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$N = 1$ domain wall solutions of massive type II supergravity as generalized geometries

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ABSTRACT

We study $N = 1$ domain wall solutions of type IIB supergravity compactified on a Calabi–Yau manifold in the presence of RR and NS electric and magnetic fluxes. We show that the dynamics of the scalar fields along the direction transverse to the domain wall is described by gradient flow equations controlled by a superpotential W. We then provide a geometrical interpretation of the gradient flow equations in terms of the mirror symmetric compactification of type IIA. They correspond to a set of generalized Hitchin flow equations of a manifold with $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure which is fibered over the direction transverse to the domain wall.

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1 Introduction

Domain wall (DW) solutions of supergravity have received a lot of attention recently which is largely due to their role in the AdS/CFT correspondence [1]. However, apart from this application they also have been studied as a class of supersymmetric ground states alternative to the commonly considered Minkowski or AdS backgrounds. In particular supergravities with non-trivial background fluxes often do not admit a stable, four-dimensional supersymmetric ground state but they do have BPS DW solutions. For example, in type IIB supergravity compactified on Calabi-Yau threefolds with non-trivial three-form fluxes it is necessary to include D-branes and orientifold planes in order to cancel the tadpoles induced by the fluxes and to obtain an $N = 1$ supersymmetric Minkowski background [2]. On the other hand without orientifold planes no four-dimensional Minkowski background is allowed. However, in this case three-dimensional $N = 1$ DW solutions do exist [3, 4].

In this paper we continue the study of such DW solutions of type IIB and generalize the previous results [3, 4] in various respects. More specifically we start from type IIB supergravity compactified on Calabi-Yau threefolds with electric and magnetic background three-form flux for both the NS three-form H_3 and the RR three-form F_3 [5]–[10]. In the presence of the magnetic fluxes the four dimensional antisymmetric tensors fields $B_{\mu\nu}$ and $C_{\mu\nu}$ become massive [10]. For this case the corresponding supergravity has only recently been constructed in refs. [11]–[14]. Using these four-dimensional $N = 2$ supergravities we study their $N = 1$ DW solutions including non-trivial magnetic fluxes. We find that the resulting DW necessarily is flat and furthermore that the background profile of the scalar fields is governed by a set of gradient flow equations expressed in terms of a single superpotential W, which is related to the superpotential suggested in [6].

The DW solutions of type IIB have their mirror analogous in type IIA. Without fluxes mirror symmetry identifies type IIB compactifications on a Calabi-Yau manifold $\tilde{Y}$ with type IIA compactified on the mirror Calabi-Yau Y [15]. In the presence of RR fluxes mirror symmetry is straightforwardly extended by also exchanging the respective flux parameters [6, 10]. For NS fluxes the situation is slightly more involved in that mirror symmetry can relate Calabi-Yau compactification with fluxes to purely geometrical compactification on a manifold $\hat{Y}$ without flux [16]–[19] or possible also to non-geometrical backgrounds [20]. For the case of geometrical backgrounds $\hat{Y}$ is no longer a Calabi-Yau manifold but rather a manifold with $SU(3)$ structure or more generally with $SU(3) \times SU(3)$-structure [21]–[30]. In such compactifications the (intrinsic) torsion of $\hat{Y}$ plays the 'mirror-role' of the fluxes.

This generalized mirror symmetry is also reflected in the DW solutions. For electric fluxes it was shown in [4] that the mirror symmetric DW can be interpreted as a solution of type IIA supergravity in a warped background $M^{1,2} \times_w X_7$. As a consequence of the $N = 1$ supersymmetry of the DW X_7 has G_2 holonomy and furthermore consists of a six-dimensional manifold $\hat{Y}$ fibered over the real line. The G_2 holonomy constrains $\hat{Y}$ to be within a special class of manifolds with $SU(3)$ structure termed 'half-flat' [21, 22]. From a mathematically point of view such fibration were studied in [21] and the DW solution precisely corresponds to the Hitchin flow equations.

For magnetic fluxes mirror symmetry is more involved. In [29, 31] it is shown that in this case $\hat{Y}$ has to be within a special class of manifolds with $SU(3) \times SU(3)$ structure. In this paper we generalize the analysis of ref. [4] and show that the mirror symmetric DW solution of type IIB with magnetic NS-flux also is of the form $M^{1,2} \times_w X_7$. However, in this case $\hat{Y}$ has to be a manifold with $SU(3) \times SU(3)$ structure which satisfies a set of generalized Hitchin flow equations given in ref. [28]. X_7 in turn has an integrable $G_2 \times G_2$ structure and is Ricci-flat as demanded by string theory.

This paper is organized as follows. In section 2 we set the stage for our analysis and recall the $N = 2$ supergravity arising as the low energy limit of type IIB string theory compactified on Calabi-Yau threefolds with background flux. In section 3.1 we study the $N = 1$ DW solutions and show that the scalar fields vary according to gradient flow equations. In section 3.2 we explicitly solve these equations and rewrite the solution in terms of mirror symmetric type IIA variables. This sets the stage for section 4 where we show that the DW solutions correspond to generalized Hitchin flow equations of a geometrical $SU(3) \times SU(3)$ background. Further details are found in two appendices.

2 $N = 2$ Supergravity with Abelian electric and magnetic charges

In order to set the stage for the discussion of the DW solutions let us briefly recall the structure of $N = 2$ supergravity with massive tensor multiplets as it arises from type IIB string theory compactified on Calabi-Yau threefolds with both electric and magnetic three-form fluxes. The $N = 2$ supergravity including massive tensor multiplets has been constructed in references [11, 12, 13, 14] where further details can be found. Here (and in appendix A) we only summarize the results needed in the following.

An $N = 2$ tensor multiplet contains $n_T \leq 3$ antisymmetric tensor, $4 - n_T$ real scalars and two Weyl fermions as its components. If the tensors are massless they can be dualized into scalars and hence a massless tensor multiplet is dual to a hypermultiplet which contains four real scalars and two Weyl fermions [32, 33]. In this dual formulation the remnant of the tensors are translational isometries acting on the dual scalars. In the standard (ungauged) $N = 2$ supergravity [34] one dualizes all tensor multiplet such that the theory contains only one gravitational multiplet, vector multiplets and hypermultiplets.

On the other hand a massive tensor is dual to a massive vector and it is often more convenient to keep the tensor multiplet in the spectrum. Such a theory can be viewed as a $N = 2$ supergravity with tensor multiplets which is deformed by Abelian electric and magnetic charges [11, 12, 13]. These charges are not related to any gauging of isometries on the residual scalar manifold. Instead the electric charges appear in Green–Schwarz type interaction of the tensors with the gauge fields while the magnetic charge appear in the Stückelberg mass terms of the tensors.

In this paper we do not discuss the general case [13] but instead focus on type IIB theories compactified on Calabi-Yau threefolds $\tilde{Y}$ in the presence of electric and magnetic three-form fluxes [12, 9, 10]. In this case the spectrum features a gravitational multiplet

$$(g_{\mu\nu},\, \psi_{A\mu},\, \psi^A_\mu,\, A^0_\mu) \tag{2.1}$$

where $g_{\mu\nu}$ is the metric, $\psi_{A\mu}$, $A = 1, 2$ are the two chiral gravitinos while A^0_μ is the graviphoton. In addition there are $n_V = h^{(1,2)}$ vector multiplets

$$(A^i_\mu,\, \lambda^{iA},\, \lambda^i_A,\, t^i)\ , \qquad i = 1, \ldots, n_V\ , \tag{2.2}$$

where A^i_μ are the gauge bosons, λ^{iA} are the doublets of chiral gaugini while t^i are complex scalars.[1] Finally there are $n_H = h^{(1,1)}$ hypermultiplets and one double tensor multiplet. Since they couple non-trivially it is convenient to combine them as

$$(\zeta_\alpha,\, \zeta^\alpha,\, q^u,\, B_{I\mu\nu})\ , \qquad \begin{array}{l} \alpha = 1, \ldots, 2n_H + 2\ , \\ u = 1, \ldots, 4n_H + 2, \quad I = 1,\, 2\ . \end{array} \tag{2.3}$$

Each of these multiplets features two chiral hyperinos which we collectively denote as ζ_α. The bosonic components of the hypermultiplets are $4n_H$ real scalars, while the double tensor multiplet contains two antisymmetric tensors $B_{I\mu\nu}$ (they are the four-dimensional part of the RR and the NSNS two–forms) together with the axion l and the four dimensional dilaton φ. We denote the scalars in the hypermultiplets and in the double tensor multiplet collectively by q^u.

The background fluxes arise from expanding both the RR three-form F_3 and the NS three-form H_3 along the third cohomology H^3 of the Calabi-Yau manifold

$$F_3 + \tau H_3 = m^\Lambda \alpha_\Lambda - e_\Lambda \beta^\Lambda\ , \qquad \Lambda = 0, \ldots, h^{(1,2)}\ , \tag{2.4}$$

where

$$e_\Lambda = e^1_\Lambda + \tau e^2_\Lambda\ , \qquad m^\Lambda = m^{\Lambda 1} + \tau m^{\Lambda 2}\ , \tag{2.5}$$

are the electric and magnetic background fluxes[2] and τ is the ten-dimensional complex type IIB dilaton $\tau = l + ie^{-\phi}$. The three-forms $(\alpha_\Lambda, \beta^\Lambda)$ denote a real, symplectic basis of H^3.

In the next section we will search for $N = 1$ DW solutions of the effective supergravity arising from type IIB compactifications. For this task we need the scalar part of the supersymmetry transformation of the fermions which

[1] Here $h^{(1,1)}$ and $h^{(1,2)}$ are the Hodge numbers of the Calabi-Yau manifold $\tilde{Y}$. Throughout the paper we denote by t^i the scalars of the vector multiplets irrespective of their geometric origin. In type IIB compactification they correspond to deformations of the complex structure while in type IIA compactification they parameterize the Kähler deformation (cf. appendix A).

[2] The nomenclature electric–magnetic is linked to the definition of the electric versus magnetic gauge bosons which arise in the expansion of the type IIB four form C_4 according to $C_4 = A^\Lambda_1 \alpha_\Lambda - \tilde{A}_{1\,\Lambda}\beta^\Lambda + \ldots$. Here A^Λ_1 are the $(h^{(1,2)}+1)$ electric gauge bosons (including the graviphoton) while $\tilde{A}_{1\,\Lambda}$ are the corresponding dual magnetic gauge bosons.

can be non–trivial along the DW. In particular, in the following we set to zero the field strengths of the vectors and the tensors. With this assumption the supersymmetry transformation of the two gravitinos $\psi_{\mu A}$ has the form [11, 12, 13]

$$\delta\psi_{\mu A} = D_\mu \varepsilon_A + i S_{AB}\gamma_\mu \varepsilon^B \ , \tag{2.6}$$

where ε_A are the two supersymmetry parameters and S_{AB} is a hermitian $SU(2)$ matrix which depends on the background fluxes (2.5). For the type IIB compactifications under consideration one finds [12]

$$S_{AB} = \frac{i}{2}\,\sigma^x_{AB}\,\omega^x_I\,\langle V, \mathcal{K}^I\rangle \ , \quad I = 1,2 \ , \tag{2.7}$$

where the quaternionic connection ω^x_I is given by [35]

$$\omega^x_1 = \delta^{x3} e^{2\varphi} \ , \qquad \omega^1_2 = -e^{2\varphi} Im\tau \ , \quad \omega^2_2 = 0 \ , \quad \omega^3_2 = e^{2\varphi} Re\tau \ . \tag{2.8}$$

Here $e^{2\varphi} = \frac{1}{8} e^{-K_H} e^{2\phi}$ is the four-dimensional real dilaton, ϕ is the ten-dimensional IIB dilaton and $\frac{1}{8}e^{-K_H}$ is the volume of $\tilde{Y}$ which is defined in (A.21). We also assembled the background fluxes into (symplectic) vectors $\mathcal{K}^I = (m^{I\Lambda}, e^I_\Lambda)$ and defined the symplectic inner product $\langle \, , \, \rangle$ as:

$$\langle V, \mathcal{K}^I\rangle = (L^\Lambda e^I_\Lambda - M_\Lambda m^{\Lambda I}) \ , \tag{2.9}$$

where $V = (L^\Lambda, M_\Lambda)$ is defined in (A.2). The electric and magnetic charges are not arbitrary, as supersymmetry in four dimensions [12] and the tadpole cancellation condition in ten dimensions [6], impose

$$\langle \mathcal{K}^1, \mathcal{K}^2\rangle = 0 \ . \tag{2.10}$$

Inserting (2.8) into (2.7) S_{AB} reads explicitly

$$S_{AB} = \frac{i}{2}\, e^{2\varphi}\Big[\sigma^3_{AB}\,(\langle V, \mathcal{K}^1\rangle + \langle V, \mathcal{K}^2\rangle \mathrm{Re}\tau) - \langle V, \mathcal{K}^2\rangle\, \mathrm{Im}\tau\, \sigma^1_{AB}\Big] \ . \tag{2.11}$$

The supersymmetry transformations of the gaugini are given by

$$\delta\lambda^{iA} = i\partial_\mu t^i \gamma^\mu \varepsilon^A + W^{iAB}\varepsilon_B \ , \tag{2.12}$$

where

$$\begin{aligned} W^{iAB} &= i g^{i\bar{\jmath}} \sigma^{AB}_x \omega^x_I \langle U_{\bar{\jmath}}, \mathcal{K}^I\rangle \\ &= i g^{i\bar{\jmath}} e^{2\varphi}\left[\sigma^{AB}_3(\langle U_{\bar{\jmath}}, \mathcal{K}^1\rangle + \langle U_{\bar{\jmath}}, \mathcal{K}^2\rangle \mathrm{Re}\tau) - \langle U_{\bar{\jmath}}, \mathcal{K}^2\rangle \mathrm{Im}\tau\, \sigma^{AB}_1\right] \ , \end{aligned} \tag{2.13}$$

and we defined $U_i = \nabla_i V \equiv (\partial_i + \frac{1}{2}\partial_i K_V)V$ where K_V is the Kähler potential of the vector multiplets defined in (A.1).

Finally the supersymmetry transformations of the hyperinos read

$$\delta\zeta_\alpha = iP_{uA\alpha}\partial_\mu q^u \gamma^\mu \varepsilon^A + N_\alpha^A \varepsilon_A \ , \tag{2.14}$$

where

$$N_A^\alpha = -2\mathcal{U}_{AI}^\alpha \langle V, \mathcal{K}^I \rangle \ . \tag{2.15}$$

The matrixes $P_{uA\alpha}$ play the rôle of a vielbein on the scalar manifold spanned by the q^u's, while $\mathcal{U}_{AI}^\alpha$ are remnants of the vielbein on the quaternionic manifold along the directions which have been dualized into scalars (see appendix A and in particular reference [11] for more details).

3 $N = 1$ Domain Wall solutions

3.1 Gradient Flow Equations

After this brief review of the $N = 2$ low energy supergravity arising in Calabi-Yau compactifications of type IIB string theory let us now turn to the main topic of this paper and study its three-dimensional $N = 1$ DW solutions. That is we study solutions of the four-dimensional $N = 2$ supergravity which preserve the three-dimensional Lorentz group $SO(1,2)$ and half of the supercharges. We split the coordinates $x^\mu, \mu = 0, \ldots, 3$ of the four-dimensional space-time into coordinates $(x^m, z), m = 0,\ 1,\ 2,$ where x^m denote the coordinates along the DW while z parameterizes the direction normal to the DW. Accordingly we split the background metric preserving Lorentz invariance as

$$g_{\mu\nu}(x^\mu)\, dx^\mu dx^\nu = e^{U(z)}\hat{g}_{mn}(x^m)\, dx^m dx^n + g_{zz}(z)\, dz dz \ . \tag{3.1}$$

where $\hat{g}_{mn}(x^m)$ is the metric of a three-dimensional space-time which we assume to have constant curvature. (In the following the 'hatted' quantities will refer to the three-dimensional un-warped metric.) Furthermore, following [3, 4] we choose to parameterize $g_{zz}(z) = -e^{-2pU(z)}$ where p is an arbitrary real number. Finally using $\mu = e^{U(z)}$ instead of z as the coordinate of the transverse space we arrive at

$$g_{\mu\nu}(x^\mu)dx^\mu dx^\nu = \mu^2 \hat{g}_{mn}(x^m)dx^m dx^n - \frac{d\mu d\mu}{\mu^2 \mathcal{W}^2(z)} = \eta_{\alpha\beta} e^\alpha_{\ \mu} e^\beta_{\ \nu} dx^\mu dx^\nu \ , \tag{3.2}$$

where

$$\mathcal{W} = \pm e^{pU(z)} U'(z) \ . \tag{3.3}$$

The non–vanishing components of the vierbein defined in (3.2) take the form

$$e_m^a = \hat{e}_m^a e^{U(z)} \ , \qquad e_z^3 = e^{-pU(z)} \delta_z^3 \ , \quad a = 0,\, 1,\, 2 \ , \tag{3.4}$$

while the non-vanishing components of the spin connection ω are found to be

$$\omega_m^{ab} = \hat{\omega}_m^{ab} \ , \qquad \omega_m^{a3} = e^{(p+1)U(z)} U'(z) \hat{e}_m^a \ . \tag{3.5}$$

Since we are interested in DW solutions which preserve four supercharges we first study the supersymmetry transformations of the fermionic fields. More precisely we solve $\delta_\epsilon$fermions $= 0$ for half of the supercharges. This is most easily done by imposing from the very beginning a relation on the two supersymmetry parameters ε_A, $A = 1, 2$ which reads [3, 4]

$$\varepsilon_A = \overline{h} A_{AB} \gamma_3 \varepsilon^B \ . \tag{3.6}$$

Here $h(z)$ is a complex function while A_{AB} is a constant matrix. Consistency of (3.6) with its hermitian conjugate implies $h\overline{h} = 1$ while $A_A^{\ B} \equiv A_{AC}\, \epsilon^{CB}$ must be a hermitian matrix which in addition satisfies

$$A_A^{\ B} A_B^{\ C} = \delta_A^C \ . \tag{3.7}$$

Thus A has to be a suitable linear combination of $(\mathbb{1}, \vec{\sigma})$ where $\vec{\sigma}$ are the Pauli matrices.

Finally, the condition of constant curvature of $\hat{g}_{mn}(x^m)$ can also be expressed as the integrability condition of [3, 36]

$$\hat{D}_m (h^{\frac{1}{2}} \varepsilon_A) = \frac{i}{\ell} \hat{e}_m^a \gamma_a \gamma_3 h^{\frac{1}{2}} \varepsilon_A \tag{3.8}$$

where $\frac{1}{\ell^2}$ is the three dimensional cosmological constant.

The next step is to look for solutions of

$$\delta \psi_{\mu A} = \delta \lambda^{iA} = \delta \zeta_\alpha = 0 \tag{3.9}$$

with (3.6) imposed. Furthermore, we only allow the scalar fields to be non-trivial in the DW background setting all other fields to zero. Since we are most interested in the values of the scalars transverse to the DW we suppose in the following that they only depend on the coordinate z and ignore their x^m dependence.

Let us first consider $\delta\psi_{Am} = 0$. Using (3.4)–(3.8) one derives

$$A^{AB} D_m \varepsilon_B = -\left(\frac{i}{\ell}e^{-U} + \frac{1}{2}e^{(p+1)U}U'\right)\overline{h}\gamma_m \varepsilon^A \ . \tag{3.10}$$

Inserted into $\delta\psi_{Am} = 0$ one obtains

$$\left(\frac{i}{\ell}e^{-U} + \frac{1}{2}e^{(p+1)U}U'\right)\overline{h}\gamma_m \varepsilon^A = iA^{AB} S_{BC}\gamma_m \varepsilon^C \ . \tag{3.11}$$

This implies that $A^{AB}S_{BC}$ is proportional to the identity or in other words

$$iA^{AB}S_{BC} = \frac{1}{2}W\delta^A_C \ , \tag{3.12}$$

where the proportionality factor defines the superpotential W. From (2.11) and (3.12) we infer the structure of $A_A^{\ B}$ to be

$$A_A^{\ B} = \frac{1}{\sqrt{2}}(-\sigma^{1\ B}_A + \sigma^{3\ B}_A) \ , \tag{3.13}$$

and the constraint

$$(Im\tau - Re\tau)\langle V, \mathcal{K}^2\rangle = \langle V, \mathcal{K}^1\rangle \ . \tag{3.14}$$

Inserting (3.12) - (3.14) into (3.11) we finally arrive at

$$\frac{i}{\ell} = \frac{1}{4}e^U(hW - \bar{h}\bar{W}) \ , \tag{3.15}$$

$$U' = \frac{1}{2}e^{-pU}(hW + \bar{h}\bar{W}) \ , \tag{3.16}$$

$$W = 4e^{\varphi}e^{K_H/2}\langle V, \mathcal{K}^2\rangle \ . \tag{3.17}$$

We see that the cosmological constant is determined by the imaginary part of hW while the derivative of the warp factor is determined by the real part. W itself is determined by the fluxes.

Before continuing let us briefly discuss the limiting cases of only RR fluxes ($\mathcal{K}^2 = 0$) and only NS fluxes ($\mathcal{K}^1 = 0$). In the first case we see from (2.11) and (3.12) that A is proportional to σ^3 and no consistency condition needs to be imposed. For only NS fluxes the consistency condition is

$$Im\tau - Re\tau = 0\ . \tag{3.18}$$

Now we look at to the solution of $\delta\psi_{Az} = 0$. It turns out that we can follow precisely the same steps as done in [3] with the only difference that we have to use the constraint (3.14). Suppressing the intermediate steps we arrive at

$$\overline{h}D_z h = \frac{2i}{\ell}\, e^{-(p+1)U}\ . \tag{3.19}$$

The solution of $\delta\lambda^{iA} = 0$ proceeds analogously. We insert (3.6) and (2.13) into (2.12) and obtain the constraint

$$(Im\tau - Re\tau)\langle U_i, \mathcal{K}^2\rangle = \langle U_i, \mathcal{K}^1\rangle\ . \tag{3.20}$$

Differentiating (3.14) with respect to z and using (3.20) we conclude

$$Im\tau - Re\tau = \alpha\ , \tag{3.21}$$

where α is a real constant ($\alpha = 0$ holds if and only if there are no RR fluxes). For our purposes we do not need to find an explicit solution for the constraints (3.14), (3.20). Note that they are satisfied, for instance, if the two vectors of electric/magnetic charges are parallel $\mathcal{K}^1 = \alpha\mathcal{K}^2$ where α is defined in equation (3.21). This is consistent with the tadpole cancellation condition (2.10) and with the two limiting cases $\mathcal{K}^1 = 0$ or $\mathcal{K}^2 = 0$.

Inserting (3.20) into (2.12) we obtain the flow equations for the scalars t^i

$$\partial_z t^i = -g^{i\bar{j}} e^{-pU(z)}\overline{h}\,\nabla_{\bar{j}}\overline{W}\ . \tag{3.22}$$

The analysis of $\delta\zeta_\alpha = 0$ proceeds analogously and one inserts (2.15) and (3.6) into (2.14). Using the quaternionic relations (A.13)–(A.16) one finds

$$\partial_z q^u = -g^{uv} e^{-pU(z)}\overline{h}\,\partial_v\overline{W}\ , \tag{3.23}$$

where g^{uv} is defined in (A.8). In addition one finds that hW has to be real or in other words $\bar{h}$ is determined as the phase of W.

$$\bar{h} = \frac{W}{|W|}\ . \tag{3.24}$$

This in turn implies that the cosmological constant on the DW must be zero, as we can see from equation (3.15). Therefore the metric $\hat{g}_{mn}$ on the DW is flat:

$$ds^2 = \mu^2 \eta_{mn} dx^m dx^n - \frac{d\mu d\mu}{\mu^2 \mathcal{W}^2} \ . \tag{3.25}$$

Note that (3.25) holds for $\mathcal{K}_1 = 0$ and in particular also for $\mathcal{K}_2 = 0$, that is in the case where just RR fluxes are present [4].

Using (3.24) we can insert (3.16) into (3.3) to arrive at

$$\mathcal{W}(z) = \pm h W = \pm |W| \ . \tag{3.26}$$

Using as a transverse coordinate $\mu(z) = e^{U(z)}$, (3.22) and (3.23) can be written as gradient flow equations

$$\begin{aligned} \mu \frac{dt^i}{d\mu} &= -g^{i\bar{\jmath}} \nabla_{\bar{\jmath}} \ln \overline{W} \ , & (3.27) \\ \mu \frac{dq^u}{d\mu} &= -g^{uv} \partial_v \ln \overline{W} \ . & (3.28) \end{aligned}$$

3.2 Solutions of the flow equations

So far we derived the gradient flow equations for an $N = 1$ BPS domain wall in type IIB supergravity compactified on a Calabi-Yau manifold $\tilde{Y}$ in the presence of electric and magnetic RR and NS fluxes. The purpose of this section is to study their solutions and to prepare for a geometrical interpretation in a mirror symmetric compactification of type IIA on some generalized manifold $\hat{Y}$.

We will not consider the most generic solution but instead follow [4] and restrict the space of scalar fields which can vary along the DW. More precisely the scalars in the vector multiplets t^i and the four-dimensional dilaton φ can be non-trivial along the DW. As we discuss in appendix A, half of the scalars in the hypermultiplets are geometrical moduli of the Calabi-Yau manifold. In type IIB compactifications they correspond to deformations of the Kähler form and we denote them by $z^a = \sigma^a + i\lambda^a$. Following [4] we only allow the λ^a to be non-trivial in the DW solution while σ^a together with the remaining scalar fields from the RR sector are kept constant.

Let us first focus on the flow equations for the hypermultiplet scalars. Inserting (3.17) and (3.24) into (3.16) and (3.23) we arrive at

$$\partial_z q^u = -2e^{-pU+\varphi+\frac{K_H}{2}} g^{uv}\partial_v(2\varphi + K_H)|\langle V, \mathcal{K}^2\rangle| \ , \tag{3.29}$$

$$\partial_z U(z) = 4e^{-pU+\varphi+\frac{K_H}{2}}|\langle V, \mathcal{K}^2\rangle| \ , \tag{3.30}$$

where K_H is defined in (A.20) and (A.21). Comparing (3.30) and (3.29) one obtains

$$\frac{dq^u}{dU} = -\frac{1}{2}g^{uv}\partial_v(2\varphi + K_H) \ . \tag{3.31}$$

This equation shows that the U–dependence of the quaternionic fields is not modified by the magnetic fluxes and thus we expect that the solution coincides with the solution derived in [4].

In order to solve equation (3.31) let us first note that on the submanifold spanned by the scalars φ and λ^a the inverse metric g^{uv} is block diagonal with the components

$$g^{\varphi\varphi} = 1 \ , \qquad g^{ab} = -\frac{2}{3}\left(d\, d^{ab} - 3\lambda^a\lambda^b\right) \ , \tag{3.32}$$

where we have evaluated g^{ab} in the large volume limit and defined

$$d = d_{abc}\lambda^a\lambda^b\lambda^c \ , \qquad d_a = d_{abc}\lambda^b\lambda^c \ , \qquad d_{ab} = d_{abc}\lambda^c \ , \tag{3.33}$$

with d^{ab} being the inverse of d_{ab}. Inserting (3.32) into (3.31) we obtain the solution

$$e^{\varphi} = C\, e^{-U(z)} \ , \qquad z^a = i\lambda^a = iD^a\, e^{2U(z)} \ , \tag{3.34}$$

where C and D^a are integration constants. From (A.21) we learn

$$e^{-K_H} = \frac{4}{3}De^{6U(z)} \ , \tag{3.35}$$

where we abbreviated $D = d_{abc}D^aD^bD^c$. Note that as expected (3.34) and (3.35) coincide with the result of reference [4].

Let us now consider the vector multiplets scalars. Also in this case it is more convenient to consider (3.22) instead of (3.27) which, following [3], we rewrite as follows

$$\partial_z \begin{pmatrix} Y^\Lambda - \overline{Y}^\Lambda \\ \mathcal{F}_\Lambda - \overline{\mathcal{F}}_\Lambda \end{pmatrix} = -i4e^{(1-p)U+\varphi+\frac{K_H}{2}} \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix} \tag{3.36}$$

where we have suppressed the label “2” on the NSNS fluxes and defined

$$\mathcal{V} \equiv h\, e^{U(z)} V = h\, e^{U(z)} \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix} \equiv \begin{pmatrix} Y^\Lambda \\ \mathcal{F}_\Lambda \end{pmatrix} \ . \tag{3.37}$$

Using the solution (3.35), choosing $D = 12C^2$ and performing the change of coordinates defined by

$$e^{(p+3)U(z)} \partial_z = \partial_w \tag{3.38}$$

equation (3.36) becomes

$$\partial_w \begin{pmatrix} Y^\Lambda - \overline{Y}^\Lambda \\ \mathcal{F}_\Lambda - \overline{\mathcal{F}}_\Lambda \end{pmatrix} = -i \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix} \ . \tag{3.39}$$

If we set $p = -3$ and $m^\Lambda = 0$ we recover the result of [4].

In order to derive further useful relations, let us display (3.39) more explicitly. Using (A.4) and the normalization $Y^0 = -\frac{i}{2}$ we infer

$$b^i = -2\mathrm{Im} Y^i \ , \qquad v^i = 2\mathrm{Re} Y^i \ , \tag{3.40}$$

where we split $\Lambda = 0, i$. Inserted into (3.39) using (A.3) we arrive at

$$0 = m^0 \tag{3.41}$$

$$\partial_w b^i = m^i \tag{3.42}$$

$$\frac{1}{2} c_{ijk} \partial_w (v^j v^k) - \frac{1}{2} c_{ijk} \partial_w (b^j b^k) = e_i \tag{3.43}$$

$$-\frac{1}{2} c_{ijk} \partial_w (b^i v^j v^k) + \frac{1}{6} c_{ijk} \partial_w (b^i b^j b^k) = e_0 \tag{3.44}$$

Solutions of equations (3.42)–(3.44) are discussed in appendix B.

Note that equations (3.15) and (3.16) can be rewritten in terms of the rescaled section $\mathcal{V}$

$$\langle \mathrm{Re}\mathcal{V}, \mathcal{K}_2 \rangle = e^{(p+2)U} \partial_w U \ , \qquad \langle \mathrm{Im}\mathcal{V}, \mathcal{K}_2 \rangle = 0 \ . \tag{3.45}$$

Using (3.40), (B.2), (B.5) and (B.6) one can easily check the second equation in (3.45) and compute the first to be

$$e^{(p+2)U} \partial_w U = \frac{1}{2} (v^i e_i^2 + c_{ijk} v^i b^j m^{2k}) \ . \tag{3.46}$$

Multiplying (3.43) by v^i and making use of (3.42) one can derive by comparison with (3.46)

$$e^{-K_V} \equiv \frac{4}{3}\, c_{ijk} v^i v^j v^k = 4e^{2U} \ , \tag{3.47}$$

where we also used (A.5). Note that the final form of K_V does not depend on the presence of the magnetic fluxes and therefore coincides with the results of [4]. Let us also observe at this point that the ten-dimensional type IIA dilaton ϕ_A defined by $e^{2\phi_A} = \frac{1}{8} e^{2\varphi - K_V}$ is given by the integration constant introduced in (3.34) $e^{\phi_A} = C$, as can be seen from (3.34) and (3.47). This will be important in the next section.

We are now in the position to formulate the DW gradient flow equations in a very compact way, in terms of the quantities $(Z^A,\, W_A)$ and $(X^\Lambda,\, F_\Lambda)$ introduced in appendix A. First notice that the relation between $(X^\Lambda,\, F_\Lambda)$ and the sections $(Y^\Lambda,\, \mathcal{F}_\Lambda)$ can be deduced from equations (A.2), (3.37) and (3.47). In particular, setting the irrelevant overall phase to zero, that is $h = 1$, we obtain

$$\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} = 2 \begin{pmatrix} Y^\Lambda \\ \mathcal{F}_\Lambda \end{pmatrix} \ , \tag{3.48}$$

and as a consequence (3.39) now reads

$$\partial_w \begin{pmatrix} \mathrm{Im} X^\Lambda \\ \mathrm{Im} F_\Lambda \end{pmatrix} = - \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix} \ . \tag{3.49}$$

Furthermore, in these variables (3.45) reads

$$\mathrm{Im} X^\Lambda e_\Lambda - \mathrm{Im} F_\Lambda m^\Lambda = 0 \ . \tag{3.50}$$

Let us return to the flow equations for the hypermultiplet scalars (3.28) or (3.29) respectively, whose solution we already gave in (3.34). However, in order to compare the solution with the Hitchin flow equation of the next section it is useful to rewrite them in a form similar to (3.49). This is achieved in terms of rescaled variables $(Z^A,\, W_A)_\eta$ given by

$$(Z^A,\, W_A) = |c|\, (Z^A,\, W_A)_\eta \ , \qquad |c|^2 \equiv e^{K_V - K_H} = \frac{D}{3} e^{4U} \ , \tag{3.51}$$

where the last equality used (3.35) and (3.47). The geometrical meaning of this rescaling will become more transparent in the next section.

Recalling the definition (A.22), the solution (3.34) and the gradient flow equation (3.45), one can easily check that

$$\partial_w \begin{pmatrix} \mathrm{Im}Z^A \\ \mathrm{Im}W_a \\ \mathrm{Im}W_0 \end{pmatrix}_\eta = -|c| \begin{pmatrix} 0 \\ 0 \\ \mathrm{Re}X^\Lambda e_\Lambda - \mathrm{Re}F_\Lambda m^\Lambda \end{pmatrix} . \tag{3.52}$$

4 The geometry of the type IIA background

The DW solution of type IIB discussed in the previous section is expected to have a mirror symmetric solution in type IIA. For RR fluxes mirror symmetry merely amounts to exchanging the flux of the RR three-form F_3 defined in (2.5) with the fluxes of the even forms F_2 and F_4 of type IIA [6, 10]. However, for the NS-form H_3 the situation is more involved in that mirror symmetry can relate H_3-flux to the torsion of a geometrical compactification [16, 17] or possibly to non-geometrical quantities [20]. For electric NS fluxes[3] e_Λ the IIA mirror symmetric solution corresponds to compactifications on half-flat manifolds $\hat{Y}_{\mathrm{hf}}$ [21, 22, 17]. More precisely, in ref. [4] it was shown that the DW solution takes the form of a warped product

$$M_{(1,2)} \times_w X_7 , \tag{4.1}$$

where the seven dimensional manifold X_7 consists a six dimensional half-flat manifold $\hat{Y}_{\mathrm{hf}}$ which is fibered over $\mathbb{R}$. Thus the metric takes the form

$$ds^2_{(7)} = dy^2 + ds^2_{(6)}(y) , \tag{4.2}$$

where $ds^2_{(6)}$ is the metric of $\hat{Y}_{\mathrm{hf}}$ and y is the coordinate of $\mathbb{R}$.

Half-flat manifolds are a special sub-class of manifolds with $SU(3)$ structure. They admit a globally defined spinor which is invariant under $SU(3)$. The existence of this spinor implies the existence of a two-form J and a complex three-form Ω_η.[4] For half-flat manifolds J and Ω_η satisfy the additional conditions [21, 22]

$$dJ^2 = 0 = d\mathrm{Im}\Omega_\eta . \tag{4.3}$$

[3] Let us recall that we suppress the index "2" for the NSNS fluxes, that is we mean $(e_\Lambda, m^\Lambda) \equiv (e^2_\Lambda, m^{2\Lambda})$.

[4] Ω is only defined up to complex rescaling. Therefore a choice of normalization is involved in the following. By Ω_η we denote the three-form constructed from a normalized spinor or equivalently a three-form which obeys $\Omega_\eta \wedge \bar{\Omega}_\eta = \frac{3i}{4} J^3$.

When $\hat{Y}_{\rm hf}$ sits inside X_7 the non-trivial fibration is expressed by the Hitchin flow equations [21, 22]

$$\frac{1}{2}\,\partial_y J^2 = -d\,\mathrm{Re}\Omega_\eta\ , \qquad \partial_y \mathrm{Im}\Omega_\eta = dJ\ . \tag{4.4}$$

They precisely ensure that X_7 has G_2 holonomy which corresponds to the $N = 1$ supersymmetry of the IIB DW solution.

In this section we suggest a generalization of the type IIA geometric compactification which also captures the mirror of non-trivial type IIB magnetic fluxes m^Λ. More precisely we check that compactifications of the form (4.1) where X_7 contains a fibered product of a six-manifold with $SU(3) \times SU(3)$ structure times the real line are mirror dual to type IIB DW solutions with electric and magnetic flux. This generalized mirror symmetry has recently been suggested in ref. [25, 29, 30, 31] and here we confirm that it also holds for the case of the DW solution constructed in the previous section.

In order to check this proposal let us briefly summarize the results of refs. [29, 31]. It was shown that the most general possible geometrical compactification of type II string theories involves manifolds with $SU(3)\times SU(3)$. Such manifolds are defined by the existence of two locally inequivalent spinors. Each of them is left invariant by an $SU(3)$ and thus together they define what is called an $SU(3) \times SU(3)$ structure [27, 28]. Compactifications on such manifolds lead to an $N = 2$ low energy effective action in four spacetime dimensions. The space of scalar fields is most conveniently expressed in terms of two pure spinors of $SO(6, 6)$ denoted by $\Phi_\pm$. Geometrically Φ_+ is a sum of even forms while Φ_- is a sum of odd forms. If one projects out all possible massive gravitino multiplets both Φ_+ and Φ_- enjoy an expansion of the form

$$\Phi_+ = X^\Lambda \omega_\Lambda - F_\Lambda\, \omega^\Lambda\ , \qquad \Phi_- = Z^A_\eta\, \alpha_A - W_{\eta A}\, \beta^A\ . \tag{4.5}$$

The $(\omega_\Lambda,\, \omega^\Lambda)$ form a (non-degenerate) symplectic basis on the space of even forms while $(\alpha_A,\, \beta^A)$ form a symplectic basis on the space of odd forms. They are normalized according to:

$$\int_Y \omega_\Lambda \wedge \omega^\Sigma = \delta^\Sigma_\Lambda\ ; \qquad \int_Y \alpha_A \wedge \beta^B = \delta^B_A\ . \tag{4.6}$$

In addition $\Phi_\pm$ satisfy a compatibility condition which in terms of the expansion (4.5) reads [29, 31]

$$(X^\Lambda \bar{F}_\Lambda - \bar{X}^\Lambda F_\Lambda) = (Z^A \bar{W}_A - \bar{Z}^A W_A)_\eta\ . \tag{4.7}$$

$\Phi_\pm$ are only defined up to arbitrary rescaling and as shown in [29] the low energy effective action or more precisely the Kähler potentials depend on the rescaled sections $(Z^A,\, W_A)$ which are related to $(Z^A,\, W_A)_\eta$ precisely by the rescaling (3.51). In terms of $(X^\Lambda,\, F_\Lambda)$ and $(Z^A,\, W_A)$ the Kähler potentials are again given by (A.1) and (A.20), respectively. Furthermore, it is possible to choose special coordinates where $X^0 = -i$, $Z^0 = 1$ holds and in these coordinates mirror symmetry is realized by imposing [30, 31]

$$d\alpha_0 = m^\Lambda \omega_\Lambda - e_\Lambda \omega^\Lambda \ , \quad d\alpha_a = d\beta^A = 0 \ , \quad d\omega_\Lambda = e_\Lambda \beta^0 \ , \quad d\omega^\Lambda = m^\Lambda \beta^0 \ . \tag{4.8}$$

One shows that for type IIA compactifications on manifolds obeying (4.8) spectrum and effective action coincide with that obtained by compactifying type IIB an Calabi-Yau threefolds with electric and magnetic NS three-form flux turned on [31]. For $m^\Lambda = 0$ one precisely obtains the half-flat manifolds discussed above. In this case one has $\Phi_+ = e^{B+iJ}$ and $\Phi_- = \Omega_\eta$, where B is the NS two-form.

What is left to study are the $SU(3)\times SU(3)$ generalizations of (4.3) and (4.4) and to show that they correspond to the DW solutions of the previous section. From a mathematical point of view the generalized flow equations have been derived in ref. [28] and (in our notation) they read

$$d\,\mathrm{Im}\Phi_- = d\,\mathrm{Im}\Phi_+ = 0 \ , \tag{4.9}$$
$$\partial_y \mathrm{Im}\Phi_+ = -d\,\mathrm{Re}\Phi_- \ , \tag{4.10}$$
$$\partial_y \mathrm{Im}\Phi_- = d\,\mathrm{Re}\Phi_+ \ . \tag{4.11}$$

Let us now show that these flow equations together with (4.8) coincide with the DW solution of the previous section. We start by computing $d\,\Phi_\pm$ and insert (4.8) into (4.5). This yields

$$d\,\Phi_+ \;=\; (X^\Lambda e_\Lambda - F_\Lambda m^\Lambda)\,\beta^0 \ , \tag{4.12}$$
$$d\,\Phi_- \;=\; |c|^{-1}(m^\Lambda \omega_\Lambda - e_\Lambda \omega^\Lambda) \ , \tag{4.13}$$

where $|c|$ is defined in (3.51). From the reality of the right hand side of (4.13) we immediately conclude $d\,\mathrm{Im}\Phi_- = 0$. Furthermore $d\,\mathrm{Im}\Phi_+ = 0$ coincides with the condition (3.50).

The next step is to compute $\partial_y \mathrm{Im}\Phi_\pm$. Using (4.5) we arrive at

$$\partial_y \mathrm{Im}\Phi_+ = (\partial_y \mathrm{Im}X^\Lambda)\,\omega_\Lambda - (\partial_y \mathrm{Im}F_\Lambda)\,\omega^\Lambda \ , \tag{4.14}$$
$$\partial_y \mathrm{Im}\Phi_- = (\partial_y \mathrm{Im}Z^A_\eta)\,\alpha_A - (\partial_y \mathrm{Im}W_{\eta\, A})\,\beta^A \ . \tag{4.15}$$

Changing coordinates according to

$$dy = |c|^{-1} dw \ , \tag{4.16}$$

we see that $\partial_y \mathrm{Im}\Phi_+ = -d\,\mathrm{Re}\Phi_-$ precisely corresponds to (3.49) and $\partial_y \mathrm{Im}\Phi_- = d\,\mathrm{Re}\Phi_+$ corresponds to (3.52). Thus we have achieved our goal and recovered the type IIB flow equations from the generalized Hitchin flow equations (4.9)–(4.11) on the type IIA side.

Our next chore is to compare the superpotentials. In (3.12) we learned that W is related to the matrix S_{AB} defined in (2.6). Precisely this quantity was computed in [29] in terms of the pure spinors $\Phi_\pm$ to be

$$W \sim e^{\frac{1}{2}(K_V+K_H)+\varphi} \int_Y d\Phi_+ \wedge \Phi_- = e^{\frac{1}{2}(K_V+K_H)+\varphi}(X^\Lambda e_\Lambda - F_\Lambda m^\Lambda) \ , \tag{4.17}$$

where we used (4.5) and (4.8). Again this type IIA quantity precisely coincides with (3.17) of type IIB if we also use (A.2). Thus the Hitchin flow equations can also be viewed as gradient flow equations of the form (3.27), (3.28) with a superpotential given by (4.17).

In summary we just showed that the DW solutions of type IIB can be expressed as generalized Hitchin flow equations for the two pure spinors $\Phi_\pm$ of a manifold with $SU(3) \times SU(3)$ structure as given in (4.9)–(4.11).

Our final task is to discuss the properties of the seven-dimensional manifold X_7. As the metric on the DW is flat and the background $M_{(1,2)} \times_w X_7$ solves the string equation of motion, we expect X_7 to be Ricci flat. For half-flat manifolds this was indeed shown in refs. [21, 22, 4]. In order to discuss the generalization at hand let us introduce the seven dimensional exterior derivative by

$$\hat{d} = d + dy\,\partial_y \,, \tag{4.18}$$

where d acts on $\hat{Y}_6$ and ∂_y is the derivative with respect to the coordinate of $\mathbb{R}$. Furthermore, following [27, 28] one can define the generalized forms ρ and $*\rho$ on X_7 which are given in terms of $\Phi_\pm$ by

$$\rho = -\mathrm{Re}\Phi_+ \wedge dy - \mathrm{Im}\Phi_- \ , \qquad *\rho = \mathrm{Re}\Phi_- \wedge dy + \mathrm{Im}\Phi_+ \ . \tag{4.19}$$

$*\rho$ is the Hodge dual of ρ with respect to the generalized metric. As noted in [27, 28] the equations (4.9)–(4.11) then correspond to

$$d\rho = *d*\rho = 0 \ , \tag{4.20}$$

and imply that X_7 has an integrable $G_2 \times G_2$ structure and is indeed Ricci-flat.

5 Conclusions and outlook

In this paper we studied three-dimensional $N = 1$ DW solutions of four-dimensional $N = 2$ supergravities which arise as the low energy limit of type IIB string theory compactified on Calabi-Yau threefolds in the presence of RR and NS three-form fluxes. An essential ingredient in our analysis was the newly constructed $N = 2$ supergravity [11]–[14] which includes massive antisymmetric tensors in the spectrum. The use of this supergravity is necessary whenever magnetic fluxes are turned on as they render antisymmetric tensors in the type IIB spectrum massive. In this respect we generalized the previous analysis of refs. [3, 4] and consistently included magnetic fluxes. We further showed that the $N = 2$ scalar fields vary according to a set of gradient flow equations and explicitly determined their solution in terms of the fluxes.

The second aspect of the paper dealt with the type IIA mirror symmetric DW solutions. Here we used the results of [29, 30, 31] and showed that the flow equations of type IIB have a mirror dual which is purely geometrical and can be understood as a set of generalized Hitchin flow equations for a particular class of manifolds with $SU(3) \times SU(3)$ structure [28]. As in refs. [21, 22, 28] these flow equations do have a seven-dimensional interpretation and can be viewed as arising from fibering a six-dimensional manifold with $SU(3) \times SU(3)$ over the real line and demanding an integrable $G_2 \times G_2$ structure of the resulting seven-dimensional manifold.

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Appendix

A The scalar σ–model of $N = 2$ supergravity

In this appendix we record some further details of the scalar fields in $N = 2$ supergravity. They can be viewed as the coordinates of some target space geometry which is constrained by $N = 2$ supersymmetry. In particular the complex scalars of the vector multiplets lead to a special Kähler geometry while the scalars in the hypermultiplets span a quaternionic manifold [34]. Let us discuss both geometries in turn.

A.1 Special Kähler geometry of the vector multiplets

The complex scalars t^i, $i = 1, \ldots, n_V$ belonging to the n_V vector multiplets span a special Kähler geometry. That is their σ-model metric is a Kähler metric with a Kähler potential

$$K_V = -\ln i\left[\bar{X}^\Lambda F_\Lambda - \bar{F}_\Lambda X^\Lambda\right] , \qquad \Lambda = 0, \ldots, n_V \ . \tag{A.1}$$

$X^\Lambda(t)$ and $F_\Lambda(t)$ depend holomorphically on the scalars t^i and are related to the covariantly holomorphic section V introduced in (2.9) by

$$V = (L^\Lambda, M_\Lambda) = e^{K_V/2}(X^\Lambda, F_\Lambda) \ . \tag{A.2}$$

For Calabi-Yau compactifications $F_\Lambda = \partial_\Lambda F(X)$ is the derivative of a prepotential F. In the large volume or large complex structure limit F is given by

$$F(X) = -\frac{1}{3!} c_{ijk} \frac{X^i X^j X^k}{X^0} , \qquad i = 1, \ldots, n_V \ , \tag{A.3}$$

where the c_{ijk} are constants. A particular set of coordinates, called special coordinates, is given by

$$t^i \equiv b^i + iv^i = \frac{X^i}{X^0} \ . \tag{A.4}$$

In these coordinates the Kähler potential (A.1) is given by

$$K_V = -\ln\left[\frac{4}{3} c_{ijk} v^i v^j v^k\right] \ . \tag{A.5}$$

A.2 Geometry of tensor- and hypermultiplets

The hypermultiplet geometry is described in terms of real scalar fields $q^{\hat{u}}$, $\hat{u} = 1, \cdots, 4n_H$, (here n_H is the number of hypermultiplets) which span a quaternionic manifold. The metric can be expressed in terms of a covariantly constant vielbein $\mathcal{U}^{A\alpha} \equiv \mathcal{U}_{\hat{u}}^{A\alpha} dq^{\hat{u}}$. More explicitly one has

$$h_{\hat{u}\hat{v}} = \mathcal{U}_{\hat{u}}^{A\alpha} \mathcal{U}_{\hat{v}}^{B\beta} \epsilon_{AB} \mathbb{C}_{\alpha\beta} \ , \qquad A, B = 1, 2 \ , \tag{A.6}$$

where $\epsilon^{AB} = -\epsilon^{BA}$ and $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ are the $SU(2)$ and $Sp(2n_H, \mathbb{R})$ invariant metrics respectively. The quaternionic vielbein obeys

$$\nabla \mathcal{U}^{A\alpha} \equiv d\mathcal{U}^{A\alpha} + \hat{\omega}^A{}_B \wedge \mathcal{U}^{B\alpha} + \hat{\Delta}^{\alpha\beta} \wedge \mathcal{U}^{A\beta} = 0 \ , \tag{A.7}$$

where $\hat{\omega}_{\hat{u}}^{AB}$, $\hat{\Delta}_{\hat{u}}^{\alpha\beta}$ are the $SU(2)$ and $Sp(2n_H, \mathbb{R})$ valued connections.

A set of scalars which parameterizes translational and commuting isometries can be dualized into a set of n_T antisymmetric rank two tensors [11]. In this case the remaining scalars q^u, $u = 1, \cdots, 4n_H - n_T$ will not parameterize a quaternionic manifold anymore. Instead their σ-model metric g_{uv} is given by

$$g_{uv} = h_{uv} - h_{Iu} M^{IJ} h_{Jv} = P_u^{A\alpha} P_v^{B\beta} \epsilon_{AB} \mathbb{C}_{\alpha\beta} \ , \qquad g^{uv} = h^{uv} \ , \tag{A.8}$$

where we decomposed the quaternionic metric as

$$h_{\hat{u}\hat{v}} = \begin{pmatrix} h_{uv} & h_{uJ} \\ h_{vI} & h_{IJ} \end{pmatrix} \ , \tag{A.9}$$

and defined M^{IJ} as the inverse of h_{JK}

$$M^{IJ} h_{JK} = \delta^I_K \ . \tag{A.10}$$

The vielbein $P_u^{A\alpha}$ of the metric g_{uv} defined in (A.8) can be expressed in terms of the quaternionic vielbein as follows

$$P_u^{A\alpha} \equiv \mathcal{U}_u^{A\alpha} - A_u^I \mathcal{U}_I^{A\alpha} \ , \qquad P^{u\,A\alpha} \equiv \mathcal{U}^{u\,A\alpha} \ , \tag{A.11}$$

where $A_u^J = h_{Iu} M^{IJ}$. Similarly the connections decompose as

$$\begin{aligned} \hat{\omega}_u^{AB} &\equiv \omega_u^{AB} + A_u^I \omega_I^{AB} \ , \quad \hat{\omega}_I^{AB} \equiv \omega_I^{AB}; \\ \hat{\Delta}_u^{\alpha\beta} &\equiv \Delta_u^{\alpha\beta} + A_u^I \Delta_I^{\alpha\beta} \ , \quad \hat{\Delta}_I^{\alpha\beta} = \Delta_I^{\alpha\beta} \ . \end{aligned} \tag{A.12}$$

The new quantities satisfy a certain number of relations [11, 33] and here we record only the ones needed in order to derive (3.23) and (3.24)

$$(P_u^{A\alpha} P_v^{B\beta} + P_v^{A\alpha} P_u^{B\beta})\mathbb{C}_{\alpha\beta} = g_{uv}\epsilon^{AB} \ , \tag{A.13}$$
$$(P_u^{A\alpha} \mathcal{U}_I^{B\beta} + \mathcal{U}_I^{A\alpha} P_u^{B\beta})\mathbb{C}_{\alpha\beta} = 0 \ , \tag{A.14}$$
$$(\mathcal{U}_I^{A\alpha} \mathcal{U}_J^{B\beta} + \mathcal{U}_J^{A\alpha} \mathcal{U}_I^{B\beta})\mathbb{C}_{\alpha\beta} = M_{IJ}\epsilon^{AB} \ , \tag{A.15}$$
$$\mathcal{U}_{I\alpha}^{(A} P_u^{B)\alpha} = \frac{1}{2}\nabla_u \omega_I^{AB} \ . \tag{A.16}$$

The covariant derivative ∇_u is defined with respect to the reduced connection ω_u^{AB}, $\Delta_u^{\alpha\beta}$.

The convention for raising and lowering the symplectic indices is as follows

$$\epsilon_{AB} T^B = T_A \ , \qquad T_B \epsilon^{BA} = T^A \ , \tag{A.17}$$
$$\mathbb{C}_{\alpha\beta} T^\beta = T_\alpha \ , \qquad T_\beta \mathbb{C}^{\beta\alpha} = T^\alpha \ . \tag{A.18}$$

A.3 Quaternionic geometry in Calabi-Yau compactifications

So far we only discussed the geometry as it appears in general in $N = 2$ supergravity. In Calabi-Yau compactifications of either type IIA or type IIB string theory only a special class of quaternionic geometries, termed 'dual quaternionic geometries', arise at the tree level [37]. This is basically a consequence of mirror symmetry and states that the quaternionic manifold of real dimension $4n_H$ necessarily has a special Kähler submanifold of real dimension $2n_H$ which is spanned by the geometrical moduli. The remaining $2n_H$ scalar fields then arise from the RR sector.

Let us be slightly more explicit. A Calabi-Yau manifold has a geometrical moduli space $\mathcal{M}$ which is product of a component $\mathcal{M}_{\rm k}$ spanned by the deformations of the Kähler form and a component $\mathcal{M}_{\rm cs}$ spanned by the deformations of the complex structure

$$\mathcal{M} = \mathcal{M}_{\rm k} \times \mathcal{M}_{\rm cs} \ . \tag{A.19}$$

Each component is a special Kähler geometry with a Kähler potential of the form (A.1), i.e. a Kähler potential which can be characterized by a holomorphic prepotential.

In compactifications of type IIA the deformations of the Kähler form reside in vector multiplets while the deformations of the complex structure are members of the hypermultiplets. In type IIB the situation is exactly reversed and the Kähler moduli sit in hypermultiplets while the complex structure moduli populate the vector multiplets. In both cases the geometrical moduli in the hypermultiplets combine with the scalar field from the RR sector to span the full quaternionic geometry.

Since we are discussing both type IIA and type IIB compactifications in the main text we choose to denote the scalar fields in the vector multiplets by t^i irrespective of their Calabi-Yau origin as Kähler or complex structure deformations. Similarly, we denote by z^a the geometrical moduli which reside in the hypermultiplets and which span the special Kähler submanifold inside the quaternionic manifold. Their Kähler potential we denote as

$$K_H = -\ln i \left[\bar{Z}^A W_A - \bar{W}_A Z^A \right] , \qquad A = 0, \ldots, n_H , \tag{A.20}$$

where $W_A(Z)$ is the second holomorphic prepotential. In the large volume or large complex structure limit K_H reduces to

$$K_H = -\ln \left[\frac{4}{3} d_{abc} \lambda^a \lambda^b \lambda^c \right] , \tag{A.21}$$

where

$$z^a = \sigma^a + i\lambda^a = \frac{Z^a}{Z^0} \tag{A.22}$$

are the special coordinates in this sector.

Finally let us also record the relation with the conventions used in ref. [35]. In this paper the quantities $\hat{K}$ and $\tilde{K}$ are used which are related to the quantities used in this paper by

$$e^{-\hat{K}} = 2e^{-K_H} , \qquad e^{-\tilde{K}} = e^{-2\varphi} , \tag{A.23}$$

where φ is the four-dimensional dilaton. Finally, the ten-dimensional dilaton can be expressed as

$$Im\tau = 4e^{\frac{\hat{K}-\tilde{K}}{2}} . \tag{A.24}$$

B Explicit solution of the flow equations

In this appendix we derive the explicit solution of the vector multiplets flow equation.

The formal integration of equation (3.39) is trivial and gives:

$$\begin{pmatrix} Y^\Lambda - \overline{Y}^\Lambda \\ \mathcal{F}_\Lambda - \overline{\mathcal{F}}_\Lambda \end{pmatrix} = -i \begin{pmatrix} m^\Lambda \\ e_\Lambda \end{pmatrix} x + \begin{pmatrix} K^\Lambda \\ K_\Lambda \end{pmatrix} . \tag{B.1}$$

Imposing (2.10), (3.45) on (B.1) one obtains the condition:

$$K^\Lambda e_\Lambda - K_\Lambda m^\Lambda = 0 \tag{B.2}$$

From the normalization $Y^0 = -\frac{i}{2}$ we infer $K^0 = 1$. Explicit integration of (3.42)–(3.44) yields

$$b^i = m^i x + K^i \tag{B.3}$$

$$c_{ijk} v^j v^k = c_{ijk} m^j m^k \, x^2 + 2(c_{ijk} m^j K^k + e_i) \, x + c_{ijk} K^j K^k + 2K_i \tag{B.4}$$

Reinserting (B.3) and (B.4) back into (3.43), (3.44) and making use of (B.2) one obtains the following set of constraints on the parameters:

$$m^0 = 0 \tag{B.5}$$

$$K^0 = 1 \tag{B.6}$$

$$c_{ijk} m^i m^j m^k = 0 \tag{B.7}$$

$$c_{ijk} m^i m^j K^k + e_i m^i = 0 \tag{B.8}$$

$$c_{ijk} m^i K^j K^k + 2K_i m^i = 0 \tag{B.9}$$

$$\frac{1}{3} c_{ijk} K^i K^j K^k + K_i K^i + K_0 = 0 \tag{B.10}$$

$$K^i e_i = K^i K_i \tag{B.11}$$

Contacting (B.4) with m^i and using (B.7)-(B.9) we further obtain:

$$c_{ijk} m^i v^j v^k = 0 \tag{B.12}$$

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