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BFKL Pomeron in string models

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Abstract

We consider scattering amplitudes in string models in the Regge limit of high energies and fixed momentum transfers with the use of the unitarity in direct channels. Intermediate states are taken in the multi-Regge kinematics corresponding to the production of resonances with fixed invariant masses and large relative rapidities. In QCD such kinematics leads to the BFKL equation for the Pomeron wave function in the leading logarithmic approximation. We derive a similar equation in the string theory and discuss its properties. The purpose of this investigation is to find a generalization of the BFKL approach to the region of small momentum transfers where non-perturbative corrections to the gluon Regge trajectory and reggeon couplings are essential. The BFKL equation in the string theory contains additional contributions coming from a linear part of the Regge trajectory and from the soft Pomeron singularity appearing already in the tree approximation. In higher dimensions in addition, a non-multi-Regge kinematics corresponding to production of particles with large masses is important. We solve the equation for the Pomeron wave function in the string theory for $D = 4$ and discuss integrability properties of analogous equations for composite states of several reggeised gluons in the multi-colour limit.

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1 Introduction

The derivation of the BFKL equation for QCD scattering amplitudes in the Regge regime of high energies $E = \sqrt{s}$ and fixed momentum transfers $q = \sqrt{-t}$ [1] is based on the fact that gluon is reggeized in perturbation theory. In the leading logarithmic approximation (LLA) the Pomeron singularity in the j-plane of the t-channel partial waves appears as a composite state of two reggeized gluons. The gluon Regge trajectory is known in two first orders of perturbation theory and the integral kernel for the BFKL equation is calculated in the next-to-leading approximation [2], which is important for the phenomenological applications [3].

It is reasonable to believe, that the gluon reggeization has a physical meaning even beyond the QCD perturbation theory, although up to now its Regge trajectory is calculated only at sufficiently large momentum transfers q where the effective coupling constant is small. For low momentum transfers we should use non-perturbative methods. For example, one can assume that the gluon trajectory in this region is approximately linear, as it takes place for the hadron trajectories. The linearity of the Regge trajectories was an important property in constructing the dual model by G. Veneziano [4]. Later a string interpretation of the dual amplitudes was developed [5]. In the Born approximation the dual hadron models include only particles lying on the secondary Regge trajectories. The Pomeron-like singularity appearing in the open string scattering amplitudes in one loop approximation was identified with a leading Regge trajectory for the closed sector. Later it was found that four dimensional string theories meet with difficulties, which were avoided in their superstring generalizations to space-time dimensions $D = 10$ [5]. Now these superstring models are considered as candidates for an unified theory of all elementary particle interactions including the gravity. Moreover, all of them are supposed to be various realizations of the same M-theory.

In the modern interpretation the closed string sector is associated with the graviton Regge family rather than with the Pomeron singularity[1], and so the Pomeron does not directly present in the string theory. In a line with the Maldacena proposal [7] for the $N = 4$ super-Yang-Mills model one might expect an appearance of a colorless composite state becoming a graviton in the t'Hooft limit $g^2 N_c \to \infty$, where g is a coupling constant and N_c is the number of colors. The Pomeron singularity seems to be a candidate for such graviton state [8]. However, in this paper we treat the Pomeron similar to the case of perturbative QCD where it is a composite state of two reggeized gluons. Namely, this singularity should appear in the diagrams where two open strings are exchanged in the t channel. Such Feynman graphs lead to the Mandelstam cut in the j-plane of the t-channel partial wave $\varphi_j(t)$. The sum of contributions from the ladder-type diagrams in the string theory corresponds to the BFKL-like equation. We hope that its string modification is a reasonable model for non-perturbative effects in the region of small momentum transfers. Indeed, the string models in extra dimensions can lead to a dual description of gauge theories including QCD [7].

Note, that in the critical dimensions $D = 10$ the multi-Regge kinematics for intermediate particles in the s-channel is not unique even for small coupling constants. Namely, one should take into account also the production of resonances having large masses at high energies, which leads in particular to the graviton contribution appearing in one loop. In the last case the

[1]Note that the Regge asymptotics was investigated also in the pure (super) gravity [6].

imaginary part of the corresponding non-planar diagram with the graviton Regge pole in the t-channel appears from the production of two resonances with masses $m \sim \sqrt{s}$. Below we does not discuss this problem in details and consider mainly the $D = 4$ case.

We use the superstring 4d model in the Ramond-Neveu-Schwarz version [5], but the supersymmetry is involved only to remove the tachion from the spectrum. It is known, that non-critical string models have difficulties related to the absence of the S-matrix unitarity in higher loops. In particular, to restore the unitarity in one loop approximation it is needed to introduce an additional $2D$ gravity field [5]. This field provides the conformal symmetry on the tree and one-loop levels, and restores the modular invariance of the one-loop closed string amplitudes [9]. Nevertheless, the modular invariance of higher loop closed string amplitudes remains to be broken [10]. Thus, the higher loop amplitudes for non-critical string models can not be constructed in a self-consistent way.

The difficulties of non-critical string models are related mainly to the closed string sector. In the perturbation theory with respect to the closed string coupling constant g_{cl} the contributions from this sector grow with energy very rapidly $\sim s^2 g_c^{2n} \ln^n(s)$. Such behavior in the case of hadron-hadron interactions is not compatible with the s -channel unitarity. So one expects that once a relevant summation over n being performed, the high asymptotics of the amplitude is reduced to $A \sim s$. It is reasonable to omit initially the closed string sector taking into account also, that g_{cl} is quadratic in the Yang-Mills coupling constant g and for $N_c \to \infty$ the open string terms in the amplitude are enhanced comparing to closed string ones. Thus, we consider here the contributions to the production amplitude only from the open string states in crossing channels leading to the Mandelstam cuts for the elastic t -channel partial wave $\varphi_j(t)$ in the angular momentum plane $j = 1 + \omega$.

In the discussed model the gluon trajectory $\omega(t)$ is given by the perturbative expansion

$$\omega(t) = \alpha' t + \omega_1(t) + \dots . \tag{1}$$

where $\omega_n(t) \sim g^{2n}$ are radiative corrections, and the Regge slope α' is a reversed square of an characteristic mass scale. Below the correction $\omega_1(t) \sim g^2$ to the trajectory is also taken into account. This correction is calculated from one-loop diagrams for the scattering amplitude. One-loop non-planar diagrams contain also a contribution destroying the unitarity, but the correction to the Regge trajectory appears only from the planar graphs, where the problem with the closed string sector does not exist. Providing that $-\alpha' t \ll 1$, the loop correction in the $D = 4$ case has the infrared divergency $\sim g^2 N_c \ln(t/\lambda^2)$ which is cancelled with the contribution from the massless particle production. For $-\alpha' t \geq 1$ the radiative correction to the Regge trajectory has a complicated form.

An important difference between QCD and the string model is related to the role of intermediate states with relatively large masses: $(\alpha')^{-1} \ll M^2 \ll s$ for produced resonances. These states are absent in QCD. In the string theory the large mass states are responsible for the appearance of the graviton contribution to the elastic scattering amplitude in the one-loop approximation. Further, the impact factors for the reggeon-particle scattering vanish for planar diagrams as a result of integration over large masses. In particular, it leads to the absence of the Mandelstam cuts in the color octet channel. In QCD the cancellation of these cuts for the t-channel with gluon quantum numbers is provided by another mechanism related to the so-called bootstrap relations for scattering amplitudes [11].

The large mass kinematics is responsible also for the additional term in the kernel of the BFKL equation corresponding to the soft pomeron contribution. It is important, that in the considered string model even in the tree approximation there is a colorless state in the t-channel with vacuum quantum numbers and a positive signature. In upper orders of the perturbation theory its Regge trajectory is renormalized. At small t this state mixes with the Mandelstam cut constructed from two reggeized gluons. The radiative corrections to its trajectory are calculated from ladder diagrams in the t-channel. The s-channel imaginary part of scattering amplitudes appears from the intermediate states in the above considered kinematics with relatively large masses: $(\alpha')^{-1} \ll M^2 \ll s$ for produced resonances. Physically the j-plane singularity with the vacuum quantum numbers in the tree approximation corresponds to the soft Pomeron which can exist together with the hard BFKL Pomeron.

Similar to the perturbative QCD, we restrict ourselves to the region $g^2 N_c \ln(s/M^2) \sim 1$. However, in the string case, the region $|\omega(t)| \ln(s/M^2) \gg 1$ is possible also because the Regge slope $\alpha' = 1/M^2$ has no g^2 smallness. Some important properties of the BFKL equation are related to this fact. In particular, we obtain that for $-\alpha' t \gg g^2 N_c$ its solution is concentrated near the saddle point $q_\perp/2$ for the reggeon transverse momenta $k_\perp$. For $D > 4$ the fluctuations of this momentum are small $\Delta(k_\perp - q_\perp/2)^2 \sim 1/(\alpha' \ln \alpha' s)$ and therefore the transverse momenta of the emitted gluons are also small $|k_\perp^g|^2 \leq \alpha' (\ln \alpha' s)^{-1}$. In the same time there are no similar restrictions on transverse momenta of the virtual gluons entering in the loop corrections to the gluon Regge trajectories. It means, that the contribution from the multiple saddle points $k_{i\perp} \approx q/2$ is suppressed by the reggeization effects.

The paper is organized as follows. In Sec. 2 the BFKL approach to the perturbative QCD is briefly reviewed. In Sec. 3 the superstring model which will be used later is introduced. In Sec. 4 the calculation of the multi-Regge asymptotics of production amplitudes is presented. In Sec. 5 the BFKL-like equation for the superstring model is derived. Also the vanishing of the impact factors for planar diagrams is demonstrated. In more details this problem is considered in Appendix D. In Sec. 6 the calculation of the BFKL kernel is performed. In Sec. 7 the equation for the case $D = 4$ is discussed. Among other things, it is explained why in the space-time $D = 10$ the non-Regge kinematics contributes to the Regge asymptotics of amplitudes. In Sec. 8 the solution of the BFKL equation at small values of $\alpha' t$ is constructed. In Sec. 9 an algebraic approach to this problem is developed and integrability properties of similar equations for composite states of several open strings in the multi-colour limit including a relation with the Heisenberg spin model are discussed. Appendices A, B and C contain some details of calculations.

2 BFKL approach in the perturbation QCD

As it was mentioned already, in the perturbative QCD the BFKL Pomeron appears as a composite state of two reggeized gluons [1]. The gluon is reggeized as a result of summing radiative corrections to the Born amplitude A_{Born} for the colored particle scattering $AB \to A'B'$ in the

Regge kinematics of large energies $\sqrt{s}$ and fixed momentum transfers $q = \sqrt{-t}$

$$A(s,t) = A_{Born}\, s^{\omega(t)}\,, \tag{2}$$

where A_{Born} is given below

$$A_{Born} = 2\,s\,g\,T^c_{A'A}\,\delta_{\lambda_{A'}\lambda_A}\frac{1}{t}\,g\,T^c_{B'B}\,\delta_{\lambda_{B'}\lambda_B}\ ,\ \ [T^c, T^{c'}] = i\,f_{cc'd}\,T^d \tag{3}$$

and $j = 1 + \omega(t)$ is the gluon Regge trajectory known in two first orders of the perturbation theory

$$\omega(t) = \omega_1(t) + \omega_2(t) + \ldots\,. \tag{4}$$

The trajectory contains logarithmic divergencies cancelled in the total cross sections with the contributions from the production of soft gluons. For example, in one loop approximation we have

$$\omega_1(-q^2) = -\frac{g^2}{16\,\pi^3}\,N_c\int d^2k\,\frac{q^2+\lambda^2}{(k^2+\lambda^2)((q-k)^2+\lambda^2)} \approx -\frac{g^2}{8\,\pi^2}\,N_c\,\ln\frac{q^2}{\lambda^2}\,, \tag{5}$$

where λ is a gluon mass introduced for the regularization of the infraredly divergent integral. On the other hand, the amplitude for the production of n gluons with momenta k_r in the multi-Regge kinematics

$$s \gg s_r = (k_{r-1}+k_r)^2 \gg q_r^2\,, \tag{6}$$

has the factorized form

$$A = 2\,s\,gT^{c_1}_{A'A}\,\delta_{\lambda_{A'}\lambda_A}\frac{s_1^{\omega(t_1)}}{t_1}g\,T^{d_1}_{c_2c_1}C(q_2,q_1)\frac{s_2^{\omega(t_2)}}{t_2}\,g\,T^{d_2}_{c_3c_2}C(q_3,q_2)...gT^c_{B'B}\,\delta_{\lambda_{B'}\lambda_B}\ , \tag{7}$$

where the effective vertex $C(q_2,q_1)$ for an emission of the gluon with a definite helicity is

$$C(q_2,q_1) = \frac{q_1\,q_2^*}{k_1}\,,\ \ k_1 = q_1 - q_2\,. \tag{8}$$

Here we introduced the complex coordinates

$$q_r = q_r^x + i\,q_r^y\,,\ \ k_r = k_r^x + i\,k_r^y \tag{9}$$

for transverse components $q_r^\perp$, $k_r^\perp$ of gluon momenta. The contribution to the elastic scattering amplitude from the intermediate state having a gluon with the momentum k_1 is proportional to the expression

$$C(q_2,q_1)\,C^*(q_2',\,q_1') + C^*(q_2,q_1)\,C(q_2',\,q_1') \tag{10}$$

and contains the pole $1/|k_1|^2$. The integration over k_1 cancels the infrared divergency in the gluon Regge trajectory appearing in the virtual corrections to the production amplitudes.

It is convenient to present the elastic amplitude for the colorless particle scattering in the form of the Mellin representation

$$A(s,t) = i\,s\int_{a-i\infty}^{a+i\infty}\frac{d\,\omega}{2\,\pi\,i}\,s^{\omega}\,f_{\omega}(t)\,, \tag{11}$$

where $f_\omega(t)$ is the t-channel partial wave analytically continued to the complex values $j = 1+\omega$ of the angular momentum. The amplitude $A(s,t)$ contains only the contribution from the t-channel state with vacuum quantum numbers and the positive signature, corresponding to the BFKL Pomeron. A positive value of the parameter a in the above representation is chosen from the condition, that all singularities of $f_\omega(t)$ are situated to the left from the integration contour.

The t-channel partial wave $f_\omega(t)$ can be expressed in terms of the gluon-gluon scattering amplitude $f_\omega(q_1,q_2;q)$ integrated with the impact-factors $\Phi(q_i, q-q_i)$

$$f_\omega(-q^2) = \int \frac{d^2q_1}{(2\pi)^2}\frac{\Phi(q_1,q-q_1)}{q_1^2\,(q-q_1)^2}\int \frac{d^2q_2}{(2\pi)^2}\,\frac{\Phi(q_2,q-q_2)}{q_2^2\,(q-q_2)^2}\,f_\omega(q_1,q_2;q)\,. \tag{12}$$

The impact-factors of colorless particles vanish at small gluon momenta

$$\Phi(0,q) = \Phi(q,0) = 0\,, \tag{13}$$

which leads to an infrared stability of $f_\omega(-q^2)$. The partial wave $f_\omega(q_1,q_2;q)$ satisfies the BFKL equation [1]

$$\omega\; f_\omega(q_1,q_2;q) \,= \omega\; f^0_\omega(q_1,q_2;q) \,-\, \frac{g^2\,N_c}{8\pi^2}\,H\,f_\omega(q_1,q_2;q)\,. \tag{14}$$

Here f^0_ω is a non-homogeneous term corresponding to the impact factor. The hamiltonian H is an integral operator, which can be defined by its action on the Pomeron wave function $f(\vec\rho_1,\vec\rho_{1'})$ in the coordinate representation [14]

$$H \,= \ln|\partial_1|^2 + \ln|\partial_2|^2 \,+\, \frac{1}{\partial_1\partial_2^*}\ln\,|\rho_{12}|^2\,\partial_1\partial_2^* \,+\, \frac{1}{\partial_1^*\partial_2}\ln\,|\rho_{12}|^2\,\partial_1^*\partial_2 - 4\Psi(1)\,, \tag{15}$$

where $\Psi(x) = (\ln\Gamma(x))'$ and we introduced the complex coordinates and momenta

$$\rho_r = x_r + iy_r\;,\;\partial_r = \frac{\partial}{\partial\rho_r}\,,\;\;\rho_{12} = \rho_1 - \rho_2\,. \tag{16}$$

The hamiltonian has the property of the Möbius invariance, which allows us to find its eigenfunctions [22]

$$E_{m,\widetilde{m}}(\vec\rho_1,\vec\rho_2;\vec\rho_0) = \left(\frac{\rho_{12}}{\rho_{10}\,\rho_{20}}\right)^m\left(\frac{\rho_{12}^*}{\rho_{10}^*\,\rho_{20}^*}\right)^{\widetilde{m}}\,, \tag{17}$$

where

$$m = \frac{1}{2}+i\nu+\frac{n}{2}\,,\;\widetilde{m} = \frac{1}{2}+i\nu-\frac{n}{2} \tag{18}$$

are conformal weights.

The high energy asymptotics of the total cross-section is parametrized by the Pomeron intercept Δ

$$\sigma_t \sim s^{\Delta} \tag{19}$$

In the leading logarithmic approximation we have

$$\Delta = -\frac{g^2 \, N_c}{8\pi^2} \, E, \tag{20}$$

where $E = -8\ln 2$ is the ground state energy of the Hamiltonian H. Therefore the cross-section σ_t violates the Froissart theorem $\sigma_t < c\ln^2(s)$. In the next-to-leading approximation the cross-section grows also, but not so rapidly (see [3]).

To verify the gluon reggeization one can use the s and u-channel unitarity constraints and dispersion relations to calculate by iterations the scattering amplitude with the color octet quantum numbers in the t-channel [1]. In LLA it is enough to consider only the multi-Regge kinematics for intermediate particles in the direct channels. In this kinematics the production amplitude has the multi-Regge form (7). The reggeization hypothesis should be in an agreement with the s- and u- channel unitarity. This requirement leads to the so-called bootstrap relations. The simplest bootstrap relation corresponds to the statement, that the scattering amplitude, obtained from the solution of the Bethe-Salpeter equation for the wave function of the composite state of two reggeized gluons in the octet channel should coincide with the Regge pole anzatz for the amplitude constructed in terms of the reggeized gluon exchange. In the momentum space the equation for the t-chanel partial wave $f^G_\omega(\overrightarrow{k}, \overrightarrow{q} - \overrightarrow{k})$ with the gluon quantum numbers has the form [1]

$$\omega \, f^G_\omega(\overrightarrow{k}, \overrightarrow{q} - \overrightarrow{k}) = \frac{1}{\overrightarrow{q}^2 + \lambda^2} - \frac{g^2}{8\pi^2} \, N_c \int \frac{d^2k'}{2\pi} \frac{\overrightarrow{q}^2 + \lambda^2}{\overrightarrow{k'}^2 + \lambda^2} \frac{f^G_\omega(\overrightarrow{k'}, \overrightarrow{q} - \overrightarrow{k'})}{(\overrightarrow{q} - \overrightarrow{k'})^2 + \lambda^2} +$$

$$\frac{g^2}{8\pi^2} \, N_c \int \frac{d^2k'}{2\pi} \left(\frac{\overrightarrow{k}^2 + \lambda^2}{\overrightarrow{k'}^2 + \lambda^2} + \frac{(\overrightarrow{q} - \overrightarrow{k})^2 + \lambda^2}{(\overrightarrow{q} - \overrightarrow{k'})^2 + \lambda^2} \right) \frac{f^G_\omega(\overrightarrow{k'}, \overrightarrow{q} - \overrightarrow{k'}) - f^G_\omega(\overrightarrow{k}, \overrightarrow{q} - \overrightarrow{k})}{(\overrightarrow{k} - \overrightarrow{k'})^2 + \lambda^2}, \tag{21}$$

where the gluon mass λ is introduced with the use of the Higgs mechanism to regularize the infrared divergencies.

It is obvious, that in an accordance with the bootstrap requirement the solution of the above equation corresponds to the Regge pole anzatz

$$f^G_\omega(\overrightarrow{k}, \overrightarrow{q} - \overrightarrow{k}) = \frac{1}{\overrightarrow{q}^2 + \lambda^2} \, \frac{1}{\omega - \omega(-\overrightarrow{q}^2)}, \tag{22}$$

where $\omega(-\overrightarrow{q}^2)$ is the gluon Regge trajectory.

3 String model

In the string and superstring models the scattering amplitude in the tree approximation satisfies the duality requirement: namely, the sum over the resonances in the t-channel related to its Regge asymptotics in the s-channel is equal to the (analytically continued) sum of resonances in the s and u-channels:

$$A(s,t,u) = A(s,t) + A(u,t) + A(s,u)\,, \;\; A(s,t) = \sum_i \frac{c_i(s)}{t - t_i} = \sum_i \frac{c_i(t)}{s - s_i}\,. \tag{23}$$

The particles with squared masses equal to t_i and integer spins $j = j_i$ lie on the linear Regge trajectories

$$j = j_0 + \alpha' t\,, \tag{24}$$

where j_0 and α' are their intercept and slope, respectively. The slope α' is universal for all excitations of the open string. For the closed strings it is equal to $\alpha'/2$. As for intercepts, in the critical dimensions $D = 26$ for the bosonic string and $D = 10$ for the superstrings, they are integer or half-integer numbers. In particular, for the intercepts of the leading bosonic Regge trajectories, corresponding to the massless vector (V) particle - "gluon" and tensor (T) particle - "graviton" we have respectively

$$j_0^V = 1\,,\; j_0^T = 2\,. \tag{25}$$

We put $j_0^V = 1$ also for the $D = 4$ model to leave the gluon on the trajectory. The "graviton" is absent in this case, instead one has a non-physical cut in the j -plane.

The Regge asymptotics of $A(s,t)$ in the dual models appears as a result of summing over the poles in the s-channel. Really at large s the contributions $\sim s^{-k}$ with integer values of k are cancelled and we can substitute approximately the sum over i by the dispersion integral

$$A(s,t) \approx \frac{1}{\pi}\int_0^\infty \frac{ds'}{s-s'}\,\Im A(s',t)\,,\; s' = s(i)\,,\; \Im A(s',t) = \pi c_i(t)\,.$$

It agrees with the Regge asymptotics $A(s,t) \sim (-s)^{j(t)}$ providing that $\Im A(s,t) \sim s^{j(t)}$.

In the Born approximation there are only stable particles in the intermediate state, but with taking into account loop corrections these particles acquire the widths due to their decay into lower mass states. As a result, $\Im A(s,t)$ has the δ-like singularities only for a finite number of stable states and the amplitude is a smooth function for large values of s. The function $A(s,t) \sim s^{1+\alpha' t}$ can be expanded in the series over the parameter $\alpha' t$ and one can interpret the corresponding term of the expansion $\sim s\,(\ln s)^n(\alpha' t)^n/n!$ as a contribution from the production of n particles in a multi-Regge kinematics. In QCD such a non-perturbative contribution $\omega(t) \sim \alpha' t$ to the Regge trajectory could appear from the integration region $k^2 \sim \Lambda^2_{QCD}$ in the loop corrections of the type of (5). In this case the common factor t would lead to the linearity of the trajectory at small t.

To begin with, let us consider the Born amplitude for the tachyon-tachyon scattering amplitude in the bosonic string theory

$$A(s,t) = g^2\,\frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(t)-\alpha(s))}\,,\;\; \alpha(t) = 1+\alpha' t\,, \tag{26}$$

where for simplicity we omitted the Chan-Paton factors. Asymptotically one obtains

$$\lim_{s\to\infty} A(s,t) = -g^2\,\alpha' s\Gamma(-\alpha(t))\,(-\alpha' s)^{\alpha' t}\,. \tag{27}$$

This result corresponds to the Regge asymptotics described by the reggeized gluon exchange in the t-channel. For other colliding particles there are additional factors depending on their spins. They are related to different residues for the corresponding Regge pole. Note, that the effective vertices for reggeized gluon interactions in QCD were obtained also from the string amplitudes in the limit $\alpha' \to 0$ [12].

For the superstring models the multiplier $\Gamma(-\alpha(t))$ in (26) is replaced by $\Gamma(-\alpha' t)$, which leads to the absence of the tachyon pole at $\alpha' t = -1$. At small momentum transfers both models give the same amplitude for the massless vector boson scattering. Taking, however, into account that one should sum over other intermediate t-channel states for the scattering amplitude with arbitrary momentum transfers, it is natural to consider only the superstring model where the tachyon disappears from the spectrum. The Regge limit of the superstring scattering amplitude is given in the end of this section (see [5]).

As it was said in Introduction, we use the Ramond-Neveu-Schwarz version of the open superstring model. In this model the interaction vertices are calculated in terms of the scalar superfield $X^M(z,\vartheta)$ where z is a world-sheet coordinate and ϑ is its superpartner. Here M labels the space-time coordinates, $M = 0, 1, \ldots, (D-1)$. The vertex $V(z,\vartheta;k,\xi)$ for the emission of a massless vector boson with its momentum $k = \{k^M\}$ and polarization vector $\xi = \{\xi^M\}$ is given below [5, 13]

$$V(z,\vartheta;k,\xi) = \xi\, D\, X\, e^{-ikX} \tag{28}$$

where $kX \equiv k_M X^M(z,\vartheta)$ and $\xi\, D\, X \equiv \xi_M\, D(z,\vartheta)\, X^M(z,\vartheta)$ are scalar products of the corresponding D-dimensional vectors. As usually, the relation $k\xi = 0$ is valid for polarizations of external vector bosons. In the contrast to the string tradition, in this paper we use the "mostly minus metrics" $ab = a_0 b_0 - \vec{a}\,\vec{b}$. The covariant super-derivative $D(z,\vartheta)$ appearing in (28) is given below

$$D(z,\vartheta) = \partial_z + \vartheta\, \partial_\vartheta\,, \tag{29}$$

where ∂_ϑ is the "left" derivative in ϑ. Note, that the gauge invariance $\xi \to \xi + ck$ of the amplitudes is valid due to the relation

$$\int dz d\vartheta\, D e^{-ikX} = 0\,. \tag{30}$$

The superfield vacuum correlator $\left\langle X^M(z,\vartheta) X^N(z',\vartheta')\right\rangle$ in super-coordinates for $z > z'$ equals

$$\left\langle X^M(z,\vartheta) X^N(z',\vartheta')\right\rangle = 2\alpha' \eta^{MN} \ln(z - z' - \vartheta\vartheta') = 2\alpha' \eta^{MN}\left[\ln(z-z') - \frac{\vartheta\vartheta'}{z-z'}\right], \tag{31}$$

where η^{MN} is the space-time metrics. The massless boson tree amplitude is obtained by integrating the vacuum expectation of the product of the vertices V_j (28) over (z_j, ϑ_j). The variables (z_j, ϑ_j) are assigned to the vertex for an emission of the boson carrying the momentum k_j and polarization ξ_j. In the amplitude we do not integrate over three of coordinates z_j using the integrand invariance under $SL(2,R)$-transformation. To conserve this symmetry after fixing the variables $(z^{(1)}, z^{(2)}, z^{(3)})$ one should include in the final expression the additional multiplier

$$r(z^{(1)}, z^{(2)}, z^{(3)}) = (z^{(1)} - z^{(2)})(z^{(1)} - z^{(3)})(z^{(2)} - z^{(3)})\,, \tag{32}$$

leading to an independence of the Born amplitude from the choice of these variables.

Thus, the open string amplitude $A_n(\{k_j,\xi_j\})$ for the interaction of n massless bosons in a tree approximation is given by

$$A_n(\{k_j,\xi_j\}) = \sum_{(r)} T_{(r)} A_n^{(r)}(\{k_j,\xi_j\}) \,, \tag{33}$$

where each a term corresponds to an ordering of the parameters z_j : $\{(r) : z_{j_1} > z_{j_2} > \ldots > z_{j_n}\}$ and the sum is taken over the configurations, which are non-equivalent under the cyclic transmutations of indices j_r. The coefficient $T_{(r)}$ is the Chan-Paton factor [5] for the given color group. Further, the expression $A^{(r)}(\{k_j,\xi_j\})$ is the integral over (z_j,ϑ_j) from the vacuum expectation of the product of interaction vertices multiplied by the factor $r(z_{j_1},z_{j_2},z_{j_n})$:

$$A_n^{(r)}(\{k_j,\xi_j\}) = g^{n-2}(z_{j_1}-z_{j_2})(z_{j_1}-z_{j_n})(z_{j_2}-z_{j_n}) \int \theta(z_{j_2}-z_{j_3}) \prod_{s=3}^{n-1} \theta(z_{j_s}-z_{j_{s+1}}) dz_{j_s}$$
$$\times \Big\langle d\vartheta_{j_1} V(z_{j_1},\vartheta_{j_1};k_{j_1},\xi_{j_1}) \ldots d\vartheta_{j_n} V(z_{j_n},\vartheta_{j_n};k_{j_n},\xi_{j_n}) \Big\rangle \,, \tag{34}$$

where $\theta(x)$ is the step function: $\theta(x)=1$ for $x>0$ and $\theta(x)=0$ for $x<0$.

Since the correlator (31) is singular at $z=z'$, the integral (34) is convergent only in a certain region of invariants constructed from external particle momenta. Each of the terms $A_n^{(r)}$ in (33) is calculated for such signs of the invariants where it is convergent, and the result is analytically continued to their physical values for the production kinematics. The integrand in (34) contains some contributions which do not contribute to the final result because they are total derivatives in integration variables. One can make their cancelation explicit using the fact, that the integrand in the superstring case is invariant under the $SL(2,R)$-SUSY transformation [13, 15]:

$$z = f(\tilde{z}) \,, \quad \vartheta = \sqrt{\frac{\partial f(\hat{z})}{\partial \hat{z}}} \left(\hat{\vartheta} + \varepsilon(\hat{z})\right) \left(1 - \frac{\beta\delta}{2}\right) \,, \quad \tilde{z} = \hat{z} + \hat{\vartheta}\varepsilon(\hat{z}) \,, \tag{35}$$

where $\varepsilon(z)$ and $f(z)$ are given below

$$\varepsilon(z) = \beta z + \delta \,, \qquad f(z) = \frac{az+b}{cz+d} \,, \quad \sqrt{\frac{\partial f(\hat{z})}{\partial \hat{z}}} = \frac{1}{cz+d} \,. \tag{36}$$

Here a, b, c, d are bosonic parameters and β, δ are their Grassmann partners. Note, that the superinterval is transformed in a simpler way

$$z - z' - \vartheta\vartheta' = Q^{-1}(\hat{z},\hat{\vartheta}) Q^{-1}(\hat{z}',\hat{\vartheta}') \Big(\hat{z} - \hat{z}' - \hat{\vartheta}\hat{\vartheta}'\Big) \,, \tag{37}$$

where

$$Q^{-1}(\hat{z},\hat{\vartheta}) = D(\hat{z},\hat{\vartheta})\vartheta \,. \tag{38}$$

Also, one can verify that

$$D(z,\vartheta) = Q(\hat{z},\hat{\vartheta}) D(\hat{z},\hat{\vartheta}). \tag{39}$$

An appropriate transformation (35) of the integration variables in (34) allows us to extract an explicit dependence from two ϑ_j, which gives a possibility to perform the integration over these variables. This symmetry is non-splitted because it mixes the Grassmann variables to bosonic ones. Note, that the step function factors in (34) lead after the symmetry transformation to the δ -function type terms which are multiplied by expressions vanishing in the kinematical region where the integral is convergent. To avoid the consideration of such terms, one can explicitly fix 5 variables (3|2) among all coordinates $(z_j|\vartheta_j)$ using the super-$SL(2,R)$ invariance. After that the integrand is multiplied by a supersymmetric generalization of the above factor $r(z_{j_1}, z_{j_2}, z_{j_n})$ [16] (for details see Appendix A). It is convenient to put $\vartheta_{j_1} = \vartheta_{j_2} = 0$. In this case the generalized factor r is $(z_{j_1} - z_{j_n})(z_{j_2} - z_{j_n})$. Thus, expression (34) is replaced by

$$A_n^{(r)}(\{k_j, \xi_j\}) = g^{n-2}(z_{j_1} - z_{j_n})(z_{j_2} - z_{j_n}) \int \theta(z_{j_2} - z_{j_3}) \prod_{s=3}^{n-1} \theta(z_{j_s} - z_{j_{s+1}}) dz_{j_s}$$

$$\times \Big\langle V(z_{j_1}, 0; k_{j_1}, \xi_{j_1}) V(z_{j_2}, 0; k_{j_2}, \xi_{j_2}) d\vartheta_{j_3} V(z_{j_3}, \vartheta_{j_3}; k_{j_3}, \xi_{j_3}) \ldots d\vartheta_{j_n} V(z_{j_n}, \vartheta_{j_n}; k_{j_n}, \xi_{j_n}) \Big\rangle . \quad (40)$$

Using relation (31) for the vacuum expectation of the product of vertices (28), one finds finally

$$A_n^{(r)}(\{k_j, \xi_j\}) = g^{n-2}(z_{j_1} - z_{j_n})(z_{j_2} - z_{j_n}) \int \theta(z_{j_2} - z_{j_3}) d\phi_{j_1} d\phi_{j_2} d\phi_{j_n} d\vartheta_{j_n} \times$$

$$\times \left(\prod_{s=3}^{n-1} \theta(z_{j_s} - z_{j_{s+1}}) dz_{j_s} d\phi_{j_s} d\vartheta_{j_s} \right)$$

$$\times \exp\Big[2\alpha' \sum_{m>n} [\xi_{jm}\phi_{jm} D(z_{jm}, \vartheta_{jm}) - ik_{jm}][\xi_{jn}\phi_{jn} D(z_{jn}, \vartheta_{jn}) - ik_{jn}] \ln(z_{jm} - z_{jn} - \vartheta_{jm}\vartheta_{jn})\Big], \quad (41)$$

where $\vartheta_{j_1} = \vartheta_{j_2} = 0$. The additional Grassmann variables ϕ_{j_s} are introduced for each of the vertices

$$V(z, \vartheta; k, \xi) = \int d\phi e^{(\phi\xi D - ik)X} . \quad (42)$$

So, the tree amplitude is presented by expression (33), where $A_n^{(r)}(\{k_j, \xi_j\})$ is given in eq. (41). Note, that under an anti-cyclic permutation the amplitude $A_n^{(r)}(\{k_j, \xi_j\})$ receives only the factor $(-1)^n$. Provided that three variables are fixed as $z_{j_1} = \infty$, $z_{j_2} = 1$ and $z_{j_n} = 0$, one can verify this property with the use of transformation (35) for the integrand in (41) choosing the functions $f(\hat{z}) = \hat{z}_{j_{n-1}}/\hat{z}$ and $\varepsilon(\hat{z}) = -\hat{\vartheta}_{j_n} - \hat{z}(\hat{\vartheta}_{j_{n-1}} - \hat{\vartheta}_{j_n})/\hat{z}_{j_{n-1}}$.

The Chan-Paton factor in (33) is given by

$$T_{(r)} = trace[\lambda_{j_1} \ldots \lambda_{j_n}] , \quad (43)$$

where λ_s is a color matrix for the corresponding group generator in the fundamental representation. Below we discuss the oriented string, for which λ_s are $U(n)$-matrices in the fundamental representation. In this case

$$trace(\lambda_r \lambda_s) = \delta_{rs}, \quad \sum_j (\lambda_j)_{ab}(\lambda_j)_{cd} = \delta_{ad}\delta_{bc} . \quad (44)$$

Hence

$$\lambda_r \lambda_s = \sum_j trace(\lambda_r \lambda_s \lambda_j)\lambda_j \,. \tag{45}$$

We take $\lambda_1 = I/\sqrt{n}$ as the $U(1)$-generator and the matrices $\lambda_2, \ldots, \lambda_n$ as generators of the $SU(n)$ group. They satisfy the following relations

$$\frac{1}{2}[\lambda_r\lambda_s - \lambda_s\lambda_r] = \sum_j f_{rsj}\lambda_j \,, \quad \frac{1}{2}[\lambda_r\lambda_s + \lambda_s\lambda_r] = \delta_{rs}n^{-1} + \sum_j d_{rsj}\lambda_j \,, \quad \sum_j d_{jjs} = 0 \,. \tag{46}$$

Obviously, the tensor d is symmetric in two first indices $d_{rsj} = d_{srj}$ and the structure constants f_{rsj} are completely anti-symmetric. Furthermore, $f_{1sj} = 0$, $d_{11j} = d_{jr1} = 0$, and, in addition, d_{rsj} is symmetric in all indices provided that both $s \neq 1$, $r \neq 1$ and $j \neq 1$. Besides, $d_{s1j} = \sqrt{1/n}\delta_{sj}$ when $s \neq 1$ and $j \neq 1$. We obtain also

$$\sum_{r,s} d_{rsj}d_{rsl} = \sum_{r,s} f_{rsj}f_{srl} = \frac{n}{2}[\delta_{jl} - \delta_{j1}\delta_{l1}] \,. \tag{47}$$

Below in the Regge kinematics ($s \gg -t \sim m^2$) we calculate the amplitude $A^{(23)}_{(14)}$ describing the scattering $a+b \to a'+b'$ of the vector massless particles (gluons) with momenta p_i ($p_i^2 = 0$). The corresponding kinematical invariants are $s = (p_a + p_b)^2 = (p_{a'} + p_{b'})^2$ and $t = (p_a - p_{a'})^2 = (p_b - p_{b'})^2$. The gauge is chosen to be $\xi_i^0 = 0$ for $i = a, a', b, b'$. In the Regge limit the amplitude $A^{(a'b')}_{(ab)}$ is (cf. (27))

$$\lim_{s\to\infty} A^{(a'b')}_{(ab)} = -2g^2\alpha' s\Gamma(-\alpha' t)(\alpha' s)^{\alpha' t}(\xi_a\xi_{a'})(\xi_b\xi_{b'})\Bigg\{\sum_s f_{j_a j_{a'} s}f_{s j_b j_{b'}}(e^{-\pi i\alpha' t} + 1)$$
$$+(e^{-\pi i\alpha' t} - 1)\left[\frac{\delta_{j_a j_{a'}}\delta_{j_b j_{b'}}}{n} + \sum_s d_{j_a j_{a'} s}d_{s j_b j_{b'}}\right]\Bigg\} \,, \tag{48}$$

where the color index j_i refers to $U(n)$-quantum numbers of the particle carrying the momentum p_i. The spin structure described by the polarization vectors ξ_i ($i = a, a', b, b'$) corresponds to the conservation of helicities for each of colliding particles. Various terms in (48) are associated with different Regge contributions. Their quantum numbers are the $SU(n)$ singlet (with the signature "+") and two adjoint $SU(n)$-representations (having the dimension n^2-1 and the signatures "+" and "-"). Note, that in QCD the Regge asymptotics of the scattering amplitude in the Born approximation contains only a contribution with the negative signature, corresponding to an exchange of the reggeized gluon. The contribution from the positive signature with octet quantum numbers appears only in upper orders of perturbation theory. Nevertheless, for large N_c the Regge trajectories with opposite signatures coincide each with another. The degeneracy of these t-channel states is important for the duality symmetry between the colorless composite states with different signatures [17]. As for the Regge contribution with vacuum quantum numbers, it also takes place in QCD only in upper orders of perturbation theory and corresponds to the BFKL Pomeron. Its appearance in the superstring model already in a tree approximation can be considered as a manifestation of the soft Pomeron having a non-perturbative nature. When $n = N_c$ is large, one can neglect this soft Pomeron contribution in expression (48).

To derive the above asymptotic behavior of the scattering amplitudes in the superstring theory we used the relation

$$\lambda_{r_i}\lambda_{r_j}e^{-i\pi(\alpha' t_{ij}+1)}+\lambda_{r_j}\lambda_{r_i}=\frac{1}{n}\delta_{r_ir_j}(e^{-i\pi(\alpha' t_{ij}+1)}+1)+\sum_s d_{r_ir_js}\lambda_s(e^{-i\pi(\alpha' t_{ij}+1)}+1)$$
$$+\sum_s f_{r_ir_js}\lambda_s(e^{-i\pi(\alpha' t_{ij}+1)}-1)\,, \qquad (49)$$

which follows from (46). Note, that the soft pomeron contribution appears also for the Chan-Paton factors corresponding to the colour group $O(n)$. Moreover, in the one-loop approximation its Regge trajectory does not contain ultraviolet divergencies for $n=32$ and $D=10$ [18]. In this model the gluon Regge trajectory $\omega_1(t)$ is finite is given below (see Appendix B)

$$\omega_1(t)=-8g^2n\int_{-1}^{1}\frac{d\lambda}{\lambda}\int_0^1 d\nu_2\,(\sin\pi\nu_2)^2\,\left(\frac{L_2}{1+L_1}\right)^{-\alpha' t}(1+L_1)^{-1}\,, \qquad (50)$$

where $N=32$, and

$$L_1=-2\sum_{n=1}^{\infty}\frac{\lambda^n\,(1-\lambda^n)^2}{(1-2\lambda^n\cos 2\pi\nu_2+\lambda^{2n})^2}\,,$$
$$L_2=\prod_{n=1}^{\infty}\frac{(1-\lambda^n)^4}{(1-2\lambda^n\cos 2\pi\nu_2+\lambda^{2n})^2}\,. \qquad (51)$$

If we consider only a contribution of the planar diagram, the low limit of integration over λ in the above expression is zero and the gluon Regge trajectory contains the logarithmic divergency at small λ, which can be removed by a renormalization of the slope α' in the Born amplitude [18]. In a similar way for the $SU(n)$ group we have

$$\omega_1(t)=-8g^2N_c\int_0^1\frac{d\lambda}{\lambda}\int_0^1 d\nu_2(\sin\pi\nu_2)^2\left[\left(\frac{L_2}{1+L_1}\right)^{-\alpha' t}(1+L_1)^{-1}-1\right]\,, \qquad (52)$$

where $n=N_c$ is the number of colors.

4 Particle production in the multi-Regge kinematics

Similar to the QCD case for string models the contribution of the ladder diagrams Fig.1 is factorized in the multi-Regge kinematics. This factorization was verified for the boson string theory [19] and it is valid also for the superstring models. We are going to calculate the kernel of the BFKL equation with the use of the s-channel unitarity by integrating the square of inelastic amplitudes over the intermediate particles in the multi-Regge kinematics.

In particular the diagram Fig.1b describes the production of one additional resonance with a fixed mass and momentum $k-k'$. In this diagram the initial particles have non-vanishing color quantum numbers whereas usually the solution of the BFKL equation for the gluon-gluon scattering should be sandwiched between the impact factors for the colorless colliding objects to avoid infrared divergencies. Note, however, that the integral kernel of the BFKL equation for the Pomeron wave function does not depend on quantum numbers of initial particles. To take into account a tower of the intermediate string states for the middle line on Fig.1b we find in this section the multi-Regge asymptotics of the amplitude for the tree diagram Fig.2. Then we calculate the sum over residues in the poles over the particle invariant mass k^2 and integrate over other kinematic variables to obtain the BFKL kernel.

As far as a large number of colors n is considered, only planar diagram contributions are important and the kernel is proportional to $n = N_c$. In the multi-Regge kinematics the momenta k_1, k_2, k_7, k_8 on Fig.2 are almost collinear. Their space components are opposite in sign to the corresponding components of the momenta k_4, k_3, k_6, k_5.

To each particle with the momentum k_j, the string coordinates z_j, ϑ_j and the color matrix λ_j are assigned. It is assumed that $z_8 < z_7 < z_6 < z_5 < z_4 < z_3 < z_2 < z_1$. We fix five variables: $z_1 = \infty$, $z_2 = 1$, $z_8 = 0$ and $\vartheta_1 = \vartheta_2 = 0$. In amplitude (33) one should sum over the contributions of the diagrams which can not be obtained from one configuration by cyclic or anti-cyclic transmutations of gluon indices. We should take into account also the Chan-Paton factors $T^{(+)}$

$$T^{(+)} = trace[\lambda_{r_8}\lambda_{r_7}\lambda_{r_6}\lambda_{r_5}\lambda_{r_4}\lambda_{r_3}\lambda_{r_2}\lambda_{r_1} + \lambda_{r_5}\lambda_{r_6}\lambda_{r_7}\lambda_{r_8}\lambda_{r_1}\lambda_{r_2}\lambda_{r_3}\lambda_{r_4}] \,. \tag{53}$$

To calculate the kernel from Fig.2 only contributions having poles in the invariant k^2 are essential. There are 16 diagrams of such type corresponding to the configuration

$$(k_1 = q_1' \,, k_2 = -p_{a'}) \,, \quad (k_3 = -p_{b'} \,, k_4 = q_2') \,, \quad (k_5 = -q_2 \,, k_6 = p_b) \,, \quad (k_7 = p_a \,, k_8 = -q_1) \tag{54}$$

and those obtained by the interchange ($k_j \rightleftharpoons k_l$) inside each of the above brackets, which leads to the signature factors. As it was pointed out already, p_a, p_b and $p_{a'}$, $p_{b'}$ are momenta of the initial and final particles, respectively. The momenta q_1, q_2, q_1' and q_2' correspond to intermediate particles. Obviously, for the calculation of a discontinuity of the elastic amplitude the relations $q_1' = -q_1$ and $q_2' = -q_2$ are valid, but temporally we distinguish between q_i and $-q_i'$ performing later an analytical continuation in the invariants $(k+q_1)^2$, $(k+q_2)^2$, $(k-q_1')^2$ and $(k-q_2')^2$ to their physical values. In the multi-Regge configuration the momentum k on Fig.2 obeys some kinematical constraints. Namely, the quantities $k^2, k_\perp^2$ and $(k^0)^2$ in the c.m. system are assumed to be much smaller than the energy invariants s, s_1 and s_2. Integral (41) for each of 16 diagrams is calculated in the kinematics where it is convergent, and subsequently the result is analytically continued to the physical region of the reaction.

In expression (41) several polarization structures arise, but only the term

$$A_s \sim (\xi_1\xi_2)(\xi_3\xi_4)(\xi_5\xi_6)(\xi_7\xi_8)$$

contributes to the multi-Regge asymptotics of the tree amplitude for Fig.2

$$A_s = g^6 \widetilde{A} T^{(+)} (\xi_1\xi_2)(\xi_3\xi_4)(\xi_5\xi_6)(\xi_7\xi_8) \,, \tag{55}$$

where the polarization vector ξ_j is associated with the momentum k_j and the Chan-Paton factor $T^{(+)}$ is given in eq. (53). Fixing the parameters as follows: $z_1 = \infty$, $z_2 = 1$, $z_8 = 0$ and

$\vartheta_1 = \vartheta_2 = 0$, one obtains from eq. (41)

$$\tilde{A} = \int \frac{\tilde{B} B \theta(1-z_3)\theta(z_7) d\vartheta_7 d\vartheta_8\, dz_7}{(z_3 - z_4 - \vartheta_3\vartheta_4)(z_5 - z_6 - \vartheta_5\vartheta_6)(z_7 - \vartheta_7\vartheta_8)} \prod_{s=3}^{6} \theta(z_s - z_{s+1}) dz_s\, d\vartheta_s\,. \tag{56}$$

Here the pre-factor $\tilde{B}$ is given below

$$\begin{aligned}
\tilde{B} \;=\; & \left[1 + \frac{2\alpha'(k_3k_4)}{z_3 - z_4}\vartheta_3\vartheta_4 + \frac{2\alpha'(k_3k_5)}{z_3 - z_5}\vartheta_3\vartheta_5 + \frac{2\alpha'(k_3k_6)}{z_3 - z_6}\vartheta_3\vartheta_6 + \frac{2\alpha'(k_3k_7)}{z_3 - z_7}\vartheta_3\vartheta_7\right. \\
& \left. + \frac{2\alpha'(k_3k_8)}{z_3 - z_8}\vartheta_3\vartheta_8\right]\left[1 + \frac{2\alpha'(k_4k_5)}{z_4 - z_5}\vartheta_4\vartheta_5 + \frac{2\alpha'(k_4k_6)}{z_4 - z_6}\vartheta_4\vartheta_6 + \frac{2\alpha'(k_4k_7)}{z_4 - z_7}\vartheta_4\vartheta_7\right. \\
& \left. + \frac{2\alpha'(k_4k_8)}{z_4 - z_8}\vartheta_4\vartheta_8\right]\left[1 + 2\frac{\alpha'(k_5k_6)}{z_5 - z_6}\vartheta_5\vartheta_6 + \frac{2\alpha'(k_5k_7)}{z_5 - z_7}\vartheta_5\vartheta_7 + \frac{2\alpha'(k_5k_8)}{z_5 - z_8}\vartheta_5\vartheta_8\right] \\
& \times\left[1 + \frac{2\alpha'(k_6k_7)}{z_6 - z_7}\vartheta_6\vartheta_7 + \frac{2\alpha'(k_6k_8)}{z_6 - z_8}\vartheta_6\vartheta_8\right]\left[1 + \frac{2\alpha'(k_7k_8)}{z_7 - z_8}\vartheta_7\vartheta_8\right]
\end{aligned} \tag{57}$$

and the expression B coincides with the integrand for a multi-tachyon scattering amplitude of the boson string theory:

$$B = \prod_{2 \le m < n \le 8} (z_m - z_n)^{-2\alpha' k_m k_n}\,. \tag{58}$$

Similar to the case of bosonic strings [20] one concludes from eq. (58) that in the multi-Regge kinematics the essential values of parameters are

$$z_3 \to 0, \quad z_3 = z_4 + x\,, \quad z_5 = z_6 + y\,, \quad x/z_6 \to 0\,, \quad y/z_6 \to 0\,, \quad z_7/z_6 \to 0\,. \tag{59}$$

In this configuration of variables the expression for B is simplified as follows

$$\begin{aligned}
B \;\approx\; & x^{-2\alpha' k_3k_4} y^{-2\alpha' k_5k_6} z_7^{-2\alpha' k_7k_8} z_4^{-2\alpha'(k_3+k_4)(k_7+k_8)} z_6^{-2\alpha'(k_5+k_6)(k_7+k_8)} \\
& \times (z_4 - z_6)^{-2\alpha'(k_3+k_4)(k_5+k_6)} \exp\Big[2\alpha' k_2k_3 x + 2\alpha' k_2k_5 y + 2\alpha' k_2(k_3+k_4) z_4 \\
& + 2\alpha' k_2(k_5+k_6) z_6 - 2\alpha' k_3k_7 \frac{z_7 x}{z_4^2} + 2\alpha' k_7(k_3+k_4)\frac{z_7}{z_4} - 2\alpha' k_5k_7 \frac{z_7 y}{z_6^2} \\
& + 2\alpha' k_7(k_5+k_6)\frac{z_7}{z_6} - 2\alpha' k_3(k_7 + ke_8)\frac{x}{z_4} - 2\alpha' k_5(k_7+k_8)\frac{y}{z_6}\Big]\,.
\end{aligned} \tag{60}$$

In the multi-Regge limit we have

$$\begin{aligned}
k_2(k_3+k_4) \;&\to\; -k_2(k_5+k_6) \to k_1 k\,, \quad k_7(k_3+k_4) \to -k_7(k_5+k_6) \to k_8 k\,, \\
k_3(k_7+k_8) \;&\to\; -k_4 k\,, \quad k_5(k_7+k_8) \to -k_6 k\,.
\end{aligned} \tag{61}$$

The integral is convergent in the following kinematical region of invariants

$$\begin{aligned}
k_2k_3 \;&<\; 0, \quad k_2k_5 < 0, \quad k_3k_7 > 0, \quad k_5k_7 > 0, \quad k_2(k_3+k_4) < 0\,, \\
k_7(k_5+k_6) \;&<\; 0\,, \; k_3(k_7+k_8) > 0\,, \quad k_5(k_7+k_8) > 0\,.
\end{aligned} \tag{62}$$

We redefine the variables as follows

$$
\begin{aligned}
z_4 &\to \frac{z_4}{-2\alpha' k_2(k_3+k_4)}, \quad z_6 \to \frac{z_6}{2\alpha' k_2(k_5+k_6)}, \quad z_7 \to \frac{z_7}{[2\alpha' k_2(k_3+k_4)][2\alpha' k_7(k_5+k_6)]}, \\
x &\to \frac{x}{-2\alpha'(k_2k_3)}, \quad y \to \frac{y}{-2\alpha'(k_2k_5)}.
\end{aligned} \tag{63}
$$

The asymptotics of $\widetilde{A}$ in expression (55) can be written as follows

$$
\widetilde{A} = GA(t_{5678}, \kappa^2, t_{3478}, t_{34}, t_{56}, t_{12}, t_{78}), \tag{64}
$$

where the factor G collects all large energy invariants, and $A(t_{5678}, \kappa^2, t_{3478}, t_{34}, t_{56}, t_{12}, t_{78})$ depends only on fixed transverse momenta. We define the energy invariants s_i and s_{jk} as follows

$$
s_i = ((k_1+k)^2 \approx 2(k_1k), \quad s_{23} = -2(k_2k_3), \quad s_{25} = -2(k_2k_5). \tag{65}
$$

Then the expression for G in (64) has the factorized form

$$
G = (-\alpha' s_1)^{\alpha' t_{12}+1}(-\alpha' s_7)^{\alpha' t_{78}+1}(-\alpha' s_4)^{\alpha' t_{34}+1}(-\alpha' s_6)^{\alpha' t_{56}+1}, \tag{66}
$$

where the fixed invariants are

$$
t_i = (k_1+k_j)^2, \quad t_{ijlm} = -(k_i+k_j+k_l+k_m)^2. \tag{67}
$$

Note, that we have the kinematical constraint

$$
t_{3456} + t_{3478} + t_{5678} = t_{12} + t_{34} + t_{56} + t_{78}. \tag{68}
$$

The fixed invariant κ^2 in (64) is given below

$$
\kappa^2 = \alpha' s_1 s_4 / s_{23} = \alpha' s_1 s_6 / s_{25} = \alpha'[(k^0)^2 - k_{||}^2), \tag{69}
$$

where $k_{||}$ is the longitudinal component of the momentum k. To simplify the last factor in (64) one can use the following relations valid in the multi-Regge kinematics due to eqs. (61)

$$
\begin{aligned}
(k_5k_7)(k_6k_8) - (k_5k_8)(k_6k_7) &= \frac{1}{2}\Big[((k_5+k_6)k_7)((k_6-k_5)k_8)) \\
+((k_5-k_6)k_7)((k_6+k_5)k_8))\Big] &= \frac{1}{2}(k_6k_8)[t_{5678} - t_{56} - t_{78} - \kappa^2]
\end{aligned} \tag{70}
$$

and

$$
(k_4k_7)(k_3k_8) - (k_4k_8)(k_3k_7) = \frac{1}{2}(k_3k_8)[t_{4378} - t_{43} - t_{78} + \kappa^2]. \tag{71}
$$

After redefinition (63) of variables in expression (56) with the use of above simplifications one can perform the Grassmann integrations. As a result, the last factor in (64) turns out to be

$$
A(t_{5678}, \kappa^2, t_{3478}, t_{34}, t_{56}, t_{12}, t_{78}) = \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz_4 \int_0^{z_4} B_s V_b \, dz_6, \tag{72}
$$

where both B_s and V_b depend on integration parameters and external variables. The expression for V_b is the same as in the bosonic string model, and the pre-factor B_s arises due to the superstring modifications. Explicitly,

$$
\begin{aligned}
V_b &= x^{-\alpha' t_{34}} y^{-\alpha' t_{56}} z_7^{-\alpha' t_{78}} z_4^{-\alpha'[t_{3478}-t_{34}-t_{78}]} z_6^{-\alpha'[t_{5678}-t_{56}-t_{78}]} (z_4 - z_6)^{-\alpha'[t_{3456}-t_{34}-t_{56}]} [xyz_7]^{-2} \\
&\times \exp\Big[-\Big(x + y + \frac{z_7 x}{z_4^2} + \frac{z_7 y}{z_6^2}\Big) - \Big(z_4 - z_6 + \frac{z_7}{z_6} - \frac{z_7}{z_4}\Big) + \kappa^2\Big(\frac{x}{z_4} + \frac{y}{z_6}\Big)\Big]. \qquad (73)
\end{aligned}
$$

The integrals in (72) are defined for $\kappa^2 < 0$, and

$$
\begin{aligned}
B_s &= (\alpha' t_{34} + 1)(\alpha' t_{56} + 1)(\alpha' t_{78} + 1) + \frac{xyz_7^2}{z_4^2 z_6^2}[\alpha' t_{35} + \alpha' t_{36} + \alpha' t_{45} + \alpha' t_{46}] \\
&- y^2 z_7^2 (\alpha' t_{34} + 1)\Big[\frac{\alpha'[t_{5678} - t_{56} - t_{78}] - \kappa^2}{y z_7 z_6^2} - \frac{1}{z_6^4}\Big] \\
&- x^2 z_7^2 (\alpha' t_{56} + 1)\Big[\frac{\alpha'[t_{3478} - t_{34} - t_{78}] + \kappa^2}{x z_7 z_4^2} - \frac{1}{z_4^4}\Big]. \qquad (74)
\end{aligned}
$$

In expression (72) one can perform easily the integration over the variables x and y

$$
\begin{aligned}
A(t_{5678}, \kappa^2, t_{3478}, t_{34}, t_{56}, t_{12}, t_{78}) &= \Gamma(-\alpha' t_{34})\Gamma(-\alpha' t_{56}) \\
&\times e^{\pi i(\alpha' t_{34} + \alpha' t_{56})} I(t_{5678}, \kappa^2, t_{3478}, t_{34}, t_{56}, t_{12}, t_{78}), \qquad (75)
\end{aligned}
$$

where the factor $I(t_{5678}, \kappa^2, t_{3478}, t_{34}, t_{56}, t_{12}, t_{78})$ is obtained from eq. (72). Its form can be essentially simplified as it is shown in Appendix C. Below we present the final result using in addition the fact that the calculated amplitude is symmetric under an interchange between the left and right parts of the considered diagram (see Section 3). Thus, taking into account the relation $t_{1256} = t_{3478}$ eq. (75) can be written as follows

$$
\begin{aligned}
A(t_{5678}, \kappa^2, t_{3478}, t_{34}, t_{56}, t_{12}, t_{78}) &= \Gamma(-\alpha' t_{12})\Gamma(-\alpha' t_{78}) e^{\pi i(\alpha' t_{12} + \alpha' t_{78})} \\
&\times I(t_{5678}, \kappa^2, t_{3478}, t_{12}, t_{78}, t_{34}, t_{56}), \qquad (76)
\end{aligned}
$$

where

$$
\begin{aligned}
& I(t_{5678}, \kappa^2, t_{3478}, t_{12}, t_{78}, t_{34}, t_{56}) = (\kappa^2)^{-\alpha' t_{12} - \alpha' t_{78} - 2} \int dy dz df e^{-f-y} f^{-\alpha' t_{34} - 1} y^{-\alpha' t_{56} - 1} \\
& \times z^{-\alpha' t_{5678} - 1} (1-z)^{-\alpha' t_{3456} + \alpha' t_{34} + \alpha' t_{56}} [f + yz - \kappa^2(1-z)]^{\alpha' t_{12}} [y + fz - \kappa^2(1-z)]^{\alpha' t_{78}} \\
& \times \Big[\alpha' t_{12} + \alpha' t_{78} - \alpha' t_{3478}(1-z) + \frac{\alpha' t_{12} y (1-z)}{f + yz - \kappa^2(1-z)} \\
& + \frac{\alpha' t_{78} f (1-z)}{y + fz - \kappa^2(1-z)} + z - (f + y)\Big]. \qquad (77)
\end{aligned}
$$

Here all integrations are performed from 0 to ∞. The above expression is convergent at $\kappa^2 < 0$. For the factor in a front of the integral we choose the condition $\arg \kappa^2 = -\pi$. Really the phase arising in this case, is compensated by a similar phase in (76) and A is real for $\kappa < 0$.

The final result is obtained by summing the contributions of 16 diagrams listed in the beginning of this Section, every term being analytically continued from the kinematical region where the corresponding integral (41) is convergent. Taking into account the spin structure (55) for each diagram and relations (64), (66) and (76), we derive the following expression for $A^{(f)}$

$$
\begin{aligned}
A^{(f)} &= g^6 s_1^{\alpha' t_{12}+\alpha' t_{78}+2} s_4^{\alpha' t_{34}+\alpha' t_{56}+2} (\xi_1\xi_2)(\xi_3\xi_4)(\xi_5\xi_6)(\xi_7\xi_8)\Gamma(-\alpha' t_{12})\Gamma(-\alpha' t_{78}) e^{\pi i(\alpha' t_{12}+\alpha' t_{78})} \\
&\times \sum_{j_1,j_2,j_3,j_4} \mathcal{F}_{r_1,r_2,j_1}(t_{12})\mathcal{F}_{r_3,r_4,j_2}(t_{34})\, \mathcal{F}_{r_5,r_6,j_3}(t_{56})\mathcal{F}_{r_7,r_8,j_3}(t_{78}) \\
&\times T^{(+)}_{j_1,j_2,j_3,j_4} I(t_{5678},\kappa^2,t_{3478},t_{12},t_{78},t_{34},t_{56})\,, \qquad (78)
\end{aligned}
$$

where

$$\mathcal{F}_{r_s,r_l,j}(t_{sl}) = trace[(\lambda_{r_s}\lambda_{r_l} e^{-\pi i(\alpha' t_{sl}+1)} + \lambda_{r_s}\lambda_{r_l})\lambda_j] \qquad (79)$$

and

$$T^{(+)}_{j_1,j_2,j_3,j_4} = trace[\lambda_{j_4}\lambda_{j_3}\lambda_{j_2}\lambda_{j_1} + \lambda_{j_1}\lambda_{j_2}\lambda_{j_3}\lambda_{j_4}]\,. \qquad (80)$$

For the group $U(n)$ the index r_i enumerates color states of the particle carrying the momentum k_i defined in (54). After an analytical continuation we put $s_1 = \pm s_7$ and $s_4 = \pm s_6$. In a similar way, $I(t_{5678},\kappa^2,t_{3478},t_{12},t_{78},t_{34},t_{56})$ in (78) is calculated using a similar continuation of expression (77) to the region $\kappa^2 > 0$. This procedure is performed by the replacement $\kappa^2 \to \kappa^2 + i\delta$ with $\delta \to +0$. We have also the condition $k^{(0)} > 0$, and therefore due to (69), our prescription corresponds to the Feynman rule for going around the singularity. After the analytic continuation the factor in front of the integral turns out to be positive.

5 BFKL equation in the string model

Omitting the impact factors of colored particles in the left and right hand sides of the contribution of the diagram Fig1.b one can obtain expressions for higher order ladder diagrams by iterating its interior part. To find the BFKL kernel in the considered string model, one should calculate from expression (78) its contribution to the t-channel partial wave for the scattering of massless particles. Also one-loop correction $\omega_1(t) \sim g^2 N_c$ to the trajectory (1) should be taken into account (see (50)). Thus, $\alpha' t_{jl}$ in (78) is replaced by the expression $\alpha' t_{jl} + \omega_1(t_{jl})$. Due to the presence of non-planar diagrams, the one-loop correction to the singlet trajectory differs from that for the octet case. However, assuming that the number of colors is large, below we neglect this difference.

The contribution to the t-channel partial wave from the diagram Fig1.b is given by the Mellin transformation in $\ln s$ applied to the imaginary part of the amplitude. To calculate it one should find in eq. (78) the sum of residues for the poles in the variable $\alpha' k^2$ and integrate the result over a relevant phase volume. Initially we put $q_1' = -q_1$, $q_2' = -q_2$, $r_1 = r_8$, $r_4 = r_5$, $\xi_1 = \xi_8$, $\xi_4 = \xi_5$ summing subsequently over indices r_1, r_4 and polarization states ξ_1 and ξ_4. The poles are situated at $\alpha' k^2 = m$, where m is an integer number changing from 0 to ∞.

Below we denote by l, l' the transverse momenta of two neighboring reggeons and by q the total momentum transfer related to the corresponding invariants as follows

$$l^2 = -t_{12}, \quad (l')^2 = -t_{34}, \quad (q-l)^2 = -t_{78}, \quad (q-l')^2 = -t_{56}\,. \tag{81}$$

With these definitions, κ^2 in (78) is given below (cf. (69))

$$\kappa^2 = \alpha'[k^2 + (l-l')^2]\,, \qquad k^2 = t_{1234} = t_{5678}\,. \tag{82}$$

The multi-Regge kinematics implies, that the inequalities $s_1/k^2 \gg 1$ and $s_4/k^2 \gg 1$ are fulfilled. The integration over this region leads to the singularities of the t -channel partial wave at $\omega = j-1$. Here j is the total angular momentum. The contribution $F^{(b)}(\omega; q^2)$ to the t-channel partial wave from the diagram Fig1.b including the correction to the trajectory $\omega_1(t)$ (1) is given below

$$\begin{aligned}
F^{(b)}(\omega; q^2) = &\sum_{r_1,r_2,r_3,r_4} \int d^{D-2}l\,\widetilde{\Phi}_{r_a,r_{a'},r_1,r_4}(q;l)\Gamma(\alpha' l^2)\Gamma(\alpha'(q-l)^2)e^{-\pi i(\alpha' l^2+\alpha'(q-l)^2)} \\
&\times \frac{g^2}{4(2\pi)^{D-1}} T^{(+)}_{r_1r_2r_3r_4} \sum_{m=0}^{\infty} \int_{s(m)}^{\infty} s^{-j-1} ds \int_{s(m)}^{\infty} \frac{ds_1\, ds_4}{s^2} (\alpha' s_1)^{-\beta(l^2)-\beta((q-l)^2)+2} \\
\times \int d^{D-2}l'\,\widetilde{I}_m(q;l,l')&(\alpha' s_4)^{-\beta((l')^2)-\beta((q-l')^2)+2}\delta(\alpha' s_1 s_4/s - \alpha' k_\perp^2 - m)\widetilde{\Phi}_{r_b,r_{b'},r_2,r_3}(q;l')\,.
\end{aligned} \tag{83}$$

Here

$$\beta(q^2) = \alpha' q^2 - \omega_1(-q^2)\,, \tag{84}$$

and D is the number of space-time dimensions. The quantity $\widetilde{I}_m(q;l,l')$ is the residue of the pole at $\alpha' k^2 = m$ in the integral $I(t_{5678}, \kappa^2, t_{3478}, t_{12}, t_{78}, t_{34}, t_{56})$ appearing in expression (78) (see also definitions (81)). Further, $s(m)$ is the low energy cut-off: $s(0) = s_0$ for $m = 0$ and $s(m) = s_0\kappa_m^2$ for $m \geq 1$. In this case $\kappa_m^2 = m + \alpha' k_\perp^2$. We impose the condition $\alpha' s_0 \gg 1$ because the production amplitude is known only in the multi-Regge kinematics. The cut-off is introduced to have a possibility to verify that the non-multi-Regge kinematics is not essential. The factor $T^{(+)}_{r_1r_2r_3r_4}$ is presented in eq. (80) and

$$\begin{aligned}
\widetilde{\Phi}_{r_a,r_{a'},r_1,r_4}(q;l) &= (\xi_a\xi_{a'})\frac{2g^2}{(2\pi)^{D-1}}\sum_r \mathcal{F}_{r_{a'},r,r_1}(t_{12})\mathcal{F}_{r_a,r,r_4}(t_{78})\,, \\
\widetilde{\Phi}_{r_b,r_{b'},r_2,r_3}(q;l') &= (\xi_b\xi_{b'})\frac{2g^2}{(2\pi)^{D-1}}\sum_r \mathcal{F}_{r_{b'},r,r_2}(t_{34})\mathcal{F}_{r_b,r,r_3}(t_{56})\,,
\end{aligned} \tag{85}$$

where the function $\mathcal{F}$ is defined in (79). Expressions (85) are massless state contributions to the impact factors. The total impact factor for the planar diagram Fig1b being the sum of a tower of string states is equal to zero, which can be verified with the use of the quasi-elastic asymptotics of the production amplitude (for more details see Appendix D). Its vanishing ensures the cancellation of the Amati-Fubini-Stangelini cuts in the j-plane for the planar diagrams.

Once the integration over s, s_1 and s_4 being performed, eq.(83) is represented as follows

$$F^{(b)}(\omega;q^2)=\sum_{r_1,r_2,r_3,r_4}\int d^{D-2}l\widetilde{\Phi}_{r_a,r_{a'},r_1,r_4}(q;l)\Gamma(\alpha' l^2)\Gamma(\alpha'(q-l)^2)e^{-\pi i(\alpha' l^2+\alpha'(q-l)^2)}$$
$$\times\frac{g^2(\alpha')^{\omega}}{4(2\pi)^{D-1}}\int d^{D-2}l' T^{(+)}_{r_1r_2r_3r_4}\frac{(\alpha' s_0)^{-\omega-\alpha' l^2-\alpha'(q-l)^2}(\alpha' s_0)^{-\omega-\alpha'(l')^2-\alpha'(q-l')^2}}{[\omega+\beta(l^2)+\beta((q-l)^2)][\omega+\beta((l')^2)-\beta((q-l')^2)]}$$
$$\times\widehat{I}(q;l,l')\widetilde{\Phi}_{r_b,r_{b'},r_2,r_3}(q;l'). \tag{86}$$

Here

$$\widehat{I}(q;l,l')=S^{(0)}_{\omega}(q;l,l')+S_{\omega}(q;l,l') \tag{87}$$

and, in turn,

$$S_{\omega}(q;l,l')=\sum_{m=1}^{\infty}(\kappa_m^2)^{-\omega-\alpha' l^2-\alpha'(q-l)^2-\alpha'(l')^2-\alpha'(q-l')^2+2}\widetilde{I}_m(q:l,l'),$$
$$S^{(0)}_{\omega}(q;l,l')=\alpha'(\kappa_0^2)^{\omega+2}\widetilde{I}_0(q;l,l'). \tag{88}$$

In this expression the factor $\widetilde{I}_m(q;l,l')$ is the same as in eq. (83). We remind, that the quantity $\kappa_m^2 = m+\alpha' k_\perp^2$ coincides with κ^2 on the mass shell $\alpha' k^2 = m$ (m is an integer number). Expression (86) is correct only in the domain where it does not depend on the cut-off s_0, which means, that the power of s_0 in this expression should be much smaller than unity. In an accordance with eq. (86), it is convenient to present the contribution $F(\omega;q^2)$ to the partial wave as a sum of contributions of the diagrams Fig.1 starting from Fig.1b written in the form

$$F(\omega;q^2) = \frac{g^2}{4(2\pi)^{D-1}}\sum_{r_1,r_2,r_3,r_4}\int d^{D-2}l\int d^{D-2}l'\widetilde{\Phi}_{r_a,r_{a'},r_1,r_2}(q;l)\Gamma(\alpha' l^2)\Gamma(\alpha'(q-l)^2)$$
$$\times e^{-\pi i(\alpha' l^2+\alpha'(q-l)^2)}\frac{R_{r_1r_2r_3r_4}(\omega;q;l,l')}{\omega+\beta((l')^2)+\beta((q-l')^2)}\widetilde{\Phi}_{r_b,r_{b'},r_3,r_4}(q;l'). \tag{89}$$

The particle-particle-reggeon vertices $\widetilde{\Phi}$ contained in eq.(89) can be extracted from eq.(48). Omitting these vertices in eq. (89), one can verify that the amplitude $R_{r_1r_2r_3r_4}(\omega;q;l,l_1)$ for Fig.1 obeys the BFKL-like equation

$$(\omega+\beta(l^2)+\beta(q-l)^2)R_{r_1,r_2,r_3,r_4}(\omega;q;l,l_1)=\widehat{I}(q:l,l_1)T^{(+)}_{r_1,r_4,r_3,r_2}$$
$$+\frac{\alpha' g^2}{4(2\pi)^{D-1}}\int d^{D-2}l'\widehat{I}(q;l,l')\sum_{r,r'}T^{(+)}_{r_1r'rr_2}R_{rr'r_3r_4}(\omega;q;l',l_1)$$
$$\times\left[\sum_{\sigma,\sigma'}[e^{\pi i(\alpha'(l')^2+1)}+(-1)^{\sigma}][e^{i\pi(\alpha'(q-l')^2+1)}+(-1)^{\sigma'}]e^{-\pi i(\alpha'(l')^2+\alpha'(q-l')^2}\right], \tag{90}$$

where the summation over (σ,σ') is associated with the signatures for the corresponding trajectories having their color group quantum numbers denoted, respectively, by (r,r'). So, σ,σ' are 0 for a positive signature and 1 for the negative one.

The number of colors is considered to be large and therefore one can neglect the color-singlet reggeons in (89). In this case r_i,r,r' coincide with color indices of the corresponding adjoint representations. Because σ and σ' take values 0 and 1, the expression inside the large square brackets in eq. (90) is equal to 4. Using eqs. (46) and (80) one finds that

$$T^{(+)}_{r_1 r' r r_2} = \tilde{T}^{(+)}_{r_1 r' r r_2} + 2\delta_{r_1 r_2}\delta_{r r'}/n \ , \qquad \sum_r \tilde{T}^{(+)}_{r_1 r r r_2} = 0 \, . \tag{91}$$

So, $\tilde{T}^{(+)}_{r_1,r',r,r_2}$ annihilates the singlet state. Furthermore,

$$R_{r_1 r_2 r_3 r_4}(\omega; q; l, l_1) = 2 f^{(0)}_\omega(q; l, l_1)\delta_{r_1 r_2}\delta_{r r'}/n + f^{(1)}_\omega(q; l, l_1)\tilde{T}^{(+)}_{r_1 r' r r_2} \, , \tag{92}$$

where $f^{(0)}_\omega(q; l, l_1)$ and $f^{(1)}_\omega(q; l, l_1)$ are partial waves for the vacuum channel and for the state belonging to the adjoint representation of the $SU(N_c)$ group, respectively.

Using expression (90) one can derive, that the partial waves $f^{(s)}_\omega(q; l, l_1)$ with $s = 0, 1$ obey the BFKL-like equation

$$\begin{aligned} & [\omega + \alpha' l^2 + \alpha'(q-l)^2 - \omega_1(-l^2) - \omega_1(-(q-l)^2)] f^{(s)}_\omega(\omega; q; l, l_1) \\ = \ & \widehat{I}(q; l, l_1) + \frac{g^2 N_c c_s}{(2\pi)^{D-1}} \int \widehat{I}(q; l, l') f^{(s)}_\omega(q; l', l_1) d^{D-2} l' \, , \end{aligned} \tag{93}$$

where $c_0 = 1$ and $c_1 = 1/2$. The integral kernel $\widehat{I}(q; l, l_1)$ is calculated in the next Section.

6 Integral kernel

With the use of (88) one can verify, that the massless state contribution to I (77) is given by

$$\begin{aligned} S^{(0)}_\omega(q; l, l')/\alpha' = & \, [\alpha'(l-l')^2]^{\omega + \alpha' l^2 + \alpha'(q-l)^2} \int df dy e^{-(y+f)} f^{\alpha'(l')^2 - 1} y^{\alpha'(q-l')^2 - 1} \\ \times [f - \alpha'(l-l')^2 - i\epsilon]^{-\alpha' l^2} & [y - \alpha'(l-l')^2 - i\epsilon]^{-\alpha'(q-l)^2} \Big[-\alpha' q^2 + \alpha'(l')^2 + \alpha'(q-l')^2 \\ & - \frac{\alpha' l^2 y}{[f - \alpha'(l-l')^2 - i\epsilon]} - \frac{\alpha'(q-l)^2 f}{[y - \alpha'(l-l')^2 - i\epsilon]} - (f+y) \Big] \, . \end{aligned} \tag{94}$$

The integral in the above expression can be written in terms of the Whittaker function $W_{\lambda,\mu}(-\kappa^2 - i\epsilon)$ defined as follows

$$J(a, b; z) \equiv \int_0^\infty e^{-t} t^a (z+t)^b dt = z^{(a+b)/2} e^{z/2} \Gamma(b+1) W_{(b-a)/2, (a-b+1)/2}(z) \tag{95}$$

which has the following representation

$$J(a, b; z) = z^{a+b+1}\Gamma(a+1)\frac{\Gamma(-a-b-1)}{\Gamma(-b)}\Phi(a+1, a+b+2, z) + \Gamma(a+b+1)\Phi(-b, -a-b, z) \tag{96}$$

as a linear combination of the confluent hypergeometric function $\Phi(a, b, z)$

$$\Phi(a, b, z) = 1 + \frac{a}{b} z + \frac{a(a+1)}{2b(b+1)} z^2 + \dots \, . \tag{97}$$

Indeed, we obtain for $\widetilde{I}_0$ (88)

$$
\begin{aligned}
(\kappa_0^2)^{\omega+2}\widetilde{I}_0(q;l,l') = [\alpha'(l-l')^2]^{\omega+\alpha' l^2+\alpha'(q-l)^2}\Big[[-\alpha' q^2+\alpha'(l')^2+\alpha'(q-l')^2]\\
\times J(\alpha'(l')^2-1,-\alpha' l^2;-\alpha'(l-l')^2)J(\alpha'(q-l')^2-1,-\alpha'(q-l)^2;-\alpha'(l-l')^2)\\
-\alpha' l^2 J(\alpha'(l')^2-1,-\alpha' l^2-1;-\alpha'(l-l')^2)J(\alpha'(q-l')^2,-\alpha'(q-l)^2;-\alpha'(l-l')^2)\\
-\alpha'(q-l)^2 J(\alpha'(l')^2,-\alpha' l^2;-\alpha'(l-l')^2)J(\alpha'(q-l')^2-1,-\alpha'(q-l)^2-1;-\alpha'(l-l')^2)\\
-J(\alpha'(l')^2,-\alpha' l^2;-\alpha'(l-l')^2)J(\alpha'(q-l')^2-1,-\alpha'(q-l)^2;-\alpha'(l-l')^2)\\
-J(\alpha'(l')^2-1,-\alpha' l^2;-\alpha'(l-l')^2)J(\alpha'(q-l')^2,-\alpha'(q-l)^2;-\alpha'(l-l')^2)\Big]\,. \qquad (98)
\end{aligned}
$$

To calculate the massive state contribution $S(q;l,l')$ to the kernel (87) it is convenient to change the integration variables $f\to\kappa^2 f$, $y\to\kappa^2 y$ in expression (77). As a result, the factor being a power of κ_m^2 is extracted from the integral

$$
\begin{aligned}
&(\kappa^2)^{-\omega-\alpha' l^2+\alpha'(q-l)^2-\alpha'(l')^2-\alpha'(q-l')^2+2} I(t_{5678},\kappa^2,t_{3478},t_{12},t_{78},t_{34},t_{56})\,\alpha'\\
=\;&\frac{1}{\Gamma(\omega+\alpha' l^2+\alpha'(q-l)^2)}\int df\,dy\,dv\,v^{\omega+\alpha' l^2+\alpha'(q-l)^2-1}e^{-\kappa^2(y+f+v)}z^{-\alpha' k^2-1}\\
&\times\left[V_1(z,f,y;q,l,l')+\frac{1}{v}[\omega+\alpha' l^2+\alpha'(q-l)^2-1]V_2(z,f,y;q,l,l')\right], \qquad (99)
\end{aligned}
$$

where $V_i(z,f,y;q,l,l')$ does not depend on κ^2

$$
\begin{aligned}
V_1(z,f,y;q,l,l') &= f^{\alpha'(l')^2-1}y^{\alpha'(q-l')^2-1}(1-z)^{\alpha' q^2-\alpha'(l')^2-\alpha'(q-l')^2}q_1^{-\alpha' l^2}q_2^{-\alpha'(q-l)^2}\\
\times\Big[&[-\alpha' q^2+\alpha'(l')^2+\alpha'(q-l')^2+\alpha' l^2+\alpha'(q-l)^2-\alpha'(l-l')^2](1-z)-\alpha' l^2\\
&-\alpha'(q-l)^2-\alpha' l^2\frac{y}{q_1}(1-z)-\alpha'(q-l)^2\frac{f}{q_2}(1-z)+z\Big], \qquad (100)\\
V_2(z,f,y;q,l,l') &= f^{\alpha'(l')^2-1}y^{\alpha'(q-l')^2-1}(1-z)^{\alpha' q^2-\alpha'(l')^2-\alpha'(q-l')^2}q_1^{-\alpha' l^2}q_2^{-\alpha'(q-l)^2}\\
&\times[(1-z)-(f+y)]\,. \qquad (101)
\end{aligned}
$$

In these expressions we denoted

$$
q_1=f+yz-(1-z)-i\epsilon\,,\quad q_2=y+fz-(1-z)-i\epsilon\,,\quad \epsilon\to 0\,, \qquad (102)
$$

where $\epsilon\to+0$. In integral (99) the residue in the pole at $k^2=m$ depends on m only through the exponent $\exp[-m(f+y+v)]$ multiplied by the derivative $\partial_z^{m-1}V_{(i)}(z,f,y;q,l,l')/(m-1)!$ calculated at $z=0$. Therefore after summing the residues over m we obtain

$$
\begin{aligned}
\sum_{m=1}^{\infty} e^{-m(f+y+v)}\partial_z^{m-1}V_{(i)}(z,f,y;q,l,l')/(m-1)!\Big|_{z=0} \;=\; & V_i(e^{-(f+y+v)},f,y;q,l,l')\\
&-V_i(0,f,y;q,l,l')\,. \qquad (103)
\end{aligned}
$$

Thus, the quantity $S_\omega(q;l,l')$ can be written as follows

$$S_\omega(q;l,l')/\alpha' = \frac{1}{\Gamma(\omega+\alpha' l^2+\alpha'(q-l)^2)}\int df dy dv v^{\omega+\alpha' l^2+\alpha'(q-l)^2-1}\times$$
$$\times e^{-\alpha'(l-l')^2(y+f+v)}\Bigg[[V_1(e^{-(f+y+v)},f,y;q,l,l')-V_1(0,f,y;q,l,l')]$$
$$+\frac{1}{v}[\omega+\alpha' l^2+\alpha'(q-l)^2-1][V_2(e^{-(f+y+v)},f,y;q,l,l')-V_2(0,f,y;q,l,l')]\Bigg]. \tag{104}$$

Integrating the last term in (104) over v by parts we obtain

$$S_\omega(q;l,l')/\alpha' = \frac{1}{\Gamma(\omega+\alpha' l^2+\alpha'(q-l)^2)}\int df dy dv v^{\omega+\alpha' l^2+\alpha'(q-l)^2-1}e^{-\alpha'(l-l')^2(y+f+v)}$$
$$\times\Bigg[f^{\alpha'(l')^2-1}y^{\alpha'(q-l')^2-1}(1-e^{-f-y-v})^{\alpha' q^2-\alpha'(l')^2-\alpha'(q-l')^2}$$
$$\times[f+ye^{-f-y-v}-(1-e^{-f-y-v})-i\epsilon]^{-\alpha' l^2}[y+fe^{-f-y-v}-(1-e^{-f-y-v})-i\epsilon]^{-\alpha'(q-l)^2}$$
$$\times\mathcal{B}(e^{-f-y-v},f,y;q,l,l')-V_1(0,f,y;q,l,l')-\alpha'(l-l')^2V_2(0,f,y;q,l,l')\Bigg], \tag{105}$$

where

$$\mathcal{B}(z,f,y;q,l,l') = -\alpha'(l-l')^2(f+y)+\alpha' l^2 y+\alpha'(q-l)^2 f+[\alpha'(q-l')^2$$
$$+\alpha'(l')^2-\alpha' q^2]\Big[1-\frac{z(f+y)}{1-z}\Big]-\frac{\alpha' l^2 fy(1-z)}{f+yz-(1-z)-i\epsilon}-\frac{\alpha'(q-l)^2 fy(1-z)}{y+fz-(1-z)-i\epsilon}. \tag{106}$$

Really the leading contribution to (105) arises from the region of small integration variables. In particular, it results in a pole at $\omega=\alpha' q^2$, as well as in a Mandelstam cut term. To find the main part of (105) we cut from below the integration variables in eq. (105) by a parameter $\lambda\ll 1$. Then from eq. (106), one can obtain, that the leading contribution to S_ω is given by the expression

$$S_\omega(q;l,l')/\alpha' \to -e^{\alpha' l^2+\alpha'(q-l)^2}\frac{[\alpha' q^2-\alpha'(l')^2-\alpha'(q-l')^2]}{\Gamma(\omega+\alpha' l^2+\alpha'(q-l)^2)}\Bigg[\int_0^\lambda df dy dv v^\omega f^{\alpha'(l')^2-1}$$
$$\times y^{\alpha'(q-l')^2-1}(v+f+y)^{\alpha' q^2-\alpha'(l')^2-\alpha'(q-l')^2-1}-\frac{1}{\alpha'(l')^2\alpha'(q-l')^2(\omega+\alpha' l^2+\alpha'(q-l)^2)}\Bigg], \tag{107}$$

where the pole term arises from two last terms in (106).

To calculate the integral (107) the integration region is divided into 6 domains: $v>f>y$, $v>y>f$, $f>v>y$, $f>y>v$, $y>v>f$ and $y>f>v$. In the first domain we replace initially $y\to fy$ and then $f\to vf$. As a result, the v-dependence of the integrand turns out to be $v^{\omega+\alpha' q^2}$. Integrating it over v we observe the pole at $\omega=-\alpha' q^2$. The similar procedure is performed in each of the rest domains. As far as, in addition, the expression $\omega+\alpha' l^2+\alpha'(q-l)^2$ is implied to be small $\sim 1/\ln s$, the factor $\exp[\alpha' l^2+\alpha'(q-l)^2]$ in (107) should be replaced by unity.

For the same reason $\Gamma(\omega+\alpha' l^2+\alpha'(q-l)^2) = \Gamma(1+\omega+\alpha' l^2+\alpha'(q-l)^2)/(\omega+\alpha' l^2+\alpha'(q-l)^2) \approx 1/(\omega+\alpha' l^2+\alpha'(q-l)^2)$. Using these simplifications expression (107) is given below

$$
\begin{aligned}
S_\omega(q;l,l')/\alpha' \approx \frac{1}{(\omega+\alpha' q^2)} & [\alpha' q^2 - \alpha' l^2 - \alpha'(q-l)^2][\alpha' q^2 - \alpha'(l')^2 - \alpha'(q-l')^2] \\
& \times \Big[\tilde{F}(\alpha'(l')^2, \alpha'(q-l')^2; \alpha' q^2 - \alpha'(l')^2 - \alpha'(q-l')^2) \\
& + \tilde{F}(1-\alpha' q^2, \alpha'(q-l')^2; \alpha' q^2 - \alpha'(l')^2 - \alpha'(q-l')^2) \\
& + \tilde{F}(1-\alpha' q^2, \alpha'(l')^2; \alpha' q^2 - \alpha'(l')^2 - \alpha'(q-l')^2) \Big], \qquad (108)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{F}(a,b,c) &= \int_0^1 df \int_0^1 dy f^{a+b-1}(1+f+fy)^{c-1}[y^{b-1}+y^{a-1}] = \\
&= \sum_{n,m=0}^{\infty} \frac{\Gamma(c)}{\Gamma(c-m-n)\Gamma(m+1)\Gamma(n+1)(a+m)(b+n)} . \qquad (109)
\end{aligned}
$$

One can verify that at small momenta $\alpha' q^2 \sim \alpha' l^2 \sim \alpha' l' \ll 1$ the first term in the large square brackets of eq. (108) gives the main contribution

$$
S_\omega^{sing}(q;l,l') = \frac{[\alpha' q^2 - \alpha' l^2 - \alpha'(q-l)^2][\alpha' q^2 - \alpha'(l')^2 - \alpha'(q-l')^2]}{\alpha'(\omega+\alpha' q^2)(l')^2(q-l')^2} . \qquad (110)
$$

Expressions (108) and (110) are correct in a neighbourhood of the pole and of zeros of the numerator with the deviations being $\sim 1/\ln s \sim N_c g^2$. As far as the numerator does not vanish at $\omega+\alpha' l^2+\alpha'(q-l)^2 = \omega+\alpha' l'^2+\alpha'(q-l')^2 = 0$, it contributes to both the Mandelstam cuts and the pole at $\omega = -\alpha' q^2$.

The pole at $\omega = -\alpha' q^2$ corresponds to the soft Pomeron which exists already in the Born expression (48) for the elastic amplitude. Relatively large masses $1 \ll \alpha' M^2 \ll \alpha' s$ of produced resonances contribute to this pole. Therefore in the box diagram Fig.1a we expect a pole of the second order from the integration over large masses of two intermediate s-channel resonances. This second order pole appears as a result of the perturbative expansion of the Pomeron Regge pole over the one-loop correction $\omega_1(t) \sim g^2$. In the two-loop approximation, corresponding to Fig.1b, we should have the third order pole with the residue proportional to $\omega_1^2(t)$. In this diagram, apart from the pole (110) there is a product of two pole singularities $1/(\omega+\alpha' q^2)$ from the integration over the large masses of resonances produced in the fragmentation regions of initial particles. In the multi-Regge kinematics one obtains also the poles $1/(\omega+\alpha' l^2+\alpha'(q-l)^2)$ and $1/(\omega+\alpha' l'^2+\alpha'(q-l')^2)$ leading after the integration over l and l' to the Mandelstam cuts (we put here $l = k_\perp$ and $l' = k'_\perp$). Because the residue of the pole (110) in the BFKL kernel is small due to the smallness of the expressions in the square brackets, it cancels approximately the neighboring poles depending on l and l' and therefore one can attempt to extract from the contribution for Fig.1b the third order pole being the second order term in the expansion of the soft Pomeron pole in $\omega_1(t)$.

Indeed, let us present the numerator of the pole in eq. (110) in the form

$$[\alpha' q^2 - \alpha' l^2 - \alpha'(q-l)^2][\alpha' q^2 - \alpha'(l')^2 - \alpha'(q-l')^2] = [\omega + \alpha' l^2 + \alpha'(q-l)^2][\omega + \alpha'(l')^2 + \alpha'(q-l')^2]$$

$$- [\omega + \alpha' q^2] \left[\omega - \alpha' q^2 + \alpha' l^2 + \alpha'(q-l)^2 + \alpha' l'^2 + \alpha'(q-l')^2\right] . \qquad (111)$$

Then the second term in the right hand side of this equality, killing the pole $1/(\omega + \alpha' q^2)$ in (110), contributes only to the Mandelstam cuts. As for the first term in (111), it corresponds to the second term of expansion for the soft Pomeron pole. Indeed, its numerator cancels the neighboring propagators for Mandelstam cuts. Therefore the corresponding integrals over the relative rapidities $\ln s_{12}$ and $\ln s_{23}$ are convergent for the large invariants s_{12} and s_{23}. So, we should calculate these integrals exactly without simplifications corresponding to the multi-Regge kinematics. It is plausible, that as a result of such calculation the pole in expression (110) together with additional poles $1/(\omega + \alpha' q^2)$ from two impact factors would reproduce the total one-loop correction $\sim \omega_1^2$ from Fig.1b in the second order expansion of the Pomeron pole.

The first term in (111) is important also for a cancellation of the singularities in (110) at $(l, l') \to 0$ and $(l, l') \to q$ leading to a convergence of the corresponding integrals over the multi-Regge region. In addition, it has a non-trivial funcional dependence containing both poles and cuts in ω. So, for the investigation of the BFKL equation in the $D = 4$ model we use the whole expression (110) without neglecting the soft Pomeron pole. Simultaneously, we add a piece from the non-multi-Regge kinematics.

For a general case of the ladder Fig.1 one can perform a decompositions similar to (111) for each kernel. The contributions appearing from the first terms in the right hand sides of (111) correspond to the particles produced in a non-multi-Regge kinematics. The form of production amplitudes in this region can not be extracted from our above results. Probably this contribution corresponds to a geometric progression appearing from an expansion of the soft Pomeron pole in $\omega_1(t)$.

Presumably one can represent the partial wave as follows $f_\omega(-q^2)$ as

$$f_\omega(-q^2) = f_\omega^{(p)}(-q^2) + f_\omega^{mr}(-q^2) , \qquad (112)$$

where the first term corresponds to the soft Pomeron contribution in the form of the geometrical progression and the term $f_\omega^{mr}(-q^2)$ results from the multi-Regge kinematics. In principle there can be a more complicated situation with an interference between the Regge pole and cut.

7 BFKL equation in the $D = 4$ string model

It follows from the above discussion that the singularities of the t=channel partial waves arise from the region where $\omega + \alpha' l'^2 + \alpha'(q - l')^2 \sim 1/\ln s$. For $D > 4$ after the integration over the region $\alpha' l'^2$ the corresponding contribution is suppressed by powers of logarithms $\sim (\ln s)^{-(D-4)/2}$ for each produced particle, which leads to a possibility to find the solution of the BFKL equation as a series in this small parameter. In principle, it is not excluded that for very large energies the number of produced particles grows so rapidly, that the averaged pair energies $s_{k,k+1}$ for these particles are not so large to justify the saddle-point method of calculations of the integrals. In this case the BFKL equation which sums contributions from the multi-Regge kinematics could have non-trivial solutions even for $D > 4$. Here, however, we restrict ourselves to the $D = 4$ case hoping to return to the discussion of other values of D in future publications. Moreover, only the amplitude with vacuum quantum numbers in the crossing channel is considered.

At $D = 4$ the BFKL equation has a non-trivial solution in terms of the function $f_\omega^{(0)}(q;l)$ defined by the relation

$$f_\omega^{(0)}(q;l) = \int f_\omega^{(0)}(q;l,l_1)\Phi(q;l_1)d^2l_1\,, \tag{113}$$

where $\Phi(q;l_1)$ is an impact factor. Generally the solution contains contributions from non-planar diagrams.

One loop correction $\omega_1(t)$ to the gluon trajectory for $D = 4$ (1) has the form

$$\omega_1(t) = -\frac{g^2N_c}{8\pi^2}\ln(q^2/\lambda^2) + \omega_1^{(m)}(q^2) \tag{114}$$

where the first contribution corresponds to massless states in the t -channel and the second term non-singular at $q^2 = 0$ appears from the massive string excitations (cf. expression (5) in QCD).

To begin with, let us discuss the region of small t, where $\alpha' q^2 \sim g^2N_c$. In this case for $D = 4$ the small gluon virtualities $\alpha' l^2 \sim g^2N_c \sim \alpha'(l')^2 \sim g^2N_c$ are important. For such momenta l and l' the pole contribution (110) dominates in $S_\omega(q:l,l')$ and the singularities of the t-channel partial wave are situated for small g^2 at $\omega \sim g^2$. Because the infra-red divergencies in the integral kernel are cancelled between the contribution from the real particle emission and one-loop correction to the Regge trajectories, the factor $[(l-l')^2]^{\omega+\alpha' l^2+\alpha'(q-l)^2}$ in the right hand side of eq. (98) can be omitted. Hence, from expression (94) we obtain the following contribution to the kernel (77) corresponding to the massless state production

$$S_\omega^{(0)}(q:l,l') = -\frac{q^2}{(l')^2(q-l')^2} + \frac{l^2}{(l-l')^2(l')^2} + \frac{(q-l)^2}{(l-l')^2(q-l')^2}\,. \tag{115}$$

Expression (115) coincides with the corresponding result [11] in QCD. The massive state term in (114) is expected to vanish at $t \to 0$. So, the radiative correction to the gluon trajectory for small momentum transfers l and $q-l$ also can be approximated by the QCD expression (5). As a result, the BFKL equation (93) for the vacuum channel at $D = 4$ and $\alpha' q^2 \sim g^2N_c$ is drastically simplified

$$\begin{aligned}
[\omega + \alpha' l^2 + \alpha'(q-l)^2]f_\omega^{(0)}(q;l) &= \Phi(q;l) + \frac{g^2N_c}{8\pi^3}\int\Bigg\{S_\omega^{(0)}(q:l,l')f_\omega^{(0)}(q;,l') \\
&-\frac{1}{(l-l')^2}\left[\frac{l^2}{[(l-l')^2+(l')^2]} + \frac{(q-l)^2}{[(l-l')^2+(q-l')^2]}\right]f_\omega^{(0)}(q;,l)\Bigg\}d^2l' \\
&+\frac{g^2N_c}{8\pi^3}\int\frac{[\alpha' q^2 - \alpha' l^2 - \alpha'(q-l)^2][q^2-(l')^2-(q-l')^2]}{(\omega+\alpha' q^2)(l')^2(q-l')^2}f_\omega^{(0)}(q;,l')d^2l'\,,
\end{aligned} \tag{116}$$

where the contribution from Fig.1a is also taken into account.

In eq. (116) we performed a relevant subtraction of the Regge trajectory contribution to obtain the integral kernel in the BFKL form (cf. [1]), and the expression for $S^{(0)}_\omega(q:l,l')$ is given in (115). Equation (116) differs from the BFKL equation in QCD only by terms linear in squared gluon momenta at its left hand side and by an additional pole term $\sim 1/(\omega+\alpha' q^2)$ in the kernel. The terms $\sim l^2$ and $\sim (q-l)^2$ improve the properties of its kernel at $l\to\infty$. As a result, unlike the QCD case in LLA, eq.(116) is expected to have a discrete spectrum at nonzero values of q^2.

Comparing the large-l behaviour of the left and right hand sides of eq.(116) we conclude, that the linear terms in the gluon trajectories in eq. (116) lead to a constant behaviour of $f^{(0)}_\omega(q;l)$ at $l\to\infty$. As a result, the integral

$$h_\omega(q)=\int\frac{[q^2-(l')^2-(q-l')^2]}{(l')^2(q-l')^2}f^{(0)}_\omega(q;,l')d^2l' \tag{117}$$

in the last term on its right hand side is divergent. Taking into account, that this term plays role of an additional inhomogenious contribution to eq.(116) we present $f^{(0)}_\omega(q;l)$ in the form

$$f^{(0)}_\omega(q;l)=\frac{g^2N_c[\alpha' q^2-\alpha' l^2-\alpha'(q-l)^2]}{8\pi^3(\omega+\alpha' q^2)[\omega+\alpha' l^2+\alpha'(q-l)^2]}h_\omega(q)+\frac{g^2N_c h_\omega(q)}{8\pi^3(\omega+\alpha' q^2)}\hat f^{(0)}_\omega(q;l)+\tilde f^{(0)}_\omega(q;l)\,, \tag{118}$$

where $h_\omega(q)$ is given by (117), while $\hat f^{(0)}_\omega(q;l)$ and $\tilde f^{(0)}_\omega(q;l)$ are determined from the equation (below $F^{(0)}_\omega(q;l)$ is denoted either by $\hat f^{(0)}_\omega(q;l)$ or $\tilde f^{(0)}_\omega(q;l)$)

$$[\omega+\alpha' l^2+\alpha'(q-l)^2]F^{(0)}_\omega(q;l)=\tilde\Phi(q;l)+\frac{g^2N_c}{8\pi^3}\int\Big\{S^{(0)}_\omega(q:l,l')F^{(0)}_\omega(q;,l')$$
$$-\frac{1}{(l-l')^2}\left[\frac{l^2}{[(l-l')^2+(l')^2]}+\frac{(q-l)^2}{[(l-l')^2+(q-l')^2]}\right]F^{(0)}_\omega(q;,l)\Big\}d^2l'\,. \tag{119}$$

Here for $F^{(0)}_\omega(q;l)=\hat f^{(0)}_\omega(q;l)$ we have

$$\tilde\Phi(q;l)=\int S^{(0)}_\omega(q:l,l')\frac{[\alpha' q^2-\alpha'(l')^2-\alpha'(q-l')^2]}{\omega+\alpha'(l')^2+\alpha'(q-l')^2}d^2l'-$$
$$\int\frac{d^2l'}{(l-l')^2}\left[\frac{l^2}{[(l-l')^2+(l')^2]}+\frac{(q-l)^2}{[(l-l')^2+(q-l')^2]}\right]\frac{[\alpha' q^2-\alpha' l^2-\alpha'(q-l)^2]}{\omega+\alpha' l^2+\alpha'(q-l)^2}\,, \tag{120}$$

and for $F^{(0)}_\omega(q;l)=\tilde f^{(0)}_\omega(q;l)$,

$$\tilde\Phi(q;l)=\Phi(q;l)\,. \tag{121}$$

Using (118) and (117) one obtains $h_\omega(q)$ as the solution of a linear equation

$$h_\omega(q)=\frac{\omega+\alpha' q^2}{\omega+\alpha' q^2-\tilde\beta(\omega,q^2)}\int\frac{[q^2-l^2-(q-l)^2]}{l^2(q-l)^2}\tilde f^{(0)}_\omega(q;,l)d^2l\,, \tag{122}$$

where

$$\tilde\beta(\omega,q^2)=\frac{g^2N_c}{8\pi^3}\Bigg[\tilde\beta_0+\left[\int_{l^2<\Lambda^2}\frac{\alpha'[q^2-l^2-(q-l)^2]^2}{[\omega+\alpha' l^2+\alpha'(q-l)^2]l^2(q-l)^2}d^2l-\pi\ln\Lambda^2\right]_{\Lambda^2\to\infty}$$
$$+\int\frac{[q^2-l^2-(q-l)^2]}{l^2(q-l)^2}\hat f^{(0)}_\omega(q;,l)d^2l\Bigg]\,. \tag{123}$$

We subtracted the logarithmic divergency from the second term in the brackets assuming that subtraction term is added to the quantity $\tilde{\beta}_0$ determined by the integration region $\alpha' k^2 \sim 1$. So, β_0 depends also on the non-multi-Regge configurations, leading to the renormalisation ω_1 of the soft Pomeron Regge trajectory. This conclusion follows from expression (86) for the production cross-section, where the kernel dependence from the cut-off s_0 is essential, and from our discussion of eq. (111). It is natural to expect that $\tilde{\beta}_0 \sim 1$. So, the solution of eq. (116) depends on the additional parameter $\tilde{\beta}_0$. The equation $\omega + \alpha' q^2 = \tilde{\beta}(\omega, q^2)$ allows to find the Regge trajectories . In addition, one can conclude from (123) that $\tilde{\beta}(\omega, q^2)$ contains the Mandelstam cuts in the ω-plane.

In the region $g^2 N_c \ll \alpha' q^2 \ll 1$ the asymptotic behaviour of the scattering amplitude is related to singularities of the integral $\int f_\omega^{(0)}(q; , l', l_1) d^2 l'$ near $\omega \approx -q^2/2$. They appear from the kinematics, in which the solution of eq. (116) is concentrated at $l = q/2$. Let us introduce the new momenta v and v' according to the definition

$$l = q/2 + v \,, \qquad l' = q/2 + v' \,, \qquad v^2 \ll q^2 \,, \quad (v')^2 \ll q^2 \,. \tag{124}$$

Leaving only leading terms, eq. (94) is simplified as follows

$$S_\omega^{(0)}(q : l, l') = 2 \, \frac{[\alpha'(l - l')^2]^{\omega + \alpha' q^2/2}}{(l - l')^2} \,, \tag{125}$$

where $l - l' = v - v'$. The numerator in (125) is different from unity only in the region $\alpha' r^2 = \alpha'(l - l')^2 \leq s_0/s$. Due to eq. (69) for a massless intermediate state the value of r^2 in the multi-Regge kinematics $s_1, s_2 > s_0 \gg 1/\alpha'$ is restricted by the condition $r^2 \geq s_0^2/s$.

However, according to the generalized Gribov theorem the gluon production amplitude for the momenta $r^2 \ll 1/\alpha'$ is also large in the quasi-elastic regions $s_1 \leq 1/\alpha'$ and $s_2 \leq 1/\alpha'$ and equals to the elastic amplitude multiplied by a bremstrahlung factor (see for example [21]). Therefore the integral over r^2 is not bounded from below by s_0^2/s being infraredly divergent. As usually, this divergency is cancelled with the contribution from the virtual corrections proportional to the gluon Regge trajectories. Thus, we substitute by unity the numerator in (125) and represent the massless contribution to the gluon trajectory correction as follows

$$g^2 N_c \ln(l^2/\lambda^2) = \int \frac{d^2 l'}{(l - l')^2} \left[\frac{l^2}{[(l - l')^2 + (l')^2]} + \frac{(q - l)^2}{[(l - l')^2 + (q - l')^2]} - \frac{4v^2}{(v^2 + (v')^2)} \right] + \int \frac{4v^2 d^2 l'}{(v^2 + (v')^2)(v - v')^2} \,. \tag{126}$$

Here l and v are related according to eqs. (124). Performing the expansion in v in the right hand side of eq. (93) one can write it as follows

$$[\omega + \alpha' q^2/2 + 2\alpha' v^2 + \frac{g^2 N_c}{4\pi^2} \ln(q^2/64 v^2)] f_\omega^{(0)}(q; v, v_1) = \Phi(q; q/2) + \frac{g^2 N_c}{8\pi^3} \int \frac{2}{(v - v')^2} \left[f_\omega^{(0)}(q; , v', v_1) - \frac{2v^2}{(v^2 + (v')^2)} f_\omega^{(0)}(q; , v, v_1) \right] d^2 l' \,. \tag{127}$$

The impact factor in (127) is taken at $l = q/2$ because it is expected to be a smooth function of l near $l = q/2$. For $\alpha' q^2 \geq 1$ the radiative correction (5) to the gluon trajectory at a small momentum transfer should be replaced by $\omega_1(t)$ taken at $t = -q^2/4$. Thus, the final equation valid for both restrictions $g^2 N_c \ll \alpha' q^2 \ll 1$ and $\alpha' q^2 \geq 1$ is obtained from (127) by the substitution $\omega \to \omega'(q^2)$ where

$$\omega'(q^2) = \omega + 2\Big[-\omega_1^{(m)}(-q^2/4) + \frac{g^2 N_c}{8\pi^2} \ln(q^2/4\lambda^2)\Big]\,. \tag{128}$$

The corresponding quantities are defined in eqs. (1) and (114). Note, that the infra-red divergency at $\lambda \to 0$ in the last term is cancelled with a similar divergency in the right hand side of eq. (126) at $v' \to v$.

8 Solution of the equation at small momentum transfers

At $q \neq 0$ the integral kernel of the BFKL equation for the string theory at $D = 4$ is non-singular at small momenta. In this case one can expect that for the t-channel partial wave the cut at $\omega = \omega_0$ disappears, and instead of a fixed singularity of $f_\omega(q^2)$ in the ω-plane there are only Regge poles. Here we demonstrate this phenomenon in the case of small values of $\alpha' \vec{q}^2$, where there exists an analytic solution of the equation in the $D = 4$ string theory. The pole contribution to the kernel corresponding to the soft Pomeron will be neglected.

In the domain of relatively small $\vec{q}^2$

$$\alpha' \vec{q}^2 \ll g^2\, N_c\,. \tag{129}$$

one can divide the region of possible values of $\vec{\rho}^2 \sim \vec{k}^{-2}$ into two subregions $\vec{\rho}^2 \sim \vec{q}^{-2}$ and $\vec{\rho}^2 \sim \alpha' (g^2\, N_c)^{-1}$, where $\rho = \rho_{12}$. In the first subregion one can use the conformal (Möbius) invariance and the eigenfunction in the mixed representation coincides with the Fourrie transformation in the c.m. coordinate $\vec{\rho}_0$ from the function $E_{m,\widetilde{m}}(\vec{\rho}_1, \vec{\rho}_2; \vec{\rho}_0)$ (17). Its asymptotics at small $\vec{\rho}^2$ has the form [22]

$$E_{m,\widetilde{m}}(\vec{q}, \vec{\rho}) \sim \rho^m\, (\rho^*)^{\widetilde{m}} + e^{i\,\delta_{m,\widetilde{m}}(\vec{q})}\, \rho^{1-m}\, (\rho^*)^{1-\widetilde{m}}\,, \tag{130}$$

where

$$e^{i\,\delta_{m,\widetilde{m}}(\vec{q})} = (-1)^n\, \left(\frac{|q|}{4}\right)^{-4i\nu}\, \left(\frac{q}{q^*}\right)^n\, \frac{\Gamma(m+\frac{1}{2})\,\Gamma(\widetilde{m}+\frac{1}{2})}{\Gamma(-m+\frac{3}{2})\,\Gamma(-\widetilde{m}+\frac{3}{2})}\,. \tag{131}$$

For simplicity we consider the case $n = 0$, where

$$e^{i\,\delta_{m,m}(\vec{q})} = \left(\frac{|q|}{4}\right)^{-4i\nu}\, \frac{\Gamma^2(1+i\nu)}{\Gamma^2(1-i\nu)} \tag{132}$$

and the wave function for small $|\rho|$

$$E_{m,m}(\vec{q}, \vec{\rho}) \sim |\vec{\rho}|^{1+2i\nu} + e^{i\,\delta_{m,m}(\vec{q})}\, |\vec{\rho}|^{1-2i\nu}\,. \tag{133}$$

After the Fourrie transformation to the momentum space we obtain

$$\Psi(\vec{q},\vec{k}) = \int d^2\rho \, e^{i\vec{\rho}\vec{k}} \, E_{m,m}(\vec{q},\vec{\rho}) \sim |k/q|^{-3-2i\nu} + e^{i\,\delta(\nu)}\,|k/q|^{-3+2i\nu} \,, \tag{134}$$

where

$$e^{i\,\delta(\nu)} = 2^{4i\nu}\,\frac{\Gamma^2(1+i\nu)\,\Gamma(-\frac{1}{2}-i\nu)\,\Gamma(\frac{3}{2}-i\nu)}{\Gamma^2(1-i\nu)\,\Gamma(-\frac{1}{2}+i\nu)\,\Gamma(\frac{3}{2}+i\nu)}\,. \tag{135}$$

On the other hand, in the region $\vec{\rho}^2 \sim \vec{k}^{-2}\alpha'(g^2\,N_c)^{-1}$ one can put $\vec{q}=0$ and after the redefinition of the wave function and its argument

$$\Psi(\vec{q},\vec{k}) = |k|^{-3}\,\phi(z)\,,\;\; z=\ln(\alpha'\vec{k}^2) \tag{136}$$

the BFKL homogeneous equation in the string model can be written as the Schrödinger equation

$$E\phi = H\phi\,,\;\; \omega = -\frac{g^2 N_c}{4\pi^2}\,E\,,\;\; H = H_{BFKL}(i\partial/\partial z) + \lambda e^z\,, \tag{137}$$

where

$$H_{BFKL}(\nu) = \psi(i\nu+1/2) + \psi(-i\nu+1/2) - 2\psi(1)\,,\;\; \lambda = \frac{4\pi^2}{g^2 N_c}\,. \tag{138}$$

The analogy with the Schrödinger equation is especially fruitful in the diffusion approximation, where

$$H_{BFKL}(i\partial/\partial z) = -4\ln 2 - 14\,\zeta(3)\,(\partial/\partial z)^2 \tag{139}$$

has the form of the non-relativistic kinetic energy. The potential energy λe^z grows rapidly at large positive z and therefore the wave function ϕ should tend to zero in this region

$$\lim_{z\to\infty}\phi(z) \sim \exp\left(-2\sqrt{\frac{\lambda}{14\,\zeta(3)}}\,e^{z/2}\right). \tag{140}$$

For $z\to-\infty$ the potential energy vanishes, which agrees with a possibility to neglect the string effects at small $\vec{k}^2$. In the momentum representation

$$\phi(p) = \int_{-\infty}^{\infty} e^{ipz}\,\phi(z)\,dz, \tag{141}$$

where $p = i\partial/\partial z$, the BFKL equation is reduced to the equation in finite differences

$$(E - H_{BFKL}(p))\phi(p) = \lambda\phi(p-i)\,. \tag{142}$$

The function $\phi(p)$ can have the singularities (poles) only in the upper semi-plane. It is analytic in the lower semi-plane to provide a rapidly decreasing behaviour of $\phi(z)$ at $z\to+\infty$. The positions of the poles is given below

$$p_r = p_0 + ir\,,\;\; (r=0,1,2,...)\,, \tag{143}$$

where the possible values of p_0 satisfy the equation

$$H_{BFKL}(p_0) = E\,. \tag{144}$$

For example, in the diffusion approximation, where

$$E - H_{BFKL}(p) = E + 4\,\ln 2 - 14\,\zeta(3)\,p^2\,, \tag{145}$$

the solution of the above recurrent relation is

$$\phi(p) = \phi_0(p)\,\left(7\,\zeta(3)\,\frac{g^2\,N_c}{2\pi^2}\right)^{ip}\,\Gamma\left(i\,p - i\sqrt{\frac{E+4\ln 2}{14\,\zeta(3)}}\right)\,\Gamma\left(i\,p + i\sqrt{\frac{E+4\ln 2}{14\,\zeta(3)}}\right) \tag{146}$$

up to a periodic function satisfying the relation $\phi_0(p) = \phi_0(p+i)$. We should substitute this function by a constant

$$\phi_0(p) = const\,, \tag{147}$$

because in an opposite case for $p \to \pm i\infty$ the wave function does not decrease sufficiently rapidly due to the additional factors $\sim \exp(\mp 2\pi i p)$. Indeed, for $\phi_0(p) = 1$ the normalization integral

$$\int_{-\infty}^{\infty} dp\,|\phi(p)|^2 = \int_{-\infty}^{\infty} dp\,\frac{\pi^2/\left(p^2 - \frac{E+4\ln 2}{14\,\zeta(3)}\right)}{\sinh\left(\pi\,p - \pi\,\sqrt{\frac{E+4\ln 2}{14\,\zeta(3)}}\right)\,\sinh\left(\pi\,p + \pi\,\sqrt{\frac{E+4\ln 2}{14\,\zeta(3)}}\right)} \tag{148}$$

is convergent at $p \to \pm\infty$. Moreover, for $\phi_0(p) = 1$ the wave functions $\phi(p)$ with different E are orthogonal. Note, that the integrand (148) contains the poles. After their appropriate regularization it leads to the δ-function $\sim \delta(E - E')$ in the orthonormality conditions.

Let us go to the z representation

$$\phi(z) = \int_{-\infty - i0}^{\infty - i0} \frac{dp}{2\pi}\,e^{-ip\,z}\,\phi(p)\,. \tag{149}$$

For large positive z the contour of the integration over p should be shifted in the lower semi-plane up to the saddle point situated at

$$z = \psi\left(i\,p - i\sqrt{\frac{E+4\ln 2}{14\,\zeta(3)}}\right) + \psi\left(i\,p + i\sqrt{\frac{E+4\ln 2}{14\,\zeta(3)}}\right) + \ln\left(7\,\zeta(3)\,\frac{g^2\,N_c}{2\pi^2}\right)$$

$$\approx \ln\left((ip)^2\,7\,\zeta(3)\,\frac{g^2\,N_c}{2\pi^2}\right)\,. \tag{150}$$

We can estimate $\phi(z)$ by the value of the integrand in (149) at this point

$$\phi(z) \approx e^{-ip\,z}\,\phi(p) \approx \exp\left(-2\,\sqrt{\frac{2\pi^2}{7\,\zeta(3)\,g^2\,N_c}}\,e^{z/2}\right) \tag{151}$$

in an accordance with eq. (140).

At small $E+4\ln 2$, where the diffusion approximation is valid, the solution near the poles at small values of p is

$$\phi(p) \sim \frac{\left(7\zeta(3)\, g^2\, N_c/(2\pi^2)\right)^{ip}}{E+4\ln 2 - 14\,\zeta(3)\, p^2}\,. \tag{152}$$

Thus, $\phi(z)$ at $z=\ln(\alpha' \vec{k}^2) \to -\infty$ behaves as follows

$$\phi(z) \sim \left(\frac{7\zeta(3)\, g^2\, N_c}{2\pi^2 \alpha' \vec{k}^2}\right)^{i\sqrt{\frac{E+4\ln 2}{14\,\zeta(3)}}} - \left(\frac{7\zeta(3)\, g^2\, N_c}{2\pi^2 \alpha' \vec{k}^2}\right)^{-i\sqrt{\frac{E+4\ln 2}{14\,\zeta(3)}}}\,. \tag{153}$$

By comparing this result with expressions (134) and (135) for small ν in the intermediate region $\alpha'/(g^2 N_c) \ll \vec{\rho}^2 \ll 1/\vec{q}^2$ we obtain the quantization of the Regge trajectories

$$2\sqrt{\frac{E_r+4\ln 2}{14\,\zeta(3)}}\ln\left(\frac{7\zeta(3)\, g^2\, N_c}{2\pi^2 \alpha' \vec{q}^2}\right) = 2\pi(r+1/2)\,,\;\; r=0,1,2,... \tag{154}$$

for $n=0$ and small

$$E+4\ln 2 = 4\,\frac{\omega_0-\omega}{g^2 N_c} \ll 1\,. \tag{155}$$

For comparatively large energies E in the diffusion approximation one can use the semiclassical approximation near the turning point $z=z_0$, where

$$\lambda e^{z_0} = E+4\ln 2\,,\;\; \lambda = \frac{4\pi^2}{g^2 N_c}\,, \tag{156}$$

corresponding to the following simplification of the solution (146) at $p \ll \sqrt{E+4\ln 2}$

$$\phi(p) \sim \left((E+4\ln 2)\,\frac{g^2\, N_c}{4\pi^2}\right)^{ip} \exp\left(-i\,\frac{14\zeta(3)}{E+4\ln 2}\,\frac{p^3}{3}\right)\,. \tag{157}$$

The Fourrie transformation to the z-representation can be performed with the use of the saddle-point method

$$\phi(z) \sim \exp\left(i\,(-\Delta z)^{3/2}\frac{2}{3}\sqrt{\frac{E+4\ln 2}{14\zeta(3)}} - i\frac{\pi}{4}\right) + \exp\left(-i\,(-\Delta z)^{3/2}\frac{2}{3}\sqrt{\frac{E+4\ln 2}{14\zeta(3)}} + i\frac{\pi}{4}\right)\,, \tag{158}$$

where $\Delta z = z - z_0$. Therefore in the diffusion approximation of small ν the wave function at $z \to -\infty$ equals

$$\phi(z) \sim e^{-i\pi/4}\left(\frac{7\zeta(3)\, g^2\, N_c}{2\pi^2 \alpha' \vec{k}^2}\right)^{i\sqrt{\frac{E+4\ln 2}{14\,\zeta(3)}}} + e^{i\pi/4}\left(\frac{7\zeta(3)\, g^2\, N_c}{2\pi^2 \alpha' \vec{k}^2}\right)^{-i\sqrt{\frac{E+4\ln 2}{14\,\zeta(3)}}} \tag{159}$$

and the quantization condition for energies is

$$2\sqrt{\frac{E_r+4\ln 2}{14\,\zeta(3)}}\ln\left(\frac{7\zeta(3)\, g^2\, N_c}{2\pi^2 \alpha' \vec{q}^2}\right) = 2\pi(r+1/4) \tag{160}$$

for large integer r.

We investigate below a general case of arbitrary ν for small $\alpha' t$ without using the diffusion approximation. To begin with, one can verify, that here in the semiclassical approach expression (157) for the wave function is also valid near the returning point $z = z_0$, where $p = 0$. The only difference with the diffusion approximation is an additional ν-dependence of the phase $\delta(\nu)$ in (135), which leads to the modified quantization condition

$$2\,|\nu_r|\ln\left(\frac{7\zeta(3)\,g^2\,N_c}{2\pi^2\alpha'\vec{q}^2}\right) = \delta(\nu_r) + 2\pi(r+1/4)\,,\;\; r = 0,1,2,... \tag{161}$$

and the corresponding quantized energies can be obtained from the relation $E = H_{BFKL}(\nu)$ (see (138)).

To derive an exact solution of the BFKL equation for small $\alpha'\vec{q}^2$ let us introduce the new variables

$$x = 2\alpha' l^2\,, \qquad x' = 2\alpha'(l')^2\,, \qquad x_1 = 2\alpha' l_1^2\,. \tag{162}$$

In these variables the inhomogeneous BFKL equation has the form

$$[\omega + x]f(x) = \hat{\Phi}(x) + c\int_0^\infty \left[\frac{f(x')}{|x-x'|} - \left[\frac{1}{|x-x'|} - \frac{1}{\sqrt{x^2+4x'^2}}\right]\frac{x}{x'}f(x)\right]dx'\,, \quad c = \frac{g^2 N_c}{4\pi^2}\,. \tag{163}$$

Here $f(x) \sim \phi(x)/\sqrt{x}$ is $F_\omega^{(0)}(0;l)/x$ averaged over the angle φ between l and l', and $\hat{\Phi}(x)$ is $\tilde{\Phi}(0;l)/x$ averaged over φ. We expect that $\tilde{\Phi}(0;l) \to 0$ at $x \to 0$ and $\hat{\Phi}(x)$ is finite at $x = 0$.

The above BFKL equation differs from that in QCD [11] by the presence of an additional term proportional to x in its left hand side. As in the QCD case, we search the solution in the form of the Mellin transformation

$$f(x) = \int_{-i\infty}^{i\infty}(x)^{\sigma-1/2}C(\sigma)\frac{d\sigma}{2\pi i}\,,\;\; \sigma = i\nu\,. \tag{164}$$

Similarly, the inhomogeneous term is presented as follows

$$\hat{\Phi}(x) = \int_{-i\infty}^{i\infty}(x)^{\sigma-1/2}\hat{\Phi}_1(\sigma)\frac{d\sigma}{2\pi i}\,. \tag{165}$$

To obtain an equation for $C(\sigma)$ one collects the terms proportional to x^σ. For the contribution $xf(x)$ the integration contour should be moved to the line $\Re\sigma = -1$ and therefore the function $C(\sigma)$ can not have any singularities inside the strip $-1 < \Re\,\sigma < 0$. If this condition is fulfilled, we have

$$C(\sigma-1) = \Phi_1(\sigma) - [c\,b(\sigma)+\omega]C(\sigma)\,, \tag{166}$$

where

$$b(\sigma) = \psi(\sigma+1/2) + \psi(-\sigma+1/2) - 2\psi(1) \tag{167}$$

and $\psi(x) = d\ln\Gamma(x)/dx$ is the derivative of the logarithm of the gamma-function.

It is convenient to introduce the new variable ξ according to the definition

$$\sigma = -\frac{\ln\xi}{2\pi i}\,. \tag{168}$$

In the new variables eq. (166) can be written as follows

$$\tilde{C}(\xi e^{-2\pi i}) = \widehat{\Phi}_1(\sigma) - [c\, b(\sigma) + \omega]\tilde{C}(\xi)\,, \tag{169}$$

where $\tilde{C}(\xi) \equiv C(\sigma(\xi))$. The calculation of $C(\xi)$ is reduced to the known mathematical problem of finding a function satisfying the requirement, that its discontinuity is proportional to the same function. Let us define an auxiliary function $\phi(\sigma, \widehat{\sigma}_1)$ being a solution of the homogeneous equation

$$\phi(\sigma - 1, \widehat{\sigma}_1) = [c\, b(\sigma) + \omega]\phi(\sigma, \widehat{\sigma}_1)\,. \tag{170}$$

with $\widehat{\sigma}_1$ being an arbitrary subtraction point, where $\Phi = 1$. Note, that the sign in the right hand side of eq. (170) is opposite in comparison with the sign in front of the corresponding term in eq. (166).

The explicit expression for such function is given below

$$\phi(\sigma, \widehat{\sigma}_1) = \exp\left[\int_{-i\infty}^{i\infty} \frac{\sin\pi(\widehat{\sigma}_1 - \sigma)\ln[c\, b(\sigma') + \omega]}{\sin\pi(\sigma' - \widehat{\sigma}_1)\sin\pi(\sigma' - \sigma)}\frac{d\sigma'}{2i}\right] \tag{171}$$

where it is implied that $\Re\sigma < 0$ and $\Re\widehat{\sigma}_1 < 0$. At $\Re\sigma > 0$ the result is obtained by an analytic continuation of (171) from the region $\Re\sigma < 0$. Furthermore, it is implied, that the solution for $\omega < \omega_0 = (g^2 N_c \ln 2)/\pi^2$ can be derived also by an analytic continuation from the region $\omega > \omega_0$, where the argument of the logarithm has two zeros situated on the imaginary axes and pinching the integration contour at $\omega \to \omega_0$.

The integral over σ' is convergent at large σ' since from (167) one obtains

$$\ln b(\sigma) \to \ln\ln|\Im\,\sigma| \tag{172}$$

at $\Im\,\sigma \to \pm\infty$.

Let us show, that indeed expression (171) is a solution of eq. (170). The pole at $\sigma' = \sigma$ is situated to the left of the integration contour and can pinch only the right singularity of the logarithm situated at the zero of its argument. The pole at $\sigma' = \sigma - 1$ being to the right from the contour pinches with the left singularity of the logarithm. It means, that the function $\phi(\sigma, \widehat{\sigma}_1)$ has no singularities in the strip $-1 \leq \Re\,\sigma \leq 0$. To verify that solution (172) satisfies eq. (170) it is enough to note that after the shift $\sigma \to \sigma - 1$ the pole at $\sigma' = \sigma + 1$ of the integrand moves to the point $\sigma' = \sigma$ which was earlier to the left from the integration contour. The initial and final expressions differ each from another by an additional term in the exponent. This term is obtained by taking the residue in the pole at $\sigma' = \sigma$. As a result, relation (170) is fulfilled.

It is useful to investigate the positions of zeroes and poles of $\phi(\sigma, \widehat{\sigma}_1)$. Both of them are obtained due to pinching the poles $1/\sin\pi(\sigma'-\sigma)$ with the singularities of the logarithm situated at zeros and poles of its argument $[c\, b(\sigma') + \omega]$. The poles are situated at $\sigma' = \pm(n+1)$, where $n = 0, 1, 2, \ldots$. The zeros are situated between these poles. We denote their position by $\sigma_m^{(+)}$ for $\Re\sigma_m > 0$ and $\sigma_m^{(-)}$ for $\Re\sigma_m < 0$. It is obvious, that $|\Re\sigma_m^{(\pm)}| < |\Re\sigma_n^{(-)}|$ for $m < n$. The function $\phi(\sigma, \widehat{\sigma}_1)$ has zeroes at $\sigma = \sigma_m^{(-)} - r$, where r is an integer or zero for $m = 1, 2, \ldots$ and $r \neq 0$ for $m = 0$. Indeed, due to the above discussion $\sigma = \sigma_0^{(-)}$ is not a singularity of the exponent.

Furthermore, $\phi(\sigma,\hat{\sigma}_1)$ has zeros in the right half-plane at $\sigma = n+1/2$, where n is an integer or zero. The poles are situated in the right half-plane at $\sigma = \sigma_m^{(+)} + n$ and in the left half-plane at $\sigma = -(n+3/2)$ for $n = 0,1,2,\ldots$. Similar to the case $\sigma_0^{(-)}$ the point $\sigma = -1/2$ does not corresponds to a singularity of the exponent. In the above discussion we used the relation

$$\frac{\sin\pi(\hat{\sigma}_1-\sigma)}{\sin\pi(\hat{\sigma}_1-\sigma')\,\sin\pi(\sigma-\sigma')} = \cot\pi(\sigma'-\hat{\sigma}_1) - \cot\pi(\sigma'-\sigma)\,. \tag{173}$$

Using expression (171) one can find for large $\sigma = r+iy$

$$\phi(\sigma,\hat{\sigma}_1) \to \exp\Big[-(\ln\ln|y|)[iy+r+1/2]\Big] \tag{174}$$

up to a phase independent from σ. Here in the essential region of integration $\sigma' \sim \sigma$ we replaced the logarithmic function $c\,b(\sigma')+\omega$ by its asymptotic value at $\sigma'=\sigma$.

In a similar way one can check that for $\omega > \omega_0$ the solution $C(\sigma)$ of the inhomogeneous equation is given by

$$C(\sigma) = \int_{-i\infty}^{i\infty} \frac{\phi(\sigma',\hat{\sigma}_1)\hat{\Phi}_1(\sigma')d\sigma'}{2i\phi(\sigma',\hat{\sigma}_1)[c\,b(\sigma')+\omega]\sin\pi(\sigma'-\sigma)}\,. \tag{175}$$

In an agreement with general arguments the continuation of the partial wave in the complex plane from the integer points is performed from the region $\omega > \omega_0$. Similar to (171) in (175) the conditions $\Re\sigma < 0$ and $\hat{\sigma}_1 < 0$ are assumed to be fulfilled and the expression in the region $\Re\sigma > 0$ are obtained by an analytic continuation. It can be written in the equivalent form

$$C(\sigma) = \int_{-i\infty}^{i\infty} \frac{\hat{\Phi}_1(\sigma')\phi(\sigma',\sigma)d\sigma'}{2i[c\,b(\sigma')+\omega]\sin\pi(\sigma'-\sigma)} \tag{176}$$

with the same conventions concerning the signs of $\Re\,\sigma$ and ω.

As in the case of QCD [11], the leading singularity in the ω-plane is situated at $\omega = \omega_0 = (g^2N_c\ln 2)/\pi^2$. It is obtained from the region $\sigma' \sim \sigma \to 0$ in eq. (176). In this limit the corresponding denominator is approximated by the diffusion expression $\omega - \omega_0 - a(\sigma')^2$. Calculating the integral at $\sigma \to 0$, one obtains

$$C(\sigma) \sim 1/(\sigma - \sqrt{\omega-\omega_0})\,, \tag{177}$$

where the omitted factor has no singularity at small σ. Thus, at $x \to 0$ the solution is $\sim x^{\sqrt{\omega-\omega_0}-1/2}$. In QCD there are singularities in both points $\sigma = \pm\sqrt{\omega-\omega_0}$, but in the string model only the singularity at $\sigma = \sqrt{\omega-\omega_0}$ survives. Another singularity is absent because at large momenta the kernel of the equation is non-singular due to the linear term in the trajectory on the left hand side of eq. (163).

In the important case of the leading singularity, where the diffusion approximation

$$c\,b(\sigma)+\omega \approx a(\sigma_0^2-\sigma^2)\,,\;\; a = -7\zeta(3)\,\frac{g^2}{2\pi^2}\,,\;\; \sigma_0^2 = -4\ln 2 - \omega/[g^2\,7\zeta(3)/2\pi^2] \tag{178}$$

is valid, the function $\phi(\sigma)$ is given by

$$\phi(\sigma) = a^{-\sigma}\Gamma(\sigma_0 - \sigma)[\Gamma(\sigma_0+\sigma+1)]^{-1}\,. \tag{179}$$

It is related to the solution $\tilde{\phi}(\sigma)$ of the homogenious equation as follows

$$\tilde{\phi}(\sigma) = -\pi\phi(\sigma)/\sin\pi(\sigma+\sigma_0)\,. \tag{180}$$

At $\omega < 0$, as it was discussed above, $\tilde{\phi}$ describes the wave function of the particle with an energy equal to $-c\omega$, which is rejected from the potential barrier e^z. In this case $z = \ln x$, and $-i\sigma$ is the momentum of the colliding particle.

According to (176), the function $C(\sigma)$ in (164) determining the solution of eq. (163) is given below

$$C(\sigma) = \frac{a^{-\sigma}\Gamma(\sigma_0-\sigma)}{\Gamma(\sigma_0+\sigma+1)}\int_{-i\infty}^{+i\infty}\frac{a^{\sigma'-1}\hat{\Phi}(\sigma')\Gamma(\sigma'+\sigma_0)\,d\sigma'}{2i\,\Gamma(\sigma_0-\sigma'+1)\sin\pi(\sigma'-\sigma)}\,. \tag{181}$$

In (181) it is implied that $\Re\sigma < 0$, and so that the pole at $\sigma' = \sigma$ is twisted with the right side. At $\Re\sigma > 0$ the result is obtained by an analytic continuation in σ. Furthermore, in (164) the pole at $\sigma = \sigma_0$ is on the right hand side from the integration contour. It is solely the solution at $\sigma_0 > 0$ because in this case (180) determines a function of x increasing at $x \to \infty$. At $\sigma_0 < 0$ the solution is not unique because (180) might be added to (181). In an agreement with general arguments one should chose the solution which is an analytical continuation of the solution (181) to the region $\sigma_0 < 0$. The result is presented by eq. (181) where the integration contour is defined in an accordance with these arguments.

9 Heisenberg spin model and integrability

To investigate the region of $\alpha'\vec{q}^2 \sim g^2 N_c$ it is convenient to use the conformal invariance of the BFKL kernel in QCD (see [22]). In the coordinate representation for the wave function describing the composite state of two reggeized gluons with the impact parameters $\vec{\rho}_1$ and $\vec{\rho}_2$ we have the expression [22] (see (17))

$$E_{m,\widetilde{m}}(\vec{\rho}_1,\vec{\rho}_2;\vec{\rho}_0) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}}\right)^m \left(\frac{\rho^*_{12}}{\rho^*_{10}\rho^*_{20}}\right)^{\widetilde{m}}\,,\ \rho_{12} = \rho_1 - \rho_2\,, \tag{182}$$

where $\vec{\rho}_0$ is the coordinate of the Pomeron, $\rho_r = x_r + iy_r$ and $\rho^*_r = x_r - iy_r$ are respectively the holomorphic and anti-holomorphic variables, $\rho_{rs} = \rho_r - \rho_s$ and

$$m = \frac{1}{2} + i\nu + \frac{n}{2}\,,\ \widetilde{m} = \frac{1}{2} + i\nu - \frac{n}{2} \tag{183}$$

are conformal weights related to the eigenvalues of the Casimir operators of the Möbius group

$$\vec{M}^2 E_{m,\widetilde{m}} = m(m-1)E_{m,\widetilde{m}}\,,\ \ \vec{M}^{*\,2} E_{m,\widetilde{m}} = \widetilde{m}(\widetilde{m}-1)E_{m,\widetilde{m}}\,,\ \vec{M}^2 = -\rho^2_{12}\frac{\partial}{\partial\rho_1}\frac{\partial}{\partial\rho_2}\,. \tag{184}$$

Note, that in (183) the conformal spin n is integer $n = 0, \pm 1, \pm 2, ...$ and the parameter ν is a real number for the principal series of the unitary representations.

The operator $\vec{M}^2$ is related to the generators of the Möbius group $\vec{M}$

$$\vec{M} = \vec{M}_1 + \vec{M}_2 \,, \;\; M_r^z = \rho_r \frac{\partial}{\partial \rho_r} \,, \;\; M_r^- = \frac{\partial}{\partial \rho_r} \,, \;\; M_r^+ = -\rho_r^2 \frac{\partial}{\partial \rho_r} \,. \tag{185}$$

The generators satisfy the following commutation relations

$$[M^z, M^\pm] = \pm M^\pm \,, \;\; [M^+, M^-] = 2M^z \,, \;\; [M^{*z}, M^{*\pm}] = \pm M^{*\pm} \,, \;\; [M^{*+}, M^{*-}] = 2M^{*z} \,. \tag{186}$$

For the solution of the BFKL equation in the string theory it is convenient to introduce also the generators

$$\vec{N} = \vec{M}_1 - \vec{M}_2 \,. \tag{187}$$

Together with the operators $\vec{M}$ they produce the Lie algebra for the Lorentz group

$$[M^z, N^\pm] = \pm N^\pm \,, \;\; [M^+, N^-] = 2N^z \,, \;\; [M^\pm, N^z] = \mp N^\pm \,, \;\; [M^-, N^+] = -2N^z \,, \tag{188}$$

$$[N^z, N^\pm] = \pm M^\pm \,, \;\; [N^+, N^-] = 2M^z \,. \tag{189}$$

We can find the representation of this algebra in the space of the functions E_m

$$E_m(\rho_1, \rho_2; \rho_0) = \left(\frac{\rho_{12}}{\rho_{10}\rho_{20}} \right)^m \tag{190}$$

as follows

$$M^z E_m = (-\rho_0 \partial_0 - m) E_m \,, \;\; M^+ E_m = (\rho_0^2 \partial_0 + 2m\rho_0) E_m \,, \;\; M^- E_m = -\partial_0 E_m \,,$$

$$N^- E_m = \frac{m(m-1)}{2m-1} \left(E_{m+1} + \frac{\partial_0^2}{(m-1)^2} E_{m-1} \right) , \;\; (N^z - \rho_0 N^-) E_m = \frac{m}{m-1} \partial_0 E_{m-1} \,,$$

$$(N^+ + 2\rho_0 N^z - \rho_0^2 N^-) E_m = -2m E_{m-1} \tag{191}$$

and analogously for the representation of $\overrightarrow{M^*}$ and $\overrightarrow{N^*}$ on functions $E^*_{\widetilde{m}}$.

The BFKL integral operator K_{BFKL} is diagonal in the $(m, \widetilde{m})$-representation and its eigenvalue has the property of the holomorphic separability

$$\omega_{BFKL} = -\frac{g^2}{8\pi^2} N_c \, \epsilon_{m,\widetilde{m}} \,, \;\; \epsilon_{m,\widetilde{m}} = \epsilon_m + \epsilon_{\widetilde{m}} \,, \tag{192}$$

where the holomorphic energies are the following functions of the conformal weights m and $\widetilde{m}$

$$\epsilon_m = \psi(m) + \psi(1-m) - 2\psi(1) \,, \;\; \epsilon_{\widetilde{m}} = \psi(\widetilde{m}) + \psi(1-\widetilde{m}) - 2\psi(1) \,. \tag{193}$$

In the case of the string theory in the eigenvalue equation for the Pomeron wave function f in the dimension $D = 4$ we have the additional contribution ΔK_{BFKL} (neglecting the pole term from the soft Pomeron)

$$\omega \, f = K \, f \,, \;\; K = K_{BFKL} + \Delta K \,, \;\; \Delta K = -\alpha' \vec{p}_1^{\,2} - \alpha' \vec{p}_2^{\,2} \,. \tag{194}$$

It is convenient to use the mixed representation ($\vec{q} = \vec{p}_1 + \vec{p}_2$, $\vec{\rho} = \vec{\rho}_{12}$), where the additional string contribution to K has the form

$$\Delta K = -\alpha' \left(\frac{\vec{q}^2}{2} - 2\, \frac{\partial^2}{(\partial \rho_\mu)^2} \right) \,,\;\; \frac{\partial^2}{(\partial \rho_\mu)^2} = N^- N^{*-} \,,\;\; N^- = \partial_1 - \partial_2 \,,\; \rho = \rho_1 - \rho_2 \,. \tag{195}$$

In this representation the Pomeron wave function in QCD can be obtained by the Fourrie transformation

$$E_{m,\widetilde{m}}(\vec{q},\vec{\rho}) = \int d^2R\, e^{i\,\vec{q}\vec{R}} \left(\frac{\rho}{(R+\frac{\rho}{2})(R-\frac{\rho}{2})} \right)^m \left(\frac{\rho^*}{(R^*+\frac{\rho^*}{2})(R^*-\frac{\rho^*}{2})} \right)^{\widetilde{m}} \,,\;\; R = \frac{\rho_1+\rho_2}{2} \,. \tag{196}$$

Let us present the solution of the BFKL homogenious equation in the string theory as a superposition of the above functions

$$f(\vec{q},\vec{\rho}) = \int_{-\infty}^{\infty} d\nu \sum_{n=-\infty}^{\infty} C_{m,\widetilde{m}}(\vec{q})\, \Gamma(m)\, \Gamma(\widetilde{m})\, E_{m,\widetilde{m}}(\vec{q},\vec{\rho}) \,. \tag{197}$$

Here we extracted the factor $\Gamma(m)\,\Gamma(\widetilde{m})$ from coefficients $C_{m,\widetilde{m}}(\vec{q})$ to simplify the relations between them. The operators N^- and N^{*-} act on the functions $E_{m,\widetilde{m}}(\vec{q},\vec{\rho})$ as follows

$$N^- E_{m,\widetilde{m}} = \frac{m(m-1)}{2m-1} \left(E_{m+1,\widetilde{m}} - \frac{q^{*\,2}}{4(m-1)^2} E_{m-1,\widetilde{m}} \right) \,, \tag{198}$$

$$N^{*-} E_{m,\widetilde{m}} = \frac{\widetilde{m}(\widetilde{m}-1)}{2\widetilde{m}-1} \left(E_{m,\widetilde{m}+1} - \frac{q^2}{4(\widetilde{m}-1)^2} E_{m,\widetilde{m}-1} \right) \,. \tag{199}$$

Therefore the function $f(\vec{q},\vec{\rho})$ is a solution of the homogeneous BFKL equation in the string theory if the coefficients $C_{m,\widetilde{m}}(\vec{q})$ in (197) satisfy the following recurrent relation

$$\left(\omega + \frac{g^2}{8\pi^2}\, N_c\, (\epsilon_m + \epsilon_{\widetilde{m}}) + \alpha' \frac{\vec{q}^2}{2} \right)\, C_{m,\widetilde{m}}(\vec{q}) =$$

$$2\alpha' \left(\frac{m-2}{2m-3}\, \frac{\widetilde{m}-2}{2\widetilde{m}-3}\, C_{m-1,\widetilde{m}-1}(\vec{q}) - \frac{m+1}{2m+1}\, \frac{\widetilde{m}-2}{2\widetilde{m}-3}\, \frac{q^{*\,2}}{4} C_{m+1,\widetilde{m}-1}(\vec{q}) \right.$$

$$\left. - \frac{m-2}{2m-3}\, \frac{\widetilde{m}+1}{2\widetilde{m}+1}\, \frac{q^2}{4} C_{m-1,\widetilde{m}+1}(\vec{q}) + \frac{m+1}{2m+1}\, \frac{\widetilde{m}+1}{2\widetilde{m}+1}\, \frac{q^{*\,2}}{4}\, \frac{q^2}{4} C_{m+1,\widetilde{m}+1}(\vec{q}) \right) \,. \tag{200}$$

By introducing the new function

$$\phi_{m,\widetilde{m}}(\vec{q}) = (2m-1)^{-1}(2\widetilde{m}-1)^{-1}\, (q/2)^{\widetilde{m}} (q^*/2)^m\, C_{m,\widetilde{m}}(\vec{q}) \tag{201}$$

one can write this recurrent relation in a simpler form

$$\left(\omega + \frac{g^2}{8\pi^2}\, N_c\, (\epsilon_m + \epsilon_{\widetilde{m}}) + \alpha' \frac{\vec{q}^2}{2} \right)\, (2m-1)(2\widetilde{m}-1) \phi_{m,\widetilde{m}}(\vec{q}) =$$

$$\alpha' \frac{\vec{q}^2}{2} \Big((m-2)\,(\widetilde{m}-2)\,\phi_{m-1,\widetilde{m}-1}(\vec{q}) - (m+1)\,(\widetilde{m}-2)\,\phi_{m+1,\widetilde{m}-1}(\vec{q})$$

$$- \,(m-2)\,(\widetilde{m}+1)\,\phi_{m-1,\widetilde{m}+1}(\vec{q}) + (m+1)\,(\widetilde{m}+1)\,\phi_{m+1,\widetilde{m}+1}(\vec{q}) \Big)\,. \tag{202}$$

One should add to this recurrent relation the information about the asymptotic behavior of the coefficients $C_{m,\widetilde{m}}(\vec{q})$ at large m and $\widetilde{m}$ corresponding to $|k| \gg |q|$ investigated above. Note, that contrary to the case of small $\alpha'\vec{q}^2$, considered in the previous section, now the eigenfunctions contain a mixture of states with different conformal spins. Expanding $\phi_{m,\widetilde{m}}$ in the basis of the functions $x^m x^{*\,\widetilde{m}}$ one can reduce the recurrent relation (202) in the diffusion approximation to a differential equation, which can be solved, for example, by the semi-classical methods similar to those used in the previous section.

In the case of the colourless state constructed from several reggeized gluons [23] the homogeneous equation for its wave function in the string theory is given in the multi-colour limit $N_c \to \infty$ below (cf. [14])

$$E\,\phi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = H\,\phi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n)\,,\ \omega = -\frac{g^2 N_c}{8\pi^2}\,E\,, \tag{203}$$

where

$$H = H^{(n)}_{BFKL} + l^2 \sum_{r=1}^{n} (\vec{p}_r)^2\,,\ l^2 = \frac{\alpha'\,8\pi^2}{g^2 N_c}\,,\ p^\mu_r = i\,\frac{\partial}{\partial \rho^\mu_r}\,. \tag{204}$$

Here $H^{(n)}_{BFKL}$ has the property of the holomorphic separability

$$H^{(n)}_{BFKL} = h^{(n)}_{BFKL} + h^{(n)*}_{BFKL}\,,\ h^{(n)}_{BFKL} = \sum_{r=1}^{n} h^{(r,r+1)}_{BFKL}\,, \tag{205}$$

$$h^{(r,r+1)}_{BFKL} = \psi(\hat{m}_{r,r+1}) + \psi(1-\hat{m}_{r,r+1}) - 2\psi(1)\,,\ \hat{m}_{r,r+1}(\hat{m}_{r,r+1}-1) = -\rho^2_{r,r+1}\partial_r\partial_{r+1} \tag{206}$$

and h^n_{BFKL} is the local hamiltonian for the integrable XXX model [24] with the spins coinciding with the generators of the Möbius group (185). Really we have two independent spin chains for holomorphic and anti-holomorphic subspaces. The term $\sim l^2$ in eq. (204) describes an additional interaction between these two spin chains because according to (185)

$$(\vec{p}_r)^2 = -4\,M^-_r\,M^{-\,*}_r\,. \tag{207}$$

This term violates the Möbius symmetry for H and leaves only its invariance under translations and rotations. Therefore the eigenvalues of H can depend on $\vec{q}^2$, which leads to the Regge trajectories for composite states of reggeized gluons. We do not know, if the corresponding Heisenberg spin model is integrable or not. But in the region $\alpha'\vec{q}^2 \ll g^2 N_c$ it is possible to apply the integrability of the QCD hamiltonian for calculating the Regge trajectories. Indeed, as in the previous section, one can divide the essential momenta in two regions $\vec{k}^2_r \sim \vec{q}^2_r$ and $\vec{k}^2_r \sim 1/\alpha'$. In the first region we can use the integrability of the BFKL hamiltonian to obtain the wave function of the composite state. For the leading singularity the integrals of motion are quantised and depend only on the conformal weights $m, \widetilde{m}$ [14, 24, 25]. Therefore the corresponding energy E_{BFKL} for this leading singularity is a function of these variables

$$E_{BFKL} = E(m, \widetilde{m})\,. \tag{208}$$

It means, that for the solution of the equation in the second region $\vec{k}^2 \sim 1/\alpha'$ we can use the same methods which were used in the previous section for the calculation of the Pomeron trajectory. We hope to return to the problem of finding the Regge trajectories for the Odderon and other gluon composite states in our future publications.

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A Conformal factor for SL(2)-SUSY transformations

If the fixed variables are $(z_1^{(0)}|\vartheta_1^{(0)})$, $(z_2^{(0)}|\vartheta_2^{(0)})$ and $z_3^{(0)}$ while the superpartner ϑ of $z_3^{(0)}$ is not fixed, then the discussed factor $H(z_1^{(0)}, z_2^{(0)}, z_3^{(0)}, \vartheta_1^{(0)}, \vartheta_2^{(0)}, \vartheta)$ turns out to be [16]

$$H(z_1^{(0)}, z_2^{(0)}, z_3^{(0)}, \vartheta_1^{(0)}, \vartheta_2^{(0)}, \vartheta) = (z_1^{(0)} - z_3^{(0)})(z_2^{(0)} - z_3^{(0)}) \left[1 - \frac{\vartheta_1^{(0)}\vartheta}{2(z_1^{(0)} - z_3^{(0)})} - \frac{\vartheta_2^{(0)}\vartheta}{2(z_2^{(0)} - z_3^{(0)})} \right]. \tag{1}$$

When $\vartheta_1^{(0)} = \vartheta_2^{(0)} = 0$ this factor is reduced to the expression given in Sec.3 of the paper. Refixing the above variables to the new values $(\hat{z}_1^{(0)}, \hat{z}_2^{(0)}, \hat{z}_3^{(0)}, \hat{\vartheta}_1^{(0)}, \hat{\vartheta}_2^{(0)})$ can be achieved by the following transformations.

Firstly, both $\vartheta_1^{(0)}$ and $\vartheta_2^{(0)}$ are pushed to vanishing values. The supersymmetric $SL(2)$ transformation (35), which preserves the variables z_1, z_2 and z_3 but adjustes to ϑ_1 and ϑ_2 the zero values, is given by

$$f(\hat{z}) = \hat{z} - \frac{(\hat{z} - z_1)(\hat{z} - z_2)}{(z_3 - z_1)(z_3 - z_2)} \hat{\vartheta}_3 \varepsilon_0(z_3)\,, \quad \varepsilon(\hat{z}) = \frac{\vartheta_1(\hat{z} - z_2)}{(z_1 - z_2)\sqrt{f'(z_1)}} - \frac{\vartheta_2(\hat{z} - z_1)}{(z_1 - z_2)\sqrt{f'(z_2)}}\,, \tag{2}$$

where $\varepsilon_0(z) = [\vartheta_1(z - z_2) - \vartheta_2(z - z_1)]/(z_1 - z_2)$. Evidently, we have $f'(z_1)f'(z_2) = 1$. Secondly, by the usual $L(2)$ transformation one changes $(z_1^{(0)}, z_2^{(0)}, z_3^{(0)})$ to new values $(\hat{z}_1^{(0)}, \hat{z}_2^{(0)}, \hat{z}_3^{(0)})$. Finally, using the change of variables inversed to transformation (2) with preserving the values $(\hat{z}_1^{(0)}, \hat{z}_2^{(0)}, \hat{z}_3^{(0)})$ one can give the new values $(\hat{z}_1^{(0)}, \hat{z}_2^{(0)})$ to the vanishing superpartners of the bosonic coordinates $(\hat{\vartheta}_1^{(0)}, \hat{\vartheta}_2^{(0)})$. To verify that with the factor (1) the amplitude is independent of the values $(z_1^{(0)}, z_2^{(0)}, z_3^{(0)}, \vartheta_1^{(0)}, \vartheta_2^{(0)})$ of the fixed world-sheet variables, one should take into account that under the Γ -transformation (35) the integrand being $SL(2)$ covariant, receives the factor $Q_\Gamma(\hat{z}, \hat{\vartheta})$ for each world sheet variable $(z|\vartheta)$, see eqs. (38) and (39). The above factor is cancelled by the factor $1/Q_\Gamma(\hat{z}, \hat{\vartheta})$ from the corresponding transformation jacobian for all variables $(z|\vartheta)$ except the fixed ones together with the superpartner ϑ of $z_3^{(0)}$, because the last jacobian is different from $1/Q_\Gamma(\hat{z}_3^{(0s)}, \hat{\vartheta})$. One can verify, that these additional extra-factors are just compensated by the corresponding change of factor (1). One can also check that the amplitude is not changed when another set of variables is fixed.

B One-loop Regge trajectory for the critical superstring

The integral for the one-loop amplitude, corresponding to the sum of the planar and non-oriented diagrams for the gluon-gluon sccattering

$$A_{pl,no} = 8K \int_{-1}^{1} \frac{d\lambda}{\lambda} \int_0^1 \left(\prod_{I=1}^{3} \theta(\nu_{I+1} - \nu_I) d\nu_I \right) R\,, \tag{1}$$

is convergent at $\lambda = 0$ [18]. In the above expression the integrand is

$$R = \left(\frac{B(\nu_1 - \nu_2, \lambda) B(\nu_3 - 1, \lambda)}{B(\nu_1 - \nu_3, \lambda) B(\nu_2 - 1, \lambda)} \right)^{-\alpha' s} \left(\frac{B(\nu_1 - 1, \lambda) B(\nu_2 - \nu_3, \lambda)}{B(\nu_1 - \nu_3, \lambda) B(\nu_2 - 1, \lambda)} \right)^{-\alpha' t} \tag{2}$$

and the function B is given below

$$B(\nu, \lambda) = \sin \pi\nu \prod_{n=1}^{\infty} \frac{1 - 2\lambda^n \cos 2\pi\nu + \lambda^{2n}}{(1 - \lambda^n)^2}. \tag{3}$$

The factor K includes the colour matrices T and the products of polarization vectors. In the Regge limit $-s \gg -t$ it equals (cf.(48))

$$K = \pi^3 g^4 N\, T\, (\alpha' s)^2 (\xi_a \xi_{a'})(\xi_b \xi_{b'}) \tag{4}$$

where $N = 32$ is the dimension of the $SO(32)$ group. In the same limit the region $\nu_{32} = \nu_3 - \nu_2 \sim 1/(\alpha' s) \ll 1$ is essential and we have the following simplifications

$$\frac{B(\nu_{12}, \lambda) B(\nu_3 - 1, \lambda)}{B(\nu_{13}, \lambda) B(\nu_2 - 1, \lambda)} \approx 1 - \frac{\sin \pi\nu_1 \sin \pi\nu_{32}}{\sin \pi\nu_2 \sin \pi\nu_{31}} - 4\pi\nu_{32}\, l_1\,, \tag{5}$$

$$\frac{B(\nu_1 - 1, \lambda) B(\nu_{23} - 1, \lambda)}{B(\nu_{13}, \lambda) B(\nu_2 - 1, \lambda)} \approx \frac{\sin \pi\nu_1 \sin \pi\nu_{32}}{\sin \pi\nu_2 \sin \pi\nu_{31}}\, l_2\,, \tag{6}$$

where

$$l_1 = \sum_{n=1}^{\infty} \left(\frac{\lambda^n \sin 2\pi\nu_{21}}{1 - 2\lambda^n \cos 2\pi\nu_{21} + \lambda^{2n}} - \frac{\lambda^n \sin 2\pi\nu_2}{1 - 2\lambda^n \cos 2\pi\nu_2 + \lambda^{2n}} \right)\,, \tag{7}$$

$$l_2 = \prod_{n=1}^{\infty} \frac{(1-2\lambda^n \cos 2\pi\nu_1 + \lambda^{2n})(1-\lambda^n)^2}{(1-2\lambda^n \cos 2\pi\nu_{31} + \lambda^{2n})(1-2\lambda^n \cos 2\pi\nu_2 + \lambda^{2n})} \,. \tag{8}$$

Instead of ν_1 it is convenient to introduce the new integration variable

$$y = \frac{\sin \pi\nu_1 \, \sin \pi\nu_{32}}{\sin \pi\nu_2 \, \sin \pi\nu_{31}} \,, \; x = 1 - y = \frac{\sin \pi\nu_3 \, \sin \pi\nu_{21}}{\sin \pi\nu_2 \, \sin \pi\nu_{31}} \tag{9}$$

with the inverse transformation

$$\tan \pi\nu_1 = \frac{(1-x)\, \sin \pi\nu_2 \sin \pi\nu_3}{\cos \pi\nu_2 \sin \pi\nu_3 - x \cos \pi\nu_3 \sin \pi\nu_2} \,. \tag{10}$$

Then the integral can be written as follows

$$A = 8K \int_{-1}^{1} \frac{d\lambda}{\lambda} \int_0^1 \frac{dx}{\pi} \int_{\nu_1}^1 d\nu_2 \int_{\nu_2}^1 d\nu_3 \, \sin \pi\nu_2 \, \sin \pi\nu_3 \, \sin \pi\nu_{23}$$
$$\times \frac{((1-x)l_2)^{-\alpha' t} \, (x - 4\pi\nu_{32} l_1)^{-\alpha' s}}{(\sin \pi\nu_3 - x \sin \pi\nu_2)^2 + 4x \, \sin \pi\nu_2 \, \sin \pi\nu_3 \, \sin^2 \frac{\pi\nu_{32}}{2}} \,. \tag{11}$$

In the Regge limit the essential region of integration over ν_{32} is

$$1 - x = y \sim (\alpha' s)^{-1} \ll \nu_{32} \ll 1 \,, \; \nu_1 \sim \frac{y}{\nu_{32}} \ll 1 \tag{12}$$

where the integral is simplified as follows

$$A = 8K \int_{-1}^{1} \frac{d\lambda}{\lambda} \frac{\Gamma(1-\alpha' t)}{\pi^2} \int_0^1 d\nu_2 \frac{(\sin \pi\nu_2)^2}{\pi} \, \ln(-1/\alpha' s)(-\alpha' s)^{-1+\alpha' t}$$
$$\times \left(\frac{L_2}{1+L_1} \right)^{-\alpha' t} \, (1+L_1)^{-1}. \tag{13}$$

Here both

$$L_1 = 4(\sin \pi\nu_2)^2 \frac{\partial l_1}{\partial(\pi\nu_1)}_{|\nu_1=\nu_{32}=0}$$

and L_2 are given explicitly by (51). As the result we obtain for the Regge trajectory eq.(50) in the text.

C Multi-Regge production amplitudes

In integral (72) we redefine $z_7 \to z_7 z_4 z_6/(z_4 - z_6)$ and introduce $f = z_4 - z_6$ instead of z_4. In addition, we replace $z_6 \to f\, z_6$. Then the integral $I(t_{5678}, \kappa^2, t_{3478}, t_{34}, t_{56}, t_{12}, t_{78})$ in (75) is given by expression

$$I(t_{5678}, \kappa^2, t_{3478}, t_{34}, t_{56}, t_{12}, t_{78}) = (\kappa^2)^{-\alpha' t_{34} - \alpha' t_{56} - 2} \int dz_7 \, dz_6 \, df \, e^{-f - z_7} f^{-\alpha' t_{12} - 2}$$
$$\times z_7^{-\alpha' t_{78}} z_6^{-\alpha' t_{5678}} (1+z_6)^{-\alpha' t_{3478}} \hat{q}_1^{\alpha' t_{34}} \hat{q}_2^{\alpha' t_{56}}$$
$$\times \Bigg[\frac{(\alpha' t_{78} + 1)\hat{q}_1 \hat{q}_2}{z_6^2 (1+z_6)^2 z_7^2} + \frac{(\alpha' t_{5678} - \alpha' t_{56} - \alpha' t_{78} - \kappa^2)\hat{q}_1}{(1+z_6) z_6^2 z_7} + \frac{\alpha' t_{56} \hat{q}_1}{z_6^2 \hat{q}_2}$$
$$+ \frac{(\alpha' t_{3478} - \alpha' t_{34} - \alpha' t_{78} + \kappa^2)\hat{q}_2}{(1+z_6)^2 z_6 z_7} + \frac{\alpha' t_{34} \hat{q}_2}{(1+z_6)^2 \hat{q}_1} + \frac{\alpha' t_{35} + \alpha' t_{36} + \alpha' t_{45} + \alpha' t_{46}}{z_6 (1+z_6)} \Bigg], \tag{1}$$

where

$$\hat{q}_1 = f(1+z_6) + z_7 z_6 - \kappa^2 - i\epsilon\,, \quad \hat{q}_2 = f z_6 + z_7(1+z_6) - \kappa^2 - i\epsilon\,, \quad \epsilon \to 0\,. \tag{2}$$

Integrating it by parts, one obtain the following result

$$\begin{aligned} I(t_{5678},\kappa^2,t_{3478},t_{34},t_{56},t_{12},t_{78}) &= (\kappa^2)^{-\alpha' t_{34}-\alpha' t_{56}-2}\int dz_7 dz_6 df e^{-f-z_7} f^{-\alpha' t_{12}-1} \\ \times z_7^{-\alpha' t_{78}-1}(1+z_6)^{-\alpha' t_{3478}} z_6^{-\alpha' t_{5678}} \hat{q}_1^{\alpha' t_{34}} \hat{q}_2^{\alpha' t_{56}} &\left[\frac{(\alpha' t_{5678}+1)}{z_6^2} + \frac{\alpha' t_{3478}+1}{(1+z_6)^2}\right. \\ &\left. -\frac{\kappa^2}{z_6(1+z_6)}\left(\frac{\alpha' t_{34}}{\hat{q}_1}+\frac{\alpha' t_{56}}{\hat{q}_2}\right) - \frac{f+z_7}{z_6(1+z_6)}\right]. \end{aligned} \tag{3}$$

To derive eq. (3), one integrates the first term in the brackets in eq. (1) over z_7 by parts. As a result, we obtain the expression similar to eq. (3) but the terms inside the brackets turn out to be

$$\begin{aligned} &-\frac{\hat{q}_1\hat{q}_2}{z_6^2(1+z_6)^2 z_7} + \frac{(\alpha' t_{5678}+1-\alpha' t_{78}-\kappa^2)\hat{q}_1}{(1+z_6)z_6^2 z_7} + \frac{\alpha' t_{56}\hat{q}_1}{z_6^2\hat{q}_2} \\ &+\frac{(\alpha' t_{3478}+1-\alpha' t_{78}+\kappa^2)\hat{q}_2}{(1+z_6)^2 z_6 z_7} + \frac{\alpha' t_{34} q_2}{(1+z_6)^2\hat{q}_1} + \frac{\alpha' t_{35}+\alpha' t_{36}+\alpha' t_{45}+\alpha' t_{46}}{z_6(1+z_6)}\,. \end{aligned} \tag{4}$$

This expression is the same as

$$\begin{aligned} &\frac{f}{z_7}\left(\frac{\alpha' t_{5678}+1-\kappa^2-\alpha' t_{78}}{z_6^2} + \frac{\alpha' t_{3478}+1+\kappa^2-\alpha' t_{78}}{(1+z_6)^2}\right) \\ &+\frac{\alpha' t_{12}-\alpha' t_{78}+2}{z_6(1+z_6)} - \frac{\kappa^2}{z_7}\left(\frac{\alpha' t_{5678}+1}{z_6^2(1+z_6)} + \frac{\alpha' t_{3478}+1}{z_6(1+z_6)^2}\right) - \frac{\hat{q}_1\hat{q}_2}{z_6^2(1+z_6)^2 z_7} \\ &+\frac{\alpha' t_{78}\kappa^2}{z_7}\left(\frac{1}{z_6^2(1+z_6)} + \frac{1}{z_6(1+z_6)^2}\right) - \frac{(-\kappa^2)^2}{z_6^2(1+z_6)^2 z_7} + \frac{\alpha' t_{56}\hat{q}_1}{z_6^2\hat{q}_2} + \frac{\alpha' t_{34}\hat{q}_2}{(1+z_6)^2\hat{q}_1}\,. \end{aligned} \tag{5}$$

Further, the terms proportional to t_{78} are integrated by parts over z_7 to remove this factor t_{78}. Analogously the term $\sim t_{12}$ is integrated by parts over f to remove the factor t_{12}. The third term is integrated by parts over z_6 to remove both nominators $(\alpha' t_{5678}+1)$ and $(\alpha' t_{3478}+1)$ in the corresponding contributions. After these transformations we obtain (3). If we shall integrate by parts the first term in eq. (5) over z_6 is possible to reduce eq. (3) to the expression

$$\begin{aligned} I(t_{5678},\kappa^2,t_{3478},t_{34},t_{56},t_{12},t_{78}) &= (\kappa^2)^{-\alpha' t_{34}-\alpha' t_{56}-2}\int dz_7 dz_6 df e^{-f-z_7} f^{-\alpha' t_{12}-1} \\ \times z_7^{-\alpha' t_{78}-1}(1+z_6)^{-\alpha' t_{3478}} z_6^{-\alpha' t_{5678}-1} \hat{q}_1^{\alpha' t_{34}} \hat{q}_2^{\alpha' t_{56}} &\left[-\frac{\alpha' t_{3478}}{(1+z_6)^2} + \frac{\alpha' t_{34}+\alpha' t_{56}}{(1+z_6)} + \frac{z_7\alpha' t_{34}}{(1+z_6)\hat{q}_1}\right. \\ &\left. +\frac{f\alpha' t_{56}}{(1+z_6)\hat{q}_2} + \frac{z_6}{(1+z_6)^2} - \frac{f+z_7}{(1+z_6)}\right]. \end{aligned} \tag{6}$$

One can introduce the variable z instead of z_6 according to the relation

$$z_6 = z/(1-z)\,, \tag{7}$$

and redenote $z_7=y$. After it (6) can be presented as follows

$$I(t_{5678},\kappa^2,t_{3478},t_{34},t_{56},t_{12},t_{78})=(\kappa^2)^{-\alpha' t_{34}-\alpha' t_{56}-2}\int dydzdf e^{-f-y}f^{-\alpha' t_{12}-1}y^{-\alpha' t_{78}-1}\times$$
$$\times z^{-\alpha' t_{5678}-1}(1-z)^{-\alpha' t_{3456}+\alpha' t_{12}+\alpha' t_{78}}(f+yz-\kappa^2(1-z)-i\epsilon)^{\alpha' t_{34}}$$
$$\times(y+fz-\kappa^2(1-z)-i\epsilon)^{\alpha' t_{56}}\Big[-\alpha' t_{3478}(1-z)+\alpha' t_{34}+\alpha' t_{56}$$
$$+\frac{\alpha' t_{34}y(1-z)}{f+yz-\kappa^2(1-z)-i\epsilon}+\frac{\alpha' t_{56}f(1-z)}{y+fz-\kappa^2(1-z)-i\epsilon}+z-(f+y)\Big]. \qquad (8)$$

D Vanishing of impact factors for planar diagrams

The impact factor for the vector particle scattering can be calculated from the asymptotics of Fig1b in the region where $s_1=\pm s_7\to\infty$ while s_3, s_4, s_5 and s_6 are finite. The impact factors for the states with the masses $\alpha' t_{34}=n_1$ and $\alpha' t_{56}=n_2$ are just proportional to the resudies in the poles at $\alpha' t_{34}=n_1$ and $\alpha' t_{56}=n_2$. One can see from expression (58) that for the discussed asymptotics the essential values of the integration variables are

$$z_3\to 0, \quad z_7/z_6\to 0\,. \qquad (1)$$

while x and y being defined by the relations $z_3=z_4+x$ and $z_5=z_6+y$ are now comparable in their values with z_4 and z_6. However, the poles $\alpha' t_{34}=n_1$ and $\alpha' t_{56}=n_2$ appear from the regions $x/z_4\to 0$ and $y/z_6\to 0$. In this kinematics one can expand the integrand in powers of x and y to obtain the poles at $\alpha' t_{34}=n_1$ and $\alpha' t_{56}=n_2$. It can be verified that the corresponding integral vanishes, and, so, the impact factor for the planar diagram is equal to zero.

For the sake of simplicity we give the corresponding proof for the boson string theory, assuming, that the external interaction states are tachyons. In this case only the leading term in x and y is needed and expression (58) can be simplified as it was done in eq. (60). Furthermore, similar to the multi-Regge limit we obtain

$$k_2(k_3+k_4)\to -k_2(k_5+k_6)\to k_1k\,,\quad k_7(k_3+k_4)\to -k_7(k_5+k_6)\to k_8k, \qquad (2)$$

but relations (61) for $k_3(k_7+k_8)$ and for $k_5(k_7+k_8)$ are not valid. It is helpfull to redefine again the variables according to eq. (63). After calculating integrals over x and y the asymptotics of $A^{(0)}$ turns out to be

$$A^{(0)}=T_p(\pm\alpha' s_1)^{\alpha' t_{12}+\alpha' t_{78}-\alpha' t_{34}-\alpha' t_{56}}(\pm\alpha' s)^{\alpha' t_{34}+\alpha' t_{56}+2}I^{(0)}$$
$$\Gamma(-\alpha' t_{34}-1)\Gamma(-\alpha' t_{56}-1) \qquad (3)$$

where $\Gamma(x)$ is the gamma function and

$$I^{(0)}=\int\frac{dz_7\,dz}{z_7^2 z}df\exp\Big[-f-z_7\Big]z_7^{-\alpha' t_{78}}(1-z)^{-\alpha'[t_{3456}-t_{12}-t_{78}]}z^{-\alpha' t_{5678}-1}f^{-\alpha' t_{12}-2}$$
$$\times\Big[f+z_7z-\frac{\pm s_1}{s}\alpha' k_3(k_7+k_8)(1-z)\Big]^{\alpha' t_{34}+1}\Big[z_7+fz-\frac{\pm s_1}{s}\alpha' k_5(k_7+k_8)(1-z)\Big]^{\alpha' t_{56}+1}. \qquad (4)$$

In the calculation of $I^{(0)}$ we performed the change of the integration variables as it was done in Appendix B: $z_7 \to z_7 z_4 z_6/(z_4 - z_6)$, $z_4 \to f = z_4 - z_6$ and $z_6 \to f\, z_6 = f\, z/(1-z)$ (see eq.(7)). The impact factor is proportional to the sum over the residues of $I^{(0)}$ in the poles $\alpha' t_{5678} = n$ for fixed values $\alpha' t_{34} = \alpha' t_{56} = -1$. The parameter $n = m - 1$ takes integer values from $n = -1$ up $n = \infty$. The result contains the factor

$$\sum_{m=0}^{\infty} \frac{d^m}{m!dz^m}(1-z)^{-\alpha'[t_{3456}-t_{12}-t_{78}]} \,. \tag{5}$$

The sum is calculated in the region $t_{3456} - t_{12} - t_{78} > 0$ where the series is covergent. Then it is continued analytically to physical values for $t_{3456} - t_{12} - t_{78}$. For $z \to 1$ this sum is equal to $(1-z)^{-\alpha'[t_{3456}-t_{12}-t_{78}]} = 0$. Thus, the impact factor for the planar diagram is zero. The vanishing of the impact factor for the higher mass states $\alpha' t_{34} = n_1$ and $\alpha' t_{56} = n_2$ is verified in a similar way. For the superstring theory one can prove also the vanishing of the impact factors for the planar diagrams.

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Captions

Fig. 1. The ladder cut determining the BFKL Pomeron in the $a + b \to a' + b'$ process. The dotted line denotes a reggion, the solid one denotes a particle.

Fig. 2. The diagram for the calculation of the BFKL kernel. The dotted line denotes massless state; the solide one denotes the tower of string states.

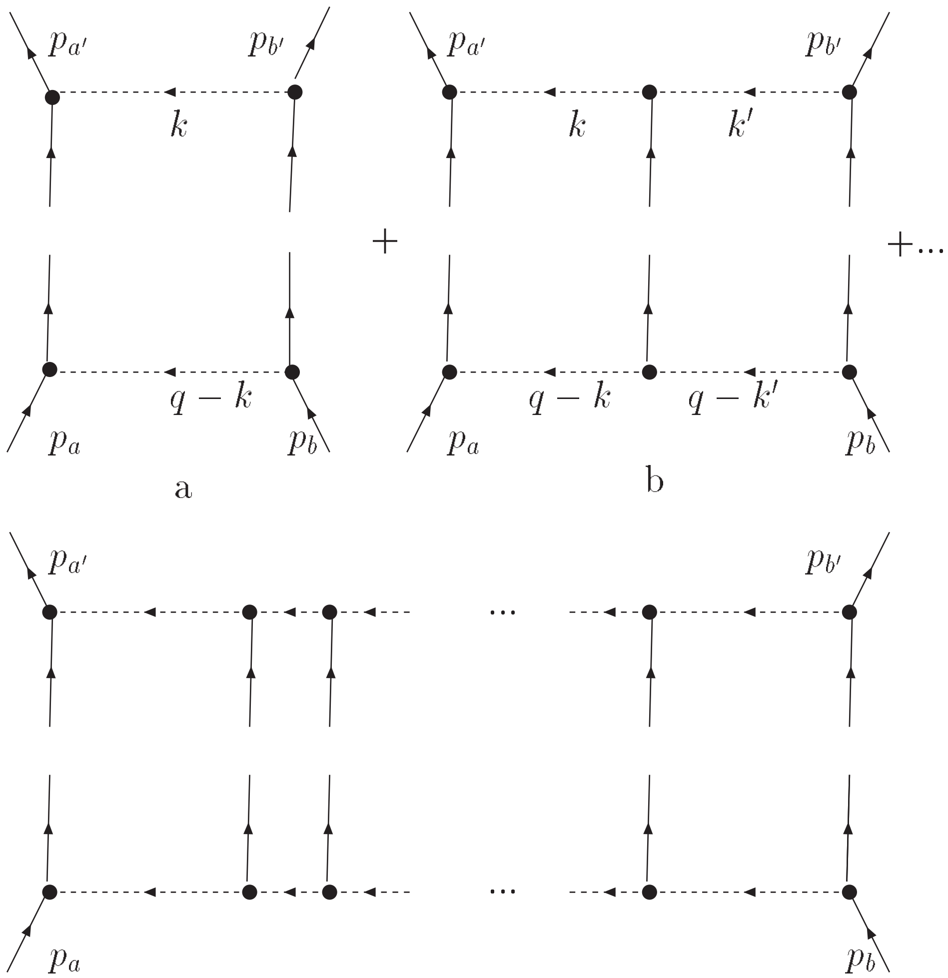

Fig.1

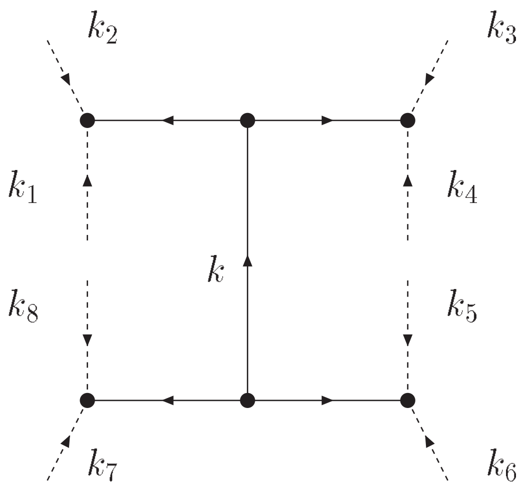

Fig.2