - \sqrt{2} \frac{g_X}{g_2} Q_2' N_{j6} U_{Ri2}^* - \frac{\lambda}{g_2} s_\beta N_{i5} U_{Ri2}^* \,, \quad (4.31)$$

The same left/right couplings determine the decays of neutralinos to charginos and charged Higgs boson $\tilde{\chi}_j^0 \rightarrow \tilde{\chi}_i^+ H^-$ ($i = 1, 2; j = 1, 2, \ldots, 6$). For the parameters chosen here, the large mass of the charged Higgs boson allows kinematically only decays of the heavier chargino $\tilde{\chi}_2^\pm$ to the lightest neutralino $\tilde{\chi}_1^0$ and $H^\pm$.

(iv) It is also possible for Higgs bosons to decay into neutralino/chargino states, for example the decays $H_i \rightarrow \tilde{\chi}_1^0 \tilde{\chi}_j^0$, $A \rightarrow \tilde{\chi}_1^0 \tilde{\chi}_j^0$ and $H^\pm \rightarrow \tilde{\chi}_1^0 \tilde{\chi}_i^\pm$. Clearly this is kinematically possible only for the heavier Higgs states. The general form of the width for these decays $\phi_i \rightarrow \tilde{\chi}_j \tilde{\chi}_k$ ($\phi_i = H_i$, A, $H^\pm$), is given by the crossing of Eq. (4.25):

$$\Gamma[\phi_i \rightarrow \tilde{\chi}_j \tilde{\chi}_k] = \mathcal{S}_{jk} \frac{g_2^2 \lambda_{\rm PS}^{1/2}}{16 \pi m_{\phi_i}} \left\{ \left(m_{\phi_i}^2 - m_{\tilde{\chi}_j}^2 - m_{\tilde{\chi}_k}^2 \right) \left(|C_{ijk}^L|^2 + |C_{ijk}^R|^2 \right) - 4 m_{\tilde{\chi}_j} m_{\tilde{\chi}_k} \, {\rm Re}\left(C_{ijk}^L C_{ijk}^{R\,*} \right) \right\} \,, \quad (4.32)$$

Higgs Decays to Neutralino Pairs
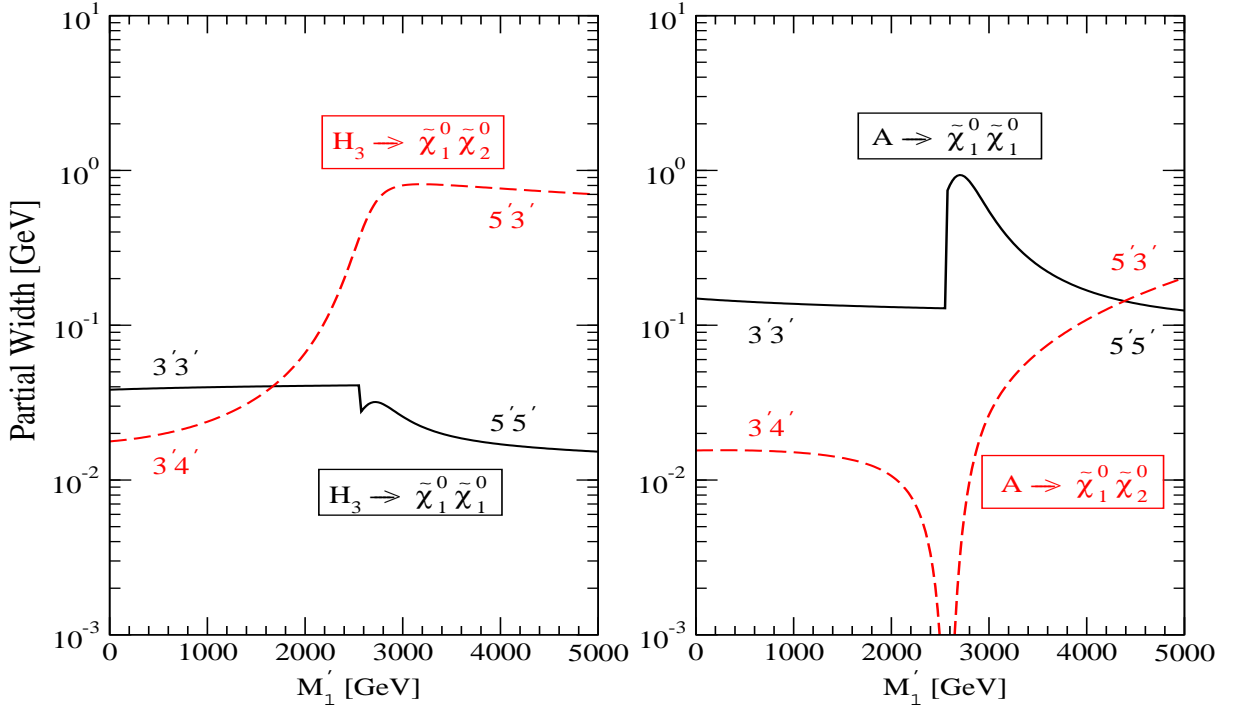
Figure 7: *The decay widths for $H_3 \rightarrow \tilde{\chi}_1^0 \tilde{\chi}_{1,2}^0$ (left) and $A \rightarrow \tilde{\chi}_1^0 \tilde{\chi}_{1,2}^0$ (right) for the same parameters as in Fig. 6.*

where $\lambda_{\mathrm{PS}} \equiv \lambda_{\mathrm{PS}}(1, m_{\tilde{\chi}_j}^2/m_{\phi_i}^2, m_{\tilde{\chi}_j}^2/m_{\phi_i}^2)$ and $\mathcal{S}_{jk} = (1 + \delta_{jk})^{-1}$ is the usual statistical factor. The couplings $C_{ijk}^{L/R}$ are related to their neutralino/chargino decay counterparts in the obvious way:

$$C_{ijk}^{L/R}(H_i \to \tilde{\chi}_j^0 \tilde{\chi}_k^0) = C_{kji}^{L/R}(\tilde{\chi}_k^0 \to \tilde{\chi}_j^0 H_i) \,, \tag{4.33}$$

$$C_{ij}^{L/R}(A \to \tilde{\chi}_i^0 \tilde{\chi}_j^0) = C_{ji}^{L/R}(\tilde{\chi}_j^0 \to \tilde{\chi}_i^0 A) \,, \tag{4.34}$$

$$C_{ij}^{L/R}(H^+ \to \tilde{\chi}_i^0 \tilde{\chi}_j^+) = C_{ji}^{L/R}(\tilde{\chi}_j^- \to \tilde{\chi}_i^0 H^-) \,. \tag{4.35}$$

As an example, the Higgs boson decays $H_3, A \to \tilde{\chi}_1^0 \tilde{\chi}_{1,2}^0$ are displayed in Fig. 7. For small M_1' the $\tilde{\chi}_{3'}^0 \tilde{\chi}_{4'}^0 H_3/A$ couplings in the decays $H_3, A \to \tilde{\chi}_1^0 \tilde{\chi}_2^0$ are suppressed while for large M_1' the $\tilde{\chi}_{5'}^0 \tilde{\chi}_{3'}^0 H_3/A$ couplings are no longer suppressed. The rapid changes in the $5'$-$4'$ cross-over zone are generated by interference effects between the Yukawa and the gauge interaction terms. Similar interference effects, though less significant, occur for the decays $H_3, A \to \tilde{\chi}_1^0 \tilde{\chi}_1^0$ near the cross-over zone.

4.4 Neutralino radiative decays

In the cross-over zones of the neutralino mass eigenvalues, the gaps between the neutralino masses become very small. As a result, standard decay channels are almost shut and photon transitions between neutralino states [37] become enhanced. These photon transitions are particularly important in the cross-over zone $4'$-$5'$ at $M_1' \simeq 2.6$ TeV [cf. Fig. 1]. The proximity of the two heavier states to the lightest neutralino dramatically reduces the rates of all other decay modes so that the radiative decays

$$\tilde{\chi}_2^0 \,, \; \tilde{\chi}_3^0 \to \tilde{\chi}_1^0 + \gamma \qquad \text{and} \qquad \tilde{\chi}_3^0 \to \tilde{\chi}_2^0 + \gamma \,, \tag{4.36}$$

become non-negligible modes. Of course, also the γ transitions are phase-space suppressed in cross-over zones but less strongly than the competing standard channels due to the vanishing photon mass, even for 3-particle decays into a lighter neutralino and lepton- or light-quark pair.

The effective couplings $g_{\tilde{\chi}_i^0 \tilde{\chi}_j^0 \gamma}$ in the partial decay widths

$$\Gamma[\tilde{\chi}_i^0 \to \tilde{\chi}_j^0 \gamma] = \frac{g_{\tilde{\chi}_i^0 \tilde{\chi}_j^0 \gamma}^2}{8\pi} \frac{(m_{\tilde{\chi}_i^0}^2 - m_{\tilde{\chi}_j^0}^2)^3}{m_{\tilde{\chi}_i^0}^5} \,, \tag{4.37}$$

are of magnetic or electric dipole type depending on the relative CP quantum numbers of $\tilde{\chi}_i^0$ and $\tilde{\chi}_j^0$. The couplings are generated by triangle graphs of sfermion/fermion, chargino/W-boson and chargino/charged Higgs-boson lines. The sum of all two-point graphs associated with the photon line and attached to the neutralino legs by a Z-boson line vanish in the non-linear R-gauge [37]. The γ transition amplitudes are finally complex combinations of mixing matrix elements with reduced triangle functions.

For the γ transitions of Eq. (4.36), the partial widths are displayed in Fig. 8 for the set of parameters chosen earlier. In this example the three lightest neutralino states are predominantly of higgsino type
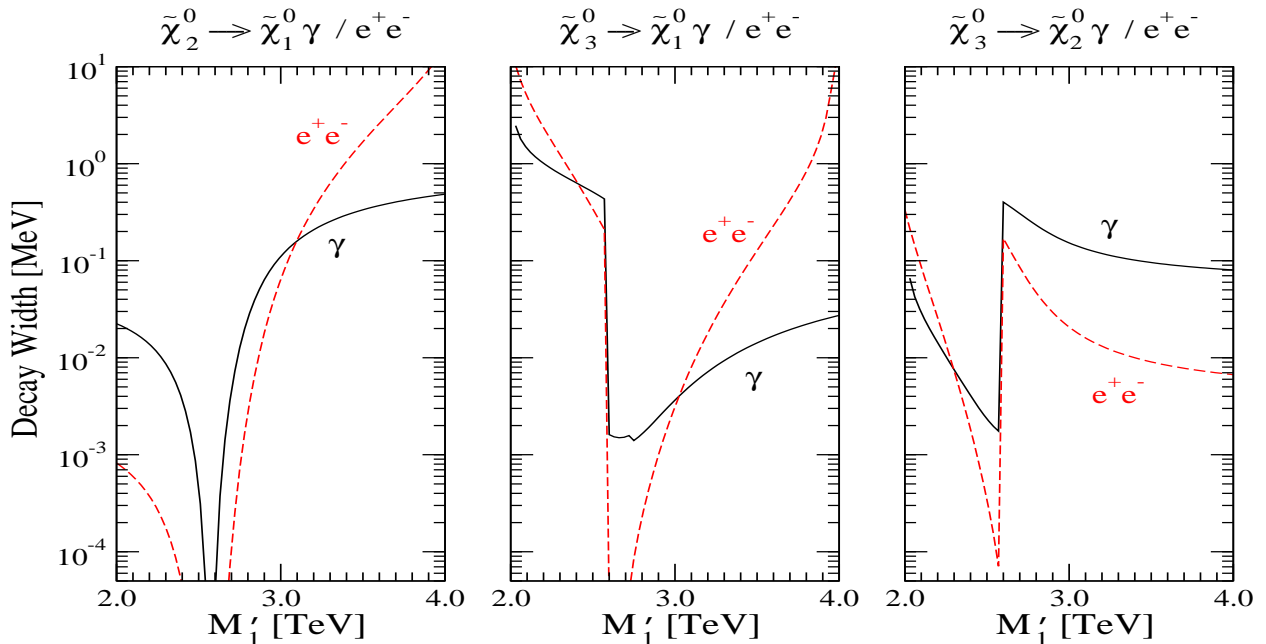
Figure 8: *Comparison between photon transitions [full lines] and ordinary lepton-pair decays [dashed lines] of neutralinos in the $3'$-$4'$-$5'$ cross-over zone of the reference point. The $U(1)_X$ gaugino mass parameter M_1' is varied between 2 and 4 TeV for comparison with the side-band wings; the charged Higgs mass, affecting the photon loop couplings is chosen 1.25 TeV and the trilinear parameters are set to 1 TeV.*

so that decays to lepton pairs are allowed through couplings mediated by virtual Z-bosons which in fact are the dominant modes. Therefore, for illustrative comparison, the partial widths for standard electron-pair decays $\tilde{\chi}_i^0 \to \tilde{\chi}_j^0 \, e^+ e^-$ are also shown. Evidently the branching fractions for radiative decays in general drastically change in the complex $4'$-$5'$ cross-over zone near $M_1' \simeq 2.6$ TeV compared with the side-band wings.

5 Summary and Conclusions

In this report we have investigated the neutralino sector in the $U(1)_X$ extension of the minimal Supersymmetric Standard Model, as suggested by many GUT and superstring models. The extended model has attractive features which solve several problems of the MSSM. It provides a natural solution of the μ-problem without creating cosmological problems. The upper limit of the light Higgs mass is somewhat increased and the growing fine tuning in this sector is reduced.

While the MSSM neutralino sector is already quite complex, the complexity increases dramatically in the extended model due to two additional degrees of freedom. However, the small coupling between the

original four MSSM states and the two new states offers an elegant analytical solution of this problem within a perturbative expansion. We have worked out this solution in detail for the mass spectrum and the mixing of the states.

The expansion in the parameter v/M_{SUSY}, the ratio of the electroweak scale v over the generic supersymmetry-breaking scale M_{SUSY}, leads to an excellent approximation of the exact solutions. Even in the cross-over zones in which two mass eigenvalues are nearly degenerate, proper adaption of the analytical formalism provides an accurate description of the system. Thus, in the limit in which M_{SUSY} is sufficiently above the electroweak scale, the neutralino system of the U(1)$_X$-extended supersymmetric standard model is under good analytical control and its features are theoretically well understood.

A few examples of mass spectra, widths for cascade decays at LHC, decays to Higgs bosons, photon transitions and production cross sections in e^+e^- collisions illustrate the characteristic features of the model.

Acknowledgments

Appendix A: Takagi diagonalization of a complex symmetric matrix

In quantum field theory, the most general neutral fermion mass matrix, M, is complex and symmetric. To identify the physical eigenstates, M must be diagonalized.[10] However, the equation that governs the identification of the physical fermion states is *not* the standard unitary similarity transformation. Instead it is a different diagonalization equation that was discovered by Takagi [23], and rediscovered many times since [24].[11] Despite this illustrious history, the mathematics of the Takagi diagonalization is relatively unknown among physicists. Thus, in this appendix we present a self-contained introduction to the Takagi diagonalization of a complex symmetric matrix. After presenting some background material

---

[10] An alternative method—the standard diagonalization of the hermitian matrix $M^\dagger M$, which is commonly advocated in the literature, fails to identify the physical states in the case of mass-degenerate fermions, as noted below Eq. (A.3).

[11] Subsequently, it was recognized in Ref. [38] that the Takagi diagonalization was first established for nonsingular complex symmetric matrices by Autonne [39].

and a constructive proof of Takagi's result, we provide, as a pedagogical example, the explicit Takagi diagonalization of an arbitrary 2×2 matrix. The latter will be particularly useful for considering cases in which there is a near-degeneracy in mass between two of the neutral fermions.

A.1 General analysis

Consider a system of n two component fermion fields $\xi \equiv (\xi_1, \xi_2, \ldots, \xi_n)^T$, whose physical masses are governed by the Lagrangian

$$\mathcal{L}_{\text{mass}} = \tfrac{1}{2} \, \xi^T M \, \xi + \text{h.c.} \tag{A.1}$$

In general, the mass matrix M is an $n \times n$ complex symmetric matrix. In order to identify the physical masses m_i and the corresponding physical fermion fields χ_i, one introduces a unitary matrix U such that $\xi = U \chi$ and demands that $\xi^T M \, \xi = \sum_i m_i \chi_i \chi_i$. This corresponds to the Takagi diagonalization of a complex symmetric matrix,[12] which is governed by the following theorem [23,24]:

Theorem: For any complex symmetric $n \times n$ matrix M, there exists a unitary matrix U such that:[13]

$$U^T M \, U = M_D = \text{diag}(m_1, m_2, \ldots, m_n), \tag{A.2}$$

where the m_k are real and non-negative.

The m_k are *not* the eigenvalues of M. Rather, the m_k are the so-called *singular values* of the symmetric matrix M, which are defined to be the non-negative square roots of the eigenvalues of $M^\dagger M$. To compute the singular values, note that:

$$U^\dagger M^\dagger M U = M_D^2 = \text{diag}(m_1^2, m_2^2, \ldots, m_n^2). \tag{A.3}$$

Since $M^\dagger M$ is hermitian, it can be diagonalized by a unitary similarity transformation. Although U can be determined from Eq. (A.3) in cases of non-degenerate singular values, the case of degenerate singular values is less straightforward. For example, if $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$, the singular value 1 is doubly-degenerate, but Eq. (A.3) yields $U^\dagger U = \mathbb{1}_{2 \times 2}$, which does not specify U. Below, we shall present a constructive method for determining U that is applicable in both the non-degenerate and the degenerate cases.

Eq. (A.2) can be rewritten as $M U = U^* M_D$, where the columns of U are orthonormal. If we denote the kth column of U by v_k, then,

$$M v_k = m_k v_k^*, \tag{A.4}$$

---

[12] If $U = N^\dagger$, we obtain the form of the Takagi diagonalization used in Eqs. (2.17) and (B.2).

[13] In Ref. [24], Eq. (A.2) is called the Takagi factorization of a complex symmetric matrix. We choose to refer to this as Takagi diagonalization to emphasize and contrast this with the more standard diagonalization of normal matrices by a unitary similarity transformation. In particular, not all *complex* symmetric matrices are diagonalizable by a similarity transformation, whereas complex symmetric matrices are *always* Takagi-diagonalizable.

where the m_k are the singular values and the vectors v_k are normalized to have unit norm. Following Ref. [25], the v_k are called the *Takagi vectors* of the symmetric complex $n \times n$ matrix M. The Takagi vectors corresponding to non-degenerate non-zero [zero] singular values are unique up to an overall sign [phase]. Any orthogonal [unitary] linear combination of Takagi vectors corresponding to a set of degenerate non-zero [zero] singular values is also a Takagi vector corresponding to the same singular value. Using these results, one can determine the degree of non-uniqueness of the matrix U. For definiteness, we fix an ordering of the diagonal elements of M_D. If the singular values of M are distinct, then the matrix U is uniquely determined up to multiplication by a diagonal matrix whose entries are either ± 1. If there are degeneracies corresponding to non-zero singular values, then within the degenerate subspace, U is unique up to multiplication on the right by an arbitrary orthogonal matrix. Finally, in the subspace corresponding to zero singular values, U is unique up to multiplication on the right by an arbitrary unitary matrix.

We shall establish the Takagi diagonalization of a complex symmetric matrix by formulating an algorithm for constructing U. A method will be provided for determining the orthonormal Takagi vectors v_k that make up the columns of U. This is achieved by rewriting the $n \times n$ complex matrix equation Eq. (A.4) [with m real and non-negative] as a $2n \times 2n$ real matrix equation [40]:

$$M_S \begin{pmatrix} \mathrm{Re}\,v \\ \mathrm{Im}\,v \end{pmatrix} \equiv \begin{pmatrix} \mathrm{Re}\,M & -\,\mathrm{Im}\,M \\ -\,\mathrm{Im}\,M & -\,\mathrm{Re}\,M \end{pmatrix} \begin{pmatrix} \mathrm{Re}\,v \\ \mathrm{Im}\,v \end{pmatrix} = m \begin{pmatrix} \mathrm{Re}\,v \\ \mathrm{Im}\,v \end{pmatrix}, \quad \text{where } m \geq 0. \quad \text{(A.5)}$$

Since $M = M^T$, the $2n \times 2n$ matrix M_S defined by Eq. (A.5) is a real symmetric matrix. In particular, M_S is diagonalizable by a real orthogonal similarity transformation, and its eigenvalues are real. Moreover, if m is an eigenvalue of M_S with eigenvector ($\mathrm{Re}\,v$, $\mathrm{Im}\,v$), then $-m$ is an eigenvalue of M_S with (orthogonal) eigenvector ($-\,\mathrm{Im}\,v$, $\mathrm{Re}\,v$). This observation proves that M_S has an equal number of positive and negative eigenvalues and an even number of zero eigenvalues.[14] Thus, Eq. (A.4) has been converted into an ordinary eigenvalue problem for a real symmetric matrix. Since $m \geq 0$, we solve the eigenvalue problem $M_S u = mu$ for the eigenvectors corresponding to the non-negative eigenvalues.[15] It is straightforward to prove that the total number of linearly independent Takagi vectors is equal to n. Simply note that the orthogonality of ($\mathrm{Re}\,v_1$, $\mathrm{Im}\,v_1$) and ($-\,\mathrm{Im}\,v_1$, $\mathrm{Re}\,v_1$) with ($\mathrm{Re}\,v_2$, $\mathrm{Im}\,v_2$) implies that $v_1^\dagger v_2 = 0$.

Thus, we have derived a constructive method for obtaining the Takagi vectors v_k. If there are degeneracies, one can always choose the v_k in the degenerate subspace to be orthonormal. The Takagi vectors then make up the columns of the matrix U in Eq. (A.2). A numerical package for performing

---

[14]Note that ($-\,\mathrm{Im}\,v$, $\mathrm{Re}\,v$) corresponds to replacing v_k in Eq. (A.4) by iv_k. However, for $m < 0$ these solutions are not relevant for Takagi diagonalization (where the m_k are by definition non-negative). The case of $m = 0$ is considered in footnote 15.

[15]For $m = 0$, the corresponding vectors ($\mathrm{Re}\,v$, $\mathrm{Im}\,v$) and ($-\,\mathrm{Im}\,v$, $\mathrm{Re}\,v$) are two linearly independent eigenvectors of M_S; but these yield only one independent Takagi vector v (since v and iv are linearly dependent). See footnote 14.

the Takagi diagonalization of a complex symmetric matrix has recently been presented in Ref. [27] (see also Refs. [25, 26] for previous numerical approaches to Takagi diagonalization).

A.2   Example: Takagi diagonalization of a 2×2 complex symmetric matrix

The Takagi diagonalization of a 2×2 complex symmetric matrix can be performed analytically.[16] The result is somewhat more complicated than the standard diagonalization of a 2×2 hermitian matrix by a unitary similarity transformation. Nevertheless, the corresponding analytic formulae for the Takagi diagonalization will prove useful in Appendix C in the treatment of nearly degenerate states. Consider the complex symmetric matrix:

$$M = \begin{pmatrix} a & c \\ c & b \end{pmatrix} , \tag{A.6}$$

where $c \neq 0$ and, without loss of generality, $|a| \leq |b|$. We parameterize the 2×2 unitary matrix U in Eq. (A.2) by [41]:

$$U = VP = \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ -e^{-i\phi}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{-i\alpha} & 0 \\ 0 & e^{-i\beta} \end{pmatrix} , \tag{A.7}$$

where $0 \leq \theta \leq \pi/2$ and $0 \leq \alpha , \beta , \phi < 2\pi$. However, we may restrict the angular parameter space further. Since the normalized Takagi vectors are unique up to an overall sign if the corresponding singular values are non-degenerate and non-zero,[17] one may restrict α and β to the range $0 \leq \alpha , \beta < \pi$ without loss of generality. Finally, we may restrict θ to the range $0 \leq \theta \leq \pi/4$. This range corresponds to one of two possible orderings of the singular values in the diagonal matrix M_D.

Using the transformation (A.7), we can rewrite the Takagi equation (A.2) as follows:

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} V = V^* \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} , \tag{A.8}$$

where

$$\sigma_1 \equiv m_1 \, e^{2i\alpha} , \qquad \text{and} \qquad \sigma_2 \equiv m_2 \, e^{2i\beta} , \tag{A.9}$$

with real and non-negative m_k. Multiplying out the matrices in Eq. (A.8) yields:

$$\sigma_1 = a - c \, e^{-i\phi} t_\theta = b \, e^{-2i\phi} - c \, e^{-i\phi} t_\theta^{-1} , \tag{A.10}$$

$$\sigma_2 = b + c \, e^{i\phi} t_\theta \quad = a \, e^{2i\phi} + c \, e^{i\phi} t_\theta^{-1} , \tag{A.11}$$

---

[16]The main results of this subsection have been obtained, e.g., in Ref. [27]. Nevertheless, we provide some of the details here, which include minor improvements over the results previously obtained.

[17]In the case of a zero singular value or a pair of degenerate of singular values, there is more freedom in defining the Takagi vectors as discussed below Eq. (A.4). These cases will be treated separately at the end of this subsection.

where $t_\theta \equiv \tan\theta$. Using either Eq. (A.10) or (A.11), one immediately obtains a simple equation for $\tan 2\theta = 2(t_\theta^{-1} - t_\theta)^{-1}$:

$$\tan 2\theta = \frac{2c}{b\,e^{-i\phi} - a\,e^{i\phi}}\,. \tag{A.12}$$

Since $\tan 2\theta$ is real, it follows that $bc^*\,e^{-i\phi} - ac^*\,e^{i\phi}$ is real and must be equal to its complex conjugate. The resulting equation can be solved for $e^{2i\phi}$:

$$e^{2i\phi} = \frac{bc^* + a^*c}{b^*c + ac^*}\,, \tag{A.13}$$

or equivalently

$$e^{i\phi} = \frac{bc^* + a^*c}{|bc^* + a^*c|}\,. \tag{A.14}$$

The (positive) choice of sign in Eq. (A.14) follows from the fact that $\tan 2\theta \geq 0$ (since by assumption, $0 \leq \theta \leq \pi/4$), which implies $0 \leq c^*(b\,e^{-i\phi} - a\,e^{i\phi}) = |c|^2(|b|^2 - |a|^2)$ after inserting the results of Eq. (A.14). Since $|b| \geq |a|$ by assumption, the asserted inequality holds as required.

Inserting the result for $e^{i\phi}$ back into Eq. (A.12) yields:

$$\tan 2\theta = \frac{2|bc^* + a^*c|}{|b|^2 - |a|^2}\,. \tag{A.15}$$

One can compute $\tan\theta$ in terms of $\tan 2\theta$ for $0 \leq \theta \leq \pi/4$:

$$\begin{aligned}
\tan\theta &= \frac{1}{\tan 2\theta}\left[\sqrt{1 + \tan^2 2\theta} - 1\right] \\
&= \frac{|a|^2 - |b|^2 + \sqrt{(|b|^2 - |a|^2)^2 + 4|bc^* + a^*c|^2}}{2|bc^* + a^*c|}\,, \tag{A.16} \\
&= \frac{2|bc^* + a^*c|}{|b|^2 - |a|^2 + \sqrt{(|b|^2 - |a|^2)^2 + 4|bc^* + a^*c|^2}}\,. \tag{A.17}
\end{aligned}$$

Starting from Eqs. (A.10) and (A.11), it is now straightforward, using Eqs. (A.14) and (A.16), to compute the squared magnitudes of σ_k:

$$m_k^2 = |\sigma_k|^2 = \frac{1}{2}\left[|a|^2 + |b|^2 + 2|c|^2 \mp \sqrt{(|b|^2 - |a|^2)^2 + 4|bc^* + a^*c|^2}\right]\,, \tag{A.18}$$

with $|\sigma_1| \leq |\sigma_2|$. This ordering of the $|\sigma_k|$ is governed by the convention that $0 \leq \theta \leq \pi/4$ (the opposite ordering would occur for $\pi/4 \leq \theta \leq \pi/2$). Indeed, one can check explicitly that the $|\sigma_k|^2$ are the eigenvalues of $M^\dagger M$, which provides the more direct way of computing the singular values.

The final step of the computation is the determination of the angles α and β from Eq. (A.9). Inserting Eqs. (A.14) and (A.17) into Eqs. (A.10) and (A.11), we end up with:

$$\alpha = \tfrac{1}{2}\arg\left\{a(|b|^2 - |\sigma_1|^2) - b^*c^2\right\}\,, \tag{A.19}$$

$$\beta = \tfrac{1}{2}\arg\left\{b(|\sigma_2|^2 - |a|^2) + a^*c^2\right\}\,. \tag{A.20}$$

If $\det M = ab - c^2 = 0$ (with $M \neq \mathbf{0}$), then there is one singular value which is equal to zero. In this case, it is easy to verify that $\sigma_1 = 0$ and $|\sigma_2|^2 = \text{Tr}\,(M^\dagger M) = |a|^2 + |b|^2 + 2|c|^2$. All the results obtained above remain valid, except that α is undefined [since in this case, the argument of arg in Eq. (A.19) vanishes]. This corresponds to the fact that for a zero singular value, the corresponding (normalized) Takagi vector is only unique up to an overall arbitrary phase [cf. footnote 17].

We provide one illuminating example of the above results. Consider the complex symmetric matrix:

$$M = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \tag{A.21}$$

The eigenvalues of M are degenerate and equal to zero. However, there is only one linearly independent eigenvector, which is proportional to $(1,\, i)$. Thus, M cannot be diagonalized by a similarity transformation [24]. In contrast, all complex symmetric matrices are Takagi-diagonalizable. The singular values of M are 0 and 2 (since these are the non-negative square roots of the eigenvalues of $M^\dagger M$), which are *not* degenerate. Thus, all the formulae derived above apply in this case. One quickly determines that $\theta = \pi/4$, $\phi = \pi/2$, $\beta = \pi/2$ and α is indeterminate (so one is free to choose $\alpha = 0$). The resulting Takagi diagonalization is $U^T M U = \text{diag}(0,\, 2)$ with:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}. \tag{A.22}$$

This example clearly indicates the distinction between the (absolute values of the) eigenvalues of M and its singular values. It also exhibits the fact that one cannot always perform a Takagi diagonalization by using the standard techniques for computing eigenvalues and eigenvectors.[18]

We end this subsection by treating the case of degenerate (non-zero) singular values, which arises when $bc^* = -a^*c$. Special considerations are required since not all the formulae derived above are applicable to this case [cf. footnote 17]. The condition $bc^* = -a^*c$ implies that $|a| = |b|$, so that $|\sigma_1|^2 = |\sigma_2|^2 = |b|^2 + |c|^2$. After noting that $a/c = -b^*/c^*$, Eq. (A.12) then yields:

$$\tan 2\theta = [\,\text{Re}\,(b/c)\,c_\phi + \text{Im}\,(b/c)\,s_\phi]^{-1}, \tag{A.23}$$

where $c_\phi \equiv \cos\phi$ and $s_\phi \equiv \sin\phi$. The reality of $\tan 2\theta$ imposes no constraint on ϕ; hence, ϕ is indeterminate [a fact that is suggested by Eq. (A.14)]. The same conclusion also follows immediately from Eq. (A.2). Namely, if $M_D = m\mathbb{1}_{2\times 2}$, then $(U\mathcal{O})^T M (U\mathcal{O}) = \mathcal{O}^T M_D \mathcal{O} = M_D$ for any real orthogonal matrix $\mathcal{O}$. In particular, ϕ simply represents the freedom to choose $\mathcal{O}$ [see, e.g., Eq. (A.28)]. Since ϕ is indeterminate, Eq. (A.23) implies that θ is indeterminate as well. In practice, it is often simplest

---

[18] For *real* symmetric matrices M, one can always find a real orthogonal V such that $V^T M V$ is diagonal. In this case the Takagi diagonalization is achieved by $U = VP$, where P is a diagonal matrix whose kk element is 1 [i] if the corresponding *eigenvalue* m_k is positive (negative). Of course, this procedure fails for complex symmetric matrices [such as M in Eq. (A.21)] that are not diagonalizable.

to choose a convenient value, say $\phi = 0$, which would then fix θ such that $\tan 2\theta = [\operatorname{Re}(b/c)]^{-1}$. For pedagogical reasons, we shall keep ϕ as a free parameter below.

Naively, it appears that α and β are also indeterminates. After all, the arguments of arg in both Eqs. (A.19) and (A.20) vanish in the degenerate limit. However, this is not a correct conclusion, as the derivation of Eqs. (A.19) and (A.20) involve a division by $|bc^* + a^*c|$, which vanishes in the degenerate limit. Thus, to determine α and β in the degenerate case, one must return to Eqs. (A.10) and (A.11). A straightforward calculation [which uses Eq. (A.23)] yields:

$$\frac{\sigma_2}{c} = -\frac{\sigma_1^*}{c^*}, \tag{A.24}$$

which implies

$$\alpha + \beta = \arg c \pm \frac{\pi}{2}. \tag{A.25}$$

Note that separately, α and β depend on the choice of ϕ (although ϕ drops out in the sum). Explicitly,

$$\sigma_1 = -c\,e^{-i\phi}\left\{\sqrt{1 + \left[c_\phi \operatorname{Re}(b/c) + s_\phi \operatorname{Im}(b/c)\right]^2} + i\left[s_\phi \operatorname{Re}(b/c) - c_\phi \operatorname{Im}(b/c)\right]\right\}, \tag{A.26}$$

$$\sigma_2 = c\,e^{i\phi}\left\{\sqrt{1 + \left[c_\phi \operatorname{Re}(b/c) + s_\phi \operatorname{Im}(b/c)\right]^2} - i\left[s_\phi \operatorname{Re}(b/c) - c_\phi \operatorname{Im}(b/c)\right]\right\}. \tag{A.27}$$

One easily verifies that Eq. (A.24) is satisfied. Moreover, using Eq. (A.9), α and β are now separately determined.

We illustrate the above results with the classic case of $M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. In this case $M^\dagger M = \mathbb{1}_{2\times 2}$, so U cannot be deduced by diagonalizing $M^\dagger M$. Setting $a = b = 0$ and $c = 1$ in the above formulae, it follows that $\theta = \pi/4$, $\sigma_1 = -e^{-i\phi}$ and $\sigma_2 = e^{i\phi}$, which yields $\alpha = -(\phi \pm \pi)/2$ and $\beta = \phi/2$. Thus, Eq. (A.7) yields:

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{i\phi} \\ -e^{-i\phi} & 1 \end{pmatrix}\begin{pmatrix} \pm i e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm i e^{i\phi/2} & e^{i\phi/2} \\ \mp i e^{-i\phi/2} & e^{-i\phi/2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix}\begin{pmatrix} \pm \cos(\phi/2) & \sin(\phi/2) \\ \mp \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}, \tag{A.28}$$

which illustrates explicitly that in the degenerate case, U is unique only up to multiplication on the right by an arbitrary orthogonal matrix.

Appendix B: The small-mixing approximation

The 6×6 USSM neutralino mass matrix of Eq. (2.12) cannot be diagonalized analytically in general. However, simple analytical expressions for masses and mixing parameters can be found, similarly as in

the NMSSM, by making use of approximations based on the natural assumption of small doublet-singlet higgsino, doublet higgsino-U(1)$_X$ gaugino mixing and kinetic gaugino mixing, i.e. for a large SUSY scale compared to the electroweak scale.

In this appendix, we provide details of the neutralino mass matrix diagonalization in the small mixing approximation, in which the weak coupling between two off-diagonal matrix blocks can be perturbatively treated. For mathematical clarity, we present the solution for a general complex $(N + M) \times (N + M)$ symmetric matrix in which the $N \times N$ and $M \times M$ submatrices are coupled weakly so that their mixing is small:

$$\mathcal{M}_{N+M} = \begin{pmatrix} \mathcal{M}_N & X_{NM} \\ X_{NM}^T & \mathcal{M}_M \end{pmatrix} \tag{B.1}$$

To obtain the corresponding physical neutralino masses, one must perform a Takagi diagonalization of $\mathcal{M}_{N+M}$:[19]

$$(N^{N+M})^* \, \mathcal{M}_{N+M} \, (N^{N+M})^\dagger = \operatorname{diag}(m_{1'}, \, m_{2'}, \, \dots, \, m_{N'+M'}), \qquad m_{k'} \geq 0, \tag{B.2}$$

where N^{N+M} is a unitary matrix.[20] The Takagi diagonalization of a general complex symmetric matrix is described in Appendix A. The non-negative $m_{k'}$ are called the singular values of M, which are defined as the non-negative square roots of the eigenvalues of $M^\dagger M$.

In Eq. (B.1), $\mathcal{M}_N$ and $\mathcal{M}_M$ are $N \times N$ and $M \times M$ complex symmetric submatrices with singular values generally of the SUSY scale, M_{SUSY}. X_{NM} is a rectangular $N \times M$ matrix whose matrix elements are generally of the electroweak scale. Assuming that the electroweak scale is significantly smaller than M_{SUSY}, one can treat X_{NM} as a perturbation as long as there are no accidental near-degeneracies between the singular values of $\mathcal{M}_N$ and $\mathcal{M}_M$, respectively. (The case of such a near-degeneracy is the subject of Appendix C.) The diagonalization of $\mathcal{M}_{N+M}$ can be performed using the following steps.

[1] In the first step, we separately perform a Takagi diagonalization of $\mathcal{M}_N$ and $\mathcal{M}_M$:

$$\overline{\mathcal{M}}_N^D = N^{N\,*} \mathcal{M}_N N^{N\,\dagger} = \operatorname{diag}(\overline{m}_{1'}, \dots \overline{m}_{N'}), \tag{B.3}$$

$$\overline{\mathcal{M}}_M^D = N^{M\,*} \mathcal{M}_M N^{M\,\dagger} = \operatorname{diag}(\overline{m}_{N'+1'}, \dots, \overline{m}_{N'+M'}), \tag{B.4}$$

where the $\overline{m}_{k'}$ are real and non-negative. The ordering of the diagonal elements above[21] is chosen according to some convenient criterion (e.g., see the discussion at the end of Sect. 2.) Analytical expressions can be obtained for the singular values and the Takagi vectors that comprise the columns of the

---

[19]In Eq. (B.2), we use primed subscripts to indicate that the corresponding neutralino states are continuously connected to the states of the unperturbed block matrix, $\operatorname{diag}(\overline{\mathcal{M}}_N^D, \overline{\mathcal{M}}_M^D)$, where the diagonal matrices $\overline{\mathcal{M}}_N^D$ and $\overline{\mathcal{M}}_M^D$ are defined in Eqs. (B.3) and (B.4).

[20]When N and M are used in subscripts and superscripts of matrices, they refer to the dimension of the corresponding square matrices. For rectangular matrices, two subscripts will be used.

[21]See footnote 19.

corresponding unitary matrices N^N and N^M for values of N, $M \leq 4$ [12].

Step [**1**] results in a partial Takagi diagonalization of $\mathcal{M}_{N+M}$:

$$\overline{\mathcal{M}}_{N+M} \equiv \begin{pmatrix} N^{N*} & \mathbb{O} \\ \mathbb{O}^T & N^{M*} \end{pmatrix} \begin{pmatrix} \mathcal{M}_N & X_{NM} \\ X_{NM}^T & \mathcal{M}_M \end{pmatrix} \begin{pmatrix} N^{N\dagger} & \mathbb{O} \\ \mathbb{O}^T & N^{M\dagger} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{\mathcal{M}}_N^D & N^{N*}X_{NM}N^{M\dagger} \\ N^{M*}X_{NM}^T N^{N\dagger} & \overline{\mathcal{M}}_M^D \end{pmatrix}. \tag{B.5}$$

where $\mathbb{O}$ is an $N \times M$ matrix of zeros. The upper left and lower right blocks of $\overline{\mathcal{M}}_{N+M}$ are diagonal with real non-negative entries, but the upper right and lower left off-diagonal blocks are non-zero.

[**2**] The ensuing $(N + M) \times (N + M)$ matrix, $\overline{\mathcal{M}}_{N+M}$, can be subsequently block-diagonalized by performing an $(N + M) \times (N + M)$ Takagi diagonalization of $\overline{\mathcal{M}}_{N+M}$. Since the elements of the off-diagonal blocks of $\overline{\mathcal{M}}_{N+M}$ are small compared to the diagonal elements $\overline{m}_{k'}$, we may treat X_{NM} as a perturbation. More precisely, X_{NM} can be treated as a perturbation if:

$$\left| \frac{\mathrm{Re}\,(N^{N*}X_{NM}N^{M\dagger})_{i'j'}}{\overline{m}_{i'} - \overline{m}_{j'}} \right| \ll 1, \tag{B.6}$$

for all choices of $i' = 1', \ldots, N'$ and $j' = N' + 1' \ldots, N' + M'$. This condition will be an output of our computation below.

The perturbative block-diagonalization is accomplished by introducing an $(N + M) \times (N + M)$ unitary matrix:

$$\overline{\mathcal{N}}_B \simeq \begin{pmatrix} \mathbb{1}_{N\times N} - \frac{1}{2}\Omega\Omega^\dagger & \Omega \\ -\Omega^\dagger & \mathbb{1}_{M\times M} - \frac{1}{2}\Omega^\dagger\Omega \end{pmatrix}, \tag{B.7}$$

where Ω is an $N \times M$ complex matrix that vanishes when X_{NM} vanishes (and hence like X_{NM} is perturbatively small). Note that $\overline{\mathcal{N}}_B\overline{\mathcal{N}}_B^\dagger = \mathbb{1}_{(N+M)\times(N+M)} + \mathcal{O}(\Omega^4)$ which is sufficiently close to the identity matrix for our purposes. Straightforward matrix multiplication then yields:

$$\overline{\mathcal{N}}_B^* \begin{pmatrix} \overline{\mathcal{M}}_N^D & \mathcal{B} \\ \mathcal{B}^T & \overline{\mathcal{M}}_M^D \end{pmatrix} \overline{\mathcal{N}}_B^\dagger = \begin{pmatrix} \mathcal{M}_N'^D + \mathcal{O}(\mathcal{B}\Omega^3) & \mathcal{B} + \Omega^*\overline{\mathcal{M}}_M^D - \overline{\mathcal{M}}_N^D\Omega + \mathcal{O}(\mathcal{B}\Omega^2) \\ \mathcal{B}^T + \overline{\mathcal{M}}_M^D\Omega^\dagger - \Omega^T\overline{\mathcal{M}}_N^D + \mathcal{O}(\mathcal{B}\Omega^2) & \mathcal{M}_M'^D + \mathcal{O}(\mathcal{B}\Omega^3) \end{pmatrix},$$

$$\tag{B.8}$$

where

$$\mathcal{B} \equiv N^{N*}X_{NM}N^{M\dagger}, \tag{B.9}$$

$$\mathcal{M}_N'^D \equiv \overline{\mathcal{M}}_N^D + \left[\Omega^*\mathcal{B}^T + \frac{1}{2}\Omega^*\overline{\mathcal{M}}_M^D\Omega^\dagger - \frac{1}{2}\overline{\mathcal{M}}_N^D\Omega\Omega^\dagger + \mathrm{transp} \right], \tag{B.10}$$

$$\mathcal{M}_M'^D \equiv \overline{\mathcal{M}}_M^D - \left[\Omega^T\mathcal{B} - \frac{1}{2}\Omega^T\overline{\mathcal{M}}_N^D\Omega + \frac{1}{2}\overline{\mathcal{M}}_M^D\Omega^\dagger\Omega + \mathrm{transp} \right], \tag{B.11}$$

and "transp" is an instruction to take the transpose of the preceding terms inside the bracket. For a consistent perturbative expansion, we may neglect all terms above that are hidden inside the various order symbols in Eq. (B.8). Hence, a successful block-diagonalization is achieved by demanding that

$$\mathcal{B} = \overline{\mathcal{M}}_N^D \Omega - \Omega^* \overline{\mathcal{M}}_M^D \,. \tag{B.12}$$

Inserting this result in Eqs. (B.10) and (B.11) and eliminating $\mathcal{B}$, we obtain:

$$\mathcal{M}_N'^D = \overline{\mathcal{M}}_N^D - \tfrac{1}{2} \left[\Omega^* \overline{\mathcal{M}}_M^D \Omega^\dagger - \overline{\mathcal{M}}_N^D \Omega \Omega^\dagger + \text{transp} \right] \,, \tag{B.13}$$

$$\mathcal{M}_M'^D = \overline{\mathcal{M}}_M^D - \tfrac{1}{2} \left[\Omega^T \overline{\mathcal{M}}_N^D \Omega - \overline{\mathcal{M}}_M^D \Omega^\dagger \Omega + \text{transp} \right] \,. \tag{B.14}$$

The results above simplify somewhat when we recall that $\overline{\mathcal{M}}_N^D$ and $\overline{\mathcal{M}}_M^D$ are diagonal matrices [see Eq. (B.3) and (B.4)]. Taking the real and imaginary parts of the matrix elements of Eq. (B.12) yields two equations for the real and imaginary parts of Ω_{ij}:

$$\text{Re}\,\Omega_{i'j'} \equiv \frac{\text{Re}\,\mathcal{B}_{i'j'}}{\overline{m}_{i'} - \overline{m}_{j'}} \,, \qquad\qquad \text{Im}\,\Omega_{i'j'} \equiv \frac{\text{Im}\,\mathcal{B}_{i'j'}}{\overline{m}_{i'} + \overline{m}_{j'}} \,, \tag{B.15}$$

with $i' = 1', \ldots, N'$ and $j' = N' + 1' \ldots, N' + M'$. Since the $\Omega_{i'j'}$ are the small parameters of the perturbation expansion, it follows that $|\,\text{Re}\,\mathcal{B}_{i'j'}/(\overline{m}_{i'} - \overline{m}_{j'})| \ll 1$, which is the perturbativity condition previously given in Eq. (B.6).

At this stage, the result of the perturbative block diagonalization is:

$$\overline{\mathcal{N}}_B^* \begin{pmatrix} \overline{\mathcal{M}}_N^D & \mathcal{B} \\ \mathcal{B}^T & \overline{\mathcal{M}}_M^D \end{pmatrix} \overline{\mathcal{N}}_B^\dagger = \begin{pmatrix} \mathcal{M}_N'^D & \mathcal{O}(\Omega^3) \\ \mathcal{O}(\Omega^3) & \mathcal{M}_M'^D \end{pmatrix} \,. \tag{B.16}$$

We can neglect the $\mathcal{O}(\Omega^3)$ terms above. Thus, the remaining task is to re-diagonalize the two diagonal blocks above. However, as long as we work self-consistently up to second order in perturbation theory, no further re-diagonalization is necessary. Indeed, the off-diagonal elements of $\mathcal{M}_N'^D$ and $\mathcal{M}_M'^D$ are of $\mathcal{O}(\Omega^2)$. However, in the Takagi diagonalization, the off-diagonal terms of the diagonal blocks only effect the corresponding diagonal elements at $\mathcal{O}(\Omega^4)$ which we neglect in this analysis. The diagonal elements of $\mathcal{M}_N'^D$ and $\mathcal{M}_M'^D$ also contain terms of $\mathcal{O}(\Omega^2)$, which generate second-order shifts of the diagonal elements relative to the $\overline{m}_{k'}$ obtained at step [1]. These are easily obtained from the diagonal matrix elements of Eqs. (B.13) and (B.14) after making use of Eq. (B.15):

$$m_{i'} \simeq \overline{m}_{i'} + \sum_{j'=N'+1'}^{N'+M'} \left\{ \frac{[\text{Re}\,\mathcal{B}_{i'j'}]^2}{\overline{m}_{i'} - \overline{m}_{j'}} + \frac{[\text{Im}\,\mathcal{B}_{i'j'}]^2}{\overline{m}_{i'} + \overline{m}_{j'}} + \frac{2\text{i}\,m_{j'}\,\text{Re}\,\mathcal{B}_{i'j'}\,\text{Im}\,\mathcal{B}_{i'j'}}{\overline{m}_{i'}^2 - \overline{m}_{j'}^2} \right\} \,, \tag{B.17}$$

$$m_{j'} \simeq \overline{m}_{j'} - \sum_{i'=1'}^{N'} \left\{ \frac{[\text{Re}\,\mathcal{B}_{ij}]^2}{\overline{m}_{i'} - \overline{m}_{j'}} - \frac{[\text{Im}\,\mathcal{B}_{i'j'}]^2}{\overline{m}_{i'} + \overline{m}_{j'}} + \frac{2\text{i}\,m_{i'}\,\text{Re}\,\mathcal{B}_{i'j'}\,\text{Im}\,\mathcal{B}_{i'j'}}{\overline{m}_{i'}^2 - \overline{m}_{j'}^2} \right\} \,, \tag{B.18}$$

with $i' = 1', .., N'$ and $j' = N' + 1', .., N' + M'$. Although the $\overline{m}_{k'}$ are real and non-negative by construction, we see that the shifted mass parameters $m_{k'}$ are in general complex. Thus, to complete the perturbative Takagi diagonalization, we perform one final step.

[3] The diagonal neutralino mass matrix is given by:

$$\mathcal{M}_{N+M}^D = \mathcal{P}^* \overline{\mathcal{N}}_B^* \begin{pmatrix} \overline{\mathcal{M}}_N^D & \mathcal{B} \\ \mathcal{B}^T & \overline{\mathcal{M}}_M^D \end{pmatrix} \overline{\mathcal{N}}_B^\dagger \mathcal{P}^\dagger = \mathrm{diag}(m_{1'}^{ph}, \ldots, m_{N'+M'}^{ph}) \, , \tag{B.19}$$

where $\mathcal{P}$ is a suitably chosen diagonal matrix of phases

$$\mathcal{P} = \mathrm{diag}(e^{-i\phi_{1'}}, \, \ldots \, , e^{-i\phi_{N'+M'}}) \, , \tag{B.20}$$

such that the elements of the diagonal mass matrix $\mathcal{M}_{N+M}^D$ (denoted by $m_{k'}^{ph}$) are real and non-negative. We identify the $m_{k'}^{ph}$ with the physical neutralino masses. The unitary neutralino mixing matrix is then identified as:

$$N^{N+M} = \mathcal{P} \overline{\mathcal{N}}_B \begin{pmatrix} N^N & \mathbb{0} \\ \mathbb{0}^{\mathbf{T}} & N^M \end{pmatrix} \, . \tag{B.21}$$

Starting from Eqs. (B.17) and (B.18), one can evaluate $\mathcal{P}$ to second order in the perturbation Ω. In particular, for $\epsilon_{1,2} \ll a$, we have $a + \epsilon_1 + i\epsilon_2 \simeq (a + \epsilon_1)e^{i\epsilon_2/a}$. From this result, we easily derive the second-order expressions for the physical neutralino masses $m_{k'}^{ph}$:

$$m_{i'}^{ph} \simeq \overline{m}_{i'} + \sum_{j'=N'+1'}^{N'+M'} \left\{ \frac{[\mathrm{Re}\,\mathcal{B}_{i'j'}]^2}{\overline{m}_{i'} - \overline{m}_{j'}} + \frac{[\mathrm{Im}\,\mathcal{B}_{i'j'}]^2}{\overline{m}_{i'} + \overline{m}_{j'}} \right\} \, , \quad [i' = 1', \ldots, N'] \, , \tag{B.22}$$

$$m_{j'}^{ph} \simeq \overline{m}_{j'} - \sum_{i'=1'}^{N'} \left\{ \frac{[\mathrm{Re}\,\mathcal{B}_{i'j'}]^2}{\overline{m}_{i'} - \overline{m}_{j'}} - \frac{[\mathrm{Im}\,\mathcal{B}_{i'j'}]^2}{\overline{m}_{i'} + \overline{m}_{j'}} \right\} \, , \quad [j' = N' + 1', \ldots, N' + M'] \, . \tag{B.23}$$

and the phases $\phi_{k'}$:

$$\phi_{i'} \simeq - \sum_{j'=N'+1'}^{N'+M'} \frac{\overline{m}_{j'}}{\overline{m}_{i'}(\overline{m}_{i'}^2 - \overline{m}_{j'}^2)} \, \mathrm{Re}\,\mathcal{B}_{i'j'} \, \mathrm{Im}\,\mathcal{B}_{i'j'} \, , \quad [i' = 1', \ldots, N'] \, , \tag{B.24}$$

$$\phi_{j'} \simeq \sum_{i'=1'}^{N'} \frac{\overline{m}_{i'}}{\overline{m}_{j'}(\overline{m}_{i'}^2 - \overline{m}_{j'}^2)} \, \mathrm{Re}\,\mathcal{B}_{i'j'} \, \mathrm{Im}\,\mathcal{B}_{i'j'} \, , \quad [j' = N' + 1', \ldots, N' + M'] \, , \tag{B.25}$$

This completes the perturbative Takagi diagonalization of the mass matrix for N-dimensional and M-dimensional subsystems of Majorana fermions weakly coupled by an off-diagonal perturbation. As noted in Eq. (B.6), the perturbation theory breaks down if any mass $\overline{m}_{i'}$ from the N-dimensional subsystem is nearly degenerate with a corresponding mass $\overline{m}_{j'}$ from the M-dimensional subsystem (assuming that the corresponding residue, $\mathrm{Re}\,\mathcal{B}_{i'j'}$, does not vanish). We provide an analytic approach to this case of near-degeneracy in Appendix C.

Appendix C: Degenerate mass eigenvalues

If the value of one of the diagonal $\overline{\mathcal{M}}_N^D$ elements, $\overline{m}_{k'}$, is nearly equal to one of the diagonal $\overline{\mathcal{M}}_M^D$ elements, say $\overline{m}_{\ell'}$, and the corresponding residue $\mathrm{Re}\,\mathcal{B}_{k'\ell'}$ does not vanish [cf. Eqs. (B.22) and (B.23)], then the techniques for degenerate states must be applied to diagonalize the full $(N+M) \times (N+M)$ matrix. We begin with the matrix $\overline{\mathcal{M}}_{N+M}$ given in Eq. (B.5), which contains off-diagonal blocks of $\mathcal{O}(X)$, which characterizes the small couplings between the original MSSM matrix and the new USSM singlino/gaugino submatrix.

We first interchange the first row and the k'th row of $\overline{\mathcal{M}}_{N+M}$ followed by an interchange of the first column and the k'th column, in order that $\overline{m}_{k'}$ occupy the $1''1''$ element of the matrix.[22] Next, we interchange the second row and the ℓ'th row followed by an interchange of the second column and the ℓ'th column, in order that $\overline{m}_{\ell'}$ occupy the $2''2''$ element of the matrix. This sequence of interchanges has the effect of grouping the two nearly degenerate diagonal elements next to each other, resulting in a new matrix $\overline{\mathcal{M}}'_{N+M}$ with the following structure:

$$\overline{\mathcal{M}}'_{N+M} = \left(\begin{array}{cc|c} \overline{m}_{1''} & \delta & \\ \delta & \overline{m}_{2''} & \Delta \\ \hline & \Delta^T & \mathcal{M}_{N+M-2} \end{array} \right) , \tag{C.1}$$

where the parameter δ and the submatrix Δ are of $\mathcal{O}(X)$. The submatrix $\mathcal{M}_{N+M-2}$ is no longer diagonal, although its new off-diagonal elements are all of $\mathcal{O}(X)$. Thus, we may perform a perturbative Takagi diagonalization using the block diagonal unitary matrix, $\mathrm{diag}(\mathbb{1}_{2 \times 2}, N^{N+M-2})$, with

$$\overline{\mathcal{M}}_{N+M-2} = (N^{N+M-2})^* \, \mathcal{M}_{N+M-2} \, (N^{N+M-2})^\dagger = \mathrm{diag}(\overline{m}'_{3''}, \overline{m}'_{4''} \ldots, \overline{m}'_{N''+M''}), \tag{C.2}$$

where the $\overline{m}'_{j''}$ $[j'' = 3'', 4'', \ldots, N'' + M'']$ are slightly shifted from the original non-degenerate $\{\overline{m}_{i''}\}$, $\{\overline{m}_{j''}\}$ by the perturbation of $\mathcal{O}(X)$.[23]

As a result of this procedure, the matrix $\overline{\mathcal{M}}'_{N+M}$ in Eq. (C.1) is modified by replacing the submatrix $\mathcal{M}_{N+M-2}$ by a diagonal matrix with perturbatively shifted diagonal elements, $\overline{\mathcal{M}}_{N+M-2}$. The off-diagonal blocks Δ and Δ^T, are perturbatively shifted as well, but these shifts can be neglected as these effects are of higher order in the perturbation X. We denote the resulting matrix by $\overline{\mathcal{M}}''_{N+M}$.

The complex parameter δ couples the two near-degenerate states with mass parameters $\overline{m}_{1''}$ and $\overline{m}_{2''}$. By definition of near-degeneracy, $|\overline{m}_{1''} - \overline{m}_{2''}| \ll \delta$, so one cannot use perturbation theory in $\delta \sim \mathcal{O}(X)$. Instead, we shall perform an exact Takagi diagonalization of the 2×2 block $\left(\begin{smallmatrix} \overline{m}_{1''} & \delta \\ \delta & \overline{m}_{2''} \end{smallmatrix} \right)$ of

---

[22]To distinguish the ordering of the physical neutralino states that arises from the manipulations performed in this appendix from the ordering of states based on Eqs. (B.3) and (B.4), we employ double-primed subscripts here.

[23]For consistency with the second-order perturbative results of Appendix B, this diagonalization should be carried out including all contributions quadratic in X.

$\overline{\mathcal{M}}_{N+M}''$, using the results of Appendix A.2:

$$\begin{pmatrix} W^* & \mathbb{O} \\ \mathbb{O}^{\mathbf{T}} & \mathbb{1} \end{pmatrix} \overline{\mathcal{M}}_{N+M}'' \begin{pmatrix} W^\dagger & \mathbb{O} \\ \mathbb{O}^{\mathbf{T}} & \mathbb{1} \end{pmatrix} = \left(\begin{array}{cc|c} \overline{m}_{1''}' & 0 & W^*\Delta \\ 0 & \overline{m}_{2''}' & \\ \hline \Delta^T W^\dagger & & \overline{\mathcal{M}}_{N+M-2} \end{array} \right) , \tag{C.3}$$

where the elements of the 2×2 unitary matrix W (which is denoted by $U^\dagger$ in Appendix A) can be determined in terms of $\overline{m}_{1''}$, $\overline{m}_{2''}$ and δ using the formulae of Appendix A.2. The (non-negative) diagonal masses $\overline{m}_{1''}'$ and $\overline{m}_{2''}'$ are obtained from Eq. (A.18):

$$\overline{m}_{1'',2''}' = \frac{1}{\sqrt{2}} \left\{ \overline{m}_{1''}^2 + \overline{m}_{2''}^2 + 2|\delta|^2 \mp \sqrt{(\overline{m}_{2''}^2 - \overline{m}_{1''}^2)^2 + 4|\overline{m}_{1''}\delta + \overline{m}_{2''}\delta^*|^2} \right\}^{1/2} . \tag{C.4}$$

Note that if δ is real, the quantity under the square root is a perfect square, in which case Eq. (C.4) reduces to the well-known expression:

$$\overline{m}_{1'',2''}' = \frac{1}{2} \left[\overline{m}_{1''} + \overline{m}_{2''} \mp \sqrt{(\overline{m}_{2''} - \overline{m}_{1''})^2 + 4\delta^2} \right] , \qquad \text{for real } \delta . \tag{C.5}$$

If δ is very small, the trajectories of the two eigenvalues nearly touch each other when the parameter M_1' moves through the cross-over zone. A non-zero δ value prevents the trajectories from crossing, keeping them at a distance $\geq \delta$. In the $4'$-$5'$ zone, δ is of first order in the ratio v/M_{SUSY}. In contrast, in the $2'$-$6'$ zone, δ vanishes at first order due to the fact that $V_{2'6'}^6 \approx V_{6'2'}^6 \approx 0$. However, as discussed below Eq. (3.39), these matrix elements acquire small non-zero corrections at higher order in v/M_{SUSY}. Thus, we have two very different behaviors for δ, leading to the characteristically different evolution of the trajectories. These two cases are illustrated by the dashed lines in the two panels of Fig. 9; on the left for $\delta \to 0$ and on the right for moderately non-zero δ values.

We may now apply the perturbative block diagonalization technique of Appendix B to complete the Takagi diagonalization of Eq. (C.3). The effect of this step is to shift the diagonal masses at second order as indicated in Eqs. (B.22) and (B.23). We finally arrive at the physical neutralino masses:

$$m_{i''}^{ph} \simeq \overline{m}_{i''}' + \sum_{j''=3''}^{N''+M''} \left\{ \frac{[\operatorname{Re}(W^*\Delta)_{i''j''}]^2}{\overline{m}_{i''}' - \overline{m}_{j''}'} + \frac{[\operatorname{Im}(W^*\Delta)_{i''j''}]^2}{\overline{m}_{i''}' + \overline{m}_{j''}'} \right\} , \quad [i'' = 1'', 2''], \tag{C.6}$$

$$m_{j''}^{ph} \simeq \overline{m}_{j''}' - \sum_{i''=1''}^{N''} \left\{ \frac{[\operatorname{Re}(W^*\Delta)_{i''j''}]^2}{\overline{m}_{i''}' - \overline{m}_{j''}'} - \frac{[\operatorname{Im}(W^*\Delta)_{i''j''}]^2}{\overline{m}_{i''}' + \overline{m}_{j''}'} \right\} , \quad [j'' = 3'', 4'', \ldots, N'' + M'']. \tag{C.7}$$

Since the appearance of $\overline{m}_{1''}'$ and $\overline{m}_{2''}'$ [given by Eq. (C.4)] takes care of the near-degeneracy via an exact diagonalization (within the near-degenerate subspace), the results for the physical masses given above provide a reliable analytic description.

The sizes of the second-order perturbative shifts in Eqs. (C.6) and (C.7) vary with the parameter M_1' as the $\overline{m}_{j''}'$ $[j'' = 3'', .., (N'' + M'')]$ depend on M_1'. The effect of these shifts can be discerned in the
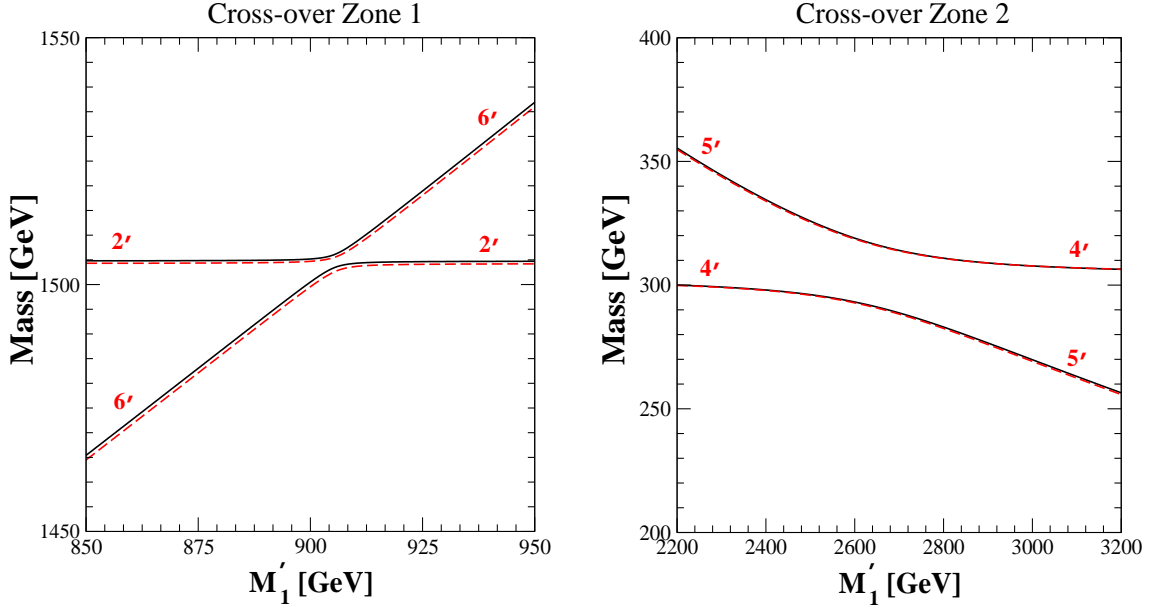
Figure 9: *The evolution of the neutralino masses near the cross-over zone 1 (left) and near the crossover zone 2 (right) when varying the $U(1)_X$ gaugino mass parameter M_1'. The red dashed lines represent the masses of the diagonalized 2×2 matrix and the black solid lines after the subsequent approximate diagonalization of the full 6×6 matrix [Eq. (C.3)].*

two cases considered above—in the cross-over zone $2'$-$6'$ with very small δ, and in the cross-over zone $4'$-$5'$ with moderately small δ, as shown by the solid line trajectories of Fig. 9.

Thus, we have demonstrated that an analytic perturbative treatment of the neutralino mass matrix can be carried out, and all of its features understood, even in the case of a pair of near-degenerate states.

References

[1] For a review, see M. Cvetič and P. Langacker, in *Perspectives on Supersymmetry*, edited by G.L. Kane (World Scientific, Singapore, 1998), pp. 312-331 [hep-ph/9707451].

[2] J.L. Hewett and T.G. Rizzo, Phys. Rept. **183** (1989) 193; M. Cvetič and P. Langacker, Phys. Rev. D **54** (1996) 3570 [hep-ph/9511378]; Mod. Phys. Lett. A **11** (1996) 1247 [hep-ph/9602424].

[3] M. Cvetič, D.A. Demir, J.R. Espinosa, L.L. Everett and P. Langacker, Phys. Rev. D **56** (1997) 2861 [hep-ph/9703317]; **58** (1997) 119905(E).

[4] D. Suematsu and Y. Yamagishi, Int. J. Mod. Phys. A **10** (1995) 4521 [hep-ph/9411239]. For the case of gauge mediated symmetry breaking, see P. Langacker, N. Polonsky and J. Wang, Phys. Rev. D **60** (1999) 115005 [hep-ph/9905252].

[5] H.P. Nilles, M. Srednicki and D. Wyler, Phys. Lett. B **120** (1983) 346; J.M. Frere, D.R.T. Jones and S. Raby, Nucl. Phys. B **222** (1983) 11; J.P. Derendinger and C.A. Savoy, Nucl. Phys. B **237** (1984) 307; M.I. Vysotsky, K.A. Ter-Martirosian, Sov. Phys. JETP **63** (1984) 307; J.R. Ellis, J.F. Gunion, H.E. Haber, L. Roszkowski, F. Zwirner, Phys. Rev. D **39** (1989) 844; U. Ellwanger, M. Rausch de Traubenberg and C.A. Savoy, Phys. Lett. B **315** (1993) 331 [hep-ph/9307322]; Nucl. Phys. B **492** (1997) 21 [hep-ph/9611251]; F. Franke and H. Fraas, Int. J. Mod. Phys. A **12** (1997) 479 [hep-ph/9512366]. For a recent summary see D.J. Miller, R. Nevzorov and P.M. Zerwas, Nucl. Phys. B **681** (2004) 3 [hep-ph/0304049].

[6] H.E. Haber and M. Sher, Phys. Rev. D **35** (1987) 2206; L. Durand and J.L. Lopez, Phys. Lett. B **217** (1989) 463; M. Drees, Int. J. Mod. Phys. A **4** (1989) 3635.

[7] J. Erler, Nucl. Phys. B **586** (2000) 73 [hep-ph/0006051].

[8] S. King, S. Moretti, R. Nevzorov, Phys. Lett. B **634** (2006) 278 [hep-ph/0511256]; Phys. Rev. D **73** (2006) 035009 [hep-ph/0510419].

[9] J. Erler, P. Langacker and T. Li, Phys. Rev. D **66** (2002) 015002 [hep-ph/0205001].

[10] D. Suematsu, Phys. Rev. D **57** (1998) 1738 [hep-ph/9708413]; S. Hesselbach, F. Franke and H. Fraas, Eur. Phys. J. C. **23** (2002) 149 [hep-ph/0107080]; F. Franke and S. Hesselbach, Phys. Lett. B **526** (2002) 370 [hep-ph/0111285]; V. Barger, P. Langacker and H.S. Lee, Phys. Lett. B **630** (2005) 85 (2005) [hep-ph/0508027]; V. Barger, P. Langacker and G. Shaughnessy, arXiv:hep-ph/0609068.

[11] D. Suematsu, Mod. Phys. Lett. A **12** (1997) 1709 [hep-ph/9705412]; Phys. Lett. B **416** (1998) 108 [hep-ph/9705405]; G.A. Moortgat-Pick, S. Hesselbach, F. Franke and H. Fraas, arXiv:hep-ph/9909549; S. Hesselbach, F. Franke and H. Fraas, arXiv:hep-ph/0003272; V. Barger, C. Kao, P. Langacker and H.S. Lee, Phys. Lett. B **600** (2004) 104 [hep-ph/0408120]; **614** (2005) 67 [hep-ph/0412136].

[12] S.Y. Choi, J. Kalinowski, G.A. Moortgat-Pick and P.M. Zerwas, Eur. Phys. J. C **22** (2001) 563 [hep-ph/0108117]; **23** (2002) 769 [hep-ph/0202039].

[13] S.Y. Choi, D.J. Miller and P.M. Zerwas, Nucl. Phys. B **711** (2005) 83 [hep-ph/0407209].

[14] ATLAS Technical Proposal, CERN/LHCC/94-43, LHCC/P2 (1994); CMS Physics, Technical Design Report, CERN/LHCC/2006/021.

[15] E. Accomando *et al.*, Phys. Rept. **299** (1998) 1, hep-ph/9705442 J.A. Aguilar-Saavedra *et al.* [ECFA/DESY LC Physics Working Group Collaboration], hep-ph/0106315; T. Abe *et al.* [American LC Working Group], hep-ex/0106055-58; K. Abe *et al.* [ACFA LC Working Group], hep-ex/0109166; E. Accomando *et al.* [CLIC Physics Working Group Collaboration], hep-ph/9412251; W. Kilian and P.M. Zerwas, Proc. 2005 Snowmass ILC Workshop, hep-ph/0601217.

[16] B. Holdom, Phys. Lett. B **166** (1986) 196.

[17] F. del Aguila, Acta. Phys. Pol. B **25** (1994) 1317 [hep-ph/9404323]; F. del Aguila, M. Cvetič and P. Langacker, Phys. Rev. D **52** (1995) 37 [hep-ph/9501390]; K.S. Babu, C. Kolda and J. March-Russell, Phys. Rev. D **54** (1996) 4635 [hep-ph/9603212]; D. Suematsu, Phys. Rev. D **59** (1999) 055017 [hep-ph/9808409].

[18] D. Bailin and A. Love, *Supersymmetric Gauge Field Theory and String Theory* (Institute of Physics Publishing, Bristol, UK, 1994).

[19] K.R. Dienes, C. Kolda, J. March-Russell, Nucl. Phys. B **492** (1997) 104 [hep-ph/9610479].

[20] S.A. Abel, J. Jaeckel, V.V. Khoze and A. Ringwald, arXiv:hep-ph/0608248.

[21] P. Abreu *et al.* [DELPHI Collaboration], Phys. Lett. B **485** (2000) 45; R. Barate *et al.* [ALEPH Collaboration], Eur. Phys. J. C **12** (2000) 183; A. Abulencia *et al.* [CDF Collaboration], hep-ph/0602045.

[22] J.A. Aguilar-Saavedra *et al.*, Eur. Phys. J. C **46** (2006) 43 [hep-ph/0511344].

[23] T. Takagi, Japan J. Math. **1** (1925) 83.

[24] R.A. Horn and C.R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, England, 1990).

[25] A. Bunse-Gerstner and W.B. Gragg, J. Comp. Appl. Math. **21** (1988) 41; W. Xu and S. Qiao, "A Divide-and-Conquer Method for the Takagi Factorization," Technical Report No. CAS 05-01-SQ, (February 2005).

[26] X. Wang and S. Qiao, in the Proceedings of the International Conference on Parallel and Distributed Processing Techniques and Applications, Vol. I, edited by H.R. Arabnia, pp. 206-212, Las Vegas, Nevada, USA, June 2002, pp. 206-212; F.T. Luk and S. Qiao, in *Advanced Signal Processing Algorithms, Architectures, and Implementations XI*, edited by F.T. Luk, Proc. SPIE **4474** (2001) 254.

[27] T. Hahn, arXiv:physics/0607103.

[28] P.A. Carruthers, J. Math. Phys. **9** (1968) 1835; *Spin and Isospin in Particle Physics* (Gordon and Breach, New York, NY, 1971); B. Kayser, Phys. Rev. D **30** (1984) 1023; B. Kayser, F. Gibrat-Debu and F. Perrier, World Sci. Lect. Notes Phys. **25** (1989) 1.

[29] J.F. Gunion and H.E. Haber, Phys. Rev. D **37** (1988) 2515; S.Y. Choi, M. Drees and B. Gaissmaier, Phys. Rev. D **70** (2004) 014010 10,2004 [hep-ph/0403054].

[30] A. Leike, Phys. Rept. **317** (1999) 143 [hep-ph/9805494]; W.M. Yao *et al.* [Particle Data Group], J. Phys. G **33** (2006) 1.

[31] H.E. Haber and G.L. Kane, Phys. Rept. **117** (1985) 75.

[32] S.Y. Choi, A. Djouadi, M. Guchait, J. Kalinowski, H.S. Song and P.M. Zerwas, Eur. Phys. J. C **14**, 535 (2000) [hep-ph/0002033].

[33] T. Han, P. Langacker and B. McElrath, Phys. Rev. D **70** (2004) 115006 [hep-ph/0405244]; V. Barger, P. Langacker, H.S. Lee and G. Shaughnessy, Phys. Rev. D **73** (2006) 115010 [hep-ph/0603247].

[34] D.A. Demir and L.L. Everett, Phys. Rev. D **69** (2004) 015008 [hep-ph/0306240].

[35] V. Barger, P. Langacker and G. Shaughnessy, arXiv:hep-ph/0611112; arXiv:hep-ph/0611239.

[36] For a recent review and references to the original literature, see e.g., M. Carena and H.E. Haber, Prog. Part. Nucl. Phys. **50** (2003) 63 [hep-ph/0208209].

[37] H.E. Haber and D. Wyler, Nucl. Phys. B **323** (1989) 267.

[38] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis* (Cambridge University Press, Cambridge, England, 1991).

[39] L. Autonne, *Sur les matrices hypohermitiennes et sur les matrices unitaire*, Annales de l'Université de Lyon, Nouvelle Série I, Fasc. **38** (1915) 1-77.

[40] S.Y. Choi and M. Drees, unpublished. This proof was inspired by the diagonalization algorithm of hermitian matrices in W.H. Williams, B.P. Flannery, S.A. Teukolsky and W.T. Vetterling, *Numerical Recipes in Fortran 77* (Cambridge University Press, Cambridge, England, 1999), section 11.4. A similar method of proof is outlined in Ref. [24], section 4.4, problem 2 (on pp. 212-213) and section 4.6, problem 15 (on p. 254).

[41] F.D. Murnaghan, *The Unitary and Rotation Groups* (Spartan Books, Washington, DC, 1962).