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Instanton Geometry and Quantum $\mathcal{A}_\infty$ structure on the Elliptic Curve

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Abstract

We first determine and then study the complete set of non-vanishing A-model correlation functions associated with the "long-diagonal branes" on the elliptic curve. We verify that they satisfy the relevant $\mathcal{A}_\infty$ consistency relations at both classical and quantum levels. In particular we find that the $\mathcal{A}_\infty$ relation for the annulus provides a reconstruction of annulus instantons out of disk instantons. We note in passing that the naive application of the Cardy-constraint does not hold for our correlators, confirming expectations. Moreover, we analyze various analytical properties of the correlators, including instanton flops and the mixing of correlators with different numbers of legs under monodromy. The classical and quantum $\mathcal{A}_\infty$ relations turn out to be compatible with such homotopy transformations. They lead to a non-invariance of the effective action under modular transformations, unless compensated by suitable contact terms which amount to redefinitions of the tachyon fields.

Contents

1. Introduction

By now, topological open string amplitudes, which determine important terms such as the superpotential in the low-energy effective action, have been well understood for single or multiple parallel D-branes. However, more general configurations, such as ones described by quiver diagrams based on intersecting branes, have not yet been investigated in comparable detail, despite their potential importance in phenomenological applications (see e.g. [1] for an overview).

For such configurations, the underlying TFT has a much richer structure, which is due to the boundary changing operators that describe open strings localized on intersecting D-branes. In fact, the sophisticated mathematical machinery of homological mirror symmetry [2, 3], which acts between certain categories of A- and B-type D-branes, gears up to full power only for this kind of geometries. However, so far there have been few applications in physics that make use of this structure, which involves new types of open string instantons, or Gromov-Witten invariants.

Indeed just a few explicit computations of boundary changing correlation functions have been presented so far (we mean here moduli-dependent, exact TFT correlators). One reason is that only two- or three-point functions can be easily computed; for example, for B-type branes via wavefunction overlaps [4] or boundary LG models based on matrix factorizations [5]. Correlators for A-type branes can then be obtained from this via mirror symmetry, or simply by directly summing up instantons [6].

On the other hand, just like in the bulk theory, correlators with more than three fields are difficult to evaluate directly, because of the presence of integrated insertions which lead to singularities that need to be regularized and may lead to contact terms. In the bulk sector, however, the situation is favorable in that the moduli space is flat and is governed by an integrable special geometry. This implies that all higher-point correlators can be obtained as derivatives from a generating function, the prepotential $\mathcal{F}(t)$ [7]. The contact terms are implicitly determined by requiring the vanishing of the Gauß-Manin connection [8]. One can rephrase this also in terms of the WDVV equations which are imposed as differential equations on $\mathcal{F}(t)$.

The situation is far more involved for the boundary sector, where in general the moduli space is obstructed and there is no notion of flatness (and correspondingly no a priori preferred coordinates). The rôle of the bulk WDVV equations is replaced by a set of generalized, open-closed WDVV equations [9], which follow from various factorization and sewing constraints of world-sheets with boundaries. The simplest ones take the form of $\mathcal{A}_\infty$ relations [10] between disk correlators; in the presence of bulk deformations, there are certain other conditions, following from bulk-boundary crossing symmetry and the factorization of the annulus amplitude. Very recently, a generalization of the $\mathcal{A}_\infty$ relations to general Riemann surfaces with h boundaries and g holes have been formulated in ref. [11] and dubbed "quantum $\mathcal{A}_\infty$ relations".

The purpose of this note is to gain insight in the interplay of the various quantum $\mathcal{A}_\infty$ relations, instanton sums and regularization ambiguities, by considering A-model correlators for the simplest brane geometry with moduli, namely for certain $D1$-branes on the elliptic curve. The point is, of course, that the elliptic curve being flat is very simple, and indeed one can solve the $\mathcal{A}_\infty$ relations in a completely geometric manner and determine the correlation functions (as well as their ambiguities) in terms of instanton sums. That is, correlators involving N boundary changing operators can be obtained by summing over the (moduli dependent) areas of N-gons, schematically:

$$C_{a_1,\ldots,a_N}(\tau) \;=\; \sum_{N-\text{gons}} e^{-Area(\tau)}$$

which correspond to world-sheet instantons whose boundaries lie on the intersecting branes under consideration.[1] This technique has been pioneered in [12–16], where it was used to prove the mirror symmetry between the Fukaya category of Lagrangian submanifolds, and the derived category of coherent sheaves on the elliptic curve.

More concretely, we will first determine the complete set of non-vanishing correlators pertaining to the "long-diagonal" branes, which have already been discussed from various perspectives in refs. [5, 17, 18]. While the three-point functions have been explicitly computed before in refs. [4–6, 12] and generic four-point correlators discussed in [13–15, 19], we will evaluate the remaining non-vanishing, higher-point disk and annulus correlators by instanton counting and verify consistency with the classical and quantum $\mathcal{A}_\infty$ constraints (for transversal as well as certain non-transversal brane configurations).

Moreover, we will discuss the analytical properties of correlators in non-technical terms, most notably singularities, "instanton flops" and "homotopy" regularization ambiguities, all of which we give a simple geometrical interpretation. Such homotopies are induced as monodromies from moving branes around the curve, and lead to a non-trivial fibration of the $\mathcal{A}_\infty$ structure over the open/closed string moduli space. The effective superpotential is thus modular only up to homotopies, which may be compensated by simultaneous field redefinitions of the tachyons, or equivalently, by adding suitable contact terms. We will also verify that homotopy transformations are compatible with both the classical and the quantum $\mathcal{A}_\infty$ constraints.

One of the most interesting results of this note concerns the annulus quantum $\mathcal{A}_\infty$ relation, for which we show that it maps certain disk instantons to annulus instantons, essentially by patching up the latter in terms of the former. This is a specific feature of open string instantons as it requires the fusion of boundaries.

Finally, in an appendix we address the question of whether the Cardy-type factorization relation (which is different to the annulus quantum $\mathcal{A}_\infty$ relation) holds or not. Although the familiar factorization of the annulus diagram into closed or open string channels is one of the fundamental axioms of open string TFT [20–22], it strictly speaking needs to apply only to correlators without integrated insertions [9]. We show, by providing a counter-example, that the Cardy-constraint does indeed not hold for general cylinder correlators on the elliptic curve.

We hope that our findings will be useful for the understanding of more complicated D-brane geometries, notably ones on Calabi-Yau threefolds.

[1] We will denote the closed string modulus, *i.e.*, the complexified Kähler parameter of the curve, by τ. The open string moduli, which correspond to brane positions and Wilson lines and which are suppressed in the above formula, will be generically denoted by u.

2. Disk instantons and tree-level correlators

2.1. Recapitulation: 3-point functions for long-diagonal branes

We will focus on the A-model and consider certain $D1$-branes wrapped around the homology cycles of the elliptic curve, Σ. More specifically, in order to make contact with previous work [5], we will consider a specific triplet of branes $\mathcal{L}_i$ with RR charges, or wrapping numbers (n,m) given by

$$\mathcal{L}_1 \sim (2,1)\,,\ \mathcal{L}_2 \sim (-1,1)\,,\ \mathcal{L}_3 \sim (-1,-2)\ . \tag{1}$$

These branes are usually referred to as "long-diagonals", which is self-explaining upon drawing the branes on the covering space of Σ (see Fig.1 below). Each brane $\mathcal{L}_i$ intersects the other ones three times within a fundamental domain. This means that every boundary changing, open string vertex operator that maps between a given pair of branes, carries an index a that labels the specific intersection at which it is located. The analysis of the cohomology (in the mirror LG-orbifold model) [5] reveals that for an open string mapping from a brane $\mathcal{L}_i$ to a brane $\mathcal{L}_j$ at the intersection a, there is a fermionic operator with R-charge $q=1/3$, which we denote by $\Psi_a^{(i,j)}$. Moreover, there is a "Serre dual" bosonic operator $\Phi_{\bar a}^{(j,i)}$ of charge $q=2/3$ for an open string going the other way. Finally, apart from the identity operator, there are fermionic, boundary preserving operators $\Omega^{(i,i)}$ of charge $q=1$, which are tied to single branes $\mathcal{L}_i$ and which are the marginal operators coupling to the brane moduli, u_i.

The simplest correlators of these operators give the topological open string metric:

$$\begin{aligned}
\rho_{1\!\!1\Omega} &:= \left\langle 1\!\!1^{(i,i)} 1\!\!1^{(i,i)} \Omega^{(i,i)} \right\rangle_{disk} = 1 \\
\rho_{a\bar a} &:= \left\langle 1\!\!1^{(i_1,i_1)} \Psi_a^{(i_1,i_2)} \Phi_{\bar a}^{(i_2,i_1)} \right\rangle_{disk} = \delta_{a\bar a}\ .
\end{aligned} \tag{2}$$

On the other hand, the simplest non-trivial correlation functions are the following three-point functions:

$$\Delta_{abc}^{(i_3 i_1 i_2)}(\tau,u_i) \ = \ \left\langle \Psi_a^{(i_3,i_1)} \Psi_b^{(i_1,i_2)} \Psi_c^{(i_2,i_3)} \right\rangle_{disk}\ , \tag{3}$$

which have been evaluated in the B-model using wavefunction overlaps in [4] or using the LG model based on matrix factorization [5]. The result, when expressed in terms of the flat coordinates τ, u_i (which coincide with the natural variables of the mirror A-model), looks:

$$\Delta_{abc}^{(i_3 i_1 i_2)}(\tau,u_i) \ = \ \delta^{(3)}_{a+b+c,0}\, \Theta \begin{bmatrix} [b-c]_3 - 3/2 \\ -3/2 \end{bmatrix} \left(3\tau \,|\, 3(u_1+u_2+u_3) \right)\ , \tag{4}$$

where $\delta^{(3)}$ is the Kronecker function defined modulo three, and $[a]_3$ denotes the mod 3 reduction of $a \in \mathbb{Z}$ to the range $\{1,2,3\}$. Furthermore,

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} \left(3\tau \,|\, 3u \right) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{6}(a+3n)^2} e^{2\pi i (u+b/9)(a+3n)}\ , \tag{5}$$

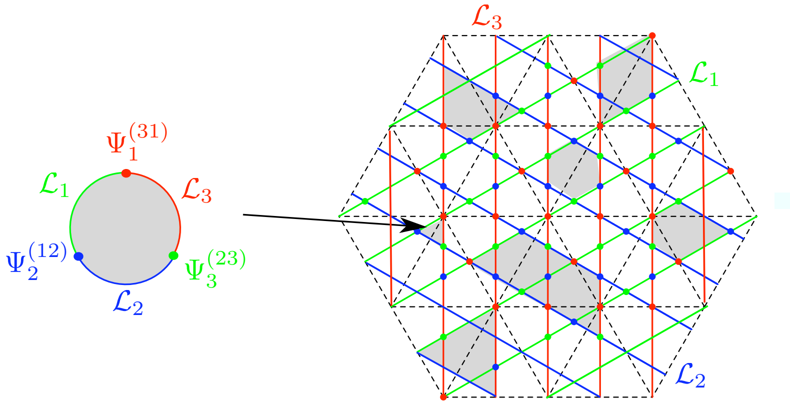

Figure 1: On the right we have drawn the long-diagonal branes $\mathcal{L}_{1,2,3}$ on the covering space of the elliptic curve, as well as some examples of polygon instantons bounded by these branes. On the left we have drawn a world-sheet disk with three boundary changing operator insertions, which gets mapped into certain triangle shaped instantons on the right. In our conventions, red, blue and green dots correspond to operators $\Psi_a^{(i,j)}$ for $a = 1, 2, 3$, respectively. Moreover, the shown brane locations correspond to our choice of origin in the open string moduli space.

is a standard theta-function whose expansion in $q \equiv e^{2\pi i\tau}$ sums up the contributions of all the triangular world-sheet instantons that are bounded by the three branes $\mathcal{L}_i$ (with moduli u_i).[2] This is completely in line with the findings of ref. [12]. We visualize the situation in Fig.1, which also serves to define our conventions for labeling branes and boundary fields.

2.2. Polygon instantons and higher-point amplitudes

The three-point functions (3) are only the first terms of the effective superpotential, and one of our purposes is to determine the complete superpotential that can be associated with the three types of branes $\mathcal{L}_i$ of (1). Due to R-charge selection rules, there is only a finite number of terms, and in our situation the maximal number of external "tachyon" legs is $N = 6$.[3]

Schematically, the effective superpotential can be written in the following form:

$$\begin{aligned}\mathcal{W}_{eff}(\tau, u_i, t_a, \xi_{\bar a}) \;&=\; \frac{1}{3}\Delta_{abc}(\tau,u)t_at_bt_c + \frac{1}{2}\mathcal{P}_{a\bar b c\bar d}(\tau,u)t_a\xi_{\bar b}t_c\xi_{\bar c} + \mathcal{T}_{ab\bar c\bar d}(\tau,u)t_at_b\xi_{\bar c}\xi_{\bar d} \\ &+\; \wp_{a\bar b\bar c\bar d\bar e}(\tau,u)t_a\xi_{\bar b}\xi_{\bar c}\xi_{\bar d}\xi_{\bar e} + \frac{1}{6}\mathcal{H}_{\bar a\bar b\bar c\bar d\bar e\bar f}(\tau,u)\xi_{\bar a}\xi_{\bar b}\xi_{\bar c}\xi_{\bar d}\xi_{\bar e}\xi_{\bar f}\;.\end{aligned} \quad (6)$$

[2] To properly describe the u-dependence of the areas, one would need to multiply these correlators with simple non-holomorphic "quantum" prefactors [4], which may be viewed as arising from a holomorphic anomaly (indeed these can be traced back to the holomorphic anomaly of the annulus amplitude). However, since TFT computations naturally yield holomorphic expressions, we suppress such factors here. They could be easily reinstated by requiring modular covariance, if one wished to do so.

[3] Allowing for general branes would involve an infinite number of terms, because the charge of the boundary changing operators is correlated with the angle of the intersecting branes, and this can become arbitrary small for generic wrapping numbers (m, n).

Note that for a given brane configuration, not all terms may contribute. (*E.g.*, if all D-branes wrap the same homology class then all terms vanish, reflecting that there is no obstruction to move the branes around; if the D-branes wrap only two homology classes then only the second term in the superpotential is non-trivial. We will see in a moment that the latter is associated with parallelogram instantons.) Above, t_a are the bosonic deformation parameters that couple to the fermionic open string tachyons Ψ_a, while the $\xi_{\bar{a}}$ are fermionic parameters coupling to the bosonic operators, $\Phi_{\bar{a}}$.[4] Note that we suppressed the labels i of the branes.

More specifically, the various inequivalent, cyclically symmetric disk correlators with $N \geq 4$ legs are defined as follows:

$$\begin{aligned}
\mathcal{T}^{(i_4 i_1 i_2 i_3)}_{ab\bar{c}\bar{d}} &= \left\langle \Psi^{(i_4,i_1)}_a \Psi^{(i_1,i_2)}_b \Phi^{(i_2,i_3)}_{\bar{c}} \Phi^{(i_3,i_4)}_{\bar{d}} \right\rangle_{disk} \\
\mathcal{P}^{(i_4 i_1 i_2 i_3)}_{a\bar{b}c\bar{d}} &= \left\langle \Psi^{(i_4,i_1)}_a \Phi^{(i_1,i_2)}_{\bar{b}} \Psi^{(i_2,i_3)}_c \Phi^{(i_3,i_4)}_{\bar{d}} \right\rangle_{disk} \\
\wp^{(i_5 i_1 i_2 i_3 i_4)}_{a\bar{b}\bar{c}\bar{d}\bar{e}} &= \left\langle \Psi^{(i_5,i_1)}_a \Phi^{(i_1,i_2)}_{\bar{b}} \Phi^{(i_2,i_3)}_{\bar{c}} \Phi^{(i_3,i_4)}_{\bar{d}} \Phi^{(i_4,i_5)}_{\bar{e}} \right\rangle_{disk} \\
\mathcal{H}^{(i_6 i_1 i_2 i_3 i_4 i_5)}_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}} &= \left\langle \Phi^{(i_6,i_1)}_{\bar{a}} \Phi^{(i_1,i_2)}_{\bar{b}} \Phi^{(i_2,i_3)}_{\bar{c}} \Phi^{(i_3,i_4)}_{\bar{d}} \Phi^{(i_4,i_5)}_{\bar{e}} \Phi^{(i_5,i_6)}_{\bar{f}} \right\rangle_{disk} ,
\end{aligned} \tag{7}$$

where it is implicitly understood that $(N-3)$ operators are integrated as topological descendants.

Correlators with N boundary changing insertions will generically get contributions of world-sheet instantons that end on N intersecting branes $\mathcal{L}_i$, which thus can be depicted as N-gons on the covering space of the curve; there are two different geometries for $N=4$, namely given by trapezoids ($\mathcal{T}$) and parallelograms ($\mathcal{P}$). Since two branes can always be fixed using translational invariance, N-point correlators involving N branes will depend on only $N-2$ independent combinations of the brane moduli u_i. Note, moreover, that the angles φ at the corners of an N-gon are related to the R-charge q of the boundary changing field, *i.e.*, $\varphi = q\pi$.

In addition to the fields in the boundary changing sectors, we allow for an arbitrary number of marginal operator insertions, $\Omega^{(i,i)}$. Since these are associated with the flat coordinates u_i [5] which are integrable, we can obtain such correlators simply by taking partial derivatives $\partial_{u_i} \equiv \frac{\partial}{\partial u_i}$, e.g.,

$$\frac{1}{(6\pi i)^n} \partial^{n_1}_{u_{i_1}} \partial^{n_2}_{u_{i_2}} \partial^{n_3}_{u_{i_3}} \Delta^{(i_3 i_1 i_2)}_{abc} = \left\langle \Psi^{(i_3,i_1)}_a \left(\Omega^{(i_1,i_1)}\right)^{n_1} \Psi^{(i_1,i_2)}_b \left(\Omega^{(i_2,i_2)}\right)^{n_2} \Psi^{(i_2,i_3)}_c \left(\Omega^{(i_3,i_3)}\right)^{n_3} \right\rangle_{disk} ,$$

where $n = n_1 + n_2 + n_3$. This readily generalizes to all the other amplitudes in (7).

The evaluation of the correlators (7) proceeds by identifying the smallest N-gon on the covering space that can be associated with the given boundary conditions and determining the area of it as well as of all other N-gons obtained by lattice translations from it. Summing all areas up we count instantons and anti-instantons with opposite orientations. The result will have the form of a generalized, indefinite theta-function [16]; this will be a section of some vector bundle over Σ, much like the ordinary theta-function (5) is a section of a line bundle, $L^{\otimes 3}$.

For example, let us consider a trapezoid associated with the correlator $\mathcal{T}_{ab\bar{c}\bar{d}}$, and label the boundary fields in a manner as shown in Fig.1. The shorter of the two parallel sides then

[4] One can view (t,ξ) as coordinates of a non-commutative superspace, see e.g., [23].

has length $l_{short} = [\bar{d}-\bar{c}+3/2]_3 + 3m$, where $m \in \mathbb{Z}$ accounts for lattice shifts. Similarly, the two sides of equal length have $l_{diag} = [b-\bar{c}]_3 + 3n$, and the longer side has $l_{long} = l_{short} + l_{diag}$. The area of the trapezoid is thus $A = 1/2([b-\bar{c}]_3 + 3n)(2[\bar{d}-\bar{c}+3/2]_3 + [b-\bar{c}]_3 + 3n + 6m)$. All-in-all, when allowing for continuous translations parametrized by u_i,[5] we obtain:

$$\mathcal{T}^{(i_4 i_1 i_2 i_3)}_{ab\bar{c}\bar{d}}(\tau, u_i) \;=\; \delta^{(3)}_{a+b,\bar{c}+\bar{d}}\, \Theta_{trap} \left[\begin{array}{c} [b-\bar{c}]_3 \\ [\bar{d}-\bar{c}+3/2]_3 \end{array} \right] (3\tau | 3(u_1+u_2+u_4), 3(u_1-u_3)) \;, \qquad (8)$$

where the trapezoidal theta-function is defined by the following indefinite series:

$$\Theta_{trap} \left[\begin{array}{c} a \\ b \end{array} \right] (3\tau|3u,3v) \;=\; \sum_{m,n\in\mathbb{Z}}^{\text{indef.}} q^{\frac{1}{6}(a+3n)(a+3n+2(b+3m))} e^{2\pi i \left((a+3n)(u-1/6) + (b+3m)v \right)} \;, \qquad (9)$$

with

$$\sum_{m,n\in\mathbb{Z}}^{\text{indef.}} \equiv \sum_{m,n=0}^{\infty} - \sum_{m,n=-1}^{-\infty} \;. \qquad (10)$$

In a similar way, one finds for the parallelogram correlators:

$$\mathcal{P}^{(i_4 i_1 i_2 i_3)}_{a\bar{b}c\bar{d}}(\tau, u_i) \;=\; \delta^{(3)}_{a+c,\bar{b}+\bar{d}}\; \Theta_{para} \left[\begin{array}{c} [c-\bar{b}]_3 \\ [\bar{d}-c]_3 \end{array} \right] (3\tau|3(u_1-u_3), 3(u_4-u_2)) \;, \qquad (11)$$

$$\Theta_{para} \left[\begin{array}{c} a \\ b \end{array} \right] (3\tau|3u,3v) \;\equiv\; \sum_{m,n\in\mathbb{Z}}^{\text{indef.}} q^{\frac{1}{3}(a+3n)(b+3m)} e^{2\pi i \left((b+3m)u + (a+3n)v \right)} \;.$$

The five-point function looks more difficult to determine, but we use a trick in order to make life simpler: there is one side of the pentagon that is not parallel to any other one - when we attach a triangle to it, the pentagon turns into a parallelogram. So we can describe the area of the pentagon as the difference of a parallelogram and a triangle (taking of course all lattice translations into account). Taking everything together, we obtain:

$$\wp^{(i_5 i_1 i_2 i_3 i_4)}_{a\bar{b}\bar{c}\bar{d}\bar{e}}(\tau, u_i) = \delta^{(3)}_{a,\bar{b}+\bar{c}+\bar{d}+\bar{e}}\; \Theta_{penta} \left[\begin{array}{c} [-b-c-d]_3 \\ [e+c+d]_3 \\ [c-d+\frac{3}{2}]_3 \end{array} \right] (3\tau|3(u_5-u_2), 3(u_1-u_4), 3(u_3+u_2+u_4)) \;, \qquad (12)$$

where

$$\Theta_{penta} \left[\begin{array}{c} a \\ b \\ c \end{array} \right] (3\tau|3u,3v,3w) \equiv$$

$$\equiv \sum_{m,n,k\in\mathbb{Z}}^{\text{indef.}} q^{\frac{1}{3}(a_> + 3(n+k))(b_> + 3(m+k)) - \frac{1}{6}(c+3k)^2} e^{2\pi i \left((a_> + 3(n+k))u + (b_> + 3(m+k))v + (c+3k)(w-1/6) \right)} \;,$$

[5]Note that this expression and analogous ones discussed below are defined only for appropriate u_i; we will address this issue in the next section. Also note, just as for the three-point function, that in order to describe the correct area dependence on the u_i, we would need to add a simple non-holomorphic prefactor that we suppress here.

where $a_> = a+3$ for $a < c$ and $a_> = a$ for $a > c$, and similarly for $b_>$. The shifts in $a_>$ and $b_>$ ensure that the sides of the parallelogram are longer than the side of the subtracted triangle. The indefinite sum is defined as

$$\sum_{m,n,k\in\mathbb{Z}}^{\text{indef.}} = \sum_{m,n,k\geq 0}^{\infty} + \sum_{m,n,k\leq -1}^{-\infty}$$

Finally, we find the six-point functions by subtracting two triangles from the acute-angled corners of a parallelogram:

$$\begin{aligned} \mathcal{H}^{(i_6 i_1 i_2 i_3 i_4 i_5)}_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}}(\tau, u_i) &= \delta^{(3)}_{0,\bar{a}+\bar{b}+\bar{c}+\bar{d}+\bar{e}+\bar{f}} \times \qquad (13) \\ \times\, \Theta_{hexa} \begin{bmatrix} [-b-c-d]_3 \\ [c+d+e]_3 \\ [c-d+\frac{3}{2}]_3 \\ [a-f+\frac{3}{2}]_3 \end{bmatrix} & (3\tau | 3(u_5-u_2), 3(u_1-u_4), 3(u_3+u_2+u_4), 3(-u_6-u_1-u_5)) \ , \end{aligned}$$

where

$$\Theta_{hexa} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} (3\tau | 3u, 3v, 3w, 3z) \equiv$$

$$\equiv \sum_{m,n,k,l\in\mathbb{Z}}^{\text{indef.}} q^{\frac{1}{3}(a+3n)(b+3m)-\frac{1}{6}(c+3k)^2-\frac{1}{6}(d+3l)^2} e^{2\pi i\left((a+3n)u+(b+3m)v+(c+3k)(w-1/6)+(d+3l)(z+1/6)\right)}.$$

The indefinite sum is given by

$$\sum_{m,n,k,l\in\mathbb{Z}}^{\text{indef.}} = \sum_{m,n\geq 0}^{\infty} \sum_{k\geq 0}^{<k_{max}} \sum_{l\geq 0}^{<l_{max}} - \sum_{m,n\leq -1}^{-\infty} \sum_{k\leq -1}^{>k_{min}} \sum_{l\leq -1}^{>l_{min}}$$

with $k_{max} = \min(a/3+n, b/3+m) - c/3$ and $l_{max} = \min(a/3+n, b/3+m) - d/3$ as well as $k_{min} = \max(a/3+n, b/3+m) - c/3$ and $l_{min} = \max(a/3+n, b/3+m) - d/3$. The restrictions in the sums ensure that the subtracted triangles are not larger than the parallelogram.

We have so far defined all topological A-model disk amplitudes in terms of instanton sums for the D-brane configuration as shown in Fig.1. In the following we proceed investigating their analytic properties.

2.3. Analytical properties of the disk amplitudes

The correlation functions we wrote down in the previous section are not completely well-defined. There are the following three inter-related issues that we need to discuss: i) singularities from colliding branes; ii) analytic continuation over the open string moduli space; iii) modular anomalies; and iv) contact term ambiguities intrinsic to the definition of the correlators. Most of these aspects have been, in one form or the other, already discussed in

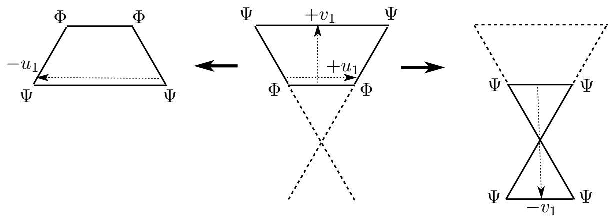

Figure 2: Degenerations of the trapezoidal instanton. For positive u_1, v_1 (middle figure), the area grows monotonically, while for negative u_1 or v_1 we formally run into the depicted, partly ill-defined geometries with negative areas. Resumming the instanton series amounts to a flop-like transition back to positive area, albeit for a different geometry.

the mathematical literature (see [13–15, 19]), although they have not yet been exhibited in the string physics literature. We found it instructive to work out, as a case study, some of these aspects for our specific brane geometry, and in order to aid the non-expert, we will present them in simple non-technical terms. We will focus on the trapezoid correlator, as it captures the relevant features, with the understanding that the other, higher-point functions can be dealt with analogously.

One basic point is that the trapezoid sum (9), as well as the other higher-point functions, is defined as a sum over instantons with positive areas; the summation (10) is equivalent to requiring $(a+3n)(a+2b+3n+6m) > 0$, *i.e.*, positive area. This assumes for the moduli that $0 \leq u_1, v_1 < 1$ where $u \equiv: u_1\tau + u_2$, $v \equiv: v_1\tau + v_2$. However, when doing large reparametrizations, such as shifts $u \to u - k\tau$, $v \to v - \ell\tau$, we may formally produce negative areas and so leave the domain of support; this is similar to the phenomenon of leaving the Kähler cone of a Calabi-Yau manifold. We thus expect that some suitable analytic continuation, describing the analog of a flop transition to a different geometry[6] with positive instanton areas, will be necessary.

To proceed, let us rewrite the trapezoid sum (9) in a form similar to an Appell-function [19] as follows:

$$\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau|3u, 3v) \;=\; e^{2\pi i v b} \sum_{n\in\mathbb{Z}} \frac{q^{\frac{1}{6}(a+3n)(a+2b+3n)} e^{2\pi i (a+3n)(u-1/6)}}{1 - q^{a+3n} e^{6\pi i v}} \,. \tag{14}$$

This way of representing it makes the singularity manifest which occurs when the parallel sides move on top of each other (and the Wilson line is tuned appropriately), *i.e.*, for $3v+\tau a \in \mathbb{Z} + 3\tau\mathbb{Z}$. The $1/(1-x)$ singularity results from summing infinitely many instantons that degenerate to zero area, and signals the appearance of extra physical states [18].

However we can go on and further continue v_1 to negative values (thereby avoiding the singularity by switching on v_2, *i.e.*, a Wilson line), by making use of the identity $\frac{x}{x-1} = -\frac{1}{(1/x)-1}$ in (14). We find:

$$\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau|3u, -3v) \;=\; -e^{-2\pi i a/3} e^{6\pi i v}\, \Theta_{trap}\left[\begin{array}{c} -a \\ -b \end{array}\right](3\tau|3(\tau-u), 3v) \,. \tag{15}$$

[6]Or "different phase" in the language of ref. [24, 25], where similar phenomena were considered for non-compact branes.

This shows how resummation of the instanton series maps back to a well-defined, however different geometry (the rôles of the boundary fields can formally change, *i.e.*, Ψ's transmute into Φ's and vice versa); one might call this phenomenon an "instanton flop". See Fig.2 for a sketch of this.

More generally, from (14) we can deduce the following behavior under shifts of the open string moduli:

$$
\begin{aligned}
\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3u,\,3v) &= e^{\mp 2\pi i a}\,\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3(u\pm 1),\,3v) \\
&= e^{\mp 2\pi i b}\,\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3u,\,3(v\pm 1))\,, \qquad (16)\\
\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3u,\,3(v\pm\tau)) &= e^{\mp 6\pi i(u-1/6)}q^{3/2}\,\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3(u\mp\tau),\,3v)\,,\\
\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3(u\pm\tau),\,3v) &= e^{\mp 6\pi i v}\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau\,|\,3u,\,3v)\\
&\mp\; e^{-2\pi i(u-\frac{1}{6})(b-\frac{3}{2}\pm\frac{3}{2})}e^{2\pi i v(b-\frac{3}{2}\mp\frac{3}{2})}q^{-\frac{1}{6}(b-\frac{3}{2}\pm\frac{3}{2})^2}\,\Theta\left[\begin{array}{c} a+b \\ -3/2 \end{array}\right](3\tau\,|\,3u).
\end{aligned}
$$

The ordinary theta function in the last equation may be viewed as an anomaly or obstruction for the trapezoid function against being (quasi-)periodic, and reflects that the Appell function, together with Θ, forms a section of a *non-split* rank two vector bundle. In physical terms, this simply means that the trapezoid function does not extend nicely over the covering space, but rather gets an extra contribution in the form of a three-point function when we translate a brane around the torus.

The mixing of different correlation functions under monodromy renders the effective superpotential (6) ambiguous and non-modular. One may remedy this by defining an invariant correlator in the following manner:

$$
\bar{\Theta}_{trap}\begin{bmatrix} a \\ b \end{bmatrix}(3\tau|3u,3v) \;\equiv\; \Theta_{trap}\begin{bmatrix} a \\ b \end{bmatrix}(3\tau|3u,3v) - P_b(q,u-v-1/6)\,\Theta\begin{bmatrix} a+b \\ -3/2 \end{bmatrix}(3\tau\,|\,3u), \qquad (17)
$$

where $P_b(x)$ is a piecewise polynomial function in $e^{2\pi i x}$ that is designed [13] to cancel the Θ function terms in (16). Explicitly, we find for our geometry ($x_1 \equiv \frac{\mathrm{Im}x}{\mathrm{Im}\tau}$):

$$
P_b(q,x) \;=\; \left\{ \begin{array}{lll} -q^{-\frac{1}{6}b^2}\sum_{n=1}^{m} q^{-\frac{3}{2}n^2+3mn+b(n-m)}\,e^{2\pi i(3n-b)x}, & m=[x_1] & \text{for } x_1\geq 0 \\ q^{-\frac{1}{6}(b-3)^2}\sum_{n=1}^{-m} q^{-\frac{3}{2}n^2-3mn+(3-b)(n+m)}\,e^{-2\pi i(3n+b-3)x}, & m=-[-x_1] & \text{for } x_1<0. \end{array} \right. \qquad (18)
$$

The correlator (17) then has indeed the desired global properties over the full open string moduli space, *i.e.*, is (quasi-)periodic:

$$
\begin{aligned}
\bar{\Theta}_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau|3(u+n+m\tau),3v) &= e^{2\pi i n a}e^{-6\pi i m v}\bar{\Theta}_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau|3u,3v) \qquad (19)\\
\bar{\Theta}_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau|3u,3(v+n+m\tau)) &= e^{2\pi i n b}q^{\frac{3}{2}m^2}e^{-6\pi i m(u-v-1/6)}\bar{\Theta}_{trap}\left[\begin{array}{c} a \\ b \end{array}\right](3\tau|3u,3v)\,,
\end{aligned}
$$

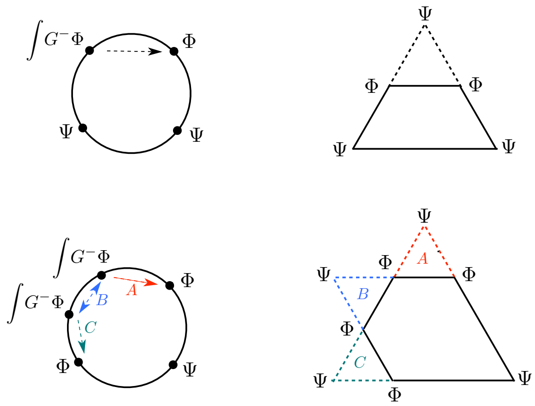

Figure 3: At top: the contact term of a trapezoid correlator is given by a fermionic insertion, generated by the collision of two bosonic operators (of which one is integrated). It gets contributions of triangular instantons as sketched on the right hand side. At bottom: the possible contact terms of the pentagon function are characterized by a parallelogram (B), two kinds of trapezoids (A or C), and a triangle (A plus C).

and always counts instantons with positive areas, but it does not extend to a meromorphic section of a line bundle over Σ^2.

One may wonder about the significance of the extra term in (17), and more generally, about ambiguities in the definition of the correlators. As noted in the introduction, in general there are contact terms arising from colliding operators which lead to regularization ambiguities. In the closed string sector, contact term ambiguities were no great deal because they were implicitly fixed by the flat structure of the moduli space. In the open string sector, no such flat structure exists for the boundary changing deformations, but we see here that we may impose other constraints to fix potential ambiguities, *e.g.*, by insisting on the (quasi-)periodicity of the correlators.

To see the relevance of contact terms, consider a redefinition of the trapezoidal correlator of the general form

$$\Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right] \longrightarrow \Theta_{trap}\left[\begin{array}{c} a \\ b \end{array}\right] + f\,\Theta\left[\begin{array}{c} a+b \\ -3/2 \end{array}\right] \tag{20}$$

for some unspecified, in general moduli-dependent function f. It can be given a simple physical interpretation as follows. A contact term arises when two boundary operators collide and form a single node. The contact term must thus behave like a three-point function, which by itself is governed by triangular instantons that are spanned between the three vertices. Concretely, the only possible contact term of a trapezoid correlator corresponds to the triangle that arises when we follow the two converging edges all the way to their intersection point (see Fig.3, upper part). From the relative angle this must correspond to a fermionic insertion, and indeed the charges are such that the collision of two bosonic operators can generate a contact term $C(\Phi,\Phi)$ of precisely the charge of a fermion; that is, $C(\Phi,\Phi) \sim \Phi(\tau_1)\int_{\tau_1} d\tau_2 G^- \Phi(\tau_2) \sim \Psi(\tau_1)$.

This simple physical picture ties nicely together with results in the mathematical literature. As was shown in [16], the origin of the trapezoidal function being multi-valued and non-modular lies in the existence of a homomorphism located at the intersection of those two converging edges. This homomorphism corresponds precisely to the physical operator content of the contact term, *i.e.*, in our situation, to Ψ.

Consequently, the parallelogram correlator (11), for which there are no converging edges, does not suffer from contact term ambiguities and modular anomalies. This is reflected by the fact that it can be written in terms of modular theta functions in the form[7] $\frac{\Theta'(0)\Theta(u+v)}{\Theta(-u)\Theta(-v)}$, and so corresponds to a well-defined, meromorphic Jacobi form [13].

The redefinition (20) has a very simple description also in terms of the effective lagrangian (6), $\mathcal{W}_{eff}$. It just corresponds to the reparametrization, $t_a \to t_a + f_a^{\bar{b}\bar{c}}\,\xi_{\bar{b}}\xi_{\bar{c}}$, of the tachyon deformation parameters, which is compatible with charges and statistics. Inserting this into $\mathcal{W}_{eff}(t_a,\xi_{\bar{a}})$ trivially reproduces (20), *i.e.*:

$$\mathcal{T}_{ab\bar{c}\bar{d}} \;\to\; \mathcal{T}_{ab\bar{c}\bar{d}} + f_{\bar{c}\bar{d}}{}^{e}\Delta_{abe}\ . \tag{21}$$

This implies, of course, corresponding simultaneous redefinitions of the higher point correlation functions as well. For example, the pentagon function will be modified in the following manner:

$$\wp_{a\bar{b}\bar{c}\bar{e}\bar{f}} \;\to\; \wp_{a\bar{b}\bar{c}\bar{e}\bar{f}} + f_{\bar{b}\bar{c}}{}^{d}\mathcal{T}_{ad\bar{e}\bar{f}} + f_{\bar{e}\bar{f}}{}^{d}\mathcal{T}_{ad\bar{b}\bar{c}} + f_{\bar{c}\bar{e}}{}^{d}\mathcal{P}_{a\bar{b}d\bar{f}} + f_{\bar{b}\bar{c}}{}^{b}f_{\bar{e}\bar{f}}{}^{c}\Delta_{abc}\ . \tag{22}$$

The origin of these terms can be visualized by means of Fig.3, lower part. On the other hand, the parallelogram function won't be modified, and this is consistent with the geometric picture as well.

Whether such a correlated redefinition of all the correlation functions is compatible with the various $\mathcal{A}_\infty$ consistency constraints, may at this point not be entirely obvious, and we will verify this below by direct computation. However, at tree (disk) level, its consistency follows also more directly from the structure of homological perturbation theory (as reviewed in [10,26,27]); basically, a redefinition of correlators by contact terms with less legs is consistent if the contact terms satisfy $\mathcal{A}_\infty$ relations by themselves. In this language, it is a homotopy transformation which by its very definition is compatible with the $\mathcal{A}_\infty$ structure.

To our knowledge, the corresponding statement has however not been proven for the $\mathcal{A}_\infty$ consistency constraints at the *quantum level*, and we will verify it for the annulus by direct computation further below.

3. Classical $\mathcal{A}_\infty$ relations

As it was shown in [9], the topological disk amplitudes that we determined in the previous sections by instanton counting, should fulfill certain algebraic equations, the (classical) $\mathcal{A}_\infty$

[7] From this we see, similar as for the trapezoid, that a singularity occurs if either pair of the parallel edges moves on top of each other, *i.e.*, if $u=0$ or $v=0$.

relations. These take the form:

$$\sum_{k<l=1}^{m}(-)^{\tilde{a}_1+\ldots+\tilde{a}_k}\mathcal{F}^{0,1}_{a_1\ldots a_k c a_{l+1}\ldots a_m}\rho^{cd}\mathcal{F}^{0,1}_{d a_{k+1}\ldots a_l}=0 \qquad \text{for} \quad m\geq 1\ . \tag{23}$$

Here $\mathcal{F}^{0,1}_{a_1\ldots a_n}$ denotes any one of the disk amplitudes given in (3) or (7). We suppress the boundary condition labels for a moment and understand that the a_i's can take values in the index set $\{1\!\!1, a, \bar{a}, \Omega\}$. Let us assume for what follows that the 'external' fields are only chosen from the index subset corresponding to boundary changing operators, *i.e.*, $a_i \in \{a, \bar{a}\}$. The remaining $\mathcal{A}_\infty$ relations, with 'external' Ω insertions, can be obtained by differentiation with respect to u_i's. The $\tilde{a}_i$'s denote the suspended $\mathbb{Z}_2$-gradings of the boundary fields [9]. Here, we have $\tilde{a}=0$ and $\tilde{\bar{a}}=1$.

From the different types of amplitudes that can appear in our specific setup it is clear that the relations (23) are non-trivial only for level $m\in\{4,\ldots,9\}$; for $m=4$ they just express associativity of the 3-point correlators. Apart from the splitting in different levels, m, we can further distinguish two classes of $\mathcal{A}_\infty$ relations according to the particular D-brane configuration:

(i) The number of D-branes that are involved in the $\mathcal{A}_\infty$ relation is maximal, *i.e.*, equal to the level m. We assume furthermore that parallel D-branes are all separated, which means that $u_k \neq u_l$ whenever $\mathcal{L}_{i_k}$ and $\mathcal{L}_{i_l}$ are in the same homology class. This is the transversality condition of ref. [15], and we will verify the $\mathcal{A}_\infty$ relations for such transversal D-brane configurations in the next section.

(ii) The number of D-branes is not maximal. This implies that at least two of the boundary condition labels i_k that appear in the $\mathcal{A}_\infty$ relations represent the same D-brane. Such situations lead to singular correlation functions in view of degenerate instantons, and we will have to introduce an appropriate regularization procedure. This configuration is not transversal in the sense of ref. [15], and we will see that it leads to new results on the $\mathcal{A}_\infty$ structure on the elliptic curve.

3.1. $\mathcal{A}_\infty$ relations for transversal D-brane configurations

The simplest non-trivial $\mathcal{A}_\infty$ relation is at level $m=5$ and involves five D-branes $\mathcal{L}_i$ for $i=1,\ldots,5$. We pick the field configuration in (23) to be

$$\{a_1,a_2,a_3,a_4,a_5\} \quad\cong\quad \{\Psi^{(i_1,i_2)}_{a_1},\Psi^{(i_2,i_3)}_{a_2},\Psi^{(i_3,i_4)}_{a_3},\Psi^{(i_4,i_5)}_{a_4},\Phi^{(i_5,i_1)}_{\bar{a}_5}\}\ , \tag{24}$$

where we assume $u_i\neq u_j$ for $i\neq j$. Naively, for $m=5$ the sum over k and l in (23) involves five terms. Charge selection dictates, however, that two of them should involve Ω-insertions, which however are not present because of the assumption of separated branes, *i.e.*, transversality. The $\mathcal{A}_\infty$ relations thus read:

$$\sum_{c=1}^{3}\mathcal{P}^{(i_1i_2i_4i_5)}_{a_1\bar{c}a_4\bar{a}_5}\ \Delta^{(i_4i_2i_3)}_{ca_2a_3}+\sum_{c=1}^{3}\mathcal{T}^{(i_1i_2i_3i_5)}_{a_1a_2\bar{c}\bar{a}_5}\ \Delta^{(i_5i_3i_4)}_{ca_3a_4}+\sum_{c=1}^{3}\Delta^{(i_1i_2i_3)}_{a_1a_2c}\ \mathcal{T}^{(i_1i_3i_4i_5)}_{\bar{c}a_3a_4\bar{a}_5}=0\ . \tag{25}$$

Insertion of all the disk amplitudes as given in Section 2 shows that these relations are indeed satisfied; this transversal situation has already been studied in [13]. One also readily verifies that (25) is compatible with homotopy reparametrizations (21) that reflect the contact term ambiguity. Indeed, while the parallelogram function does not allow for a contact term, the contributions from the trapezoid functions just sum up to:

$$\sum_{c,e}\Big(f_{\bar{c}\bar{a}_5}{}^{e}\Delta_{a_1a_2e}\Delta_{ca_3a_4}-f_{\bar{a}_5\bar{c}}{}^{e}\Delta_{a_1a_2c}\Delta_{a_3a_4e}\Big) \tag{26}$$

and thus cancel for cyclic coefficients, $f_{\bar{a}\bar{b}}{}^{c}=f_{\bar{b}\bar{c}}{}^{a}$. The latter condition ensures that the cyclic invariance of disk amplitudes is preserved (accordingly, the transformations (21) and (22) satisfying this condition are called *cyclic* homotopies [15, 27]).

Let us consider the next level, $m=6$. A generic $\mathcal{A}_\infty$ relation will then involve also pentagon amplitudes. We will come to those in a moment, but first ask whether there exist relations at level $m=6$ that do not involve pentagon amplitudes. Indeed, there are, and they correspond to the field configuration:

$$\{a_1,a_2,a_3,a_4,a_5,a_6\} \;\cong\; \{\Psi_{a_1}^{(i_1,i_2)},\Phi_{\bar{a}_2}^{(i_2,i_3)},\Psi_{a_3}^{(i_3,i_4)},\Phi_{\bar{a}_4}^{(i_4,i_5)},\Psi_{a_5}^{(i_5,i_6)},\Phi_{\bar{a}_6}^{(i_6,i_1)}\}\ ,$$

where all $\mathcal{L}_{i_k}$ for odd k belong to the same homology class. The same is true for all $\mathcal{L}_{i_k}$ for even k. The $\mathcal{A}_\infty$ relation takes an intriguing form that involves only parallelogram amplitudes:

$$\sum_{c=1}^{3}\mathcal{P}_{a_1\bar{c}a_5\bar{a}_6}^{(i_1i_2i_5i_6)}\ \mathcal{P}_{c\bar{a}_2a_3\bar{a}_4}^{(i_5i_2i_3i_4)}-\sum_{c=1}^{3}\mathcal{P}_{a_1\bar{a}_2c\bar{a}_6}^{(i_1i_2i_3i_6)}\ \mathcal{P}_{\bar{c}a_3\bar{a}_4a_5}^{(i_6i_3i_4i_5)}-\sum_{c=1}^{3}\mathcal{P}_{a_1\bar{a}_2a_3\bar{c}}^{(i_1i_2i_3i_4)}\mathcal{P}_{c\bar{a}_4a_5\bar{a}_6}^{(i_1i_4i_5i_6)}=0\ . \tag{27}$$

Notice that it is manifestly homotopy invariant. In fact, this relation was interpreted in [28] as an associative Yang–Baxter equation, for which the R-matrix is essentially given by the parallelogram amplitude. In particular, this link was used to construct elliptic solutions to the classical Yang–Baxter equation for $sl_n(\mathbb{C})$. The integer n is an intersection number, in our situation given by $n=\chi(\mathcal{L}_{i_{2k}},\mathcal{L}_{i_{2k'+1}})=3$.

There are three further choices for the external fields in equations (23) for level $m=6$. Two of them, *i.e.*,

$$\{a_1,a_2,a_3,a_4,a_5,a_6\} \;\cong\; \begin{array}{l}\{\Psi_{a_1}^{(i_1,i_2)},\Psi_{a_2}^{(i_2,i_3)},\Phi_{\bar{a}_3}^{(i_3,i_4)},\Psi_{a_4}^{(i_4,i_5)},\Phi_{\bar{a}_5}^{(i_5,i_6)},\Phi_{\bar{a}_6}^{(i_6,i_1)}\}\ ,\\ \{\Psi_{a_1}^{(i_1,i_2)},\Phi_{\bar{a}_2}^{(i_2,i_3)},\Psi_{a_3}^{(i_3,i_4)},\Psi_{a_4}^{(i_4,i_5)},\Phi_{\bar{a}_5}^{(i_5,i_6)},\Phi_{\bar{a}_6}^{(i_6,i_1)}\}\ ,\end{array}$$

give rise to similar $\mathcal{A}_\infty$ relations that include pentagon as well as lower-point amplitudes. We present only one of them here:

$$\begin{aligned}&\sum_{c=1}^{3}\Delta_{a_1a_2c}^{(i_1i_2i_3)}\ \wp_{\bar{c}\bar{a}_3a_4\bar{a}_5\bar{a}_6}^{(i_1i_3i_4i_5i_6)}+\sum_{c=1}^{3}\mathcal{T}_{a_1c\bar{a}_5\bar{a}_6}^{(i_1i_2i_5i_6)}\ \mathcal{P}_{\bar{c}a_2\bar{a}_3a_4}^{(i_5i_2i_3i_4)}+\\+\ &\sum_{c=1}^{3}\mathcal{T}_{a_1a_2\bar{c}\bar{a}_6}^{(i_1i_2i_3i_6)}\ \mathcal{P}_{c\bar{a}_3a_4\bar{a}_5}^{(i_6i_3i_4i_5)}-\sum_{c=1}^{3}\mathcal{T}_{a_1a_2\bar{a}_3\bar{c}}^{(i_1i_2i_3i_4)}\mathcal{T}_{ca_4\bar{a}_5\bar{a}_6}^{(i_1i_4i_5i_6)}=0\ .\end{aligned} \tag{28}$$

The final $\mathcal{A}_\infty$ relation at level $m=6$ corresponds to the fields

$$\{a_1,a_2,a_3,a_4,a_5,a_6\} \;\cong\; \{\Psi_{a_1}^{(i_1,i_2)},\Psi_{a_2}^{(i_2,i_3)},\Psi_{a_3}^{(i_3,i_4)},\Phi_{\bar{a}_4}^{(i_4,i_5)},\Phi_{\bar{a}_5}^{(i_5,i_6)},\Phi_{\bar{a}_6}^{(i_6,i_1)}\}\ ,$$

and reads:

$$\sum_{c=1}^{3} \Delta^{(i_1 i_2 i_3)}_{a_1 a_2 c}\ \wp^{(i_1 i_3 i_4 i_5 i_6)}_{\bar{c} a_3 \bar{a}_4 \bar{a}_5 \bar{a}_6} + \sum_{c=1}^{3} \wp^{(i_1 i_2 i_4 i_5 i_6)}_{a_1 \bar{c} \bar{a}_4 \bar{a}_5 \bar{a}_6}\ \Delta^{(i_4 i_2 i_3)}_{c a_2 a_3}$$
$$+ \sum_{c=1}^{3} \mathcal{T}^{(i_1 i_2 i_5 i_6)}_{a_1 c \bar{a}_5 \bar{a}_6}\ \mathcal{T}^{(i_5 i_2 i_3 i_4)}_{\bar{c} a_2 a_3 \bar{a}_4} + \sum_{c=1}^{3} \mathcal{T}^{(i_1 i_2 i_3 i_6)}_{a_1 a_2 \bar{c} \bar{a}_6}\ \mathcal{T}^{(i_6 i_3 i_4 i_5)}_{c a_3 \bar{a}_4 \bar{a}_5} = 0\ . \tag{29}$$

Plugging in the instanton sums we verified that all three relations are indeed satisfied. We can also easily verify invariance of the level $m = 6$ relations (28) and (29) under the combined homotopy transformations (21) and (22); these map the equations into $\mathcal{A}_\infty$ relations at lower levels, (26) and (25), that we have already checked to vanish before.

We conclude this section by presenting the $\mathcal{A}_\infty$ relation at level $m = 7$ that contains hexagon amplitudes (13). For the fields

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\} \quad \cong \quad \{\Psi^{(i_1,i_2)}_{a_1}, \Psi^{(i_2,i_3)}_{a_2}, \Phi^{(i_3,i_4)}_{\bar{a}_3}, \Phi^{(i_4,i_5)}_{\bar{a}_4}, \Phi^{(i_5,i_6)}_{\bar{a}_5}, \Phi^{(i_6,i_7)}_{\bar{a}_6}, \Phi^{(i_7,i_1)}_{\bar{a}_7}\}\ ,$$

we get:

$$\sum_{c=1}^{3} \Delta^{(i_1 i_2 i_3)}_{a_1 a_2 c}\ \mathcal{H}^{(i_1 i_3 i_4 i_5 i_6 i_7)}_{\bar{c} \bar{a}_3 \bar{a}_4 \bar{a}_5 \bar{a}_6 \bar{a}_6} + \sum_{c=1}^{3} \wp^{(i_1 i_2 i_5 i_6 i_7)}_{a_1 \bar{c} \bar{a}_5 \bar{a}_6 \bar{a}_7}\ \mathcal{T}^{(i_5 i_2 i_3 i_4)}_{c a_2 \bar{a}_3 \bar{a}_4} + \tag{30}$$
$$+ \sum_{c=1}^{3} \mathcal{T}^{(i_1 i_2 i_6 i_7)}_{a_1 c \bar{a}_6 \bar{a}_7}\ \wp^{(i_6 i_2 i_3 i_4 i_5)}_{\bar{c} a_2 \bar{a}_3 \bar{a}_4 \bar{a}_5} + \sum_{c=1}^{3} \mathcal{T}^{(i_1 i_2 i_3 i_7)}_{a_1 a_2 \bar{c} \bar{a}_7}\ \wp^{(i_7 i_3 i_4 i_5 i_6)}_{c \bar{a}_3 \bar{a}_4 \bar{a}_5 \bar{a}_6} - \sum_{c=1}^{3} \mathcal{T}^{(i_1 i_2 i_3 i_4)}_{a_1 a_2 \bar{a}_3 \bar{c}}\ \wp^{(i_1 i_4 i_5 i_6 i_7)}_{c \bar{a}_4 \bar{a}_5 \bar{a}_6 \bar{a}_7} = 0\ .$$

The other two $\mathcal{A}_\infty$ relations at level $m = 7$ involve only four- and five-point amplitudes. These relations as well as the missing ones at level $m = 8$ and 9 can easily be deduced from (23).

3.2. $\mathcal{A}_\infty$ relations for non-transversal D-brane configurations

Let us consider situation *(ii)* where the number of D-branes is not maximal, that is, smaller than m. Take an $\mathcal{A}_\infty$ relation (23) at level $m = 5$ for four boundary condition labels, say $\mathcal{L}_{i_1}$, $\mathcal{L}_{i_2}$, $\mathcal{L}_{i_3}$ and $\mathcal{L}_{i_5}$, and the collection of fields:

$$\{a_1, a_2, a_3, a_4, a_5\} \quad \cong \quad \{\Psi^{(i_1,i_2)}_{a_1}, \Psi^{(i_2,i_3)}_{a_2}, \Psi^{(i_3,i_1)}_{a_3}, \Psi^{(i_1,i_5)}_{a_4}, \Phi^{(i_5,i_1)}_{\bar{a}_5}\}\ ,$$

where we assume $u_i \neq u_j$ for $i \neq j$.[8]

The $\mathcal{A}_\infty$ relations look similar to (25), but now we do not have a transversal configuration, so that an additional term appears, where a disk with the fields $\langle \mathbb{1}^{(i_1,i_1)} \Psi^{(i_1,i_5)}_{a_4} \Phi^{(i_5,i_1)}_{\bar{a}_5} \rangle$ bubbles off and gives rise to an amplitude with an insertion of the boundary condition preserving field $\Omega^{i_1 i_1}$. Naively we get:

$$\sum_{c=1}^{3} \mathcal{P}^{(i_1 i_2 i_1 i_5)}_{a_1 \bar{c} a_4 \bar{a}_5} \Delta^{(i_1 i_2 i_3)}_{c a_2 a_3} + \sum_{c=1}^{3} \mathcal{T}^{(i_1 i_2 i_3 i_5)}_{a_1 a_2 \bar{c} \bar{a}_5} \Delta^{(i_5 i_3 i_1)}_{c a_3 a_4} + \sum_{c=1}^{3} \Delta^{(i_1 i_2 i_3)}_{a_1 a_2 c}\ \mathcal{T}^{(i_1 i_3 i_1 i_5)}_{\bar{c} a_3 a_4 \bar{a}_5} = -\frac{\rho_{a_4 \bar{a}_5}}{6\pi i} \partial_{u_1} \Delta^{(i_1 i_2 i_3)}_{a_1 a_2 a_3}\ .$$

[8] Note that this configuration is similar to (24), only the label i_4 was substituted by i_1.

There is however an important subtlety. The 4-point functions on the left-hand side that bear the boundary condition $\mathcal{L}_{i_1}$ twice can become singular due to an infinite sum over degenerate instantons. This happens for instance in the amplitude $\mathcal{P}^{(i_1 i_2 \tilde{\imath}_1 i_5)}_{a_1 \bar{c} a_4 \bar{a}_5}$ when opposite sides of a parallelogram collide in the limit $\tilde{u}_1 \to u_1$. We therefore need to regularize the singular 4-point functions. We do this by point-splitting in the following way. Whenever two sides of a parallelogram (or a trapezoid) bear the *same* D-brane $\mathcal{L}_{i_1}$ at position u_1, we formally set one of the two sides at position $\tilde{u}_1$ and take the limit $\tilde{u}_1 \to u_1$. In order to track the (formal) $\tilde{u}_1$-dependence of the four-point correlators in the $\mathcal{A}_\infty$ relation, we introduce the index $\tilde{\imath}_1$ and denote the amplitudes with point-splitting regularization by $\widetilde{\mathcal{P}}^{(i_1 i_2 \tilde{\imath}_1 i_5)}_{a_1 \bar{c} a_4 \bar{a}_5}$ and $\widetilde{\mathcal{T}}^{(i_1 i_3 \tilde{\imath}_1 i_5)}_{\bar{c} a_3 a_4 \bar{a}_5}$. The $\mathcal{A}_\infty$ relations then become:

$$\lim_{\tilde{u}_1 \to u_1} \sum_{c=1}^{3} \left(\widetilde{\mathcal{P}}^{(i_1 i_2 \tilde{\imath}_1 i_5)}_{a_1 \bar{c} a_4 \bar{a}_5} \Delta^{(i_1 i_2 i_3)}_{c a_2 a_3} + \mathcal{T}^{(i_1 i_2 i_3 i_5)}_{a_1 a_2 \bar{c} \bar{a}_5} \Delta^{(i_5 i_3 i_1)}_{c a_3 a_4} + \Delta^{(i_1 i_2 i_3)}_{a_1 a_2 c} \; \widetilde{\mathcal{T}}^{(i_1 i_3 \tilde{\imath}_1 i_5)}_{\bar{c} a_3 a_4 \bar{a}_5} \right) = -\frac{\rho_{a_4 \bar{a}_5}}{6\pi i} \partial_{u_1} \Delta^{(i_1 i_2 i_3)}_{a_1 a_2 a_3} \; . \tag{31}$$

Using the transversal $\mathcal{A}_\infty$ relation (25) with $i_4 = \tilde{\imath}_1$ we see that the singularities of the right-hand side of (31) mutually cancel. So we can safely take the limit $\tilde{u}_1 \to u_1$ and verify the relation.

Analogously, there is a non-transversal version of the level $m = 6$ relation (27) that includes only parallelograms. For this, let us consider the field configuration:

$$\{a_1, a_2, a_3, a_4, a_5, a_6\} \quad \cong \quad \{\Psi^{(i_1,i_2)}_{a_1}, \Phi^{(i_2,i_1)}_{\bar{a}_2}, \Psi^{(i_1,i_4)}_{a_3}, \Phi^{(i_4,i_5)}_{\bar{a}_4}, \Psi^{(i_5,i_6)}_{a_5}, \Phi^{(i_6,i_1)}_{\bar{a}_6}\} \; ,$$

Following the same regularization procedure as above we obtain the $\mathcal{A}_\infty$ relations:

$$\lim_{\tilde{u}_1 \to u_1} \sum_{c=1}^{3} \left(\mathcal{P}^{(i_1 i_2 i_5 i_6)}_{a_1 \bar{c} a_5 \bar{a}_6} \; \mathcal{P}^{(i_5 i_2 i_1 i_4)}_{c \bar{a}_2 a_3 \bar{a}_4} - \widetilde{\mathcal{P}}^{(i_1 i_2 \tilde{\imath}_1 i_6)}_{a_1 \bar{a}_2 c \bar{a}_6} \; \mathcal{P}^{(i_6 i_1 i_4 i_5)}_{\bar{c} a_3 \bar{a}_4 a_5} - \widetilde{\mathcal{P}}^{(i_1 i_2 \tilde{\imath}_1 i_4)}_{a_1 \bar{a}_2 a_3 \bar{c}} \mathcal{P}^{(i_1 i_4 i_5 i_6)}_{c \bar{a}_4 a_5 \bar{a}_6} \right) = \frac{\rho_{a_1 \bar{a}_2}}{6\pi i} \partial_{u_1} \mathcal{P}^{(i_1 i_4 i_5 i_6)}_{a_3 \bar{a}_4 a_5 \bar{a}_6} \; ,$$

which are manifestly homotopy invariant.

We refrain from going through the list of remaining non-transversal $\mathcal{A}_\infty$ relations here, the general picture should be clear from the cases that we presented so far.

4. Quantum $\mathcal{A}_\infty$ relations: the annulus

In order to get a handle on higher genus topological string amplitudes with multiple boundary components, we can take advantage of the quantum $\mathcal{A}_\infty$ relations of [11] which follow from factorizations of higher genus amplitudes. For the elliptic curve, the charges of the boundary operators are such that the only non-vanishing open topological string amplitudes beyond tree level appear at one loop, *i.e.*, for annulus world-sheets. The charge selection rule for the topological amplitude is quite restrictive and implies for the D-brane geometry at hand that there are only two non-trivial factorization relations for the annulus. All others can be obtained by differentiation with respect to the boundary moduli u_i.

The first non-trivial relation is diagrammatically depicted in Fig.4. Explicitly, denoting by $\mathcal{F}^{0,1}$ and $\mathcal{F}^{0,2}$ the generic topological string amplitude on the disk and annulus, respec-

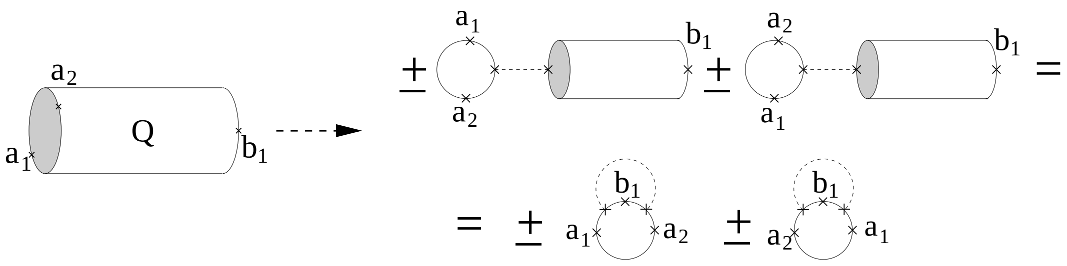

Figure 4: A graphical representation of the quantum $\mathcal{A}_\infty$ relation, given in (32).

tively, this quantum $\mathcal{A}_\infty$ relation takes the form [11]:

$$\begin{aligned}&\sum_{c,d}\Big((-)^{\tilde a_1+\tilde d\tilde a_2}\ \mathcal{F}^{0,1}_{a_1ca_2}\ \rho^{cd}\ \mathcal{F}^{0,2}_{d|\ b_1}+(-)^{\tilde a_1+\tilde a_2}\ \mathcal{F}^{0,1}_{a_1a_2c}\ \rho^{cd}\ \mathcal{F}^{0,2}_{d|\ b_1}\Big)\\ =\ &\sum_{c,d}\Big((-)^{\tilde a_1+\tilde b_1(\tilde d+\tilde a_2)}\rho^{cd}\mathcal{F}^{0,1}_{a_1cb_1da_2}+(-)^{\tilde a_1+\tilde a_2+\tilde b_1\tilde d}\rho^{cd}\mathcal{F}^{0,1}_{a_1a_2cb_1d}\Big)\ .\end{aligned}\tag{32}$$

Here, the labels a_i subsume both field and boundary condition labels, and ρ^{ab} denotes the inner product of the boundary fields (2). Without loss of generality we pick the following choice of boundary operators in (32):

$$a_1\sim\Psi_a^{(i_1,i_2)}\qquad a_2\sim\Phi_{\bar b}^{(i_2,i_1)}\qquad b_1\sim\Omega^{(i_3,i_3)}\ .$$

The reason why we insert the boundary preserving modulus $\Omega^{(i_3,i_3)}$ on the second boundary, is that the factorization problem of open higher genus amplitudes is well-defined only if there is at least one topological observable on each boundary component [11]. Otherwise the boundary without insertions bubbles off in the closed string channel and gives rise to a non-stable disk amplitude. This corresponds to a noncompact direction in the moduli space of the Riemann surface and, as we will discuss below, indicates divergences in the amplitudes.

From the field configuration it is clear that $\mathcal{L}_{i_1}$ and $\mathcal{L}_{i_2}$ wrap different homology classes. Introducing the following notation for annulus correlators:

$$\mathcal{A}^{(i_1i_1\ldots|i_2i_2\ldots)}_{\Omega\ldots|\Omega\ldots}\ =\ \big\langle\Omega^{(i_1,i_1)}\ldots|\Omega^{(i_2,i_2)}\ldots\big\rangle_{ann}\ ,\tag{33}$$

the annulus $\mathcal{A}_\infty$ relation (32) then simplifies to

$$\begin{aligned}\rho_{a\bar b}\left(\mathcal{A}^{(i_1i_1|i_3i_3)}_{\Omega|\ \Omega}-\mathcal{A}^{(i_2i_2|i_3i_3)}_{\Omega|\ \Omega}\right)\ &=\ \sum_{c=1}^{3}\big(\mathcal{T}^{i_1i_2i_3i_3i_2}_{ac\Omega\bar c\bar b}-\mathcal{P}^{i_1i_2i_1i_3i_3}_{a\bar bc\Omega\bar c}\big)+\\ &+\ \sum_{c=1}^{3}\big(\mathcal{P}^{i_1i_2i_3i_3i_2}_{a\bar c\Omega c\bar b}-\mathcal{T}^{i_3i_1i_2i_1i_3}_{ca\bar b\bar c\Omega}\big)\ .\end{aligned}\tag{34}$$

Note that this equation is compatible with homotopy transformations, *i.e.*, contact term redefinitions of the trapezoid amplitudes as given in (21). Specifically, a homotopy transformation adds

$$\partial_{u_3}\Big(\sum_{c,e}\big(f_{\bar c\bar b}{}^{e}\Delta_{ace}-f_{\bar b\bar c}{}^{e}\Delta_{cae}\big)\Big)$$

to the right-hand side of (34), which vanishes for cyclic coefficients, *i.e.*, $f_{\bar{a}\bar{b}}{}^{c} = f_{\bar{b}\bar{c}}{}^{a}$.

For the third homology class i_3 we have two choices: *(i)* either $\mathcal{L}_{i_3}$ wraps a homology class different from both, $\mathcal{L}_{i_1}$ and $\mathcal{L}_{i_2}$, or *(ii)* it wraps the same class as $\mathcal{L}_{i_1}$ (or $\mathcal{L}_{i_2}$). We will now discuss these two cases separately:

(i) When all three D-branes wrap different homology classes, the quantum $\mathcal{A}_\infty$ relations become:

$$\begin{aligned}\rho_{a\bar{b}}\left(\partial_{u_3}\mathcal{A}^{(i_1 i_1|i_3)}_{\Omega|\ \cdot} - \partial_{u_3}\mathcal{A}^{(i_2 i_2|i_3)}_{\Omega|\ \cdot}\right) &= \sum_{c=1}^{3}\left(\partial_{u_3}\mathcal{T}^{i_1 i_2 i_3 i_2}_{ac\bar{c}\bar{b}} - \partial_{u_3}\mathcal{T}^{i_3 i_1 i_2 i_1}_{ca\bar{b}\bar{c}}\right) = \\ &= \sum_{c=1}^{3}\rho_{a\bar{b}}\left(\partial_{u_3}\mathcal{T}^{i_1 i_2 i_3 i_2}_{ac\bar{c}\bar{a}} - \partial_{u_3}\mathcal{T}^{i_3 i_1 i_2 i_1}_{ca\bar{a}\bar{c}}\right) = 0\ .\end{aligned} \tag{35}$$

Here we wrote the field insertion $\Omega^{(i_3 i_3)}$ in terms of derivatives with respect to u_3. In the second line we used the Kronecker deltas from (8) and (11). The last step follows from explicitly inserting the trapezoid amplitudes (8). By considering analogous relations for other choices for the $\mathcal{L}_i$'s it follows readily that

$$\mathcal{A}^{(i_1 i_1|i_2 i_2)}_{\Omega|\ \Omega} = \mathcal{A}^{(i_2 i_2|i_3 i_3)}_{\Omega|\ \Omega} = \mathcal{A}^{(i_3 i_3|i_1 i_1)}_{\Omega|\ \Omega} = \frac{1}{(6\pi i)^2} f_{\mathcal{A}}(\tau)\ ,$$

for some $f_{\mathcal{A}}(\tau)$. Since this function is u_i-independent, it cannot be an instanton series and so must be simple. This fact is also clear from the geometric picture, *i.e.*, it is not possible to span an annulus between non-parallel D-branes. In principle, we could determine $f_{\mathcal{A}}(\tau)$ via imposing modular invariance, but we will identify it below by comparison with a known result.

(ii) If, say, $\mathcal{L}_{i_1}$ and $\mathcal{L}_{i_3}$ wrap the same homology class, then the annulus factorization condition (34) simplifies to

$$\rho_{a\bar{b}}\left(\partial_{u_3}\mathcal{A}^{(i_1 i_1|i_3)}_{\Omega|\ \cdot} - \frac{f_{\mathcal{A}}(\tau)}{6\pi i}\right) = \sum_{c=1}^{3}\rho_{a\bar{b}}\partial_{u_3}\mathcal{P}^{i_1 i_2 i_3 i_2}_{a\bar{c}c\bar{a}}\ , \tag{36}$$

which is manifestly homotopy invariant. The function $\partial_{u_3}\mathcal{A}^{(i_2 i_2|i_3)}_{\Omega|\ \cdot} = 1/(6\pi i) f_{\mathcal{A}}(\tau)$ appears here because $\mathcal{L}_{i_2}$ and $\mathcal{L}_{i_3}$ wrap different homology classes.

Notice that in the disk correlators on the right-hand side of (36), one pair of parallel sides of the parallelograms corresponds to the same D-brane, $\mathcal{L}_{i_2}$; we thus encounter a non-transversal configuration and need to regularize the correlators in order to evaluate the sum. Its divergent part is, however, u_3-independent and gets annihilated by the u_3-derivative, so that the right-hand side of (36) is well-defined. Had we not inserted the boundary modulus $\Omega^{(i_3,i_3)}$ in the first place and thus had considered the integrated version of (36), the (non-cancelling) divergent pieces of the parallelogram correlators would not have been killed; this divergence reflects the non-stable closed string degeneration channel where a disk with a "bare" boundary bubbles off [11].[9]

[9]In a sense, infinitely many degenerate parallelogram instantons on the right hand side of (36) conspire to

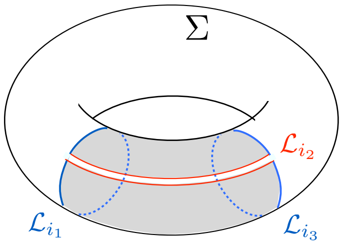

Figure 5: Shown is how annulus instantons can be obtained from disk instantons via the fusion of boundary components. This is what underlies geometrically the quantum $\mathcal{A}_\infty$ relation (36). The divergence arising from coinciding boundary components disappears if we consider a well-defined stable degeneration limit, by choosing suitable operator insertions.

Since $\mathcal{L}_1$ and $\mathcal{L}_3$ are parallel, we expect world-sheet instantons with the topology of an annulus to contribute to $\mathcal{A}_{\Omega|\,\cdot}^{(i_1 i_1|i_3)}$. Indeed, if we insert in (36) the parallelogram series $\mathcal{P} \sim \Theta_{para}$ as given in (11), we get

$$\begin{aligned}
\partial_{u_3}\mathcal{A}_{\Omega|\,\cdot}^{(i_1 i_1|i_3)} &= \frac{f_{\mathcal{A}}(\tau)}{6\pi i} + \sum_{c=1}^{3} \partial_{u_3}\Theta_{para}\begin{bmatrix} 3 \\ [a-c]_3 \end{bmatrix}(3\tau|0,3(u_1-u_3)) \\
&= \frac{f_{\mathcal{A}}(\tau)}{6\pi i} + \sum_{b=1}^{3} \partial_{u_3}\Theta_{para}\begin{bmatrix} 3 \\ b \end{bmatrix}(3\tau|0,3(u_1-u_3)) \\
&= \frac{f_{\mathcal{A}}(\tau)}{6\pi i} + \sum_{b=1}^{3} \partial_{u_3} \sum_{n\neq -1, m\in\mathbb{Z}}^{\text{indef.}} q^{(n+1)(b+3m)} e^{6\pi i(n+1)(u_1-u_3)} \\
&= \frac{f_{\mathcal{A}}(\tau)}{6\pi i} + \partial_{u_3} \sum_{n\neq 0, m\in\mathbb{Z}}^{\text{indef.}} q^{nm} e^{6\pi i n(u_1-u_3)} \ .
\end{aligned}$$

The interpretation of this series is obvious: it should describe the second derivative of the annulus instanton sum for parallel D-branes, $\mathcal{L}_{i_k}$ and $\mathcal{L}_{i_l}$. Up to an integration constant, we can read it off as follows:

$$\mathcal{A}_{\cdot|\cdot}^{(i_k|i_l)} = -\frac{1}{2} f_A(\tau)(u_k-u_l)^2 + \sum_{n\neq 0, m\in\mathbb{Z}}^{\text{indef.}} \frac{1}{n} q^{nm} e^{6\pi i n(u_k-u_l)} \ . \tag{37}$$

This indeed coincides with the result given in [3] for the annulus partition function in the holomorphic limit, $\mathcal{F}^{(0,2)}$, provided we identify:

$$f_A(\tau) \;=\; \text{const.}\tau \ .$$

This term is thus a remnant of the holomorphic anomaly of the annulus amplitude.

reproduce the singularity arising from a non-stable degeneration, *i.e.*, from not having fixed the isometries of a disk that pinches off.

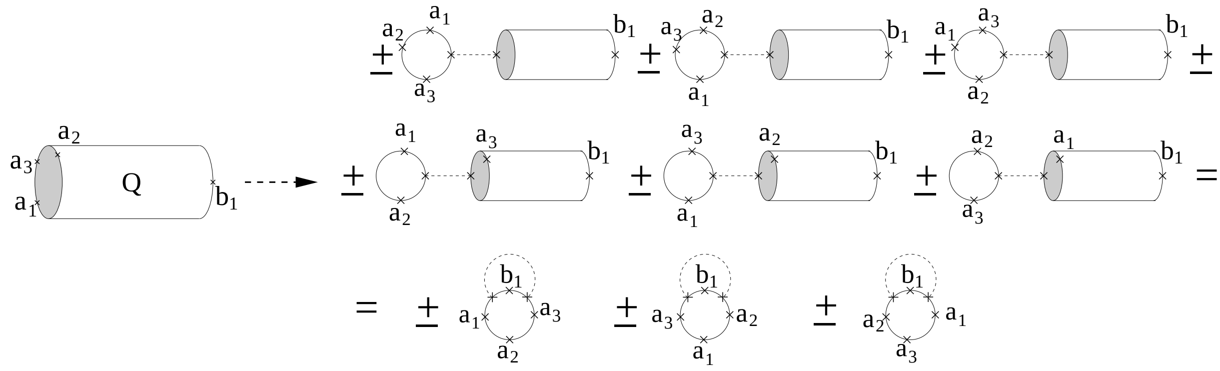

Figure 6: A graphical realization of the quantum $\mathcal{A}_\infty$ relation that leads to (38). In our situation only two diagrams on the right-hand side are associated with non-vanishing amplitudes.

Note that the annulus instanton series (37) was obtained, via the $\mathcal{A}_\infty$ factorization relation (32), from disk correlators, which by themselves were determined by counting disk instantons. Geometrically, this implies that the annulus instanton contributions can be patched together in terms of disk instantons, in a similar spirit as the $\mathcal{A}_\infty$ relations on the disk imply the patching up of higher N-gons in terms of smaller N-gons. Indeed, there is a very simple geometrical picture that describes the instanton geometry underlying eq. (36), and this is schematically shown in Fig.5.

The second non-trivial annulus factorization condition on the torus is schematically depicted in Fig.6. Before we present the explicit quantum $\mathcal{A}_\infty$ relation, let us pick the boundary operators to be:

$$a_1 \sim \Phi^{(i_1,i_2)}_{\bar{a}_1} \qquad a_2 \sim \Phi^{(i_2,i_3)}_{\bar{a}_2} \qquad a_2 \sim \Phi^{(i_3,i_1)}_{\bar{a}_3} \qquad b_1 \sim \Omega^{(i_4,i_4)} \ .$$

From the charge selection rule it is quite straightforward to see that all factorization channels, where disks bubble off from the annulus, must vanish. Another immediate consequence of the field configuration at hand is that $\mathcal{L}_{i_k}$ for $k = 1, 2, 3$ must wrap three different homology classes. Let us choose the fourth D-brane $\mathcal{L}_{i_4}$ to be parallel to, say, $\mathcal{L}_{i_1}$.

Then the algebraic relation associated to Fig.6 simplifies considerably and we get the following constraint on pentagon correlation functions:

$$\sum_c \left(\partial_{u_4} \wp^{(i_1 i_2 i_3 i_4 i_3 i_1)}_{\bar{a}_1 \bar{a}_2 \bar{c} c \bar{a}_3} - \partial_{u_4} \wp^{(i_1 i_2 i_4 i_2 i_3 i_1)}_{\bar{a}_1 c \bar{c} \bar{a}_2 \bar{a}_3} \right) = 0 \ , \tag{38}$$

where we have already substituted the proper labels for the operators in question. Inserting the instanton sum (12) for the five-point amplitudes we learn that relation (38) is indeed satisfied. Moreover, it is easily shown to be homotopy invariant for cyclic coefficients $f_{\bar{a}\bar{b}}{}^c$, provided we use the annulus relations (35) and (36).

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A. Heat equation from the Cardy relation?

In [9] another kind of annulus factorization was discussed, in the spirit of the topological "Cardy-constraint". It essentially equates open and closed string channels of the annulus diagram, and reads, in the notation introduced above:

$$\begin{aligned} &\partial_i \mathcal{F}^0_{a_0\ldots a_n} \eta^{ij} \partial_j \mathcal{F}^0_{b_0\ldots b_m} = \\ &= \sum_{\substack{0\le n_1\le n_2\le n \\ 0\le m_1\le m_2\le m}} (-1)^{(\tilde{c}_1+\tilde{a}_0)(\tilde{c}_2+\tilde{b}_0)+\tilde{c}_1+\tilde{c}_2} \rho^{c_1 d_1} \rho^{c_2 d_2} \mathcal{F}^{0,1}_{a_0\ldots a_{n_1} d_1 b_{m_1+1}\ldots b_{m_2} c_2 a_{n_2+1}\ldots a_n} \mathcal{F}^{0,1}_{b_0\ldots b_{m_1} c_1 a_{n_1+1}\ldots a_{n_2} d_2 b_{m_2+1}\ldots b_m} , \end{aligned} \tag{39}$$

where $\mathcal{F}^0$ and η^{ij} are prepotential and metric of the closed string sector, respectively. However, as already pointed out in [9], while the topological Cardy condition is one of the basic axioms of boundary TFT [20–22], it needs to apply only to correlators without integrated insertions (for example, to the topological intersection amplitude). Whenever there are integrated insertions, so that we deal with topological strings rather than with TFT, the proof of the independence of the correlator of the annulus metric does not necessarily go through, and so it may be somewhat unclear whether the Cardy constraint should be imposed in such a situation or not.

In fact, there are results pointing in either way: it was shown in [29] that for boundary LG models with arbitrarily deformed univariate superpotentials, the Cardy constraint is satisfied. Moreover, it was shown in [9] that the Cardy constraint can be imposed on the correlators of the A-series of boundary minimal models and this does in fact lead precisely to the correct effective action [30]. On the other hand, recent results [31] indicate that the Cardy condition cannot be consistently imposed on correlators of minimal models other than the A-series.

One of the original motivations for our present work was to see whether the Cardy condition can be imposed on correlators on the elliptic curve. This question appeared to be potentially interesting also from a different perspective, namely from the heat equation that is satisfied by the three-point correlators:[10]

$$\left(\frac{\partial}{\partial \tau} + \frac{i}{12\pi} \frac{\partial^2}{\partial {u_i}^2} \right) \Delta_{abc}(\tau|u_i) \;=\; 0 \; . \tag{40}$$

The question is, whether, as discussed in [5], this equation simply reflects the underlying operator algebra of the model, or whether there is a deeper reason behind it – such as some form of background (in-)dependence. From this point of view, one may interpret (40) as

[10] This is one of the defining equations for the ordinary theta-functions; analogous equations hold for the higher point correlators which are given in terms of indefinite theta-functions.

telling how a change of open string background (u_i) can be compensated by a change of closed string background (τ), or vice versa.

Our initial observation was that the heat equation (40) may be linked to the Cardy condition (39) as follows. Consider relation (39) with $n=2$, $m=0$, and

$$\{a_0, a_1, a_2, b_0\} \cong \{\Psi_a^{(i_1,i_2)}, \Psi_b^{(i_2,i_3)}, \Psi_c^{(i_3,i_1)}, \Omega^{(i_1,i_1)}\} \ .$$

Note that the boundary condition $\mathcal{L}_{i_1}$ appears on both sides of the annulus; we encounter a non-transversal configuration as in Section 3.2, which will require some regularization. Taking everything together, the Cardy condition (39) then reduces to the following equation:

$$\begin{aligned} -\frac{2}{3\pi i}\frac{\partial}{\partial\tau}\Delta_{abc}(u_1+u_2+u_3) \ &+\ 2\,\Delta_{abc\Omega\Omega}(u_1+u_2+u_3) \\ &= \sum_{e,f} \pm\widetilde{\mathcal{T}}_{abd\bar{e}}(u_1+u_2+u_3, u_1-\tilde{u}_1)\,\Delta_{dce\Omega}(u_2+u_3+u_1)\ , \end{aligned} \tag{41}$$

with the understanding that we need to take the limit $\tilde{u}_1 \to u_1$. Converting the Ω-insertions into u-derivatives, we see that the LHS of this equation indeed coincides with the heat equation, provided the RHS vanishes. By inserting the explicit expressions for the correlators, it however turns out that the RHS does not vanish. One may be tempted to make use of homotopy transformations of the form (21) to remove it, but by their nature as theta-functions they cannot cancel the singularity of the trapezoidal instanton sum, *i.e.*, the Appell function. Thus, by presenting a counter-example, we conclude that the Cardy constraint (39) does not hold for the elliptic curve, and specifically that the heat equation (40) is not implied by it.

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